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**$p$ -ADIC ASAI  $L$ -FUNCTIONS FOR QUADRATIC HILBERT EIGENFORMS**

GIADA GROSSI, DAVID LOEFFLER, AND SARAH LIVIA ZERBES

ABSTRACT. We construct  $p$ -adic Asai  $L$ -functions for cuspidal automorphic representations  $\Pi$  of  $\text{Res}_{F/\mathbb{Q}}\text{GL}_2$ , where  $F$  is a real quadratic field in which  $p$  splits. Our method relies on higher Hida theory for Hilbert modular surfaces with Iwahori level at one prime above  $p$ .

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## 1. INTRODUCTION

The study of critical values of  $L$ -functions is central in many areas of modern number theory. The Bloch–Kato and Beilinson conjectures relate critical values of complex  $L$ -functions to certain arithmetic data; and much progress towards such conjectures has been made by  $p$ -adic methods, in which a fundamental role is played by  $p$ -adic  $L$ -functions, interpolating  $p$ -adically the algebraic parts of critical values of a given complex  $L$ -function or family of  $L$ -functions. In this paper, we construct such a  $p$ -adic counterpart for the *Asai*, or *twisted tensor*,  $L$ -function attached to a Hilbert modular form over a real quadratic field  $F$ .

**1.1. Higher Hida theory for  $\mathfrak{p}_1$ -ordinary cohomology.** The main tool we shall use is a version of *higher Hida theory* for Hilbert modular forms at primes  $p = \mathfrak{p}_1\mathfrak{p}_2$  split in  $F$ . Such a theory was initially introduced in the first author’s paper [Gro21]; in this paper we develop a variant assuming only ordinarity at a single prime above  $p$ , and we study functoriality properties of this theory for the embedding  $\text{GL}_2 \hookrightarrow \text{Res}_{F/\mathbb{Q}}\text{GL}_2$ , defining pullback and pushforward maps between  $p$ -adic cohomology spaces for these two groups.

For our first result, let  $E$  be a (sufficiently large) finite extension of  $\mathbb{Q}_p$  with ring of integers  $R$ , and write  $X_{G,0}(\mathfrak{p}_1)$  for the compactified Hilbert modular surface with Iwahori level at  $\mathfrak{p}_1$  and spherical level at  $\mathfrak{p}_2$  (and some fixed prime-to- $p$  level structure  $K_G^{(p)}$ ). We also fix  $(k_1, k_2, w) \in \mathbb{Z}^3$  with  $k_1 = k_2 = w \pmod{2}$ .

**Theorem A.** *There exists a perfect complex  $M_G^\bullet(k_1 + 2\kappa, k_2, w)$  of modules over  $\Lambda = R[[\mathbb{Z}_p^\times]]$ , with an action of prime-to- $p$  Hecke operators, such that for all integers  $a \in \mathbb{Z}$  with  $k_1 + 2a \geq 2$ , we have a Hecke-equivariant isomorphism*

$$M_G^\bullet(k_1 + 2\kappa, k_2, w) \otimes_{\Lambda, a}^{\mathbb{L}} E \simeq e(U_{\mathfrak{p}_1}) \text{R}\Gamma \left( X_{G,0}(\mathfrak{p}_1)_E, \underline{\omega}^{(k_1+2a, k_2, w)} \right).$$

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There is also an analogous complex  $M_G^\bullet(k_1 + 2\kappa, k_2, w; -D)$  interpolating the cohomology of the cuspidal sheaves  $\underline{\omega}^{(k_1 + 2a, k_2, w)}(-D)$ .

See theorem 3.2.3 below. Here  $e(U_{\mathfrak{p}_1})$  is the ordinary projector associated to the Hecke operator at  $\mathfrak{p}_1$ ; we do not impose any ordinarity conditions at  $\mathfrak{p}_2$ . For our second result, let  $M_H(\tau)$  denote the space of  $\Lambda$ -adic elliptic modular forms of weight  $\tau : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$  and tame level  $K_H^{(p)} = K_G^{(p)} \cap H$ .

**Theorem B.** *There exists a pushforward map of  $\Lambda$ -modules*

$$\iota_* : M_H(k_1 - k_2 + 2\kappa) \rightarrow H^1(M_G^\bullet(k_1 + 2\kappa, 2 - k_2, w))$$

which interpolates, via the isomorphism of theorem A, the projection to the  $U_{\mathfrak{p}_1}$ -ordinary parts of the pushforward maps in coherent cohomology in all weights  $(k_1 + 2a, 2 - k_2, w)$  with  $k_1 + 2a \geq 2$ .

Note that the integer-weight specialisations of forms in the space  $M_H(\tau)$  can be classical holomorphic modular forms of Iwahori level at  $p$ , but may also include holomorphic forms of higher  $p$ -power levels, or nearly-holomorphic modular forms (via the  $p$ -adic unit root splitting); and we show that the specialisations of our  $p$ -adic pushforward map are compatible with algebraic pushforward maps on these larger spaces. This is crucial for our applications to  $p$ -adic  $L$ -functions, since the  $\mathrm{GL}_2$  Eisenstein series we consider are only nearly-holomorphic in general.

**Remark.** In this paper we shall apply Theorem B to pushforwards of Eisenstein families, but the case of pushforwards of cuspidal families is also interesting, since these pushforwards are related to the square roots of twisted triple-product  $L$ -functions. We hope to investigate this case in future work.  $\diamond$

**1.2. One-variable  $p$ -adic  $L$ -functions.** Our first application of the above theory is to construct a one-variable  $p$ -adic Asai  $L$ -function attached to a single cuspidal automorphic representation  $\Pi$  of  $G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ , interpolating critical values of the Asai  $L$ -function of  $\Pi$  and its twists by  $p$ -power-conductor Dirichlet characters.

Our conventions are such that  $\Pi$  is unitary, and its central character  $\chi_\Pi$  is a finite-order Grössencharacter of  $F$ . The weight of  $\Pi$  is given by a pair of positive integers  $(k_1, k_2)$ ; we shall only consider “parituous” weights (i.e. with  $k_1 = k_2 \pmod{2}$ ), and in order to have critical values for the Asai  $L$ -function, we need to assume  $k_1 \neq k_2$ . Without loss of generality we suppose  $k_1 > k_2 \geq 1$ . We suppose  $p$  is a prime split in  $F$  and coprime to the level of  $\Pi$ ; we fix an isomorphism  $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ , and label the primes above  $p$  as  $\mathfrak{p}_1, \mathfrak{p}_2$  so that the prime  $\mathfrak{p}_i$  corresponds to the infinite place  $\sigma_i$ .

We write  $L_{\mathrm{As}}(\Pi, s)$  for the Asai  $L$ -function (normalised in the analytic fashion, so the centre of the functional equation is at  $s = \frac{1}{2}$ ), and  $L_{\mathrm{As}}(\Pi, \chi, s)$  for the  $L$ -function twisted by a Dirichlet character  $\chi$ . Here we define the local  $L$ -factors at bad primes via the local Langlands correspondence, or equivalently as the common denominator of the Asai zeta integral defined in §8.2.1 (for the equivalence of these definitions see [LW20, Remark 2.14]). The critical points of  $L_{\mathrm{As}}(\Pi, \chi, s)$ , for any  $\chi$ , are given by

$$\text{(crit)} \quad s \in \mathbb{Z}, \quad \frac{2 - k_1 + k_2}{2} \leq s \leq \frac{k_1 - k_2}{2}.$$

The general conjectures of Coates–Perrin-Riou [CPR89] and Panchishkin [Pan94] on the expected form of  $p$ -adic  $L$ -functions associated to motives over  $\mathbb{Q}$ , in this case, predict the following: if a suitable “ordinarity” condition (the Panchishkin condition) holds, the critical  $L$ -values of  $L_{\mathrm{As}}(\Pi, \chi, s)$ , for Dirichlet characters  $\chi$  of  $p$ -power conductor, can be interpolated by a  $p$ -adic measure after multiplying by a certain Euler factor  $\Phi_p(\mathrm{As}(\Pi), \chi, s)$ . See [Pan94, Definition 5.5] for the definition of Panchishkin’s condition; in our setting (with  $k_1 > k_2$ ) the Panchishkin condition amounts to ordinarity of  $\Pi$  at  $\mathfrak{p}_1$ , while the stronger ordinarity condition assumed in [CPR89] amounts to ordinarity at both  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$ .

**Notation 1.2.1.** For  $i = 1, 2$ , we write  $\alpha_i^\circ, \beta_i^\circ$  for the Satake parameters of  $\Pi_{\mathfrak{p}_i}$  (unitarily normalised, so that  $|\alpha_i^\circ| = |\beta_i^\circ| = 1$ ), and define

$$\alpha_i = p^{(k_i - 1)/2} \alpha_i^\circ, \quad \beta_i = p^{(k_i - 1)/2} \beta_i^\circ.$$

We will see below that  $\alpha_1, \beta_1$  are  $p$ -adically integral (for our chosen embedding into  $\overline{\mathbb{Q}}_p$ ), and the ordinarity condition at  $\mathfrak{p}_1$  is equivalent to precisely one of  $\alpha_1$  and  $\beta_1$  being a  $p$ -adic unit, which we may take to be  $\alpha_1$ .

**Definition 1.2.2.** Suppose  $\Pi$  is ordinary at  $\mathfrak{p}_1$ . For  $s \in \mathbb{Z}$  and  $\nu$  a Dirichlet character of  $p$ -power conductor, define

$$\mathcal{E}_p(\text{As}(\Pi), \nu, s) = \begin{cases} \left(1 - \frac{p^{s-1}}{\alpha_1^\circ \alpha_2^\circ}\right) \left(1 - \frac{p^{s-1}}{\alpha_1^\circ \beta_2^\circ}\right) \left(1 - \frac{\beta_1^\circ \alpha_2^\circ}{p^s}\right) \left(1 - \frac{\beta_1^\circ \beta_2^\circ}{p^s}\right) & \text{if } \nu = 1, \\ \frac{1}{G(\nu)^2} \cdot \left(\frac{p^{2s}}{(\alpha_1^\circ)^2 \alpha_2^\circ \beta_2^\circ}\right)^r & \text{if } \nu \text{ has conductor } p^r, r \geq 1. \end{cases}$$

where  $G(\nu) = \sum_{a \in (\mathbb{Z}/p^r)^\times} \nu(a) e^{2\pi i a/p^r}$  is the Gauss sum. This is exactly the modified Euler factor predicted by the conjectures of [CPR89, Pan94].

**Theorem C.** *Suppose  $\Pi$  is ordinary at  $\mathfrak{p}_1$  and its level is coprime to  $p$ . Then there exists a  $p$ -adic measure  $\mathcal{L}_{p, \text{As}}(\Pi) \in \Lambda_E$ , for  $E/\mathbb{Q}_p$  a sufficiently large finite extension, and a period  $\Omega(\Pi) \in \mathbb{C}^\times$ , such that for every  $s$  satisfying (crit) and Dirichlet character  $\nu$  of conductor  $p^r$ , for  $r \geq 0$ , we have*

$$\int_{\mathbb{Z}_p^\times} \nu^{-1}(x) x^s d\mathcal{L}_{p, \text{As}}(\Pi) = \mathcal{E}_p(\text{As}(\Pi), \nu, s) \cdot \frac{\Gamma(s + \frac{k_1+k_2-2}{2})\Gamma(s + \frac{k_1-k_2}{2})}{2^{(k_1-2)} i^{(1-k_2)} (-2\pi i)^{(2s+k_1-1)}} \cdot \frac{L_{\text{As}}(\Pi, \nu, s)}{\Omega(\Pi)}.$$

Note that both sides of the equality of theorem C are algebraic (and lie in  $L(\nu)$  for a number field  $L$  depending only on  $\Pi$ ), which is an extension of existing algebraicity results due to Shimura [Shi81].

**1.3. Two-variable  $p$ -adic  $L$ -functions.** Our second main result involves a slight modification of the Asai  $L$ -function, denoted  $L_{\text{As}}^{\text{imp}}(\Pi, \chi, s)$ , which we call the *imprimitive* Asai  $L$ -function. Unlike the primitive Asai  $L$ -function, it is given by a straightforward formula in terms of Hecke eigenvalues: if  $a_{\mathfrak{m}}^\circ(\Pi)$  is the normalised Hecke eigenvalue of the new vector of  $\Pi$  at an ideal  $\mathfrak{m} \trianglelefteq \mathcal{O}_F$ , then the imprimitive Asai  $L$ -function is given by  $(\star) \cdot \sum_{n \in \mathbb{N}} a_{n\mathcal{O}_F}^\circ(\Pi) n^{-s}$  where  $(\star)$  is a Dirichlet  $L$ -function. (See §8.3 for a more precise, representation-theoretic definition). This differs from  $L_{\text{As}}(\Pi, \chi, s)$  by a finite product of Euler factors, namely

$$L_{\text{As}}^{\text{imp}}(\Pi, \chi, s) = L_{\text{As}}(\Pi, \chi, s) \cdot \prod_{\ell | N_{\mathfrak{m}}(\mathfrak{N})} C_\ell(\ell^{-s} \chi(\ell)),$$

where  $\mathfrak{N} \trianglelefteq \mathcal{O}_F$  is the conductor of  $\Pi$ , and  $C_\ell(X)$  is a polynomial dividing the Euler factor of  $L_{\text{As}}(\Pi, \chi, s)$  at  $\ell$ . Since we can clearly define measures interpolating the factors  $C_\ell(-)$ , it follows *a fortiori* from Theorem B that there exists a measure  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi)$  interpolating critical values of  $L_{\text{As}}^{\text{imp}}(\Pi, \chi, s)$ . We shall show that these “imprimitive  $p$ -adic  $L$ -functions” vary analytically when  $\Pi$  itself is allowed to vary in a family.

**Theorem D.** *Let  $\Pi$  be ordinary at  $\mathfrak{p}_1$ , with level coprime to  $p$ , and with weight  $(k_1, k_2)$  such that  $k_1 = k_2 \pmod{2}$  and  $k_2 \geq 2$ . (We no longer assume that  $k_1 > k_2$ ).*

- (1) *There exists a  $p$ -adic family  $\underline{\Pi}$  of automorphic representations parametrised by some affinoid neighbourhood  $\mathcal{U}$  of 0 in the  $p$ -adic weight space  $\mathcal{W}$ , with weight  $(k_1 + 2\kappa_{\mathcal{U}}, k_2)$  where  $\kappa_{\mathcal{U}}$  is the universal character of  $\mathcal{U}$ , whose specialisation at  $0 \in \mathcal{U}$  is  $\Pi$ .*
- (2) *There exists an element  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}) \in \mathcal{O}(\mathcal{U}) \hat{\otimes} \Lambda_E$ , whose specialisation at any integer  $a \in \mathcal{U} \cap \mathbb{Z}$  with  $k_1 + 2a > k_2$  is a non-zero multiple of  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi[a])$ , where  $\Pi[a]$  is the specialisation of  $\underline{\Pi}$  in weight  $(k_1 + 2a, k_2)$ .*

In particular, this allows us to define the one-variable  $p$ -adic  $L$ -function  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi)$  without assuming  $k_1 > k_2$ , since the interpolating property of the two-variable measure uniquely determines the fibre at  $a = 0$ , even when the Asai  $L$ -function has no critical values. This includes the important case of base-changes of elliptic modular forms.

**Remark 1.3.1.** It should be possible to interpolate the “primitive”  $p$ -adic  $L$ -functions  $\mathcal{L}_{p, \text{As}}(\Pi[a])$  as  $a$  varies, not just their imprimitive versions, but this would require a careful study of local test vectors at primes  $\ell \neq p$  in  $p$ -adic families (analogous to the analysis carried out in [CH20] for Rankin–Selberg  $L$ -functions), and we shall not undertake this here for reasons of space.  $\diamond$

**1.4. Relation to other work.** The results described above are in many ways parallel to the results of [LPSZ21] in the case of spin  $L$ -functions of genus 2 Siegel modular forms. In particular, both here and in *op.cit.*, the underlying  $L$ -value formula requires us to work with coherent cohomology classes in intermediate degrees (not corresponding to either holomorphic or anti-holomorphic forms) and hence the use of higher Hida theory is essential. This contrasts with earlier works in which the theory of ordinary  $p$ -adic families of

modular forms has been used to construct  $p$ -adic  $L$ -functions, such as for  $\mathrm{GL}_2 \times \mathrm{GL}_2$  [Hid85, Pan82], and the  $\mathrm{GL}_2$  triple product [HT01]. Classical Hida theory is sufficient for those cases since one works with products of the modular curve and it suffices to vary  $p$ -adically the degree zero cohomology group. Another case where higher Hida theory plays a role in constructing  $p$ -adic  $L$ -functions is  $U(2, 1)$ , which has been treated by Oh [Oh23].

In a companion paper [GLZ23] (in preparation), we shall relate the values of the  $p$ -adic Asai  $L$ -functions constructed here, at points *outside* the range of interpolation, to the Euler system classes in Galois cohomology constructed in [LLZ18]; this will be used in subsequent work to prove an explicit reciprocity law for these Euler system classes, and hence obtain new cases of the Bloch–Kato conjecture and Iwasawa main conjecture for Asai motives.

The main result of [LW20] is a counterpart of theorem C for automorphic representations of  $\mathrm{GL}_2$  over *imaginary* quadratic fields. However, the two constructions use very different techniques; indeed, the construction of the  $p$ -adic  $L$ -function in the imaginary case is more strongly related to the construction of the Euler system of [LLZ18] in the real-quadratic case, rather than that of the  $p$ -adic  $L$ -function carried out here.

## 2. HIGHER HIDA THEORY FOR $\mathfrak{p}_1$ -ORDINARY COHOMOLOGY

We adapt the constructions of [Gro21] to the case Hilbert modular varieties of Iwahori level at one of the primes above  $p$ . Assume that the prime  $p$  splits in  $F$ . Let us write  $p = \mathfrak{p}_1 \cdot \mathfrak{p}_2$ , and let  $\sigma_i$  be the complex embedding corresponding to  $\mathfrak{p}_i$  for our fixed isomorphism  $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ .

**Remark 2.0.1.** More generally, we could choose an arbitrary totally real field  $F$  with  $p$  completely split, and choose an arbitrary subset of primes above  $p$ ; we stick to the case  $[F : \mathbb{Q}] = 2$  for simplicity, and leave the generalisation to the interested reader.  $\diamond$

### 2.1. Preliminaries.

2.1.1. *Hilbert modular surfaces.* For  $n \geq 1$ , we consider the level group  $K_n \subset \mathrm{GL}_2(\mathbb{A}_{F,f})$  defined by

$$K_n = K^{(p)} \cdot K_{\mathfrak{p}_1, n} \cdot \mathrm{GL}_2(\mathcal{O}_{F, \mathfrak{p}_2}).$$

where  $K_{\mathfrak{p}_1, n} = \{g \in \mathrm{GL}_2(\mathcal{O}_{F, \mathfrak{p}_1}) : g \equiv \begin{pmatrix} * & \\ 0 & * \end{pmatrix} \pmod{p^n}\}$  and  $K^{(p)}$  is a sufficiently small open compact subgroup of  $G(\mathbb{A}_f^{(p)})$ . We let  $X_{\mathbb{Q}} = X_0(\mathfrak{p}_1^n)_{\mathbb{Q}}$  denote a toroidal compactification of the Hilbert modular variety of level  $K_n$ , as an algebraic surface over  $\mathbb{Q}$ . The toroidal compactification depends on a choice of polyhedral cone decomposition; we can, and do, choose this so that  $X_{\mathbb{Q}}$  is smooth and projective.

Its connected components are obtained as quotient by a finite group  $\Delta_K$  of totally positive units of  $\mathcal{O}_F$  of the connected components of the compactified moduli space  $\mathcal{M}_{\mathbb{Q}}$  parametrising tuples  $(A, \iota, \lambda, \alpha_{K^{(\mathfrak{p}_1)}}, H)$ , where  $(A, \iota, \lambda, \alpha_{K^{(\mathfrak{p}_1)}})$  is a (suitably) polarised abelian surface with real multiplication by  $F$  and  $K^{(\mathfrak{p}_1)}$ -level structure  $\alpha_{K^{(\mathfrak{p}_1)}}$  and  $H$  is a cyclic subgroup of  $A[\mathfrak{p}_1^n]$  of order  $p^n$ , compatible with the other data (see [Gro21, §2.1] for more details).

For  $n = 1$ , which is the main cases of interest for us, there are canonical  $\mathbb{Z}_{(p)}$ -models  $\mathcal{M}$  and  $X$  of  $\mathcal{M}_{\mathbb{Q}}$  and  $X_{\mathbb{Q}}$ , with  $\mathcal{M}$  being a compactified moduli space for tuples  $(A, \iota, \lambda, \alpha_{K^{(\mathfrak{p}_1)}}, H)$  over  $\mathbb{Z}_{(p)}$ -algebras (see [Pap95]). Moreover,  $\mathcal{M}$  and  $X$  are both semistable over  $\mathbb{Z}_{(p)}$ , although not smooth; we shall recall the geometry of their special fibres in more detail below.

2.1.2. *Automorphic sheaves and integral models.* Let  $L$  denote the field  $F(\sqrt{\epsilon} : \epsilon \in \mathcal{O}_{F,+}^{\times})$  (which is Galois over  $\mathbb{Q}$ ) and  $\mathcal{O}_L$  be its ring of integers. Fix a noetherian  $\mathcal{O}_{L,(p)}$ -algebra  $R$ , and suppose that either  $n = 1$  or  $p$  is invertible in  $R$ , so we have natural  $R$ -models of  $\mathcal{M}$  and  $X$ . Let  $\mathcal{A} \rightarrow \mathcal{M}$  be the semi-abelian scheme extending the universal abelian surface with real multiplication by  $\mathcal{O}_F$ . Let  $e : \mathcal{M} \rightarrow \mathcal{A}$  be the unit section and

$$\underline{\omega}_{\mathcal{A}} := e^* \Omega_{\mathcal{A}/\mathcal{M}}^1;$$

this is a  $(\mathcal{O}_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathcal{O}_F)$ -module locally free of rank 1. We can write

$$(2.1) \quad \underline{\omega}_{\mathcal{A}} = \omega_{\mathcal{A},1} \oplus \omega_{\mathcal{A},2},$$

where  $\omega_{\mathcal{A},i}$  is the direct summand on which the  $\mathcal{O}_F$ -action is given by the composition of the embedding  $\sigma_i$  with the structure morphism  $\mathcal{O}_{L,(p)} \rightarrow R$ . Let  $\mathcal{H}_{\mathcal{A}}^1$  be the canonical extension of the relative de Rham cohomology  $\mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{M})$ . It is a  $(\mathcal{O}_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathcal{O}_F)$ -module locally free of rank 2 and we can decompose it with respect to the  $\mathcal{O}_F$ -action as above.

