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# DLR equations for the superstable Bose gas at any temperature and activity

G. BELLOT,<sup>\*†</sup> D. DEREUDRE<sup>\*‡</sup> AND M. MAÏDA<sup>\*§</sup>

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## Abstract

We construct a thermodynamic limit for the grand canonical Bose gas (in its Feynman-Kac representation) with superstable interaction. Although we do not prove the presence of infinite cycles, our infinite volume model is naturally a distribution over configurations of finite loops and interlacements. We prove the limiting process to be solution of DLR equations. We will work within the framework of Dirichlet and periodic boundary conditions, for any inverse temperature  $\beta > 0$ , chemical potential  $\mu \in \mathbb{R}$  and dimension  $d \geq 1$ .

**Keywords:** *Gibbs point process, thermodynamic limit, entropy, random permutations, interlacements*

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## 1 Introduction and results

Since the celebrated lesson of Ginibre in Les Houches [Gin70], it has been formally established that, under broad hypotheses, a canonical ensemble of  $N$  bosons in a domain  $\Lambda \subset \mathbb{R}^d$  at thermal equilibrium at inverse temperature  $\beta > 0$  interacting through a potential  $U : \Lambda^N \rightarrow \mathbb{R} \cup \{+\infty\}$  can be modeled as a point process. More precisely, if we denote as  $\mathbb{S}_N$  the  $N$ 'th permutation group and  $W_{x,y}^{\beta, \mathcal{C}\Lambda}$  the un-normalized Wiener measure over Brownian bridges  $w : [0; \beta] \rightarrow \Lambda$  going from  $x$  to  $y$  then the function

$$(x_1 \dots x_N) \mapsto \sum_{\sigma \in \mathbb{S}_N} \int \left( \bigotimes_{i=1}^n W_{x_i, x_{\sigma(i)}}^{\beta, \mathcal{C}\Lambda}(dw_i) \right) \exp \left[ - \int_0^\beta U(w_1(s) \dots w_N(s)) ds \right]$$

is proportional to the square modulus of the Bose gas wave function. This is the Feynman-Kac formula in the case of Dirichlet boundary condition. By extension, the process sampling the interacting Brownian bridges  $w_1 \dots w_N$  realizing the permutation  $\sigma$  is called the Feynman-Kac (FK) representation of the Bose gas.

For technical reasons, like some other authors in the literature ([AFY21], [Vog23], *etc*), we will prefer studying the grand canonical ensemble, where the number of points in the point process is random, and the average density is controlled by a parameter  $\mu \in \mathbb{R}$  (the chemical potential).

A standard approach to the study of the Bose gas from point processes consists in taking its thermodynamic limit, which means inflating the domain grow to infinity  $\Lambda \uparrow \mathbb{R}^d$ . One may then hope to deduce facts about the physical Bose gas from the properties of this infinite volume model. In particular, the community is most interested in proving when and how much the Bose gas undergoes Bose-Einstein condensation (BEC). In the non-interacting case ( $U = 0$ ), a lot of this work has already been done and the literature provides a good picture of the free Bose gas. Sütö ([Süt93], [Süt02]) has proven Feynman's conjecture [Fey53] that the emergence of infinite cycles in the permutation (interlacements) in infinite volume is equivalent to BEC. Later, it has then been proven in various ways and frameworks ([BU09], [AFY21], [Vog23]) that these interlacements appear through a saturation effect, beyond some critical density  $\rho_c > 0$ .

The publications of Sütö proved the equivalence of interlacements and BEC for the free and mean-field case, but it is expected to fail in the non-ideal gas. Nonetheless, figuring out whether or not the thermodynamic limit samples infinite cycles is considered an interesting question. First, interlacements do not only appear in Bose gas models and are a more general notion [Szn09]. But more importantly, understanding exactly how Feynman's prediction fails necessitates investigating both when BEC and interlacements happen.

Of course the community has been interested with the more realistic picture of the interacting Bose gas but it is much more difficult than the free case. Even taking the thermodynamic

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limit – although it really is the very first step – is not trivial. Several authors have investigated general interacting Bose gases with various strategies ([ACK11], [SKS20], [BV23]). But drawing conclusive results becomes significantly harder at high densities. This is why some ingenious approximations were proposed in the literature ([CJK22], [DV24]) to approach the behavior of the true interacting Bose gas and provide some precious insight.

We believe a major reason why the general approach had trouble at high densities is due to its usual *loop soup* formulation. Indeed, some authors ([AFY21], [BV23], [DV24], *etc*) prefer sampling the Brownian cycles entirely instead of generating the points and the permutation beforehand, and only then drawing the Brownian bridges. One may get rid of that complicated procedure and directly sample the whole loops as marks of a point process. Although much more mathematically elegant, this method has a drawback: loops are fundamentally non-local objects. At high enough densities, they are expected to become very large and this comes at odds with the desire to control their behavior. Indeed, the widely used [GZ93]’s theorem only states that the integral of *local* functionals is preserved by taking the limit. Furthermore, in the event that one really could control those loops and conclude to an infinite volume model, this limit model would only sample finite loops, by construction. This will be illustrated in section 2.1. Of course this broad description should be amended. Vogel [Vog23] for example used a loop soup model to conclude to the presence of interlacements at the thermodynamic limit in the free case.

As titled, the major assumption we will make on the interaction between bosons is its super-stability, which is a very standard hypothesis. For the *classical* superstable gas, Ruelle originally provided some probability estimates and general results ([Rue70]). Park successfully extended these bounds on the point-wise part of the quantum bosonic gas several decades ago ([Par84], [Par85]). In our work, we were able to preserve the cycle structure in our thermodynamic limit, not just the point-wise part. Our setting includes every  $\beta > 0$  and  $\mu \in \mathbb{R}$ , which could potentially allow the Bose gas to reach any density. Therefore, our results provide some hope to eventually detect infinite cycles in infinite volume at high enough densities.

Our results are stated in the well-known (FK) representation. As it is not very tractable, we constructed an original *marked point* view of the Bose gas which serves as an encoding of the (FK) representation. Even though it does not appear in the title, nor in the results, we believe our main contribution really is this encoding. It turns out this idea of using a custom marked point process to model the Bose gas was first thought of by Georgii and mentioned in private exchanges. Our new formulation, although convoluted, was tailor-made for the crucial advantage to have a uniformly bounded entropy along some Poisson point process. This allows the use of standard entropic tools and the encoded Bose gas does go through a thermodynamic limit. Then we decode the newly defined infinite volume model and prove it really is the thermodynamic limit of the original (FK) model, thanks to the encoding being local-wise. Our approach is purely probabilistic and does not use operator theory.

Once the thermodynamic limit is stated in its full generality, we establish DLR equations with little additional work. More precisely, the difficulty to write those DLR equations is due to the nature of the sampled objects (curves instead of points) rather than the range of the interaction, as it is assumed finite (we only add this finite range hypothesis for the DLR equations).

## 1.1 Interaction and assumptions

**Definition 1.1.1** — *Let us denote the set of finite point configurations in  $\mathbb{R}^d$  ( $d \geq 1$ ) as*

$$\text{Conf}_{<\infty} := \left\{ \xi \subset \mathbb{R}^d \mid \#\xi < +\infty \right\}$$

---

where  $\#\xi$  is the cardinal of  $\xi$ . We equip this space with the smallest  $\sigma$ -algebra  $\mathcal{C}_{<\infty}$  making measurable the maps  $\xi \mapsto \#(\xi \cap E)$  for every Borelian  $E \subseteq \mathbb{R}^d$ .

Let  $U : \text{Conf}_{<\infty} \rightarrow \mathbb{R} \cup \{+\infty\}$  measurable be an interaction.

**Hypothesis I** — We assume the interaction to be non-degenerate  $U(\emptyset) < +\infty$ .

**Hypothesis II** — We assume the interaction to be superstable [Rue70], that is to say there exist  $A \geq 0$ ,  $B > 0$  and  $r > 0$  such that

$$\forall \xi \in \text{Conf}_{<\infty}, U(\xi) \geq -A\#\xi + B \sum_{z \in r\mathbb{Z}^d} \#[\xi \cap (z + \Lambda_r)]^2$$

where  $\Lambda_r := [-r/2; r/2]^d$ .

**REMARK 1.1.2**

Superstability is a very standard assumption for an interaction. We refer to Proposition 1.2 from [Rue70] for criteria for a pairwise interaction to be superstable.

If the interaction is superstable for some constants  $(A, B, r)$  then it is not true it will satisfy superstability's inequality for  $(A, B, r')$  for all  $0 < r' < r$ . But we were not able to find interesting counterexamples. In the following pages, we will consider domains  $\Lambda_L = [-L/2; L/2]^d$  only for  $L$  multiple of  $r$  but this really is a technical artifact rather than a limitation.

We are going to investigate the Bose gas in Dirichlet (Dir) and periodic (per) boundary conditions. An indeterminate boundary condition among those two will be denoted (bc).

The boundary condition we work in will have consequences on the interaction we do calculations with. In Dirichlet boundary condition, we need to add an infinite exterior potential restraining the particles inside the domain  $\Lambda_L$ , while in periodic boundary condition, the interaction needs to be periodized. We define below the relevant adaptations of the interaction.

Our conclusions still hold for several other boundary conditions, including Neumann's. But we think presenting our results in full generality would have damaged readability without enriching significantly the theorems. We believe presenting the proofs for both Dirichlet and periodic boundary conditions already paints a complete enough picture.

**Definition 1.1.3** — Let  $L \in r\mathbb{N}$ . We define  $U_{(\text{bc}),L} : \text{Conf}_{<\infty} \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$U_{(\text{Dir}),L}(\xi) := \begin{cases} U(\xi) & \text{if } \xi \subset \Lambda_L \\ +\infty & \text{otherwise} \end{cases}$$

$$U_{(\text{per}),L}(\xi) := \lim_{K \rightarrow +\infty} \frac{1}{(2K+1)^d} U \left[ \bigcup_{k \in \mathbb{Z}^d} (\xi + Lk) \cap \Lambda_{(2K+1)L} \right]$$

if the limit exists in  $\mathbb{R} \cup \{+\infty\}$ .

---

**Hypothesis III** — *If the boundary condition is periodic, we assume the limit*

$$\lim_{K \rightarrow +\infty} \frac{1}{(2K+1)^d} U \left[ \bigcup_{k \in \mathbb{Z}^d} (\xi + Lk) \cap \Lambda_{(2K+1)L} \right] \in \mathbb{R} \cup \{+\infty\}$$

to exist for any  $L \in r\mathbb{N}$  and any  $\xi \in \text{Conf}_{<\infty}$ .

It is not obvious this way of defining  $U_{(\text{per}),L}$  in full generality is relevant. The following proposition helps to support its validity.

**Proposition 1.1.4** — *Let us assume the interaction is pairwise, that is to say there exists a radial potential  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$  such that*

$$\forall \xi \in \text{Conf}_{<\infty}, U(\xi) = \sum_{\substack{x,y \in \xi \\ x \neq y}} \Phi(x-y).$$

If there exists a decreasing function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that

$$\int_0^{+\infty} r^{d-1} \psi(r) dr < +\infty \quad \text{and} \quad \forall x \in \mathbb{R}^d, |\Phi(x)| \leq \psi(|x|)$$

then for any  $L > 0$  the periodic version of the interaction  $U_{(\text{per}),L}$  is well defined. Furthermore, it is pairwise up to a linear term

$$U_{(\text{per}),L}(\xi) = \sum_{\substack{x,y \in \xi \\ x \neq y}} \Phi_{(\text{per}),L}(x-y) + \#\xi [\Phi_{(\text{per}),L}(0) - \Phi(0)]$$

where

$$\Phi_{(\text{per}),L}(x) := \sum_{k \in \mathbb{Z}^d} \Phi(x + Lk).$$

The hypotheses of this subsection will always be considered throughout the following pages and not stated anymore. Similarly, the numbers  $A \geq 0$ ,  $B > 0$  and  $r > 0$  will always refer to the superstability constants of hypothesis II. Other assumptions will be added when needed.

## 1.2 Feynman-Kac state space

**Definition 1.2.1** — *For any  $t > 0$ , we denote as  $\Omega_t$  the set of continuous functions from  $[0; t]$  to  $\mathbb{R}^d$ . This set is equipped with the topology  $\mathcal{W}_t$  associated to the uniform norm  $\|\cdot\|_\infty$ . We denote as  $\mathcal{W}_t$  the associated Borel  $\sigma$ -algebra.*

*On the set  $\Omega_t$ , we consider the Wiener measure  $W_{x,y}^t$  weighing Brownian bridges going from  $x$  to  $y$  in time  $t$ , whose finite-dimensional distributions are given by*

$$dW_{x,y}^t[\omega(s_1) = z_1 \dots \omega(s_n) = z_n] = \prod_{i=0}^n [2\pi(s_{i+1} - s_i)]^{-d/2} \exp\left[-\frac{\|z_{i+1} - z_i\|^2}{2(s_{i+1} - s_i)}\right] dz_1 \dots dz_n$$

---

with  $0 = s_0 < \dots < s_{n+1} = t$  and the convention  $z_0 = x$  and  $z_{n+1} = y$ . This measure is unnormalized, as

$$W_{x,y}^t(\Omega_t) = (2\pi t)^{-d/2} \exp\left(-\frac{1}{2t} \|y - x\|^2\right).$$

For any  $L > 0$  multiple of  $r$ , we define a version of the Wiener measure for both boundary conditions on  $\Lambda_L$

$$W_{x,y,(\text{Dir}),L}^t := W_{x,y}^t \quad \text{and} \quad W_{x,y,(\text{per}),L}^t := \sum_{k \in \mathbb{Z}^d} W_{x,y+Lk}^t.$$

Since we are studying the Bose gas at thermal equilibrium at inverse temperature  $\beta > 0$ , we will be focusing on the  $t = \beta$  case. The parameter  $\beta > 0$  is fixed in the sequel and omitted in most notations.

**Definition 1.2.2** — We denote the set of Feynman-Kac configurations (FK) as

$$\text{Conf}^{(\text{FK})} := \left\{ \gamma \subset \Omega_\beta \mid \gamma \text{ is locally finite for } \mathbb{W}_\beta \right\}$$

and we equip this configuration set with the smallest  $\sigma$ -algebra  $\mathcal{C}^{(\text{FK})}$  making measurable the maps  $\gamma \mapsto \#(\gamma \cap E)$  for every bounded  $E \in \mathcal{W}_\beta$ .

**Definition 1.2.3** — A Feynman-Kac configuration  $\gamma \in \text{Conf}^{(\text{FK})}$  is said to be permutation-wise if the following condition is satisfied

$$\forall w \in \gamma, \begin{cases} \exists! w' \in \gamma, w'(0) = w(\beta) \\ \exists! w'' \in \gamma, w(0) = w''(\beta). \end{cases}$$

We denote

$$\text{ConfPerm}^{(\text{FK})} := \left\{ \gamma \in \text{Conf}^{(\text{FK})} \mid \gamma \text{ is permutation-wise} \right\}.$$

For any  $\gamma \in \text{ConfPerm}^{(\text{FK})}$ , we define the permutation

$$\sigma^{(\text{FK})}(\gamma, \cdot): \begin{array}{ccc} \gamma & \longrightarrow & \gamma \\ w & \longmapsto & w' \text{ such that } w'(0) = w(\beta). \end{array}$$

In finite volume, all probability measures will be defined so that they are supported on permutation-wise configurations. We will later prove the thermodynamic limits to have the same property.

### 1.3 Thermodynamic limit

**Definition 1.3.1** — For each boundary condition, we define a Hamiltonian over finite configurations  $\gamma \in \text{Conf}^{(\text{FK})}$  with

$$H_{(\text{bc}),L}^{(\text{FK})}(\gamma) := \int_0^\beta U_{(\text{bc}),L}[\{w(s), w \in \gamma\}] ds \in \mathbb{R} \cup \{+\infty\}.$$

---

We will justify in remark 3.1.3 this integral is well defined.

**Definition 1.3.2** — For any compact  $\Delta \subset \mathbb{R}^d$ , we denote as  $\Pi_\Delta$  the Poisson point process over  $\Delta$  with intensity 1.

We introduce the Feynman-Kac representation of the grand canonical Bose gas interacting through the potential  $U$  at inverse temperature  $\beta > 0$  and chemical potential  $\mu \in \mathbb{R}$ . The parameter  $\mu$  controls the average density of the Bose gas at some given temperature. Just like  $\beta$ , the parameter  $\mu$  is fixed in the sequel and omitted in the notations.

**Lemma-definition 1.3.3 (proof: see remark 3.1.4)** — Let  $L \in r\mathbb{N}$ ,  $\beta > 0$  and  $\mu \in \mathbb{R}$ .

In both boundary conditions, there exists a well-defined probability measure  $\mathbb{P}_{(bc),L}^{(FK)}$  over  $\text{Conf}^{(FK)}$  which satisfies for any measurable  $f : \text{Conf}^{(FK)} \rightarrow \mathbb{R}^+$  the following equality

$$\int f d\mathbb{P}_{(bc),L}^{(FK)} := \frac{1}{Z_{(bc),L}} \int e^{\beta\mu\#\xi} \Pi_{\Lambda_L}(d\xi) \sum_{\sigma \in \mathbb{S}(\xi)} \int e^{-H_{(bc),L}^{(FK)}(\gamma)} f(\gamma_{(bc),L}) \left( \bigotimes_{x \in \xi} W_{x,\sigma(x),(bc),L}^\beta \right) (d\gamma)$$

where  $\mathbb{S}(\xi)$  is the set of permutations over  $\xi$  and

$$\gamma_{(\text{Dir}),L} := \gamma \quad \gamma_{(\text{per}),L} := \bigcup_{k \in \mathbb{Z}^d} (\gamma + Lk)$$

with

$$Z_{(bc),L} := \int e^{\beta\mu\#\xi} \Pi_{\Lambda_L}(d\xi) \sum_{\sigma \in \mathbb{S}(\xi)} \int e^{-H_{(bc),L}^{(FK)}(\gamma)} \left( \bigotimes_{x \in \xi} W_{x,\sigma(x),(bc),L}^\beta \right) (d\gamma).$$

We illustrate for each boundary condition the kind of configurations sampled by  $\mathbb{P}_{(bc),L}^{(FK)}$  in fig. 1. The points of  $\xi$  are indicated with “•” and we represent smooth trajectories instead of Brownian bridges, both for readability purposes and by lack of expertise in the usage of TikZ. We also did not represent bridges that intersect each others because, although common, this situation could make the drawings less clear.

**REMARK 1.3.4**

By construction, it is clear that  $\mathbb{P}_{(bc),L}^{(FK)}(\text{ConfPerm}^{(FK)}) = 1$ .

By periodicity of the (FK) configuration under  $\mathbb{P}_{(\text{per}),L}^{(FK)}$ , outside of degenerate cases (like the interaction  $U_{(\text{per}),L}$  only authorizing empty configurations), there is a positive probability for the permutation  $\sigma^{(FK)}(\gamma, \cdot)$  to include infinite cycles. This situation is illustrated in fig. 1b. This is amusing as the sampled  $\sigma \in \mathbb{S}(\xi)$  is only made up of finite cycles. Unfortunately, we have not found a way to conclude to the presence of infinite cycles in infinite volume from this fact.



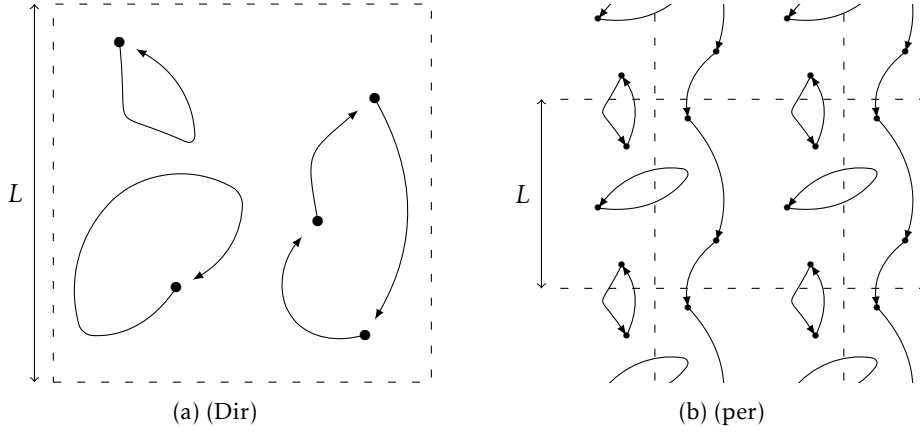


Figure 1: Example of (FK) configurations

**Definition 1.3.5** — For any  $L \in r\mathbb{N}$ , we define the empirical field  $\widetilde{\mathbb{P}}_{(bc),L}^{(FK)}$  over  $\text{Conf}^{(FK)}$  by

$$\int f d\widetilde{\mathbb{P}}_{(bc),L}^{(FK)} := \frac{1}{L^d} \int_{\Lambda_L} dv \int f(\gamma + v) \mathbb{P}_{(bc),L}^{(FK)}(d\gamma)$$

for any measurable  $f : \text{Conf}^{(FK)} \rightarrow \mathbb{R}^+$ .

One should think the empirical field as a partially stationarized version of the probability  $\mathbb{P}_{(bc),L}^{(FK)}$ . In the case of periodic boundary conditions, by stationarity of the original model,

$$\widetilde{\mathbb{P}}_{(per),L}^{(FK)} = \mathbb{P}_{(per),L}^{(FK)}.$$

According to [GZ93], the functionals whose integral is compatible with the thermodynamic limit are *local* functionals. For marked point processes, locality just means the function can only depend on points inside some compact. But in our setting this is not so simple.

**Definition 1.3.6** — Let  $\Delta \subset \mathbb{R}^d$  be a compact. We define the following projections  $\text{Conf}^{(FK)} \rightarrow \text{Conf}^{(FK)}$

$$\begin{aligned} \text{Proj}_{\in\Delta}(\gamma) &:= \{w \in \gamma / w(0) \in \Delta\} \\ \text{Proj}_{\cap\Delta}(\gamma) &:= \{w \in \gamma / w \cap \Delta \neq \emptyset\}. \end{aligned}$$

For any integer  $n \geq 1$ , we also define

$$\begin{aligned} \text{Proj}_{\cap_+^n \Delta}(\gamma) &:= \left\{ w \in \gamma \left/ \begin{array}{l} \exists k \in \llbracket 1; n \rrbracket, \\ \exists w_0 \dots w_{k-1} \in \gamma \\ w_k = w \end{array} \right. , \begin{array}{l} w_0 \cap \Delta \neq \emptyset \\ w_k \cap \Delta = \emptyset \\ \forall i \in \llbracket 0; k-1 \rrbracket, w_i(\beta) = w_{i+1}(0) \end{array} \right\} \\ \text{Proj}_{\cap_-^n \Delta}(\gamma) &:= \left\{ w \in \gamma \left/ \begin{array}{l} \exists k \in \llbracket 1; n \rrbracket, \\ \exists w_0 \dots w_{k-1} \in \gamma \\ w_k = w \end{array} \right. , \begin{array}{l} w_0 \cap \Delta \neq \emptyset \\ w_k \cap \Delta = \emptyset \\ \forall i \in \llbracket 0; k-1 \rrbracket, w_{i+1}(\beta) = w_i(0) \end{array} \right\}. \end{aligned}$$

Finally, for any  $n \geq 0$  we define

$$\text{Proj}_{\cap^n \Delta}(\gamma) := \begin{cases} \text{Proj}_{\cap \Delta}(\gamma) & \text{if } n = 0 \\ \text{Proj}_{\cap \Delta}(\gamma) \cup \text{Proj}_{\cap_+^n \Delta}(\gamma) \cup \text{Proj}_{\cap_-^n \Delta}(\gamma) & \text{if } n \geq 1. \end{cases}$$

The following notions of locality ensue.

**Definition 1.3.7** — A function  $f$  defined over  $\text{Conf}^{(\text{FK})}$  is said to be  $\in$ -local if there exists a compact  $\Delta \subset \mathbb{R}^d$  such that

$$\text{Proj}_{\in \Delta}(\gamma) = \text{Proj}_{\in \Delta}(\gamma') \implies f(\gamma) = f(\gamma').$$

A function  $f$  defined over  $\text{ConfPerm}^{(\text{FK})}$  with the same property will also be called  $\in$ -local.

Let  $n \geq 0$ . Similarly, a function  $f$  defined over  $\text{Conf}^{(\text{FK})}$  is said to be  $\cap^n$ -local if there exists a compact  $\Delta \subset \mathbb{R}^d$  such that

$$\text{Proj}_{\cap^n \Delta}(\gamma) = \text{Proj}_{\cap^n \Delta}(\gamma') \implies f(\gamma) = f(\gamma').$$

We also need to define some notions of tameness which are compatible with our respective definitions of locality.

**Definition 1.3.8** — For any trajectory  $w \in \Omega_\beta$  and  $\delta > 0$ , we define its Wiener sausage of thickness  $\delta$  as

$$\text{Saus}_\delta(w) := \left\{ x \in \mathbb{R}^d \mid \exists s \in [0; \beta], \|x - w(s)\| \leq \delta \right\}$$

whose volume we denote  $|\text{Saus}_\delta(w)|$ .

A function  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  is said to be  $\in$ -tame if there exists  $a, \delta > 0$ ,  $\alpha \in ]0; 2[$ , and a compact  $\Delta \subset \mathbb{R}^d$  such that for any  $\gamma \in \text{Conf}^{(\text{FK})}$ ,

$$a|f(\gamma)| \leq 1 + \sum_{w \in \text{Proj}_{\in \Delta}(\gamma)} |\text{Saus}_\delta(w)|^\alpha.$$

A function  $f : \text{ConfPerm}^{(\text{FK})} \rightarrow \mathbb{R}$  with the analogous property will also be called  $\in$ -tame.

Let  $n \geq 0$ . A function  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  is said to be  $\cap^n$ -tame if there exists  $a, \delta > 0$ ,  $\alpha \in ]0; 2[$ , and a compact  $\Delta \subset \mathbb{R}^d$  such that for any  $\gamma \in \text{Conf}^{(\text{FK})}$ ,

$$a|f(\gamma)| \leq 1 + \sum_{w \in \text{Proj}_{\cap^n \Delta}(\gamma)} |\text{Saus}_\delta(w)|^\alpha.$$

A function  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  is said to be  $\cap^n$ -regular if there exists  $a, \delta > 0$ ,  $\alpha \in ]0; 1[$ , and a compact  $\Delta \subset \mathbb{R}^d$  such that for any  $\gamma \in \text{ConfPerm}^{(\text{FK})}$  and  $\gamma' \subseteq \gamma$ ,

$$\begin{aligned} a|f(\gamma) - f(\gamma \setminus \gamma')| &\leq \sum_{w \in \text{Proj}_{\in \Delta}(\gamma) \cap \gamma'} |\text{Saus}_\delta(w)|^{1+\alpha} + \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma) \cap \gamma'} |\text{Saus}_\delta(w)|^\alpha \\ &+ \sum_{w \in \text{Proj}_{\cap_+^n \Delta}(\gamma) \cap \gamma'} \mathbb{1}_{\sigma^{(\text{FK})}(\gamma, \cdot)^{-1}(w) \notin \gamma'} |\text{Saus}_\delta(w)|^{1+\alpha} \\ &+ \sum_{w \in \text{Proj}_{\cap_-^n \Delta}(\gamma) \cap \gamma'} \mathbb{1}_{\sigma^{(\text{FK})}(\gamma, w) \notin \gamma'} |\text{Saus}_\delta(w)|^{1+\alpha}. \end{aligned}$$

---

**REMARK 1.3.9**

A function  $f$  being  $\cap^n$ -regular should be thought of as some kind of Lipschitz property for  $f$ ; if the configuration  $\gamma$  is slightly modified then the value  $f(\gamma)$  only marginally changes.

**EXAMPLE 1.3.10**

We provide a few examples of functions which are local, tame or regular in various ways.

- $f_1 : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  defined by

$$f_1(\gamma) = \sum_{w \in \gamma, w(0) \in [0;1]^d} \|w(\beta) - w(0)\|$$

is  $\in$ -local and  $\in$ -tame.

Locality is clear. Tameness comes from the following fact: a cylinder whose axis goes from  $w(0)$  to  $w(\beta)$  with radius  $\delta$  has a smaller volume than  $\text{Saus}_\delta(w)$ . Therefore

$$c_{d-1} \delta^{d-1} \|w(\beta) - w(0)\| \leq |\text{Saus}_\delta(w)|$$

where  $c_{d-1}$  is the volume of a  $d - 1$  dimensional unit ball.

- $f_2 : \text{ConfPerm}^{(\text{FK})} \rightarrow \mathbb{R}$  defined by

$$f_2(\gamma) = \sum_{w \in \gamma, w \subset [0;1]^d} \frac{1}{\inf\left\{j \geq 1 \mid \left[\sigma^{(\text{FK})}(\gamma, \cdot)\right]^j(w) = w\right\}} \mathbb{1}_{\forall j \in \mathbb{Z}, [\sigma^{(\text{FK})}(\gamma, \cdot)]^j(w) \subset [0;1]^d}$$

is  $\in$ -local and  $\in$ -tame.

Indeed, the function  $f_2$  counts the number of disjoint cycles in the cycle structure of  $\gamma$  which are completely included inside  $[0;1]^d$ . Locality is then intuitive. Tameness is clear because  $f_2(\gamma) \leq \#\text{Proj}_{\in[0;1]^d}(\gamma)$ .

- $f_3 : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  defined by

$$f_3(\gamma) = \#\left\{w \in \gamma \mid w \cap [0;1]^d \neq \emptyset\right\} \cdot \mathbb{1}_{\#\{w \in \gamma \mid w \cap [0;1]^d \neq \emptyset\} \text{ is even}}$$

is  $\cap^0$ -local and  $\cap^0$ -tame, but not  $\cap^n$ -regular for any  $n \geq 0$ .

There is no hope of regularity because the variations of  $f_3$  can be arbitrarily large.

- $f_4 : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  defined by

$$f_4(\gamma) = \#\left\{w \in \text{Proj}_{\in[0;1]^d}(\gamma) \mid \exists w' \in \gamma, \begin{array}{l} w'(0) = w(\beta) \\ w'(\beta) = w(0) \end{array} \right\}$$

is  $\cap^1$ -local,  $\in$ -tame and  $\cap^1$ -regular.

If  $\gamma \in \text{ConfPerm}^{(\text{FK})}$ , then  $f_4(\gamma)$  counts the number of bridges  $w \in \gamma$  starting in  $[0;1]^d$  which are part of a cycle of length 1 or 2.

---

None of the examples above are bounded. So we will state the thermodynamic limit in the most general possible way, without restricting ourselves to bounded functions. This will allow us to state corollary 1.3.15.

**REMARK 1.3.11**

If a function  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  is  $\cap^n$ -regular, then its restriction to  $\text{ConfPerm}^{(\text{FK})}$  is  $\cap^n$ -local because by definition

$$\forall \gamma \in \text{ConfPerm}^{(\text{FK})}, |f(\gamma) - f[\text{Proj}_{\cap^n \Delta}(\gamma)]| = 0.$$

**Theorem 1.3.12 (Thermodynamic limit, proof pp. 48,55)**

Under hypotheses I to III, for any inverse temperature  $\beta > 0$ , chemical potential  $\mu \in \mathbb{R}$ , boundary condition  $(\text{bc}) = (\text{Dir})$  or  $(\text{per})$ , there exists a stationary probability measure  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}$  over  $\text{Conf}^{(\text{FK})}$  and an increasing sequence  $(L_m)_{m \geq 0}$  of multiples of  $r$  such that,

- for any measurable  $f : \text{ConfPerm}^{(\text{FK})} \rightarrow \mathbb{R}$  which is  $\in$ -local and  $\in$ -tame
- for any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  which is  $\cap^n$ -regular and  $\cap^n$ -tame for some  $n \geq 0$

then

$$\lim_{m \rightarrow +\infty} \int f d\widetilde{\mathbb{P}}_{(\text{bc}),L_m}^{(\text{FK})} = \int f d\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}.$$

In the following, we will abbreviate this fact as

$$\lim_{L \rightarrow +\infty} \int f d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})} = \int f d\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}.$$

**REMARK 1.3.13**

The first part of theorem 1.3.12 concerns functions defined on  $\text{ConfPerm}^{(\text{FK})}$ . This is not a limitation, quite the contrary. Any function defined on the whole  $\text{Conf}^{(\text{FK})}$  can be restricted to  $\text{ConfPerm}^{(\text{FK})}$ . But the reverse is not as trivial: extending  $f_2$  from example 1.3.10 to the whole configuration space while keeping locality and tameness would be unpleasant.

The second part of the theorem unfortunately forces us to go through this obnoxious exercise. This is why we defined  $f_4$  from example 1.3.10 on the entire  $\text{Conf}^{(\text{FK})}$ .

