



**HAL**  
open science

# On the Bloch-Kato conjecture and Iwasawa main conjecture for $GO(4)$

Giada Grossi, Sarah Zerbes, David Loeffler

► **To cite this version:**

Giada Grossi, Sarah Zerbes, David Loeffler. On the Bloch-Kato conjecture and Iwasawa main conjecture for  $GO(4)$ . 2024. ⟨hal-04727885⟩

**HAL Id: hal-04727885**

**<https://hal.science/hal-04727885v1>**

Preprint submitted on 9 Oct 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# ON THE BLOCH–KATO CONJECTURE AND IWASAWA MAIN CONJECTURE FOR $GO_4$

GIADA GROSSI, DAVID LOEFFLER, AND SARAH LIVIA ZERBES

ABSTRACT. We prove the Bloch–Kato conjecture for critical values of Asai  $L$ -functions of  $p$ -ordinary Hilbert modular forms over quadratic fields (with  $p$  split); and one inclusion in the Iwasawa main conjecture for these  $L$ -functions (up to a power of  $p$ ).

Nous prouvons la conjecture de Bloch–Kato pour la fonction  $L$  Asai pour les formes modulaires Hilbert  $p$ -ordinaires associées à une extension réelle quadratique dans laquelle  $p$  se décompose. Également nous obtenons une inclusion de la Conjecture Principale d’Iwasawa.

## 1. INTRODUCTION

**1.1. Setting.** The Bloch–Kato conjecture, which relates the dimensions of Selmer groups of geometric Galois representations to the order of vanishing of their  $L$ -functions, is one of the outstanding open problems of number theory. One of the most successful approaches to this problem so far has been via the method of Euler systems.

In this paper, we use this method to prove one inclusion of the cyclotomic Iwasawa Main Conjecture (Theorem 6.3.2) and the Bloch–Kato conjecture (Theorem 6.3.4) for the Asai  $L$ -functions of Hilbert modular forms over real quadratic fields (under various technical hypotheses, see below).

Via a theorem of Ramakrishnan (defining an “Asai transfer” from automorphic representations of  $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$  to automorphic representations of  $\mathrm{GL}_4$ ), these  $L$ -functions can also be identified as  $L$ -functions of automorphic representations of  $\mathrm{GL}_4$ . Conversely, one knows that any cuspidal automorphic representation of  $\mathrm{GL}_4/\mathbf{Q}$  which is *essentially self-dual* (i.e. there exists a character  $\chi$  such that  $\pi \cong \pi^\vee \times \chi$ ) is in the image of one of the following functorial transfers (which are not mutually exclusive):

- (a) the spinor transfer from  $\mathrm{GSp}_4$ ;
- (b) the Rankin–Selberg transfer from  $\mathrm{GL}_2 \times \mathrm{GL}_2$ ;
- (c) the Asai transfer from  $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$  for  $F$  a quadratic field extension.

(Case (a) corresponds to Langlands parameters factoring through  $\mathrm{GSp}_4$ , and cases (b) and (c) correspond to parameters factoring through  $GO_4$ , with the projection to  $GO_4/\mathrm{GSO}_4 \cong \{\pm 1\}$  being trivial in case (b) and the quadratic character associated to  $F$  in case (c).)

The theory of Euler systems has been successfully applied to the first two cases in [KLZ17] and [LZ20] respectively, so it is natural to focus on case (c). We shall also suppose that  $\pi$  is *regular algebraic*; then  $F$  must be a real quadratic field (rather than imaginary), and  $\pi$  arises from a Hilbert modular automorphic representation  $\Pi$  of  $\mathrm{GL}_2/F$  of weight  $(k_1 + 2, k_2 + 2) \geq (2, 2)$  with  $k_1 \neq k_2$ . Moreover, we shall suppose  $k_1 = k_2 \pmod{2}$ , so that  $\Pi$  is “paritious”. This implies that the central value of the  $L$ -function is at a half-integer rather than an integer; so the critical values are automatically non-vanishing, and the Bloch–Kato conjecture becomes the assertion that the Selmer groups of the critical twists of the Galois representation of  $\pi$  are zero. This is what we show in

**1.2. Main results.** The main theorems of this paper are the following:

**Theorem A.** *Let  $\Pi$  be an automorphic representation of  $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$  of weight  $(k_1 + 2, k_2 + 2)$ , for some  $k_1 > k_2 \geq 0$  with  $k_1 = k_2 \pmod{2}$ , and level  $\mathcal{N}_\Pi$ ; and  $\mathfrak{P}$  a prime of the coefficient field of  $\Pi$ , above some rational prime  $p > 2$ . Suppose that*

- $p$  is split in  $F$ ,
- $\Pi$  is unramified and ordinary at the primes above  $p$ ,
- the Galois representation satisfies a “big image” condition (see Assumptions 6.1.1),
- the local Euler factor  $P_p(\Pi, X)$  does not vanish at  $p^j$  for any  $j \in \mathbf{Z}$ .

Then:

---

Supported by European Research Council Consolidator Grant No. 101001051 (SHIMBSD) (Loeffler) and US National Science Foundation Grant No. DMS-1928930 (all authors).

- (Theorem 6.3.4) For each  $j$  with  $k_2 + 1 \leq j \leq k_1$ , the Bloch–Kato Selmer group of  $V_p^{\text{As}}(\Pi)^*(-j)$  is zero.
- (Theorem 6.3.2) The characteristic ideal of the dual Selmer group of  $\Pi$  over  $\mathbf{Q}(\mu_{p^\infty})$  divides  $p^r \cdot \mathcal{L}_{p,\text{As}}(\Pi)$  for some  $r \geq 0$ , where  $\mathcal{L}_{p,\text{As}}(\Pi)$  is the  $p$ -adic Asai  $L$ -function for  $\Pi$  constructed in [GLZ23a].

For more details of the notations and the definitions of the objects involved, we refer to the body of the paper.

Although the statements of these theorems involve only a single automorphic representation  $\Pi$ , our proof relies crucially on the use of  $p$ -adic families of automorphic representations (Hida families). More precisely, we consider a two-parameter family  $\underline{\Pi}$  of automorphic representations specialising to  $\Pi$ , with both components of the weight independently varying, and we prove the following theorem:

**Theorem B** (Theorem 4.5.1). *There exists a 3-variable  $p$ -adic Asai  $L$ -function for  $\underline{\Pi}$ , interpolating all critical values of the  $L$ -functions for all specialisations of  $\underline{\Pi}$ .*

On the other hand, Sheth [She24] has shown that the Euler system construction of [LLZ18] also extends to such families; and the regulator formula that we proved in [GLZ23b] gives a relation between these objects after specialising at suitable (non-classical) points. In order to derive from this an explicit reciprocity law relating the Euler system and  $L$ -function in the critical range, we need to show that the  $p$ -adic Eichler–Shimura comparison isomorphism, relating étale and de Rham cohomology, “interpolates in Hida families” in a suitable sense (analogous to results of Ohta [Oht00] and Andreatta–Iovita–Stevens [AIS15] for  $\text{GL}_2/\mathbf{Q}$ ). In our setting, we would hope for an isomorphism between two free rank 1 modules over  $\mathcal{O}(\mathcal{U})$ , where  $\mathcal{U}$  is a small neighbourhood in weight space over which  $\underline{\Pi}$  is defined: one module arising from higher Hida theory which interpolates coherent  $H^1$  of Hilbert modular varieties, and another which is  $\mathbf{D}_{\text{cris}}$  of a rank-1 subquotient of the Asai Galois representation of  $\underline{\Pi}$ . The Eichler–Shimura map gives canonical isomorphisms between the fibres of these modules at classical points, and we would hope for a single  $\mathcal{O}(\mathcal{U})$ -module isomorphism having all of these “pointwise” isomorphisms as specialisations. What we actually prove is something a little weaker:

**Theorem C.** *There exists an isomorphism between the above modules after base-extending to  $\text{Frac } \mathcal{O}(\mathcal{U})$ , interpolating the Eichler–Shimura isomorphisms for all but finitely many classical points. Moreover, the composite of this isomorphism and Perrin–Riou’s regulator map sends the Euler system to the  $p$ -adic  $L$ -function.*

Note that the proofs of both halves of the theorem are intertwined: we *use* the Euler system and regulator formula in order to derive the existence of the meromorphic Eichler–Shimura isomorphism.