For  $(\underline{k}, w) \in \mathbb{Z}^2 \times \mathbb{Z}$  such that  $k_i \equiv w \pmod{2}$ . Let

$$(2.2) \quad \underline{\omega}^{(\underline{k}, w)} := \bigotimes_{1 \leq i \leq 2} \left( (\wedge^2 \mathcal{H}_{\mathcal{A}, i}^1)^{\frac{w-k_i}{2}} \otimes \omega_{\mathcal{A}, i}^{k_i} \right)$$

Using the descent datum of [Gro21, Definition 2.3.3] for the map  $\mathcal{M} \rightarrow X$ , we can descend  $\underline{\omega}^{(\underline{k}, w)}$  to a sheaf over  $X$ . The cohomology of this sheaf has an action of the prime-to- $p$  Hecke algebra of level  $K^{(p)}$ , and the double coset of  $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$  for  $x \in F^{\times+}$  acts as multiplication by  $x \mapsto \text{Nm}_{F/\mathbb{Q}}(x)^w$ .

We also have the semi-abelian surface  $\mathcal{A}/H \rightarrow \mathcal{M}$  extending the universal abelian surface isogenous to  $\mathcal{A}$ . They also come with unit sections (which we still denote by  $e$  by abuse of notation) and give integral sheaves

$$(2.3) \quad \underline{\omega}_{\mathcal{A}/H} := e^* \Omega_{(\mathcal{A}/H)/\mathcal{M}}^1 = \omega_{\mathcal{A}/H, 1} \oplus \omega_{\mathcal{A}/H, 2}.$$

(Note that the natural pullback map  $\underline{\omega}_{\mathcal{A}/H} \rightarrow \underline{\omega}_{\mathcal{A}}$  is an isomorphism after inverting  $p$ , but is not an isomorphism in characteristic  $p$ .)

2.1.3. *Geometry of the special fibre for  $n = 1$ .* From now on, we assume  $R$  is a local ring with maximal ideal  $\mathfrak{p} \ni p$ . Denote by  $X_{\mathbb{F}_p}$  the special fibre of  $X$  and let  $\mathfrak{X}$  be the formal completion of  $X$  along  $X_{\mathbb{F}_p}$ . It is the limit of the schemes  $X \times_{\text{Spec } R} \text{Spec}(R/\mathfrak{p}^n)$ . We denote by  $\mathfrak{X}^{1\text{-ord}}$  the ordinary locus of  $\mathfrak{X}$  for  $\mathfrak{p}_1$ . We use analogous notation for the formal completion  $\mathfrak{M}$  of  $\mathcal{M}$  along  $\mathcal{M}_{\mathbb{F}_p}$  and its ordinary locus  $\mathfrak{M}^{1\text{-ord}}$ . We can decompose the  $\mathfrak{p}_1$ -ordinary locus of the special fibre as follows:

$$X_{\mathbb{F}_p}^{1\text{-ord}} = X_{\mathbb{F}_p}^{1\text{-ét}} \cup X_{\mathbb{F}_p}^{1\text{-m}},$$

where  $X_{\mathbb{F}_p}^{1\text{-}\bullet}$ , for  $\bullet \in \{\text{ét}, \text{m}\}$  denotes the locus where the subgroup  $H \subset A[\mathfrak{p}_1^n]$  is isomorphic to  $\mathbb{Z}/p^n\mathbb{Z}$  if  $\bullet = \text{ét}$  and to  $\mu_{p^n}$  if  $\bullet = \text{m}$ . Note that the closures  $\overline{X_{\mathbb{F}_p}^{1\text{-ét}}}$  and  $\overline{X_{\mathbb{F}_p}^{1\text{-m}}}$  are both smooth over  $\mathbb{F}_p$ , and they intersect transversely in the  $\mathfrak{p}_1$ -supersingular locus  $X_{\mathbb{F}_p}^{1\text{-ss}}$ .

2.2. **The  $U_{\mathfrak{p}_1}$ -operator at level  $K_{\mathfrak{p}_1, 1}$ .** In this section we work at level  $K_{\mathfrak{p}_1, 1}$  and we define a cohomological correspondence on  $X$ . Let  $\mathcal{M}(\mathfrak{p}_1)$  be the (toroidal compactification of the) moduli space parametrising tuples  $(A, \iota, \lambda, \alpha_{K(\mathfrak{p}_1)}, H')$ , where  $(A, \iota, \lambda, \alpha_{K(\mathfrak{p}_1)})$  is as above and  $H' \subset A[\mathfrak{p}_1^2]$  is a subgroup admitting locally in the flat topology an isomorphism with  $\mathbb{Z}/p^2\mathbb{Z}$  and admitting a generator in the sense of [KM85].

Note that this is isomorphic to the moduli space of  $\mathfrak{p}_1$ -isogenies “between points of  $\mathcal{M}$ ”, in particular the point  $(A, \iota, \lambda, \alpha_{K(\mathfrak{p}_1)}, H')$  corresponds to the isogeny:

$$\eta : A/H'[\mathfrak{p}_1] \rightarrow A, \quad H'/H'[\mathfrak{p}_1] \xrightarrow{\cong} H'[\mathfrak{p}_1]$$

We will denote by  $L$  the kernel of the isogeny.

Since the  $\mathfrak{p}_1$ -torsion of the universal abelian variety is a  $p$ -divisible group of dimension 1 and height 2, the moduli scheme  $\mathcal{M}(\mathfrak{p}_1)$  admits a description analogue to the one of the modular curve of level  $\Gamma_0(p^2)$  in [KM85]. After quotienting by the action of the suitable finite group  $\Delta_{K(\mathfrak{p}_1)}$ , this gives an integral model over  $\mathbb{Z}_{(p)}$  of the (toroidal compactification of the) Hilbert modular surface of level  $K_2 = K^{(\mathfrak{p}_1)}K_{\mathfrak{p}_1, 2}$ , which we can then base-change to  $R$ . We denote it by  $C(\mathfrak{p}_1)$ . We have two maps  $p_1, p_2 : \mathcal{M}(\mathfrak{p}_1) \rightarrow \mathcal{M}$ . The first one is induced by the morphism  $(A, \iota, \lambda, \alpha_{K(\mathfrak{p}_1)}, H') \mapsto (A/H'[\mathfrak{p}_1], \iota', \lambda', \alpha'_{K(\mathfrak{p}_1)}, H'/H[\mathfrak{p}_1])$ , whereas the second is induced by  $(A, \iota, \lambda, \alpha_{K(\mathfrak{p}_1)}, H') \mapsto (A, \iota, \lambda, \alpha_{K(\mathfrak{p}_1)}, H'[\mathfrak{p}_1])$ ,  $\lambda'$  is the induced polarisation (see the discussion before [Gro21, (2.3)]). Here, as in [Gro21], we choose a decomposition  $\mathcal{M}_K^{\text{tor}} := \coprod_{[c] \in C_{\mathbb{F}_p}^+} \mathcal{M}^{c, \text{tor}}$  with all representatives  $c$  coprime to  $p$ . One can then check these maps are equivariant for the actions of  $\Delta_K$  and hence they induce maps

$$p_1, p_2 : C(\mathfrak{p}_1) \rightarrow X.$$

We want to study the cohomological correspondence given by

$$(2.4) \quad \begin{array}{ccc} & C(\mathfrak{p}_1) & \\ p_1 \swarrow & & \searrow p_2 \\ X & & X. \end{array}$$

More precisely, we will proceed as in [BP23] and consider a correspondence

$$(2.5) \quad \begin{array}{ccc} & C = C(\mathfrak{p}_1) \setminus D_C & \\ p_1 \swarrow & & \searrow p_2 \\ X \setminus D & & X \setminus D, \end{array}$$

for certain divisors  $D_C, D$  and then show that (2.4) gives a strict dynamical compactification of it. In particular the results of *op. cit.* show that the (ordinary part) of the cohomology associated to this correspondence does not depend on the compactification as it is uniquely determined by the behaviour over  $X \setminus D$ . The characterisation of the full integral model of  $C(\mathfrak{p}_1)$  is therefore not needed.

We will make use of the universal  $\mathfrak{p}_1$ -isogeny  $\eta : p_1^* \mathcal{A} \rightarrow p_2^* \mathcal{A}$  of kernel  $L$ . This isogeny induces a rational map

$$p_2^* \underline{\omega}_{\mathcal{A}} \xrightarrow{\eta^*} p_1^* \underline{\omega}_{\mathcal{A}},$$

which respects the decompositions (2.1) and (2.3), giving rational maps

$$p_2^* \omega_{\mathcal{A},j} \xrightarrow{\eta_j^*} p_1^* \omega_{\mathcal{A},j},$$

for  $j \in \{1, 2\}$ . One can similarly obtain maps for the sheaves  $\wedge^2 \mathcal{H}_{A,j}^1$ . All these maps of sheaves over  $\mathcal{M}(\mathfrak{p}_1)$  are equivariant under the action of  $\Delta_K$ , hence we get an analogous map of sheaves over  $C(\mathfrak{p}_1)$ .

Similarly as for  $X$ , we can consider the ordinary  $\mathfrak{p}_1$ -locus  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ord}}$  of the special fibre  $C(\mathfrak{p}_1)_{\mathbb{F}_p}$  of  $C(\mathfrak{p}_1)$ . Recall that the universal isogeny  $\eta$  is isogeny dual to  $A \rightarrow A/H'[\mathfrak{p}_1]$ . If  $A$  is ordinary, from the classification of  $p$ -groups schemes,  $H'$  can either be a twisted form of  $\mu_{p^2}$ , of  $\mathbb{Z}/p^2$  or an extension of  $\mathbb{Z}/p\mathbb{Z}$  by  $\mu_p$ . Therefore we can decompose  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ord}}$  as follows

$$C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ord}} = C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m,m}} \cup C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,m}} \cup C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,ét}},$$

where

- $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m,m}}$  is the locus mapped to  $X_{\mathbb{F}_p}^{1-m}$  by  $p_1$  and to  $X_{\mathbb{F}_p}^{1-m}$  by  $p_2$ , i.e.  $H' \simeq \mu_{p^2}$  and  $H'[\mathfrak{p}_1] \simeq \mu_p$ ;
- $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,m}}$  is the locus mapped to  $X_{\mathbb{F}_p}^{1-\text{ét}}$  by  $p_1$  and to  $X_{\mathbb{F}_p}^{1-m}$  by  $p_2$ , i.e.  $H'$  is an extension of  $\mathbb{Z}/p\mathbb{Z}$  by  $\mu_p$  and  $H'[\mathfrak{p}_1] \simeq \mu_p$ ;
- $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,ét}}$  is the locus mapped to  $X_{\mathbb{F}_p}^{1-\text{ét}}$  by  $p_1$  and to  $X_{\mathbb{F}_p}^{1-\text{ét}}$  by  $p_2$ , i.e.  $H' \simeq \mathbb{Z}/p^2\mathbb{Z}$  and  $H'[\mathfrak{p}_1] \simeq \mathbb{Z}/p\mathbb{Z}$ .

From the above description, the universal isogeny  $p_1^* \mathcal{A} \rightarrow p_2^* \mathcal{A}$  is étale on the first two components and multiplicative on the third one.

Let us now consider  $W$  to be the union of the generic fibre  $C(\mathfrak{p}_1)_{\mathbb{Q}_p}$  of  $C(\mathfrak{p}_1)$  and  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m,m}}$ . In other words, we are removing from the special fibre the divisor  $D_C$  obtained as union of the *bad locus*  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,m}} \cup C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ss}}$ , where  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ss}}$  is the  $\mathfrak{p}_1$ -supersingular locus, and  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,ét}}$ .

Let  $D^{1-\text{ét}}$  be the Cartier divisor in  $X$  (for a more precise definition see §2.3) such that  $X \setminus D^{1-\text{ét}} = X_{\mathbb{Q}_p} \cup X_{\mathbb{F}_p}^{1-m}$ , which by abuse of notation we will denote by  $X^{1-m}$ . The restrictions of the maps in (2.4) give a correspondence

$$(2.6) \quad \begin{array}{ccc} & W & \\ p_1 \swarrow & & \searrow p_2 \\ X^{1-m} & & X^{1-m}. \end{array}$$

We now consider the map, called "fundamental class", constructed more generally in [Pil20, §4.2] (see also [BCGP, §3.8.11-3.8.16]). It is a map

$$\Theta : p_1^* \mathcal{O}_X \rightarrow p_1^! \mathcal{O}_X.$$

Note that the fundamental class is uniquely determined by its definition on  $V$ , the set generic points of the special fibre. There it is defined to be the determinant of the map  $dp_1 : \Omega_U^1 \otimes \mathcal{O}_V \rightarrow \Omega_V^1$ , where  $U$  is a smooth open subset of  $X^{1-\text{ét}}$  such that  $p_1(V) \subset U$ . Therefore, by [Pil20, Lemma 4.2.3.1], the fundamental class is given on  $V$  by the trace map  $\text{Tr}_{p_1} : \mathcal{O}_V \rightarrow p_1^! \mathcal{O}_U$ .

Tensoring  $p_2^* \omega^{(k,w)} \dashrightarrow p_1^* \omega^{(k,w)}$  with  $\Theta$ , we get the *naive* cohomological correspondence over  $W$

$$U_{\mathfrak{p}_1, (k,w)}^{\text{naive}} : p_2^* \underline{\omega}^{(k,w)} \dashrightarrow p_1^! \underline{\omega}^{(k,w)}.$$



We finally normalise it letting

$$U_{\mathfrak{p}_1,(\underline{k},w)} := p^{-\inf\left\{\frac{w+k_1}{2}, \frac{w+(2-k_1)}{2}\right\}} U_{\mathfrak{p}_1,(\underline{k},w)}^{\text{naive}} = \begin{cases} p^{-\frac{w-k_1}{2}-1} U_{\mathfrak{p}_1,(\underline{k},w)}^{\text{naive}} & \text{if } k_1 \geq 1 \\ p^{-\frac{w+k_1}{2}} U_{\mathfrak{p}_1,(\underline{k},w)}^{\text{naive}} & \text{if } k_1 \leq 1. \end{cases}$$

To simplify the notation, when clear from the context, we denote simply by  $U_{\mathfrak{p}_1}$  the operator  $U_{\mathfrak{p}_1,(\underline{k},w)}$  for the automorphic sheaf  $\underline{\omega}^{(\underline{k},w)}$ .

**Remark 2.2.1.** Note that if  $\Pi$  is a cuspidal automorphic representation of weight  $(k_1, k_2)$ , with  $k_1, k_2 \geq 1$ , as in the introduction, then the representation  $\Pi \otimes \|\cdot\|^{-w/2}$  appears in the cohomology of  $X_{\overline{\mathbb{Q}}}$  with coefficients in each of the four sheaves  $\underline{\omega}^{(1 \pm (k_1-1), 1 \pm (k_2-1), w)}$  for small enough tame levels. For any of these choices of signs, the the eigenvalues of  $U_{\mathfrak{p}_1}^{\text{naive}}$  on this eigenspace are  $p^{(1+w)/2} \alpha_1^\circ$  and  $p^{(1+w)/2} \beta_1^\circ$ , while the eigenvalues of  $U_{\mathfrak{p}_1}$  are  $p^{(k_1-1)/2} \alpha_1^\circ = \alpha_1$  and  $p^{(k_1-1)/2} \beta_1^\circ = \beta_1$ . In particular, the action of  $U_{\mathfrak{p}_1}$  is independent of the choice of  $w$  (while the prime-to- $p$  Hecke action and  $U_{\mathfrak{p}_1}^{\text{naive}}$  are not).  $\diamond$

**Proposition 2.2.2.**  $U_{\mathfrak{p}_1} : p_2^* \underline{\omega}^{(\underline{k},w)} \rightarrow p_1^! \underline{\omega}^{(\underline{k},w)}$  is a well-defined optimally integral<sup>1</sup> cohomological correspondence over  $W$ . Moreover it is supported on  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m,m}}$  if  $k_1 > 1$ .

*Proof.* The proof follows as in [Gro21, Proposition 3.1.1].  $\square$

**2.3. Twisted vector bundles.** We now consider a twisted integral structure on  $\underline{\omega}^{(\underline{k},w)}$  in order to obtain a sheaf for which the  $U_{\mathfrak{p}_1}$ -operator extends to  $C(\mathfrak{p}_1)$ . Consider the dual isogenies:

$$(2.7) \quad \lambda_1 : \mathcal{A} \rightarrow \mathcal{A}/H, \quad \lambda_1^\vee : \mathcal{A}/H \rightarrow \mathcal{A}.$$

These are the universal isogenies defined over the moduli space  $\mathcal{M}^{\text{tor}}$ , where once more we choose a decomposition  $\mathcal{M}^{\text{tor}} := \coprod_{[\mathfrak{c}]} \mathcal{M}^{\mathfrak{c},\text{tor}}$  with all representatives  $\mathfrak{c}$  coprime to  $p$ ; this allows to use the polarisations on  $\mathcal{A} = \coprod_{[\mathfrak{c}]} \mathcal{A}^{\mathfrak{c}}$  and on  $\mathcal{A}/H$  to see the dual isogeny  $\lambda_1^\vee : (\mathcal{A}/H)^\vee \rightarrow \mathcal{A}^\vee$  as an isogeny  $\lambda_1^\vee : \mathcal{A}/H \rightarrow \mathcal{A}$ .

The pullbacks on differentials induce, in particular, maps for every  $i \in \{1, 2\}$  as follows:

$$(2.8) \quad (\lambda_1^*)_i : \omega_{\mathcal{A}/H,i} \rightarrow \omega_{\mathcal{A},i}, \quad ((\lambda_1^\vee)^*)_i : \omega_{\mathcal{A},i} \rightarrow \omega_{\mathcal{A}/H,i}$$

We consider the rank one sheaf

$$\delta := \omega_{\mathcal{A}/H,1} \otimes \omega_{\mathcal{A},1}^{-1}.$$

From the maps  $(\lambda_1^*)_1$  and  $((\lambda_1^\vee)^*)_1$  respectively, we obtain global sections of  $\delta^{-1}$  and  $\delta$  respectively:

$$s_1 : \mathcal{O}_X \rightarrow \delta^{-1}, \quad s_1^\vee : \mathcal{O}_X \rightarrow \delta.$$

We denote by  $D^{1-\text{m}}$  and  $D^{1-\text{ét}}$  respectively the associated divisors. These are the Cartier divisor associated to the determinant complex as in [KM76, Chapter II]. We have  $X_{\mathbb{F}_p} - D^{1-\text{m}} = X_{\mathbb{F}_p}^{1-\text{ét}}$  and  $X_{\mathbb{F}_p} - D^{1-\text{ét}} = X_{\mathbb{F}_p}^{1-\text{ét}}$  respectively, since the maps in (2.8) factor through multiplication by a positive power of  $p$  when the corresponding isogenies in (2.7) are of multiplicative type.

**Lemma 2.3.1.** *The divisors  $p_2^* D^{1-\text{m}} - p_1^* D^{1-\text{m}}$ ,  $p_1^* D^{1-\text{ét}} - p_2^* D^{1-\text{ét}}$  are effective and their support is  $p_2^* D^{1-\text{m}} \cap p_1^* D^{1-\text{ét}}$ . More precisely, we have*

$$p_1^* D^{1-\text{m}} \leq s' p_2^* D^{1-\text{m}}, \quad p_2^* D^{1-\text{ét}} \leq s p_1^* D^{1-\text{ét}}$$

for some  $0 < s, s' < 1$  in the language of [BP23, Definition 2.1.15].

*Proof.* We have generically étale maps

$$p_1^* D^{1-\text{m}} = \overline{C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m,m}}} \longrightarrow p_2^* D^{1-\text{m}} = \overline{C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m,m}} \cup C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,m}}}, \quad p_1^* H \rightarrow p_2^* H$$

$$p_2^* D^{1-\text{ét}} = \overline{C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,ét}}} \longrightarrow p_1^* D^{1-\text{ét}} = \overline{C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,ét}} \cup C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,m}}}, \quad p_2^* H^D \rightarrow p_1^* H^D$$

induced by  $\eta : p_1^* \mathcal{A} \rightarrow p_2^* \mathcal{A}$  and  $\lambda_1^\vee : p_2^* \mathcal{A}/H \rightarrow p_1^* \mathcal{A}/H$  respectively. The maps  $p_1^* H \rightarrow p_2^* H, p_2^* H^D \rightarrow p_1^* H^D$  are isomorphisms on  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m,m}}$  and  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{ét,ét}}$  respectively. The statements then follow from the fact that the considered divisors arise from short exact sequences of complexes induced by generically étale maps. Divisors

<sup>1</sup>Here *optimally integral* is intended as in [BP22], i.e. it is integral and not divisible by a power of  $p$  on the special fibre.



behave well with respect to exact sequences (see [KM76, Chapter II] and [Far11] for the case of  $p$ -divisible groups). Note that in the language of [KM76, Chapter II] we have  $D^{1-m} = \delta_H, D^{1-\acute{e}t} = \delta_{H^D}$ .

In order to prove the statement about the "strict inclusion" of divisors, we can proceed similarly as in the proof of [BP23, Lemma 4.4.2]. Note that the situation here is simpler as for each inequality we can choose a local generator  $f$  of the divisor  $D^{1-\acute{e}t}$  (or  $D^{1-m}$  respectively). The first part of the lemma shows that we can write  $p_1^* f = p_2^* f \cdot h$  for some  $h$  vanishing outside  $p_2^* D^{1-m} \cap p_1^* D^{1-\acute{e}t}$ . Choosing another divisor with disjoint support with respect to  $D^{1-\acute{e}t}$  and with local generator  $f'$ , we can write  $af + bf' = 1$  and from the equality  $p_1^* f = p_1^* f \cdot p_2^*(af + bf')$  combined with  $p_1^* f = p_2^* f \cdot h$ , we find that  $p_1^* f$  is divisible by  $(p_2^* f)^m$  with  $m \geq 2$  yielding the final claim.  $\square$

We can rephrase the above result in the language of [BP23]:

**Corollary 2.3.2.** *The correspondence (2.4) is a strict dynamic compactification of the correspondance of Proposition 2.2.2 strictly expanding the divisor  $D^{1-\acute{e}t}$ .*

We can therefore apply the results of [BP23, §2.5] to extend the operator  $U_{p_1}$  from  $W$  to  $C(p_1)$ . From now on we hence see all the objects in the category of solid  $\mathcal{O}_W$  and  $\mathcal{O}_{X^{1-m}}$ -modules (or  $\mathcal{O}_{C(p_1)}$  and  $\mathcal{O}_X$ -modules respectively) as in [CS]. Recall that the derived category of quasi-coherent modules can be fully-faithfully mapped to this category. The advantage of working in the larger category is that, thanks to the six functor formalism it comes with, one has a well-defined action of the Hecke operator on the cohomology of  $X$ , but also on its cohomology *with support*, that can be defined as limits that do exist once one works in such category. Moreover, using the solid tensor product, one has a well-defined notion of ordinary part of these cohomology complexes. All such formalism has been developed in [BP23] and we refer the interested reader to it for more details.