**Proposition 1.3.14 (proof: see remark 3.5.2)** —  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(\text{ConfPerm}^{(\text{FK})}) = 1.$

The event “ $\gamma \in \text{ConfPerm}^{(\text{FK})}$ ” is not local in any of the senses from definition 1.3.7 so this is not a trivial result.

Unfortunately, we were not able to prove the induced permutation  $\sigma^{(\text{FK})}(\gamma, \cdot)$  to comprise infinite cycles at low enough temperature (or high enough chemical potential). But we believe our construction naturally includes this possibility, because it is a local to global construction, rather than a cycle-wise definition like in loop soup models.

---

**Corollary 1.3.15** — *The proportion of Brownian bridges which are part of an infinite cycle in the limiting process is the limiting proportion of bridges which are part of an arbitrarily large cycle in finite volume. More precisely,*

$$\begin{aligned} & \int \#\left\{w \in \text{Proj}_{\in[0;1]^d}(\gamma) \mid \forall j \geq 1, [\sigma^{(\text{FK})}(\gamma, \cdot)]^j(w) \neq w\right\} \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(d\gamma) \\ &= \lim_{n \rightarrow +\infty} \lim_{L \rightarrow +\infty} \int \#\left\{w \in \text{Proj}_{\in[0;1]^d}(\gamma) \mid \forall j \in \llbracket 1; n \rrbracket, [\sigma^{(\text{FK})}(\gamma, \cdot)]^j(w) \neq w\right\} \widehat{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(d\gamma). \end{aligned}$$

To prove this corollary, we apply the second half of theorem 1.3.12 to the functions  $f_n : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$ ,  $n \geq 1$ , defined by

$$\begin{aligned} f_n(\gamma) = & \#\text{Proj}_{\in[0;1]^d}(\gamma) \\ & - \#\left\{w \in \text{Proj}_{\in[0;1]^d}(\gamma) \mid \exists j \in \llbracket 1; n \rrbracket, \exists w_0 \dots w_j \in \gamma, \forall i \in \llbracket 0; j-1 \rrbracket, w_{i+1}(0) = w_i(\beta)\right\} \end{aligned}$$

which makes it  $\cap^n$ -regular and  $\in$ -tame.

On  $\text{ConfPerm}^{(\text{FK})}$ , the quantity  $f_n(\gamma)$  is the number of bridges starting in  $[0;1]^d$  which are part of a cycle of length larger than  $n$ .

## 1.4 DLR equations

In the previous section, we stated the existence of an infinite volume model  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}$  for each boundary condition. Since this probability measure is a thermodynamic limit, we can hope to calculate the probability of some events as limits, but we did not provide any information on this infinite volume distribution itself, apart from proposition 1.3.14. Unfortunately, it is not possible to describe the probability  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}$  as simply as we did in finite volume in lemma-definition 1.3.3, because an infinite volume Hamiltonian would always value any infinite configuration to an infinite energetic cost. The solution is to write DLR (Dobrushin-Lanford-Ruelle) equations, that is to say, write the conditional distribution of the infinite configuration inside some compact  $\Delta$ , given the configuration outside  $\Delta$ .

**Hypothesis IV** — *We assume the interaction to be stationary*

$$\forall \xi \in \text{Conf}_{<\infty}, \forall v \in \mathbb{R}^d, U(\xi + v) = U(\xi).$$

**Hypothesis V** — *We assume the interaction to be finite range with range  $R > 0$ , that is to say for any compact  $\Delta \subset \mathbb{R}^d$ , there exists a local interaction  $U_\Delta : \text{Conf}_{<\infty} \rightarrow \mathbb{R} \cup \{+\infty\}$  such that*

$$\forall \xi \in \text{Conf}_{<\infty}, U(\xi) = U_\Delta[\xi \cap (\Delta + \Lambda_R)] + U(\xi \cap \Delta^c)$$

where  $\Delta + \Lambda_R$  is the Minkowski sum of those two sets.

The existence of such a  $U_\Delta$  implies heredity of the original interaction  $U$ .

---

**Proposition 1.4.1** — *Let us assume the interaction is pairwise (see proposition 1.1.4) with potential  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  such that*

$$\forall x \in \mathbb{R}^d, \|x\| > R \implies \Phi(x) = 0.$$

*Then the interaction is finite range with range  $R$  and for any compact  $\Delta \subset \mathbb{R}^d$ ,*

$$\forall \xi \in \text{Conf}_{<\infty}, U_\Delta(\xi) = \sum_{\substack{x,y \in \xi \cap \Delta \\ x \neq y}} \Phi(x-y) + 2 \sum_{x \in \xi \cap \Delta} \sum_{y \in \xi \cap [(\Delta + \Lambda_R) \setminus \Delta]} \Phi(x-y).$$

**Hypothesis VI** — *We assume the interaction to be uniformly regular from below, that is to say, for any compact  $\Delta \subset \mathbb{R}^d$  and integer  $N \geq 0$ , there exists  $C_{\Delta,N} \in \mathbb{R}$  such that*

$$\forall \xi \in \text{Conf}_{<\infty}, \#[\xi \cap (\Delta + \Lambda_R)] = N \implies U(\xi) \geq U(\xi \cap \Delta^c) + C_{\Delta,N}.$$

In other words, we assume the energetic value of a configuration  $\xi$  (with a given number of points close to  $\Delta$ ) not to decrease by an arbitrarily large value when adding a given number of points inside  $\Delta$ .

This could probably be guaranteed by some weak regularity criteria (hence the name of hypothesis VI) because we were not able to find an example of interaction which would check hypotheses I to V but not the VI'th. A finite range pairwise interaction satisfies the hypothesis as long as the potential is bounded from below, which is guaranteed by superstability.

**REMARK 1.4.2 (PROOF P. 58)**

Thanks to hypothesis VI, we can assume without any loss of generality that for any compact  $\Delta \subset \mathbb{R}^d$  and integer  $N \geq 0$ ,

$$\forall \xi \in \text{Conf}_{<\infty}, \#[\xi \cap (\Delta + \Lambda_R)] = N \implies U_\Delta[\xi \cap (\Delta + \Lambda_R)] \geq C_{\Delta,N}.$$

**Definition 1.4.3** — *For any compact  $\Delta \subset \mathbb{R}^d$ , we define the local Hamiltonian  $H_\Delta^{(\text{FK})}$  over*

$$\left\{ \gamma \in \text{Conf}^{(\text{FK})} \mid \sup_{s \in [0;\beta]} \#[\{w(s), w \in \gamma\} \cap (\Delta + \Lambda_R)] < +\infty \right\}$$

by

$$H_\Delta^{(\text{FK})}(\gamma) = \int_0^\beta U_\Delta[\{w(s), w \in \gamma\} \cap (\Delta + \Lambda_R)] ds \in \mathbb{R} \cup \{+\infty\}.$$

According to remark 1.4.2, the integrand is bounded from below by  $\min_{0 \leq k \leq N} C_{\Delta,k}$  where

$$N = \sup_{s \in [0;\beta]} \#[\{w(s), w \in \gamma\} \cap (\Delta + \Lambda_R)]$$

so the integral is well-defined.

---

**Lemma-definition 1.4.4 (proof p. 64)** — Let  $\Delta \subset \mathbb{R}^d$  be a compact and  $\gamma \in \text{Conf}^{(\text{FK})}$ .  
We define the exterior configuration relatively to  $\Delta$  as

$$\gamma_{\Delta}^{\text{ext}} := \{w \in \gamma / w \not\subset \Delta\}.$$

If  $\sup_{s \in [0; \beta]} \#\left[\{w(s), w \in \gamma_{\Delta}^{\text{ext}}\} \cap (\Delta + \Lambda_R)\right]$  is finite, then one can define the constant

$$Z_{\Delta}(\gamma) := \int \Pi_{\Delta}(d\zeta) e^{\beta \mu \# \zeta} \sum_{\sigma_{\Delta}^{\text{int}} \in \mathbb{S}(\partial\gamma_{\Delta}^{\text{in}} \cup \zeta \rightarrow \partial\gamma_{\Delta}^{\text{out}} \cup \zeta)} \int \left( \bigotimes_{x \in \partial\gamma_{\Delta}^{\text{in}} \cup \zeta} W_{x, \sigma_{\Delta}^{\text{int}}(x)}^{\beta, \zeta, \Delta} \right) (d\eta) e^{-H_{\Delta}^{(\text{FK})}(\eta \cup \gamma_{\Delta}^{\text{ext}})}$$

where we denote

$$\mathbb{S}(X \rightarrow Y) := \{\sigma : X \rightarrow Y / \sigma \text{ is bijective}\}$$

the inward and outward boundaries are the finite point configurations

$$\begin{aligned} \partial\gamma_{\Delta}^{\text{in}} &:= \left\{ x \in \Delta \left| \begin{array}{l} \exists w \in \gamma_{\Delta}^{\text{ext}}, x = w(\beta) \\ \forall w \in \gamma_{\Delta}^{\text{ext}}, x \neq w(0) \end{array} \right. \right\} \\ \partial\gamma_{\Delta}^{\text{out}} &:= \left\{ x \in \Delta \left| \begin{array}{l} \exists w \in \gamma_{\Delta}^{\text{ext}}, x = w(0) \\ \forall w \in \gamma_{\Delta}^{\text{ext}}, x \neq w(\beta) \end{array} \right. \right\} \end{aligned}$$

and we define the measure

$$W_{x,y}^{\beta, \zeta, \Delta}(dw) := \mathbb{1}_{w \subset \Delta} W_{x,y}^{\beta}(dw).$$

The constant  $Z_{\Delta}(\gamma)$  is well defined, positive and finite  $\mathbb{P}_{(\text{bc}), L}^{(\text{FK})}$  almost surely for any  $L \in \mathbb{N}$  such that  $\Delta + \Lambda_R \subseteq \Lambda_L$ . It is also true  $\mathbb{P}_{(\text{bc}), \infty}^{(\text{FK})}$  almost surely.

If  $Z_{\Delta}(\gamma) \in ]0; +\infty[$  then we define the probability measure  $\mathbb{P}_{\Delta}^{(\text{FK})}(\cdot | \gamma)$  over  $\text{Conf}^{(\text{FK})}$  by

$$\begin{aligned} \mathbb{P}_{\Delta}^{(\text{FK})}(d\eta | \gamma) &:= \frac{1}{Z_{\Delta}(\gamma)} \int \Pi_{\Delta}(d\xi_{\Delta}) e^{\beta \mu \# \xi_{\Delta}} \sum_{\sigma_{\Delta}^{\text{int}} \in \mathbb{S}(\partial\gamma_{\Delta}^{\text{in}} \cup \xi_{\Delta} \rightarrow \partial\gamma_{\Delta}^{\text{out}} \cup \xi_{\Delta})} \\ &\quad \left( \bigotimes_{x \in \partial\gamma_{\Delta}^{\text{in}} \cup \xi_{\Delta}} W_{x, \sigma_{\Delta}^{\text{int}}(x)}^{\beta, \zeta, \Delta} \right) (d\eta) e^{-H_{\Delta}^{(\text{FK})}(\eta \cup \gamma_{\Delta}^{\text{ext}})}. \end{aligned}$$

Otherwise we define the measure by  $\mathbb{P}_{\Delta}^{(\text{FK})}(\cdot | \gamma) := 0$ .

We illustrate in fig. 2 the detail of exterior and interior configurations.

Only points inside  $\Delta$  and bridges intersecting the compact are represented. The bridges of  $\gamma_{\Delta}^{\text{ext}}$  are drawn in dashed lines and those of  $\gamma_{\Delta}^{\text{int}} := \gamma \setminus \gamma_{\Delta}^{\text{ext}}$  in solid lines. Points of  $\partial\gamma_{\Delta}^{\text{in}}$  are represented by “■”, points of  $\partial\gamma_{\Delta}^{\text{out}}$  by “▲”.

Points which both start and end solid bridges are represented by “●”, those which start and end dashed bridges are represented by “◆”.

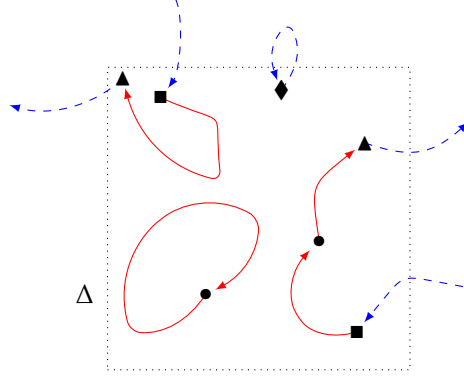


Figure 2: Exterior and interior configurations relative to  $\Delta$

**REMARK 1.4.5**

If the exterior configuration  $\gamma_{\Delta}^{\text{ext}}$  is far enough from the compact  $\Delta$  :

$$\forall w \in \gamma_{\Delta}^{\text{ext}}, w \cap (\Delta + \Lambda_R) = \emptyset$$

then the conditional distribution does not depend on the exterior configuration :

$$\mathbb{P}_{\Delta}^{(\text{FK})}(\cdot | \gamma) = \mathbb{P}_{\Delta}^{(\text{FK})}(\cdot | \emptyset).$$

In fact, it coincides with a finite volume model on  $\Delta$  with Dirichlet boundary conditions.

**Theorem 1.4.6 (DLR equations, proof p. 65)**

Let  $\Delta \subset \mathbb{R}^d$  be a compact. Under hypotheses I to VI, for any inverse temperature  $\beta > 0$ , chemical potential  $\mu \in \mathbb{R}$ , boundary condition (bc) = (Dir) or (per), for any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}^+$ ,

$$\int f(\gamma) \mathbb{P}_{(\text{bc}), \infty}^{(\text{FK})}(d\gamma) = \int \left[ \int f(\eta \cup \gamma_{\Delta}^{\text{ext}}) \mathbb{P}_{\Delta}^{(\text{FK})}(d\eta | \gamma) \right] \mathbb{P}_{(\text{bc}), \infty}^{(\text{FK})}(d\gamma).$$

The resampling of the interior configuration in the DLR equations consists in the following: we erase  $\bullet$  points and **solid bridges**, then sample new interior  $\bullet$  points, a new interior bijection  $\sigma_{\Delta}^{\text{int}} \in \mathbb{S}(\blacksquare \cup \bullet \rightarrow \blacktriangle \cup \bullet)$  and the associated **solid bridges**. This is illustrated in fig. 3.

## 2 Equivalent models

As we mentioned previously, there is more than one formulation of the Bose gas. The expression closest to the physics is Feynman-Kac's and we chose to express our main results in this setting. We will also properly introduce 3 other equivalent models. Transitions from these various points of view are summed up in fig. 4.

Of course, the respective definitions will make the diagram of fig. 4 commute.

As we announced in the general introduction, the *marked point* (mp) framework will be essential to establish the thermodynamic limit. The *path soup* (ps) framework will be necessary



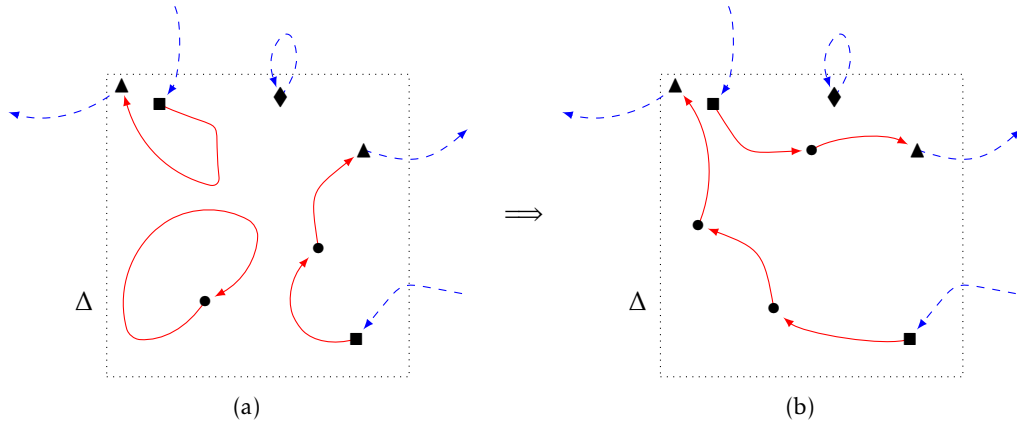


Figure 3: Resampling in the DLR equations

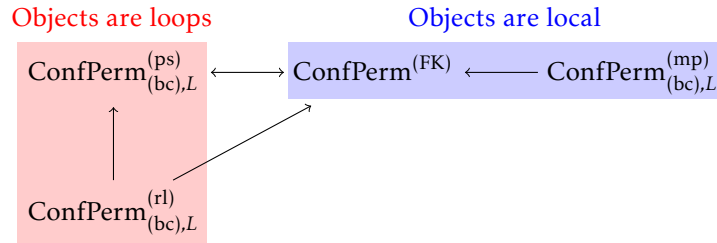


Figure 4: Equivalence of models in  $\Lambda_L$

because of proposition 2.2.8 which states the invariance of the Bose gas under a time-shift of the Brownian bridges in the configuration. This will be useful in the proof of the (mp) entropic bound. The *rooted loops* (rl) framework, commonly referred to as loop soup in the literature, is more secondary. It is necessary to introduce the (ps) model, and it will be interesting to ascertain how incompatible the topologies of (rl) and (FK) thermodynamic limits are.

## 2.1 Rooted loops

A usual formulation of the Bose gas is the loop soup one ([BV23], [DV24], *etc*). Instead of sampling separately  $N$  different Brownian bridges which happen to draw a permutation  $\sigma \in \mathbb{S}(N)$ , this model samples directly the cycles of  $\sigma$  as Brownian loops  $w : \mathbb{R} \rightarrow \mathbb{R}^d$ . This model can be naturally interpreted as a process sampling marked points: the position is the root  $w(0)$  of the loop, and the mark is the loop itself. This is why we will call this representation the *rooted loop* (rl) model of the Bose gas. We will see this representation has some flaws that prevent us from studying interlacements.

**Definition 2.1.1** — For any  $\omega \in \Omega_t$ , we define the continuation  $\tilde{\omega} : \mathbb{R} \rightarrow \mathbb{R}^d$  of the trajectory

$$\tilde{\omega}(s + tq) := \omega(s) + q[\omega(t) - \omega(0)]$$

for  $q \in \mathbb{Z}$  and  $s \in [0; t[$ .

---

We introduce the set

$$\text{Loop}_t^{(\text{root})} := \{\tilde{\omega}, \omega \in \Omega_t\}.$$

The map  $\omega \mapsto \tilde{\omega}$  induces the respective topology  $\mathfrak{W}_t^{(\text{root})}$  and  $\sigma$ -algebra  $\mathcal{W}_t^{(\text{root})}$ .

We understand it is not reasonable to call *loops* the trajectories in  $\text{Loop}_t^{(\text{root})}$ , but we are going to investigate specific subsets of  $\text{Loop}_t^{(\text{root})}$ , so we believe this notation is fairly justified.

For Dirichlet boundary condition, we will call rooted loops continuous trajectories  $w : \mathbb{R} \rightarrow \mathbb{R}^d$  which are  $\beta j$ -periodic for some  $j \geq 1$ .

In periodic boundary condition, a loop is not forced to return to its starting point  $w(0)$ . It can go to some periodic copy  $w(0) + Lk$ ,  $k \in \mathbb{Z}^d$  instead.

**Definition 2.1.2** — We define the length  $\ell(w)$  of a given  $w \in \bigcup_{j \geq 1} \text{Loop}_{\beta j}^{(\text{root})}$  to be the integer

$$\ell(w) := \inf \left\{ j \geq 1 \mid w \in \text{Loop}_{\beta j}^{(\text{root})} \right\}.$$

We define

$$\begin{aligned} \text{Loop}_{(\text{Dir}),L}^{(\text{root})} &:= \bigcup_{j \geq 1} \left\{ w \in \text{Loop}_{\beta j}^{(\text{root})} \mid \begin{array}{l} \ell(w) = j \\ w(\beta j) = w(0) \end{array} \right\} \\ \text{Loop}_{(\text{per}),L}^{(\text{root})} &:= \bigcup_{j \geq 1} \left\{ w \in \text{Loop}_{\beta j}^{(\text{root})} \mid \begin{array}{l} \ell(w) = j \\ w(\beta j) - w(0) \in L\mathbb{Z}^d \end{array} \right\}. \end{aligned}$$

Trajectories in a set  $\text{Loop}_{(\text{bc}),L}^{(\text{root})}$  will be called rooted loops.

**Lemma-definition 2.1.3** — Defining  $\text{Loop}_{(\text{bc}),L}^{(\text{root})}$  as a discrete disjoint union of subsets induces a topology and  $\sigma$ -algebra over this space, which we will denote  $\mathfrak{W}_{(\text{bc}),L}^{(\text{root})}$  and  $\mathcal{W}_{(\text{bc}),L}^{(\text{root})}$  respectively.

The topological space  $\left( \text{Loop}_{(\text{bc}),L}^{(\text{root})}, \mathfrak{W}_{(\text{bc}),L}^{(\text{root})} \right)$  is Polish for the distance

$$\delta(w, w') := \begin{cases} \sup_{s \in [0; \beta j]} \|w(s) - w'(s)\| & \text{if } j = \ell(w) = \ell(w') \\ +\infty & \text{otherwise.} \end{cases}$$

It is clear  $\delta : \text{Loop}_{(\text{bc}),L}^{(\text{root})} \times \text{Loop}_{(\text{bc}),L}^{(\text{root})} \rightarrow \mathbb{R} \cup \{+\infty\}$  metrizes the topology of the spaces  $\text{Loop}_{\beta j}^{(\text{root})}$  for all  $j \geq 1$ . Hence it metrizes the topology of the subsets of  $\text{Loop}_{\beta j}^{(\text{root})}$ ,  $j \geq 1$ , in the definition of  $\text{Loop}_{(\text{bc}),L}^{(\text{root})}$ . Finally,  $\delta$  sets a  $+\infty$  distance between those subsets, which is coherent with a discrete disjoint union.

The property of being Polish is necessary to ensure that a Poisson point process can be defined on this space.

---

**Definition 2.1.4** — For each boundary condition, we call a rooted loops configuration (rl) a set  $\eta$  in

$$\text{Conf}_{(\text{bc}),L}^{(\text{rl})} := \left\{ \eta \subset \text{Loop}_{(\text{bc}),L}^{(\text{root})} \mid \#\eta < +\infty \right\}$$

and we equip this configuration set with the smallest  $\sigma$ -algebra  $\mathcal{C}_{(\text{bc}),L}^{(\text{rl})}$  making measurable the maps  $\eta \mapsto \#(\eta \cap E)$  for every bounded  $E \in \mathcal{W}_{(\text{bc}),L}^{(\text{root})}$ .

**Definition 2.1.5** — On the set of rooted loops, we introduce a measure for Dirichlet and periodic boundary conditions

$$W_{(\text{Dir}),L}^{(\text{root})} := \int_{\Lambda_L} dx \sum_{j \geq 1} \frac{e^{-\beta j}}{j} W_{x,x}^{\beta j}$$

$$W_{(\text{per}),L}^{(\text{root})} := \int_{\Lambda_L} dx \sum_{j \geq 1} \frac{e^{-\beta j}}{j} \sum_{k \in \mathbb{Z}^d} W_{x,x+Lk}^{\beta j}.$$

We denote by  $\Pi_{(\text{bc}),L}^{(\text{rl})}$  the Poisson point process over  $\text{Conf}_{(\text{bc}),L}^{(\text{rl})}$  with intensity measure  $W_{(\text{bc}),L}^{(\text{root})}$ .

The weight  $e^{-\beta j}$  is not standard but we add it to make these measures finite. This additional factor will be compensated by increasing the chemical potential of the Gibbs model we will introduce in lemma-definition 2.1.7.

**Definition 2.1.6** — For both boundary conditions, we define a Hamiltonian for rooted loops configurations

$$H_{(\text{bc}),L}^{(\text{rl})}(\eta) := \int_0^\beta U_{(\text{bc}),L}[\{w(\beta j + s), 0 \leq j < \ell(w), w \in \eta\}] ds.$$

**Lemma-definition 2.1.7 (proof: see remark 3.1.4)** — Let  $L \in r\mathbb{N}$ . The probability measure over  $\text{Conf}_{(\text{bc}),L}^{(\text{rl})}$

$$\mathbb{P}_{(\text{bc}),L}^{(\text{rl})}(d\eta) := \frac{e^{-L^d + W_{(\text{bc}),L}^{(\text{root})}(\text{Loop}_{(\text{bc}),L}^{(\text{root})})}}{Z_{(\text{bc}),L}} \exp \left[ \beta(\mu + 1) \sum_{w \in \eta} \ell(w) - H_{(\text{bc}),L}^{(\text{rl})}(\eta) \right] \Pi_{(\text{bc}),L}^{(\text{rl})}(d\eta).$$

where  $Z_{(\text{bc}),L}$  is the same normalization constant as in lemma-definition 1.3.3, is well defined. We call this probability the rooted loops model of the Bose gas.

**Definition 2.1.8** — For each boundary condition, we define the map  $\varphi_{(\text{bc}),L}^{(\text{FK}) \leftarrow (\text{rl})} : \text{Conf}_{(\text{bc}),L}^{(\text{rl})} \rightarrow \text{Conf}^{(\text{FK})}$  of transition from a rooted loops configuration into a Feynman-Kac configuration as

$$\varphi_{(\text{Dir}),L}^{(\text{FK}) \leftarrow (\text{rl})}(\eta) := \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto w(\beta j + s), \quad 0 \leq j < \ell(w), w \in \eta \end{array} \right\}$$

$$\varphi_{(\text{per}),L}^{(\text{FK}) \leftarrow (\text{rl})}(\eta) := \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto w(\beta j + s) + Lk, \quad 0 \leq j < \ell(w), w \in \eta, k \in \mathbb{Z}^d \end{array} \right\}.$$

---

Furthermore, we denote

$$\text{ConfPerm}_{(\text{bc}),L}^{(\text{rl})} := \left( \begin{matrix} (\text{FK})^{\leftarrow(\text{rl})} \\ \varphi_{(\text{bc}),L} \end{matrix} \right)^{-1} \left( \text{ConfPerm}^{(\text{FK})} \right).$$

The following proposition states the Feynman-Kac and rooted loops models are equivalent.

**Proposition 2.1.9 (proof: see remark 3.1.4)** — For any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}^+$ ,

$$\int f \circ \begin{matrix} (\text{FK})^{\leftarrow(\text{rl})} \\ \varphi_{(\text{bc}),L} \end{matrix} d\mathbb{P}_{(\text{bc}),L}^{(\text{rl})} = \int f d\mathbb{P}_{(\text{bc}),L}^{(\text{FK})}$$

We will see in remark 2.1.17 this result may only be true in finite volume.

**REMARK 2.1.10**

In the case of Dirichlet boundary condition, the state space does not depend on the size  $L$  of the domain. So from then on, we are going to drop the  $L$  in the notation of the space  $\text{Loop}_{(\text{Dir})}^{(\text{root})}$ , its topology  $\mathcal{W}_{(\text{Dir})}^{(\text{root})}$ , its  $\sigma$ -algebra  $\mathcal{W}_{(\text{Dir})}^{(\text{root})}$  and  $\text{Conf}_{(\text{Dir})}^{(\text{rl})}$ . Additionally, we are going to introduce a more general configuration space to construct a rooted loops thermodynamic limit.

**Definition 2.1.11** — We define a set of rooted loops configurations as

$$\text{Conf}_{\infty}^{(\text{rl})} := \left\{ \eta \subset \text{Loop}_{(\text{Dir})}^{(\text{root})} \mid \eta \text{ is locally finite in } \mathcal{W}_{(\text{Dir})}^{(\text{root})} \right\}$$

and we equip this space with the smallest  $\sigma$ -algebra  $\mathcal{C}_{\infty}^{(\text{rl})}$  making measurable the maps  $\eta \mapsto \#(\eta \cap E)$  for every bounded  $E \in \mathcal{W}_{(\text{Dir})}^{(\text{root})}$ .

We also define the transition map infinite volume  $\begin{matrix} (\text{FK})^{\leftarrow(\text{rl})} \\ \varphi_{(\text{Dir}),\infty} \end{matrix} : \text{Conf}_{\infty}^{(\text{rl})} \rightarrow \text{Conf}^{(\text{FK})}$  by

$$\begin{matrix} (\text{FK})^{\leftarrow(\text{rl})} \\ \varphi_{(\text{Dir}),\infty} \end{matrix}(\eta) := \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto w(\beta j + s) \end{array} , 0 \leq j < \ell(w), w \in \eta \right\}.$$

**REMARK 2.1.12**

Trivially, we have the inclusion  $\text{Conf}_{(\text{Dir})}^{(\text{rl})} \subseteq \text{Conf}_{\infty}^{(\text{rl})}$ , so we can consider the models  $\mathbb{P}_{(\text{bc}),L}^{(\text{rl})}$  to be probability measures over  $\text{Conf}_{\infty}^{(\text{rl})}$ .

**Definition 2.1.13** — For any  $L \in r\mathbb{N}$ , we define the empirical field  $\widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(\text{rl})}$  over  $\text{Conf}_{\infty}^{(\text{rl})}$  by

$$\int f d\widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(\text{rl})} := \frac{1}{L^d} \int_{\Lambda_L} dv \int f(\gamma + v) \mathbb{P}_{(\text{Dir}),L}^{(\text{rl})}(d\gamma)$$

for any measurable  $f : \text{Conf}_{\infty}^{(\text{rl})} \rightarrow \mathbb{R}$ .

---

**Definition 2.1.14** — A function  $f : \text{Conf}_\infty^{(\text{rl})} \rightarrow \mathbb{R}$  is said to be (rl)-tame local if there exists  $a > 0$  and a compact  $\Delta \subset \mathbb{R}^d$  such that

$$\forall \eta \in \text{Conf}_\infty^{(\text{rl})}, f(\eta) = f(\{w \in \eta / w(0) \in \Delta\})$$

and

$$\forall \eta \in \text{Conf}_\infty^{(\text{rl})}, a|f(\eta)| \leq 1 + \#\{w \in \eta / w(0) \in \Delta\}.$$

**Theorem 2.1.15 (proof p. 34)**

There exists a stationary probability measure  $\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{rl})}$  over  $\text{Conf}_\infty^{(\text{rl})}$  and an increasing sequence  $(L_m)_{m \geq 0}$  of multiples of  $r$  such that, for any (rl)-tame local measurable  $f : \text{Conf}_\infty^{(\text{rl})} \rightarrow \mathbb{R}$ ,

$$\lim_{m \rightarrow +\infty} \int f \, d\tilde{\mathbb{P}}_{(\text{Dir}, L_m)}^{(\text{rl})} = \int f \, d\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{rl})}.$$

**Corollary 2.1.16** — By construction,  $\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{rl})}(\eta \subset \text{Loop}_{(\text{Dir})}^{(\text{root})}) = 1$ .

**REMARK 2.1.17**

With corollary 2.1.16, one may conclude that the Feynman-Kac model  $\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{FK})}$ , just as  $\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{rl})}$ , only produces finite cycles. But this is not so trivial. The infinite volume models  $\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{FK})}$  and  $\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{rl})}$  may *not* be equivalent, as we conjecture :

$$\mathbb{P}_{(\text{Dir}, \infty)}^{(\text{rl})} \left[ \left( \mathcal{P}_{(\text{Dir}, \infty)}^{(\text{FK}) \leftarrow (\text{rl})} \right)^{-1} (E) \right] \neq \mathbb{P}_{(\text{Dir}, \infty)}^{(\text{FK})} (E)$$

for some event  $E \in \mathcal{C}^{(\text{FK})}$ , because the topologies of convergence for Feynman-Kac and rooted loops models are fundamentally incompatible. At first glance the class of functions on which rooted loops models converge seems strictly stronger. Indeed, (rl)-local functions can depend on whole loops, whereas  $\cap^n$ -local functions can only see  $n$  Brownian bridges beyond the boundary of some compact  $\Delta$ . However (rl)-local functions are limited as they can only see loops whose root is inside  $\Delta$ , contrary to  $\cap^n$ -local functions. Those two modes of convergence can not be compared and we believe this is not just a technicality.

We could have chosen a more general notion of tameness for the (rl) framework but this would not have changed the discussion above because the problem comes from locality.

## 2.2 Path soup

The information of which point is the root of the cycle is not native. In the (FK) model, any point could be considered the root of its cycle; this additional information is not natural. When presented with a rooted loop  $w$ , we are not interested about knowing which one of the points  $w(\beta_j)$ ,  $0 \leq j < \ell(w)$  is its starting point  $w(0)$  (its root). We were inspired by the work from [AFY21] and refer to it for a similar presentation, used to inquire the Bose gas without interaction.