Theorem C, together with the results of [LLZ18], immediately imply Theorem A as long as  $\Pi$  is not one of the finitely many “bad” specialisations. We conjecture that such bad specialisations do not actually exist, but we cannot rule it out with the presently available techniques<sup>1</sup>. Hence we carry out a *leading term argument*, adapting methods developed in [LZ20] for the standard (rather than Asai)  $L$ -functions of quadratic Hilbert modular forms, to show that one can recover a modified version of the Euler system which is related to the  $L$ -function even at “bad” specialisations. This gives Theorem A.

*Acknowledgements.* Parts of this paper were written while the authors were in residence at MSRI (now known as Simons Laufer Mathematical Institute) for the 2023 semester program “Algebraic Cycles,  $L$ -Values, and Euler Systems”. We are grateful to MSRI for offering such a fantastic working environment.

## 2. SETUP: HILBERT MODULAR FORMS

**2.1. Notations.** Let  $F$  be a real quadratic field of discriminant  $D$ , and let  $G = \text{Res}_{\mathbf{Q}}^F \text{GL}_2$ . We fix a numbering of the embeddings  $F \hookrightarrow \mathbf{R}$  as  $\{\sigma_1, \sigma_2\}$ . The Weyl group of  $G$  is isomorphic to  $(\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$ , with generators  $w_1$  and  $w_2$  (corresponding to the  $\sigma_i$ ).

Let  $H = \text{GL}_2$ , with  $\iota : H \hookrightarrow G$  the natural embedding. We write  $B_H$  and  $B_G$  for the upper-triangular Borel subgroups.

Let  $J \in \{G, H\}$ . For a neat open subgroup  $K$  of  $J(\mathbf{A}_f)$ , write  $Y_{J,K}$  for the Shimura variety of  $J$  of level  $K$ ,  $X_{J,K}$  for a toroidal compactification, and  $D_J = X_J - Y_J$  for the boundary divisor.

A *weight* for  $G$  will mean a tuple of integers  $(k_1, k_2; w)$  with  $w = k_1 = k_2 \pmod{2}$ . If  $\mu$  is a weight, we have a locally-free sheaf  $\omega^\mu$  on  $X_{G,K}/F$ , whose sections are Hilbert modular forms of weight  $\mu$ .

<sup>1</sup>This would follow from work in progress by Birkbeck and Williams, aiming to prove an interpolation of  $p$ -adic Eichler–Shimura maps in higher Hida, or Coleman, families of Hilbert modular forms using perfectoid methods.

*Note 2.1.1.* As an abstract sheaf  $\omega_\mu$  does not depend on  $w$ ; but the action of Hecke operators on it does. With our conventions, if  $\mathfrak{q}$  is a prime of  $F$  which is trivial in the narrow ray class group modulo  $Z_G \cap K$ , the action of  $(\varpi_{\mathfrak{q}})$  on  $R\Gamma(X_{G,K}/F, \omega^\mu)$  is multiplication by  $(\#\mathcal{O}_F/\mathfrak{q})^w$ .  $\diamond$

## 2.2. Asai $L$ -functions.

**Definition 2.2.1.** *Let  $\Pi$  be the unitary cuspidal automorphic representation of  $G(\mathbf{A})$  generated by a holomorphic Hilbert modular newform of weight  $(k_1 + 2, k_2 + 2)$ , for some integers  $k_1, k_2 \geq 0$ , and some level  $\mathcal{N}_\Pi \trianglelefteq \mathcal{O}_F$ .*

We define  $a_{\mathfrak{n}}^\circ(\Pi)$ , for each ideal  $\mathfrak{n} \trianglelefteq \mathcal{O}_F$ , to be the Hecke eigenvalues of the new vector of  $\Pi$ , normalised in the analytic fashion, so that  $|a_{\mathfrak{q}}^\circ(\Pi)| \leq 2$  for each prime  $\mathfrak{q} \nmid \mathcal{N}_\Pi$ . For each such prime  $\mathfrak{q}$  we let  $\alpha_{\mathfrak{q}}^\circ$  and  $\beta_{\mathfrak{q}}^\circ$  be the Satake parameters of  $\Pi_{\mathfrak{p}}$  (so that  $|\alpha_{\mathfrak{q}}^\circ| = |\beta_{\mathfrak{q}}^\circ| = 1$  and  $a_{\mathfrak{q}}^\circ(\Pi) = \alpha_{\mathfrak{q}}^\circ + \beta_{\mathfrak{q}}^\circ$ ).

*Note 2.2.2.* By a rationality theorem due to Shimura [Shi86, Proposition 1.2], if we choose  $w$  with  $k_1 = k_2 = w \pmod{2}$  (so that  $\mu = (k_1, k_2; w)$  is a weight), there exists a number field  $L \subset \mathbf{C}$  (depending on  $\Pi$ ) such that the quantities  $N(\mathfrak{n})^{(w+1)/2} a_{\mathfrak{n}}^\circ$  lie in  $L$ , for all integral ideals  $\mathfrak{n}$ . Extending  $L$  if necessary, we may assume that  $L$  also contains the images of the embeddings  $\sigma_i : F \hookrightarrow \mathbf{R}$ ; this is automatic if  $k_1 \neq k_2$ .  $\diamond$

**Definition 2.2.3.** *We define the imprimitive Asai  $L$ -series of  $\Pi$  to be the function*

$$L_{\text{As}}^{\text{imp}}(\Pi, s) = L_{(N)}(\chi, 2s) \sum_{n \in \mathbf{Z}_{\geq 1}} \frac{a_n^\circ(\Pi)}{n^s},$$

where  $\chi$  is the restriction to  $\mathbf{A}_{\mathbf{Q}}^\times$  of the central character of  $\Pi$ , and  $(N)$  denotes omitting the Euler factors at the primes dividing  $N = \mathcal{N}_\Pi \cap \mathbf{Z}$ .

*Note 2.2.4.* The imprimitive  $L$ -function  $L_{\text{As}}^{\text{imp}}(\Pi, s)$  may differ by finitely many Euler factors from the “true” Asai  $L$ -series  $L_{\text{As}}(\Pi, s)$  whose local factors are defined using the local Langlands correspondence.  $\diamond$

**Notation 2.2.5.** *We write  $\mathcal{L}_{\text{As}}(\Pi, s)$  for the shifted  $L$ -function*

$$\mathcal{L}_{\text{As}}(\Pi, s) = L_{\text{As}}(\Pi, s - \frac{k_1 + k_2 + 2}{2}),$$

and similarly for the imprimitive version.