**Lemma 2.3.3.** *There exist  $t, m \in \mathbb{Z}$  big enough such that for every weight  $(\underline{k}, w)$  with  $k_1 \geq 2$  the cohomological correspondence  $U_{p_1} : p_2^* \underline{\omega}_{|W}^{(\underline{k}, w)} \rightarrow p_1^* \underline{\omega}_{|W}^{(\underline{k}, w)}$  can be extended for every  $n \geq m$  to a map*

$$U_{p_1} : p_2^* \underline{\omega}^{(\underline{k}, w)}(ntD^{1-\acute{e}t}) \rightarrow p_1^* \underline{\omega}^{(\underline{k}, w)}(ntD^{1-\acute{e}t})$$

in the category of solid  $\mathcal{O}_{C(p_1)}$ -modules. Moreover it factors as

$$\begin{array}{ccc} p_2^* \underline{\omega}^{(\underline{k}, w)}((n+1)tD^{1-\acute{e}t}) & & \\ \uparrow & \dashrightarrow^{U_{p_1}} & \\ p_2^* \underline{\omega}^{(\underline{k}, w)}(ntD^{1-\acute{e}t}) & \xrightarrow{U_{p_1}} & p_1^* \underline{\omega}^{(\underline{k}, w)}(ntD^{1-\acute{e}t}). \end{array}$$

*Proof.* This follows from the above corollary and Lemma 2.5.3 in [BP23].  $\square$

**Definition 2.3.4.** For any weight  $(\underline{k}, w)$  with  $k_1 \geq 2$ , and  $t_1 = tm$ , with  $t, m$  the integers in the previous lemma, we let

$$\tilde{\omega}^{(\underline{k}, w)} := \underline{\omega}^{(\underline{k}, w)}(t_1 D^{1-\acute{e}t}) \cong \underline{\omega}^{(\underline{k}, w)} \otimes (\delta^{-1})^{t_1}.$$

In other words, the component at  $i = 1$  becomes  $(\wedge^2 \mathcal{H}_{\mathcal{A}, 1}^1)^{\frac{w-k_1}{2}} \otimes \omega_{\mathcal{A}, 1}^{k_1+t_1} \otimes \omega_{\mathcal{A}/H_{1,1}}^{-t_1}$ . By the previous lemma and [BP23, §2.2],  $U_{p_1}$  induces a map

$$U_{p_1} : \mathrm{R}\Gamma(X, \tilde{\omega}^{(\underline{k}, w)}) \rightarrow \mathrm{R}\Gamma(X, \tilde{\omega}^{(\underline{k}, w)}).$$

**Remark 2.3.5.** Since  $\tilde{\omega}^{(\underline{k}, w)}$  agrees with the naive  $\underline{\omega}^{(\underline{k}, w)}$  on the generic fibre, this shows that there exists an  $R$ -lattice in  $H^*(X_{R[1/p]}, \underline{\omega}^{(\underline{k}, w)})$  which is preserved by  $U_{p_1}$ . In particular,  $\alpha_1$  and  $\beta_1$  are  $p$ -adically integral, as claimed above.  $\diamond$

We can now deduce the following result. We denote by  $\mathfrak{X}^{1-m}$  the formal completion of  $X - D^{1-\acute{e}t}$  along its special fibre. Let

$$\mathrm{R}\Gamma(X_{R/\varpi^m}^{1-m}, \tilde{\omega}^{(\underline{k}, w)}) = \operatorname{colim}_{n \geq 0} \mathrm{R}\Gamma(X_{R/\varpi^m}, \tilde{\omega}^{(\underline{k}, w)}(nD^{1-\acute{e}t})),$$

<sup>2</sup>In the notation of *op. cit.*, we are saying that the correspondence  $(C(p_1), X, p_1, p_2, D = D^+ = D^{1-\acute{e}t}, D^- = \emptyset)$  is a strict dynamic compactification of  $(W, X^{1-m}, (p_1)_{|W}, (p_2)_{|W})$ .

where the limit are taken in the category of solid  $\mathcal{O}_{X_{R/\varpi^m}}$ -modules. Such limit exists and ‘‘computes’’ the cohomology  $\mathrm{R}\Gamma_{D^{1-\acute{e}t}, \emptyset}(X_{R/\varpi^m}, \underline{\omega}^{(k,w)})$  (see [BP23, Corollary 2.3.11]). Finally, let

$$\mathrm{R}\Gamma(\mathfrak{X}^{1-m}, \underline{\tilde{\omega}}^{(k,w)}) = \varinjlim_{m \geq 1} \mathrm{R}\Gamma(X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(k,w)}).$$

Now note that by base change we have a cohomological correspondence, which by abuse of notation we still denote by  $U_{\mathfrak{p}_1}$ , over  $C(\mathfrak{p}_1)_{R/\varpi^m}$ . Hence the operator  $U_{\mathfrak{p}_1}$  acts on each of the terms in the first colimit, inducing an action on  $\mathrm{R}\Gamma(X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(k,w)})$ . Recall from [BP23, §2.4] the following definition:

**Definition 2.3.6.** If  $M$  is an object in the category of solid  $A$ -modules  $D(A_{\blacksquare})$  and  $U$  is an operator acting on  $M$ , the ordinary part  $M^{U\text{-ord}} \in D(A[U, U^{-1}]_{\blacksquare})$  of  $M$  is defined by

$$M^{U\text{-ord}} = M \otimes_{(A[U], A)} A[U, U^{-1}].$$

By [BP23, Lemma 2.4.4], we find that the  $U_{\mathfrak{p}_1}$ -ordinary part of  $\mathrm{R}\Gamma(X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(k,w)})$  is equal to the limit of the ordinary part of  $\mathrm{R}\Gamma(X_{R/\varpi^m}, \underline{\tilde{\omega}}^{(k,w)}(nD^{1-\acute{e}t}))$ . Moreover using the base change formula [BP23, Proposition 2.6.13], we also have

$$(2.9) \quad \mathrm{R}\Gamma(\mathfrak{X}^{1-m}, \underline{\tilde{\omega}}^{(k,w)})^{U_{\mathfrak{p}_1}\text{-ord}} \otimes R/\varpi^m = \mathrm{R}\Gamma(X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(k,w)})^{U_{\mathfrak{p}_1}\text{-ord}}.$$

**Corollary 2.3.7.** *If  $k_1 \geq 2$ , pullback along the inclusion  $\mathfrak{X}^{1-m} \hookrightarrow \mathfrak{X}$  gives an isomorphism*

$$\mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)})^{U_{\mathfrak{p}_1}\text{-ord}} \simeq \mathrm{R}\Gamma(\mathfrak{X}^{1-m}, \underline{\tilde{\omega}}^{(k,w)})^{U_{\mathfrak{p}_1}\text{-ord}} = \mathrm{R}\Gamma(\mathfrak{X}^{1-m}, \underline{\omega}^{(k,w)})^{U_{\mathfrak{p}_1}\text{-ord}}.$$

*In particular, they are all bounded complexes of  $R$ -modules.*

*Proof.* By Lemma 2.3.3, the horizontal maps in the commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)}(ntD^{1-\acute{e}t})) & \longrightarrow & \mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)}((n+1)tD^{1-\acute{e}t})) \\ \downarrow U_{\mathfrak{p}_1} & \swarrow U_{\mathfrak{p}_1} & \downarrow U_{\mathfrak{p}_1} \\ \mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)}(ntD^{1-\acute{e}t})) & \longrightarrow & \mathrm{R}\Gamma(X_1, \underline{\tilde{\omega}}^{(k,w)}((n+1)tD^{1-\acute{e}t})). \end{array}$$

become quasi-isomorphisms on the ordinary part, namely if  $k_1 \geq 2$ ,  $n \geq 0$  and  $m \geq 1$  then we have quasi-isomorphisms

$$\mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)}(nD^{1-\acute{e}t})/\varpi^m)^{U_{\mathfrak{p}_1}\text{-ord}} \xrightarrow{\simeq} \mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)}((n+1)D^{1-\acute{e}t})/\varpi^m)^{U_{\mathfrak{p}_1}\text{-ord}}.$$

Therefore, the maps in the direct limit defining  $\mathrm{R}\Gamma(X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(k,w)})$  are isomorphisms after taking the ordinary part and the first equality follows using the base change (2.9). The second follows from the fact that  $\underline{\omega}^{(k,w)}$  and  $\underline{\tilde{\omega}}^{(k,w)}$  coincide after restriction to  $\mathfrak{X}^{1-m}$ . The statement about boundedness follows then from [BP23, Lemma 2.4.3].  $\square$

**Remark 2.3.8.** Note that this is the analogue of [BP23, Corollary 4.5.4], where the condition on the weight ( $C(\kappa) = \{w\}$  in *op.cit.*) translates in  $\kappa = (k, w)$  such that  $k_1 \geq 2$ .  $\diamond$

By [BP23, Lemmas 2.4.3-2.4.4], the  $U_{\mathfrak{p}_1}$ -ordinary parts of the complexes in the above corollary are direct summands of the respective complexes. From now on we will write  $e(U_{\mathfrak{p}_1})$  for the projector to the ordinary part and we can rewrite the quasi-isomorphism above as

$$e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)}) \simeq e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(\mathfrak{X}^{1-m}, \underline{\tilde{\omega}}^{(k,w)}).$$

The same arguments apply also with  $\underline{\omega}^{(k,w)}$  replaced by its ‘‘cuspidal’’ variant  $\underline{\omega}^{(k,w)}(-D)$ , where  $D$  is the toroidal boundary divisor, giving an isomorphism

$$e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(X, \underline{\tilde{\omega}}^{(k,w)}(-D)) \simeq e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(\mathfrak{X}^{1-m}, \underline{\omega}^{(k,w)}(-D))$$

if  $t_1$  is sufficiently large.

**2.4. Compact support.** Similarly as in [BP23, BP22, Gro21], we can also consider compactly-supported cohomology. Recall that  $(U_{\mathfrak{p}_1, (\underline{k}, w)})|_W$  was defined using the pullback by  $\eta$ . Similarly, one can consider the rational map

$$p_1^* \omega_{\mathcal{A}, 1} \xrightarrow{(\eta^\vee)^*} p_2^* \omega_{\mathcal{A}, 1},$$

Tensoring with the maps on the second component induced by the same pullback  $(\eta^\vee)^*$  (which are isomorphisms), and with the trace map for  $p_2$ , we obtain a map

$$U_{\mathfrak{p}_1, (\underline{k}, w)}^{t, \text{naive}} : p_1^* \underline{\omega}^{(\underline{k}, w)} \dashrightarrow p_2^! \underline{\omega}^{(\underline{k}, w)},$$

which we can normalise as before letting

$$U_{\mathfrak{p}_1, (\underline{k}, w)}^t := p^{-\inf\left\{\frac{w-k_1}{2}+1, \frac{w+k_1}{2}\right\}} U_{\mathfrak{p}_1, (\underline{k}, w)}^{t, \text{naive}}.$$

Note that  $(U_{\mathfrak{p}_1, (\underline{k}, w)}^t)|_W$  is now regular, and supported on  $C(\mathfrak{p}_1)_{\mathbb{F}_p}^{\text{m}, \text{m}}$  if  $k_1 < 1$ . So if we define  $\underline{\tilde{\omega}}^{(\underline{k}, w)}$  for  $k_1 \leq 0$  by

$$\underline{\tilde{\omega}}^{(\underline{k}, w)} := \underline{\omega}^{(\underline{k}, w)}(-t'_1 D^{1-\epsilon t}) \cong \underline{\omega}^{(\underline{k}, w)} \otimes \delta^{t'_1}$$

for some  $t'_1 \gg 0$ , we have the following. Recall that the compactly-supported cohomology is defined by

$$\text{R}\Gamma_c \left( X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(\underline{k}, w)} \right) := \lim_{n \geq 0} \text{R}\Gamma \left( X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(\underline{k}, w)}(-n D^{1-\epsilon t}) \right),$$

and

$$\text{R}\Gamma_c \left( \mathfrak{X}^{1-m}, \underline{\tilde{\omega}}^{(\underline{k}, w)} \right) := \lim_{m \geq 1} \text{R}\Gamma_c \left( X_{R/\varpi^m}^{1-m}, \underline{\tilde{\omega}}^{(\underline{k}, w)} \right).$$

**Corollary 2.4.1.** *If  $k_1 \leq 0$ , pushforward along the natural inclusion  $\mathfrak{X}^{1-m} \hookrightarrow \mathfrak{X}$  gives an isomorphism*

$$e(U_{\mathfrak{p}_1}^t) \text{R}\Gamma_c(\mathfrak{X}^{1-m}, \underline{\omega}^{(\underline{k}, w)}) = e(U_{\mathfrak{p}_1}^t) \text{R}\Gamma_c(\mathfrak{X}^{1-m}, \underline{\tilde{\omega}}^{(\underline{k}, w)}) \simeq e(U_{\mathfrak{p}_1}^t) \text{R}\Gamma(X, \underline{\tilde{\omega}}^{(\underline{k}, w)}).$$

**Remark 2.4.2.** This correspondance contracts the divisor  $D_- = D^{1-\epsilon t}$ , and the condition  $C(\kappa) = \{w\}$  in [BP23] translates to considering weights  $\kappa = (\underline{k}, w)$  such that  $k_1 \leq 0$ .  $\diamond$

**2.5. Duality.** We now investigate a duality relation between the operators  $U_{\mathfrak{p}_1}$  and  $U_{\mathfrak{p}_1}^t$ :

**Lemma 2.5.1.** *We have  $D(U_{\mathfrak{p}_1, (\underline{k}, w)}) = U_{\mathfrak{p}_1, (2-\underline{k}, -w)}^t$  acting on the cohomology of the automorphic sheaf  $\underline{\omega}^{(2-\underline{k}, -w)}$  over  $X^{1-m}$ .*

*Proof.* This follows similarly as in [BP22, Lemma 3.8] and [Gro21, Prop. 3.1.2]. We need to compare  $U_{\mathfrak{p}_1, (2-\underline{k}, -w)}^{t, \text{naive}}$  with  $D(U_{\mathfrak{p}_1, (\underline{k}, w)}^{\text{naive}})$  where  $U_{\mathfrak{p}_1, (\underline{k}, w)}^{\text{naive}}$  is the operator in weight  $(\underline{k}, w)$ . It dualises to an operator

$$D(\text{R}\Gamma(X^{1-m}, \underline{\omega}^{(\underline{k}, w)})) \xrightarrow{p_1^*} \text{R}\Gamma(W, p_1^* D(\underline{\omega}^{(\underline{k}, w)})) \xrightarrow{D(U_{\mathfrak{p}_1}^{\text{naive}})} \text{R}\Gamma(W, p_2^! D(\underline{\omega}^{(\underline{k}, w)})) \xrightarrow{\text{Tr}_{p_2}} D(\text{R}\Gamma(X^{1-m}, \underline{\omega}^{(\underline{k}, w)})),$$

where  $D(U_{\mathfrak{p}_1}^{\text{naive}})$  can be written as the composition

$$D(U_{\mathfrak{p}_1}^{\text{naive}}) : p_1^*(\omega^{(-\underline{k}, -w)} \otimes \Omega_{X/\mathbb{Z}_p}^n) \xrightarrow{\text{id} \otimes \text{Tr}_{p_1}} p_1^!(\omega^{(-\underline{k}, -w)} \otimes \Omega_{X/\mathbb{Z}_p}^n) \rightarrow p_2^!(\omega^{(-\underline{k}, -w)} \otimes \Omega_{X/\mathbb{Z}_p}^n).$$

Here, by abuse of notation, we write  $\text{Tr}_{p_1}$  for the fundamental class  $\Theta$  and, similarly  $\text{Tr}_{p_2}$  for the fundamental class associated to the morphism  $p_2$ . In order to describe the second map, recall that we have the Kodaira–Spencer isomorphism  $KS : \underline{\omega}^{(2,0)}(-D) \rightarrow \Omega_{X/\mathbb{Z}_p}^n$ , so that we can identify the sheaf  $\underline{\omega}^{(-\underline{k}, -w)} \otimes \Omega_{X/\mathbb{Z}_p}^n$  with  $\underline{\omega}^{(2-\underline{k}, -w)}(-D) = \underline{\omega}^{(-\underline{k}, -w)} \otimes \underline{\omega}^{(2,0)}(-D)$ . The map  $p_1^!(\omega^{(-\underline{k}, -w)} \otimes \Omega_{X/\mathbb{Z}_p}^n) \rightarrow p_2^!(\omega^{(-\underline{k}, -w)} \otimes \Omega_{X/\mathbb{Z}_p}^n)$  can be written as  $p_1^*(\omega^{(-\underline{k}, -w)} \otimes \Omega_{X/\mathbb{Z}_p}^n) \rightarrow p_2^*(\omega^{(-\underline{k}, -w)} \otimes \Omega_{C(\mathfrak{p}_1)/\mathbb{Z}_p}^n)$  and is equal to  $\eta_{(-\underline{k}, -w)}^{-1} \otimes \text{id}$ . Applying the Kodaira–Spencer isomorphism, the identity

$$\Omega_{X_0(\mathfrak{p})/\mathbb{Z}_p}^n = p_1^! \mathcal{O}_X \otimes p_1^* \underline{\omega}^{(2,0)}(-D) \rightarrow \Omega_{X_0(\mathfrak{p})/\mathbb{Z}_p}^n \otimes p_2^* \underline{\omega}^{(2,0)}(-D)$$

can be decomposed as  $\text{Tr}_{p_2}(\text{Tr}_{p_1}^{-1}) \otimes \pi_{(2,0)}^D$  (see for example [BP22, Lemma 3.7]). All in all, we find

$$D(U_{\mathfrak{p}_1}^{\text{naive}}) = \text{Tr}_{p_2} \otimes (\eta_{(2,0)}^D \otimes \eta_{(-\underline{k}, -w)}^{-1}).$$

Moreover, we have

$$\eta_{(\underline{k}, w)} \circ \eta_{(\underline{k}, w)}^D = p^{k_p + 2(\frac{w-k_p}{2})}, \quad \eta_{(-\underline{k}, -w)} \circ \eta_{(-\underline{k}, -w)}^D = p^{-w}, \quad \eta_{(2,0)} \circ \eta_{(2,0)}^D = 1$$

Hence we obtain that  $D(U_{\mathfrak{p}_1}^{\text{naive}}) = p^w U_{\mathfrak{p}_1}^{t, \text{naive}}$  and from the definitions of the normalisation factors in the different weights, we obtain the claimed result.  $\square$

Since  $W$  contains the whole generic fibre of  $C(\mathfrak{p}_1)$ , we recover the well-known result that  $U_{\mathfrak{p}_1}$  and  $U_{\mathfrak{p}_1}^t$  are transposes of each other with respect to the (perfect) Serre duality pairing

$$H^1\left(X_{R[1/p], \underline{\omega}^{(k,w)}}\right) \times H^1\left(X_{R[1/p], \underline{\omega}^{(2-k, -w)}}(-D)\right) \rightarrow R[1/p],$$

and in particular we have a perfect duality

$$(\dagger) \quad e(U_{\mathfrak{p}_1})H^1\left(X_{R[1/p], \underline{\omega}^{(k,w)}}\right) \times e(U_{\mathfrak{p}_1}^t)H^1\left(X_{R[1/p], \underline{\omega}^{(2-k, -w)}}(-D)\right) \rightarrow R[1/p]$$

(and similarly with the  $-D$  on the first term rather than the second). On the other hand, we also have an  $R$ -linear pairing

$$e(U_{\mathfrak{p}_1})H^1\left(\mathfrak{X}^{1-m}, \underline{\omega}^{(k,w)}\right) \times e(U_{\mathfrak{p}_1}^t)H_c^1\left(\mathfrak{X}^{1-m}, \underline{\omega}^{(2-k, -w)}(-D)\right) \rightarrow R$$

arising from Serre duality for the coherent cohomology of the reductions  $X_{R/\varpi^n}^{1-m}$ , which are smooth schemes over  $R/\varpi^n$ . (Compare [BP23, Theorem 2.7.4] for a more general treatment.) This pairing is not necessarily perfect (there is a perfect duality in the derived category, but since the cohomology groups are not necessarily flat over  $R$ , we do not necessarily obtain a perfect pairing on  $H^1$ 's). However, all we need for our application is that the pairing is well-defined, and that it is compatible with the pairing  $(\dagger)$  after extending to the generic fibre via the maps of corollary 2.3.7 and corollary 2.4.1. This last compatibility follows from comparing both pairings with Grothendieck–Serre duality for the semistable model  $X_R$ ; this is not smooth over  $R$ , but it is Gorenstein, so it has a well-defined dualising sheaf, and this coincides with  $\underline{\omega}^{(2,2)}(-D)$  away from the  $\mathfrak{p}_1$ -supersingular locus of the special fibre.

**2.6. Recap of the classicality isomorphisms.** For the purpose of this paper, we will be interested in the degree one cohomology of a certain complex for the compactified Hilbert modular surface of spherical level at  $\mathfrak{p}_2$  and Iwahori level at  $\mathfrak{p}_1$ :

$$X_0(\mathfrak{p}_1) = X_{K(\mathfrak{p}_1)K_{\mathfrak{p}_1, 1}}.$$

For  $(k_1, k_2, w)$  with  $k_1 = k_2 = w \pmod{2}$  and  $k_i \geq 1$ , we define two complexes of  $R$ -modules

$$M_{\mathfrak{m}, \emptyset}^\bullet(k_1, 2 - k_2, w) := e(U_{\mathfrak{p}_1}) \text{R}\Gamma\left(\mathfrak{X}_0(\mathfrak{p}_1)^{1-m}, \underline{\omega}^{(k_1, 2-k_2, w)}\right)$$

and

$$M_{\mathfrak{m}, c}^\bullet(2 - k_1, k_2, w) := e(U_{\mathfrak{p}_1}^t) \text{R}\Gamma_c\left(\mathfrak{X}_0(\mathfrak{p}_1)^{1-m}, \underline{\omega}^{(2-k_1, k_2, w)}\right).$$

(We consider weights  $(k_1, 2 - k_2)$  and  $(2 - k_1, k_2)$  to ensure that the Hecke eigensystems associated to Hilbert modular forms of weight  $(k_1, k_2)$  appear in  $H^1$  of both complexes.)

**2.6.1. Weight  $\geq 2$  at  $\sigma_1$  and  $\leq 1$  at  $\sigma_2$ .** Suppose  $k_1 \geq 2$ . In order for the Hecke operators to be well defined as in the previous sections, we fix the following twisted integral model (see Definition 2.3.4) of our automorphic sheaf: let  $\delta := \omega_{\mathcal{A}/H, 1} \otimes \omega_{\mathcal{A}, 1}^{-1}$ , and

$$(2.10) \quad \tilde{\omega}^{(k_1, 2-k_2, w)} := \underline{\omega}^{(k_1, 2-k_2, w)}(t_1 D^{1-\epsilon t}) = \underline{\omega}^{(k_1, 2-k_2, w)} \otimes \delta^{-t_1},$$

for  $t_1$  sufficiently large. We obtain from Corollary 2.3.7 the following isomorphism:

$$M_{\mathfrak{m}, \emptyset}^\bullet(k_1, 2 - k_2, w) \cong e(U_{\mathfrak{p}_1}) \text{R}\Gamma(\mathfrak{X}_0^{1-m}(\mathfrak{p}_1), \tilde{\omega}^{(k_1, 2-k_2, w)}) \cong e(U_{\mathfrak{p}_1}) \text{R}\Gamma(X_0(\mathfrak{p}_1), \tilde{\omega}^{(k_1, 2-k_2, w)})$$

and similarly for its cuspidal version. The  $H^1$  of this complex is a  $p$ -adic avatar of Hilbert modular forms of weight  $(k_1, k_2)$  holomorphic at  $\sigma_1$  and anti-holomorphic at  $\sigma_2$ .