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**Definition 2.2.1** — We define  $\pi_{(\text{bc}),L} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  as

$$\pi_{(\text{Dir}),L}(x) := x \quad \pi_{(\text{per}),L}(x) \in \Lambda_L \text{ such that } \pi_{(\text{per}),L}(x) - x \in LZ^d$$

and introduce an equivalence relation defined over  $\text{Loop}_{(\text{bc}),L}^{(\text{root})}$  in finite volume as

$$w \equiv_{(\text{bc}),L} w' \iff \exists j \in \mathbb{Z}, \forall s \in \mathbb{R}, \pi_{(\text{bc}),L}[w'(s)] = \pi_{(\text{bc}),L}[w(s + \beta j)].$$

In the quotient space  $\text{Loop}_{(\text{bc}),L} := \text{Loop}_{(\text{bc}),L}^{(\text{root})} / \equiv_{(\text{bc}),L}$ , we denote as  $\bar{w}$  the equivalence class of a rooted path  $w$ . We will abuse notations and denote as  $\ell(\bar{w})$  the common length of representatives  $w \in \bar{w}$ .

That is to say, in Dirichlet boundary condition, 2 rooted loops are identified if they are constituted of the same set of trajectories  $[0; \beta] \rightarrow \mathbb{R}^d$ .

**Lemma-definition 2.2.2** — We define  $\text{dist}_{(\text{bc}),L} : \text{Loop}_{(\text{bc}),L}^{(\text{root})} \times \text{Loop}_{(\text{bc}),L}^{(\text{root})} \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$\text{dist}_{(\text{Dir}),L}(w, w') := \begin{cases} \inf_{i \in \mathbb{Z}} \sup_{s \in [0; \beta j]} \|w(s + \beta i) - w'(s)\| & \text{if } j = \ell(w) = \ell(w') \\ +\infty & \text{otherwise} \end{cases}$$

$$\text{dist}_{(\text{per}),L}(w, w') := \begin{cases} \inf_{k \in \mathbb{Z}^d} \inf_{i \in \mathbb{Z}} \sup_{s \in [0; \beta j]} \|w(s + \beta i) - w'(s) + Lk\| & \text{if } \begin{matrix} w(t) - w(0) = w'(t) - w'(0) \\ j = \ell(w) = \ell(w') \end{matrix} \\ +\infty & \text{otherwise.} \end{cases}$$

The function  $\text{dist}_{(\text{bc}),L}$  is compatible with the equivalence relationship  $\equiv_{(\text{bc}),L}$ . We then define the quotient function  $\text{dist}_{(\text{bc}),L} : \text{Loop}_{(\text{bc}),L} \times \text{Loop}_{(\text{bc}),L} \rightarrow \mathbb{R} \cup \{+\infty\}$  which is a distance.

This distance makes the space  $\text{Loop}_{(\text{bc}),L}$  Polish and the quotient map

$$\begin{aligned} \left( \text{Loop}_{(\text{bc}),L}^{(\text{root})}, \delta \right) &\longrightarrow \left( \text{Loop}_{(\text{bc}),L}, \text{dist}_{(\text{bc}),L} \right) \\ w &\longmapsto \bar{w} \end{aligned}$$

continuous.

We denote as  $\mathcal{W}_{(\text{bc}),L}$  and  $\mathcal{W}_{(\text{bc}),L}$  the respective topology and Borelian  $\sigma$ -algebra induced on  $\text{Loop}_{(\text{bc}),L}$  by  $\text{dist}_{(\text{bc}),L}$ .

We define  $W_{(\text{bc}),L}$  the quotient measure of  $W_{(\text{bc}),L}^{(\text{root})}$  over  $\text{Loop}_{(\text{bc}),L}$ .

**Definition 2.2.3** — In finite volume, we call a path soup configuration (ps) a finite set  $\eta$  in

$$\text{Conf}_{(\text{bc}),L}^{(\text{ps})} := \left\{ \eta \subset \text{Loop}_{(\text{bc}),L} \mid \#\eta < +\infty \right\}.$$

We equip this space with the smallest  $\sigma$ -algebra  $\mathcal{C}_{(\text{bc}),L}^{(\text{ps})}$  making measurable the maps  $\eta \mapsto \#(\eta \cap E)$  for every bounded  $E \in \mathcal{W}_{(\text{bc}),L}$ .

We denote the Poisson point process on  $\text{Conf}_{(\text{bc}),L}^{(\text{ps})}$  with intensity measure  $W_{(\text{bc}),L}$  by  $\Pi_{(\text{bc}),L}^{(\text{ps})}$ .

---

**Definition 2.2.4** — The quotient map  $\text{Loop}_{(\text{bc}),L}^{(\text{root})} \rightarrow \text{Loop}_{(\text{bc}),L}$  induces a quotient over configuration spaces

$$\begin{array}{ccc} \text{Conf}_{(\text{bc}),L}^{(\text{rl})} & \longrightarrow & \text{Conf}_{(\text{bc}),L}^{(\text{ps})} \\ \varphi_{(\text{bc}),L}^{(\text{ps})\leftarrow(\text{rl})} : \eta & \longmapsto & \{\bar{w}, w \in \eta\} \end{array} .$$

For all  $s \in [0; \beta]$ , the map  $\eta \mapsto U_{(\text{bc}),L}[\{w(\beta j + s), 0 \leq j < \ell(w), w \in \eta\}]$  is compatible with this quotient over configurations so we define the quotient map

$$\begin{array}{ccc} \text{Conf}_{(\text{bc}),L}^{(\text{ps})} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ \eta & \longmapsto & U_{(\text{bc}),L}[\{\bar{w}(\beta j + s), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}] \end{array}$$

and the path soup Hamiltonian by

$$H_{(\text{bc}),L}^{(\text{ps})}(\eta) := \int_0^\beta U_{(\text{bc}),L}[\{\bar{w}(\beta j + s), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}] ds.$$

We did everything necessary to define yet another equivalent model of the Bose gas.

**Lemma-definition 2.2.5 (proof: see remark 3.1.4)** — Let  $L \in r\mathbb{N}$ . For both boundary conditions, the probability measure

$$\mathbb{P}_{(\text{bc}),L}^{(\text{ps})}(d\eta) := \frac{e^{-L^d + W_{(\text{bc}),L}(\text{Loop}_{(\text{bc}),L})}}{Z_{(\text{bc}),L}} \exp\left[\beta(\mu + 1) \sum_{\bar{w} \in \eta} \ell(\bar{w}) - H_{(\text{bc}),L}^{(\text{ps})}(\eta)\right] \Pi_{(\text{bc}),L}^{(\text{ps})}(d\eta).$$

where  $Z_{(\text{bc}),L}$  is the same normalization constant as in lemma-definition 1.3.3, is well defined.

**Definition 2.2.6** — We define the transition map

$$\begin{array}{ccc} \text{Conf}_{(\text{bc}),L}^{(\text{ps})} & \longrightarrow & \text{Conf}^{(\text{FK})} \\ \varphi_{(\text{bc}),L}^{(\text{FK})\leftarrow(\text{ps})} : \eta & \longmapsto & \bigcup_{\bar{w} \in \eta} \left\{ \begin{array}{ccc} [0; \beta] & \longrightarrow & \mathbb{R}^d \\ s & \longmapsto & w(s) \end{array} , w \in \bar{w} \right\} \end{array}$$

and the set

$$\text{ConfPerm}_{(\text{bc}),L}^{(\text{ps})} := \left( \varphi_{(\text{bc}),L}^{(\text{FK})\leftarrow(\text{ps})} \right)^{-1} \left( \text{ConfPerm}^{(\text{FK})} \right).$$

Let  $\gamma \in \text{ConfPerm}^{(\text{FK})}$ . We write the permutation  $\sigma^{(\text{FK})}(\gamma, \cdot)$  as a product of disjoint cycles  $c_i$ ,  $i \in I$

$$\sigma^{(\text{FK})}(\gamma, \cdot) = \bigcirc_{i \in I} c_i.$$

For anyone of these  $c_i$ , there are many possible labelings  $\text{lab}_i = (w_{i,j}, j \in \mathbb{Z})$  of its support in a way compatible with the permutation :

$$\sigma^{(\text{FK})}(\gamma, w_{i,j}) = w_{i,j+1}.$$

For any cycle labeling  $\text{lab}_i$ , we define the rooted loop  $w_{\text{lab}_i} : \mathbb{R} \rightarrow \mathbb{R}^d$  as

$$\begin{cases} w_{\text{lab}_i}(0) & := w_{i,0}(0) \\ w_{\text{lab}_i}(s + \beta j) & := w_{\text{lab}_i}(\beta j) + w_{i,j}(s) - w_{i,j}(0) & \text{for } j \geq 0, s \in ]0; \beta] \\ w_{\text{lab}_i}(s + \beta j) & := w_{\text{lab}_i}[\beta(j + 1)] + w_{i,j}(s) - w_{i,j}(\beta) & \text{for } j \leq -1, s \in [0; \beta[. \end{cases}$$

---

Finally we define

$$\begin{aligned} \begin{matrix} \text{ConfPerm}^{(\text{ps})\leftarrow(\text{FK})} \\ \varphi_{(\text{bc}),L} \end{matrix} : \text{ConfPerm}^{(\text{FK})} &\longrightarrow \text{ConfPerm}_{(\text{bc}),L}^{(\text{ps})} \\ \gamma &\longmapsto \left( \bigcup_{i \in I} \{w_{\text{lab}_i}, \text{lab}_i \text{ labeling of } c_i\} \right) / \equiv_{(\text{bc}),L}. \end{aligned}$$

The advantage of the root-less vision is to be able to define a transition  $\varphi_{(\text{bc}),L}^{(\text{ps})\leftarrow(\text{FK})}$  without the issue of having to choose which  $w(\beta j)$ ,  $j \in \mathbb{Z}$  is the root.

**Proposition 2.2.7 (proof: see remark 3.1.4)** — For any measurable  $f : \text{Conf}_{(\text{bc}),L}^{(\text{ps})} \rightarrow \mathbb{R}^+$ ,

$$\int f \circ \begin{matrix} \text{ConfPerm}^{(\text{ps})\leftarrow(\text{rl})} \\ \varphi_{(\text{bc}),L} \end{matrix} \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{rl})} = \int f \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{ps})}$$

Furthermore, for any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}^+$ ,

$$\int f \circ \begin{matrix} \text{ConfPerm}^{(\text{FK})\leftarrow(\text{ps})} \\ \varphi_{(\text{bc}),L} \end{matrix} \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{ps})} = \int f \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{FK})}$$

and for any measurable  $f : \text{Conf}_{(\text{bc}),L}^{(\text{ps})} \rightarrow \mathbb{R}^+$ ,

$$\int f \circ \begin{matrix} \text{ConfPerm}^{(\text{ps})\leftarrow(\text{FK})} \\ \varphi_{(\text{bc}),L} \end{matrix} \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{FK})} = \int f \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{ps})}$$

**Proposition 2.2.8 (proof p. 29)** — The path soup model is time-shift invariant. More precisely, for any  $s \in \mathbb{R}$  and measurable  $f : \text{Conf}_{(\text{bc}),L}^{(\text{ps})} \rightarrow \mathbb{R}^+$ ,

$$\int f \circ T_s^{(\text{ps})} \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{ps})} = \int f \mathbf{d}\mathbb{P}_{(\text{bc}),L}^{(\text{ps})}$$

with the time-shift operator  $T_s^{(\text{ps})}$  defined as

$$\begin{aligned} T_s^{(\text{ps})} : \text{Conf}_{(\text{bc}),L}^{(\text{ps})} &\longrightarrow \text{Conf}_{(\text{bc}),L}^{(\text{ps})} \\ \eta &\longmapsto \{w(\cdot + s), w \in \bar{w}, \bar{w} \in \eta\}. \end{aligned}$$

### 2.3 Marked points

We present now the framework in which most of the work has been done. But it unfortunately also is the least elegant one. We encode the Feynman Kac representation into a configuration of *marked points* (mp). The goal is to localize the global information of the permutation  $\sigma$  into marks, so that the mark of each point  $x$  is enough to know  $\sigma(x)$ .

**Definition 2.3.1** — We define

$$\text{Conf}^{(\text{mp})} := \left\{ \gamma = (x, p_x, u_x, \omega_x)_{x \in \xi} \subset \mathbb{R}^d \times \mathbb{Z}^d \times [0; 1] \times \Omega_1 \mid \xi \text{ is locally finite in } \mathbb{R}^d \right\}.$$



and for any  $\gamma \in \text{Conf}^{(\text{mp})}$  we denote its spatial component

$$\dot{\gamma} := \{x, (x, p, u, \omega) \in \gamma\} \subset \mathbb{R}^d.$$

For any finite  $\gamma \in \text{Conf}^{(\text{mp})}$ , we denote

$$\dot{\gamma}_{(\text{Dir}),L} := \dot{\gamma} \quad \text{and} \quad \dot{\gamma}_{(\text{per}),L} := \bigcup_{k \in \mathbb{Z}^d} (\dot{\gamma} + Lk)$$

and

$$N_{(\text{bc}),L}^r(\dot{\gamma}, z) := \#[\dot{\gamma}_{(\text{bc}),L} \cap (z + \Lambda_r)].$$

We define the set of authorized configurations as

$$\text{ConfAuth}_{(\text{bc}),L}^{(\text{mp})} := \left\{ \gamma \in \text{Conf}^{(\text{mp})} \mid \dot{\gamma} \text{ is finite, simple and } \forall (x, p, u, \omega) \in \gamma, N_{(\text{bc}),L}^r(\dot{\gamma}, x + rp) \geq 1 \right\}.$$

For any authorized configuration  $\gamma \in \text{ConfAuth}_{(\text{bc}),L}^{(\text{mp})}$ , we define the map  $\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, \cdot) : \dot{\gamma} \rightarrow \dot{\gamma}_{(\text{bc}),L}$  such that for any  $(x, p, u, \omega) \in \gamma$ ,

$$\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x) := \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, x + rp) \cdot u \right] \text{'th element of } \dot{\gamma}_{(\text{bc}),L} \cap (x + rp + \Lambda_r)$$

if we order them lexicographically. Finally we call an authorized configuration permutation-wise if it satisfies

$$\forall y \in \dot{\gamma}, \exists! x \in \dot{\gamma}, \pi_{(\text{bc}),L} \left[ \sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x) \right] = \pi_{(\text{bc}),L}(y)$$

and we denote

$$\text{ConfPerm}_{(\text{bc}),L}^{(\text{mp})} := \left\{ \gamma \in \text{ConfAuth}_{(\text{bc}),L}^{(\text{mp})} \mid \gamma \text{ is permutation-wise} \right\}.$$

**Definition 2.3.2** — We define the transition map  $\varphi_{(\text{bc}),L}^{(\text{FK}) \leftarrow (\text{mp})} : \text{ConfPerm}_{(\text{bc}),L}^{(\text{mp})} \rightarrow \text{ConfPerm}^{(\text{FK})}$  by

$$\begin{aligned} \varphi_{(\text{Dir}),L}^{(\text{FK}) \leftarrow (\text{mp})}(\gamma) &:= \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto x + \frac{s}{\beta} \left[ \sigma_{(\text{Dir}),L}^{(\text{mp})}(\gamma, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), \end{array} \right. (x, p, u, \omega) \in \gamma \\ \varphi_{(\text{per}),L}^{(\text{FK}) \leftarrow (\text{mp})}(\gamma) &:= \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto x + \frac{s}{\beta} \left[ \sigma_{(\text{per}),L}^{(\text{mp})}(\gamma, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) + Lk, \end{array} \right. (x, p, u, \omega) \in \gamma, k \in \mathbb{Z}^d \end{aligned}$$

**Definition 2.3.3** — Let  $\nu$  be a probability measure over  $\mathbb{Z}^d$  such that  $\forall p \in \mathbb{Z}^d, \nu(p) > 0$ .

We also denote as  $\mathbb{W}_{0,0}^1$  the probability measure  $\mathbb{W}_{0,0}^1 := \frac{W_{0,0}^1}{W_{0,0}^1(\Omega_1)}$ .

We denote as  $\Pi_L^{(\text{mp})}$  the Poisson point process with intensity measure  $\text{Leb}^{(d)} \otimes \nu \otimes \text{Leb}^{(1)} \otimes \mathbb{W}_{0,0}^1$  over configurations  $\gamma \in \text{Conf}^{(\text{mp})}$  such that  $\dot{\gamma} \subset \Lambda_L$ .

We chose the reference measure  $\mathbb{W}_{0,0}^1$  so that the state space  $\Omega_1$  does not depend on  $\beta$ , unlike the (rl) and (ps) frameworks, and chose a normalized measure for mathematical elegance (the intensity measure's mass is exactly  $L^d$ ).

---

**Lemma-definition 2.3.4 (proof: see remark 3.1.5)** — We define a Hamiltonian over finite marked points configurations as follows

if  $\gamma \in \text{ConfPerm}_{(\text{bc}),L}^{(\text{mp})}$

$$H_{(\text{bc}),L}^{(\text{mp})}(\gamma) := \sum_{(x,p,u,\omega) \in \gamma} \left\{ \frac{d}{2} \log(2\pi\beta) + \frac{1}{2\beta} \left\| \sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x) - x \right\|^2 + \log[v(p)] - \log[N_{(\text{bc}),L}^r(\dot{\gamma}, x + rp)] \right\} \\ + \beta \int_0^1 U_{(\text{bc}),L} \left[ \left\{ x + s \left( \sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x) - x \right) + \sqrt{\beta} \omega(s), (x, p, u, \omega) \in \gamma \right\} \right] ds$$

else  $H_{(\text{bc}),L}^{(\text{mp})}(\gamma) := +\infty$ .

Then for any  $L \in r\mathbb{N}$ , the probability measure over  $\text{Conf}^{(\text{mp})}$

$$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(d\gamma) := \frac{1}{Z_{(\text{bc}),L}} \exp \left[ \beta \mu \# \gamma - H_{(\text{bc}),L}^{(\text{mp})}(\gamma) \right] \Pi_L^{(\text{mp})}(d\gamma)$$

where  $Z_{(\text{bc}),L}$  is the same normalization constant as in lemma-definition 1.3.3, is well defined.

**REMARK 2.3.5**

The model  $\mathbb{P}_{(\text{bc}),L}^{(\text{mp})}$  does not depend on the chosen density  $v : \mathbb{Z}^d \rightarrow \mathbb{R}$ .

This horrendous formulation of the Bose gas is equivalent to the previously defined models.

**Proposition 2.3.6 (proof: see remark 3.1.5)** — For any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}^+$ ,

$$\int f \circ \varphi_{(\text{bc}),L}^{(\text{FK}) \leftarrow (\text{mp})} d\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} = \int f d\mathbb{P}_{(\text{bc}),L}^{(\text{FK})}$$

**Definition 2.3.7** — A function  $f : \text{Conf}^{(\text{mp})} \rightarrow \mathbb{R}$  is said to be (mp)-tame local if there exists  $a, \delta > 0$ ,  $\alpha \in [0; 2[$  and a compact  $\Delta \subset \mathbb{R}^d$  such that

$$\forall \gamma \in \text{Conf}^{(\text{mp})}, f(\gamma) = f[\{(x, p, u, \omega) \in \gamma / x \in \Delta\}]$$

and

$$\forall \gamma \in \text{Conf}^{(\text{mp})}, a |f(\gamma)| \leq 1 + \sum_{(x,p,u,\omega) \in \gamma} \left| \text{Saus}_\delta \left( \begin{array}{ccc} [0; \beta] & \rightarrow & \mathbb{R}^d \\ s & \mapsto & \frac{s}{\beta} rp + \sqrt{\beta} \omega\left(\frac{s}{\beta}\right) \end{array} \right) \right|^\alpha.$$

**Definition 2.3.8** — For any  $L \in r\mathbb{N}$ , we define the empirical field  $\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})}$  over  $\text{Conf}^{(\text{mp})}$  by

$$\int f(\gamma) \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})}(d\gamma) = \frac{1}{L^d} \int_{\Lambda_L} dv \int f(\gamma + v) \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(d\gamma).$$

---

**Theorem 2.3.9 (proof p. 40)**

There exists a stationary probability measure  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}$  over  $\text{Conf}^{(\text{mp})}$  and an increasing sequence  $(L_m)_{m \geq 0}$  of multiples of  $r$  such that, for any (mp)-tame local measurable  $f : \text{Conf}^{(\text{mp})} \rightarrow \mathbb{R}$ ,

$$\lim_{m \rightarrow +\infty} \int f d\widetilde{\mathbb{P}}_{(\text{bc}),L_m}^{(\text{mp})} = \int f d\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}.$$

This theorem is the one unlocking everything. The probability  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}$  will be used to directly define  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}$  through an infinite volume transition map  $\varphi_\infty^{(\text{FK}) \leftarrow (\text{mp})}$  we will introduce later.

### 3 Proofs

#### 3.1 Equivalence of models

The following proposition establishes a link between (FK) and (rl) frameworks. It proves that, if the models (FK) and (rl) are well defined, then they are equivalent.

**Proposition 3.1.1** — For any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}^+$ ,

$$\begin{aligned} & e^{L^d} \int \Pi_{\Lambda_L}(d\xi) \sum_{\sigma \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{x \in \xi} W_{x,\sigma(x),(\text{bc}),L}^\beta \right) (d\gamma) \\ &= e^{W_{(\text{bc}),L}^{(\text{root})}(\text{Loop}_{(\text{bc}),L}^{(\text{root})})} \int \left( f \circ \varphi_{(\text{bc}),L}^{(\text{FK}) \leftarrow (\text{rl})} \right) (\eta) \exp \left[ \beta \sum_{w \in \eta} \ell(w) - H_{(\text{bc}),L}^{(\text{rl})}(\eta) \right] \Pi_{(\text{bc}),L}^{(\text{rl})}(d\eta). \end{aligned}$$

where we remind

$$\gamma_{(\text{Dir}),L} := \gamma \quad \text{and} \quad \gamma_{(\text{per}),L} := \bigcup_{k \in \mathbb{Z}^d} (\gamma + Lk).$$

**PROOF (PROPOSITION 3.1.1)**

The following proof is heavily inspired by the proof of Lemma 2.1 p. 358 from [Gin70].

By definition of the standard Poisson point process,

$$\begin{aligned} & \int \Pi_{\Lambda_L}(d\xi) \sum_{\sigma \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{x \in \xi} W_{x,\sigma(x),(\text{bc}),L}^\beta \right) (d\gamma) \\ &= e^{-L^d} \sum_{N=0}^{+\infty} \frac{1}{N!} \int_{\Lambda_L} dx^{\otimes N} \sum_{\sigma \in \mathbb{S}(N)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^N W_{x_i, x_{\sigma(i)}, (\text{bc}),L}^\beta \right) (d\gamma). \end{aligned}$$

Let  $N \geq 0$  and  $\sigma \in \mathbb{S}(N)$ . We denote as  $\ell_i$ ,  $1 \leq i \leq n$  the respective cycle lengths of  $\sigma$ .

We can re-order the variables  $(x_i)_{1 \leq i \leq N}$  into  $(x_{i,j})_{0 \leq j < \ell_i, 1 \leq i \leq n}$  such that

$$\sigma(x_{i,j}) = x_{i,j+1}$$

with the convention  $x_{i,\ell_i} = x_{i,0}$ . Then

$$\begin{aligned} & \int_{\Lambda_L} dx^{\otimes N} \int f(\gamma_{(\text{bc},L)}) e^{-H_{(\text{bc},L)}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^N W_{x_i, x_{\sigma(i)}, (\text{bc},L)}^\beta \right) (d\gamma) \\ &= \int_{\Lambda_L} \bigotimes_{i=1}^n dx_i^{\otimes \ell_i} \int f(\gamma_{(\text{bc},L)}) e^{-H_{(\text{bc},L)}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^n \bigotimes_{j=0}^{\ell_i-1} W_{x_{i,j}, x_{i,j+1}, (\text{bc},L)}^\beta \right) (d\gamma) \end{aligned}$$

In the case of Dirichlet boundary condition, for any  $g : \Omega_\beta \times \Omega_\beta \rightarrow \mathbb{R}^+$ ,

$$\int_{\Lambda_L} dy g(w_1, w_2) \mathbb{1}_{w_1 \subset \Lambda_L} W_{x,y}^\beta(dw_1) \mathbb{1}_{w_2 \subset \Lambda_L} W_{y,z}^\beta(dw_2) = g(\mathbb{w}|_{[0;\beta]}, \mathbb{w}|_{[\beta;2\beta]}) \mathbb{1}_{w \subset \Lambda_L} W_{x,z}^{2\beta}(dw).$$

In the case of periodic boundary condition, for any  $g : \Omega_\beta \times \Omega_\beta \rightarrow \mathbb{R}^+$  which is  $L$ -periodic in each variable,

$$\int_{\Lambda_L} dy g(w_1, w_2) W_{x,y,(\text{per}),L}^\beta(dw_1) W_{y,z,(\text{per}),L}^\beta(dw_2) = g(\mathbb{w}|_{[0;\beta]}, \mathbb{w}|_{[\beta;2\beta]}) W_{x,z,(\text{per}),L}^{2\beta}(dw).$$

Therefore, by integrating over  $\Lambda_L$ , we can assemble the respective Brownian bridges into Brownian loops, along the cycle structure of  $\sigma$

$$\begin{aligned} & \int_{\Lambda_L} \bigotimes_{i=1}^n dx_i^{\otimes \ell_i} \int f(\gamma_{(\text{bc},L)}) e^{-H_{(\text{bc},L)}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^n \bigotimes_{j=0}^{\ell_i-1} W_{x_{i,j}, x_{i,j+1}, (\text{bc},L)}^\beta \right) (d\gamma) \\ &= \int_{\Lambda_L} dx^{\otimes n} \int \left( f \circ \begin{matrix} (\text{FK}) \leftarrow (\text{rl}) \\ \varphi_{(\text{bc},L)} \end{matrix} \right) (\eta) e^{-H_{(\text{bc},L)}^{(\text{rl})}(\eta)} \left( \bigotimes_{i=1}^n W_{x_i, x_i, (\text{bc},L)}^{\beta \ell_i} \right) (d\eta). \end{aligned}$$

For any given sequence  $\delta \in \mathbb{N}^{\mathbb{N}}$  such that  $\sum_{j \geq 1} j \delta_j = N$ , there are  $N! \cdot \prod_{j \geq 1} \frac{1}{\delta_j! j^{\delta_j}}$  permutations  $\sigma \in \mathbb{S}(N)$  which have exactly  $\delta_j$  cycles of length  $j$ , for any  $j \geq 1$ .

Furthermore, for any  $g : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}^+$ ,

$$\sum_{(\delta_j)_{j \geq 1}} \left( \prod_{j \geq 1} \frac{1}{\delta_j!} \right) g(\delta) = \sum_{n \geq 0} \frac{1}{n!} \sum_{j_1 \geq 1} \cdots \sum_{j_n \geq 1} g \left( \begin{matrix} \mathbb{N} & \xrightarrow{\quad} & \mathbb{N} \\ q & \mapsto & \#\{i \in \llbracket 1; n \rrbracket / j_i = q\} \end{matrix} \right).$$

where  $\sum_{(\delta_j)_{j \geq 1}}$  is a summation over sequences of integers with a finite number of non-zero entries.

We conclude

$$\begin{aligned} & \sum_{N=0}^{+\infty} \frac{1}{N!} \int_{\Lambda_L} dx^{\otimes N} \sum_{\sigma \in \mathbb{S}(N)} \int f(\gamma_{(\text{bc},L)}) e^{-H_{(\text{bc},L)}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^N W_{x_i, x_{\sigma(i)}, (\text{bc},L)}^\beta \right) (d\gamma) \\ &= \sum_{n \geq 0} \frac{1}{n!} \sum_{j_1 \geq 1} \frac{1}{j_1} \cdots \sum_{j_n \geq 1} \frac{1}{j_n} \int_{\Lambda_L} dx^{\otimes n} \int \left( f \circ \begin{matrix} (\text{FK}) \leftarrow (\text{rl}) \\ \varphi_{(\text{bc},L)} \end{matrix} \right) (\eta) e^{-H_{(\text{bc},L)}^{(\text{rl})}(\eta)} \left( \bigotimes_{i=1}^n W_{x_i, x_i, (\text{bc},L)}^{\beta j_i} \right) (d\eta) \\ &= \sum_{n \geq 0} \frac{1}{n!} \left( f \circ \begin{matrix} (\text{FK}) \leftarrow (\text{rl}) \\ \varphi_{(\text{bc},L)} \end{matrix} \right) (\eta) e^{\beta \sum_{i=1}^n j_i - H_{(\text{bc},L)}^{(\text{rl})}(\eta)} \left( \bigotimes_{i=1}^n \int_{\Lambda_L} dx_i \sum_{j_i \geq 1} \frac{e^{-\beta j_i}}{j_i} W_{x_i, x_i, (\text{bc},L)}^{\beta j_i} \right) (d\eta) \\ &= e^{W_{(\text{bc},L)}^{(\text{root})}(\text{Loop}_{(\text{bc},L)}^{(\text{root})})} \int \left( f \circ \begin{matrix} (\text{FK}) \leftarrow (\text{rl}) \\ \varphi_{(\text{bc},L)} \end{matrix} \right) (\eta) \exp \left[ \beta \sum_{w \in \eta} \ell(w) - H_{(\text{bc},L)}^{(\text{rl})}(\eta) \right] \Pi_{(\text{bc},L)}^{(\text{rl})}(d\eta) \end{aligned}$$

because the measure  $W_{(bc),L}^{(\text{root})}$  has finite mass.  $\square$

As mentioned previously, proposition 3.1.1 proved neither (FK) nor (rl) models to be well defined. This is what lemma 3.1.2 is for.

**Lemma 3.1.2** — For all  $L \in r\mathbb{N}$ ,  $\beta > 0$  and  $\mu \in \mathbb{R}$ ,

$$Z_{(bc),L} := \int e^{\beta\mu\#\xi} \Pi_{\Lambda_L}(\mathbf{d}\xi) \sum_{\sigma \in \mathbb{S}(\xi)} \int e^{-H_{(bc),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{x \in \xi} W_{x,\sigma(x),(bc),L}^\beta \right) (\mathbf{d}\gamma) \in ]0; +\infty[.$$

**PROOF (LEMMA 3.1.2)**

According to proposition 3.1.1,

$$Z_{(bc),L} = e^{-L^d + W_{(bc),L}^{(\text{root})}(\text{Loop}_{(bc),L}^{(\text{root})})} \int \exp \left[ \beta(\mu + 1) \sum_{w \in \eta} \ell(w) - H_{(bc),L}^{(\text{rl})}(\eta) \right] \Pi_{(bc),L}^{(\text{rl})}(\mathbf{d}\eta).$$

If  $L$  is a multiple of  $r$ , we deduce immediately from hypothesis II and definition 1.1.3 that for any  $\xi \in \text{Conf}_{<\infty}$ ,

$$U_{(bc),L}(\xi) \geq -A\#\xi + B \sum_{\substack{z \in r\mathbb{Z}^d \\ (z+\Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(bc),L}^r(\xi, z)^2.$$

In particular,

$$(\mu + 1)\#\xi - U_{(bc),L}(\xi) \leq \sum_{\substack{z \in r\mathbb{Z}^d \\ (z+\Lambda_r) \cap \Lambda_L \neq \emptyset}} (A + \mu + 1)N_{(bc),L}^r(\xi, z) - B \cdot N_{(bc),L}^r(\xi, z)^2.$$

So for any  $\beta > 0$  and  $\mu \in \mathbb{R}$ ,

$$(\mu + 1)\#\xi - U_{(bc),L}(\xi) \leq \left(\frac{L}{r}\right)^d \frac{(A + \mu + 1)^2}{4B}. \quad (1)$$

Let  $\eta \in \text{Conf}_{(bc),L}^{(\text{rl})}$ . By applying inequality (1) to sets  $\{w(\beta j + s), 0 \leq j < \ell(w), w \in \eta\}$  for any  $s \in [0; \beta]$  and integrating it, we conclude

$$\beta(\mu + 1) \sum_{w \in \eta} \ell(w) - H_{(bc),L}^{(\text{rl})}(\eta) \leq \beta \left(\frac{L}{r}\right)^d \frac{(A + \mu + 1)^2}{4B}. \quad (2)$$

Since the measure  $W_{(bc),L}^{(\text{root})}$  has finite mass,

$$Z_{(bc),L} \leq e^{-L^d + W_{(bc),L}^{(\text{root})}(\text{Loop}_{(bc),L}^{(\text{root})})} \exp \left[ \beta \left(\frac{L}{r}\right)^d \frac{(A + \mu + 1)^2}{4B} \right] < +\infty.$$

Since the interaction is non-degenerate,

$$Z_{(bc),L} \geq \exp[-L^d - \beta U_{(bc),L}(\emptyset)] > 0. \quad (3)$$

$\square$

---

**REMARK 3.1.3**

If we apply inequality (1) for  $\mu = -1$ , it is clear the interaction  $U_{(\text{bc}),L}$  is bounded from below by a finite quantity. The integral from the formula of  $H_{(\text{bc}),L}^{(\text{FK})}$  is then well defined.

**REMARK 3.1.4**

Together, proposition 3.1.1 and lemma 3.1.2 are enough to prove lemma-definition 1.3.3, lemma-definition 2.1.7 and proposition 2.1.9. Moreover, the fact the path soup model (ps) is well defined (lemma-definition 2.2.5) and equivalent to the previous models (proposition 2.2.7) directly follows from this, as it is basically defined as the quotient of the (rl) model. The second and third equalities of proposition 2.2.7 are corollaries of proposition 2.1.9 and the first equality.