This shift ensures that the Dirichlet-series coefficients of  $\mathcal{L}_{\text{As}}(\Pi, s)$  and  $\mathcal{L}_{\text{As}}^{\text{imp}}(\Pi, s)$  are algebraic integers; it corresponds to the conventional normalisation for Rankin–Selberg  $L$ -functions (which are “Asai  $L$ -functions for the degenerate quadratic field  $\mathbf{Q} \times \mathbf{Q}$ ”).

*Note 2.2.6.* The critical values of  $\mathcal{L}_{\text{As}}(\Pi, s)$  are the integers satisfying  $k_2 + 2 \leq s \leq k_1 + 1$  if  $k_1 > k_2$ , and similarly if  $k_1 < k_2$ . (There are no critical values if  $k_1 = k_2$ .)  $\diamond$

**2.3. Hecke eigensystems.** We let  $U_1(\mathcal{N}_\Pi)$  denote the open compact subgroup  $\{g \in G(\mathbf{A}_f) : g = \begin{pmatrix} * & \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{N}_\Pi}\}$ , so that the space of  $U_1(\mathcal{N}_\Pi)$ -invariants of  $\Pi_f$  (the new subspace of  $\Pi_f$ ) is one-dimensional.

**Definition 2.3.1.** *Let  $S$  be a finite set of primes containing all those dividing  $\text{disc}_{F/\mathbf{Q}} \cdot \text{Nm}_{F/\mathbf{Q}}(\mathcal{N}_\Pi)$ ; and write  $\mathbb{T}^S$  for the abstract Hecke algebra  $L[G(\mathbf{A}_f^S) // G(\hat{\mathbf{Z}}^S)] = L[\{T_{\mathfrak{q}}, S_{\mathfrak{q}}, S_{\mathfrak{q}}^{-1} : \mathfrak{q} \notin S_F\}]$  (where  $S_F$  is the set of primes of  $F$  above  $S$ ).*

*Note 2.3.2.* If we choose  $\Pi$  of weight  $(k_1, k_2)$  as above, and also a  $w$  such that  $(k_1, k_2; w)$  is a weight, then  $\Pi_f^S \otimes \|\cdot\|^{-w/2}$  is definable over  $L$  as a  $G(\mathbf{A}_f^S)$ -module, and hence the action of the Hecke algebra on its  $G(\mathbf{Z}^S)$ -invariants gives a ring homomorphism

$$\lambda_{\Pi}^S : \mathbb{T}^S \rightarrow L, \quad T_{\mathfrak{q}} \mapsto \text{Nm}(\mathfrak{q})^{(w+1)/2} a_{\mathfrak{q}}^\circ(\Pi), \quad S_{\mathfrak{q}} \mapsto \text{Nm}(\mathfrak{q})^w \chi_{\Pi}(\varpi_{\mathfrak{q}}),$$

whose kernel we denote  $I_{\Pi}^S$ .  $\diamond$

Now let  $p \notin S$  be a prime, and assume that  $p$  is split in  $F$ , say  $\langle p \rangle = \mathfrak{p}_1 \mathfrak{p}_2$ . We can, and do, fix a prime  $\mathfrak{P}$  of our coefficient field  $L$  above  $p$  such that  $\mathfrak{p}_i$  is the preimage under  $\sigma_i$  of the prime  $\mathfrak{P}$ .

**Notation 2.3.3.** *Let  $\text{Iw}(p) = \{g \in G(\mathbf{Z}_p) : g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p}\}$ , and for  $i = 1, 2$  let  $U_{\mathfrak{p}_i}$  and  $U'_{\mathfrak{p}_i}$  be the  $\text{Iw}(p)$ -double cosets of  $\begin{pmatrix} \varpi_{\mathfrak{p}_i} & \\ & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & \\ & \varpi_{\mathfrak{p}_i} \end{pmatrix}$  respectively, for any uniformiser  $\varpi_{\mathfrak{p}_i}$  at  $\mathfrak{p}_i$ .*

**Definition 2.3.4.** *By a  $p$ -stabilisation of  $\Pi$ , we mean a choice of one of the Satake parameters (which we denote WLOG by  $\alpha_i^\circ$ ) at each of the primes  $\mathfrak{p}_i \mid p$ .*

*Remark 2.3.5.* The  $p$ -stabilisation defines choices of eigenvalues for the operators  $U_{\mathfrak{p}_i}$ , or their transposes  $U'_{\mathfrak{p}_i}$ , at level  $\text{Iw}(p)$ . We normalise these operators in such a way that their eigenvalues are

$$\alpha_i := p^{(k_i+1)/2} \alpha_i^\circ \quad (\text{not } p^{(w+1)/2} \alpha_i^\circ).$$

This normalisation is less canonical (it depends on the choice of ideal  $\mathfrak{P} \mid p$  of  $L$ ), but will work better for  $p$ -adic interpolation; in particular, the valuation of  $\alpha_i$  lies in the range  $[0, k_i + 1]$ .  $\diamond$

**Definition 2.3.6.** For  $i = 1, 2$ , we say that  $\Pi$  is **ordinary at  $\mathfrak{p}_i$**  if  $v_p(\alpha_i) = 0$ ; and  $\Pi$  is **ordinary at  $p$**  if it is ordinary at both  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$ .

**Definition 2.3.7.** Let  $\mathbb{T}_{\text{Iw}, w_1}^S$  denote the product of  $\mathbb{T}^{S \cup \{p\}}$  and the subalgebra of  $L[G(\mathbf{Q}_p) // \text{Iw}(p)]$  generated by  $U'_{\mathfrak{p}_1}$  and  $U_{\mathfrak{p}_2}$  [sic; the roles of the primes are not symmetric].

As before, the choice of a  $p$ -stabilisation defines a character  $\lambda_{\Pi, \text{Iw}, w_1}^S$  of  $\mathbb{T}_{\text{Iw}, w_1}^S$  (sending  $U'_{\mathfrak{p}_1}$  to  $\alpha_1$  and  $U_{\mathfrak{p}_2}$  to  $\alpha_2$ ), and we let  $I_{\Pi, \text{Iw}, w_1}^S$  be its kernel.

#### 2.4. Coherent cohomology and periods.

**Notation 2.4.1.** For  $\star \in \{\text{sph}, \text{Iw}\}$ , write  $Y_{G, \star}$  for the Hilbert modular variety  $Y_G(U_1(\mathcal{N}_\Pi)^{(p)} K_{p, \star})$ , where

$$K_{p, \star} = \begin{cases} G(\mathbf{Z}_p) & \text{when } \star = \text{sph} \\ \text{Iw}(p) \subset G(\mathbf{Z}_p) & \text{when } \star = \text{Iw} \end{cases}$$

We assume, for simplicity, that  $\mathcal{N}_\Pi$  does not divide  $6 \text{disc}_{F/\mathbf{Q}}$ , so that  $U_1(\mathcal{N}_\Pi)$  is *sufficiently small* in the sense of [LLZ18, Definition 2.2.1]. Then  $Y_{G, \text{sph}}$  and  $Y_{G, \text{Iw}}$  are smooth  $\mathbf{Q}$ -varieties. Let  $X_{G, \star}$  be a smooth projective toroidal compactification of  $Y_{G, \star}$ .

**Proposition 2.4.2.** *The space*

$$\mathcal{S}_{w_1}(\Pi) := H^1 \left( X_{G, \text{sph}}/L, \omega^{(-k_1, k_2+2; w)}(-D_G) \right) [I_{\Pi}^S]$$

is 1-dimensional over  $L$ , and is independent of the choice of  $S$  (and independent of the toroidal boundary data up to a canonical isomorphism).  $\square$

(The label  $w_1$  refers to the first generator of the Weyl group  $(\mathbf{Z}/2) \times (\mathbf{Z}/2)$ ; the identity element would correspond to holomorphic Hilbert modular forms.)