2.6.2. *Weight  $\leq 0$  at  $\sigma_1$  and  $\geq 1$  at  $\sigma_2$ .* Dually, defining  $\tilde{\omega}^{(k_1, 2-k_2, w)} := \underline{\omega}^{(k_1, 2-k_2, w)}(-t'_1 D^{1-\acute{e}t})$  for  $t'_1 \gg 0$ , we have

$$(2.11) \quad M_{m,c}^\bullet(2-k_1, k_2, w) \cong e(U_{\mathfrak{p}_1}^t) \mathrm{R}\Gamma_c(\mathfrak{X}_0^{1-m}(\mathfrak{p}_1), \tilde{\omega}^{(2-k_1, k_2, -w)}) \cong e(U_{\mathfrak{p}_1}^t) \mathrm{R}\Gamma(X_0(\mathfrak{p}_1), \tilde{\omega}^{(2-k_1, k_2, -w)}).$$

The  $H^1$  is now a  $p$ -adic avatar of Hilbert modular forms of weight  $(k_1, k_2)$  anti-holomorphic at  $\sigma_1$  and holomorphic at  $\sigma_2$ .

Since the sheaves  $\tilde{\omega}^{(k,w)}$  coincide with the usual  $\underline{\omega}^{(k,w)}$  on the generic fibre, we have in particular

$$M_{m,\emptyset}^\bullet(k_1, 2-k_2, w) \otimes R[1/p] \cong e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(X_0(\mathfrak{p}_1)_{R[1/p]}, \underline{\omega}^{(k_1, 2-k_2, w)})$$

and

$$M_{m,c}^\bullet(2-k_1, k_2, w) \otimes R[1/p] \cong e(U_{\mathfrak{p}_1}^t) \mathrm{R}\Gamma(X_0(\mathfrak{p}_1)_{R[1/p]}, \underline{\omega}^{(2-k_1, k_2, w)}).$$

There are also cuspidal variants  $M_{m,\emptyset}^\bullet(\underline{k}, w; -D)$  and  $M_{m,c}^\bullet(\underline{k}, w; -D)$  satisfying analogous comparisons with the classical cohomology of the sheaves  $\underline{\omega}^{(k_1, 2-k_2, w)}(-D)$ ,  $\underline{\omega}^{(2-k_1, k_2, w)}(-D)$ , where  $D$  denotes the toroidal boundary divisor.

**2.7. Hecke operators at deeper Iwahori level.** Note that in the previous sections we worked at Iwahori level. As the  $\mathrm{GL}_2$   $p$ -adic Eisenstein series we will work with will have deeper level at  $p$ , we need to understand the action of the  $U_{\mathfrak{p}_1}$ -operator on the cohomology of the Shimura variety

$$X_0(\mathfrak{p}_1^n) = X_{\Gamma_0(\mathfrak{p}_1^n)}, \quad \text{where } \Gamma_0(\mathfrak{p}_1^n) = K^{(\mathfrak{p}_1)} K_{\mathfrak{p}_1, n} \quad n \geq 1.$$

Since the geometry of integral models of  $X_0(\mathfrak{p}_1^n)$  for  $n > 1$  is complicated, we shall only consider this variety after inverting  $p$ ; on the cohomology of this variety we shall define  $U_{\mathfrak{p}_1}^{\mathrm{naive}}$  in weight  $(k_1, k_2, w)$  as the double-coset operator  $[\Gamma_0(\mathfrak{p}_1^n) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(\mathfrak{p}_1^n)]$ , and  $U_{\mathfrak{p}_1} = p^{-\inf\left\{\frac{w+k_1}{2}, \frac{w+(2-k_1)}{2}\right\}} U_{\mathfrak{p}_1}^{\mathrm{naive}}$ .

**Proposition 2.7.1.** *On the cohomology of the generic fibre  $X_0(\mathfrak{p}_1)_{\mathbb{Q}_p}$ , the action of the operator  $U_{\mathfrak{p}_1}$  just defined coincides with the definition in the previous sections. Moreover, for general  $n \geq 2$ , the action of  $(U_{\mathfrak{p}_1})^{n-1}$  on the cohomology of  $X_0(\mathfrak{p}_1^n)_{\mathbb{Q}_p}$  factors through the image via  $(\pi_n)^*$  of the cohomology of  $X_0(\mathfrak{p}_1)_{\mathbb{Q}_p}$ .  $\square$*

One has a similar result for  $U_{\mathfrak{p}_1, n}^t$  using the pushforward  $(\pi_n)_*$ . Let  $E/\mathbb{Q}_p$  be a finite extension. We now consider a cuspidal automorphic representation  $\Pi$  of  $\mathrm{GL}_2(\mathbb{A}_F)$  of tame level  $K^{(\mathfrak{p}_1)}$  and ordinary at  $\mathfrak{p}_1$ , so that (for large enough  $E$ )  $\Pi_{\mathfrak{f}} \otimes |\cdot|^{-w/2}$  contributes to the degree one cohomology of the sheaf  $\underline{\omega}^{(2-k_1, k_2, w)}(-D)$ , i.e.

$$H^1(X_0(\mathfrak{p}_1)_E, \underline{\omega}^{(2-k_1, k_2, w)}(-D))[\Pi_{\mathfrak{f}}] \neq 0.$$

Similarly as in [LPSZ21, Definition 5.8], we have the following.

**Definition 2.7.2.** Let  $\eta = \eta_1 \in H^1(X_0(\mathfrak{p}_1)_E, \underline{\omega}^{(2-k_1, k_2, w)}(-D))[\Pi_{\mathfrak{f}}]$  and let  $n \geq 1$ . We let

$$\eta_n \in H^1(X_0(\mathfrak{p}_1^n)_E, \underline{\omega}^{(2-k_1, k_2, w)}(-D))[\Pi_{\mathfrak{f}}]$$

be the unique element such that the pairing

$$\langle -, \eta_n \rangle : H^1(X_0(\mathfrak{p}_1^n)_E, \underline{\omega}^{(k_1, 2-k_2, -w)})[\Pi_{\mathfrak{f}}^\vee]$$

vanishes on the  $U_{\mathfrak{p}_1} = 0$  eigenspace and it agrees with  $\langle -, \eta_1 \rangle$  on the image of  $H^1(X_0(\mathfrak{p}_1)_{\mathbb{Q}_p}, \underline{\omega}^{(k_1, 2-k_2, -w)})[\Pi_{\mathfrak{f}}^\vee]$ , i.e. the following diagram commutes:

$$(2.12) \quad \begin{array}{ccc} H^1(X_0(\mathfrak{p}_1^n)_E, \underline{\omega}^{(k_1, 2-k_2, -w)})[\Pi_{\mathfrak{f}}^\vee] & \xrightarrow{\langle -, \eta_n \rangle} & \\ \pi_n^* \uparrow & \searrow & \\ H^1(X_0(\mathfrak{p}_1)_E, \underline{\omega}^{(k_1, 2-k_2, -w)})[\Pi_{\mathfrak{f}}^\vee] & \xrightarrow{\langle -, \eta_1 \rangle} & E. \end{array}$$

It follows from the definition that the  $\eta_n$ 's are compatible under the trace map as follows:

$$\eta_n = p^{-1} \cdot \sum_{g \in \Gamma_0(\mathfrak{p}_1^n) \backslash \Gamma_0(\mathfrak{p}_1^{n+1})} g \cdot \eta_{m+1}$$

**Proposition 2.7.3.** *If  $\eta_1$  lies in the  $U_{\mathfrak{p}_1}^t = \alpha$  eigenspace, then for any  $n \geq 1$ ,  $\eta_n$  is an eigenvector for  $U_{\mathfrak{p}_1}^t$  at level  $\Gamma_0(\mathfrak{p}_1^n)$  with the same eigenvalue.  $\square$*

Via the ‘‘canonical subgroup’’ map, we can identify the rigid-analytic generic fibre  $\mathfrak{X}_0(\mathfrak{p}_1)_E^{1-m}$  with an open subspace of the analytification of  $X_0(\mathfrak{p}_1)_E^n$ , for any  $n \geq 1$ . This gives a map

$$H^1(X_0(\mathfrak{p}_1)_E^n, \underline{\omega}^{(k_1, 2-k_2, -w)}) \rightarrow H^1(\mathfrak{X}_0(\mathfrak{p}_1)_E^{1-m}, \underline{\omega}^{(k_1, 2-k_2, -w)}).$$

**Proposition 2.7.4.** *If  $\eta_1$  lies in the  $U_{\mathfrak{p}_1}^t = \alpha$  eigenspace, and  $\alpha$  is a  $p$ -adic unit, then the map*

$$\begin{aligned} H^1(\mathfrak{X}_0(\mathfrak{p}_1)_E^{1-m}, \underline{\omega}^{(k_1, 2-k_2, -w)}) &\xrightarrow{e(U_{\mathfrak{p}_1})} e(U_{\mathfrak{p}_1})H^1(\mathfrak{X}_0(\mathfrak{p}_1)_E^{1-m}, \underline{\omega}^{(k_1, 2-k_2, -w)}) \\ &\cong e(U_{\mathfrak{p}_1})H^1(X_0(\mathfrak{p}_1)_E, \underline{\omega}^{(k_1, 2-k_2, -w)}) \\ &\xrightarrow{\langle -, \eta_1 \rangle} E \end{aligned}$$

coincides with  $\langle -, \eta_n \rangle$  on the image of  $H^1(X_0(\mathfrak{p}_1)_E^n, \underline{\omega}^{(k_1, 2-k_2, -w)})$ , for any  $n \geq 1$ .

*Proof.* This follows from the fact that  $U_{\mathfrak{p}_1}^{n-1}$  factors through  $\pi_n^*$ .  $\square$

### 3. IGUSA TOWERS AND $p$ -ADIC FAMILIES

**3.1. Igusa tower sheaves.** We now show that the automorphic line bundles can be interpolated into families, with  $k_1$  varying, over the  $\mathfrak{p}_1$ -multiplicative locus. We suppose here that  $R$  is the ring of integers of a finite extension of  $\mathbb{Q}_p$ .

**Theorem 3.1.1.** *Let  $(k_1, k_2, w) \in \mathbb{Z}^3$  be given with  $k_1 = k_2 = w \pmod{2}$ ; and let  $\Lambda = R[[\mathbb{Z}_p^\times]]$  with its canonical character  $\kappa : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$ . Then there exists an invertible sheaf of  $\Lambda \otimes \mathcal{O}_{\mathfrak{X}^{1-m}(\mathfrak{p}_1)}$ -modules  $\Omega^{(k_1+2\kappa, k_2, w)}$ , whose specialisation at the character  $x \mapsto x^n$  of  $\Lambda$ , for any  $n \in \mathbb{Z}$ , is  $\underline{\omega}^{(k_1+2n, k_2, w)}|_{\mathfrak{X}^{1-m}(\mathfrak{p}_1)}$ .*

We first work over the moduli space  $\mathfrak{M}^m$ , the formal completion of the multiplicative locus of  $\mathcal{M}$ . Similarly as in [Gro21], we can consider the  $\mathbb{Z}_p^\times$ -torsor

$$\pi_1 : \mathfrak{Jg}(\mathfrak{p}_1^\infty) = \text{Isom}(\mu_{p^\infty}, T_{\mathfrak{p}_1}(\mathcal{A})^\circ) \rightarrow \mathfrak{M}^m,$$

We then define the sheaf

$$\Omega_1^\kappa := (\pi_{1,*} \mathcal{O}_{\mathfrak{Jg}(\mathfrak{p}_1^\infty)} \hat{\otimes} \Lambda)^{\mathbb{Z}_p^\times},$$

where  $\mathbb{Z}_p^\times$  acts on  $\Lambda$  via the universal character  $\kappa : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$ , and on  $\pi_{1,*} \mathcal{O}_{\mathfrak{Jg}(\mathfrak{p}_1^\infty)}$  via the action on  $\mathfrak{Jg}(\mathfrak{p}_1^\infty)$ . As in [Gro21, Lemma 4.2.1], for any  $k \in \mathbb{Z}$  the specialisation of this sheaf at the character  $x \mapsto x^k$  of  $\Lambda$  is given by

$$\Omega_1^\kappa \otimes_k R \simeq \omega_{\mathcal{A}, 1}^k.$$

We also need to interpolate the powers of  $\wedge^2 \mathcal{H}_{\mathcal{A}, 1}^1$ . Similarly as in [Gro21, Definition 4.2.2] (where instead the Igusa varieties at all primes above  $p$  are considered), we can also introduce the  $\mathbb{Z}_p^\times$ -torsor  $\pi_1' : \mathfrak{Jg}^\vee(\mathfrak{p}_1^\infty) = \text{Isom}(T_{\mathfrak{p}_1}(\mathcal{A}^\vee)^{\text{ét}}, \mathbb{Z}_p) \rightarrow \mathfrak{M}^m$  and consider

$$(\wedge^2 \mathcal{H}_{\mathbb{A}, 1}^1)^\kappa := ((\pi_1 \times \pi_1')_* \mathcal{O}_{\mathfrak{Jg}(\mathfrak{p}_1^\infty) \times \mathfrak{Jg}^\vee(\mathfrak{p}_1^\infty)} \hat{\otimes} \Lambda)^\mathbb{T},$$

where  $\mathbb{T}$  denotes the maximal torus of  $\text{GL}_2(\mathbb{Z}_p)$ , acting via its natural action on each component of  $(\pi_1 \times \pi_1')_* \mathcal{O}_{\mathfrak{Jg}(\mathfrak{p}_1^\infty) \times \mathfrak{Jg}^\vee(\mathfrak{p}_1^\infty)}$ , and on  $\Lambda$  via the composite of the determinant  $\mathbb{T} \rightarrow \mathbb{Z}_p^\times$  and the universal character. One can show (interpreting  $\wedge^2 \mathcal{H}_{\mathcal{A}, 1}^1$  in terms of a torsor as in [Gro21, §2.3.1]) that the specialisation of the above sheaf at the character  $x \mapsto x^k$  of  $\Lambda$  is given by

$$(\wedge^2 \mathcal{H}_{\mathbb{A}, 1}^1)^\kappa \simeq (\wedge^2 \mathcal{H}_{\mathcal{A}, 1}^1)^k,$$

justifying the notation. We now define

$$(3.1) \quad \Omega^{(2\kappa, 0, 0)} := \Omega_1^{2\kappa} \otimes (\wedge^2 \mathcal{H}_{\mathcal{A}, 1}^1)^{-\kappa},$$

which is a sheaf of  $\Lambda \otimes \mathcal{O}_{\mathfrak{M}^m}$ -modules; by construction we have

$$\Omega^{(2\kappa, 0, 0)} \otimes_{\Lambda, n} R = \underline{\omega}^{(2n, 0, 0)}|_{\mathfrak{M}^m}$$

as sheaves on  $\mathfrak{M}^m$ .

We have an action of  $x \in \mathcal{O}_{F,+}^\times$  on  $\Omega^{(2\kappa, 0)}$  via the tautological isomorphism (since the construction of  $\Omega^{(2\kappa, 0)}$  does not use the polarisation)

$$x^* \Omega^{(2\kappa, 0, 0)} \simeq \Omega^{(2\kappa, 0, 0)}.$$

Note that the action factors through the quotient of  $\mathcal{O}_{F,+}^\times$  by  $(K \cap \mathcal{O}_F^\times)^2$ . Since  $\epsilon^*(A, \iota, \lambda, \alpha_K, \varphi, \psi) = (A, \iota, \epsilon^2 \lambda, \alpha_K, \epsilon \varphi, \epsilon^2 \psi)$  and, by definition, global sections of  $\Omega^{(2\kappa, 0, 0)}$  satisfy  $f((A, \iota, \lambda, \alpha_K, x \cdot \varphi, x^2 \cdot \psi)) = f((A, \iota, \lambda, \alpha_K, \varphi, \psi))$  for any  $x \in \mathbb{Z}_p^\times$ , embedding  $\mathcal{O}_F^\times$  in  $\mathbb{Z}_p^\times$  via the embedding induced by the prime  $\mathfrak{p}_1$ , we have

$$f(\epsilon^2 \cdot (A, \iota, \lambda, \alpha_K, \varphi, \psi)) = f\left(\left(\epsilon^{-1}\right)^*(A, \iota, \lambda, \alpha_K, \epsilon^{-1} \varphi, \epsilon^{-2} \psi)\right) = f((A, \iota, \lambda, \alpha_K, \varphi, \psi))$$

This definition allows to descend the sheaf  $\Omega^{(2\kappa, 0, 0)}$  to a sheaf of  $\Lambda \otimes \mathcal{O}_{\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)}$ -modules, compatibly with the descent of the classical sheaves, so we have isomorphisms

$$(3.2) \quad \Omega^{(2\kappa, 0, 0)} \otimes_{\Lambda, n} R \simeq \underline{\omega}^{(2n, 0, 0)}|_{\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)},$$

for any  $n \in \mathbb{Z}$ .

Finally we consider  $k_1 \equiv k_2 \equiv w \pmod{2}$  and consider the sheaf  $\underline{\omega}^{(k_1, k_2, w)} = \omega_{\mathcal{A}, 1}^{k_1} \otimes (\wedge^2 \mathcal{H}_{\mathcal{A}, 1}^1)^{(w-k_1)/2} \otimes \omega_{\mathcal{A}, 2}^{k_2} \otimes (\wedge^2 \mathcal{H}_{\mathcal{A}, 2}^1)^{(w-k_2)/2}$ , let

$$(3.3) \quad \Omega^{(k_1+2\kappa, k_2, w)} := \Omega^{(2\kappa, 0, 0)} \otimes_{\mathcal{O}_{\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)}} \underline{\omega}^{(k_1, k_2, w)}.$$

This way the isomorphism (3.2) induces an isomorphism  $\Omega^{(k_1+2\kappa, k_2, w)} \otimes_{\Lambda, a} R \simeq (\underline{\omega}^{(k_1+2a, k_2, w)})|_{\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)}$  for all  $a \in \mathbb{Z}$ .

**Remark 3.1.2.** Note that if  $k'_1 = k_1 + 2t$  for some integer  $t$ , then  $\Omega^{(k'_1+2\kappa, k_2, w)}$  is isomorphic to the base-change of  $\Omega^{(k_1+2\kappa, k_2, w)}$  via the map  $[x] \mapsto x^t[x]$  on  $\Lambda$ .  $\diamond$

**3.2.  $U_{\mathfrak{p}_1}$ -action on the Igusa tower and classicality isomorphism varying the weight.** We now define an Hecke operator acting on the cohomology of the sheaf  $\Omega^{(k_1+2\kappa, k_2, w)}$ , which will be compatible with the normalised operator  $U_{\mathfrak{p}_1}$  defined above on the classical specialisations.

We consider the pullback of the formal completion of  $C(\mathfrak{p}_1)^{m, m}$  to the Igusa tower, i.e. let  $\mathfrak{C}_n$  be the formal scheme defined by the following diagram

$$(3.4) \quad \begin{array}{ccccc} & & \mathfrak{C}_n & & \\ & \swarrow & \downarrow & \searrow & \\ \mathfrak{I}g(\mathfrak{p}_1^n) & & \mathfrak{C}(\mathfrak{p}_1)^{m, m} & & \mathfrak{I}g(\mathfrak{p}_1^n) \\ \downarrow \pi_{1, n} & \swarrow p_1 & & \searrow p_2 & \downarrow \pi_{1, n} \\ \mathfrak{M}^m & & & & \mathfrak{M}^m \end{array}$$

and let  $\mathfrak{C}$  be the limit of the  $\mathfrak{C}_n$ . We obtain maps

$$\begin{array}{ccc} & \mathfrak{C} & \\ & \swarrow p_1 & \searrow p_2 \\ \mathfrak{I}g(\mathfrak{p}_1^\infty) & & \mathfrak{I}g(\mathfrak{p}_1^\infty) \end{array}$$

which we still denote by  $p_1, p_2$  by abuse of notation. By the property of the maps over  $\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)$ , we deduce that  $p_1$  is ramified of degree  $p$  and  $p_2$  is an isomorphism. Let us consider the universal étale isogeny  $\eta : p_1^* \mathcal{A} \rightarrow p_2^* \mathcal{A}$ . It induces a morphism between the Igusa towers  $\mathfrak{I}g(\mathfrak{p}_1^\infty)$  over  $\mathfrak{M}^m$  given by  $(A, \iota, \lambda, \alpha_K, \varphi : \mu_{p^\infty} \simeq T_{\mathfrak{p}_1}(A)^\circ) \mapsto (A/L, \iota', \lambda', \alpha'_K, \varphi' : \mathbb{Z}_p \simeq T_{\mathfrak{p}_1}(A/L)^\circ)$ , where  $\varphi' : \mu_{p^\infty} \xrightarrow{\varphi} T_{\mathfrak{p}_1}(A)^\circ \simeq T_{\mathfrak{p}_1}(A/L)^\circ$  and the last isomorphism is induced by the isogeny  $\lambda$ . Note that the map changes the polarisation type (and hence the connected components), but one can patch everything together arguing as in [Gro21] (see the discussion after Lemma 4.2.6).

We also have an analogous picture for the Igusa tower  $\mathfrak{I}g^\vee(\mathfrak{p}_1^\infty)$ , where the induced map is obtained as follows: the dual isogeny  $\lambda^\vee : (A/L)^\vee \rightarrow A^\vee$  induces an isomorphism  $T_{\mathfrak{p}_1}((A/L)^\vee)^{\text{ét}} \rightarrow T_{\mathfrak{p}_1}(A^\vee)^{\text{ét}}$ . Composing this with the rigidification  $\varphi : T_{\mathfrak{p}}(A^\vee)^{\text{ét}} \simeq \mathbb{Z}_p$ , we obtain an isomorphism  $\varphi \circ \lambda^\vee : T_{\mathfrak{p}}((A/L)^\vee)^{\text{ét}} \rightarrow \mathbb{Z}_p$ .