Now we quickly justify the time-shift invariance property of the (ps) model.

**PROOF (PROPOSITION 2.2.8)**

First, it is clear that both measures  $\mathbb{1}_{\bar{w} \subset \Lambda_L} W_{(\text{Dir}),L}(\text{d}\bar{w})$  and  $W_{(\text{per}),L}$  are time-shift invariant :  
for any measurable  $f : \text{Loop}_{(\text{Dir}),L} \rightarrow \mathbb{R}^+$ ,

$$\forall s \in \mathbb{R}, \int f[\bar{w}(\cdot + s)] \mathbb{1}_{\bar{w} \subset \Lambda_L} W_{(\text{Dir}),L}(\text{d}\bar{w}) = \int f(\bar{w}) \mathbb{1}_{\bar{w} \subset \Lambda_L} W_{(\text{Dir}),L}(\text{d}\bar{w})$$

and for any measurable  $f : \text{Loop}_{(\text{per}),L} \rightarrow \mathbb{R}^+$ ,

$$\forall s \in \mathbb{R}, \int f[\bar{w}(\cdot + s)] W_{(\text{per}),L}(\text{d}\bar{w}) = \int f(\bar{w}) W_{(\text{per}),L}(\text{d}\bar{w})$$

where we denote

$$\bar{w}(\cdot + s) := \{w(\cdot + s), w \in \bar{w}\}.$$

We deduce immediately that the processes  $\mathbb{1}_{\eta \subset \Lambda_L} \Pi_{(\text{Dir}),L}^{(\text{ps})}(\text{d}\eta)$  and  $\Pi_{(\text{per}),L}^{(\text{ps})}$  are time-shift invariant too :

for any measurable  $f : \text{Conf}_{(\text{Dir}),L}^{(\text{rl})} \rightarrow \mathbb{R}^+$ ,

$$\forall s \in \mathbb{R}, \int (f \circ T_s^{(\text{ps})})(\eta) \mathbb{1}_{\eta \subset \Lambda_L} \Pi_{(\text{Dir}),L}^{(\text{ps})}(\text{d}\eta) = \int f(\eta) \mathbb{1}_{\eta \subset \Lambda_L} \Pi_{(\text{Dir}),L}^{(\text{ps})}(\text{d}\eta)$$

and for any measurable  $f : \text{Conf}_{(\text{per}),L}^{(\text{rl})} \rightarrow \mathbb{R}^+$ ,

$$\forall s \in \mathbb{R}, \int f \circ T_s^{(\text{ps})} \text{d}\Pi_{(\text{per}),L}^{(\text{ps})} = \int f \text{d}\Pi_{(\text{per}),L}^{(\text{ps})}$$

Since the Hamiltonian  $H_{(\text{bc}),L}^{(\text{ps})}$  is also time-shift invariant :

$$\forall s \in \mathbb{R}, H_{(\text{bc}),L}^{(\text{ps})} \circ T_s^{(\text{ps})} = H_{(\text{bc}),L}^{(\text{ps})}$$

we conclude on the probability measure  $\mathbb{P}_{(\text{bc}),L}^{(\text{ps})}$ . □

---

**REMARK 3.1.5**

Proving the (mp) model to be both well defined (lemma-definition 2.3.4) and equivalent to the (FK) model (proposition 2.3.6) can be done with only the proposition below.

**Proposition 3.1.6** — For any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}^+$ ,

$$\begin{aligned} & \int \Pi_{\Lambda_L}(\mathrm{d}\xi) \sum_{\sigma \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{x \in \xi} W_{x, \sigma(x), (\text{bc}),L}^\beta \right) (\mathrm{d}\gamma) \\ &= \int \left( f \circ \begin{matrix} (\text{FK}) \leftarrow (\text{mp}) \\ \varphi_{(\text{bc}),L} \end{matrix} \right) (\gamma) e^{-H_{(\text{bc}),L}^{(\text{mp})}(\gamma)} \Pi_L^{(\text{mp})}(\mathrm{d}\gamma). \end{aligned}$$

**PROOF (PROPOSITION 3.1.6)**

Let  $\xi = \{x_1 \dots x_N\} \subset \Lambda_L$  be a finite simple configuration.

Sampling uniformly a permutation  $\sigma \in \mathbb{S}(\xi)$  is the same as sampling independently and uniformly the image of each  $x \in \xi$  among all possible images, conditioned to the whole map being indeed a permutation over  $\xi$ . Therefore

$$\begin{aligned} & \sum_{\sigma \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^N W_{x_i, \sigma(x_i), (\text{bc}),L}^\beta \right) (\mathrm{d}\gamma) \\ &= \sum_{y_1 \in \xi} \dots \sum_{y_N \in \xi} \mathbb{1}_{(x_i \mapsto y_i) \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^N W_{x_i, y_i, (\text{bc}),L}^\beta \right) (\mathrm{d}\gamma) \end{aligned}$$

The end point of a Brownian bridge sampled by the measure  $W_{x,y,(\text{bc}),L}^\beta$  is not always  $y$ , depending on the boundary condition. We take this into account

$$= \sum_{y_1 \in \xi_{(\text{bc}),L}} \dots \sum_{y_N \in \xi_{(\text{bc}),L}} \mathbb{1}_{[x_i \mapsto \pi_{(\text{bc}),L}(y_i)] \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^N W_{x_i, y_i}^\beta \right) (\mathrm{d}\gamma)$$

where  $\xi_{(\text{Dir}),L} := \xi$  and  $\xi := \bigcup_{k \in \mathbb{Z}^d} (\xi + Lk)$ .

We then introduce the encoding of the permutation of the (mp) model. First the  $p$  mark :

$$\begin{aligned} &= \sum_{p_1 \in \mathbb{Z}^d} \mathbb{1}_{\xi_{(\text{bc}),L} \cap (x_1 + rp_1 + \Lambda_r) \neq \emptyset} \sum_{y_1 \in \xi_{(\text{bc}),L} \cap (x_1 + rp_1 + \Lambda_r)} \dots \sum_{p_N \in \mathbb{Z}^d} \mathbb{1}_{\xi_{(\text{bc}),L} \cap (x_N + rp_N + \Lambda_r) \neq \emptyset} \sum_{y_N \in \xi_{(\text{bc}),L} \cap (x_N + rp_N + \Lambda_r)} \\ & \mathbb{1}_{[x_i \mapsto \pi_{(\text{bc}),L}(y_i)] \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left( \bigotimes_{i=1}^N W_{x_i, y_i}^\beta \right) (\mathrm{d}\gamma). \end{aligned}$$

Next, we add the  $u$  mark so that the map  $\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, \cdot)$  can be defined

$$\begin{aligned} &= \sum_{p_1 \in \mathbb{Z}^d} N_{(\text{bc}),L}^r(\xi, x_1 + rp_1) \int_0^1 \mathrm{d}u_1 \dots \sum_{p_N \in \mathbb{Z}^d} N_{(\text{bc}),L}^r(\xi, x_N + rp_N) \int_0^1 \mathrm{d}u_N \\ & \mathbb{1}_{\pi_{(\text{bc}),L} \circ \sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, \cdot) \in \mathbb{S}(\xi)} \int f(\gamma_{(\text{bc}),L}) e^{-H_{(\text{bc}),L}^{(\text{FK})}(\gamma)} \left[ \bigotimes_{i=1}^N W_{x_i, \sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x_i)}^\beta \right] (\mathrm{d}\gamma). \end{aligned}$$

---

Finally, all the additional factors are assembled into the (mp) Hamiltonian

$$= \sum_{p_1 \in \mathbb{Z}^d} \nu(p_1) \int_0^1 du_1 \int \mathbb{W}_{0,0}^1(d\omega_1) \dots \sum_{p_N \in \mathbb{Z}^d} \nu(p_N) \int_0^1 du_N \int \mathbb{W}_{0,0}^1(d\omega_N) \left( f \circ \begin{matrix} \text{(FK)} \leftarrow \text{(mp)} \\ \varphi_{(\text{bc}),L} \end{matrix} \right) (\gamma) e^{-H_{(\text{bc}),L}^{(\text{mp})}(\gamma)}$$

where  $\gamma = \{(x_i, p_i, u_i, \omega_i), 1 \leq i \leq N\}$ . □

### 3.2 Entropic method

Unfortunately, to the best of our knowledge, there is no reference in the literature fully exposing the entropic method as we are using it here, much less proving it from start to finish. We state here the plan we follow to establish the thermodynamic limits of the (mp) and (rl) models and quickly justify its validity.

Let  $\mathbb{M}$  be a Polish space, called the mark space. We denote  $\mathbb{X} := \mathbb{R}^d \times \mathbb{M}$ . We define the configuration space

$$\text{Conf}(\mathbb{X}) := \left\{ \gamma = (x, y_x)_{x \in \xi} \mid \xi \subset \mathbb{R}^d \text{ is locally finite} \right\}$$

which we equip with the smallest  $\sigma$ -algebra  $\mathcal{C}(\mathbb{X})$  making measurable the maps  $\gamma \mapsto \#(\gamma \cap E)$  for all bounded Borelian  $E \subset \mathbb{X}$ .

Let  $\nu$  be a finite measure over  $\mathbb{M}$ . We denote by  $\Pi_L^{\text{Leb}^{(d)} \otimes \nu}$  the Poisson point process over  $\Lambda_L \times \mathbb{M}$  with intensity measure  $\text{Leb}^{(d)} \otimes \nu$ .

**Definition 3.2.1** — *Let  $\mathbb{P}$  and  $\mathbb{Q}$  be measures over  $\text{Conf}(\mathbb{X})$ . We define the relative entropy of  $\mathbb{P}$  over  $\mathbb{Q}$  by*

$$I(\mathbb{P}|\mathbb{Q}) := \begin{cases} \int \log\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{P} & \text{if } \mathbb{P} \ll \mathbb{Q} \\ +\infty & \text{otherwise.} \end{cases}$$

We also define the specific entropy of  $\mathbb{P}$  with reference measure  $\nu$  by

$$I_\nu(\mathbb{P}) = \sup_{L>0} \frac{1}{L^d} I\left(\mathbb{P}|_{\Lambda_L} \mid \Pi_L^{\text{Leb}^{(d)} \otimes \nu}\right)$$

where  $\mathbb{P}|_{\Lambda_L}$  is the restriction of  $\mathbb{P}$  to  $\Lambda_L$  defined by

$$\int f d\mathbb{P}|_{\Lambda_L} := \int f[\{(x, y) \in \gamma \mid x \in \Lambda_L\}] \mathbb{P}(d\gamma).$$

Let  $\psi : \mathbb{M} \rightarrow \mathbb{R}$  such that

$$\forall \lambda > 0, \int e^{\lambda \psi(y)} \nu(dy) < +\infty.$$

We call a function  $f : \text{Conf}(\mathbb{X}) \rightarrow \mathbb{R}$  local and tame if there exists  $a > 0$  and a compact  $\Delta \subset \mathbb{R}^d$  such that

$$\forall \gamma \in \text{Conf}(\mathbb{X}), f[\{(x, y) \in \gamma \mid x \in \Delta\}] = f(\gamma)$$



---

and

$$\forall \gamma \in \text{Conf}(\mathbb{X}), a|f(\gamma)| \leq 1 + \sum_{(x,y) \in \gamma, x \in \Delta} \psi(y).$$

Let  $(\mathbb{P}_L)_{L>0}$  be a family of probabilities over  $\text{Conf}(\mathbb{X})$  such that

$$\sup_{L>0} \mathbb{I} \left( \mathbb{P}_L \middle| \Pi_L^{\text{Leb}^{(d)} \otimes \nu} \right)$$

We define for all  $L > 0$  the associated empirical field  $\widetilde{\mathbb{P}}_L$  by

$$\int f d\widetilde{\mathbb{P}}_L := \frac{1}{L^d} \int_{\Lambda_L} dv \int f(\gamma + v) \mathbb{P}_L(d\gamma)$$

for any measurable  $f : \text{Conf}(X) \rightarrow \mathbb{R}^+$ .

We also define the associated stationary field  $\overline{\mathbb{P}}_L$  by

$$\int f d\overline{\mathbb{P}}_L := \frac{1}{L^d} \int_{\Lambda_L} dv \int f \left( \bigcup_{k \in \mathbb{Z}^d} \gamma_k + Lk + v \right) \bigotimes_{k \in \mathbb{Z}^d} \mathbb{P}_L(d\gamma_k)$$

for any measurable  $f : \text{Conf}(X) \rightarrow \mathbb{R}^+$ .

**Lemma 3.2.2** — *For any  $V, \varepsilon > 0$ , there exists some  $m \geq 1$  such that for any  $L > 0$  multiple of  $r$  and any compact  $\Delta$  of volume  $V$ ,*

$$\int \text{Thresh}_m \left[ \sum_{(x,y) \in \gamma, x \in \Delta} \psi(y) \right] \widetilde{\mathbb{P}}_L(d\gamma) \leq \varepsilon$$

where we define the threshold function

$$\text{Thresh}_m(x) := \begin{cases} 0 & \text{if } x \leq m \\ x & \text{if } x > m. \end{cases}$$

**PROOF (LEMMA 3.2.2)**

First let us justify inequality

$$\int \text{Thresh}_m \left[ \sum_{(x,y) \in \gamma, x \in \Delta} \psi(y) \right] \overline{\mathbb{P}}_L(d\gamma) \leq \varepsilon.$$

We are merely restating Lemma 5.2 from [GZ93] with one subtle difference: the compact  $\Delta$  can depend on  $L$ , as long as its volume remains constant. This slight generalization is not exclusive to our model and could be stated for any Gibbs point process of marked points. Indeed the proof of Lemma 5.2 from [GZ93] never involves the shape of  $\Delta$  and exclusively uses its volume.

From there, it is clear

$$\begin{aligned}
\int \text{Thresh}_m \left[ \sum_{(x,y) \in \gamma, x \in \Delta} \psi(y) \right] \widetilde{\mathbb{P}}_L(d\gamma) &\leq \frac{1}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_m \left[ \sum_{(x,y) \in \gamma+v, x \in \Delta} \psi(y) \right] \mathbb{P}_L(d\gamma) \\
&\leq \frac{1}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_m \left[ \sum_{k \in \mathbb{Z}^d} \sum_{\substack{(x,y) \in \gamma_k + Lk + v \\ x \in \Delta}} \psi(y) \right] \bigotimes_{k \in \mathbb{Z}^d} \mathbb{P}_L(d\gamma_k) \\
&\leq \int \text{Thresh}_m \left[ \sum_{(x,y) \in \gamma, x \in \Delta} \psi(y) \right] \overline{\mathbb{P}}_L(d\gamma) \leq \varepsilon. \quad \square
\end{aligned}$$

**Theorem 3.2.3**

The family  $(\overline{\mathbb{P}}_L)_{L>0}$  is sequentially compact. More precisely, for any sequence  $(L_n)_{n \in \mathbb{N}}$ , there exists a subsequence  $(L'_n)_{n \in \mathbb{N}}$  and a probability distribution  $\mathbb{P}_\infty$  over  $\text{Conf}(\mathbb{X})$  such that for any local tame  $f : \text{Conf}(\mathbb{X}) \rightarrow \mathbb{R}$ ,

$$\lim_{n \rightarrow +\infty} \int f d\widetilde{\mathbb{P}}_{L_n} = \int f d\mathbb{P}_\infty.$$

Furthermore, for any compact  $\Delta$ ,

$$\int \sum_{(x,y) \in \gamma, x \in \Delta} \psi(y) \mathbb{P}_\infty(d\gamma) < +\infty.$$

**PROOF (THEOREM 3.2.3)**

According to Proposition 15.52 (p. 330) from [Geo11],

$$I_v(\overline{\mathbb{P}}_L) = \frac{1}{L^d} I(\mathbb{P}_L | \Pi_L^{\text{Leb}^{(d)} \otimes v}).$$

The setting in [Geo11] is discrete, but the result is still valid in a continuous setting. We deduce immediately

$$\sup_{L>0} I_v(\overline{\mathbb{P}}_L) < +\infty.$$

According to Proposition 2.6 from [GZ93], this is enough to prove sequential compactness of the family  $(\overline{\mathbb{P}}_L)_{L>0}$ , which we will abusively write as

$$\lim_{L \rightarrow +\infty} \int f d\overline{\mathbb{P}}_L = \int f d\mathbb{P}_\infty$$

for all local tame functions  $f$ . Furthermore, according to Lemma 5.2 from [GZ93] again,

$$\int \sum_{(x,y) \in \gamma} \psi(y) \mathbb{P}_\infty(d\gamma) < +\infty.$$

According to sections 2.1 and 2.2 from [Der19], the empirical field has the same limit

$$\lim_{L \rightarrow +\infty} \int f d\widetilde{\mathbb{P}}_L = \int f d\mathbb{P}_\infty$$

for all local *bounded* functions  $f$ .

Thanks to lemma 3.2.2, we can extend this limit to any local *tame* function. □

### 3.3 Entropic bounds

In this section, we prove the thermodynamic limits of theorem 2.1.15 and theorem 2.3.9. This will be done by uniformly bounding the relative entropy of the finite volume probabilities.

**PROOF (THEOREM 2.1.15)**

The goal of this proof is to establish

$$\sup_{L \in r\mathbb{N}} \frac{1}{L^d} \mathbb{I} \left( \mathbb{P}_{(bc),L}^{(rl)} \middle| \Pi_{(bc),L}^{(rl)} \right) < +\infty.$$

According to inequations (2) and (3) from the proof of lemma 3.1.2,

$$\mathbb{I} \left( \mathbb{P}_{(bc),L}^{(rl)} \middle| \Pi_{(bc),L}^{(rl)} \right) \leq W_{(bc),L}^{(\text{root})} \left( \text{Loop}_{(bc),L}^{(\text{root})} \right) + \beta U_{(bc),L}(\emptyset) + \beta \left( \frac{L}{r} \right)^d \frac{(A + \mu + 1)^2}{4B}.$$

In the case of Dirichlet boundary condition,

$$\begin{aligned} W_{(\text{Dir}),L}^{(\text{root})} \left( \text{Loop}_{(\text{Dir}),L}^{(\text{root})} \right) &\leq L^d \sum_{j \geq 1} \frac{e^{-\beta j}}{j} W_{0,0}^{\beta j}(\Omega_\beta) \\ &\leq L^d \frac{1}{(2\pi\beta)^{d/2}} \sum_{j \geq 1} \frac{e^{-\beta j}}{j^{d/2+1}}. \end{aligned}$$

For periodic boundary condition,

$$\begin{aligned} W_{(\text{per}),L}^{(\text{root})} \left( \text{Loop}_{(\text{per}),L}^{(\text{root})} \right) &\leq L^d \sum_{j \geq 1} \frac{e^{-\beta j}}{j} \sum_{k \in \mathbb{Z}^d} \frac{1}{(2\pi\beta j)^{d/2}} e^{-\frac{L^2}{2\pi\beta j} \|k\|^2} \\ &\leq L^d \sum_{j \geq 1} \frac{e^{-\beta j}}{j} \cdot \frac{1}{(2\pi\beta j)^{d/2}} \left( 1 + \frac{\sqrt{2\pi\beta j}}{L} \right)^d. \end{aligned}$$

Either way, there exists  $W_0 > 0$  such that for any  $L \geq r$ ,  $W_{(bc),L}^{(\text{root})} \left( \text{Loop}_{(bc),L}^{(\text{root})} \right) \leq W_0 L^d$ .

Therefore

$$\limsup_{L \in r\mathbb{N}} \frac{1}{L^d} \mathbb{I} \left( \mathbb{P}_{(bc),L}^{(rl)} \middle| \Pi_{(bc),L}^{(rl)} \right) \leq W_0 + \beta \frac{1}{r^d} \cdot \frac{(A + \mu + 1)^2}{4B} < +\infty.$$

According to theorem 3.2.3, this is enough to prove the thermodynamic limit. The class of tame functions is defined by the choice  $\psi = 1$ .  $\square$

To prove the thermodynamic limit of the (mp) model, we first need the following technical result.

**Lemma 3.3.1** — *Let  $L > 0$  be a multiple of  $r$ . For any  $\gamma \in \text{ConfPerm}_{(bc),L}^{(\text{mp})}$  such that  $\dot{\gamma} \subset \Lambda_L$ ,*

$$6^{-d} \sum_{(x,p,u,\omega) \in \gamma} \log \left[ N_{(bc),L}^r(\dot{\gamma}, x + rp) \right] \leq \log(2^d) \# \gamma + \sum_{\substack{z \in r\mathbb{Z}^d \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(bc),L}^r(\dot{\gamma}, z) \cdot \log \left[ N_{(bc),L}^r(\dot{\gamma}, z) \right].$$

---

**PROOF (LEMMA 3.3.1)**

Let  $(x, p, u, \omega) \in \gamma$ . It is clear that

$$N_{(\text{bc}),L}^r(\dot{\gamma}, x + rp) = N_{(\text{bc}),L}^r[\dot{\gamma}, \pi_{(\text{bc}),L}(x + rp)].$$

If for any  $y \in \mathbb{R}^d$  we denote as  $\lfloor y \rfloor$  the closest point to  $y$  in the lattice  $r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1)$ , then

$$N_{(\text{bc}),L}^r(\dot{\gamma}, y) \leq N_{(\text{bc}),L}^{2r}(\dot{\gamma}, \lfloor y \rfloor)$$

Furthermore, let  $z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1)$  such that  $(z + \Lambda_r) \cap \Lambda_L \neq \emptyset$ , then  $z$  is the closest point of that lattice to some  $\pi_{(\text{bc}),L}(x + rp)$  at most  $N_{(\text{bc}),L}^r(\dot{\gamma}, z)$  times. Thus

$$\begin{aligned} \sum_{(x,p,u,\omega) \in \gamma} \log\{N_{(\text{bc}),L}^r[\dot{\gamma}, \pi_{(\text{bc}),L}(x + rp)]\} &\leq \sum_{(x,p,u,\omega) \in \gamma} \log\{N_{(\text{bc}),L}^{2r}[\dot{\gamma}, \lfloor \pi_{(\text{bc}),L}(x + rp) \rfloor]\} \\ &\leq \sum_{\substack{z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1) \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^{2r}(\dot{\gamma}, z) \cdot \log[N_{(\text{bc}),L}^r(\dot{\gamma}, z)]. \end{aligned}$$

Since for any  $z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1)$ ,

$$\frac{1}{2^d} N_{(\text{bc}),L}^{2r}(\dot{\gamma}, z) = \frac{1}{2^d} \sum_{\varepsilon \in \{\pm 1\}^d} N_{(\text{bc}),L}^r(\dot{\gamma}, z + \frac{r}{2}\varepsilon)$$

by convexity of  $x \mapsto x \log(x)$  then

$$\begin{aligned} &\sum_{\substack{z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1) \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^{2r}(\dot{\gamma}, z) \cdot \log[N_{(\text{bc}),L}^{2r}(\dot{\gamma}, z)] \\ &\leq \sum_{\substack{z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1) \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} \left\{ \log(2^d) N_{(\text{bc}),L}^{2r}(\dot{\gamma}, z) + \sum_{\varepsilon \in \{\pm 1\}^d} N_{(\text{bc}),L}^r(\dot{\gamma}, z + \frac{r}{2}\varepsilon) \cdot \log[N_{(\text{bc}),L}^r(\dot{\gamma}, z + \frac{r}{2}\varepsilon)] \right\}. \end{aligned}$$

We take care of the first term

$$\sum_{\substack{z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1) \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^{2r}(\dot{\gamma}, z) \leq \sum_{\substack{z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1) \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} \sum_{\varepsilon \in \{\pm 1\}^d} N_{(\text{bc}),L}^r(\dot{\gamma}, z + \frac{r}{2}\varepsilon).$$

As illustrated in fig. 5, for each  $z \in r\mathbb{Z}^d$  such that  $(z + \Lambda_{2r}) \cap \Lambda_L \neq \emptyset$ , the term  $N_{(\text{bc}),L}^r(\dot{\gamma}, z)$  appears in the summation at most  $2^d$  times.

Thus

$$\begin{aligned} \sum_{\substack{z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1) \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^{2r}(\dot{\gamma}, z) &\leq 2^d \sum_{\substack{z \in r\mathbb{Z}^d \\ (z + \Lambda_{2r}) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^r(\dot{\gamma}, z) \\ &\leq 2^d \#[\dot{\gamma}_{(\text{bc}),L} \cap (\Lambda_L + \Lambda_{2r})] \leq 2^d \#(\dot{\gamma}_{(\text{bc}),L} \cap \Lambda_{3L}) \leq 6^d \#\gamma. \end{aligned}$$

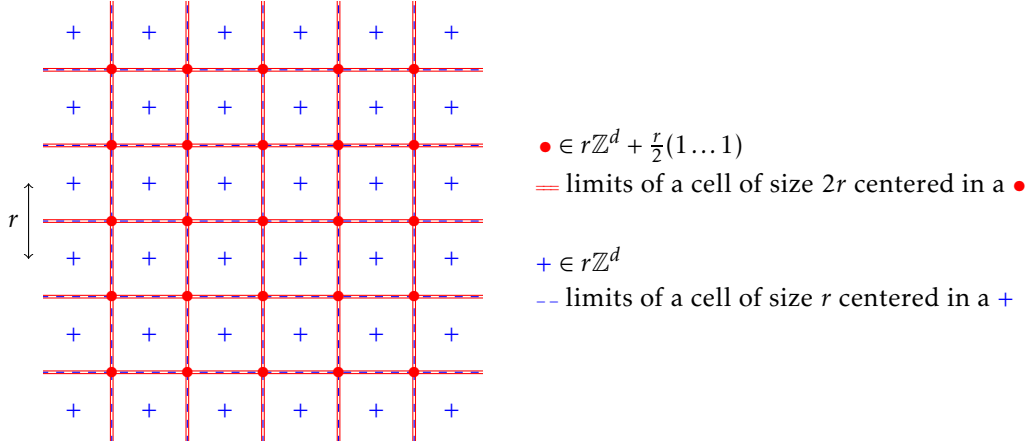


Figure 5: Counting the  $N_{(\text{bc}),L}^r(\dot{\gamma}, z)$ ,  $z \in r\mathbb{Z}^d$

We control similarly the second term

$$\begin{aligned} & \sum_{\substack{z \in r\mathbb{Z}^d + \frac{r}{2}(1 \dots 1) \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} \sum_{\varepsilon \in \{\pm 1\}^d} N_{(\text{bc}),L}^r(\dot{\gamma}, z + \frac{r}{2}\varepsilon) \cdot \log \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, z + \frac{r}{2}\varepsilon) \right] \\ & \leq 2^d \sum_{\substack{z \in r\mathbb{Z}^d \\ (z + \Lambda_{2r}) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^r(\dot{\gamma}, z) \cdot \log \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, z) \right]. \end{aligned}$$

For any  $z \in r\mathbb{Z}^d$ , the condition  $(z + \Lambda_{2r}) \cap \Lambda_L \neq \emptyset$  implies  $(z + \Lambda_r) \cap \Lambda_{3L} \neq \emptyset$ . Therefore

$$\sum_{\substack{z \in r\mathbb{Z}^d \\ (z + \Lambda_{2r}) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^r(\dot{\gamma}, z) \cdot \log \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, z) \right] \leq \sum_{\substack{z \in r\mathbb{Z}^d \\ (z + \Lambda_r) \cap \Lambda_{3L} \neq \emptyset}} N_{(\text{bc}),L}^r(\dot{\gamma}, z) \cdot \log \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, z) \right].$$

Since  $\dot{\gamma} \subset \Lambda_L$ , we conclude

$$\sum_{\substack{z \in r\mathbb{Z}^d \\ (z + \Lambda_r) \cap \Lambda_{3L} \neq \emptyset}} N_{(\text{bc}),L}^r(\dot{\gamma}, z) \cdot \log \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, z) \right] \leq 3^d \sum_{\substack{z \in r\mathbb{Z}^d \\ (z + \Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^r(\dot{\gamma}, z) \cdot \log \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, z) \right]. \quad \square$$

As previously, proving the thermodynamic limit is done by uniformly bounding the relative entropy.

**Proposition 3.3.2** — *Let  $\kappa \in ]0; 1[$ . Assume*

$$v(p) = \left[ \frac{r^2(1-\kappa)}{2\pi\beta} \right]^{d/2} \int_{p+\Lambda_1} \exp \left[ -\frac{r^2}{2\beta}(1-\kappa)\|x\|^2 \right] dx.$$

*Then, there exists  $\lambda > 0$  such that*

$$\sup_{L \in \mathbb{N}} \frac{1}{L^d} \mathbb{I} \left( \mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \middle| \Pi_L^{\lambda \text{m}} \right) < +\infty$$

---

where  $\Pi_L^{\lambda\mathfrak{m}}$  is the Poisson process of intensity measure  $\lambda\mathfrak{m}$  with  $\mathfrak{m} := \text{Leb}^{(d)} \otimes \nu \otimes \text{Leb}^{(1)} \otimes \mathbb{W}_{0,0}^1$ .

**PROOF (PROPOSITION 3.3.2)**

First, we know that

$$\begin{aligned}\mathbb{P}_{(\text{bc},L)}^{(\text{mp})}(\text{d}\gamma) &= \frac{1}{Z_{(\text{bc},L)}} \exp\left[\beta\mu\#\gamma - H_{(\text{bc},L)}^{(\text{mp})}(\gamma)\right] \Pi_L^{\mathfrak{m}}(\text{d}\gamma) \\ &= \frac{1}{Z_{(\text{bc},L)}} \exp\left\{\lambda + [\beta\mu - \log(\lambda)]\#\gamma - H_{(\text{bc},L)}^{(\text{mp})}(\gamma)\right\} \Pi_L^{\lambda\mathfrak{m}}(\text{d}\gamma).\end{aligned}$$

We will uniformly bound the relative entropy by dealing separately with each non-trivial term of the density

- $-\log(Z_{(\text{bc},L)})$
- $-\sum_{(x,p,u,\omega)\in\gamma} \left\{ \frac{1}{2\beta} \left\| \sigma_{(\text{bc},L)}^{(\text{mp})}(\gamma, x) - x \right\|^2 + \log[\nu(p)] \right\}$
- $\sum_{(x,p,u,\omega)\in\gamma} \log\left[N_{(\text{bc},L)}^r(\dot{\gamma}, x + rp)\right]$  – interaction term

We will proceed in this order then conclude.

- According to inequation (3) from the proof of lemma 3.1.2,

$$-\log(Z_{(\text{bc},L)}) \leq L^d + \beta U_{(\text{bc},L)}(\emptyset).$$

- For any  $\gamma \in \text{ConfPerm}_{(\text{bc},L)}^{(\text{mp})}$  and  $(x, p, u, \omega) \in \gamma$ ,

$$\left\| \sigma_{(\text{bc},L)}^{(\text{mp})}(\gamma, x) - x \right\| \geq r(\|p\| - \sqrt{d}).$$

Given the admitted formula of  $\nu$ ,

$$\log[\nu(p)] \geq \frac{d}{2} \log\left[\frac{r^2(1-\kappa)}{2\pi\beta}\right] - \frac{r^2}{2\beta}(1-\kappa)(\|p\| + \sqrt{d})^2.$$

Thus there exists  $C_\kappa \in \mathbb{R}$  such that for any  $(x, p, u, \omega) \in \gamma$ ,

$$\frac{1}{2\beta} \left\| \sigma_{(\text{bc},L)}^{(\text{mp})}(\gamma, x) - x \right\|^2 + \log[\nu(p)] \geq C_\kappa.$$

- According to lemma 3.3.1,

$$\sum_{(x,p,u,\omega)\in\gamma} \log\left[N_{(\text{bc},L)}^r(\dot{\gamma}, x + rp)\right] \leq 6^d \left\{ \log(2^d)\#\gamma + \sum_{\substack{z \in r\mathbb{Z}^d \\ (z+\Lambda_r) \cap \Lambda_L = \emptyset}} N_{(\text{bc},L)}^r(\dot{\gamma}, z) \cdot \log\left[N_{(\text{bc},L)}^r(\dot{\gamma}, z)\right] \right\}.$$

To deal with that problematic  $x \log(x)$  term, we will use the interaction's superstability.

By the equivalence of models (mp), (FK) and (ps) from proposition 2.3.6 and proposition 2.2.7,

$$\begin{aligned} & \int \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\text{d}\gamma) \int_0^\beta \text{d}s \left( -U_{(\text{bc}),L} \left[ \left\{ x + \frac{s}{\beta} \left( \sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x) - x \right) + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), (x, p, u, \omega) \in \gamma \right\} \right] \right) \\ &= \int \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) \int_0^\beta \text{d}s \left[ -U_{(\text{bc}),L}(\{w(s), w \in \gamma\}) \right] \\ &= \int \mathbb{P}_{(\text{bc}),L}^{(\text{ps})}(\text{d}\eta) \int_0^\beta \text{d}s \left( -U_{(\text{bc}),L}[\{\bar{w}(\beta j + s), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}] \right). \end{aligned}$$

We remind that the meaning of  $\eta \mapsto U_{(\text{bc}),L}[\{\bar{w}(\beta j + s), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}]$  is that of a quotient function. We refer to definition 2.2.4 for more details.