**Definition 2.4.3.** Let  $\eta_\Pi$  be a basis of the space  $\mathcal{S}_{w_1}(\Pi)$  over  $L$ .

*Remark 2.4.4.* Over  $\mathbf{C}$  this space has a canonical basis given by the comparison between coherent cohomology and automorphic forms. The ratio between this basis and  $\eta_\Pi$ , uniquely defined as an element of  $\mathbf{C}^\times / L^\times$ , is Harris' *occult period*  $\Omega_\infty(\Pi)$  for  $\Pi$ . See [GLZ23a, §6.2].  $\diamond$

Similarly, we define

$$\mathcal{S}_{w_1, \text{Iw}}(\Pi) := H^1 \left( X_{G, \text{Iw}}, \omega^{(-k_1, k_2+2; w)}(-D_G) \right) [I_{\Pi, \text{Iw}, w_1}^S].$$

This is also 1-dimensional, and there is an isomorphism  $\mathcal{S}_{w_1}(\Pi) \rightarrow \mathcal{S}_{w_1, \text{Iw}}(\Pi)$  given by

$$(2.1) \quad \left( 1 - \frac{\beta_1}{U'_{\mathfrak{p}_1}} \right) \left( 1 - \frac{\beta_2}{U_{\mathfrak{p}_2}} \right) \text{pr}^*$$

where  $\text{pr} : X_{G, \text{Iw}} \rightarrow X_{G, \text{sph}}$  is the natural degeneracy map (c.f. [GLZ23a, §6.1]).

**2.5.  $p$ -adic  $L$ -functions.** We recall the construction of the 1-variable Asai  $p$ -adic  $L$ -function from [GLZ23a].

**Theorem 2.5.1.** *Assume that  $k_1 > k_2$  and  $\Pi$  is ordinary at  $\mathfrak{p}_1$ . Then there exists  $\mathcal{L}_{p, \text{As}}(\Pi; \eta) \in \Lambda_L(\mathbf{Z}_p^\times)$  such that for every  $s \in \mathbf{Z}$  with  $k_2 + 2 \leq s \leq k_1 + 1$ , and every Dirichlet character  $\nu$  of  $p$ -power conductor, we have*

$$\int_{\mathbf{Z}_p^\times} \nu^{-1}(x) x^s d\mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi; \eta) = \mathcal{E}_p(\text{As}(\Pi), \nu, s - \frac{k_1 + k_2 + 2}{2}) \cdot \frac{\Gamma(s) \Gamma(s - k_2 - 1)}{2^{k_1} i^{-1 - k_2} (-2\pi i)^{2s - k_2 + 1}} \cdot \frac{\mathcal{L}_{\text{As}}(\Pi, \nu, s)}{\Omega(\Pi)}.$$

Here,  $\mathcal{E}_p(\text{As}(\Pi), \nu, s)$  is the modified Euler factor as defined in [GLZ23a, Def. 1.2.2]. There is also an imprimitive variant  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi)$  interpolating the values of  $\mathcal{L}_{\text{As}}^{\text{imp}}(\Pi)$ .

*Proof.* This is Theorem C in *op.cit.* (with  $s$  substituted by  $s - \frac{k_1 + k_2 + 2}{2}$ , and  $(k_1, k_2)$  by  $(k_1 + 2, k_2 + 2)$ , corresponding to the shifts of normalisations for this paper).  $\square$

## 2.6. De Rham and étale cohomology.

### Definition 2.6.1.

(1) Let  $V_{\mathfrak{p}}^{\text{As}}(\Pi)$  denote the 4-dimensional Asai Galois representation associated to  $\Pi$ , defined as

$$V_{\mathfrak{p}}^{\text{As}}(\Pi) := H_{\text{ét},c}^2(Y_{G,\text{sph}}/\overline{\mathbf{Q}}, \mathcal{V}_{\text{ét},L_{\mathfrak{p}}})[I_{\Pi}^S]$$

where  $\mathcal{V}_{\text{ét},L_{\mathfrak{p}}}$  is the étale local system of  $L_{\mathfrak{p}}$ -vector spaces determined by  $(k_1, k_2; w)$ . We twist this by a power of the cyclotomic character so it is independent of  $w$  and has Hodge–Tate weights<sup>2</sup>  $\{0, -1 - k_1, -1 - k_2, -2 - k_1 - k_2\}$  (c.f. [LLZ18, Def. 4.4.2]).

(2) Let

$$D_{\mathfrak{p}}^{\text{As}}(\Pi) = L_{\mathfrak{p}} \otimes_L H_{\text{dR},c}^2(Y_{G,\text{sph}}/L, \mathcal{V}_{\text{dR},L})[I_{\Pi}^S],$$

where  $\mathcal{V}_{\text{dR},L}$  is the denotes the vector bundle with connection determined by  $(k_1, k_2; w)$  (c.f. [GLZ23b, §2.2]).

As  $p \notin S$ , the restriction of  $V = V_{\mathfrak{p}}^{\text{As}}(\Pi)$  to the decomposition group at  $p$  is crystalline; and we have a canonical comparison isomorphism

$$\mathbf{D}_{\text{cris}}(\mathbf{Q}_p, V_{\mathfrak{p}}^{\text{As}}(\Pi)) \cong D_{\mathfrak{p}}^{\text{As}}(\Pi),$$

compatible with the Hodge filtration.

We also have additional *plectic structures* on these spaces (cf. [LZ23]). Firstly, the Hodge filtration can be refined to a “ $\mathbf{Z}^2$ -filtration” in the sense of [NS16]; concretely, this means that there are two 2-dimensional subspaces  $\text{Fil}_1^+$  and  $\text{Fil}_2^+$  of  $D_{\mathfrak{p}}^{\text{As}}(\Pi)$ , whose sum is  $\text{Fil}^1$  and whose intersection is  $\text{Fil}^{k_1+k_2+2}$ . Then  $\mathcal{S}_{w_1}(\Pi) \otimes L_{\mathfrak{p}}$  is identified with  $\text{Fil}_2^+ / (\text{Fil}_1^+ \cap \text{Fil}_2^+)$ .

*Remark 2.6.2.* Note that if  $k_1 > k_2$ , then  $\text{Fil}_1^+$  is a step in the Hodge filtration, but  $\text{Fil}_2^+$  is not; and vice versa if  $k_1 < k_2$ . If  $k_1 = k_2$  then neither of them are.  $\diamond$

Secondly, we have *partial Frobenii*  $\varphi_1$  and  $\varphi_2$  acting on  $D_{\mathfrak{p}}^{\text{As}}(\Pi)$ , with  $\varphi_1\varphi_2 = \varphi_2\varphi_1 = \varphi$ ; and the eigenvalues of  $\varphi_i$  are  $\alpha_i$  and  $\beta_i$ .