Hence the pullback by  $\eta$  yields, applying the cartesianity of (3.4) and using the fact the maps are well defined up to units and  $\mathbb{Z}_p^\times$ -invariant, a map

$$\eta_{\mathfrak{I}g} : p_2^* \Omega \rightarrow p_1^* \Omega.$$



**Definition 3.2.1.** We define the operator  $U_{\mathfrak{p}_1, (k_1+2\kappa, k_2, w)}$  acting on the cohomology of the sheaf  $\Omega^{(k_1+2\kappa, k_2, w)}$  as the product

$$\frac{1}{p} \text{Tr}_{p_1} \otimes (\eta_{\mathfrak{I}\mathfrak{g}} \otimes \eta_{(k_1, k_2, w)}) \in \text{End R}\Gamma \left( \mathfrak{X}_0^{1-m}(\mathfrak{p}_1), \Omega^{(k_1+2\kappa, k_2, w)} \right),$$

where  $\eta_{(k_1, k_2, w)} := p^{-(w-k_1)/2} \eta^* : p_2^*(\underline{\omega}^{(k_1, k_2, w)}) \rightarrow p_1^*(\underline{\omega}^{(k_1, k_2, w)})$  is the normalised map induced by the universal isogeny of the correspondence  $\mathfrak{C}(\mathfrak{p}_1)^{\text{m}, \text{m}}$  on the sheaf of weight  $(k_1, k_2, w)$ . Note that  $\eta_{(k_1, k_2, w)}$  once tensored with  $\frac{1}{p} \text{Tr}_{p_1}$  is equal to  $(\tilde{U}_{\mathfrak{p}_1, (k_1, k_2, w)})|_{\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)}$ , the Hecke operator defined above for the sheaf  $\underline{\omega}^{(k_1, k_2, w)}$ .

**Proposition 3.2.2.** *Let  $a \in \mathbb{Z}_{\geq 0}$ . The operator  $U_{\mathfrak{p}_1, (k_1+2\kappa, k_2, w)}$  specialised in weight  $a$  is equal to the operator  $(U_{\mathfrak{p}_1, (k_1+2a, k_2, w)})|_{\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)}$  acting on the cohomology of  $\underline{\omega}^{(k_1+2a, k_2, w)}$  under the isomorphism induced by (3.2).*

*Proof.* We proceed similarly as in the proof of [Gro21, Lemma 4.2.8]. Let  $m \leq n$  and recall the torsor over the reduction mod  $\varpi^m$  of  $\mathcal{M}$  defined by  $\mathcal{T}_{\mathfrak{p}_1} = \text{Isom}_{\mathcal{M}_m}(\mathcal{O}_{\mathcal{M}_m}, (e^* \Omega_{\mathcal{A}}^1)_{\mathfrak{p}_1})$ . We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{I}\mathfrak{g}(\mathfrak{p}_1^n)_m & \xrightarrow{\eta} & \mathfrak{I}\mathfrak{g}(\mathfrak{p}_1^n)_m & (A, \iota, \lambda, \alpha_K, \varphi_n) & \xrightarrow{\eta} & (A/L, \iota', \lambda', \alpha'_K, \eta \circ \varphi_n) \\ \downarrow \text{HT}_{m,n} & & \downarrow \text{HT}_{m,n} & \downarrow \text{HT}_{m,n} & & \downarrow \text{HT}_{m,n} \\ \mathcal{T}_{\mathfrak{p}_1 | \mathcal{M}_m^{\text{m}}} & \xrightarrow{(\eta^*)^{-1}} & \mathcal{T}_{\mathfrak{p}_1 | \mathcal{M}_m^{\text{m}}} & (A, \iota, \lambda, \alpha_K, (\varphi_n^*)^{-1}) & \xrightarrow{(\eta^*)^{-1}} & (A/L, \iota', \lambda', \alpha'_K, (\eta^*)^{-1} \circ (\varphi_n^*)^{-1}) \end{array}$$

where  $(\eta^*)^{-1}$  is the inverse of the isomorphism induced by the pullback  $\Omega_{A/L}^1 \rightarrow \Omega_{\mathcal{A}}^1$ .

Similarly, recall  $\mathcal{T}'_{\mathfrak{p}_1} = \text{Isom}_{\mathcal{M}_m}(\mathcal{O}_{\mathcal{M}_m}, (e^* \Omega_{\mathcal{A}^\vee, 1}^1)^\vee)$ . We have the commutative diagram

$$\begin{array}{ccc} \mathfrak{I}\mathfrak{g}^\vee(\mathfrak{p}_1^n)_m & \xrightarrow{\eta} & \mathfrak{I}\mathfrak{g}^\vee(\mathfrak{p}_1^n)_m & (A, \iota, \lambda, \alpha_K, \varphi_n) & \xrightarrow{\eta^\vee} & (A/L, \iota', \lambda', \alpha'_K, \varphi_n \circ \eta^\vee) \\ \downarrow \text{HT}_{m,n} & & \downarrow \text{HT}_{m,n} & \downarrow \text{HT}_{m,n} & & \downarrow \text{HT}_{m,n} \\ \mathcal{T}'_{\mathfrak{p}_1 | \mathcal{M}_m^{\text{m}}} & \xrightarrow{\eta^\vee} & \mathcal{T}'_{\mathfrak{p}_1 | \mathcal{M}_m^{\text{m}}} & (A, \iota, \lambda, \alpha_K, \varphi_{n,D}^*) & \xrightarrow{(\eta_D^\vee)^*} & (A/L, \iota', \lambda', \alpha'_K, \varphi_{n,D}^* \circ (\eta_D^\vee)^*) \end{array}$$

where  $\varphi_{n,D}^*$  denotes the pre-composition with the map  $e^* \Omega_{\mathcal{A}^\vee, 1}^1 \rightarrow e^* \Omega_{\mu_{p^n}}^1$  induced by the pullback of the Cartier dual of  $\varphi_n$ , denoted by  $\varphi_{n,D} : A^\vee[\mathfrak{p}_1^n]^\circ \rightarrow \mu_{p^n}$ . Similarly,  $\eta^\vee$  induces an isomorphism  $\eta^\vee : (A/L)^\vee[\mathfrak{p}_1^n]^{\text{ét}} \rightarrow A^\vee[\mathfrak{p}_1^n]^{\text{ét}}$  and we denote by  $\eta_D^\vee : A^\vee[\mathfrak{p}_1^n]^\circ \rightarrow (A/L)^\vee[\mathfrak{p}_1^n]^\circ$  its Cartier dual. The induced pullback map  $(\eta_D^\vee)^*$  coincide with the pullback by the isogeny  $\eta^D$  making the following diagram commutative

$$(3.5) \quad \begin{array}{ccccc} A/L \otimes_{\mathcal{O}_F} \mathfrak{c} & \longrightarrow & A \otimes_{\mathcal{O}_F} \mathfrak{c}' & \xrightarrow{\eta} & A/L \otimes_{\mathcal{O}_F} \mathfrak{c} \\ \downarrow \lambda' & & \downarrow \lambda & & \downarrow \lambda' \\ (A/L)^\vee & \xrightarrow{\eta^\vee} & A^\vee & \xrightarrow{\eta^D} & (A/L)^\vee, \end{array}$$

and where the composition of both the top and bottom horizontal maps is the isogeny of kernel  $A/L[\mathfrak{p}_1]$  and  $(A/L)^\vee[\mathfrak{p}_1]$  respectively (here we are crucially using the fact that the polarisation types  $\mathfrak{c}, \mathfrak{c}'$  can be chosen to be coprime to  $p$ , see for example [Gro21, discussion after (2.2)]).

Passing to the limits, we obtain commutative diagrams

$$(3.6) \quad \begin{array}{ccc} \mathfrak{I}\mathfrak{g}(\mathfrak{p}_1^\infty) & \xrightarrow{\eta} & \mathfrak{I}\mathfrak{g}(\mathfrak{p}_1^\infty) & \mathfrak{I}\mathfrak{g}^\vee(\mathfrak{p}_1^\infty) & \xrightarrow{\eta^\vee} & \mathfrak{I}\mathfrak{g}^\vee(\mathfrak{p}_1^\infty) \\ \downarrow \text{HT} & & \downarrow \text{HT} & \downarrow \text{HT} & & \downarrow \text{HT} \\ \mathcal{T}_{|\mathfrak{M}^\infty} & \xrightarrow{(\eta^*)^{-1}} & \mathcal{T}_{|\mathfrak{M}^\infty} & \mathcal{T}'_{|\mathfrak{M}^\infty} & \xrightarrow{(\eta^D)^*} & \mathcal{T}'_{|\mathfrak{M}^\infty}. \end{array}$$

The pullback map  $\eta^*$  decomposes as

$$\begin{array}{ccc} \wedge^2 \mathcal{H}_{\text{dR}}^1(\mathcal{A}/L)_1 & \xrightarrow{\eta^*} & \wedge^2 \mathcal{H}_{\text{dR}}^1(\mathcal{A})_1 \\ \downarrow \simeq & & \downarrow \simeq \\ (e^* \Omega_{\mathcal{A}/L}^1)_1 \otimes \text{Lie}((\mathcal{A}/L)^\vee)_1 & \xrightarrow{\eta^* \otimes \text{Lie}(\eta^\vee)} & (e^* \Omega_{\mathcal{A}}^1)_1 \otimes \text{Lie}(\mathcal{A}^\vee)_1 \end{array}$$

and, under the isomorphism [Kat78, (1.0.13)], we can write  $\mathrm{Lie}(\eta^\vee) = (\eta^\vee)^* : (e^* \Omega_{(\mathcal{A}/L)^\vee}^1)^\vee \simeq \mathrm{Lie}((\mathcal{A}/L)^\vee) \rightarrow (e^* \Omega_{\mathcal{A}^\vee}^1)^\vee \simeq \mathrm{Lie}(\mathcal{A}^\vee)$ , where here  $(\eta^\vee)^*$  denotes the map obtained composing with  $(\eta^\vee)^* : e^* \Omega_{\mathcal{A}^\vee}^1 \rightarrow e^* \Omega_{(\mathcal{A}/L)^\vee}^1$ . So, since the composition  $(\eta^\vee)^* \circ (\eta^D)^*$  is given by multiplication by  $p$  on  $(e^* \Omega_{\mathcal{A}^\vee/L}^1)_1$ , we obtain

$$\eta_{\mathfrak{g}} = \eta^* \otimes [\eta^* \otimes ((\eta^D)^*)^{-1}] = p^a \eta^* : \omega_{\mathcal{A}/L,1}^{2a} \otimes (\wedge^2 \mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}/L)_1)^{-a} \rightarrow \omega_{\mathcal{A},1}^{2a} \otimes (\wedge^2 \mathcal{H}_{\mathrm{dR}}^1(\mathcal{A})_1)^{-a}$$

Tensoring with  $\eta_{(k_1, k_2, w)}$  on the component  $\underline{\omega}^{(k_1, k_2, w)}$  and composing with the normalised trace map, we obtain

$$U_{\mathfrak{p}_1, (k_1+2\kappa, k_2, w)} = p^{-\frac{w-k_1}{2}+a-1} \tilde{U}_{\mathfrak{p}_1}^{\mathrm{naive}}$$

which is the claimed result.  $\square$

With this proposition in hand, we can write  $U_{\mathfrak{p}_1}$  for this operator (dropping the additional subscripts) without risk of confusion. We can now construct  $\Lambda$ -module complexes interpolating the complexes  $M_{\mathfrak{m}, \emptyset}^\bullet$  above:

$$\begin{aligned} M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa, k_2, w) &:= e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(\mathfrak{X}_0^{1-m}(\mathfrak{p}_1), \Omega^{(k_1+2\kappa, k_2, w)}) \\ M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa, k_2, w; -D) &:= e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(\mathfrak{X}_0^{1-m}(\mathfrak{p}_1), \Omega^{(k_1+2\kappa, k_2, w)}(-D)). \end{aligned}$$

From the interpolation property of the sheaves  $\Omega^{(k_1+2\kappa, k_2, w)}$ , for any  $a \in \mathbb{Z}$  we have an isomorphism

$$M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa, k_2, w) \otimes_{\Lambda, a}^{\mathbb{L}} R \cong M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2a, k_2, w)$$

and likewise with  $-D$ . Combining the above results yields the following.

**Theorem 3.2.3.** *The complex  $M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa, k_2, w)$  is a perfect complex of  $\Lambda$ -modules; and for any  $a \in \mathbb{Z}$  with  $k_1 + 2a \geq 2$ , we have*

$$M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa, k_2, w) \otimes_{\Lambda, a}^{\mathbb{L}} R \simeq e(U_{\mathfrak{p}_1}) \mathrm{R}\Gamma(X_0(\mathfrak{p}_1), \underline{\tilde{\omega}}^{(k_1+2a, k_2, w)});$$

and similarly for the cuspidal version.

*Proof.* To get the desired isomorphism one applies firstly the above Proposition and the isomorphism (3.2). Then note that the sheaves  $\underline{\tilde{\omega}}^{(k_1+2a, k_2, w)}$  and  $\underline{\omega}^{(k_1+2a, k_2, w)}$  are isomorphic over  $\mathfrak{X}_0^{1-m}(\mathfrak{p}_1)$ . Hence the result follows from Corollary 2.3.7.  $\square$

Since the sheaves  $\underline{\tilde{\omega}}^{(k_1+2a, k_2, w)}$  and  $\underline{\omega}^{(k_1+2a, k_2, w)}$  are isomorphic after inverting  $p$ , this implies theorem A.

**Remark 3.2.4.** One can show that the image of  $X^{1-m}$  in the minimal compactification is covered by two affine subsets. It follows that  $M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa, k_2, w; -D)$  is represented by a perfect complex concentrated in degrees  $[0, 1]$ . We expect that if  $k_2 \leq 0$  this complex is represented by a single projective  $\Lambda$ -module in degree 1 only (analogous to the results proved in [LPSZ21] in the  $\mathrm{GSp}_4$  case); but we do not need this here.  $\diamond$

**Remark 3.2.5.** Even if we do not give the details here, note that one can similarly construct *compactly supported* complexes interpolating the  $U_{\mathfrak{p}_1}^t$ -ordinary part of classical cohomology when  $k_1 + 2a \leq 0$ . More precisely, one needs to define a normalised  $U_{\mathfrak{p}_1}^t$  operator acting on the cohomology of the sheaf  $\Omega^{(k_1+2\kappa, k_2, w)}$  replacing  $\eta$  with  $\eta^\vee$  in Definition 3.2.1. One can then define the complexes

$$\begin{aligned} M_{\mathfrak{m}, c}^\bullet(k_1 + 2\kappa, k_2, w) &:= e(U_{\mathfrak{p}_1}^t) \mathrm{R}\Gamma_c(\mathfrak{X}_0^{1-m}(\mathfrak{p}_1), \Omega^{(k_1+2\kappa, k_2, w)}) \\ M_{\mathfrak{m}, c}^\bullet(k_1 + 2\kappa, k_2, w; -D) &:= e(U_{\mathfrak{p}_1}^t) \mathrm{R}\Gamma_c(\mathfrak{X}_0^{1-m}(\mathfrak{p}_1), \Omega^{(k_1+2\kappa, k_2, w)}(-D)). \end{aligned}$$

One then expects an analogue of Proposition 3.2.2 for  $U_{\mathfrak{p}_1}^t$  and can apply the classically isomorphism of (2.11) to prove that for  $k_1 + 2a \leq 0$ , there is an isomorphism

$$M_{\mathfrak{m}, c}^\bullet(k_1 + 2\kappa, k_2, w) \otimes_{\Lambda, a}^{\mathbb{L}} R \simeq e(U_{\mathfrak{p}_1}^t) \mathrm{R}\Gamma(X_0(\mathfrak{p}_1), \underline{\tilde{\omega}}^{(k_1+2a, k_2, w)}),$$

as well as for the cuspidal version.  $\diamond$

**3.3. Families of eigenclasses.** As above we let  $E$  be a finite extension of  $\mathbb{Q}_p$ .

**Notation 3.3.1.** Let  $\mathbb{T}$  denote the product of the prime-to- $\mathfrak{p}_1$  Hecke algebra  $\mathbb{T}^{(\mathfrak{N}_{\mathfrak{p}_1})}$  (with  $E$ -coefficients) and the polynomial ring in a formal variable  $U_{\mathfrak{p}_1}$  and its inverse.

**Definition 3.3.2.** Let  $\mathcal{U} \subset \mathcal{W} = (\mathrm{Spf} \Lambda)_E^{\mathrm{rig}}$  be an open affinoid containing 0; and let  $\kappa_{\mathcal{U}} : \mathbb{Z}_p^\times \rightarrow \mathcal{O}(\mathcal{U})$  be the universal weight. A  $\mathfrak{p}_1$ -adic family of eigensystems over  $\mathcal{U}$  of weight  $(k_1 + 2\kappa_{\mathcal{U}}, k_2, w)$  and level  $\mathfrak{N}$  is a homomorphism  $\mathbb{T} \rightarrow \mathcal{O}(\mathcal{U})$  with the following property:

- For every  $a \in \mathcal{U} \cap \mathbb{Z}$  with  $k_1 + 2a \geq 2$ , the composite homomorphism  $\mathbb{T} \rightarrow \mathcal{O}(\mathcal{U}) \rightarrow E$  given by evaluation at  $a$  is the Hecke eigenvalue system associated to a  $\mathfrak{p}_1$ -stabilisation of  $\Pi[a]$  (or more precisely  $\Pi[a] \otimes \|\cdot\|^{-w/2}$ ), for some cuspidal automorphic representation  $\Pi[a]$  of  $\mathrm{GL}_2/F$  of level  $\mathfrak{N}$  and weight  $(k_1 + 2a, k_2)$  and some embedding of its coefficient field into  $E$ .

We denote such a family by  $\underline{\Pi}$ . We say that  $\underline{\Pi}$  is  $\mathfrak{p}_1$ -ordinary if the Hecke eigenvalues of  $U_{\mathfrak{p}_1}$  and its inverse acting on  $\underline{\Pi}$  are power-bounded.

By standard results due to Hida, we see that for any given  $\Pi$  of weight  $(k_1, k_2)$  and conductor  $\mathfrak{N}$ , with  $k_1, k_2 \geq 2$ , ordinary at  $\mathfrak{p}_1$ , we may find a  $\mathfrak{p}_1$ -ordinary family  $\underline{\Pi}$  over some sufficiently small  $\mathcal{U}$  passing through  $\Pi$  (i.e. with  $\Pi[0] = \Pi$ ), and this is unique up to further shrinking  $\mathcal{U}$ .

**Proposition 3.3.3.** *Let  $\underline{\Pi}$  be a  $\mathfrak{p}_1$ -ordinary family over  $\mathcal{U}$ , and define*

$$M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa_{\mathcal{U}}, 2 - k_2, w) := M_{\mathfrak{m}, \emptyset}^\bullet(k_1 + 2\kappa, 2 - k_2, w) \otimes_{\Lambda} \mathcal{O}(\mathcal{U}).$$

*Then  $H^1$  of this complex contains a free rank 1  $\mathcal{O}(\mathcal{U})$ -direct summand  $H_{\mathfrak{m}, \emptyset}^1(k_1 + 2\kappa_{\mathcal{U}}, 2 - k_2, w)[\underline{\Pi}^\vee]$  whose specialisation at every  $a \in \mathcal{U} \cap \mathbb{Z}$  with  $k_1 + 2a \geq 2$  is the eigenspace in  $e(U_{\mathfrak{p}_1})H^1(X_0(\mathfrak{p}_1), \underline{\omega}^{(k_1+2a, 2-k_2, w)})$  on which  $\mathbb{T}$  acts via the ordinary  $\mathfrak{p}_1$ -stabilisation of  $\Pi[a]^\vee$ .*

*Proof.* Both this result, and the existence of the family  $\underline{\Pi}$  through a given  $\Pi$ , follow from the fact that each automorphic representation of regular weight  $k_1, k_2 \geq 2$  contributes to coherent cohomology with four distinct coefficient sheaves, each of them in a single degree. This, combined with the classicity results for the complexes  $M_{\mathfrak{m}, \emptyset}^\bullet$  (theorem 3.2.3), shows that the cohomology groups of  $M_{\mathfrak{m}, \emptyset}^\bullet$  are locally free over  $\mathcal{O}(\mathcal{U})$  after localising a classical Hecke eigensystem.  $\square$

**Remark 3.3.4.** In more geometric language, for each  $k_2 \geq 2$ , there is a “partial eigenvariety” parametrising  $\mathfrak{p}_1$ -adic families of weight  $(k_1 + 2\kappa, k_2, w)$ ; and the map from the partial eigenvariety to  $\mathcal{W}$  is étale at the point corresponding to  $\Pi$ , so we may write down a section over a small enough neighbourhood of its image. (See forthcoming work of C.-Y. Hsu for a more systematic investigation of partial eigenvarieties for Hilbert modular forms.) Note that this step would fail if  $k_2 = 1$ , since in this case  $\Pi^\vee$  contributes to both  $H^0$  and  $H^1$  of  $\underline{\omega}^{(k_1, 1, w)}$ ; we do not expect partial eigenvarieties for partial weight 1 forms to be étale (or flat) over the weight space.  $\diamond$

#### 4. PUSHFORWARDS

We now consider relations between the higher Hida theory spaces for the group  $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$  developed in the previous section, and spaces of  $p$ -adic automorphic forms for its subgroup  $H = \mathrm{GL}_2$ . To avoid confusion between the two groups, we will use subscripts  $(-)_G$  or  $(-)_H$  as appropriate, so  $X_G$  denotes the Hilbert modular variety which we previously denoted simply by  $X$ .

**4.1. Classical pushforward map.** Let  $H = \mathrm{GL}_2$  and  $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$  as above. We have the canonical inclusion  $H \subset G$  (corresponding to the diagonal embedding  $H \hookrightarrow H \times H$ ) inducing a finite map of  $\mathbb{Z}_p$ -schemes

$$\iota : X_H = X_{H,0}(p) \hookrightarrow X_G = X_{G,0}(\mathfrak{p}_1),$$

where the Shimura varieties are taken to be of Iwahori level at  $p$  and  $\mathfrak{p}_1$  respectively.

Note that the preimage under  $\iota$  of the locus  $\mathfrak{X}_G^{1-m}$  of the modular surface  $G$  (where the subgroup at  $\mathfrak{p}_1$  is multiplicative) is the multiplicative locus  $\mathfrak{X}_H^m$ . In particular, we obtain a finite map

$$\iota : \mathfrak{X}_H^m = \mathfrak{X}_{H,0}(p)^m \hookrightarrow \mathfrak{X}_G^{1-m} = \mathfrak{X}_{G,0}^{1-m}(\mathfrak{p}_1).$$

Moreover, since the pullback of the universal abelian surface  $\mathcal{A}$  is isogenous to  $\mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_F$ , we have isomorphisms for any  $n_1, n_2, a, b \in \mathbb{Z}$

$$\omega_{\mathcal{E}}^{n_1+n_2} \otimes (\wedge^2 \mathcal{H}_{\mathcal{E}}^1)^{a+b} \rightarrow \iota^* \left( \omega_{\mathcal{A},1}^{n_1} \otimes \omega_{\mathcal{A},2}^{n_2} \otimes (\wedge^2 \mathcal{H}_{\mathcal{A},1}^1)^a \otimes (\wedge^2 \mathcal{H}_{\mathcal{A},2}^1)^b \right),$$

where  $\omega_{\mathcal{E}}$  denotes the pullback along the unit sections of the sheaf of differentials of the universal elliptic curve  $\mathcal{E}/X_H$  and one can define similarly the sheaf  $\wedge^2 \mathcal{H}_{\mathcal{E}}^1$ .