By choosing  $\mu = -1$  in inequality 1 from the proof of lemma 3.1.2, the function  $-U_{(\text{bc}),L}$  is bounded, thus integrable. So

$$\begin{aligned} & \int \mathbb{P}_{(\text{bc}),L}^{(\text{ps})}(\text{d}\eta) \int_0^\beta \text{d}s \left( -U_{(\text{bc}),L}[\{\bar{w}(\beta j + s), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}] \right) \\ &= \int_0^\beta \text{d}s \int \mathbb{P}_{(\text{bc}),L}^{(\text{ps})}(\text{d}\eta) \left( -U_{(\text{bc}),L}[\{\bar{w}(\beta j + s), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}] \right). \end{aligned}$$

By time-shift stationarity, this equals

$$\begin{aligned} & \int_0^\beta \text{d}s \int \mathbb{P}_{(\text{bc}),L}^{(\text{ps})}(\text{d}\eta) \left( -U_{(\text{bc}),L}[\{\bar{w}(\beta j), 0 \leq j < \ell(\bar{w}), \bar{w} \in \cdot\}] \circ T_s^{(\text{ps})} \right)(\eta) \\ &= \beta \int \mathbb{P}_{(\text{bc}),L}^{(\text{ps})}(\text{d}\eta) \left( -U_{(\text{bc}),L}[\{\bar{w}(\beta j), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}] \right). \end{aligned}$$

By equivalence of models again,

$$\begin{aligned} & \int \mathbb{P}_{(\text{bc}),L}^{(\text{ps})}(\text{d}\eta) \left( -U_{(\text{bc}),L}[\{\bar{w}(\beta j), 0 \leq j < \ell(\bar{w}), \bar{w} \in \eta\}] \right) \\ &= \int \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\text{d}\gamma) \left[ -U_{(\text{bc}),L}(\dot{\gamma}) \right] \\ &\leq \int \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\text{d}\gamma) \left[ A \# \gamma - B \sum_{\substack{z \in r\mathbb{Z}^d \\ (z+\Lambda_r) \cap \Lambda_L \neq \emptyset}} N_{(\text{bc}),L}^r(\dot{\gamma}, z)^2 \right]. \end{aligned}$$

There exists  $C^{(0)} \in \mathbb{R}$  such that  $6^d x \log(x) - B\beta x^2 \leq C^{(0)}$  for any  $x \geq 0$ . We deduce

$$\sum_{(x,p,u,\omega) \in \gamma} \log \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, x + rp) \right] - \text{interaction term} \leq \left[ 6^d \log(2^d) + A\beta \right] \# \gamma - \left( \frac{L}{r} \right)^d C^{(0)}.$$

Therefore,

$$\begin{aligned} \mathbb{I} \left( \mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \middle| \Pi_L^{\lambda \text{m}} \right) &\leq \lambda + \beta U_{(\text{bc}),L}(\emptyset) + \left( 1 - \frac{C^{(0)}}{rd} \right) L^d \\ &\quad + \int \left[ \beta \mu - \log(\lambda) + \frac{d}{2} \log(2\pi\beta) - C_\kappa + 6^d \log(2^d) + A\beta \right] \# \gamma \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\text{d}\gamma). \end{aligned}$$

For  $\lambda > 0$  large enough, the integrand is non-positive and we conclude

$$\sup_{L \in r\mathbb{N}} \frac{1}{L^d} \mathbb{1} \left( \mathbb{P}_{(bc),L}^{(mp)} \Big| \Pi_L^{\lambda m} \right) < +\infty. \quad \square$$

We need to justify the choice

$$\psi(p, u, \omega) = \left| \text{Saus}_\delta \left( \begin{array}{l} [0; \beta] \rightarrow \mathbb{R}^d \\ s \mapsto \frac{s}{\beta} r p + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \end{array} \right) \right|^\alpha$$

for all  $\alpha \in [0; 2[$ .

**Lemma 3.3.3** — *Let  $\alpha \in [0; 2[$  and  $\delta > 0$ . There exists  $\kappa \in ]0; 1[$  small enough such that for any  $\lambda > 0$ ,*

$$\int \exp \left[ \lambda \left| \text{Saus}_\delta \left( \begin{array}{l} [0; \beta] \rightarrow \mathbb{R}^d \\ s \mapsto \frac{s}{\beta} r p + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \end{array} \right) \right|^\alpha \right] \nu_\kappa(dp) \mathbb{W}_{0,0}^1(d\omega) < +\infty$$

with

$$\nu_\kappa(p) = \left[ \frac{r^2(1-\kappa)}{2\pi\beta} \right]^{d/2} \int_{p+\Lambda_1} \exp \left[ -\frac{r^2}{2\beta}(1-\kappa)\|x\|^2 \right] dx.$$

**PROOF (LEMMA 3.3.3)**

For any  $y \in \mathbb{R}^d$ , we denote as  $\lfloor y \rfloor$  the point of the lattice  $\mathbb{Z}^d$  closest to  $x$ .

If a random vector  $Y \in \mathbb{R}^d$  has a Gaussian distribution given by

$$G_{r^2(1-\kappa)}(dy) = \left[ \frac{r^2(1-\kappa)}{2\pi\beta} \right]^{d/2} \exp \left[ -\frac{r^2}{2\beta}(1-\kappa)\|y\|^2 \right] dy$$

then  $\lfloor Y \rfloor$ 's distribution is  $\nu_\kappa$ . Therefore

$$\begin{aligned} & \int \exp \left[ \lambda \left| \text{Saus}_\delta \left( s \mapsto \frac{s}{\beta} r p + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right) \right|^\alpha \right] \nu_\kappa(dp) \mathbb{W}_{0,0}^1(d\omega) \\ & \leq \int \exp \left[ \lambda \left| \text{Saus}_{\delta+r\sqrt{d}} \left( s \mapsto \frac{s}{\beta} r y + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right) \right|^\alpha \right] G_{r^2(1-\kappa)}(dy) \mathbb{W}_{0,0}^1(d\omega) \\ & \leq \int \exp \left[ \lambda \left| \text{Saus}_{\delta+r\sqrt{d}} \left( s \mapsto \frac{s}{\beta} y + \omega(s) \right) \right|^\alpha \right] G_{1-\kappa}(dy) \mathbb{W}_{0,0}^\beta(d\omega) \\ & \leq (1-\kappa)^{d/2} \int e^{\frac{\kappa}{2\beta}\|y\|^2} \exp \left[ \lambda \left| \text{Saus}_{\delta+r\sqrt{d}} \left( s \mapsto \frac{s}{\beta} y + \omega(s) \right) \right|^\alpha \right] G_1(dy) \mathbb{W}_{0,0}^\beta(d\omega) \end{aligned}$$

If  $Y$  is a random vector with Gaussian distribution of variance  $\beta$  and  $\omega$  is a Brownian bridge from 0 to 0 in time  $\beta$ , then the trajectory  $s \mapsto \frac{s}{\beta} Y + \omega(s)$  has the distribution of a Brownian motion, up to time  $\beta$ . So

$$\begin{aligned} & \int e^{\frac{\kappa}{2\beta}\|y\|^2} \exp \left[ \lambda \left| \text{Saus}_{\delta+r\sqrt{d}} \left( s \mapsto \frac{s}{\beta} y + \omega(s) \right) \right|^\alpha \right] g_1(dy) \mathbb{W}_{0,0}^\beta(d\omega) \\ & \leq \mathbb{E}_\omega \left\{ e^{\frac{\kappa}{2\beta}\|\omega(\beta)\|^2} \exp \left[ \lambda \left| \text{Saus}_{\delta+r\sqrt{d}}(\omega) \right|^\alpha \right] \right\}. \end{aligned}$$



where the expectancy is taken along the standard Brownian motion.

We repeat the reasoning we made for  $f_1$  in example 1.3.10. The cylinder whose axis goes from 0 to  $\omega(\beta)$  with radius  $\delta + r\sqrt{d}$  has bigger volume than the Wiener sausage. Then

$$\|\omega(\beta)\| \leq \frac{1}{c_{d-1}(\delta + r\sqrt{d})^{d-1}} |\text{Saus}_{\delta+r\sqrt{d}}(\omega)|$$

where we denote as  $c_{d-1}$  the volume of the  $d-1$  dimensional ball of radius 1.

Thus

$$\begin{aligned} & \mathbb{E}_\omega \left\{ e^{\frac{\kappa}{2\beta} \|\omega(\beta)\|^2} \exp[\lambda |\text{Saus}_{\delta+r\sqrt{d}}(\omega)|^\alpha] \right\} \\ & \leq \mathbb{E}_\omega \left\{ \exp \left[ \frac{\kappa}{2\beta c_{d-1}^2 (\delta + r\sqrt{d})^{2d-2}} |\text{Saus}_{\delta+r\sqrt{d}}(\omega)|^2 + \lambda |\text{Saus}_{\delta+r\sqrt{d}}(\omega)|^\alpha \right] \right\}. \end{aligned}$$

To prove this integral is finite, it is sufficient to prove

$$\mathbb{E}_\omega \left\{ \exp \left[ \frac{\kappa}{2\beta c_{d-1}^2 (\delta + r\sqrt{d})^{2d-2}} |\text{Saus}_{\delta+r\sqrt{d}}(\omega)|^2 \right] \right\} < +\infty$$

which is true for  $\kappa$  small enough, according to theorem A.o.2.  $\square$

#### PROOF (THEOREM 2.3.9)

Finally, according to lemma 3.2.2, the uniform bound from proposition 3.3.2 on the entropy is enough to prove the existence of the thermodynamic limit  $\mathbb{P}_{(bc),\infty}^{(mp)}$ . More precisely, this is enough to prove tightness of the sequence  $\tilde{\mathbb{P}}_{(bc),L}^{(mp)}$ ,  $L \in \mathbb{N}$ , hence the existence of a converging subsequence.

The class of tame functions is justified by lemma 3.3.3.  $\square$

### 3.4 Permutation in infinite volume

In section 2.3 we constructed the (mp) framework to encode the permutation of the (FK) model into the  $p$ ,  $u$  and  $\omega$  marks. We just proved the (mp) model to have a thermodynamic limit, but does the encoding pass to the limit? Is it only possible to define an infinite volume permutation with those marks? Thankfully the answer is yes.

**Definition 3.4.1** — For any  $\gamma \in \text{Conf}^{(mp)}$  and  $z \in \mathbb{R}^d$ , we denote

$$N_\infty^r(\dot{\gamma}, z) := \#[\dot{\gamma} \cap (z + \Lambda_r)].$$

In infinite volume, we define the set of authorized configurations as

$$\text{ConfAuth}_\infty^{(mp)} := \left\{ \gamma \in \text{Conf}^{(mp)} \mid \dot{\gamma} \text{ is simple and } \forall (x, p, u, \omega) \in \gamma, N_\infty^r(\dot{\gamma}, x + rp) \geq 1 \right\}.$$

---

For any authorized configuration  $\gamma \in \text{ConfAuth}_\infty^{(\text{mp})}$ , we define the map  $\sigma_\infty^{(\text{mp})}(\gamma, \cdot) : \dot{\gamma} \rightarrow \dot{\gamma}$  such that for any  $(x, p, u, \omega) \in \gamma$ ,

$$\sigma_\infty^{(\text{mp})}(\gamma, x) := \left[ N_\infty^r(\dot{\gamma}, x + rp) \cdot u \right]' \text{th element of } \dot{\gamma} \cap (x + rp + \Lambda_r)$$

if we order them lexicographically. Finally, in infinite volume, we call an authorized configuration permutation-wise if it satisfies

$$\forall y \in \dot{\gamma}, \exists ! x \in \dot{\gamma}, \sigma_\infty^{(\text{mp})}(\gamma, x) = y$$

and we denote

$$\text{ConfPerm}_\infty^{(\text{mp})} := \left\{ \gamma \in \text{ConfAuth}_\infty^{(\text{mp})} \mid \gamma \text{ is permutation-wise} \right\}.$$

**Definition 3.4.2** — We define the transition map  $\varphi_\infty^{(\text{FK}) \leftarrow (\text{mp})} : \text{ConfPerm}_\infty^{(\text{mp})} \rightarrow \text{ConfPerm}^{(\text{FK})}$  by

$$\varphi_\infty^{(\text{FK}) \leftarrow (\text{mp})}(\gamma) := \left\{ \begin{array}{ll} [0; \beta] & \rightarrow \mathbb{R}^d \\ s & \mapsto x + \frac{s}{\beta} \left[ \sigma_\infty^{(\text{mp})}(\gamma, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), \quad (x, p, u, \omega) \in \gamma \end{array} \right\}.$$

**Proposition 3.4.3** —  $\mathbb{P}_{(\text{bc}), \infty}^{(\text{mp})}(\text{ConfPerm}_\infty^{(\text{mp})}) = 1$ .

**PROOF (PROPOSITION 3.4.3)**

If we are given a configuration  $\gamma$ , the map  $\sigma_\infty^{(\text{mp})}(\gamma, \cdot)$  is a well-defined bijection if and only if

- o. The marked point configuration  $\gamma$  is simple.
1.  $\forall (x, p, u, \omega) \in \gamma, N_\infty^r(\dot{\gamma}, x + rp) \geq 1$  so that the mark is not pointing to an empty region of space. The map  $\sigma_\infty^{(\text{mp})}(\gamma, \cdot)$  is then well defined.
2.  $\forall (x, p, u, \omega) \in \gamma, \sum_{(x', p', u', \omega') \in \gamma} \mathbb{1}_{\sigma_\infty^{(\text{mp})}(\gamma, x') = x} \leq 1$  so that the map is injective.
3. The map  $\sigma_\infty^{(\text{mp})}(\gamma, \cdot)$  is surjective.

We will check that those 4 properties hold  $\mathbb{P}_{(\text{bc}), \infty}^{(\text{mp})}$  almost surely.

**Step o** Let us assume  $\mathbb{P}_{(\text{bc}), \infty}^{(\text{mp})}(\gamma \text{ is not simple}) > 0$ . So there exists a compact  $\Delta \subset \mathbb{R}^d$  such that

$$\mathbb{P}_{(\text{bc}), \infty}^{(\text{mp})}(\gamma \cap \Delta \text{ is not simple}) > 0.$$

This event is local so there exists  $L > 0$  such that

$$\widetilde{\mathbb{P}}_{(\text{bc}), L}^{(\text{mp})}(\gamma \cap \Delta \text{ is not simple}) = \frac{1}{L^d} \int_{\Lambda_L} dv \mathbb{P}_{(\text{bc}), L}^{(\text{mp})}[(\gamma + v) \cap \Delta \text{ is not simple}] > 0$$

which is wrong.

---

**Step 1** Let us denote as  $(x_1, p_1, u_1, \omega_1) \in \gamma$  the closest point to 0 in the configuration. We assume

$$\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})} \left[ N_{\infty}^r(\dot{\gamma}, x_1 + rp_1) = 0 \right] > 0.$$

Then there exists  $\varepsilon > 0$  and  $D > 0$  such that

$$\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})} \left[ N_{\infty}^r(\dot{\gamma}, x_1 + rp_1) = 0 \text{ and } \|x_1\| \leq D \text{ and } \|p_1\| \leq \frac{D}{r} \right] > \varepsilon.$$

This event is local so there exists  $L_0 > 0$  such that for any  $L > L_0$ ,

$$\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})} \left[ N_{\infty}^r(\dot{\gamma}, x_1 + rp_1) = 0 \text{ and } \|x_1\| \leq D \text{ and } \|p_1\| \leq \frac{D}{r} \right] > \frac{\varepsilon}{2}.$$

By definition of the empirical field,

$$\begin{aligned} & \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})} \left[ N_{\infty}^r(\dot{\gamma}, x_1 + rp_1) = 0 \text{ and } \|x_1\| \leq D \text{ and } \|p_1\| \leq \frac{D}{r} \right]_{L \rightarrow +\infty} = \\ & \frac{1}{L^d} \int_{\Lambda_{L-2D-r}} dv \mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ N_{\infty}^r(\dot{\gamma} + v, x_{1,v} + rp_{1,v}) = 0 \text{ and } \|x_{1,v}\| \leq D \text{ and } \|p_{1,v}\| \leq \frac{D}{r} \right] + o(1) \end{aligned}$$

where we denote as  $(x_{1,v}, p_{1,v}, u_{1,v}, \omega_{1,v})$  the point of  $\gamma + v$  which is closest to 0.

So there exists  $L > L_0$ ,  $\delta > 0$  and  $v \in \Lambda_{L-2D-r}$  such that

$$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ N_{\infty}^r(\dot{\gamma} + v, x_{1,v} + rp_{1,v}) = 0 \text{ and } \|x_{1,v}\| \leq D \text{ and } \|p_{1,v}\| \leq \frac{D}{r} \right] > \delta.$$

Since  $v \in \Lambda_{L-2D-r}$ , if  $\|x_{1,v}\| \leq D$  and  $\|p_{1,v}\| \leq \frac{D}{r}$ , then we have

$$\begin{aligned} (\dot{\gamma}_{(\text{bc}),L} + v) \cap (x_{1,v} + rp_{1,v} + \Lambda_r) &= [\dot{\gamma}_{(\text{bc}),L} \cap (x_{1,v} - v + rp_{1,v} + \Lambda_r)] + v \\ &= [\dot{\gamma} \cap (x_{1,v} - v + rp_{1,v} + \Lambda_r)] + v \\ &= (\dot{\gamma} + v) \cap (x_{1,v} + rp_{1,v} + \Lambda_r). \end{aligned}$$

Thus

$$N_{\infty}^r(\dot{\gamma} + v, x_{1,v} + rp_{1,v}) = N_{(\text{bc}),L}^r(\dot{\gamma} + v, x_{1,v} + rp_{1,v}).$$

Therefore

$$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ N_{(\text{bc}),L}^r(\dot{\gamma} + v, x_{1,v} + rp_{1,v}) = 0 \right] > \delta.$$

If we denote as  $(x_1^v, p_1^v, u_1^v, \omega_1^v)$  the point of  $\gamma$  which is closest to  $-v$ , this translates to

$$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ N_{(\text{bc}),L}^r(\dot{\gamma}, x_1^v + rp_1^v) = 0 \right] > \delta$$

which is not true because  $\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left( \text{ConfAuth}_{(\text{bc}),L}^{(\text{mp})} \right) = 1$ .

We have proven

$$\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})} \left[ N_{\infty}^r(\dot{\gamma}, x_1 + rp_1) \geq 1 \right] = 1$$

and with the same procedure, we can prove this equality for the  $n$ 'th closest point to 0 in the configuration, for any  $n \geq 1$ . Finally

$$\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})} \left[ \forall n \geq 1, N_{\infty}^r(\dot{\gamma}, x_n + rp_n) \geq 1 \right] = 1.$$

---

**Step 2** We will follow a similar proof path in step 2. We assume

$$\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})} \left[ \sum_{(x',p',u',\omega') \in \gamma} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma,x')=x_1} > 1 \right] > 0.$$

Then there exists  $\varepsilon > 0$  and  $D > 0$  such that

$$\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})} \left[ \sum_{\substack{(x',p',u',\omega') \in \gamma \\ \|x'\| \leq D, \|p'\| \leq D/r}} \mathbb{1}_{N_{\infty}^r(\dot{\gamma},x'+rp') \geq 1} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma,x')=x_1} > 1 \text{ and } \|x_1\| \leq D \right] > \varepsilon.$$

Since this event is local, there exists  $L_0 > 0$  such that for any  $L > L_0$ ,

$$\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})} \left[ \sum_{\substack{(x',p',u',\omega') \in \gamma \\ \|x'\| \leq D, \|p'\| \leq D/r}} \mathbb{1}_{N_{\infty}^r(\dot{\gamma},x'+rp') \geq 1} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma,x')=x_1} > 1 \text{ and } \|x_1\| \leq D \right] > \frac{\varepsilon}{2}.$$

By definition of the empirical field,

$$\begin{aligned} & \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})} \left[ \sum_{\substack{(x',p',u',\omega') \in \gamma \\ \|x'\| \leq D, \|p'\| \leq D/r}} \mathbb{1}_{N_{\infty}^r(\dot{\gamma},x'+rp') \geq 1} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma,x')=x_1} > 1 \text{ and } \|x_1\| \leq D \right] \stackrel{=}{=} \\ & \frac{1}{L^d} \int_{\Lambda_{L-2D-r}} dv \mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ \sum_{\substack{(x',p',u',\omega') \in \gamma+v \\ \|x'\| \leq D, \|p'\| \leq D/r}} \mathbb{1}_{N_{\infty}^r(\dot{\gamma}+v,x'+rp') \geq 1} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma+v,x')=x_{1,v}} > 1 \text{ and } \|x_{1,v}\| \leq D \right] + o(1) \end{aligned}$$

where we denote as  $(x_{1,v}, p_{1,v}, u_{1,v}, \omega_{1,v})$  the point of  $\gamma + v$  which is closest to 0.

So there exists  $L > L_0$ ,  $\delta > 0$  and  $v \in \Lambda_{L-2D-r}$  such that

$$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ \sum_{\substack{(x',p',u',\omega') \in \gamma+v \\ \|x'\| \leq D, \|p'\| \leq D/r}} \mathbb{1}_{N_{\infty}^r(\dot{\gamma}+v,x'+rp') \geq 1} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma+v,x')=x_{1,v}} > 1 \text{ and } \|x_{1,v}\| \leq D \right] > \delta.$$

Since  $v \in \Lambda_{L-2D-r}$ , if  $\|x'\| \leq D$  and  $\|p'\| \leq \frac{D}{r}$ , then we have

$$\begin{aligned} (\dot{\gamma}_{(\text{bc}),L} + v) \cap (x' + rp' + \Lambda_r) &= [\dot{\gamma}_{(\text{bc}),L} \cap (x' + rp' - v + \Lambda_r)] + v \\ &= [\dot{\gamma} \cap (x' + rp' - v + \Lambda_r)] + v \\ &= (\dot{\gamma} + v) \cap (x' + rp' + \Lambda_r). \end{aligned}$$

Therefore

$$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ \sum_{\substack{(x',p',u',\omega') \in \gamma+v \\ \|x'\| \leq D, \|p'\| \leq D/r}} \mathbb{1}_{N_{(\text{bc}),L}^r(\dot{\gamma}+v,x'+rp')} \geq 1} \mathbb{1}_{\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma+v,x')=x_{1,v}} > 1 \right] > \delta.$$

If we denote as  $(x_1^v, p_1^v, u_1^v, \omega_1^v)$  the point of  $\gamma$  which is closest to  $-v$ , this translates to

$$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})} \left[ \sum_{\substack{(x',p',u',\omega') \in \gamma \\ \|x'\| \leq D, \|p'\| \leq D/r}} \mathbb{1}_{N_{(\text{bc}),L}^r(\dot{\gamma},x'+rp')} \geq 1} \mathbb{1}_{\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma,x')=x_1^v} > 1 \right] > \delta$$

which is not true because  $\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, \cdot)$  is  $\mathbb{P}_{(\text{bc}),L}^{(\text{mp})}$  almost surely injective.

Once again, we conclude by generalizing the procedure for any  $n$ 'th closest point to 0 of the configuration.

**Step 3** We will prove a seemingly weaker, yet sufficient result :

$$\forall k \in \mathbb{Z}^d, \quad \sum_{(x,p,u,\omega) \in \gamma} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma,x) \in rk + \Lambda_r} = \#\left[\dot{\gamma} \cap (rk + \Lambda_r)\right].$$

By stationarity, it is enough to prove it for  $k = 0$ .

By injectivity, we already know

$$\sum_{(x,p,u,\omega) \in \gamma} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma,x) \in \Lambda_r} \leq \#\left(\dot{\gamma} \cap \Lambda_r\right)$$

To prove the equality more easily, we introduce a discretized version of the configuration

$$\begin{aligned} \gamma &\longrightarrow \lfloor \gamma \rfloor \\ (x,p,u,\omega) &\longmapsto \left( \lfloor x \rfloor, \left[ \sigma_{\infty}^{(\text{mp})}(\gamma,x) \right] - \lfloor x \rfloor \right) \end{aligned}$$

where, for  $x \in \mathbb{R}^d$ , the point  $\lfloor x \rfloor$  is the closest point of  $r\mathbb{Z}^d$  to  $x$ .

Then

$$\begin{aligned} \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}(\text{d}\gamma) \sum_{(x,p,u,\omega) \in \gamma} \mathbb{1}_{\sigma_{\infty}^{(\text{mp})}(\gamma,x) \in \Lambda_r} &= \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}(\text{d}\gamma) \sum_{(z,k) \in \lfloor \gamma \rfloor} \mathbb{1}_{z+k=0} \\ &= \sum_{j \in \mathbb{Z}^d} \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}(\text{d}\gamma) \sum_{(z,k) \in \lfloor \gamma \rfloor} \mathbb{1}_{z=rj} \mathbb{1}_{k=-rj}. \end{aligned}$$

---

The probability measure  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}$  is stationary so  $[\gamma]$ 's distribution is  $r\mathbb{Z}^d$ -stationary. Thus

$$\begin{aligned} &= \sum_{j \in \mathbb{Z}^d} \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}(\mathrm{d}\gamma) \sum_{(z,k) \in [\gamma]} \mathbb{1}_{z=0} \mathbb{1}_{k=-rj} \\ &= \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}(\mathrm{d}\gamma) \sum_{(z,k) \in [\gamma]} \mathbb{1}_{z=0} \\ &= \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}(\mathrm{d}\gamma) \#(\dot{\gamma} \cap \Lambda_r). \end{aligned}$$

We conclude  $\sum_{(x,p,u,\omega) \in \gamma} \mathbb{1}_{\sigma_\infty^{(\text{mp})}(\gamma,x) \in \Lambda_r} = \#(\dot{\gamma} \cap \Lambda_r)$  is true  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}$  almost surely.

We know each cell of the lattice receives the good number of marks pointing to it. By injectivity of the map  $\sigma_\infty^{(\text{mp})}(\gamma, \cdot)$ , it guarantees all points in the cell are reached.  $\square$

### 3.5 Thermodynamic limit

Section 3.3 was dedicated to the thermodynamics limits of (mp) and (rl) models. We then proved (section 3.4) the marks of the (mp) infinite volume model to still have meaning and still encode a (FK) representation.

**Definition 3.5.1** — We define the probability measure  $\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}$  over  $\text{Conf}^{(\text{FK})}$  by

$$\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(E) := \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})} \left[ \left( \varphi_\infty^{(\text{FK}) \leftarrow (\text{mp})} \right)^{-1} (E \cap \text{ConfPerm}^{(\text{FK})}) \right]$$

for any event  $E \in \mathcal{C}^{(\text{FK})}$ .

But is this new (FK) model the thermodynamic limit of the finite volume (FK) model we presented in lemma-definition 1.3.3 ? We claimed this is true in our main thermodynamic limit result (theorem 1.3.12) and the current section will prove this statement.

**REMARK 3.5.2**

Proposition 1.3.14 is a direct corollary of proposition 3.4.3.

**Corollary 3.5.3** — For any  $V, \delta, \varepsilon > 0$  and  $\alpha \in [0; 2]$ , there exists  $m \geq 1$  such that for any  $L > 0$  multiple of  $r$  and compact  $\Delta$  of volume  $V$ ,

$$\int \text{Thresh}_m \left\{ \sum_{(x,p,u,\omega) \in \gamma, x \in \Delta} \left| \text{Saus}_\delta \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \right\} \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\mathrm{d}\gamma) \leq \varepsilon.$$

Furthermore

$$\int \sum_{(x,p,u,\omega) \in \gamma, x \in \Delta} \left| \text{Saus}_\delta \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}(\mathrm{d}\gamma) < +\infty.$$

---

**PROOF**

This is immediate from lemma 3.2.2 and lemma 3.3.3 (see section 3.2).  $\square$

We establish a result analogous to corollary 3.5.3 for the (FK) models.

**Proposition 3.5.4** — *For any  $V, \delta, \varepsilon > 0$ ,  $\alpha \in [0; 2[$  and  $K \in \mathbb{N}$ , there exists  $m > 0$  such that for any  $L > 0$  multiple of  $r$  and compact  $\Delta$  of volume  $V$ , if  $\Delta \subseteq \Lambda_{KL}$  then*

$$\int \text{Thresh}_m \left[ \sum_{w \in \gamma, w(0) \in \Delta} |\text{Saus}_\delta(w)|^\alpha \right] \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \leq \varepsilon.$$

Furthermore

$$\int \sum_{w \in \gamma, w(0) \in \Delta} |\text{Saus}_\delta(w)|^\alpha \mathbb{P}_{(bc),\infty}^{(FK)}(d\gamma) < +\infty.$$

**PROOF (PROPOSITION 3.5.4)**

By definition,

$$\begin{aligned} & \int \text{Thresh}_m \left[ \sum_{w \in \gamma, w(0) \in \Delta} |\text{Saus}_\delta(w)|^\alpha \right] \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \\ &= \frac{1}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_m \left[ \sum_{w \in \gamma+v, w(0) \in \Delta} |\text{Saus}_\delta(w)|^\alpha \right] \mathbb{P}_{(bc),L}^{(FK)}(d\gamma) \\ &= \frac{1}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_m \left[ \sum_{w \in \gamma, w(0) \in \Delta-v} |\text{Saus}_\delta(w)|^\alpha \right] \mathbb{P}_{(bc),L}^{(FK)}(d\gamma). \end{aligned} \quad (4)$$

**Dir** Trivially, this is equal to

$$\frac{1}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_m \left[ \sum_{w \in \gamma, w(0) \in \pi_{(\text{Dir}),L}(\Delta-v)} |\text{Saus}_\delta(w)|^\alpha \right] \mathbb{P}_{(\text{Dir}),L}^{(FK)}(d\gamma).$$

**per** Since  $\Delta + v \subseteq \Lambda_{(K+1)L}$  for all  $v \in \Lambda_L$ , each term  $w(0) \in \pi_{(\text{per}),L}(\Delta - v)$  of the sum appears at most  $(K+1)^d$  times. This makes the expression from (4) smaller than

$$\frac{1}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_m \left[ (K+1)^d \sum_{w \in \gamma, w(0) \in \pi_{(\text{per}),L}(\Delta-v)} |\text{Saus}_\delta(w)|^\alpha \right] \mathbb{P}_{(\text{per}),L}^{(FK)}(d\gamma).$$

So in all cases, the expression from (4) is smaller than

$$\frac{(K+1)^d}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_{\frac{m}{(K+1)^d}} \left[ \sum_{w \in \gamma, w(0) \in \pi_{(bc),L}(\Delta-v)} |\text{Saus}_\delta(w)|^\alpha \right] \mathbb{P}_{(bc),L}^{(FK)}(d\gamma).$$

According to proposition 2.3.6 and the definition of the map  $\varphi_{(bc),L}^{(FK)\leftarrow(mp)}$ , this equals

$$\frac{(K+1)^d}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_{\frac{m}{(K+1)^d}} \left\{ \sum_{\substack{(x,p,u,\omega) \in \gamma \\ x \in \pi_{(bc),L}(\Delta-v)}} \left| \text{Saus}_\delta \left[ s \mapsto \frac{s}{\beta} \left( \sigma_{(bc),L}^{(mp)}(\gamma, x) - x \right) + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \right\} \mathbb{P}_{(bc),L}^{(mp)}(d\gamma).$$

Then

$$\leq \frac{(K+1)^d}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_{\frac{m}{(K+1)^d}} \left\{ \sum_{\substack{(x,p,u,\omega) \in \gamma \\ x \in \bar{\Delta}_L - v}} \left| \text{Saus}_\delta \left[ s \mapsto \frac{s}{\beta} \left( \sigma_{(bc),L}^{(mp)}(\gamma, x) - x \right) + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \right\} \mathbb{P}_{(bc),L}^{(mp)}(d\gamma)$$

where  $\bar{\Delta}_L := \bigcup_{k \in \mathbb{Z}^d, \|k\|_\infty \leq K} (\Delta + Lk)$ .

By approximating the trajectory by  $s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right)$ ,

$$\begin{aligned} &\leq \frac{(K+1)^d}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_{\frac{m}{(K+1)^d}} \left\{ \sum_{\substack{(x,p,u,\omega) \in \gamma \\ x \in \bar{\Delta}_L - v}} \left| \text{Saus}_{\delta+r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \right\} \mathbb{P}_{(bc),L}^{(mp)}(d\gamma) \\ &\leq \frac{(K+1)^d}{L^d} \int_{\Lambda_L} dv \int \text{Thresh}_{\frac{m}{(K+1)^d}} \left\{ \sum_{\substack{(x,p,u,\omega) \in \gamma+v \\ x \in \Delta_L}} \left| \text{Saus}_{\delta+r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \right\} \mathbb{P}_{(bc),L}^{(mp)}(d\gamma) \\ &\leq \frac{(K+1)^d}{L^d} \int \text{Thresh}_{\frac{m}{(K+1)^d}} \left\{ \sum_{\substack{(x,p,u,\omega) \in \gamma \\ x \in \bar{\Delta}_L}} \left| \text{Saus}_{\delta+r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \right\} \widetilde{\mathbb{P}}_{(bc),L}^{(mp)}(d\gamma). \end{aligned}$$

Finally we just need corollary 3.5.3 to conclude.