**Proposition 2.6.3** (see [LZ23, Theorem 2.2]). *If  $\alpha_1$  has sufficiently small slope, then the intersection  $\text{Fil}_2^+ D_{\mathfrak{p}}^{\text{As}}(\Pi) \cap D_{\mathfrak{p}}^{\text{As}}(\Pi)^{(\varphi_1=\alpha_1)}$  is 1-dimensional, and maps isomorphically to*

$$\frac{\text{Fil}_2^+}{\text{Fil}_1^+ \cap \text{Fil}_2^+} \cong \mathcal{S}_{w_1}(\Pi) \otimes_L L_{\mathfrak{p}}.$$

If  $\Pi$  is ordinary at  $p$ , then the subspace  $D_{\mathfrak{p}}^{\text{As}}(\Pi)^{\varphi_1=\alpha_1}$  is weakly admissible, so arises from a 2-dimensional subrepresentation  $\mathcal{F}_1^+ V \subset V$ , and similarly for  $\mathcal{F}_2^+ V$ ; the intersection of these is the unique one-dimensional unramified subspace. Hence the subspace  $\text{Fil}_2^+ D_{\mathfrak{p}}^{\text{As}}(\Pi) \cap D_{\mathfrak{p}}^{\text{As}}(\Pi)^{(\varphi_1=\alpha_1)}$  also maps isomorphically to  $\mathbf{D}_{\text{cris}}(\mathcal{F}_1^+ V) / \mathbf{D}_{\text{cris}}(\mathcal{F}_1^+ V \cap \mathcal{F}_2^+ V)$ , and we obtain an isomorphism

$$(2.2) \quad \text{ES}_{\Pi, w_1} : \mathcal{S}_{w_1}(\Pi) \cong \mathbf{D}_{\text{cris}} \left( \frac{\mathcal{F}_1^+ V}{\mathcal{F}_1^+ V \cap \mathcal{F}_2^+ V} \right)$$

Dually, we let  $\mathcal{F}_i^+ V^* \subset V^*$  be the annihilator of  $\mathcal{F}_i^+ V$ , so we can regard  $\text{ES}_{\Pi, w_1}(\eta_{\Pi})$  as an isomorphism

$$\mathbf{D}_{\text{cris}} \left( \frac{\mathcal{F}_1^+ V^* + \mathcal{F}_2^+ V^*}{\mathcal{F}_1^+ V^*} \right) \xrightarrow{\cong} L_{\mathfrak{p}}.$$

## 3. HIDA FAMILIES

**3.1. Families of eigensystems.** Let us choose an “initial” weight  $(k_1, k_2; w)$ , with  $k_i \geq 0$ . We shall consider families of eigensystems around a neighbourhood of this weight. Let  $E$  be a finite extension of  $\mathbf{Q}_p$ .

**Notation 3.1.1.** Let  $\mathbb{T}$  denote the product of the prime-to- $\mathfrak{N}p$  Hecke algebra  $\mathbb{T}^{(\mathfrak{N}p)}$  (with  $E$ -coefficients) and the polynomial ring in formal variables  $U_{\mathfrak{p}_1}, U_{\mathfrak{p}_2}$  and their inverses.

**Definition 3.1.2.** For  $i = 1, 2$  let  $\mathcal{U}_i \subset \mathcal{W} = (\text{Spf } \Lambda)_E^{\text{rig}}$  be an open affinoid containing 0; and let

$$\kappa_{\mathcal{U}_i} : \mathbf{Z}_p^{\times} \rightarrow \mathcal{O}(\mathcal{U}_i)$$

be the universal weight. A  $p$ -adic family of eigensystems over  $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$  of weight  $(k_1 + 2\kappa_{\mathcal{U}_1}, k_2 + 2\kappa_{\mathcal{U}_2}; w)$  and level  $\mathfrak{N}$  is a homomorphism  $\mathbb{T} \rightarrow \mathcal{O}(\mathcal{U}_1 \times \mathcal{U}_2)$  with the following property:

<sup>2</sup>For us the Hodge–Tate weight of the cyclotomic character is 1.

- For every  $P = (a_1, a_2) \in \mathcal{U} \cap \mathbf{Z}^2$  with  $k_i + 2a_i \geq 0 \forall i$ , the composite homomorphism

$$\mathbb{T} \rightarrow \mathcal{O}(\mathcal{U}) \rightarrow E$$

given by evaluation at  $P$  is the Hecke eigenvalue system associated to a  $p$ -stabilisation of  $\Pi[P]$ , for some cuspidal automorphic representation  $\Pi[P]$  of  $\mathrm{GL}_2/F$  of level  $\mathfrak{N}$  and weight  $(k_1+2a, k_2+2a_2; w)$  and some embedding of its coefficient field into  $E$ .

We denote such a family by  $\underline{\Pi}$ . We say that  $\underline{\Pi}$  is  $p$ -ordinary if the Hecke eigenvalues of  $U'_{\mathfrak{p}_1}$ ,  $U_{\mathfrak{p}_2}$  and their inverses acting on  $\underline{\Pi}$  are power-bounded.

By standard results due to Hida, we see that for any given  $\Pi$  of weight  $(k_1, k_2; w)$  and conductor  $\mathfrak{N}$ , with  $k_1, k_2 \geq 0$ , ordinary at  $p$ , we may find a  $p$ -ordinary family  $\underline{\Pi}$  over some sufficiently small  $\mathcal{U}_1 \times \mathcal{U}_2$  passing through  $\Pi$  (i.e. with  $\Pi[0] = \Pi$ ), and this is unique up to further shrinking  $\mathcal{U}_1 \times \mathcal{U}_2$ .

**3.2. Interpolation sets.** Let  $\underline{\Pi}$  be a family as above. We now define certain special sets of points in  $\mathcal{U}$  and in the product  $\mathcal{U} \times \mathcal{W}$ .

**Notation 3.2.1.** We write  $\Sigma_{\mathrm{cl}}$  for the set of  $P = (a_1, a_2) \in \mathcal{U} \cap \mathbf{Z}^2$  with  $k_i + 2a_i \geq 0 \forall i$ , and we call these points classical points.

We shall also need to consider various loci in the 3-dimensional space  $\mathcal{U} \times \mathcal{W}$ :

**Notation 3.2.2.** Let  $(P, Q) \in \mathcal{U} \times \mathcal{W}$ , with  $P = (a_1, a_2) \in \Sigma_{\mathrm{cl}}$ .

- We say  $(P, Q)$  is geometric if  $Q = j$  is an integer with  $0 \leq j \leq \min(k_1 + 2a_1, k_2 + 2a_2)$ .
- We say  $(P, Q)$  is 1-dominant critical if  $Q = j + \chi$ , with  $\chi$  a finite-order character, and  $j$  an integer such that  $k_2 + 2a_2 < j \leq k_1 + 2a_1$ .

We write  $\tilde{\Sigma}_{\mathrm{geom}}$ , respectively  $\tilde{\Sigma}_{\mathrm{crit}}$ , for the sets of points  $(P, Q)$  satisfying these conditions.

*Remark 3.2.3.* Note that  $\tilde{\Sigma}_{\mathrm{crit}}$  will be the locus where our  $p$ -adic  $L$ -function interpolates critical complex  $L$ -values, and  $\tilde{\Sigma}_{\mathrm{geom}}$  the locus where it interpolates regulators of Euler-system classes. Crucially, both loci are Zariski-dense in  $\mathcal{U} \times \mathcal{W}$ .

However, while the intersection of  $\tilde{\Sigma}_{\mathrm{crit}}$  with  $\mathcal{U}_1 \times \{0\} \times \mathcal{W}$  is Zariski-dense in this smaller space, this is **not** true for  $\tilde{\Sigma}_{\mathrm{geom}}$ . This is why we need to introduce the second weight variable in order to prove our reciprocity law.  $\diamond$

#### 4. A THREE-VARIABLE $p$ -ADIC ASAI $L$ -FUNCTION

We now prove a strengthening of the main result of [GLZ23a] by constructing a 3-variable  $p$ -adic Asai  $L$ -function over  $\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{W}$ , for a Hida family over  $\mathcal{U}_1 \times \mathcal{U}_2$ .