**Definition 4.1.1.** For any  $n \in 2\mathbb{Z}$ , we let

$$\underline{\omega}_H^n := \omega_{\mathcal{E}}^n \otimes (\wedge^2 \mathcal{H}_{\mathcal{E}}^1)^{-\frac{n}{2}},$$

so the centre of  $H(\mathbb{A}_f)$  acts on the cohomology of this sheaf via finite-order characters.

**Remark 4.1.2.** Note that we have a canonical trivialisation of  $\wedge^2 \mathcal{H}_{\mathcal{E}}^1$  (since an elliptic curve has a unique principal polarisation). However, this trivialisation induces a twist of the Hecke action: we have an isomorphism of  $H(\mathbb{A})$ -modules

$$H^0(X_H, \omega_{\mathcal{E}}^n \otimes (\wedge^2 \mathcal{H}_{\mathcal{E}}^1)^m) \simeq H^0(X_H, \omega_{\mathcal{E}}^n) \otimes |\det(-)|^{-m}.$$

where  $|\cdot|$  denotes the adèle norm character (mapping a uniformizer at  $p$  to  $1/p$ ). Since we shall only consider modular forms for  $H$  of even weight in this paper, we can use this freedom to make the central character unitary.  $\diamond$

Let  $(k_1, k_2, w)$  be as above, with  $k_1 > k_2 > 0$ , we get a pullback map:

$$\iota^* : H^1 \left( X_G, \underline{\omega}_G^{(2-k_1, k_2, -w)}(-D_G) \right) \longrightarrow H^1 \left( X_H, \underline{\omega}_H^{2-k_1+k_2}(-D_H) \right) \otimes |\det|^w$$

Dually, we find

$$(4.1) \quad \iota_* : H^0 \left( X_H, \underline{\omega}_H^{k_1-k_2} \right) \otimes |\det|^{-w} \longrightarrow H^1 \left( X_G, \underline{\omega}_G^{(k_1, 2-k_2, w)} \right)$$

where  $\underline{\omega}_G^{(k_1, 2-k_2, w)}$  is defined with the local choice of models according to (2.2) (recall that the dualising sheaf for  $X_G$  is  $\underline{\omega}_G^{(2, 2, 0)}(-D_G)$ , where  $D_G$  denoted the cuspidal divisor of  $X_G$ , and similarly for  $H$ ).

Now we note that, since  $\iota^*(D^{1-\acute{e}t}) = D^{\acute{e}t}$ , the étale locus of  $X_H$ , for any integer  $n$ , we have

$$(4.2) \quad \iota^*(\mathcal{O}_{X_G}(-nD^{1-\acute{e}t})) = \mathcal{O}_{X_H}(-nD^{\acute{e}t})$$

Moreover, since  $\mathcal{O}_{X_H}(-D^{\acute{e}t}) \simeq \mathcal{O}_{X_H}(D^m)$ , we obtain from (4.1), a map

$$\iota_* : H^0 \left( X_H, \underline{\omega}_H^{k_1-k_2}(-t_1 - 2)D^m \right) \otimes (\wedge^2 \mathcal{H}_{\mathcal{E}}^1)^w \rightarrow H^1 \left( X_G, \tilde{\omega}_G^{(k_1, 2-k_2, w)} \right),$$

where  $\tilde{\omega}_G^{(k_1, 2-k_2, w)}$  is our choice of integral model given in (2.10).

We choose as integral model for  $\underline{\omega}_H^{k_1-k_2} \otimes (\wedge^2 \mathcal{H}_{\mathcal{E}}^1)^w$  the following

$$(4.3) \quad \tilde{\omega}_H = \tilde{\omega}_H^{k_1-k_2} = \underline{\omega}_H^{k_1-k_2}(-t_1 D^m) \otimes (\wedge^2 \mathcal{H}_{\mathcal{E}}^1)^w = \omega_{\mathcal{E}}^{k_1-k_2} \otimes (\omega_{\mathcal{E}} \otimes \omega_{\mathcal{E}/H}^{-1})^{t_1} \otimes (\wedge^2 \mathcal{H}_{\mathcal{E}}^1)^w.$$

Therefore (4.1) gives a map

$$(4.4) \quad \iota_* : H^0 \left( X_H, \tilde{\omega}_H(-D_H) \right) \rightarrow H^1 \left( X_G, \tilde{\omega}_G(-D_G) \right)$$

**4.2.  $p$ -adic pushforward map.** We now want to define a pushforward map with target the cohomology of the complex  $M_{G, m, \emptyset}^{\bullet}(k_1, 2-k_2, w)$  of section 2.6. Combining (4.2) and (4.4), for every  $n \geq 0$ , we obtain

$$(4.5) \quad \iota_*^n : H^0 \left( X_H, \tilde{\omega}_H(nD^{\acute{e}t}) \right) \rightarrow H^1 \left( X_G, \tilde{\omega}_G(nD^{1-\acute{e}t}) \right).$$

We obtain the following diagram

$$\begin{array}{ccc}
 H^0(X_H, \tilde{\omega}_H(nD^{\acute{e}t})) & \xrightarrow{\quad} & H^0(X_H, \tilde{\omega}_H((n+1)D^{\acute{e}t})) \\
 \downarrow f_n \circ \iota_{p,\star}^n & \searrow g_n & \swarrow g_{n+1} \\
 & \text{colim}_{m \geq 0} H^0(X_H, \tilde{\omega}_H(mD^{\acute{e}t})) & \\
 & \downarrow \iota_{p,\star} & \\
 & \text{colim}_{n \geq 0} H^1(X_G, \tilde{\omega}_G(nD^{1-\acute{e}t})) & \\
 & \swarrow f_{n+1} \circ \iota_{p,\star}^{n+1} & \\
 & & 
 \end{array}$$

where the horizontal map is the natural transition map,  $g_n$  and  $f_n$  are the universal morphisms of the direct limits in the second and third row respectively and, since all these maps are compatible, we denoted by  $\iota_{p,\star}$  the map obtained by the universal property of the direct limit in the second row, which we call the  $p$ -adic pushforward map:

$$(4.6) \quad \iota_{p,\star} : H^0(\mathfrak{X}_H^m, \tilde{\omega}_H) \rightarrow H^1(\mathfrak{X}_G^{1-m}, \tilde{\omega}_G)$$

From the construction it is clear that the following diagram commutes

$$\begin{array}{ccc}
 H^0(X_H, \tilde{\omega}_H) & \xrightarrow{\iota_{p,\star}} & H^1(X_G, \tilde{\omega}_G) \\
 \downarrow & & \downarrow \\
 H^0(\mathfrak{X}_H^m, \tilde{\omega}_H) & \xrightarrow{\iota_{p,\star}} & H^1(\mathfrak{X}_G^{1-m}, \tilde{\omega}_G).
 \end{array}$$

where the vertical arrows are the natural map of the direct limits.

**4.3. Compatibility of the  $p$ -adic pushforward with image of deeper level cohomology.** We will take the image under the above defined pushforward maps of  $p$ -adic modular forms for  $\text{GL}_2/\mathbb{Q}$  coming from classical modular forms of level  $\Gamma_0(p^n)$  for some  $n \geq 1$ . By definition of (4.6), we have a commutative diagram:

$$\begin{array}{ccc}
 H^0(X_{H,0}(p^n)_{\mathbb{Q}_p}, \tilde{\omega}_H) & \longrightarrow & H^1(X_{G,0}(\mathfrak{p}_1^n)_{\mathbb{Q}_p}, \tilde{\omega}_G) \\
 \downarrow & & \downarrow \\
 H^0(\mathfrak{X}_{H,\mathbb{Q}_p}^m, \tilde{\omega}_H) & \xrightarrow{\iota_{p,\star}} & H^1(\mathfrak{X}_{G,\mathbb{Q}_p}^{1-m}, \tilde{\omega}_G)
 \end{array}$$

where the top arrow is the level  $p^n$ -analogue of the classical pushforward (4.4). Combining the above diagram with (2.12), we find, for  $\eta_n$  as in Definition 2.7.2, the following commutative diagram

$$(4.7) \quad \begin{array}{ccc}
 H^0(X_{H,0}(p^n)_{\mathbb{Q}_p}, \tilde{\omega}_H) & \longrightarrow & H^1(X_{G,0}(\mathfrak{p}_1^n)_{\mathbb{Q}_p}, \tilde{\omega}_G) \\
 \downarrow & & \downarrow \\
 H^0(\mathfrak{X}_{H,\mathbb{Q}_p}^m, \tilde{\omega}_H) & \xrightarrow{\iota_{p,\star}} & H^1(\mathfrak{X}_{G,\mathbb{Q}_p}^{1-m}, \tilde{\omega}_G) \xrightarrow{e(U_{\mathfrak{p}_1})} e(U_{\mathfrak{p}_1})H^1((\mathfrak{X}_{G,\mathbb{Q}_p}^{1-m}, \tilde{\omega}_G) \xrightarrow{\cong} e(U_{\mathfrak{p}_1})H^1(X_{G,\mathbb{Q}_p}, \tilde{\omega}_G) \xrightarrow{\langle -, \eta_n \rangle} \mathbb{Q}_p^\times.
 \end{array}$$

**4.4. Functoriality of Igusa towers.** Consider the  $\text{GL}_2$  Igusa tower  $\pi_H : \mathfrak{I}\mathfrak{g}_H(p^\infty) \rightarrow \mathfrak{X}_H^m$  parametrising trivialisations of  $T_p(\mathcal{E})^\circ$ .

**Proposition 4.4.1.** *The natural morphism  $\mathfrak{I}\mathfrak{g}_H(p^n) \xrightarrow{\iota_n} \mathfrak{I}\mathfrak{g}_G(\mathfrak{p}_1^n)$  is finite for every  $n \geq 1$ . Moreover the following diagram is Cartesian:*

$$\begin{array}{ccc}
 \mathfrak{I}\mathfrak{g}_H(p^n) & \xrightarrow{\iota_n} & \mathfrak{I}\mathfrak{g}_G(\mathfrak{p}_1^n) \\
 \downarrow & & \downarrow \\
 \mathfrak{X}_H^m & \xrightarrow{\iota'} & \mathfrak{M}^{1-m}.
 \end{array}$$

*Proof.* Similar to [LPSZ21, Prop. 4.6, Lemma 4.7]. □

Thanks to the above proposition, we obtain

$$(4.8) \quad \iota^*(\pi_{1,*}\mathcal{O}_{\mathfrak{Jg}_G(\mathfrak{p}_1^\infty)}) = \pi_{H,*}\mathcal{O}_{\mathfrak{Jg}_H(p^\infty)}$$

compatibly with the action of  $\mathbb{Z}_p^\times$  on both sides. There is a similar picture for the dual torsors  $\mathfrak{Jg}_H^\vee(p^\infty)$  and  $\mathfrak{Jg}_G^\vee(p^\infty)$ , and we have a canonical isomorphism between  $\mathfrak{Jg}_H(p^\infty)$  and  $\mathfrak{Jg}_H^\vee(p^\infty)$  given by the polarisation of the universal elliptic curve (cf. remark 4.1.2).

Proceeding for  $H$  as we did for  $G$ , we can define a sheaf of  $\Lambda \otimes \mathfrak{X}_H^m$ -modules  $\Omega_H^{2\kappa}$  interpolating the sheaves  $\omega_H^{2a}$  for all  $a \in \mathbb{Z}$ ; and from (4.8) we obtain pushforward maps of  $\Lambda$ -modules

$$(4.9) \quad \iota_{p,*}^\kappa : H^0(\mathfrak{X}_H^m, \Omega_H^{k_1-k_2+2\kappa}) \rightarrow H^1(\mathfrak{X}_G^{1-m}, \Omega_G^{(k_1+2\kappa, 2-k_2, w)}),$$

interpolating the map of (4.6) in weight  $(k_1 + 2a, k_2, w)$  for any  $a \in \mathbb{Z}$  (and likewise for cuspidal sheaves).

## 5. NEARLY SHEAVES

**5.1. Sheaves of nearly holomorphic modular forms.** Let  $H = \mathrm{GL}_2/\mathbb{Q}$ , let  $X_H$  denote the compactification of the modular curve  $Y_H$  of some fixed level, and let  $\mathcal{H}_\mathcal{E}^1$  denote the canonical extension to  $X_H$  of the first relative de Rham cohomology sheaf of the universal elliptic curve  $\mathcal{E}/Y_H$ .

**Definition 5.1.1.** Given an even integer  $k \geq 0$ , we define the following sheaves over  $X_H$ :

- $\mathcal{N}_H^0 := \mathrm{Sym}^k(\mathcal{H}_{\mathrm{dR}}^1(\mathcal{E}/X_H)) \otimes (\wedge^2 \mathcal{H}_\mathcal{E}^1)^{-\frac{k}{2}}$ ;
- $\mathcal{N}_H^1 := \mathrm{Sym}^k(\mathcal{H}_{\mathrm{dR}}^1(\mathcal{E}/X_H)) \otimes \omega_\mathcal{E}^2(-D_H) \otimes (\wedge^2 \mathcal{H}_\mathcal{E}^1)^{-1-\frac{k}{2}}$ .

Note that these sheaves are Serre dual to each other, and the action of the centre of  $\mathrm{GL}_2$  is trivial on both (see Remark 4.1.2 for the reason of the choice of the twist). Since we can identify  $\omega_\mathcal{E}^k$  with the top graded piece in the Hodge filtration of  $\mathcal{H}_H^k$ , and dually  $\omega_\mathcal{E}^{-k}$  is identified with the bottom graded piece, we have maps

$$(5.1) \quad \omega_H^k \hookrightarrow \mathcal{N}_{H,n}^0, \quad \mathcal{N}_{H,n}^1 \twoheadrightarrow \omega_H^{2-k}(-D_H),$$

where for any  $n \in 2\mathbb{Z}$ , we let  $\omega_H^n := \omega_\mathcal{E}^n \otimes (\wedge^2 \mathcal{H}_\mathcal{E}^1)^{-\frac{n}{2}}$ . Note that these maps are transposes of each other with respect to Serre duality.

As described in [Urb14] for example, the map  $\omega_H^k \hookrightarrow \mathcal{N}_H^0$  has a  $C^\infty$  section over  $\mathbb{R}$ , the *Hodge splitting* (which identifies  $H^0(X_H, \mathcal{N}_H^0)$ , with image the space of *nearly-holomorphic modular forms* in the sense of Shimura). If the level group at  $p$  is of the form  $K_0(p^n)$ , it also has a  $p$ -adic section over the multiplicative locus  $\mathfrak{X}_H^m$ , the *unit-root splitting*.

**5.2. Sheaves of (partially) nearly holomorphic Hilbert modular forms.** We want to proceed analogously for the Hilbert modular surface. Recall that over the moduli space  $\mathcal{M}$  we can consider the sheaf  $\mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{M}) = \mathcal{H}_{\mathcal{A},1}^1 \oplus \mathcal{H}_{\mathcal{A},2}^1$ .

Now fix  $(k_1, k_2, w)$  with  $k_i \geq 0$  and  $k_1 = k_2 = w \pmod{2}$ , and set

$$\wedge^2 \mathcal{H}_{0,1} := \left( (\wedge^2 \mathcal{H}_{\mathcal{A},1}^1)^{\frac{w-k_1}{2}} \right) \otimes \left( (\wedge^2 \mathcal{H}_{\mathcal{A},2}^1)^{\frac{w-2+k_2}{2}} \right) \quad \text{and} \quad \wedge^2 \mathcal{H}_{1,0} := \left( (\wedge^2 \mathcal{H}_{\mathcal{A},1}^1)^{\frac{k_1+2-w}{2}} \right) \otimes \left( (\wedge^2 \mathcal{H}_{\mathcal{A}/H_2,2}^1)^{\frac{-k_2-w}{2}} \right).$$

**Definition 5.2.1.** We define the following sheaves over  $X_G$ :

- $\mathcal{N}_G^{0,1} := (\mathrm{Sym}^{k_1} \mathcal{H}_{\mathcal{A},1}) \otimes \omega_{\mathcal{A},2}^{2-k_2} \otimes \wedge^2 \mathcal{H}_{0,1}$ ;
- $\mathcal{N}_G^{1,0} := (\mathrm{Sym}^{k_1} \mathcal{H}_{\mathcal{A},1} \otimes \omega_{\mathcal{A},1}^2) \otimes \omega_{\mathcal{A},2}^{k_2} \wedge^2 \mathcal{H}_{1,0}(-D_G)$ .

(Note that these sheaves do indeed descend to  $X_G$ , since the centre acts as  $\mathrm{Nm}^w$  on  $\mathcal{N}_G^{0,1}$  and as  $\mathrm{Nm}^{-w}$  on  $\mathcal{N}_G^{1,0}$ .)

We call an element in  $H^1(X_G, \mathcal{N}_G^{0,1})$  a Hilbert modular form nearly holomorphic in the first variable and anti-holomorphic in the second variable. As before, the Serre dual of this group is  $H^1(X_G, \mathcal{N}_G^{1,0})$ .

As in (5.1), we obtain natural maps

$$(5.2) \quad \omega_G^{(k_1, 2-k_2, w)} \hookrightarrow \mathcal{N}_G^{0,1}, \quad \mathcal{N}_G^{1,0} \twoheadrightarrow \omega_G^{(2-k_1, k_2, -w)}(-D_G)$$

which are Serre duals of each other.

**5.3. Nearly pushforward maps and unit root splittings.** Finally, we want to extend  $\iota_*$  to the sheaves considered in §5. From the construction we obtain a map of sheaves over  $X_H$

$$\iota^*(\mathcal{N}_{G,n}^{1,0}) \rightarrow \mathcal{N}_{H,n}^1,$$

yielding a pullback map  $\iota_{\text{nearly}}^* : H^1(X_G, \mathcal{N}_G^{1,0}) \rightarrow H^1(X_H, \mathcal{N}_H^1)$ . Dually we find

$$\iota_{\star}^{\text{nearly}} : H^0(X_H, \mathcal{N}_H^0) \rightarrow H^1(X_G, \mathcal{N}_G^{0,1}).$$

Twisting the model of the sheaves  $\mathcal{N}_H^0, \mathcal{N}_G^{0,1}$  similarly as we did for  $\underline{\omega}_H^{k_1-k_2}, \underline{\omega}_G^{(k_1, 2-k_2, w)}$  in (4.3) and (2.10), we let

$$\tilde{\mathcal{N}}_H^0 = \mathcal{N}_H^0(t_1 D^{\text{ét}}), \quad \tilde{\mathcal{N}}_G^{0,1} = \mathcal{N}_G^{0,1}(t_1 D^{1-\text{ét}})$$

for  $t_1 \gg 0$ , and therefore obtain

$$(5.3) \quad \iota_{\star}^{\text{nearly}} : H^0(X_H, \tilde{\mathcal{N}}_H^0) \rightarrow H^1(X_G, \tilde{\mathcal{N}}_G^{0,1}).$$

This map is compatible with the classical pushforward map and the maps (5.1), (5.2).

We can also perform analogous constructions as in the previous section to obtain a map:

$$(5.4) \quad \iota_{p,\star}^{\text{nearly}} : H^0(\mathfrak{X}_H^m, \tilde{\mathcal{N}}_H^0) \rightarrow H^1(\mathfrak{X}_G^{1-m}, \tilde{\mathcal{N}}_G^{0,1}),$$

which is compatible with the classical pushforward map (not only for  $X_H(p)$  and  $X_G(\mathfrak{p}_1)$ , but also for higher  $p$ -power levels as in (4.7)).

**Proposition 5.3.1.** *We have  $p$ -adic unit root splittings  $u_H, u_G$  of (5.1), (5.2) making the following commutative*

$$\begin{array}{ccc} H^0(\mathfrak{X}_H^m, \tilde{\mathcal{N}}_H^0) & \xrightarrow{\iota_{p,\star}^{\text{nearly}}} & H^1(\mathfrak{X}_G^{1-m}, \tilde{\mathcal{N}}_G^{0,1}) \\ \downarrow u_H & & \downarrow u_G \\ H^0(\mathfrak{X}_H^m, \tilde{\omega}_H) & \xrightarrow{\iota_{p,\star}} & H^1(\mathfrak{X}_G^{1-m}, \tilde{\omega}_G) \end{array}$$

*Proof.* It suffices to show that the unit-root splittings  $u_H$  and  $u_G$  are compatible via  $\iota^*$ . The unit-root splitting  $u_H$  follows from the fact that the unit-root subspace for the Frobenius map on the fibres is a canonical complement of  $\omega_{\mathcal{E}}$  inside  $\mathcal{H}_{\mathcal{E}}^1$ . The splitting  $u_G$  has a similar description using the action of Frobenius on  $\mathcal{H}_{\mathcal{A},1}^1$ ; so it suffices to note that the isomorphism  $\iota^*(\mathcal{H}_{\mathcal{A},1}^1) \cong \mathcal{H}_{\mathcal{E}}^1$  is compatible with the Frobenii.  $\square$

## 6. COHERENT CLASSES ATTACHED TO CUSP FORMS

We now change tack slightly, and introduce a specific cup-product in coherent cohomology which is related to  $L$ -values. For the present section  $p$  plays no role (but we shall later use the theory of the previous sections to interpolate these cup-products  $p$ -adically).

Throughout this section  $\Pi$  is a cuspidal automorphic representation of weight  $(k_1, k_2)$ , where  $k_i \geq 1$  and  $k_1 \equiv k_2 \pmod{2}$ , and conductor  $\mathfrak{N}$ . Thus  $\Pi_f$  has 1-dimensional invariants under the subgroup  $K_1(\mathfrak{N}) = \{g = \begin{pmatrix} * & \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{N}}\}$ .

### 6.1. Whittaker models.

**Notation 6.1.1** (Additive characters).

- (i)  $\psi$  denotes the additive character of  $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$  satisfying  $\psi_{\infty}(x_{\infty}) = \exp(-2\pi i x_{\infty})$  for  $x_{\infty} \in \mathbb{R}$ .
- (ii)  $\psi_F$  denotes the additive character of  $\mathbb{A}_F/F$  given by  $\psi_F(y) = \psi(\text{Tr}_{F/\mathbb{Q}}(\frac{y}{\sqrt{D}}))$ , where  $\sqrt{D}$  denotes the unique square root in  $F$  of the discriminant  $D = \text{disc}_F$  such that  $\sigma_1(\sqrt{D}) > 0$ .