With a very analog procedure as in finite volume, we can show that

$$\begin{aligned} &\int \sum_{w \in \gamma, w(0) \in \Delta} |\text{Saus}_\delta(w)|^\alpha \mathbb{P}_{(bc),\infty}^{(FK)}(d\gamma) \\ &\leq \int \sum_{(x,p,u,\omega) \in \gamma, x \in \Delta} \left| \text{Saus}_{\delta+r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \mathbb{P}_{(bc),\infty}^{(mp)}(d\gamma). \end{aligned}$$

which is finite by corollary 3.5.3.  $\square$

We can now prove the first part of theorem 1.3.12, that is, for any  $f : \text{ConfPerm}^{(FK)} \rightarrow \mathbb{R}$   $\epsilon$ -local and  $\epsilon$ -tame,

$$\lim_{L \rightarrow +\infty} \int f d\widetilde{\mathbb{P}}_{(bc),L}^{(FK)} = \int f d\mathbb{P}_{(bc),\infty}^{(FK)}.$$



---

**PROOF (THEOREM 1.3.12, 1/2)**

Let  $\varepsilon > 0$  and  $f : \text{ConfPerm}^{(\text{FK})} \rightarrow \mathbb{R}$   $\varepsilon$ -local such that

$$\forall \gamma \in \text{ConfPerm}^{(\text{FK})}, a|f(\gamma)| \leq 1 + \sum_{w \in \text{Proj}_{\varepsilon\Delta}(\gamma)} |\text{Saus}_\delta(w)|^\alpha.$$

We assume without any loss of generality that  $\delta > r\sqrt{d}$ .

For any  $m \geq 1$ , we define  $f_m : \text{ConfPerm}^{(\text{FK})} \rightarrow \mathbb{R}$  by

$$f_m(\gamma) := \begin{cases} f(\gamma) & \text{if } \sum_{w \in \text{Proj}_{\varepsilon\Delta}(\gamma)} |\text{Saus}_\delta(w)|^\alpha < \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m}{2} \right]^\alpha \\ 0 & \text{otherwise.} \end{cases}$$

For  $m \geq 1$  large enough, the error between  $f$  and  $f_m$  will be proven to be arbitrarily small. Our goal is to write the integrals of the  $f_m$  as integrals of local bounded  $h_m : \text{Conf}^{(\text{mp})} \rightarrow \mathbb{R}$ . By using the thermodynamic limit of the (mp) model, this will be enough to conclude.

The functions  $f_m$ ,  $m \geq 1$  are uniformly dominated by the  $\varepsilon$ -tame bound of  $f$ , which is integrable under  $\mathbb{P}_{(\text{bc},\infty)}^{(\text{FK})}$  (see proposition 3.5.4). So by dominated convergence theorem there exists  $m_0 \geq 1$  large enough such that

$$\forall m \geq m_0, \int |f_m - f| d\mathbb{P}_{(\text{bc},\infty)}^{(\text{FK})} \leq \varepsilon.$$

Furthermore, according to proposition 3.5.4, there exists  $m_1 \geq m_0$  such that

$$\forall L \in r\mathbb{N}, \Delta \subseteq \Lambda_L \implies \int |f_{m_1} - f| d\widetilde{\mathbb{P}}_{(\text{bc},L)}^{(\text{FK})} \leq \varepsilon.$$

Then for all  $L \in r\mathbb{N}$  such that  $\Delta \subseteq \Lambda_L$ , the error term is

$$\left| \int f d\mathbb{P}_{(\text{bc},\infty)}^{(\text{FK})} - \int f d\widetilde{\mathbb{P}}_{(\text{bc},L)}^{(\text{FK})} \right| \leq 2\varepsilon + \left| \int f_{m_1} d\mathbb{P}_{(\text{bc},\infty)}^{(\text{FK})} - \int f_{m_1} d\widetilde{\mathbb{P}}_{(\text{bc},L)}^{(\text{FK})} \right|$$

We assume without any loss of generality that  $\Delta + \Lambda_r \subseteq \Lambda_{m_1}$ . Since the function  $f_{m_1}$  is bounded, we have

$$\int f_{m_1} d\widetilde{\mathbb{P}}_{(\text{bc},L)}^{(\text{FK})} \underset{L \rightarrow +\infty}{=} \frac{1}{L^d} \int_{\Lambda_{L-2m_1}} dv \int f_{m_1}(\gamma + v) \mathbb{P}_{(\text{bc},L)}^{(\text{FK})}(d\gamma) + o(1) \quad (5)$$

We define the function  $g_{m_1} : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  by

$$g_{m_1}(\gamma) = \begin{cases} f_{m_1}(\gamma_0) & \text{if } \exists \gamma_0 \in \text{ConfPerm}^{(\text{FK})}, \gamma = \text{Proj}_{\varepsilon\Delta}(\gamma_0) \\ 0 & \text{otherwise.} \end{cases}$$

This is well defined thanks to  $\varepsilon$ -locality of  $f_{m_1}$  and it is clear that

$$\forall \gamma \in \text{ConfPerm}^{(\text{FK})}, (g_{m_1} \circ \text{Proj}_{\varepsilon\Delta})(\gamma) = f_{m_1}(\gamma).$$

Let  $L \geq 2m_1$ . Since the map  $\overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}}$  commutes with any translation  $\gamma \mapsto \gamma + v$ , we deduce

$$\frac{1}{L^d} \int_{\Lambda_{L-2m_1}} dv \int f_{m_1}(\gamma+v) \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) = \frac{1}{L^d} \int_{\Lambda_{L-2m_1}} dv \int \left( g_{m_1} \circ \text{Proj}_{\in\Delta} \circ \overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}} \right) (\gamma+v) \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma). \quad (6)$$

Let  $\gamma \in \text{ConfPerm}_{(\text{bc}),L}^{(\text{mp})}$  such that  $\dot{\gamma} \subset \Lambda_{2L-2m_1}$ . Since  $\Delta \subseteq \Lambda_{m_1}$ , we have  $\dot{\gamma}_{(\text{bc}),L} \cap \Delta = \dot{\gamma} \cap \Delta$ . Then

$$\begin{aligned} & \left( \text{Proj}_{\in\Delta} \circ \overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}} \right) (\gamma) \\ &= \left\{ \begin{array}{l} [0; \beta] \rightarrow \mathbb{R}^d \\ s \mapsto x + \frac{s}{\beta} \left[ \sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), \quad (x, p, u, \omega) \in \gamma \\ \text{s.t. } x \in \Delta \end{array} \right\} \\ &= \left\{ \begin{array}{l} [0; \beta] \rightarrow \mathbb{R}^d \\ s \mapsto x + \frac{s}{\beta} \left[ \sigma_{\infty}^{(\text{mp})}(\gamma_{(\text{bc}),L}, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), \quad (x, p, u, \omega) \in \gamma \\ \text{s.t. } x \in \Delta \end{array} \right\}. \end{aligned}$$

In the following, we will make a slight abuse of notations by writing  $(x, p, u, \omega) \in \gamma \cap \Delta$ .

**Case 1** We assume the condition

$$\sum_{(x,p,u,\omega) \in \gamma \cap \Delta} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha > \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m_1}{2} \right]^\alpha.$$

The map  $\text{Proj}_{\in\Delta} \circ \overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}}$  induces a bijection between the set of marked points  $(x, p, u, \omega) \in \gamma \cap \Delta$  and the image set of bridges  $w \in \left( \text{Proj}_{\in\Delta} \circ \overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}} \right) (\gamma)$ . Then

$$\begin{aligned} \sum_{w \in \left( \text{Proj}_{\in\Delta} \circ \overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}} \right) (\gamma)} |\text{Saus}_\delta(w)|^\alpha &\geq \sum_{(x,p,u,\omega) \in \gamma \cap \Delta} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \\ &> \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m_1}{2} \right]^\alpha. \end{aligned}$$

Thus

$$\left( g_{m_1} \circ \text{Proj}_{\in\Delta} \circ \overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}} \right) (\gamma) = f_{m_1} \left[ \overset{(\text{FK})\leftarrow(\text{mp})}{\varphi_{(\text{bc}),L}} (\gamma) \right] = 0.$$

**Case 2** We assume

$$\sum_{(x,p,u,\omega) \in \gamma \cap \Delta} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \leq \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m_1}{2} \right]^\alpha.$$

We use the same reasoning as in example 1.3.10: the cylinder whose axis goes from 0 to  $rp$  with diameter  $\delta - r\sqrt{r}$  has a smaller volume than the Wiener sausage

$$\forall (x, p, u, \omega) \in \gamma \cap \Delta, r\|p\| \leq \frac{1}{c_{d-1} (\delta - r\sqrt{d})^{d-1}} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right| \leq \frac{m_1}{2}.$$

Then, for all  $(x, p, u, \omega) \in \gamma \cap \Delta$ , the expression  $\sigma_\infty^{(\text{mp})}(\gamma_{(\text{bc}),L}, x)$  only depends on

$$\gamma_{(\text{bc}),L} \cap (\Delta + \Lambda_{m_1} + \Lambda_r) \subseteq \gamma_{(\text{bc}),L} \cap \Lambda_{2m_1} = \gamma \cap \Lambda_{2m_1}.$$

We will slightly abuse notations by writing

$$\begin{aligned} & \left( \text{Proj}_{\in \Delta} \circ \overset{(\text{FK}) \leftarrow (\text{mp})}{\varphi_{(\text{bc}),L}} \right) (\gamma) \\ &= \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto x + \frac{s}{\beta} \left[ \sigma_\infty^{(\text{mp})}(\gamma \cap \Lambda_{2m_1}, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), \end{array} \right. (x, p, u, \omega) \in \gamma \cap \Delta \end{aligned}$$

In all cases, we denote

$$\varphi_\Delta^*(\gamma) := \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto x + \frac{s}{\beta} \left[ \sigma_\infty^{(\text{mp})}(\gamma \cap \Lambda_{2m_1}, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), \end{array} \right. \begin{array}{l} (x, p, u, \omega) \in \gamma \cap \Delta \text{ s.t.} \\ N_\infty^r(\dot{\gamma} \cap \Lambda_{2m_1}, x + rp) \geq 1 \end{array}$$

We define  $h_{m_1} : \text{Conf}^{(\text{mp})} \rightarrow \mathbb{R}$  by

$$h_{m_1}(\gamma) := \begin{cases} (g_{m_1} \circ \varphi_\Delta^*)(\gamma) & \text{if } \sum_{(x,p,u,\omega) \in \gamma \cap \Delta} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \leq \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m_1}{2} \right]^\alpha \\ 0 & \text{otherwise.} \end{cases}$$

In all cases,  $h_{m_1}$  equals  $g_{m_1} \circ \text{Proj}_{\in \Delta} \circ \overset{(\text{FK}) \leftarrow (\text{mp})}{\varphi_{(\text{bc}),L}}$  if  $\dot{\gamma} \subset \Lambda_{2L-2m_1}$ . Therefore

$$\frac{1}{L^d} \int_{\Lambda_{L-2m_1}} dv \int \left( g_{m_1} \circ \text{Proj}_{\in \Delta} \circ \overset{(\text{FK}) \leftarrow (\text{mp})}{\varphi_{(\text{bc}),L}} \right) (\gamma+v) \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\mathbf{d}\gamma) = \frac{1}{L^d} \int_{\Lambda_{L-2m_1}} dv \int h_{m_1}(\gamma+v) \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\mathbf{d}\gamma).$$

By combining with equations (5) and (6), we get

$$\begin{aligned} \int f_{m_1} d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})} &\underset{L \rightarrow +\infty}{=} \frac{1}{L^d} \int_{\Lambda_{L-2m_1}} dv \int h_{m_1}(\gamma+v) \mathbb{P}_{(\text{bc}),L}^{(\text{mp})}(\mathbf{d}\gamma) + o(1) \\ &\underset{L \rightarrow +\infty}{=} \int h_{m_1} d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})} + o(1) \end{aligned}$$

because  $h_{m_1}$  is bounded.

To sum things up,

$$\begin{aligned} \left| \int f d\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})} - \int f d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})} \right| &\underset{L \rightarrow +\infty}{\leq} 2\varepsilon + \left| \int f_{m_1} d\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})} - \int f_{m_1} d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})} \right| \\ &\underset{L \rightarrow +\infty}{\leq} 2\varepsilon + \left| \int f_{m_1} d\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})} - \int h_{m_1} d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})} \right| + o(1). \end{aligned}$$

Since  $\varphi_\Delta^* : \text{Conf}^{(\text{mp})} \rightarrow \text{Conf}^{(\text{FK})}$  is  $\varepsilon$ -local relatively to  $\Lambda_{2m_1}$ , the function  $h_{m_1}$  satisfies the conditions of theorem 2.3.9 thus

$$\lim_{L \rightarrow +\infty} \int h_{m_1} d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{mp})} = \int h_{m_1} d\mathbb{P}_{(\text{bc}),\infty}^{(\text{mp})}$$

so

$$\left| \int f \, d\mathbb{P}_{(bc),\infty}^{(FK)} - \int f \, d\widetilde{\mathbb{P}}_{(bc),L}^{(FK)} \right|_{L \rightarrow +\infty} \leq 2\varepsilon + \left| \int f_{m_1} \, d\mathbb{P}_{(bc),\infty}^{(FK)} - \int h_{m_1} \, d\mathbb{P}_{(bc),\infty}^{(mp)} \right| + o(1).$$

According to definition 3.5.1,

$$\int f_{m_1} \, d\mathbb{P}_{(bc),\infty}^{(FK)} = \int f_{m_1} \circ \varphi_\infty^{(FK)\leftarrow(mp)} \, d\mathbb{P}_{(bc),\infty}^{(mp)}$$

so the final step is to establish the equality

$$h_{m_1} = f_{m_1} \circ \varphi_\infty^{(FK)\leftarrow(mp)}$$

$\mathbb{P}_{(bc),\infty}^{(mp)}$  almost surely and the proof will be complete.

**Case 1** We assume

$$\sum_{(x,p,u,\omega) \in \gamma \cap \Delta} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha > \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m_1}{2} \right]^\alpha.$$

Then of course  $h_{m_1}(\gamma) = 0$ .

Just like previously, the map  $\text{Proj}_{\in\Delta} \circ \varphi_\infty^{(FK)\leftarrow(mp)}$  induces a bijection between the set of marked points  $(x, p, u, \omega) \in \gamma \cap \Delta$  and the image set of bridges  $w \in \left( \text{Proj}_{\in\Delta} \circ \varphi_\infty^{(FK)\leftarrow(mp)} \right)(\gamma)$ .

Then

$$\begin{aligned} \sum_{w \in \left( \text{Proj}_{\in\Delta} \circ \varphi_\infty^{(FK)\leftarrow(mp)} \right)(\gamma)} |\text{Saus}_\delta(w)|^\alpha &\geq \sum_{(x,p,u,\omega) \in \gamma \cap \Delta} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \\ &> \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m_1}{2} \right]^\alpha. \end{aligned}$$

Therefore

$$\left( f_{m_1} \circ \varphi_\infty^{(FK)\leftarrow(mp)} \right)(\gamma) = 0 = h_{m_1}(\gamma).$$

**Case 2** We assume

$$\sum_{(x,p,u,\omega) \in \gamma \cap \Delta} \left| \text{Saus}_{\delta-r\sqrt{d}} \left[ s \mapsto \frac{s}{\beta} rp + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \right] \right|^\alpha \leq \left[ c_{d-1} (\delta - r\sqrt{d})^{d-1} \frac{m_1}{2} \right]^\alpha.$$

Then

$$\begin{aligned} h_{m_1}(\gamma) &= (g_{m_1} \circ \varphi_\Delta^*)(\gamma) \\ &= g_{m_1} \left( \left\{ \begin{array}{l} [0; \beta] \rightarrow \mathbb{R}^d \\ s \mapsto x + \frac{s}{\beta} \left[ \sigma_\infty^{(mp)}(\gamma \cap \Lambda_{2m_1}, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right) \end{array} \right\} \right) \end{aligned}$$

$(x, p, u, \omega) \in \gamma \cap \Delta \text{ s.t. } N_\infty^r(\dot{\gamma} \cap \Lambda_{2m_1}, x + rp) \geq 1$

Just as we argued in the previous case 2, we know

$$\forall (x, p, u, \omega) \in \gamma \cap \Delta, r\|p\| \leq \frac{m_1}{2}.$$

Then for all  $(x, p, u, \omega) \in \gamma \cap \Delta$ , the expressions  $N_\infty^r(\dot{\gamma}, x + rp)$  and  $\sigma_\infty^{(\text{mp})}(\gamma, x)$  only depends on  $\gamma \cap \Lambda_{2m_1}$  and

$$h_{m_1}(\gamma) = g_{m_1} \left( \left\{ \begin{array}{l} [0; \beta] \rightarrow \mathbb{R}^d \\ s \mapsto x + \frac{s}{\beta} \left[ \sigma_\infty^{(\text{mp})}(\gamma, x) - x \right] + \sqrt{\beta} \omega \left( \frac{s}{\beta} \right), \quad (x, p, u, \omega) \in \gamma \cap \Delta \text{ s.t. } \\ N_\infty^r(\dot{\gamma}, x + rp) \geq 1 \end{array} \right\} \right)$$

Since  $\mathbb{P}_{(\text{bc}), \infty}^{(\text{mp})}(\text{ConfPerm}_\infty^{(\text{mp})}) = 1$  (see proposition 3.4.3), it is clear the condition  $N_\infty^r(\dot{\gamma}, x + rp) \geq 1$  is almost surely true.

Therefore

$$h_{m_1}(\gamma) = \left( g_{m_1} \circ \text{Proj}_{\in \Delta} \begin{array}{c} (\text{FK}) \leftarrow (\text{mp}) \\ \varphi_\infty \end{array} \right) (\gamma).$$

If  $\gamma \in \text{ConfPerm}_\infty^{(\text{mp})}$  then  $\begin{array}{c} (\text{FK}) \leftarrow (\text{mp}) \\ \varphi_\infty \end{array} (\gamma) \in \text{ConfPerm}^{(\text{FK})}$  and

$$\left( g_{m_1} \circ \text{Proj}_{\in \Delta} \circ \begin{array}{c} (\text{FK}) \leftarrow (\text{mp}) \\ \varphi_\infty \end{array} \right) (\gamma) = \left( f_{m_1} \circ \begin{array}{c} (\text{FK}) \leftarrow (\text{mp}) \\ \varphi_\infty \end{array} \right) (\gamma).$$

Again, according to proposition 3.4.3, the equality is true  $\mathbb{P}_{(\text{bc}), \infty}^{(\text{FK})}$  almost surely.  $\square$

### 3.6 Extension of locality

locality in the sense of  $\text{Proj}_{\in \Delta}$  is natural but it does not preserve the cycle structure. This is what the second part of theorem 1.3.12 is for.

In section 3.5, we made use of proposition 3.5.4 in the proof of the first part of the thermodynamic limit. Similarly, we will need the following proposition.

**Proposition 3.6.1** — *For any  $D, \delta, \varepsilon > 0$  and  $\alpha \in [0; 1[$ , there exists  $m > 0$  such that for any  $L \geq 4 \max(\delta, D)$  multiple of  $r$  and compact  $\Delta$  whose diameter is  $\sup_{x \in \Delta} \|x\| \leq D$ ,*

$$\int \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w(0) \notin \Lambda_m} |\text{Saus}_\delta(w)|^\alpha \widetilde{\mathbb{P}}_{(\text{bc}), L}^{(\text{FK})}(d\gamma) \leq \varepsilon.$$

Furthermore,

$$\int \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} |\text{Saus}_\delta(w)|^\alpha \mathbb{P}_{(\text{bc}), \infty}^{(\text{FK})}(d\gamma) < +\infty.$$

#### REMARK 3.6.2

Accessing the bridges that only intersect  $\Delta$  made us lose a power in the volume of the sausage. This can be seen at the end of the proof below: an exponent  $1 + \alpha$  appears.

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**PROOF (PROPOSITION 3.6.1)**  
For any  $L > 0$  multiple of  $r$ ,

$$\begin{aligned}
& \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \gamma} \mathbb{1}_{w \cap \Lambda \neq \emptyset} \mathbb{1}_{w(0) \notin \Lambda_m} |\text{Saus}_\delta(w)|^\alpha \\
& \leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \gamma} \mathbb{1}_{w \cap \Lambda \neq \emptyset} \mathbb{1}_{w(0) \in i + \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha \\
& \leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \gamma} \mathbb{1}_{0 \in \text{Saus}_\delta(w)} \mathbb{1}_{w(0) \in i + \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha.
\end{aligned}$$

We assume without any loss of generality  $\delta \geq D$ . Then

$$\leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \gamma} \mathbb{1}_{0 \in \text{Saus}_\delta(w)} \mathbb{1}_{w(0) \in i + \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha. \quad (7)$$

**per** By stationarity of  $\widetilde{\mathbb{P}}_{(\text{per}),L}^{(FK)}$ , this is

$$\leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(\text{per}),L}^{(FK)}(d\gamma) \sum_{w \in \gamma} \mathbb{1}_{-i \in \text{Saus}_\delta(w)} \mathbb{1}_{w(0) \in \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha.$$

**Dir** We directly bound the expression from (7) by replacing the configuration by its periodized version

$$\leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(FK)}(d\gamma) \sum_{w \in \gamma_{(\text{per}),L}} \mathbb{1}_{0 \in \text{Saus}_\delta(w)} \mathbb{1}_{w(0) \in i + \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha$$

where we remind  $\gamma_{(\text{per}),L} = \bigcup_{k \in \mathbb{Z}^d} (\gamma + Lk)$ .

If  $\gamma$  is distributed along  $\widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(FK)}$ , then the configuration  $\gamma_{(\text{per}),L}$  has a stationary distribution. Thus

$$\begin{aligned}
& \leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(FK)}(d\gamma) \sum_{w \in \gamma_{(\text{per}),L}} \mathbb{1}_{-i \in \text{Saus}_\delta(w)} \mathbb{1}_{w(0) \in \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha \\
& \leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(FK)}(d\gamma) \sum_{k \in \mathbb{Z}^d} \sum_{w \in \gamma} \mathbb{1}_{-i \in \text{Saus}_\delta(w) + Lk} \mathbb{1}_{w(0) + Lk \in \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha.
\end{aligned}$$

Under  $\widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(FK)}$ , the set  $\{w(0), w \in \gamma\}$  is included inside  $\Lambda_{2L}$ . Plus, we know  $\Lambda_{4\delta} \subseteq \Lambda_L$ . So the condition  $w(0) + Lk \in \Lambda_{4\delta}$  allows us to restrict the sum to  $k \in \{-1; 0; 1\}^d$

$$\leq \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(\text{Dir}),L}^{(FK)}(d\gamma) \sum_{k \in \{-1; 0; 1\}^d} \sum_{w \in \gamma} \mathbb{1}_{-i \in \text{Saus}_\delta(w) + Lk} \mathbb{1}_{w(0) + Lk \in \Lambda_{4\delta}} |\text{Saus}_\delta(w)|^\alpha.$$

In all cases, we can bound the expression from (7) by

$$\sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{k \in \{-1;0;1\}^d} \sum_{w \in \gamma} \mathbb{1}_{-i \in \text{Saus}_\delta(w) + Lk} \mathbb{1}_{w(0) \in \Lambda_{4\delta-Lk}} |\text{Saus}_\delta(w)|^\alpha.$$

If  $-i \in \text{Saus}_\delta(w) + Lk$  then

$$|(-i + \Lambda_{4\delta}) \cap [\text{Saus}_\delta(w) + Lk]| \geq c_d \delta^d$$

where  $c_d$  is the volume of the  $d$ -dimensional unit ball. We deduce

$$\begin{aligned} \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} \mathbb{1}_{-i \in \text{Saus}_\delta(w) + Lk} &\leq \frac{1}{c_d \delta^d} \sum_{i \in 4\delta\mathbb{Z}^d \cap \Lambda_{m-4\delta}^c} |(-i + \Lambda_{4\delta}) \cap [\text{Saus}_\delta(w) + Lk]| \\ &\leq \frac{1}{c_d \delta^d} |\Lambda_{m-8\delta}^c \cap [\text{Saus}_\delta(w) + Lk]|. \end{aligned}$$

Therefore

$$\begin{aligned} &\int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \gamma} \mathbb{1}_{w \cap \Delta \neq \emptyset} \mathbb{1}_{w(0) \in \Lambda_m} |\text{Saus}_\delta(w)|^\alpha \\ &\leq \frac{1}{c_d \delta^d} \sum_{k \in \{-1;0;1\}^d} \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \text{Proj}_{\in \Lambda_{4\delta-Lk}(\gamma)}^c} |\Lambda_{m-8\delta}^c \cap [\text{Saus}_\delta(w) + Lk]| \cdot |\text{Saus}_\delta(w)|^\alpha \\ &\leq \frac{1}{c_d \delta^d} \sum_{k \in \{-1;0;1\}^d} \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \text{Proj}_{\in \Lambda_{4\delta-Lk}(\gamma)}^c} \mathbb{1}_{(\Lambda_{m-8\delta}^c - Lk) \cap \text{Saus}_\delta(w) \neq \emptyset} \cdot |\text{Saus}_\delta(w)|^{1+\alpha} \\ &\leq \frac{1}{c_d \delta^d} \sum_{k \in \{-1;0;1\}^d} \int \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) \sum_{w \in \text{Proj}_{\in \Lambda_{4\delta-Lk}(\gamma)}^c} \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1} \delta^{d-1} (m-9\delta)} \cdot |\text{Saus}_\delta(w)|^{1+\alpha}. \end{aligned}$$

Thanks to proposition 3.5.4, this is enough to conclude on the first part of the proposition. Infinite volume is managed in a similar way.  $\square$

**Proposition 3.6.3** — *The (FK) model is time-reversal invariant. More precisely, for any measurable  $f : \text{Conf}^{(FK)} \rightarrow \mathbb{R}^+$ ,*

$$\int f \circ R^{(FK)} d\mathbb{P}_{(bc),L}^{(FK)} = \int f d\mathbb{P}_{(bc),L}^{(FK)}$$

where we define the time-reversal operator as

$$R^{(FK)} : \begin{array}{ccc} \text{Conf}^{(FK)} & \longrightarrow & \text{Conf}^{(FK)} \\ \gamma & \longmapsto & \left\{ \begin{array}{ll} [0; \beta] & \longrightarrow \mathbb{R}^d \\ s & \longmapsto w(\beta - s) \end{array} \right\}, w \in \gamma \end{array}$$

**PROOF (PROPOSITION 3.6.3)**

One just needs to replace  $\sigma \in \mathbb{S}(\xi)$  by  $\sigma^{-1}$  in the definition of  $\mathbb{P}_{(bc),L}^{(FK)}$ .  $\square$

---

We can now prove the second part of theorem 1.3.12, that is, for any  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R} \cap^n$ -regular and  $\cap^n$ -tame,

$$\lim_{L \rightarrow +\infty} \int f d\tilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})} = \int f d\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}.$$

**PROOF (THEOREM 1.3.12, 2/2)**

Let  $\varepsilon, a, \delta > 0$ ,  $\alpha \in [0; 1[$  and a compact  $\Delta$ . We denote  $D = \sup_{x \in \Delta} \|x\|$ . According to proposition 3.5.4 and proposition 3.6.1, there exists  $m_0 \geq 1$  such that for any  $L \geq 4 \max(D, \delta)$  multiple of  $r$ ,

$$\begin{aligned} & \int \tilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(d\gamma) \text{Thresh}_{(c_{d-1}\delta^{d-1}m_0)^{\alpha+1}} \left[ \sum_{w \in \gamma, w(0) \in \Delta} |\text{Saus}_\delta(w)|^{\alpha+1} \right] \leq \frac{\varepsilon}{2} \\ & \int \tilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(d\gamma) \text{Thresh}_{(c_{d-1}\delta^{d-1}m_0)^\alpha} \left[ \sum_{w \in \gamma, w(0) \in \Delta} |\text{Saus}_\delta(w)|^\alpha \right] \leq \frac{\varepsilon}{2} \\ & \int \tilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(d\gamma) \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w(0) \notin \Lambda_{m_0}} |\text{Saus}_\delta(w)|^\alpha \leq \varepsilon \end{aligned}$$

where  $c_{d-1}$  is the volume of the  $d-1$  dimensional unit ball.

For any  $x_1 \dots x_n \geq 0$ , it is clear that for any  $b > 0$

$$\sum_{i=1}^n \mathbb{1}_{x_i \geq b} x_i \leq \text{Thresh}_b \left( \sum_{i=1}^n x_i \right).$$

Then

$$\begin{aligned} & \int \tilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(d\gamma) \sum_{w \in \gamma, w(0) \in \Delta} \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1}\delta^{d-1}m_0} [|\text{Saus}_\delta(w)|^{\alpha+1} + |\text{Saus}_\delta(w)|^\alpha] \leq \varepsilon \\ & \int \tilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(d\gamma) \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w(0) \notin \Lambda_{m_0}} |\text{Saus}_\delta(w)|^\alpha \leq \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(d\gamma) \sum_{w \in \gamma, w(0) \in \Delta} \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1}\delta^{d-1}m_0} [|\text{Saus}_\delta(w)|^{\alpha+1} + |\text{Saus}_\delta(w)|^\alpha] \leq \varepsilon \\ & \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(d\gamma) \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w(0) \notin \Lambda_{m_0}} |\text{Saus}_\delta(w)|^\alpha \leq \varepsilon. \end{aligned}$$

Recursively, according to proposition 3.6.1 again, there exists a family of positive  $m_i$ ,  $1 \leq i \leq n$  such that for any  $i$ , for any  $L > 0$  multiple of  $r$ ,

$$\begin{aligned} & \int \tilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(d\gamma) \sum_{w \in \gamma, w(0) \in \Lambda_{m_0 + \dots + m_{i-1}}} \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1}\delta^{d-1}m_i} |\text{Saus}_\delta(w)|^{1+\alpha} \leq \varepsilon \\ & \int \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(d\gamma) \sum_{w \in \gamma, w(0) \in \Lambda_{m_0 + \dots + m_{i-1}}} \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1}\delta^{d-1}m_i} |\text{Saus}_\delta(w)|^{1+\alpha} \leq \varepsilon. \end{aligned}$$



For any  $\gamma \in \text{ConfPerm}^{(\text{FK})}$ , we define

$$\begin{aligned} \gamma_m^0 &:= \left\{ w \in \text{Proj}_{\cap \Delta}(\gamma) \mid w \subset \Lambda_{m_0} \right\} \\ \gamma_m^+ &:= \left\{ w \in \text{Proj}_{\cap_+^n \Delta}(\gamma) \mid \exists 0 < k \leq n, \right. \\ &\quad \left. \begin{array}{l} w \subset \Lambda_{m_0 + \dots + m_k} \\ \sigma^{(\text{FK})}(\gamma, \cdot)^{-k}(w) \cap \Delta \neq \emptyset \\ \forall 0 \leq i < k, \sigma^{(\text{FK})}(\gamma, \cdot)^{-i}(w) \cap \Delta = \emptyset \end{array} \right\} \\ \gamma_m^- &:= \left\{ w \in \text{Proj}_{\cap_-^n \Delta}(\gamma) \mid \exists 0 < k \leq n, \right. \\ &\quad \left. \begin{array}{l} w \subset \Lambda_{m_0 + \dots + m_k} \\ \sigma^{(\text{FK})}(\gamma, \cdot)^k(w) \cap \Delta \neq \emptyset \\ \forall 0 \leq i < k, \sigma^{(\text{FK})}(\gamma, \cdot)^i(w) \cap \Delta = \emptyset \end{array} \right\} \\ \gamma_m &:= \gamma_m^0 \cup \gamma_m^+ \cup \gamma_m^- \end{aligned}$$

The function  $\gamma \mapsto f(\gamma_m)$  is  $\in$ -local relatively to the compact  $\Lambda_{m_0 + \dots + m_n}$  and is  $\in$ -tame relatively to the same compact. Therefore, according to the first part of theorem 1.3.12,

$$\lim_{L \rightarrow +\infty} \int f(\gamma_m) \widetilde{\mathbb{P}}_{(\text{bc}), L}^{(\text{FK})}(\text{d}\gamma) = \int f(\gamma_m) \mathbb{P}_{(\text{bc}), \infty}^{(\text{FK})}(\text{d}\gamma).$$

Let  $\gamma \in \text{ConfPerm}^{(\text{FK})}$ . By  $\cap^n$ -regularity,

$$\begin{aligned} a|f(\gamma) - f(\gamma_m)| &\leq \sum_{w \in \text{Proj}_{\in \Delta}(\gamma)} \mathbb{1}_{w \notin \gamma_m} |\text{Saus}_\delta(w)|^{\alpha+1} + \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w \notin \gamma_m} |\text{Saus}_\delta(w)|^\alpha \\ &\quad + \sum_{w \in \text{Proj}_{\cap_+^n \Delta}(\gamma)} \mathbb{1}_{w \notin \gamma_m} \mathbb{1}_{\sigma^{(\text{FK})}(\gamma, \cdot)^{-1}(w) \in \gamma_m} |\text{Saus}_\delta(w)|^{\alpha+1} \\ &\quad + \sum_{w \in \text{Proj}_{\cap_-^n \Delta}(\gamma)} \mathbb{1}_{w \notin \gamma_m} \mathbb{1}_{\sigma^{(\text{FK})}(\gamma, w) \in \gamma_m} |\text{Saus}_\delta(w)|^{\alpha+1}. \end{aligned}$$

Let  $w \in \text{Proj}_{\cap \Delta}(\gamma)$ . If  $w \notin \gamma_m$  then  $w \not\subset \Lambda_{m_0}$ , and either  $w(0)$  is in  $\Delta$  either it is not

$$\sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w \notin \gamma_m} |\text{Saus}_\delta(w)|^\alpha \leq \sum_{w \in \text{Proj}_{\in \Delta}(\gamma)} \mathbb{1}_{w \not\subset \Lambda_{m_0}} |\text{Saus}_\delta(w)|^\alpha + \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w(0) \in \Delta} |\text{Saus}_\delta(w)|^\alpha.$$

Let  $w \in \text{Proj}_{\cap_+^n \Delta}(\gamma)$  such that  $w \notin \gamma_m$  and  $\sigma^{(\text{FK})}(\gamma, \cdot)^{-1}(w) \in \gamma_m$ .