**4.1. Geometry of the Iwahori-level Hilbert modular surface.** Let  $\mathbb{Y}_{G, \mathrm{Iw}}$  denote the canonical  $\mathbf{Z}_p$ -model of  $Y_{G, \mathrm{Iw}}$ , and write  $Y_{G, \mathrm{Iw}, 0}$  denote its special fibre. This has a stratification as a disjoint union of 9 smooth strata  $Y_{G, \mathrm{Iw}, 0}^{\diamond, \heartsuit}$ , where  $\diamond, \heartsuit \in \{m, \alpha, \acute{\mathrm{e}}\}$ . We use the notation

$$Y_{G, \mathrm{Iw}, 0}^{\bullet, m} := Y_{G, \mathrm{Iw}, 0}^{m, m} \cup Y_{G, \mathrm{Iw}, 0}^{m, \alpha} \cup Y_{G, \mathrm{Iw}, 0}^{m, \acute{\mathrm{e}}\mathrm{t}}$$

which is open in  $Y_{G, \mathrm{Iw}, 0}$ . This stratification extends naturally to  $X_{G, \mathrm{Iw}, 0}$ .

*Remark 4.1.1.* There exists a finite covering of  $\mathbb{Y}_{G, \mathrm{Iw}}$  which is a moduli space for Hilbert–Blumenthal abelian surfaces  $A$  over  $\mathbf{Z}_p$ -algebras, equipped with level groups  $C_1 \subset A[\mathfrak{p}_1]$  and  $C_2 \subset A[\mathfrak{p}_2]$  (along with a prime-to- $p$  polarisation and level structure). The strata  $Y_{G, \mathrm{Iw}, 0}^{\diamond, \heartsuit}$  correspond to the regions where  $C_1$  is (étale-locally) of type  $\diamond$  and  $C_2$  of type  $\heartsuit$ .  $\diamond$

*Note 4.1.2.* The embedding  $\iota$  gives rise to finite morphisms

$$\iota_{\mathrm{sph}} : Y_{H, \mathrm{sph}} \rightarrow Y_{G, \mathrm{sph}} \quad \text{and} \quad \iota_{\mathrm{Iw}} : Y_{H, \mathrm{Iw}} \rightarrow Y_{G, \mathrm{Iw}}. \quad \diamond$$

**4.2. Summary of higher Hida theory.** Let  $\mathcal{X}_{G,\text{Iw}}$  denote the generic fibre of  $\mathbb{X}_{G,\text{Iw}}$  as rigid analytic space. We now state the classicality isomorphism of higher Hida theory at Iwahori level generalizing [GLZ23a, Corollary 2.3.6-Corollary 2.4.1].

Recall that in *op. cit.* we developed the theory at  $\mathfrak{p}_1$ -level: we constructed Hecke operators  $U_{\mathfrak{p}_1}$  and  $U'_{\mathfrak{p}_1}$  and certain integral models of the sheaves  $\omega^{(k_1,k_2,w)}(-D_G), \omega^{(k_1,k_2,w)}$ . After tensoring with  $\mathbb{Q}_p$ , Corollary 2.3.6 (respectively Corollary 2.4.1) in *op. cit.* state that the  $U_{\mathfrak{p}_1}$ -ordinary part (the  $U'_{\mathfrak{p}_1}$ -ordinary part respectively) of the cohomology of the  $\mathfrak{p}_1$ -multiplicative locus (of the  $\mathfrak{p}_1$ -multiplicative locus with compact support towards the  $\mathfrak{p}_1$ -supersingular locus respectively) is isomorphic to the corresponding ordinary part of the cohomology of  $\mathcal{X}_{G,\mathfrak{p}_1-\text{Iw}}$  of the sheaves  $\omega^{(k_1,k_2,w)}(-D_G), \omega^{(k_1,k_2,w)}$  when  $k_1 \geq 2$  (when  $k_1 \leq 0$  respectively). The theory applies *verbatim* for  $\mathfrak{p}_2$  and we can define similarly the Hecke operators  $U_{\mathfrak{p}_2}$  and  $U'_{\mathfrak{p}_2}$ . We denote by  $e_{\mathfrak{p}_i}$  and  $e'_{\mathfrak{p}_i}$  the ordinary projectors for  $U_{\mathfrak{p}_i}$  and  $U'_{\mathfrak{p}_i}$  respectively.

For  $i = 1, 2$  let  $D^{i-\acute{e}t}$  be the Cartier divisor such that

$$X_{G,\text{Iw},0} - D^{1-\acute{e}t} = X_{G,\text{Iw},0}^{\mathfrak{m},\bullet} \quad \text{and} \quad X_{G,\text{Iw},0} - D^{2-\acute{e}t} = X_{G,\text{Iw},0}^{\bullet,\mathfrak{m}}.$$

**Definition 4.2.1.** We define the following complex

$$\begin{aligned} M_{\mathfrak{m},\mathfrak{m}}^{\bullet}(k_1, k_2) &= \text{R}\Gamma_{w1}(\mathfrak{X}_{G,\text{Iw}}^{\mathfrak{m},\mathfrak{m}}, \omega^{(-k_1,k_2+2,w)}(-D_G)) \\ &= \varprojlim_m \varinjlim_{n_2} \varprojlim_{n_1} \text{R}\Gamma(\mathbb{X}_{G,\text{Iw},R/\mathfrak{o}^m}, \omega^{(-k_1,k_2+2,w)}(-D_G - n_1 D^{1-\acute{e}t} + n_2 D^{2-\acute{e}t})), \end{aligned}$$

and let

$$H_{w1}^1(\mathfrak{X}_{G,\text{Iw}}^{\mathfrak{m},\mathfrak{m}}, \omega^{(-k_1,k_2+2,w)}(-D_G)) = H^1(M_{\mathfrak{m},\mathfrak{m}}^{\bullet}(k_1, k_2)).$$

We then obtain the following result.

**Theorem 4.2.2.** If  $k_1, k_2 \geq 0$ , then we have an isomorphism

$$e'_{\mathfrak{p}_1} e_{\mathfrak{p}_2} H_{w1}^1(\mathfrak{X}_{G,\text{Iw}}^{\mathfrak{m},\mathfrak{m}}, \omega^{(-k_1,k_2+2,w)}(-D_G)) \cong e'_{\mathfrak{p}_1} e_{\mathfrak{p}_2} H^1(X_{G,\text{Iw}}, \tilde{\omega}^{-k_1,k_2+2}(-D_G)),$$

for some coherent sheaf  $\tilde{\omega}^{(-k_1,k_2+2)}(-D_G)$  isomorphic to  $\omega^{(-k_1,k_2+2)}(-D_G)$  on the generic fibre of  $X_{G,\text{Iw}}$ .