Hence  $\psi_F$  is trivial on  $\mathbb{A}_{\mathbb{Q}} \subset \mathbb{A}_F$ , and its conductor is 1 (i.e. it is invariant under  $\widehat{\mathcal{O}}_F$ , and not under any larger fractional ideal). Since any cuspidal automorphic representation of  $\text{GL}_n$  is generic, every  $\phi \in \Pi$  has a Whittaker expansion

$$(6.1) \quad \phi(g) = \sum_{\alpha \in F^{\times}} W_{\phi}(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g), \quad W_{\phi}(g) = \int_{F \backslash \mathbb{A}_F} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) \psi_F(-x) dx$$

where  $dx$  is the Haar measure on  $\mathbb{A}_F$  giving  $F \backslash \mathbb{A}_F$  volume 1.

**Definition 6.1.2.**



- (i) We define the *normalised Whittaker function at  $\infty$*  as the function on  $\mathrm{GL}_2(F \otimes \mathbb{R})$  which is supported on the identity component and satisfies

$$W_\infty^{\mathrm{ah},1} \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t \cos \theta & t \sin \theta \\ -t \sin \theta & t \cos \theta \end{pmatrix} \right) = \mathrm{sgn}(t)^k \cdot y^{k/2} \cdot e^{i(-k_1 \theta_1 + k_2 \theta_2)} \cdot e^{\frac{2\pi i}{\sqrt{D}}((-x_1 + iy_1) + (x_2 + iy_2))};$$

for  $x, y, t, \theta \in F \otimes \mathbb{R}$  with  $y \gg 0$ . This is a basis of the minimal  $K$ -type subspace in the Whittaker model of the  $\sigma_1$ -antiholomorphic,  $\sigma_2$ -holomorphic part of  $\Pi_\infty$ .

- (ii) Casselman's theory of the new vector shows that the 1-dimensional space of  $K_1(\mathfrak{N})$ -invariants in the Whittaker model of  $\Pi_f$  has a unique basis  $W_f^{\mathrm{new}}$  with  $W_f^{\mathrm{new}}(1) = 1$ .  
 (iii) We let  $\phi_{\mathrm{new}}^{\mathrm{ah},1}$  be the unique function in  $\Pi$  whose Whittaker function factors as  $W_f^{\mathrm{new}} W_\infty^{\mathrm{ah},1}$ .

(If we identify Hilbert modular forms with functions on  $\mathrm{GL}_2(\mathbb{A}_{F,f}) \times \mathcal{H}_F$  where  $\mathcal{H}_F$  is a product of upper half-planes, this  $\phi_{\mathrm{new}}^{\mathrm{ah},1}$  corresponds to the form  $\mathcal{F}^{\mathrm{ah},1}$  defined in [LLZ18].)

**6.2. Periods.** We choose  $w$  with  $k_1 \equiv k_2 \equiv w \pmod{2}$ , and a number field  $L$  such that  $\Pi_f \otimes \|\det\|^{-w/2}$  is definable over  $L$ . If  $X_G^{\mathrm{sph}}$  denotes the Hilbert modular variety of level  $K_1(\mathfrak{N})$ , the  $L$ -vector space

$$H^1 \left( X_{G,L}^{\mathrm{sph}}, \underline{\omega}_G^{(2-k_1, k_2, w)} \right) [\Pi_f]$$

is 1-dimensional; and we may choose a basis  $\nu_\Pi$  of this space. Note that the choice of  $\nu_\Pi$  determines an isomorphism of  $\mathrm{GL}_2(\mathbb{A}_{F,f})$ -representations

$$(6.2) \quad \mathcal{W}(\Pi_f) \otimes \|\cdot\|^{-w/2} \rightarrow \varinjlim_K H^1(X_G(K)_\mathbb{C}, \underline{\omega}^{(2-k_1, k_2, w)})[\Pi_f],$$

characterised by sending  $W_f^{\mathrm{new}}$  to  $\nu_\Pi$ .

After tensoring with  $\mathbb{C}$ , there is a canonical isomorphism between  $H^1 \left( X_{G,L}^{\mathrm{sph}}, \underline{\omega}_G^{(2-k_1, k_2, w)} \right)$  and the space of Hilbert modular forms of weight  $(k_1, k_2)$  and level  $K_1(\mathfrak{N})$  which are anti-holomorphic at  $\sigma_1$  and holomorphic at  $\sigma_2$ . By comparing Hecke eigenvalues, the image of  $\nu_\Pi$  must be a scalar multiple of  $\phi_{\mathrm{new}}^{\mathrm{ah},1}$ .

**Definition 6.2.1.** We define the period  $\Omega(\Pi) \in \mathbb{C}^\times$  (the *occult period* for  $\Pi$  at  $\sigma_1$  as in [Har90]) to be the scalar such that  $\phi_{\mathrm{new}}^{\mathrm{ah},1} = \Omega(\Pi) \cdot \nu_\Pi$ . (This is uniquely determined once  $\nu_\Pi$  is chosen; its class in  $\mathbb{C}^\times/L^\times$  does not depend on  $\nu_\Pi$ .)

**6.3. Siegel sections and adelic Eisenstein series.** We now recall a very general construction of adelic Eisenstein series (as for example in [Jac72]). For a Schwartz function  $\Phi : \mathbb{A}^2 \rightarrow \mathbb{C}$ , and  $\chi$  a Dirichlet character, we define a global Siegel section and associated Eisenstein series by

$$(6.3) \quad f^\Phi(g; \chi, s) := \|\det g\|^s \int_{\mathbb{A}^\times} \Phi((0, a)g) \hat{\chi}(a) \|a\|^{2s} d^\times a,$$

$$(6.4) \quad E^\Phi(g; \chi, s) := \sum_{\gamma \in B(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{Q})} f^\Phi(\gamma g; \chi, s).$$

Here  $\hat{\chi}$  is the character of  $\mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0}^\times$  corresponding to  $\chi$  as in [LPSZ21, §2.2]. The sum converges absolutely and uniformly on any compact subset of  $\{s : \mathrm{Re}(s) > 1\}$ , it has a meromorphic continuation in  $s$ , and defines a function on the quotient  $\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})$  that transforms under the center by  $\hat{\chi}^{-1}$ .

We now make a choice of the Schwartz function at  $\infty$  which recovers familiar holomorphic or nearly-holomorphic Eisenstein series. For  $k \in \mathbb{Z}_{\geq 1}$ , let  $\Phi_\infty^{(k)} \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$  be the function given by

$$(6.5) \quad \Phi_\infty^{(k)}(x, y) := 2^{1-k} (x + iy)^k e^{-\pi(x^2 + y^2)}.$$

Then, for  $\Phi = \Phi_f \cdot \Phi_\infty^{(k)}$ , where  $\Phi_f \in \mathcal{S}(\mathbb{A}_f^2, \mathbb{C})$  is arbitrary, and  $1 - \frac{k}{2} \leq s \leq \frac{k}{2}$  with  $s = \frac{k}{2} \pmod{\mathbb{Z}}$ , the function  $E^{k, \Phi_f}(-; \chi, s) := E^{\Phi_f \cdot \Phi_\infty^{(k)}}(-; \chi, s)$  is a nearly-holomorphic modular form. Moreover, if  $\Phi_f$  and  $\chi$  take values in a number field  $L$ , then this form is defined over  $L$  as a coherent cohomology class.

#### 6.4. The global zeta integral.

**Definition 6.4.1.** Let  $\nu$  be a Dirichlet character. For  $\Pi$  as in section 6,  $\phi \in \Pi$ , and  $\Phi \in \mathcal{S}(\mathbb{A}_f^2)$ , we define

$$Z(\phi, \Phi; \nu, s) = \int_{Z_H(\mathbb{A})H(\mathbb{Q}) \backslash H(\mathbb{A})} \phi(h) \hat{\nu}(\det h)^{-1} E^\Phi(h; \chi\nu^{-2}, s) dh,$$

where  $dh$  is the Haar measure on  $H(\mathbb{A})$  (normalised as in [HK92, Remark 1.6]), and  $\hat{\nu}$  the adelic character associated to  $\nu$ .

Let  $W_\phi$  be the Whittaker function of  $\phi$  as in (6.1). We suppose  $W_\phi = \prod_v W_v$  and  $\Phi = \prod_v \Phi_v$  are pure tensors. Then the global integral factors as  $Z(\phi, \Phi; \nu, s) = \prod_v Z(W_v, \Phi_v; \hat{\nu}_v, s)$ , where

$$Z(W_v, \Phi_v; \hat{\nu}_v, s) := \int_{(Z_H N_H \backslash H)(\mathbb{Q}_v)} W_v(h_v) \hat{\nu}_v(\det h_v)^{-1} f^{\Phi_v}(h_v; \hat{\chi}_v \hat{\nu}_v^{-2}, s) dh_v.$$

**Proposition 6.4.2.** *Suppose  $\ell$  is a finite place and  $\nu$  is unramified at  $\ell$ . For each  $W_\ell$  and  $\Phi_\ell$ , there exists a polynomial  $C_\ell(W_\ell, \Phi_\ell; X) \in \mathbb{C}[X, X^{-1}]$  such that*

$$Z(W_\ell, \Phi_\ell; \hat{\nu}_\ell, s) = L_{\text{As}}(\Pi_\ell \times \hat{\nu}_\ell^{-1}, s) \cdot C_\ell(W_\ell, \Phi_\ell; \nu(\ell)^{-1} \ell^{-s}).$$

Moreover,

- (i) for any  $\ell$ , the ideal of  $\mathbb{C}[X, X^{-1}]$  generated by the  $C_\ell(W_\ell, \Phi_\ell; X)$  is the unit ideal;
- (ii) if  $\Pi_\ell$  is unramified and  $W_\ell, \Phi_\ell$  are the normalised spherical data, then  $C_\ell(W_\ell, \Phi_\ell; X) = 1$ .

*Proof.* Since the character  $\hat{\nu}_\ell$  of  $\mathbb{Q}_\ell^\times$  is unramified, we may reduce to the case  $\hat{\nu}_\ell = 1$  by replacing  $s$  with  $s + \alpha$  for any  $\alpha \in \mathbb{C}$  such that  $\ell^\alpha = \nu(\ell)$ . Then we obtain the standard local Asai zeta integral studied in [Gro20] among other works; see [LW20, Remark 2.14] for the equality of the lowest common denominator of the values of the Asai zeta-integral with the Langlands–Deligne  $L$ -factor.  $\square$

### 7. $p$ -ADIC EISENSTEIN MEASURE AND INTERPOLATION OF PERIOD INTEGRALS

**7.1.  $\mathfrak{p}_1$ -stabilisation.** We now assume  $\Pi$  is ordinary at  $\mathfrak{p}_1$ , and that its field of definition  $L$  is contained in our  $p$ -adic coefficient field  $E$  (via our isomorphism  $\mathbb{C} \cong \overline{\mathbb{Q}_p}$ ). Let  $\alpha_1, \beta_1$  be the two roots of the Hecke polynomial at  $\mathfrak{p}_1$ , ordered so that  $v_p(\alpha_1) = 0$ .

There is a natural pullback map

$$\text{pr}^* : H^1 \left( X_{G,E}^{\text{sph}}, \underline{\omega}_G^{(2-k_1, k_2, w)} \right) \rightarrow H^1 \left( X_{G,0}(\mathfrak{p}_1)_E, \underline{\omega}_G^{(2-k_1, k_2, w)} \right)$$

We let

$$(7.1) \quad \nu_{\Pi, \mathfrak{p}_1} := \left( 1 - \frac{\beta_1}{U_{\mathfrak{p}_1}^t} \right) \text{pr}^*(\nu_\Pi).$$

By definition it lies in  $e(U_{\mathfrak{p}_1}^t)H^1 \left( X_{G,0}(\mathfrak{p}_1)_E, \underline{\omega}_G^{(2-k_1, k_2, w)}(-D_G) \right) [\Pi_f]$ ; and it corresponds to the basis vector  $\left( 1 - \frac{\beta_1}{U_{\mathfrak{p}_1}^t} \right) W_f^{\text{new}}$  of the ordinary  $U_{\mathfrak{p}_1}^t$ -eigenspace in  $\mathcal{W}(\Pi_f)^{\Gamma_0(\mathfrak{p}_1) \times K_1(\mathfrak{N})^{(p)}}$ .

**7.2. Katz  $p$ -adic Eisenstein measure.** Let us now fix the choice of a Schwartz function  $\Phi^{(p)} \in \mathcal{S}((\mathbb{A}_f^{(p)})^2, \mathbb{C})$  at places *away from*  $p$ , and  $\chi^{(p)}$  a Dirichlet character of conductor coprime to  $p$  and such that  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \Phi^{(p)} = \hat{\chi}^{(p)}(a)^{-1} \Phi^{(p)}$  for  $a \in (\hat{\mathbb{Z}}^{(p)})^\times$ .

Now let us consider a pair  $\mu, \nu$  of Dirichlet characters of  $p$ -power conductor. We will now attach to such pair an element  $\Phi_{p, \mu, \nu} \in \mathcal{S}(\mathbb{Q}_p^2, \mathbb{C})$ . First recall that if  $v$  is any place of  $\mathbb{Q}$ ,  $\phi \in \mathcal{S}(\mathbb{Q}_v, \mathbb{C})$  is a Schwartz function in one variable, and  $\psi_v$  is our fixed choice of additive character on  $\mathbb{Q}_v$ , the usual Fourier transform  $\hat{\phi}$  of  $\phi$  is given by

$$\hat{\phi}(x) = \int_{\mathbb{Q}_v} \phi(y) \psi_v(yx) dy.$$

For any character  $\xi : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ , let  $\phi_\xi$  be the Schwartz function defined by  $\phi_\xi(x) := \text{ch}_{\mathbb{Z}_p^\times}(x) \xi(x)$ . We let

$$(7.2) \quad \Phi_{p, \mu, \nu}(x, y) = \phi_\mu(x) \hat{\phi}_\nu(y).$$

We regard this as taking values in  $\overline{\mathbb{Q}_p}$  via our fixed choice of isomorphism from  $\mathbb{C}$ .

**Theorem 7.2.1.** *Let  $\Lambda'_E = \Lambda_E \hat{\otimes} \Lambda_E$ , where  $E$  is a  $p$ -adic field containing  $L$ , and let  $(\kappa_1, \kappa_2)$  be the two canonical characters into  $\Lambda'_E$ . Then for each  $\Phi^{(p)}$  taking values in  $L$ , there exists a two-variable measure*

$$\mathcal{E}^{\Phi^{(p)}}(\kappa_1, \kappa_2; \chi^{(p)}) \in H^0(\mathfrak{X}_H^m, \Omega_H^\kappa(-D_H))_{\Lambda, \kappa_1 + \kappa_2 + 1} \hat{\otimes} \Lambda'_E$$

such that its specialisation at  $(a + \mu, b + \nu)$  for any  $a, b \geq 0$  is the  $p$ -adic modular form associated to the algebraic nearly-holomorphic modular form

$$g \mapsto \widehat{\nu}(\det g)^{-1} \cdot E^{(a+b+1, \Phi^{(p)} \Phi_{p, \mu, \nu})} \left( g; \chi^{(p)} \mu \nu^{-1}, \frac{b-a+1}{2} \right) \in M_{a+b+1}^{\text{nh}}.$$

Moreover, let  $\Lambda_E = \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \otimes E$  with canonical character  $\kappa$ , and let  $t \in \mathbb{Z}_{\geq 2}$  be even. Then for each  $\Phi^{(p)}$  taking values in  $L$ , there exists an element

$$\mathcal{E}_t^{\Phi^{(p)}}(\kappa; \chi^{(p)}) \in H^0(\mathfrak{X}_H^m, \underline{\omega}_H^t(-D_H))_{\mathbb{Z}_p} \hat{\otimes} \Lambda_L$$

such that its specialisation at  $s + \nu$  for any integer  $s$  with  $1 - \frac{t}{2} \leq s \leq \frac{t}{2}$  is the  $p$ -adic modular form associated to the algebraic nearly-holomorphic modular form

$$g \mapsto \widehat{\nu}(\det g)^{-1} \cdot E^{(t, \Phi^{(p)} \Phi_{p, \nu^{-1}, \nu})} \left( g; \chi^{(p)} \nu^{-2}, s \right) \in M_t^{\text{nh}}.$$

*Proof.* The first statement is just a consequence of [LPSZ21, Theorem 7.6], which in turn is a reformulation of the existence of Katz' Eisenstein measure. The second statement follows from the first one, as the one-variable object we are looking for is obtained letting

$$(7.3) \quad \mathcal{E}_t^{\Phi^{(p)}}(\kappa; \chi^{(p)}) := \mathcal{E}^{\Phi^{(p)}} \left( \frac{t}{2} - \kappa, \kappa + \frac{t}{2} - 1; \chi^{(p)} \right). \quad \square$$

**Remark 7.2.2.** In order to construct the one-variable  $p$ -adic  $L$ -function, we are going to consider the measure  $\mathcal{E}_{k_1 - k_2}^{\Phi^{(p)}}(\kappa; \chi^{(p)})$ . The interpolation property tells us that, fixing the  $p$ -power conductor character  $\nu$ , we can get the Eisenstein series  $\widehat{\nu}(\det g)^{-1} \cdot E^{(k_1 - k_2, \Phi_{\nu^{-1}, \nu})}(g, \tau; \chi^{(p)} \nu^{-2}, s)$  for values of  $s$  in the range

$$1 - \frac{k_1 - k_2}{2} \leq s \leq \frac{k_1 - k_2}{2}.$$

Note this is exactly the range (crit) from the introduction.  $\diamond$

**7.3. One-variable interpolation of zeta-integrals.** Let  $\Pi$  be a cuspidal automorphic representation of  $\text{GL}_2(\mathbb{A}_F)$ , of weight  $(k_1, k_2)$  with  $k_1 = k_2 \pmod{2}$  and  $k_1 > k_2 \geq 1$ , and of level prime to  $p$  (as in theorem C).

We choose the following local data away from  $p\infty$ :

- a Whittaker function  $W_f^{(p)} \in W(\Pi_f^{(p)})$  which is defined over  $L$ ;
- a Schwartz function  $\Phi^{(p)}$  defined over  $L$ ;
- a level group  $K_G^{(p)}$  such that  $K_G^{(p)}$  fixes  $W_f^{(p)}$  and  $K_H^{(p)} := K_G^{(p)} \cap H$  fixes  $\Phi^{(p)}$ .

We let  $W_{f, \alpha_1} = W_f^{(p)} \cdot W_{\mathfrak{p}_1, \alpha_1} \cdot W_{\mathfrak{p}_2, \text{sph}} \in W(\Pi_f)$ , and we let  $\check{\nu}_{\Pi, \alpha}$  be its image under the map (6.2).

**Definition 7.3.1.** We define

$$\mathcal{Z}_\Lambda \left( W_f^{(p)}, \Phi^{(p)}; \kappa \right) = \text{Vol}(K_H^{(p)}) \cdot \left\langle \check{\nu}_{\Pi, \alpha}, \iota_\star^p \left( \mathcal{E}_{k_1 - k_2}^{\Phi^{(p)}} \right) \right\rangle_{\mathfrak{X}_{G, 0}(\mathfrak{p}_1)^{1-m}} \in \Lambda_E.$$

Here the cup-product is taken in the cohomology of the Shimura variety of prime-to- $p$  level  $K_G^{(p)}$ ; the volume factor makes the product independent of the choice of level group. Since the cup-product for coherent cohomology classes is given by integration over the Shimura variety (see e.g. [Har90], we see that for any locally-algebraic character  $s + \nu$ , with  $s$  in the critical range, we have

$$(7.4) \quad \mathcal{Z}_\Lambda \left( W_f^{(p)}, \Phi^{(p)}; s + \nu \right) = \frac{1}{\Omega_\infty(\Pi)} Z \left( W_{f, \alpha_1} W_\infty, \Phi^{(p)} \Phi_{p, \nu^{-1}, \nu} \Phi_\infty^{(k_1 - k_2)}; \nu, s \right).$$

From the results of the previous section, we can write this as

$$\frac{L_{\text{As}}(\Pi, \nu^{-1}, s)}{\Omega_\infty(\Pi)} \cdot \frac{Z(W_{p, \alpha_1}, \Phi_{p, \nu^{-1}, \nu}; \hat{\nu}_p, s)}{L_{\text{As}}(\Pi_p, \nu_p^{-1}, s)} \cdot Z(W_\infty; \Phi_\infty^{k_1 - k_2}, s) \cdot \prod_{\ell \neq p} C_\ell(W_\ell, \Phi_\ell; \ell^{-s} \nu(\ell)^{-1}).$$

(Note that the local integral at  $\infty$  is independent of  $\nu$ , and all but finitely many of the local terms  $C_\ell$  are 1.)

We shall consider the following two inputs to  $\mathcal{Z}_\Lambda$ :

7.3.1. *Primitive test data.* From part (i) of proposition 6.4.2 we know that for each  $\ell$ , the ideal of  $\mathbb{C}[X, X^{-1}]$  generated by the local terms  $C_\ell(\dots)$  is 1. Mapping this into  $\Lambda_E$  via the map  $X \mapsto [\ell]^{-1}$ , we obtain a  $\Lambda_E$ -linear combination of values  $\mathcal{Z}_\Lambda(\dots)$  whose value at  $s + \nu$ , for all critical  $s$  and finite-order  $\nu$ , is given by

$$\frac{L_{\text{As}}(\Pi \times \nu^{-1}, s)}{\Omega_\infty(\Pi)} \cdot \frac{Z(W_{p, \alpha_1}, \Phi_{p, \nu^{-1}, \nu}; \nu_p, s)}{L_{\text{As}}(\Pi_p \times \nu_p^{-1}, s)} \cdot Z(W_\infty; \Phi_\infty^{(k_1 - k_2)}, s).$$

**Definition 7.3.2.** We define  $\mathcal{L}_{p, \text{As}}(\Pi)$  to be the above element of  $\Lambda_E$ , multiplied by the factor

$$(7.5) \quad \frac{p+1}{p} \cdot (\sqrt{D})^{1 - (k_1 + k_2)/2 - \sigma} (-1)^\sigma \in \Lambda_E^\times,$$

where  $\sigma : \mathbb{Z}_p^\times \rightarrow \Lambda_E^\times$  is the universal character of  $\Lambda_E$ .

We shall evaluate the local factors at  $p$  and  $\infty$  below, and this will give the interpolation property of theorem C. The factor (7.5) will serve to cancel out various unwanted terms arising from the local zeta-integrals.

7.3.2. *Imprimitive test data.* Since we do not know how the primitive test data behave as we vary  $\Pi$  over the specialisations of a  $p$ -adic family, we will also work with a less optimal, but explicit, alternative set of data. These ‘‘imprimitive test data’’ are the inputs  $W_f^{(p)}$  and  $\Phi^{(p)}$  given by the following recipe, where  $N$  is the integer generating  $\mathfrak{N} \cap \mathbb{Z}$ .