Since  $w \in \text{Proj}_{\cap_+^n \Delta}(\gamma)$ , there exists  $k \in \llbracket 1; n \rrbracket$  such that

$$\sigma^{(\text{FK})}(\gamma, \cdot)^{-k}(w) \cap \Delta \neq \emptyset \quad \text{and} \quad \forall 0 \leq i < k, \sigma^{(\text{FK})}(\gamma, \cdot)^{-i}(w) \cap \Delta = \emptyset.$$

Since  $w \notin \gamma_m$ , by definition,

$$w \not\subset \Lambda_{m_0 + \dots + m_k}.$$

Since  $\sigma^{(\text{FK})}(\gamma, \cdot)^{-1}(w) \in \gamma_m$ , we know

$$\left[ \sigma^{(\text{FK})}(\gamma, \cdot) \right]^{-1}(w) \not\subset \Lambda_{m_0 + \dots + m_{k-1}}.$$

We deduce

$$\begin{aligned} &\sum_{w \in \text{Proj}_{\cap_+^n \Delta}(\gamma)} \mathbb{1}_{w \notin \gamma_m} \mathbb{1}_{\sigma^{(\text{FK})}(\gamma, \cdot)^{-1}(w) \in \gamma_m} |\text{Saus}_\delta(w)|^{\alpha+1} \\ &\leq \sum_{w \in \text{Proj}_{\cap_+^n \Delta}(\gamma)} \sum_{k=1}^n \mathbb{1}_{w \not\subset \Lambda_{m_0 + \dots + m_k} \text{ and } \sigma^{(\text{FK})}(\gamma, \cdot)^{-1}(w) \subset \Lambda_{m_0 + \dots + m_{k-1}}} |\text{Saus}_\delta(w)|^{\alpha+1}. \end{aligned}$$

We take care of the term in  $\text{Proj}_{\cap^u \Delta}(\gamma)$  similarly.

Then

$$\begin{aligned}
a|f(\gamma) - f(\gamma_m)| &\leq \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w \notin \Lambda_{m_0}} \left[ |\text{Saus}_\delta(w)|^{\alpha+1} + |\text{Saus}_\delta(w)|^\alpha \right] + \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w \in \Lambda_{m_0}} |\text{Saus}_\delta(w)|^\alpha \\
&+ \sum_{w \in \text{Proj}_{\cap^u \Delta}(\gamma)} \sum_{k=1}^n \mathbb{1}_{w \notin \Lambda_{m_0+\dots+m_k} \text{ and } \sigma^{(\text{FK})}(\gamma, \cdot)^{-1}(w) \subset \Lambda_{m_0+\dots+m_{k-1}}} |\text{Saus}_\delta(w)|^{\alpha+1} \\
&+ \sum_{w \in \text{Proj}_{\cap^u \Delta}(\gamma)} \sum_{k=1}^n \mathbb{1}_{w \notin \Lambda_{m_0+\dots+m_k} \text{ and } \sigma^{(\text{FK})}(\gamma, w) \subset \Lambda_{m_0+\dots+m_{k-1}}} |\text{Saus}_\delta(w)|^{\alpha+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
a|f(\gamma) - f(\gamma_m)| &\leq \sum_{\substack{w \in \gamma \\ w(0) \in \Delta}} \mathbb{1}_{w \notin \Lambda_{m_0}} \left[ |\text{Saus}_\delta(w)|^{\alpha+1} + |\text{Saus}_\delta(w)|^\alpha \right] + \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w(0) \notin \Lambda_{m_0}} |\text{Saus}_\delta(w)|^\alpha \\
&+ \sum_{k=1}^n \sum_{\substack{w \in \gamma \\ w(0) \in \Lambda_{m_0+\dots+m_{k-1}}} } \mathbb{1}_{w \notin \Lambda_{m_0+\dots+m_k}} |\text{Saus}_\delta(w)|^{\alpha+1} \\
&+ \sum_{k=1}^n \sum_{\substack{w \in \gamma \\ w(\beta) \in \Lambda_{m_0+\dots+m_{k-1}}} } \mathbb{1}_{w \notin \Lambda_{m_0+\dots+m_k}} |\text{Saus}_\delta(w)|^{\alpha+1}.
\end{aligned}$$

The conditions in the indicator functions can be weakened as

$$\begin{aligned}
a|f(\gamma) - f(\gamma_m)| &\leq \sum_{\substack{w \in \gamma \\ w(0) \in \Delta}} \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1} \delta^{d-1} m_0} \left[ |\text{Saus}_\delta(w)|^{\alpha+1} + |\text{Saus}_\delta(w)|^\alpha \right] \\
&+ \sum_{w \in \text{Proj}_{\cap \Delta}(\gamma)} \mathbb{1}_{w(0) \notin \Lambda_{m_0}} |\text{Saus}_\delta(w)|^\alpha \\
&+ \sum_{k=1}^n \sum_{\substack{w \in \gamma \\ w(0) \in \Lambda_{m_0+\dots+m_{k-1}}} } \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1} \delta^{d-1} m_k} |\text{Saus}_\delta(w)|^{\alpha+1} \\
&+ \sum_{k=1}^n \sum_{\substack{w \in \gamma \\ w(\beta) \in \Lambda_{m_0+\dots+m_{k-1}}} } \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1} \delta^{d-1} m_k} |\text{Saus}_\delta(w)|^{\alpha+1}.
\end{aligned}$$

Given the definitions of  $m_0 \dots m_n$ , the three first terms are uniformly small under  $\widetilde{\mathbb{P}}_{(\text{bc}), L}^{(\text{FK})}$ ,  $L > 0$  and  $\mathbb{P}_{(\text{bc}), \infty}^{(\text{FK})}$ . The fourth term is managed thanks to time-reversal invariance (see proposition 3.6.3)

$$\begin{aligned}
&\int \widetilde{\mathbb{P}}_{(\text{bc}), L}^{(\text{FK})}(\mathrm{d}\gamma) \sum_{\substack{w \in \gamma \\ w(\beta) \in \Lambda_{m_0+\dots+m_{k-1}}} } \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1} \delta^{d-1} m_k} |\text{Saus}_\delta(w)|^{\alpha+1} \\
&= \int \widetilde{\mathbb{P}}_{(\text{bc}), L}^{(\text{FK})}(\mathrm{d}\gamma) \sum_{\substack{w \in \gamma \\ w(0) \in \Lambda_{m_0+\dots+m_{k-1}}} } \mathbb{1}_{|\text{Saus}_\delta(w)| \geq c_{d-1} \delta^{d-1} m_k} |\text{Saus}_\delta(w)|^{\alpha+1}.
\end{aligned}$$

We conclude

$$\left| \lim_{L \rightarrow +\infty} \int f(\gamma) \widetilde{\mathbb{P}}_{(bc),L}^{(FK)}(d\gamma) - \int f(\gamma) \mathbb{P}_{(bc),\infty}^{(FK)}(d\gamma) \right|_{L \rightarrow +\infty} \leq \frac{2}{a}(2\varepsilon + 2n\varepsilon) + o(1). \quad \square$$

### 3.7 Technical lemmas

We will now prove any limiting model  $\mathbb{P}_{(bc),\infty}^{(FK)}$  to be Gibbs, that is to say, to be solution of DLR equations. From now on, we will be assuming all hypotheses I to VI.

Most of the technicalities in this section are due to the fact we are stating DLR equations over trajectories rather than points. Indeed, cutting down the configuration into interior and exterior configurations respectively to some compact  $\Delta$  is not as straightforward as for points.

Firstly, we make use below of hypothesis V and VI to justify  $H_{\Delta}^{(FK)}$  is well defined and write the Hamiltonian  $H_{(bc),L}^{(FK)}$  as a sum of 2 distinct terms. Concerning stationarity, its necessity is linked to the empirical field. The probability  $\widetilde{\mathbb{P}}_{(bc),L}^{(FK)}$  can be seen as sampling a configuration along  $\mathbb{P}_{(bc),L}^{(FK)}$  with the extra randomness of not knowing where is the origin in  $\Lambda_L$ . To write DLR equations in infinite volume, we need some coherence between the distribution of the empirical field and the original model, and stationarity will ensure that sampling a configuration inside a compact  $\Delta$  is the same as sampling a configuration inside  $\Delta + v$ , then shifting it by  $-v$ . This is intuitive, but not guaranteed without stationarity.

**PROOF (REMARK 1.4.2)**

Let  $\xi \in \text{Conf}_{<\infty}$ . We denote  $N = \#\{\xi \cap (\Delta + \Lambda_R)\}$ .

**Case 1**  $\forall \xi' \in \text{Conf}_{<\infty}, \xi \cap (\Delta + \Lambda_R) = \xi' \cap (\Delta + \Lambda_R) \implies U(\xi' \cap \Delta^c) = +\infty$

Then the equality defining  $U_{\Delta}$  from hypothesis V becomes  $+\infty = +\infty$ . In other words, we can assume without any loss of generality that  $U_{\Delta}[\xi \cap (\Delta + \Lambda_R)] = C_{\Delta,N}$ .

**Case 2**  $\exists \xi' \in \text{Conf}_{<\infty}, \xi \cap (\Delta + \Lambda_R) = \xi' \cap (\Delta + \Lambda_R)$  and  $U(\xi' \cap \Delta^c) < +\infty$

Then for any such  $\xi'$ , the equality defining  $U_{\Delta}$  from hypothesis V becomes

$$U_{\Delta}[\xi \cap (\Delta + \Lambda_R)] = U(\xi') - U(\xi' \cap \Delta^c)$$

Furthermore, according to hypothesis VI,

$$U(\xi') - U(\xi' \cap \Delta^c) \geq C_{\Delta,N}. \quad \square$$

**Lemma 3.7.1** — *Let  $\Delta \subset \mathbb{R}^d$  be a compact be such that  $\Delta + \Lambda_R \subseteq \Lambda_L$ . Let  $\gamma \in \text{Conf}^{(FK)}$  be a finite configuration. Then*

$$\sup_{s \in [0;\beta]} \#\left[\left\{w(s), w \in \gamma_{(bc),L}\right\} \cap (\Delta + \Lambda_R)\right] < +\infty.$$

Furthermore

$$H_{(bc),L}^{(FK)}(\gamma) = H_{\Delta}^{(FK)}(\gamma_{(bc),L}) + H_{\Delta^c, (bc),L}^{(FK)}(\gamma)$$

where

$$H_{\Delta^c, (bc), L}^{(FK)}(\gamma) = \int_0^\beta U_{(bc), L}[\{w(s), w \in \gamma\} \cap (\Delta_{(bc), L})^c] ds$$

with

$$\Delta_{(\text{Dir}), L} = \Delta \quad \text{and} \quad \Delta_{(\text{per}), L} = \bigcup_{k \in \mathbb{Z}^d} (\Delta + Lk).$$

**PROOF (LEMMA 3.7.1)**

Let us quickly establish the first part of the lemma for each boundary condition.

**Dir** The configuration  $\gamma_{(\text{Dir}), L}$  is finite so the conclusion is trivial.

**per** The bridges of  $\gamma$  are continuous trajectories, so there exists  $M > 0$  such that the ball centered in 0 with radius  $M$  contains the whole  $\gamma$ . It is then clear only finitely many bridges of  $\gamma_{(\text{per}), L}$  ever intersect  $\Delta + \Lambda_R$ .

Let  $\xi \in \text{Conf}_{< \infty}$ .

**Dir** Since  $\Delta \subseteq \Lambda_L$ , by hypothesis V, it is clear that

$$U_{(\text{Dir}), L}(\xi) = U_\Delta[\xi \cap (\Delta + \Lambda_R)] + U_{(\text{Dir}), L}(\xi \cap \Delta^c).$$

where we remind that  $R$  is the range of the interaction.

**per** By definition,

$$U_{(\text{per}), L}(\xi) = \lim_{K \rightarrow +\infty} \frac{1}{(2K+1)^d} U(\xi_{(\text{per}), L} \cap \Lambda_{KL}).$$

By applying hypothesis V successively for the compacts  $\Delta + Lk$  for all  $k \in \mathbb{Z}^d$  such that  $\|k\|_\infty \leq K$ ,

$$\begin{aligned} U(\xi_{(\text{per}), L} \cap \Lambda_{KL}) &= \sum_{k \in \mathbb{Z}^d, \|k\|_\infty \leq K} U_{\Delta + Lk}[\xi_{(\text{per}), L} \cap \Lambda_{KL} \cap (\Delta + Lk + \Lambda_R)] \\ &\quad + U\left[\xi_{(\text{per}), L} \cap \Lambda_{KL} \cap \bigcap_{k \in \mathbb{Z}^d, \|k\|_\infty \leq K} (\Delta + Lk)^c\right]. \end{aligned}$$

Since for all  $k \in \mathbb{Z}^d$ ,  $\|k\|_\infty \leq K$  we have  $\Delta + \Lambda_R + Lk \subseteq \Lambda_{KL}$ , this simplifies as

$$\sum_{k \in \mathbb{Z}^d, \|k\|_\infty \leq K} U_{\Delta + Lk}[\xi_{(\text{per}), L} \cap (\Delta + Lk + \Lambda_R)] + U\left[\xi_{(\text{per}), L} \cap \bigcap_{k \in \mathbb{Z}^d} (\Delta + Lk)^c \cap \Lambda_{KL}\right].$$

By stationarity of the interaction, this equals

$$\begin{aligned} &(2K+1)^d U_\Delta[\xi_{(\text{per}), L} \cap (\Delta + \Lambda_R)] + U\left[\xi_{(\text{per}), L} \cap \bigcap_{k \in \mathbb{Z}^d} (\Delta + Lk)^c \cap \Lambda_{KL}\right] \\ &= (2K+1)^d U_\Delta[\xi_{(\text{per}), L} \cap (\Delta + \Lambda_R)] + U\left[\xi_{(\text{per}), L} \cap (\Delta_{(\text{per}), L})^c \cap \Lambda_{KL}\right] \\ &= (2K+1)^d U_\Delta[\xi_{(\text{per}), L} \cap (\Delta + \Lambda_R)] + U\left[\xi \cap (\Delta_{(\text{per}), L})^c\right]_{(\text{per}), L} \cap \Lambda_{KL}. \end{aligned}$$

Therefore

$$U_{(\text{per}),L}(\xi) = U_{\Delta}[\xi_{(\text{per}),L} \cap (\Delta + \Lambda_R)] + U_{(\text{per}),L}[\xi \cap (\Delta_{(\text{per}),L})^c].$$

So in both boundary conditions,

$$U_{(\text{bc}),L}(\xi) = U_{\Delta}[\xi_{(\text{bc}),L} \cap (\Delta + \Lambda_R)] + U_{(\text{bc}),L}[\xi \cap (\Delta_{(\text{bc}),L})^c].$$

We conclude by applying this equality over sets  $\{w(s), w \in \gamma\}$  and integrate it over  $s \in [0; \beta]$ .  $\square$

Then, we need to be able to cut down the permutation part of  $\mathbb{P}_{(\text{bc}),L}^{(\text{FK})}$  into the shuffling of interior points and exterior points respectively.

**Lemma 3.7.2** — *Let  $X$  be a finite set and  $Y \subseteq X$ . For any function  $f : \mathbb{S}(X) \rightarrow \mathbb{R}$ ,*

$$\sum_{\sigma \in \mathbb{S}(X)} f(\sigma) = \sum_{Z_1 \subseteq Y} \sum_{Z_2 \subseteq X \setminus Y} \sum_{\sigma^{\text{int}} \in \mathbb{S}(Y \rightarrow Z_2 \cup (Y \setminus Z_1))} \sum_{\sigma^{\text{ext}} \in \mathbb{S}(X \setminus Y \rightarrow Z_1 \cup X \setminus (Z_2 \cup Y))} f(\sigma^{\text{int}} \cup \sigma^{\text{ext}})$$

where  $\sigma^{\text{int}} \cup \sigma^{\text{ext}}$  is a notation for the permutation induced on the whole set  $X$  by the interior (relatively to  $Y$ ) bijection  $\sigma^{\text{int}}$  and the exterior bijection  $\sigma^{\text{ext}}$ .

**PROOF (LEMMA 3.7.2)**

For any  $Z_1, Z_2 \subseteq X \setminus Y$ , we denote

$$\mathbb{S}_{X,Y,Z_1,Z_2} := \left\{ \sigma \in \mathbb{S}(X) \mid \begin{array}{l} Z_1 = \sigma(X \setminus Y) \cap Y \\ Z_2 = \sigma(Y) \setminus Y \end{array} \right\}.$$

Then we can define the natural map

$$\varphi_{X,Y,Z_1,Z_2} : \begin{array}{ccc} \mathbb{S}_{X,Y,Z_1,Z_2} & \longrightarrow & \mathbb{S}(Y \rightarrow Z_2 \cup (Y \setminus Z_1)) \times \mathbb{S}(X \setminus Y \rightarrow Z_1 \cup X \setminus (Z_2 \cup Y)) \\ \sigma & \longmapsto & (\sigma^{\text{int}}, \sigma^{\text{ext}}) \end{array}$$

by

$$\sigma^{\text{int}} : \begin{array}{ccc} Y & \longrightarrow & Z_2 \cup (Y \setminus Z_1) \\ x & \longmapsto & \sigma(x) \end{array} \quad \text{and} \quad \sigma^{\text{ext}} : \begin{array}{ccc} X \setminus Y & \longrightarrow & Z_1 \cup X \setminus (Z_2 \cup Y) \\ x & \longmapsto & \sigma(x). \end{array}$$

This map turns out to be bijective. Furthermore, the sets  $\mathbb{S}_{X,Y,Z_1,Z_2}$  for  $Z_1 \subseteq Y$  and  $Z_2 \subseteq X \setminus Y$  form a partition of  $\mathbb{S}(X)$ .  $\square$

We need one last easy result about the sampling of subsets from a Poisson point process.

**Lemma 3.7.3** — *For any measurable  $f : \text{Conf}_{z_\infty}^2 \rightarrow \mathbb{R}^+$ ,*

$$\int \Pi_{\Delta}(d\xi) \sum_{\zeta \subseteq \xi} f(\zeta, \xi \setminus \zeta) = e^{|\Lambda|} \int \Pi_{\Delta}(d\xi_1) \Pi_{\Delta}(d\xi_2) f(\xi_1, \xi_2).$$

**PROOF (LEMMA 3.7.3)**

For any  $\xi \in \text{Conf}_{<\infty}$ ,

$$\begin{aligned} \sum_{\zeta \subseteq \xi} f(\zeta, \xi \setminus \zeta) &= \sum_{n=0}^{+\infty} \sum_{\zeta \subseteq \xi, \#\zeta=n} f(\zeta, \xi \setminus \zeta) \\ &= f(\emptyset, \xi) + \sum_{n=1}^{+\infty} \frac{1}{n!} \sum_{x_1 \in \xi} \dots \sum_{x_n \in \xi \setminus \{x_1 \dots x_{n-1}\}} f(\{x_1 \dots x_n\}, \xi \setminus \{x_1 \dots x_n\}). \end{aligned}$$

By Mecke formula (see Theorem 4.1 p. 27 in [LP17]),

$$\begin{aligned} \int \Pi_{\Delta}(\mathrm{d}\xi) \sum_{\zeta \subseteq \xi} f(\zeta, \xi \setminus \zeta) &= \int \Pi_{\Delta}(\mathrm{d}\xi) \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Delta^n} \mathrm{d}x^{\otimes n} f(\{x_1 \dots x_n\}, \xi) \\ &= e^{|\Delta|} \int \Pi_{\Delta}(\mathrm{d}\xi) \Pi_{\Delta}(\mathrm{d}\zeta) f(\zeta, \xi). \quad \square \end{aligned}$$

### 3.8 DLR equations

Now, we make use of the previous lemmas to introduce an auxiliary measure which takes care of all the exterior sampling (relatively to a compact  $\Delta$ ).

**Definition 3.8.1** — We define the measure  $\mathbb{Q}_{\Delta,L}$  over  $\text{Conf}^{(\text{FK})}$  by stating the following equality for any measurable  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}^+$

$$\begin{aligned} \int f(\gamma) \mathbb{Q}_{\Delta,L}(\mathrm{d}\gamma) &:= e^{2|\Delta|} \int \Pi_{\Lambda_L}(\mathrm{d}\xi) \Pi_{\Delta}(\mathrm{d}\zeta_1) e^{\beta \mu^{\#}(\xi \cup \zeta_1)} \sum_{\zeta_2 \subseteq \xi} \sum_{\sigma_{\Delta}^{\text{ext}} \in \mathcal{S}(\xi \rightarrow \zeta_1 \cup (\xi \setminus \zeta_2))} \\ &\int \left( \bigotimes_{x \in \xi} W_{x, \sigma(x), (\text{bc}), L}^{\beta, \zeta \Delta} \right) (\mathrm{d}\gamma) e^{-H_{\Delta^c, (\text{bc}), L}^{(\text{FK})}(\gamma)} f(\gamma_{(\text{bc}), L}) \end{aligned}$$

where

$$W_{x, \sigma(x), (\text{bc}), L}^{\beta, \zeta \Delta} := W_{x, \sigma(x), (\text{bc}), L}^{\beta} - W_{x, \sigma(x)}^{\beta, \subset \Delta}$$

In the following proposition, we decompose the formula of  $\mathbb{P}_{(\text{bc}), L}^{(\text{FK})}$  into an exterior and an interior sampling.

**Proposition 3.8.2 (non-normalized DLR equations)** — Let  $\Delta \subset \mathbb{R}^d$  be a compact. Let  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  be a bounded function,  $\epsilon$ -local relatively to some compact  $\Delta_f$ . We assume  $(\Delta + \Lambda_R) \cup \Delta_f \subseteq \Lambda_L$ . Then

$$\begin{aligned} \int \Pi_{\Lambda_L}(\mathrm{d}\xi) e^{\beta \mu^{\#} \xi} \sum_{\sigma \in \mathcal{S}(\xi)} \int \left( \bigotimes_{x \in \xi} W_{x, \sigma(x), (\text{bc}), L}^{\beta} \right) (\mathrm{d}\gamma) e^{-H_{(\text{bc}), L}^{(\text{FK})}(\gamma)} f(\gamma_{(\text{bc}), L}) &= \\ \int \mathbb{Q}_{\Delta,L}(\mathrm{d}\gamma) \int \Pi_{\Delta}(\mathrm{d}\zeta) e^{\beta \mu^{\#} \zeta} \sum_{\sigma_{\Delta}^{\text{int}} \in \mathcal{S}(\partial \gamma_{\Delta}^{\text{in}} \cup \zeta \rightarrow \partial \gamma_{\Delta}^{\text{out}} \cup \zeta)} \int \left( \bigotimes_{x \in \partial \gamma_{\Delta}^{\text{in}} \cup \zeta} W_{x, \sigma_{\Delta}^{\text{int}}(x)}^{\beta, \subset \Delta} \right) (\mathrm{d}\eta) e^{-H_{\Delta}^{(\text{FK})}(\eta \cup \gamma_{\Delta}^{\text{ext}})} f(\eta \cup \gamma_{\Delta}^{\text{ext}}). \end{aligned}$$

---

**PROOF (PROPOSITION 3.8.2)**

Since  $\Delta_f \subseteq \Lambda_L$ ,

$$\begin{aligned} & \int \Pi_{\Lambda_L}(d\xi) e^{\beta\mu\#\xi} \sum_{\sigma \in \mathbb{S}(\xi)} \int \left( \bigotimes_{x \in \xi} W_{x, \sigma(x), (bc), L}^\beta \right) (d\gamma) e^{-H_{(bc), L}^{(FK)}(\gamma)} f(\gamma_{(bc), L}) \\ &= \int \Pi_{\Lambda_L}(d\xi) e^{\beta\mu\#\xi} \sum_{\sigma \in \mathbb{S}(\xi)} \int \left( \bigotimes_{x \in \xi} W_{x, \sigma(x), (bc), L}^\beta \right) (d\gamma) e^{-H_{(bc), L}^{(FK)}(\gamma)} f(\gamma). \end{aligned}$$

Since  $\Delta \subseteq \Lambda_L$ , we rewrite the measure over Brownian bridges as

$$= \int \Pi_{\Lambda_L}(d\xi) e^{\beta\mu\#\xi} \sum_{\sigma \in \mathbb{S}(\xi)} \int \sum_{\zeta \subseteq \xi} \left( \bigotimes_{x \in \zeta} W_{x, \sigma(x)}^{\beta, c\Delta} \right) (d\eta) \left( \bigotimes_{x \in \xi \setminus \zeta} W_{x, \sigma(x), (bc), L}^{\beta, \varrho\Delta} \right) (d\gamma) e^{-H_{(bc), L}^{(FK)}(\eta \cup \gamma)} f(\eta \cup \gamma).$$

We can restrict the summation over subsets of  $\xi$  to subsets of  $\xi_\Delta = \xi \cap \Delta$

$$\begin{aligned} &= \int \Pi_{\Lambda_L \setminus \Delta}(d\xi_{\Delta^c}) \Pi_{\Delta}(d\xi_{\Delta}) e^{\beta\mu\#(\xi_{\Delta^c} \cup \xi_{\Delta})} \sum_{\zeta \subseteq \xi_{\Delta}} \sum_{\sigma \in \mathbb{S}(\xi_{\Delta^c} \cup \xi_{\Delta})} \\ & \int \left( \bigotimes_{x \in \zeta} W_{x, \sigma(x)}^{\beta, c\Delta} \right) (d\eta) \left( \bigotimes_{x \in \xi_{\Delta^c} \cup \xi_{\Delta} \setminus \zeta} W_{x, \sigma(x), (bc), L}^{\beta, \varrho\Delta} \right) (d\gamma) e^{-H_{(bc), L}^{(FK)}(\eta \cup \gamma)} f(\eta \cup \gamma). \end{aligned}$$

According to lemma 3.7.3 this equals

$$\begin{aligned} &= e^{|\Delta|} \int \Pi_{\Lambda_L \setminus \Delta}(d\xi_{\Delta^c}) \Pi_{\Delta}(d\xi'_{\Delta}) \Pi_{\Delta}(d\zeta) e^{\beta\mu\#(\xi_{\Delta^c} \cup \xi'_{\Delta} \cup \zeta)} \sum_{\sigma \in \mathbb{S}(\xi_{\Delta^c} \cup \xi'_{\Delta} \cup \zeta)} \\ & \int \left( \bigotimes_{x \in \zeta} W_{x, \sigma(x)}^{\beta, c\Delta} \right) (d\eta) \left( \bigotimes_{x \in \xi_{\Delta^c} \cup \xi'_{\Delta}} W_{x, \sigma(x), (bc), L}^{\beta, \varrho\Delta} \right) (d\gamma) e^{-H_{(bc), L}^{(FK)}(\eta \cup \gamma)} f(\eta \cup \gamma) \\ &= e^{|\Delta|} \int \Pi_{\Lambda_L}(d\xi) \Pi_{\Delta}(d\zeta) e^{\beta\mu\#(\xi \cup \zeta)} \sum_{\sigma \in \mathbb{S}(\xi \cup \zeta)} \\ & \int \left( \bigotimes_{x \in \zeta} W_{x, \sigma(x)}^{\beta, c\Delta} \right) (d\eta) \left( \bigotimes_{x \in \xi} W_{x, \sigma(x), (bc), L}^{\beta, \varrho\Delta} \right) (d\gamma) e^{-H_{(bc), L}^{(FK)}(\eta \cup \gamma)} f(\eta \cup \gamma) \end{aligned}$$

According to lemma 3.7.2 this equals

$$\begin{aligned} &= e^{|\Delta|} \int \Pi_{\Lambda_L}(d\xi) \Pi_{\Delta}(d\zeta) e^{\beta\mu\#(\xi \cup \zeta)} \sum_{\zeta_1 \subseteq \zeta} \sum_{\zeta_2 \subseteq \xi} \sum_{\sigma_{\Delta}^{\text{int}} \in \mathbb{S}(\zeta \rightarrow \zeta_2 \cup \zeta_1)} \sum_{\sigma_{\Delta}^{\text{ext}} \in \mathbb{S}(\xi \rightarrow \zeta_1 \cup \xi \setminus \zeta_2)} \\ & \int \left( \bigotimes_{x \in \zeta} W_{x, \sigma_{\Delta}^{\text{int}}(x)}^{\beta, c\Delta} \right) (d\eta) \left( \bigotimes_{x \in \xi} W_{x, \sigma_{\Delta}^{\text{ext}}(x), (bc), L}^{\beta, \varrho\Delta} \right) (d\gamma) e^{-H_{(bc), L}^{(FK)}(\eta \cup \gamma)} f(\eta \cup \gamma) \end{aligned}$$

According to lemma 3.7.3 again, this equals

$$\begin{aligned}
&= e^{2|\Delta|} \int \Pi_{\Lambda_L}(d\xi) \Pi_{\Delta}(d\zeta') \Pi_{\Delta}(d\zeta_1) e^{\beta\mu^{\#}(\xi \cup \zeta' \cup \zeta_1)} \sum_{\zeta_2 \subseteq \xi} \sum_{\sigma_{\Delta}^{\text{int}} \in \mathbb{S}(\zeta_1 \cup \zeta' \rightarrow \zeta_2 \cup \zeta')} \sum_{\sigma_{\Delta}^{\text{ext}} \in \mathbb{S}(\xi \rightarrow \zeta_1 \cup (\xi \setminus \zeta_2))} \\
&\int \left( \bigotimes_{x \in \zeta_1 \cup \zeta'} W_{x, \sigma_{\Delta}^{\text{int}}(x)}^{\beta, \subset \Delta} \right) (d\eta) \left( \bigotimes_{x \in \xi} W_{x, \sigma_{\Delta}^{\text{ext}}(x), (\text{bc}), L}^{\beta, \supset \Delta} \right) (d\gamma) e^{-H_{(\text{bc}), L}^{\text{(FK)}}(\eta \cup \gamma)} f(\eta \cup \gamma). \tag{8}
\end{aligned}$$

Since  $f$  is  $\epsilon$ -local relatively to  $\Delta_f \subseteq \Lambda_L$ , we have

$$f(\eta \cup \gamma) = f(\eta \cup \gamma_{(\text{bc}), L}).$$

Let  $\eta, \gamma \in \text{Conf}^{\text{(FK)}}$  be finite configurations. We assume  $\eta$  is only made up of bridges included in  $\Delta$ . According to lemma 3.7.1,

$$\begin{aligned}
H_{(\text{bc}), L}^{\text{(FK)}}(\eta \cup \gamma) &= H_{\Delta}^{\text{(FK)}}[(\eta \cup \gamma)_{(\text{bc}), L}] + H_{\Delta^c, (\text{bc}), L}^{\text{(FK)}}(\eta \cup \gamma) \\
&= \int_0^{\beta} U_{\Delta}[\{w(s), (\eta \cup \gamma)_{(\text{bc}), L}\} \cap (\Delta + \Lambda_R)] ds + \int_0^{\beta} U_{(\text{bc}), L}[\{w(s), w \in \eta \cup \gamma\} \cap (\Delta_{(\text{bc}), L})^c] ds \\
&= \int_0^{\beta} U_{\Delta}[\{w(s), \eta \cup \gamma_{(\text{bc}), L}\} \cap (\Delta + \Lambda_R)] ds + \int_0^{\beta} U_{(\text{bc}), L}[\{w(s), w \in \gamma\} \cap (\Delta_{(\text{bc}), L})^c] ds \\
&= H_{\Delta}^{\text{(FK)}}(\eta \cup \gamma_{(\text{bc}), L}) + H_{\Delta^c, (\text{bc}), L}^{\text{(FK)}}(\gamma).
\end{aligned}$$

Then equation (8) becomes

$$\begin{aligned}
&= e^{2|\Delta|} \int \Pi_{\Lambda_L}(d\xi) \Pi_{\Delta}(d\zeta') \Pi_{\Delta}(d\zeta_1) e^{\beta\mu^{\#}(\xi \cup \zeta' \cup \zeta_1)} \sum_{\zeta_2 \subseteq \xi} \sum_{\sigma_{\Delta}^{\text{int}} \in \mathbb{S}(\zeta_1 \cup \zeta' \rightarrow \zeta_2 \cup \zeta')} \sum_{\sigma_{\Delta}^{\text{ext}} \in \mathbb{S}(\xi \rightarrow \zeta_1 \cup (\xi \setminus \zeta_2))} \\
&\int \left( \bigotimes_{x \in \zeta_1 \cup \zeta'} W_{x, \sigma_{\Delta}^{\text{int}}(x)}^{\beta, \subset \Delta} \right) (d\eta) \left( \bigotimes_{x \in \xi} W_{x, \sigma_{\Delta}^{\text{ext}}(x), (\text{bc}), L}^{\beta, \supset \Delta} \right) (d\gamma) e^{-H_{\Delta}^{\text{(FK)}}(\eta \cup \gamma_{(\text{bc}), L}) - H_{\Delta^c, (\text{bc}), L}^{\text{(FK)}}(\gamma)} f(\eta \cup \gamma_{(\text{bc}), L}).
\end{aligned}$$

By definition of the measure  $\mathbb{Q}_{\Delta, L}$ , this equals

$$\begin{aligned}
&= \int \mathbb{Q}_{\Delta, L}(d\gamma) \int \Pi_{\Delta}(d\zeta') e^{\beta\mu^{\#}\zeta'} \sum_{\sigma_{\Delta}^{\text{int}} \in \mathbb{S}(\partial\gamma_{\Delta}^{\text{in}} \cup \zeta' \rightarrow \partial\gamma_{\Delta}^{\text{out}} \cup \zeta')} \\
&\int \left( \bigotimes_{x \in \partial\gamma_{\Delta}^{\text{in}} \cup \zeta'} W_{x, \sigma_{\Delta}^{\text{int}}(x)}^{\beta, \subset \Delta} \right) (d\eta) e^{-H_{\Delta}^{\text{(FK)}}(\eta \cup \gamma)} f(\eta \cup \gamma).
\end{aligned}$$

We have  $\mathbb{Q}_{\Delta, L}$  almost everywhere the equality  $\gamma = \gamma_{\Delta}^{\text{ext}}$ , which is enough to conclude.  $\square$

We make use of proposition 3.8.2 to prove the probability  $\mathbb{P}_{\Delta}^{\text{(FK)}}(\cdot | \gamma)$  is well defined almost surely.