*Proof.* The proof follows as in [GLZ23a]. In particular, one shows that the Hecke operators factor as follows:

$$\begin{array}{ccc} \text{R}\Gamma(X, \omega_{n_1}(n_2 t_2 D^{2-\acute{e}t})) & \longrightarrow & \text{R}\Gamma(X, \omega_{n_1}((n_2+1)t_2 D^{2-\acute{e}t})) \\ \downarrow U_{\mathfrak{p}_2} & \swarrow \text{---} U_{\mathfrak{p}_2} & \downarrow U_{\mathfrak{p}_2} \\ \text{R}\Gamma(X, \omega_{n_1}(n_2 t_2 D^{2-\acute{e}t})) & \longrightarrow & \text{R}\Gamma(X, \omega_{n_1}((n_2+1)t_2 D^{2-\acute{e}t})), \end{array}$$

for some  $t_2 \in \mathbb{Z}$  (see [GLZ23a, Lemma 2.3.3 and Definition 2.3.4]) and where  $\omega_{n_1} = \omega^{(-k_1,k_2+2,w)}(-D_G - n_1 D^{1-\acute{e}t})$ . And

$$\begin{array}{ccc} \text{R}\Gamma(X, \omega_{n_2}(-(n_1+1)t_1 D^{1-\acute{e}t})) & \longrightarrow & \text{R}\Gamma(X, \omega_{n_2}(-n_1 t_1 D^{1-\acute{e}t})) \\ \downarrow U'_{\mathfrak{p}_1} & \swarrow \text{---} U'_{\mathfrak{p}_1} & \downarrow U'_{\mathfrak{p}_1} \\ \text{R}\Gamma(X, \omega_{n_2}(-(n_1+1)t_1 D^{1-\acute{e}t})) & \longrightarrow & \text{R}\Gamma(X, \omega_{n_2}(-n_1 t_1 D^{1-\acute{e}t})), \end{array}$$

for some  $t_1 \in \mathbb{Z}$  and where  $\omega_{n_2} = \omega^{(-k_1,k_2+2,w)}(-D_G + n_2 D^{2-\acute{e}t})$ . Using the base change formula as in *op. cit.*, we hence deduce that the maps in the direct and inverse limits defining  $M_{\mathfrak{m},\mathfrak{m}}^{\bullet}(k_1, k_2)$  are quasi-isomorphisms, from which the theorem follows (after fixing a certain choice of twist of the sheaf depending on  $t_1, t_2$ ). Moreover note that the proof also yields the fact that the cohomology of the complex  $M_{\mathfrak{m},\mathfrak{m}}^{\bullet}(k_1, k_2)$  is independent on the order for which we take the inverse and direct limit for  $n_1, n_2$ .  $\square$

Let  $\Lambda = \Lambda(\mathbf{Z}_p^\times)$ , and denote by  $\kappa_1, \kappa_2 : \mathbf{Z}_p^\times \rightarrow (\Lambda^\times)^2$  the universal characters associated to the two factors of  $(\mathbf{Z}_p^\times)^2$ . We consider the following sheaf

$$(4.1) \quad \tilde{\Omega}^{(\kappa_1, \kappa_2)} := ((\pi_* \mathcal{O}_{\mathfrak{J}\mathfrak{G}} \otimes (\pi_{\mathfrak{J}\mathfrak{G} \times \mathfrak{J}\mathfrak{G}^\vee})_* \mathcal{O}_{\mathfrak{J}\mathfrak{G} \times \mathfrak{J}\mathfrak{G}^\vee}) \hat{\otimes} \Lambda^2)^{(\mathbf{Z}_p^\times)^6},$$

where  $\mathfrak{J}\mathfrak{G}$  denotes the Igusa tower over the (formal completion of the) multiplicative locus and each copy  $(\mathbf{Z}_p^\times)^2$  acts on  $\pi_* \mathcal{O}_{\mathfrak{J}\mathfrak{G}}$  via the action on  $\mathfrak{J}\mathfrak{G} = \mathfrak{J}\mathfrak{G}(\mathfrak{p}_1^\infty) \times \mathfrak{J}\mathfrak{G}(\mathfrak{p}_2^\infty)$  as in [Gro21, Definition 4.2.2]. Moreover the action of  $((x_{\mathfrak{p}_i})_i, (z_{\mathfrak{p}_i})_i, (t_{\mathfrak{p}_i})_i) \in (\mathbf{Z}_p^\times)^6$  is given on  $\Lambda^2$  by

$$\kappa_1(x_{\mathfrak{p}_1}^{-2} t_{\mathfrak{p}_1}) \kappa_2(x_{\mathfrak{p}_2}^2 t_{\mathfrak{p}_2}^{-1}).$$

We remark that here we do not vary the *norm variable*  $w$  as in [Gro21]. In particular, the specialisation  $\tilde{\Omega}^{(\kappa_1, \kappa_2)}[P]$  of  $\tilde{\Omega}^{(\kappa_1, \kappa_2)}$  at an integer point  $P = (a_1, a_2)$  is isomorphic to  $\omega^{(-2a_1, 2a_2; 0)}|_{\mathfrak{X}_{G, \text{Iw}}^{\text{m, m}}}$ .

*Remark 4.2.3.* Note that this sheaf of  $\Lambda$ -modules is defined over the moduli space, but it can be descended to a sheaf on the (multiplicative loci of) the Shimura variety. We have an action of  $x \in \mathcal{O}_{F, +}^\times$  on  $\tilde{\Omega}^{(\kappa_1, \kappa_2)}$  via the tautological isomorphism (since the construction of  $\tilde{\Omega}^{(\kappa_1, \kappa_2)}$  does not use the polarisation)

$$x^* \tilde{\Omega}^{(\kappa_1, \kappa_2)} \simeq \tilde{\Omega}^{(\kappa_1, \kappa_2)}.$$

This action factors through the quotient of  $\mathcal{O}_{F, +}^\times$  by  $(K \cap \mathcal{O}_F^\times)^2$  (c.f. [Gro21, 4.2.2]).  $\diamond$

Finally, for our fixed choice of weight  $(-k_1, k_2 + 2, w)$  as above, we let

$$(4.2) \quad \Omega^{(\kappa_1, \kappa_2)} := \tilde{\Omega}^{(\kappa_1, \kappa_2)} \otimes_{\mathcal{O}_{\mathfrak{X}_{G, \text{Iw}}^{\text{m, m}}}} \omega^{(-k_1, k_2 + 2, w)}.$$

The sheaf  $\Omega^{(\kappa_1, \kappa_2)}$  has the following interpolation property:

**Proposition 4.2.4.** *The sheaf  $\Omega^{(\kappa_1, \kappa_2)}$  is an invertible sheaf of  $\mathcal{O}(\mathfrak{X}_{G, \text{Iw}}^{\text{m, m}}) \otimes \Lambda^2$ -modules, and its specialisation at an integer point  $P = (a_1, a_2)$  is given by*

$$\Omega^{(\kappa_1, \kappa_2)}[P] = \omega^{(-k_1 - 2a_1, k_2 + 2a_2 + 2; w)}|_{\mathfrak{X}_{G, \text{Iw}}^{\text{m, m}}}.$$

We can now define Hecke operators  $U_{\mathfrak{p}_1}^{(\kappa_1, \kappa_2)}$  and  $U_{\mathfrak{p}_2}^{(\kappa_1, \kappa_2)}$  acting on the cohomology of the sheaf  $\Omega^{(\kappa_1, \kappa_2)}$  and specialising at an integer point  $P = (a_1, a_2)$  to  $U_{\mathfrak{p}_1}'$  and  $U_{\mathfrak{p}_2}'$  acting on the cohomology of  $\omega^{(-k_1 - 2a_1, k_2 + 2a_2 + 2; w)}$  (under the isomorphism of the above proposition). The operator  $U_{\mathfrak{p}_2}^{(\kappa_1, \kappa_2)}$  is defined as in [GLZ23a, Definition 3.2.1], namely considering the pullback induced by the universal étale  $\mathfrak{p}_2$ -isogeny (as in (3.4) of *op. cit.*) and tensoring it with the trace map for  $\mathfrak{p}_1$  and multiplying by the normalisation factor  $p^{-1 - \frac{w - k_2 - 2}{2}}$ . The operator  $U_{\mathfrak{p}_1}^{(\kappa_1, \kappa_2)}$  is defined similarly but considering the pullback by the dual  $\mathfrak{p}_1$ -isogeny (the pullback on the Igusa tower of the map considered in [GLZ23a, § 2.4]), tensoring it with the trace map for  $\mathfrak{p}_2$  and multiplying by the normalisation factor  $p^{-\frac{w - k_1}{2}}$ . The fact that they specialise to the correct Hecke operators follows as in [GLZ23a, Proposition 3.2.2].