- $W_f^{(p)}$  is the normalised new-vector,
- $\Phi^{(p)} = \prod_{\ell \neq p} \Phi_\ell$ , where  $\Phi_\ell = \text{ch}(\mathbb{Z}_\ell^2)$  if  $\ell \nmid N$ , and if  $v_\ell(N) = r > 0$ , we define

$$\Phi_\ell = (\ell^2 - 1)\ell^{2r-2} \cdot \text{ch}(\ell^r \mathbb{Z}_\ell, 1 + \ell^r \mathbb{Z}_\ell).$$

Thus we may take  $K_G^{(p)} = K_{G,1}(\mathfrak{N})^{(p)}$ . The imprimitive (one-variable)  $L$ -function  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi)$  is defined as the zeta-integral  $\mathcal{Z}_\Lambda(W_f^{(p)}, \Phi^{(p)}; \kappa)$  for this choice of data, multiplied by the correction factor  $\frac{p+1}{p} \cdot \sigma(\sqrt{D})^{-1}$  as before.

7.4. **Two-variable interpolation.** We now consider the case of families. Given a  $\mathfrak{p}_1$ -ordinary family  $\Pi$  of level  $\mathfrak{N}$  over some affinoid  $\mathcal{U}$ , as above, we choose an arbitrary isomorphism

$$\nu_\Pi : H_{\mathfrak{m}, \emptyset}^1(k_1 + 2\kappa_{\mathcal{U}}, 2 - k_2, w)[\Pi^\vee] \cong \mathcal{O}(\mathcal{U}).$$

The specialisation of  $\nu_\Pi$  at an integer weight  $a$  is a basis of the  $\Pi[a]$ -eigenspace in cohomology over our  $p$ -adic field  $E$ , but it may not necessarily descend to a number field. So we choose an algebraic basis  $\nu_{\Pi[a]}$  defined over the coefficient field  $L_a$  of  $\Pi[a]$  (which is a subfield of  $E$ ). This defines a *pair* of periods

$$\Omega_p(\Pi[a]) \in E^\times \quad \text{and} \quad \Omega_\infty(\Pi[a]) \in \mathbb{C}^\times$$

with

$$(\nu_\Pi)_a = \Omega_p(\Pi[a]) \cdot \nu_{\Pi[a], \mathfrak{p}_1}, \quad \phi_{\text{new}, \Pi[a]}^{\text{ah}, 1} = \Omega_\infty(\Pi[a]) \cdot \nu_{\Pi[a]}.$$

**Definition 7.4.1.** We let the two-variable measure  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}) \in \mathcal{O}(\mathcal{U}) \hat{\otimes} \Lambda_E$  be defined by

$$\mathcal{L}_{p, \text{As}}^{\text{imp}}(\kappa, \sigma) = (\star) \cdot \left\langle \nu_\Pi, \iota_{p, \star}^{k_1} \left( \mathcal{E}^{\Phi^{(p)}} \left( \frac{k_1 - k_2}{2} + \kappa_{\mathcal{U}} - \sigma, \sigma + \kappa_{\mathcal{U}} + \frac{k_1 - k_2}{2} - 1; \chi^{(p)} \right) \right) \right\rangle,$$

where  $(\star)$  is the factor from (7.5) (with  $k_1 + 2\kappa$  in place of  $k_1$ ),  $\chi^{(p)}$  is the common central character of all the  $\Pi[a]$ , and  $\Phi^{(p)}$  is determined by  $\mathfrak{N}$  as above.

By construction we have the specialisation formula

$$\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi})|_{\kappa_{\mathcal{U}}=a} = \Omega_p(\Pi[a]) \cdot \mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi[a]).$$

Combining this result with proposition 3.3.3 (which gives the existence of a family through any given  $\mathfrak{p}_1$ -ordinary  $\Pi$ ) gives theorem D.

## 8. EVALUATING LOCAL ZETA-INTEGRALS

We now compute the classical specialisations of the measure defined above and prove the interpolation property of Theorem C.

**8.1. Non-archimedean case: setup.** Let  $K$  be a nonarchimedean local field, of residue characteristic  $\ell$ , and  $L$  a finite étale  $K$ -algebra of degree 2. Thus  $L$  is either  $K \times K$ , or a quadratic field extension. (In our applications,  $K$  will be  $\mathbb{Q}_\ell$ , and  $L$  will be  $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$  for a real quadratic field  $F$ .) We write  $q_K$ , resp.  $q_L$ , for the orders of the residue fields of  $K$  and  $L$  (understood as  $q_K = q_L$  if  $L = K \times K$ ).

We fix a choice of nontrivial character  $\psi : L \rightarrow \mathbb{C}^\times$ ; we can and do assume that  $\psi$  is trivial on  $K \subset L$ , and that  $\psi$  is unramified (i.e. it is trivial on  $\mathcal{O}_L$  but not on any larger fractional ideal).

We set  $H = \mathrm{GL}_2(K)$ , and  $G = \mathrm{GL}_2(L)$ , so  $H \subset G$ . Write  $B_H, N_H, Z_H$  etc for the usual subgroups of  $H$ , and similarly for  $G$ .

**8.1.1. Godement–Siegel sections.** For  $\chi$  a smooth character of  $K^\times$ , and  $\Phi \in \mathcal{S}(K^2, \mathbb{C})$  and  $h \in H$ , recall

$$f^\Phi(h; \chi, s) = |\det h|_K^s \int_{K^\times} \Phi((0, a)h) \chi(a) |a|_K^{2s} d^\times a.$$

This defines a meromorphic section of the family of principal series representations  $I(|\cdot|_K^{s-1/2}, |\cdot|_K^{1/2-s} \chi^{-1})$  of  $H$ .

**8.1.2. Whittaker models, new vector theory.** Let  $\pi$  be a generic irreducible smooth representation of  $G$ , which we suppose to be unitary and tempered. Then we have a Whittaker model  $\mathcal{W}(\pi, \psi)$  (we shall omit  $\psi$  henceforth).

**Theorem 8.1.1** (Casselman). *There exists an ideal  $\mathfrak{c} = \mathfrak{c}(\pi)$  of  $\mathcal{O}_L$  with the following property: for  $\mathfrak{a}$  any ideal of  $\mathcal{O}_L$ , the invariants of  $\pi$  under the subgroup  $K_1(\mathfrak{a}) := \{g \in \mathrm{GL}_2(\mathcal{O}_L) : g = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{a}}\}$  are nonzero iff  $\mathfrak{c} \mid \mathfrak{a}$ . The invariants at level  $K_1(\mathfrak{c})$  are one-dimensional, and in the Whittaker model  $\mathcal{W}(\pi, \psi)$ , there is a unique generator  $W^{\mathrm{new}}$  of this subspace satisfying  $W^{\mathrm{new}}(1) = 1$ .*

The values of  $W^{\mathrm{new}}$  along the maximal torus can be extracted from the  $\mathrm{GL}_2$  standard zeta integral:

**Proposition 8.1.2.**

(1) *Suppose  $L$  is a field. Then we have*

$$\sum_{n \geq 0} W^{\mathrm{new}} \left( \begin{pmatrix} \varpi_L^n & 0 \\ 0 & 1 \end{pmatrix} \right) q_L^{(1/2-s)n} = L^{\mathrm{std}}(\pi, s)$$

where  $L^{\mathrm{std}}(-, s)$  denotes the standard  $L$ -factor.

(2) *Suppose  $L = K \times K$ , and let  $\varpi_1 = (\varpi_K, 1)$  and  $\varpi_2 = (1, \varpi_K)$  be uniformizers of the two maximal ideals of  $\mathcal{O}_L$ . Then*

$$\sum_{n_1, n_2 \geq 0} W^{\mathrm{new}} \left( \begin{pmatrix} \varpi_1^{n_1} \varpi_2^{n_2} & 0 \\ 0 & 1 \end{pmatrix} \right) q_K^{(1/2-s)(n_1+n_2)} = L^{\mathrm{std}}(\pi_1, s) L^{\mathrm{std}}(\pi_2, s).$$

**8.2. The Asai zeta integral.**

**8.2.1. Definitions.** For  $W \in \mathcal{W}(\pi)$  and  $\Phi \in \mathcal{S}(K^2)$ , we define

$$Z(W, \Phi; s) := \int_{Z_H N_H \backslash H} W(h) f^\Phi(h; \chi, s) dh$$

where  $\chi = \omega_\pi|_{K^\times}$ . This is well-defined for  $\Re(s) \gg 0$  and has analytic continuation as a meromorphic function of  $q_K^s$ ; as usual, we define  $L^{\mathrm{As}}(\pi, s)$  as the unique  $L$ -factor which generates the fractional ideal of  $\mathbb{C}[q_K^{\pm s}]$  given by the values of this integral as  $W$  and  $\Phi$  vary.

**8.2.2. Torus integral.** We define for  $W \in \mathcal{W}(\pi)$  a “partial zeta integral”

$$Z(W; s) = \int_{K^\times} W \left( \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) |x|_K^{s-1} d^\times x.$$

Cf. [Gro21, Definition 3.3.1]. Again, the integral is convergent for  $\Re(s) \gg 0$  and has analytic continuation as a meromorphic function of  $q_K^s$ .

**Proposition 8.2.1.** *The function  $Z(hW; \chi, s)$  transforms as an element of the induced representation  $I(|\cdot|^{1/2-s}, |\cdot|^{s-1/2} \chi)$ , i.e. the dual of the representation in which the Siegel section lies; and we have*

$$\int_{B_H \backslash H} Z(hW; s) f^\Phi(h; \chi, s) dh = Z(W, \Phi, s).$$

*Proof.* This follows by splitting up the integral over  $Z_H N_H \backslash H$  into an integral over  $Z_H N_H \backslash B_H$  followed by an integral over  $B_H \backslash H$ .  $\square$

Note that it suffices to take the integral over  $K_H = \mathrm{GL}_2(\mathcal{O}_K)$ , since  $B_H K_H = H$ .

8.2.3. *A special Schwartz function.* Let  $r \geq 1$ , and consider the Schwartz function

$$\Phi_r = (q_K^2 - 1)q_K^{2r-2} \cdot \mathrm{ch}(\mathfrak{p}_K^r, 1 + \mathfrak{p}_K^r).$$

From [LSZ22, Lemma 3.2.5], if  $\chi$  is trivial on  $1 + \mathfrak{p}^r$ , then  $f^{\Phi_r}(-; \chi, s)$  vanishes outside the coset  $B_H \backslash B_H K_0(\mathfrak{p}_K^r)$  (the ‘‘small cell’’ modulo  $\mathfrak{p}^r$ ), and we have

$$f^{\Phi_r}(1; \chi, s) = (q_K + 1)q_K^{r-1},$$

which is also the volume of the small cell for the unramified Haar measure on  $B_H \backslash H$ . (On the other hand, if  $\chi$  is nontrivial on  $1 + \mathfrak{p}^r$  then  $f^{\Phi_r}(-; \chi, s)$  is identically zero.)

Since the small cell modulo  $\mathfrak{p}^r$  is the orbit of the identity under the subgroup  $\begin{pmatrix} 1 & 0 \\ \mathfrak{p}^r & 1 + \mathfrak{p}^r \end{pmatrix}$ , we have the following: if  $W \in \mathcal{W}(\pi)$  is invariant under this subgroup, the integrand  $h \mapsto Z(hW; s)f^{\Phi_r}(h; \chi, s)$  is equal to  $f^{\Phi_r}(1)Z(W, s)$  on the small cell in  $(B_H \cap K_H) \backslash K_H$ , and vanishes everywhere else on  $K_H$ . So for any  $W$  invariant under this subgroup we have

$$Z(W, \Phi_r, s) = Z(W; s).$$

### 8.3. Imprimitive Asai $L$ -factors.

**Definition 8.3.1.** We define the *imprimitive Asai  $L$ -factor* by

$$L_{\mathrm{As}}^{\mathrm{imp}}(\pi, s) = \begin{cases} Z(W^{\mathrm{new}}, s) & \text{if } \pi \text{ is ramified} \\ L(\chi, 2s) \cdot Z(W^{\mathrm{new}}, s) & \text{if } \pi \text{ is unramified.} \end{cases}$$

(Note that ‘‘unramified’’ here means ‘‘has invariants under  $\mathrm{GL}_2(\mathcal{O}_L)$ ’’, even when  $L$  is a ramified extension.)

**Proposition 8.3.2.** *Let  $r_0$  be such that  $\varpi_K^{r_0} = \mathfrak{c} \cap \mathcal{O}_K$ , where  $\mathfrak{c} \trianglelefteq \mathcal{O}_L$  is the conductor of  $\pi$ . Then we have*

$$L_{\mathrm{As}}^{\mathrm{imp}}(\pi, s) = Z(W^{\mathrm{new}}, \Phi_{r_0}, s).$$

*If  $r_0 \geq 1$  then we have in fact  $L_{\mathrm{As}}^{\mathrm{imp}}(\pi, s) = Z(W^{\mathrm{new}}, \Phi_r, s)$  for any  $r \geq r_0$ .*

8.3.1. *Unramified case.* Suppose  $\pi$  has trivial conductor (but we allow  $L$  to be a ramified extension). Then the new vector is simply the normalised spherical Whittaker function  $W^{\mathrm{sph}}$ ; and the function  $h \mapsto Z(hW^{\mathrm{sph}}, s)$  is invariant under  $K_H$ , so we want to take  $r = 0$ . In this case  $f^{\Phi_r}(1; \chi, s) = L(\chi, s)$ , so by a well-known explicit computation we find

$$L_{\mathrm{As}}^{\mathrm{imp}}(\pi, s)Z(W^{\mathrm{sph}}, \Phi_0, s) = L(\chi, s)Z(W^{\mathrm{sph}}, s) = L_{\mathrm{As}}(\pi, s).$$

**Remark 8.3.3.** This computation is well-known if  $L = K \times K$  or  $L$  is an unramified quadratic extension, in which case  $L_{\mathrm{As}}(\pi, s)$  is a degree 4  $L$ -factor. However the computation is also valid for  $L$  a ramified field extension, in which case  $L_{\mathrm{As}}$  is a degree 3  $L$ -factor (and coincidentally agrees with the symmetric square  $L$ -factor of  $\pi$ ).  $\diamond$

8.3.2. *Explicit formulae.* We give explicit formulae for the imprimitive Asai  $L$ -factor defined above.

**Proposition 8.3.4.** *Suppose  $L$  is a field extension (ramified or unramified), and  $\pi$  has non-trivial conductor. Let  $C$  be the multiset of nonzero complex numbers such that*

$$L^{\mathrm{std}}(\pi, s) = \prod_{\alpha \in C} (1 - \alpha q_L^{-s})^{-1}.$$

*Then*

$$L_{\mathrm{As}}^{\mathrm{imp}}(\pi, s) = \prod_{\alpha \in C} (1 - \alpha^{e_{L/K}} q_K^{-s})^{-1}$$

*where  $e_{L/K}$  is the ramification index (1 or 2).*  $\square$

**Proposition 8.3.5.** *Suppose  $L = K \times K$  (so again  $q_K = q_L =: q$ ), and  $\pi = \pi_1 \times \pi_2$  is arbitrary (ramified or unramified). For  $i = 1, 2$  let  $C_i$  be the multiset of complex numbers such that  $L^{\mathrm{std}}(\pi_i, s) = \prod_{\alpha_i \in C_i} (1 - \alpha_i q^{-s})^{-1}$ . Then we have*

$$L_{\mathrm{As}}^{\mathrm{imp}}(\pi, s) = \prod_{\alpha_1 \in C_1} \prod_{\alpha_2 \in C_2} (1 - \alpha_1 \alpha_2 q^{-s})^{-1}.$$

8.3.3. *Relation to the primitive Asai  $L$ -factor.* In all cases there exists an  $L$ -factor  $P(s)$  such that

$$L_{\text{As}}(\pi, s) = P(s) \cdot L_{\text{As}}^{\text{imp}}(\pi, s).$$

We have seen that if  $\pi$  is unramified, then  $P(s) = 1$ . One checks similarly that if  $L = K \times K$ , then we have  $P(s) = 1$  in any of the following cases:

- at least one of  $\pi_1, \pi_2$  is unramified;
- one of the  $\pi_i$  is supercuspidal, but the other is not.

8.4. **Zeta integrals with  $U^t$ -eigenvectors.** We now consider the local zeta integral at the split finite place  $v = p$ . Since we want to consider twists by ramified characters, we no longer assume the character  $\nu$  is trivial.

We can identify  $\mathcal{W}(\Pi_1, \psi_p) \otimes \mathcal{W}(\Pi_2, \psi_p)$  with  $\mathcal{W}(\Pi_p, \psi_{F,p})$  via mapping  $W_1 \otimes W_2$  to the function  $g \mapsto W_1\left(\begin{smallmatrix} 1/\sqrt{D} & \\ & 1 \end{smallmatrix}\right) g_1 W_2\left(\begin{smallmatrix} -1/\sqrt{D} & \\ & 1 \end{smallmatrix}\right) g_2$  (so normalised new vectors go to normalised new vectors). A straightforward substitution gives

$$Z(W, \Phi; \nu, s) = \hat{\nu}_p^{-1}(-\sqrt{D}) \cdot \int_{\mathbb{Q}_p^\times} W_1\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right) g W_2(g) \hat{\nu}_p^{-1}(\det g) f^\Phi(g; \chi \nu^{-2}, s) dg,$$

and the integral on the right-hand side is a Rankin–Selberg zeta integral. We are interested in the following situation:

- $\pi_1$  and  $\pi_2$  are unramified;
- $\nu$  is a Dirichlet character of  $p$ -power conductor (possibly trivial), so  $\hat{\nu}_p^{-1}(-\sqrt{D}) = \nu(-\sqrt{D})$ ;
- $\Phi = \Phi_{p, \nu^{-1}, \nu}$  as defined above;
- $W_2$  is the normalised spherical vector;
- $W_1$  is the  $U_{\mathfrak{p}_1}^t$ -eigenvector given by  $(1 - \frac{\beta_1}{U_{\mathfrak{p}_1}^t}) W_1^{\text{sph}}$ , where  $W_1^{\text{sph}}$  is the normalised spherical vector.

**Proposition 8.4.1.** *With the above choices, we have*

$$Z(W, \Phi; \nu, s) = \frac{p}{p+1} \nu(-\sqrt{D}) \mathcal{E}_p(\text{As } \Pi, \nu^{-1}, s) L_{\text{As}}(\Pi_p \times \hat{\nu}_p^{-1}, s).$$

This is a rather standard computation (compare e.g. [CH20]) so we shall not give the proof in detail here.

8.5. **Archimedean case.** It remains to compute the Archimedean zeta integral, for which we use the formula of definition 6.1.2 for  $W_\infty$ . Since  $\hat{\nu}_\infty$  is trivial on the support of  $W_\infty$ , we need to compute

$$Z(W_\infty, \Phi_\infty^{(k_1-k_2)}, s) := \int_{Z(\mathbb{R})N(\mathbb{R}) \backslash \text{GL}_2(\mathbb{R})} W_\infty(h) f^{\Phi_\infty}(h; 1, s) dh.$$

As shown in [Jac72, p.85], since the integrand is invariant under  $\mathbb{R}^\times \times \text{SO}_2(\mathbb{R})$ , we can write this as

$$f^{\Phi_\infty^{(k_1-k_2)}}(1; 1, s) \cdot \int_{\mathbb{R}_+^\times} y^{(k_1+k_2)/2} e^{-4\pi y/\sqrt{D}} y^{s-1} d^\times y.$$

We have

$$\int_{\mathbb{R}_+^\times} y^{(k_1+k_2)/2} e^{-4\pi y/\sqrt{D}} y^{s-1} d^\times y = \frac{1}{2} \cdot \left(\frac{\sqrt{D}}{2}\right)^{s-1 + \frac{k_1+k_2}{2}} \cdot \Gamma_{\mathbb{C}}\left(s - 1 + \frac{k_1+k_2}{2}\right),$$

where  $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s)$ , and similarly

$$f^{\Phi_\infty^{(k_1-k_2)}}(1; 1, s) = i^{k_1-k_2} 2^{s - \frac{k_1-k_2}{2}} \Gamma_{\mathbb{C}}\left(s + \frac{k_1-k_2}{2}\right),$$

so we conclude that

$$Z(W_\infty, \Phi_\infty^{(k_1-k_2)}, s) = i^{k_1-k_2} 2^{-k_1} (\sqrt{D})^{s-1 + (k_1+k_2)/2} \cdot \Gamma_{\mathbb{C}}\left(s - 1 + \frac{k_1+k_2}{2}\right) \Gamma_{\mathbb{C}}\left(s + \frac{k_1-k_2}{2}\right).$$



**8.6. The global zeta integral.** We can finally put together the zeta integrals of the above sections to prove our main result.

**Theorem 8.6.1.** *Let  $\Pi$  be an automorphic representation as above, with central character  $\chi_\Pi$ . Let  $\mathcal{L}_\Pi$  the one-variable measure defined in Definition 7.3.2. Then for any locally-algebraic  $s + \nu$  of  $\mathbb{Z}_p^\times$ , with  $s$  an integer such that  $\frac{1-k_1+k_2}{2} \leq s \leq \frac{k_1-k_2}{2}$  and  $\nu$  of finite order, we have*

$$\mathcal{L}_{p,\text{As}}(\Pi)(s + \nu) = \mathcal{E}_p(\text{As}(\Pi), \nu^{-1}, s) \cdot \frac{\Gamma(s + \frac{k_1+k_2-2}{2})\Gamma(s + \frac{k_1-k_2}{2})}{2^{(k_1-2)i(1-k_2)}(-2\pi i)^{(2s+k_1-1)}} \cdot \frac{L_{\text{As}}(\Pi, \nu^{-1}, s)}{\Omega(\Pi)}.$$

The same holds with  $\mathcal{L}_{p,\text{As}}^{\text{imp}}(\Pi)$  and  $L_{\text{As}}^{\text{imp}}(\Pi, \nu^{-1}, s)$  in place of the primitive  $L$ -functions.

*Proof.* We first consider the primitive  $L$ -function. By definition and the discussion in section 7.3, we have

$$\mathcal{L}_{p,\text{As}}(\Pi)(s + \nu) = (\star) \cdot \prod_v \int_{Z_H(\mathbb{Q}_v)N_H(\mathbb{Q}_v)\backslash H(\mathbb{Q}_v)} \nu^{-1}(\det(h_v))W_v(h_v)f^{\Phi_v}(h_v; \chi_v\nu^{-2}, s) dh_v.$$

where  $(\star)$  is the term from (7.5). We have chosen our test data so that the terms in this product for all places  $v \notin \{p, \infty\}$  are the local  $L$ -factors  $L_{\text{As}}(\Pi_v \times \nu_v^{-1}, s)$ . From proposition 8.4.1, the local term at  $p$  is given by

$$\frac{p}{p+1}\nu(-\sqrt{D}) \cdot L_{\text{As}}(\Pi_p \times \nu_p^{-1}, s) \cdot \mathcal{E}_p(\text{As } \Pi, \nu^{-1}, s).$$

The local integral at  $v = \infty$  was computed in the previous section, and combining these gives the stated result.

In the case of the imprimitive  $p$ -adic  $L$ -function, we have seen above that our choice of local data at the ramified primes (the “imprimitive test data” defined in section 7.3) gives the imprimitive Asai  $L$ -factor, and the computation at  $p$  and  $\infty$  is the same as before.  $\square$

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