---

**PROOF (LEMMA-DEFINITION 1.4.4)**

Let  $L \in r\mathbb{N}$  such that  $\Delta + \Lambda_R \subseteq \Lambda_L$ .

According to the first part of lemma 3.7.1 and the definition of the measure  $\mathbb{Q}_{\Delta,L}$ , it is clear that  $\sup_{s \in [0; \beta]} \#\{w(s), w \in \gamma\} \cap (\Delta + \Lambda_R)$  is finite  $\mathbb{Q}_{\Delta,L}$  almost everywhere. So the quantity  $Z_{\Delta}(\gamma)$  is well defined  $\mathbb{Q}_{\Delta,L}$  almost everywhere.

We know from proposition 3.8.2 that

$$Z_{(\text{bc}),L} = \int Z_{\Delta}(\gamma) \mathbb{Q}_{\Delta,L}(d\gamma) < +\infty.$$

We also know from lemma 3.1.2 that  $Z_{(\text{bc}),L} < +\infty$ . Therefore  $Z_{\Delta}(\gamma)$  is finite  $\mathbb{Q}_{\Delta,L}$  almost everywhere. Then

$$\mathbb{P}_{(\text{bc}),L}^{(\text{FK})}[Z_{\Delta}(\gamma) \text{ is well defined}] = \frac{1}{Z_{(\text{bc}),L}} \int \mathbb{1}_{Z_{\Delta}(\gamma) \text{ is well defined}} Z_{\Delta}(\gamma) \mathbb{Q}_{\Delta,L}(d\gamma) = 1.$$

Similarly,

$$\begin{aligned} \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}[Z_{\Delta}(\gamma) = +\infty] &= \frac{1}{Z_{(\text{bc}),L}} \int \mathbb{1}_{Z_{\Delta}(\gamma) = +\infty} Z_{\Delta}(\gamma) \mathbb{Q}_{\Delta,L}(d\gamma) \\ &= +\infty \cdot \mathbb{Q}_{\Delta,L}[Z_{\Delta}(\gamma) = +\infty] = 0. \end{aligned}$$

and

$$\mathbb{P}_{(\text{bc}),L}^{(\text{FK})}[Z_{\Delta}(\gamma) = 0] = \frac{1}{Z_{(\text{bc}),L}} \int \mathbb{1}_{Z_{\Delta}(\gamma) = 0} Z_{\Delta}(\gamma) \mathbb{Q}_{\Delta,L}(d\gamma) = 0.$$

In infinite volume, since  $Z_{\Delta}(\gamma)$  is  $\cap^0$ -local,

$$\begin{aligned} &\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}[Z_{\Delta}(\gamma) \text{ is well defined, positive and finite}] \\ &= \lim_{L \rightarrow +\infty} \overline{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}[Z_{\Delta}(\gamma) \text{ is well defined, positive and finite}] \\ &= \lim_{L \rightarrow +\infty} \frac{1}{L^d} \int_{\Lambda_L} dv \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}[Z_{\Delta}(\gamma + v) \text{ is well defined, positive and finite}] \end{aligned}$$

By stationarity of the interaction, we have  $Z_{\Delta}(\gamma + v) = Z_{\Delta-v}(\gamma)$  so

$$\begin{aligned} &\mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}[Z_{\Delta}(\gamma) \text{ is well defined, positive and finite}] \\ &= \frac{1}{L^d} \int_{\Lambda_L} dv \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}[Z_{\Delta-v}(\gamma) \text{ is well defined, positive and finite}] + o(1) \\ &= \frac{1}{L^d} \int_{\Delta-v \subseteq \Lambda_L} dv \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}[Z_{\Delta-v}(\gamma) \text{ is well defined, positive and finite}] + o(1) \\ &= 1 + o(1) \end{aligned}$$

from the finite volume property we proved beforehand. □

We establish the DLR equations in the following last proof, mainly based of proposition 3.8.2 and the second part of theorem 1.3.12.

---

**PROOF (THEOREM 1.4.6)**

Let  $f : \text{Conf}^{(\text{FK})} \rightarrow \mathbb{R}$  be bounded and  $\epsilon$ -local relatively to some compact  $\Delta_f$ .

We assume  $(\Delta + \Lambda_R) \cup \Delta_f \subseteq \Lambda_L$ . According to proposition 3.8.2,

$$\int f(\gamma) \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) = \frac{1}{Z_{(\text{bc}),L}} \int \mathbb{Q}_{\Delta,L}(\text{d}\gamma) Z_{\Delta}(\gamma) \int \mathbb{P}_{\Delta}^{(\text{FK})}(\text{d}\eta|\gamma) f(\eta \cup \gamma_{\Delta}^{\text{ext}}).$$

According to proposition 3.8.2 again,

$$\int f(\gamma) \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) = \int \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) \int \mathbb{P}_{\Delta}^{(\text{FK})}(\text{d}\eta|\gamma) f(\eta \cup \gamma_{\Delta}^{\text{ext}}).$$

We call this equality the DLR equations in finite volume.

We establish the DLR equations for the empirical field

$$\begin{aligned} & \int \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) \int \mathbb{P}_{\Delta}^{(\text{FK})}(\text{d}\eta|\gamma) f(\eta \cup \gamma_{\Delta}^{\text{ext}}) \\ & \stackrel{L \rightarrow +\infty}{=} \frac{1}{L^d} \int_{[(\Delta + \Lambda_R) \cup \Delta_f] - v \subseteq \Lambda_L} \text{d}v \int \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) \int \mathbb{P}_{\Delta}^{(\text{FK})}(\text{d}\eta|\gamma + v) f[\eta \cup (\gamma + v)_{\Delta}^{\text{ext}}] + o(1). \end{aligned}$$

By stationarity of the interaction, if  $\eta$  is distributed along  $\mathbb{P}_{\Delta}^{(\text{FK})}(\cdot | \gamma + v)$  then  $\eta - v$  is distributed along  $\mathbb{P}_{\Delta - v}^{(\text{FK})}(\cdot | \gamma)$ . Therefore

$$\begin{aligned} & \stackrel{L \rightarrow +\infty}{=} \frac{1}{L^d} \int_{[(\Delta + \Lambda_R) \cup \Delta_f] - v \subseteq \Lambda_L} \text{d}v \int \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) \int \mathbb{P}_{\Delta - v}^{(\text{FK})}(\text{d}\eta|\gamma) f[(\eta + v) \cup (\gamma + v)_{\Delta}^{\text{ext}}] + o(1) \\ & \stackrel{L \rightarrow +\infty}{=} \frac{1}{L^d} \int_{[(\Delta + \Lambda_R) \cup \Delta_f] - v \subseteq \Lambda_L} \text{d}v \int \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) \int \mathbb{P}_{\Delta - v}^{(\text{FK})}(\text{d}\eta|\gamma) f[(\eta \cup \gamma_{\Delta - v}^{\text{ext}}) + v] + o(1). \end{aligned}$$

By the DLR equations in finite volume for the compact  $\Delta - v$  and the function  $f(\cdot + v)$ ,

$$\begin{aligned} & \stackrel{L \rightarrow +\infty}{=} \frac{1}{L^d} \int_{[(\Delta + \Lambda_R) \cup \Delta_f] - v \subseteq \Lambda_L} \text{d}v \int \mathbb{P}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) f(\gamma + v) + o(1) \\ & \stackrel{L \rightarrow +\infty}{=} \int f(\gamma) \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) + o(1). \end{aligned}$$

Since the function  $f$  is  $\epsilon$ -local and bounded,

$$\lim_{L \rightarrow +\infty} \int f(\gamma) \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) = \int f(\gamma) \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(\text{d}\gamma). \quad (9)$$

We denote

$$f_{\Delta}(\gamma) := \int \mathbb{P}_{\Delta}^{(\text{FK})}(\text{d}\eta|\gamma) f(\eta \cup \gamma_{\Delta}^{\text{ext}}).$$

The function  $f_{\Delta}$  is  $\cap^0$ -local relatively to the compact  $\Delta_f \cup (\Delta + \Lambda_R)$  and bounded, thus

$$\lim_{L \rightarrow +\infty} \int f_{\Delta}(\gamma) \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{FK})}(\text{d}\gamma) = \int f_{\Delta}(\gamma) \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(\text{d}\gamma). \quad (10)$$

According to the DLR equations for the empirical field and equations (9) and (10),

$$\int f(\gamma) \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(\text{d}\gamma) = \int f_{\Delta}(\gamma) \mathbb{P}_{(\text{bc}),\infty}^{(\text{FK})}(\text{d}\gamma).$$

---

We just proved the equality of measures  $\mathbb{P}_{(bc),\infty}^{(FK)}$  and  $E \mapsto \int \mathbb{P}_{(bc),\infty}^{(FK)}(d\gamma) \int \mathbb{P}_{\Delta}^{(FK)}(d\eta|\gamma) \mathbb{1}_E(\eta \cup \gamma_{\Delta}^{\text{ext}})$  over the ring of sets of  $\epsilon$ -local events. So by Carathéodory's extension theorem, the two measures coincide.  $\square$

## A Wiener sausage

**Definition A.o.1** — Let  $T, \delta > 0$ . Let  $\omega = (\omega_t)_{t \geq 0}$  be a  $d$  dimensional Brownian motion. We denote the Wiener sausage of thickness  $\delta > 0$  and length  $T > 0$  the set

$$\text{Saus}_{\delta,T}(\omega) := \left\{ x \in \mathbb{R}^d \mid \exists t \in [0; T], \|x - \omega_t\| \leq \delta \right\}$$

and its volume as  $|\text{Saus}_{\delta,T}(\omega)|$ .

**Theorem A.o.2**

Let  $T, \delta > 0$ . Let  $\omega = (\omega_t)_{t \geq 0}$  be a  $d$  dimensional Brownian motion. There exists  $\epsilon > 0$  such that

$$\mathbb{E} \left[ e^{\epsilon |\text{Saus}_{\delta,T}(\omega)|^2} \right] < +\infty.$$

The proof of this theorem is heavily based on the proof from [Szn87] of finite exponential moments for the Wiener sausage.

**PROOF**

We define the sequence of stopping times  $(\tau_i)_{i \in \mathbb{N}}$  by  $\begin{cases} \tau_0 := 0 \\ \tau_{i+1} := \inf\{t \geq 0 \mid \|\omega_{\tau_i+t} - \omega_{\tau_i}\|_{\infty} > \delta\} \end{cases}$  and the random variable  $N_T := \inf\{n \in \mathbb{N} \mid \sum_{i=1}^n \tau_i > T\}$ .

It is clear that

$$\text{Saus}_{\delta,T}(\omega) = \bigcup_{t=0}^T \mathbb{B}(\omega_t, \delta) \subseteq \bigcup_{i=0}^{N_T-1} \left[ \bigcup_{t=\tau_0+\dots+\tau_i}^{\tau_0+\dots+\tau_{i+1}} \mathbb{B}(\omega_t, \delta) \right] \subseteq \bigcup_{i=0}^{N_T-1} \left[ \bigcup_{t=\tau_0+\dots+\tau_i}^{\tau_0+\dots+\tau_{i+1}} (\omega_t + \Lambda_{2\delta}) \right] \subseteq \bigcup_{i=0}^{N_T-1} (\omega_{\tau_i} + \Lambda_{4\delta})$$

thus

$$|\text{Saus}_{\delta,T}(\omega)|^2 \leq N_T^2 (4\delta)^{2d}.$$

For any  $k \geq 0$  and  $\lambda > 0$ , by Markov inequality,

$$\mathbb{P}(N_T \geq k) = \mathbb{P}\left(\sum_{i=1}^k \tau_i \leq T\right) \leq e^{\lambda T} \mathbb{E}\left(e^{-\lambda \sum_{i=1}^k \tau_i}\right)$$

Since the stopping times  $\tau_i$ ,  $i \geq 1$  are iid, we deduce

$$\mathbb{P}(N_T \geq k) \leq e^{\lambda T} \mathbb{E}\left(e^{-\lambda \tau_1}\right)^k.$$

For any non-negative random variable  $X$  and  $\lambda > 0$ ,

$$\mathbb{E}(e^{-\lambda X}) = \lambda \int_0^{+\infty} e^{-\lambda x} [1 - \mathbb{P}(X > x)] dx.$$

Furthermore, if we denote as  $\omega^{(i)}$ ,  $1 \leq i \leq d$  the  $d$  spatial components of the Brownian motion  $\omega$ , then we can write

$$\tau_1 = \min_{1 \leq i \leq d} \tau_1^{(i)} \quad \text{where} \quad \tau_1^{(i)} := \inf \left\{ t \geq 0 \mid \left| \omega_t^{(i)} \right| > \delta \right\}.$$

Therefore

$$\mathbb{E}(e^{-\lambda \tau_1}) = \lambda \int_0^{+\infty} e^{-\lambda x} [1 - \mathbb{P}(\tau_1 > x)] dx = \lambda \int_0^{+\infty} e^{-\lambda x} \left[ 1 - \mathbb{P}(\tau_1^{(1)} > x)^d \right] dx.$$

By concavity, it is clear that  $\forall p \geq 0$ ,  $1 - p^d \leq d(1 - p)$  so

$$\mathbb{E}(e^{-\lambda \tau_1}) \leq d \lambda \int_0^{+\infty} e^{-\lambda x} [1 - \mathbb{P}(\tau_1^{(1)} > x)] dx \leq d \mathbb{E}(e^{-\lambda \tau_1^{(1)}}).$$

According to the *Handbook of Brownian Motion - Facts and Formulae* [BS02] (3.o.1 p218),

$$\mathbb{E}(e^{-\lambda \tau_1^{(1)}}) = \frac{1}{\cosh(\delta \sqrt{2\lambda})} \leq 2e^{-\delta \sqrt{2\lambda}}.$$

For  $\lambda = \varepsilon k^2$ , we get

$$\mathbb{P}(N_T \geq k) \leq e^{\varepsilon T k^2} (2d)^k e^{-\sqrt{2\varepsilon} \delta k^2}.$$

We conclude

$$\mathbb{E} \left[ e^{\varepsilon |\text{Saus}_{\delta, T}(\omega)|^2} \right] \leq \mathbb{E} \left( e^{\varepsilon (4\delta)^{2d} N_T^2} \right) \leq \sum_{k=1}^{+\infty} e^{\varepsilon (4\delta)^{2d} k^2} \mathbb{P}(N_T \geq k) \leq \sum_{k=1}^{+\infty} e^{\varepsilon (4\delta)^{2d} k^2} e^{\varepsilon T k^2} (2d)^k e^{-\sqrt{2\varepsilon} \delta k^2}$$

which converges for  $\varepsilon > 0$  small enough. □

## B Notation table

We order symbols in an approximate alphabetical order.

Symbol	Definition	Page
$A$	Superstability constant (see hypothesis II)	4
$B$	Superstability constant (see hypothesis II)	4
$\beta$	Inverse temperature	-
...	...	...

Symbol	Definition	Page
$\mathcal{C}_{<\infty}$	$\sigma$ -algebra over $\text{Conf}_{<\infty}$ (see definition 1.1.1)	3
$\mathcal{C}^{(\text{FK})}$	$\sigma$ -algebra over $\text{Conf}^{(\text{FK})}$ (see definition 1.2.2)	6
$\mathcal{C}^{(\text{mp})}$	$\sigma$ -algebra over $\text{Conf}^{(\text{mp})}$ (see definition 2.3.1)	23
$\mathcal{C}_{(\text{bc}),L}^{(\text{ps})}$	$\sigma$ -algebra over $\text{Conf}_{(\text{bc}),L}^{(\text{ps})}$ (see definition 2.2.3)	21
$\mathcal{C}_{(\text{bc}),L}^{(\text{rl})}$	$\sigma$ -algebra over $\text{Conf}_{(\text{bc}),L}^{(\text{rl})}$ (see definition 2.1.4)	18
$\text{Conf}_{<\infty}$	$\{\xi \subset \mathbb{R}^d \mid \#\xi < +\infty\}$	3
$\text{Conf}^{(\text{FK})}$	$\{\gamma \subset \Omega_\beta \mid \gamma \text{ is locally finite for } \mathbb{W}_\beta\}$	6
$\text{Conf}^{(\text{mp})}$	$\{\gamma = (x, p_x, u_x, \omega_x)_{x \in \xi} \subset \mathbb{R}^d \times \mathbb{Z}^d \times [0;1] \times \Omega_1 \mid \xi \subset \mathbb{R}^d \text{ loc. finite}\}$	23
$\text{Conf}_{(\text{bc}),L}^{(\text{ps})}$	$\{\eta \subset \text{Loop}_{(\text{bc}),L} \mid \#\eta < +\infty\}$	21
$\text{Conf}_{(\text{bc}),L}^{(\text{rl})}$	$\{\eta \subset \text{Loop}_{(\text{bc}),L}^{(\text{root})} \mid \#\eta < +\infty\}$	18
$\text{Conf}_\infty^{(\text{rl})}$	$\{\eta \subset \text{Loop}_{(\text{Dir})}^{(\text{root})} \mid \eta \text{ is locally finite in } \mathbb{W}_{(\text{Dir})}^{(\text{root})}\}$ (see remark 2.1.10)	19
$\text{ConfAuth}_{(\text{bc}),L}^{(\text{mp})}$	$\{\gamma \in \text{Conf}^{(\text{mp})} \mid \forall (x, p, u, \omega) \in \gamma, N_{(\text{bc}),L}^r(\dot{\gamma}, x + rp) \geq 1\}$	23
$\text{ConfAuth}_\infty^{(\text{mp})}$	$\{\gamma \in \text{Conf}^{(\text{mp})} \mid \forall (x, p, u, \omega) \in \gamma, N_\infty^r(\dot{\gamma}, x + rp) \geq 1\}$	40
$\text{ConfPerm}_{(\text{bc}),L}^{(\text{mp})}$	$\{\gamma \in \text{ConfAuth}_{(\text{bc}),L}^{(\text{mp})} \mid \forall y \in \dot{\gamma}, \exists! x \in \dot{\gamma}, \pi_{(\text{bc}),L}[\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, x)] = \pi_{(\text{bc}),L}(y)\}$	23
$\text{ConfPerm}_\infty^{(\text{mp})}$	$\{\gamma \in \text{ConfAuth}_\infty^{(\text{mp})} \mid \forall y \in \dot{\gamma}, \exists! x \in \dot{\gamma}, \sigma_\infty^{(\text{mp})}(\gamma, x) = y\}$	40
$\text{ConfPerm}^{(\text{FK})}$	$\{\gamma \in \text{Conf}^{(\text{FK})} \mid \gamma \text{ is permutation-wise}\}$	6
$\text{ConfPerm}_{(\text{bc}),L}^{(\text{ps})}$	$\left( \begin{smallmatrix} (\text{FK}) \leftarrow (\text{ps}) \\ \varphi_{(\text{bc}),L} \end{smallmatrix} \right)^{-1} (\text{ConfPerm}^{(\text{FK})})$	22
$\text{ConfPerm}_{(\text{bc}),L}^{(\text{rl})}$	$\left( \begin{smallmatrix} (\text{FK}) \leftarrow (\text{rl}) \\ \varphi_{(\text{bc}),L} \end{smallmatrix} \right)^{-1} (\text{ConfPerm}^{(\text{FK})})$	18
$\partial\gamma_\Delta^{\text{in}}$	See lemma-definition 1.4.4	14
$\partial\gamma_\Delta^{\text{out}}$	See lemma-definition 1.4.4	14
$\dot{\gamma}$	$\{x, (x, p, u, \omega) \in \gamma\}$	23
$\gamma_{(\text{bc}),L}$	$\gamma_{(\text{Dir}),L} := \gamma$ and $\gamma_{(\text{per}),L} := \bigcup_{k \in \mathbb{Z}^d} (\gamma + Lk)$	7
$\gamma_\Delta^{\text{ext}}$	$\{w \in \gamma \mid w \not\subset \Delta\}$	14
$H_{(\text{bc}),L}^{(\text{FK})}$	$\gamma \mapsto \int_0^\beta U_{(\text{bc}),L}[\{w(s), w \in \gamma\}] ds$	6
$H_{(\text{bc}),L}^{(\text{mp})}$	Hamiltonian in (mp) framework (see lemma-definition 2.3.4)	25
$H_{(\text{bc}),L}^{(\text{ps})}$	Hamiltonian in (ps) framework (see definition 2.2.4)	22
$H_{(\text{bc}),L}^{(\text{rl})}$	$\eta \mapsto \int_0^\beta U_{(\text{bc}),L}[\{w(\beta j + s), 0 \leq j < \ell(w), w \in \eta\}] ds$	18
$H_\Delta^{(\text{FK})}$	$\gamma \mapsto \int_0^\beta U_\Delta[\{w(s), w \in \gamma\}] ds$	14
$\mathbb{I}(\cdot)$	See definition 3.2.1	31
...	...	...

Symbol	Definition	Page
$\ell$	Length of a loop (see definition 2.1.2 and definition 2.2.1)	-
$\Lambda_L$	$[-L/2; L/2]^d$	4
$\text{Leb}^{(d)}$	$d$ dimensional Lebesgue measure	24
$\text{Loop}_t^{(\text{root})}$	Subset of continuous trajectories $\mathbb{R} \rightarrow \mathbb{R}^d$ (see definition 2.1.1)	16
$\text{Loop}_{(\text{bc}),L}^{(\text{root})}$	Space of rooted loops (see definition 2.1.2)	17
$\text{Loop}_{(\text{bc}),L}$	Quotient space (see definition 2.2.1)	21
$\mu$	Chemical potential	-
$N_{(\text{bc}),L}^r(\dot{\gamma}, \cdot)$	$z \mapsto \#\left[\dot{\gamma}_{(\text{bc}),L} \cap (z + \Lambda_r)\right]$	23
$N_\infty^r(\dot{\gamma}, \cdot)$	$z \mapsto \#\left[\dot{\gamma} \cap (z + \Lambda_r)\right]$	40
$\nu$	Probability measure over $\mathbb{Z}^d$ such that $\forall p \in \mathbb{Z}^d, \nu(p) > 0$	24
$\Omega_t$	Set of continuous $[0; t] \rightarrow \mathbb{R}^d$	5
$\varphi_{(\text{bc}),L}^{(\text{FK}) \leftarrow (\text{ps})}$	Transition map from (ps) to (FK) (see definition 2.2.6)	22
$\varphi_{(\text{bc}),L}^{(\text{FK}) \leftarrow (\text{rl})}$	Transition map from (rl) to (FK) (see definition 2.1.8)	18
$\varphi_{(\text{bc}),L}^{(\text{mp}) \leftarrow (\text{FK})}$	Transition map from (FK) to (mp) (see definition 2.3.2)	24
$\varphi_{(\text{bc}),L}^{(\text{ps}) \leftarrow (\text{FK})}$	Transition map from (FK) to (ps) (see definition 2.2.6)	22
$\varphi_{(\text{bc}),L}^{(\text{rl}) \leftarrow (\text{ps})}$	Transition map from (ps) to (rl) (see definition 2.2.4)	22
$\varphi_\infty^{(\text{FK}) \leftarrow (\text{mp})}$	Transition map from (mp) to (FK) in infinite volume (see def. 3.4.2)	41
$\varphi_\infty^{(\text{FK}) \leftarrow (\text{rl})}$	Transition map from (rl) to (FK) in infinite volume (see def. 2.1.11)	19
$\pi_{(\text{bc}),L}$	See definition 2.2.1	21
$\Pi_{\Lambda_L}$	Poisson point process over $\Lambda_L$ with intensity 1	7
$\Pi_L^{(\text{mp})}$	Poisson process over $\Lambda_L \times \mathbb{Z}^d \times [0; 1] \times \Omega_1$ (see definition 2.3.3)	24
$\Pi_{(\text{bc}),L}^{(\text{ps})}$	Poisson process with intensity measure $W_{(\text{bc}),L}$	21
$\Pi_{(\text{bc}),L}^{(\text{rl})}$	Poisson process with intensity measure $W_{(\text{bc}),L}^{(\text{root})}$	21
$\mathbb{P}_{(\text{bc}),L}^{(\text{FK})}$	Probability measure over $\text{Conf}^{(\text{FK})}$ (see lemma-definition 1.3.3)	7
$\mathbb{P}_{(\text{bc}),L}^{(\text{mp})}$	Probability measure over $\text{Conf}^{(\text{mp})}$ (see lemma-definition 2.3.4)	25
$\mathbb{P}_{(\text{bc}),L}^{(\text{ps})}$	Probability measure over $\text{Conf}_{(\text{bc}),L}^{(\text{ps})}$ (see lemma-definition 2.2.5)	22
$\mathbb{P}_{(\text{bc}),L}^{(\text{rl})}$	Probability measure over $\text{Conf}_{(\text{bc}),L}^{(\text{rl})}$ (see lemma-definition 2.1.7)	18
$\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{xx})}$	$\int f d\widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{xx})} := \frac{1}{L^d} \int_{\Lambda_L} dv \int f(\gamma + v) \widetilde{\mathbb{P}}_{(\text{bc}),L}^{(\text{xx})}(d\gamma)$ for (xx)=(FK), (mp) or (rl)	8
$\mathbb{P}_{(\text{bc}),\infty}^{(\text{xx})}$	Infinite volume (xx) model (see theorems 1.3.12, 2.3.9 and 2.1.15)	-
$\mathbb{P}_\Delta^{(\text{FK})}(\cdot   \gamma)$	Conditional measure over $\text{Conf}^{(\text{FK})}$ (see lemma-definition 1.4.4)	14
...	...	...

Symbol	Definition	Page
$\text{Proj}_{?\Delta}$	Projection map over $\text{Conf}^{(\text{FK})}$ , for ? being $\in, \cap, \dots$ (see def. 1.3.6)	8
$\mathbb{Q}_{\Delta,L}$	Measure over $\text{Conf}^{(\text{FK})}$ (see definition 3.8.1)	61
$r$	Superstability constant (see hypothesis II)	4
$R$	Range of the interaction $U$ (see hypothesis V)	12
$R^{(\text{FK})}$	$\gamma \mapsto \{s \mapsto w(\beta - s), w \in \gamma\}$	54
$\mathbb{S}(X)$	Set of permutations over $X$	7
$\mathbb{S}(X \rightarrow Y)$	Set of bijections from $X$ to $Y$	14
$\sigma^{(\text{FK})}(\gamma, \cdot)$	Natural permutation over $\gamma \in \text{ConfPerm}^{(\text{FK})}$ (see definition 1.2.3)	6
$\sigma_{(\text{bc}),L}^{(\text{mp})}(\gamma, \cdot)$	Encoded permutation in finite volume (see definition 2.3.1)	23
$\sigma_{\infty}^{(\text{mp})}(\gamma, \cdot)$	Encoded permutation in infinite volume (see definition 3.4.1)	40
$T_s^{(\text{ps})}$	$\eta \mapsto \{w(\cdot + s), w \in \overline{w}, \overline{w} \in \eta\}$	23
$\text{Thresh}_m$	$x \mapsto x \cdot \mathbb{1}_{x > m}$	32
$U$	$\text{Conf}_{<\infty} \rightarrow \mathbb{R} \cup \{+\infty\}$	4
$U_{(\text{bc}),L}$	Adaptation of the interaction in (bc) (see definition 1.1.3)	4
$U_{\Delta}$	Local interaction (see hypothesis V)	12
$W_{x,y}^t$	Wiener measure over Brownian bridges (see definition 1.2.1)	5
$W_{x,y}^{\beta, \triangleleft \Delta}$	$W_{x,y}^{\beta, \triangleleft \Delta}(\text{dw}) := \mathbb{1}_{w \subset \Delta} W_{x,y}^{\beta}(\text{dw})$	14
$W_{x,y}^t$	$W_{x,y}^t$	5
$W_{x,y,(\text{bc}),L}^t$	$W_{x,y,(\text{Dir}),L}^t := W_{x,y}^t$ and $W_{x,y,(\text{per}),L}^t := \sum_{k \in \mathbb{Z}^d} W_{x,y+Lk}^t$	5
$W_{x,\sigma(x),(\text{bc}),L}^{\beta, \triangleleft \Delta}$	$W_{x,\sigma(x),(\text{bc}),L}^{\beta} - W_{x,\sigma(x)}^{\beta, \triangleleft \Delta}$	61
$W_{(\text{bc}),L}^{(\text{root})}$	$\int_{\Lambda_L} \text{d}x \sum_{j \geq 1} \frac{e^{-\beta j}}{j} W_{x,x,(\text{bc}),L}^{\beta j}$	18
$W_{(\text{bc}),L}$	Quotient measure (see lemma-definition 2.2.2)	21
$\mathbb{W}_{0,0}^1$	$W_{0,0}^1 / W_{0,0}^1(\Omega_1)$	24
$\mathcal{W}_t$	Borelian $\sigma$ -algebra over $\Omega_t$ associated to the uniform norm	5
$\mathcal{W}_{(\text{bc}),L}^{(\text{root})}$	$\sigma$ -algebra over $\text{Loop}_{(\text{bc}),L}^{(\text{root})}$ (see lemma-definition 2.1.3)	17
$\mathcal{W}_{(\text{bc}),L}$	$\sigma$ -algebra over $\text{Loop}_{(\text{bc}),L}$ (see lemma-definition 2.2.2)	21
$\mathfrak{W}_t$	Topology over $\Omega_t$ associated to the uniform norm	5
$\mathfrak{W}_{(\text{bc}),L}^{(\text{root})}$	Topology over $\text{Loop}_{(\text{bc}),L}^{(\text{root})}$ (see lemma-definition 2.1.3)	17
$\mathfrak{W}_{(\text{bc}),L}$	Topology over $\text{Loop}_{(\text{bc}),L}$ (see lemma-definition 2.2.2)	21
$Z_{(\text{bc}),L}$	Partition function (see lemma-definition 1.3.3)	7
$Z_{\Delta}(\gamma)$	Local partition function (see lemma-definition 1.4.4)	14

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