By abuse of notation, we will use  $e'_{\mathfrak{p}_1}, e_{\mathfrak{p}_2}$  for the ordinary part with respect to  $U_{\mathfrak{p}_1}^{(\kappa_1, \kappa_2)}$  and  $U_{\mathfrak{p}_2}^{(\kappa_1, \kappa_2)}$ .

**Definition 4.2.5.** *Define*

$$M_{\text{m, m}}^\bullet(\kappa_1, \kappa_2) := e'_{\mathfrak{p}_1} e_{\mathfrak{p}_2} R\Gamma_{w_1}(\mathfrak{X}_{G, \text{Iw}}^{\text{m, m}}, \Omega^{(\kappa_1, \kappa_2)}(-D_G)).$$

The complex  $M_{\text{m, m}}^\bullet(\kappa_1, \kappa_2)$  is a perfect complex of  $\Lambda^2$ -modules, and it has the following interpolation property (c.f. [GLZ23a, Theorem 3.2.3], [Gro21, Theorem 4.2.13]):

**Proposition 4.2.6.** *For  $a_1, a_2 \in \mathbf{Z}$  with  $k_1 + 2a_1, k_2 + 2a_2 \geq 0$ , we have*

$$M_{\text{m, m}}^\bullet(\kappa_1, \kappa_2) \otimes_{\Lambda^2, a_1, a_2}^{\mathbb{L}} R \simeq e'_{\mathfrak{p}_1} e_{\mathfrak{p}_2} R\Gamma(X_{G, \text{Iw}}, \tilde{\omega}^{-k_1 - 2a_1, k_2 + 2a_2 + 2}(-D_G)).$$

We have a similar picture over  $\mathfrak{X}_{H, \Gamma_0(p)}^m$ : write  $\lambda : \mathbf{Z}_p^\times \rightarrow \Lambda^\times$  for the universal character, and let  $k$  be any weight. Then there exists a sheaf  $\Omega_H = \underline{\omega}_H(\lambda)$  of  $\mathcal{O}(\mathfrak{X}_{H, \Gamma_0(p)}^m) \otimes \Lambda$ -modules whose specialisation at an integer point  $a$  is given by

$$\Omega_H(a) = \omega^{k + 2a}.$$

Then choosing  $\lambda = \kappa_2 - \kappa_1$ , pullback along  $\iota_{\text{Iw}}$  induces a morphism

$$(4.3) \quad \iota_{\text{Iw}}^* : R\Gamma_{w_1}(\mathfrak{X}_{G, \text{Iw}}^{(m, m)}, \Omega_G(\kappa_1, \kappa_2)(-D_G)) \rightarrow R\Gamma_c(\mathfrak{X}_{H, \Gamma_0(p)}^m, \Omega_H(\kappa_2 - \kappa_1) \otimes \Omega_H^1(-D_H)) \otimes \|\cdot\|^{-w}.$$

### 4.3. $p$ -adic families.

**Proposition 4.3.1.** *Let  $\underline{\Pi}$  be a  $p$ -ordinary family over  $\mathcal{U}_1 \times \mathcal{U}_2$ , and define*

$$M_{\text{m, m}}^\bullet(\kappa_{\mathcal{U}_1}, \kappa_{\mathcal{U}_2}) := M_{\text{m, m}}^\bullet(\kappa_1, \kappa_2) \otimes_{\Lambda^2} \mathcal{O}(\mathcal{U}_1 \times \mathcal{U}_2).$$

Then  $H^1$  of this complex contains a free rank 1  $\mathcal{O}(\mathcal{U}_1 \times \mathcal{U}_2)$ -direct summand  $H_{\text{m, m}}^1(\kappa_{\mathcal{U}_1}, \kappa_{\mathcal{U}_2})(\underline{\Pi})$  whose specialisation at every  $(a_1, a_2) \in (\mathcal{U}_1 \times \mathcal{U}_2) \cap \mathbf{Z}^2$  with  $k_i + 2a_i \geq 0$  is the eigenspace in

$$e'_{\mathfrak{p}_1} e_{\mathfrak{p}_2} H^1(X_{G, \text{Iw}}, \tilde{\omega}^{(-k_1 - 2a_1, k_2 + 2a_2 + 2)}(-D_G)) \otimes_{\mathbf{Z}_p} E$$

on which  $\mathbb{T}$  acts via the ordinary  $p$ -stabilisation of  $\Pi[a]$ .

**Definition 4.3.2.** Write

$$\mathcal{S}(\underline{\Pi}) = H_{m,m}^1(\kappa_{\mathcal{U}_1}, \kappa_{\mathcal{U}_2})[\underline{\Pi}],$$

and let  $\underline{\eta}$  be a basis of  $\mathcal{S}(\underline{\Pi})$  as a  $\mathcal{O}(\mathcal{U}_1 \times \mathcal{U}_2)$ -module.

*Note 4.3.3.* The specialisation of  $\underline{\eta}$  at  $P = (a_1, a_2)$  is a basis of the space  $\mathcal{S}_{w_1, \text{Iw}}(\Pi[P]) \otimes_{L_P} E$ , which can be identified with  $\mathcal{S}_{w_1}(\Pi[P]) \otimes_{L_P} E$  via (2.1), where  $L_P \subset \overline{\mathbf{Q}}$  is the coefficient field of  $\Pi[P]$ . However, it may not descend to a basis of this space over  $L_P$ . If we choose an algebraic basis  $\eta_{\Pi[P]}$  defined over  $L_P$  (see Remark 2.4.4), then this defines<sup>3</sup> a pair of periods

$$\Omega_p(\Pi[P]) \in E^\times \quad \text{and} \quad \Omega_\infty(\Pi[P]) \in \mathbf{C}^\times. \quad \diamond$$

**4.4. The  $p$ -adic  $L$ -function.** Consider the dual interpolation sheaf  $\Omega_H^\vee(\lambda)$ , whose specialisation at an integer  $a$  specializes to  $\omega_H^{-k-2a}$  with  $k = k_2 - k_1$ . We then have a cup product pairing

$$\langle \ , \ \rangle : H^0(\mathfrak{X}_{H, \Gamma_0(p)}^m, \Omega_H^\vee(\kappa_{\mathcal{U}_1} - \kappa_{\mathcal{U}_2})) \times H_c^1(\mathfrak{X}_{H, \Gamma_0(p)}^m, \Omega_H(\kappa_{\mathcal{U}_2} - \kappa_{\mathcal{U}_1}) \otimes \Omega_H^1(-D_H)) \rightarrow \mathcal{O}(U).$$

Write  $\mathbf{j} : \mathbf{Z}_p^\times \rightarrow \mathcal{O}(W)^\times$  for the universal character. We consider Katz's Eisenstein measure

$$\mathcal{E} \left( \frac{k_1 - k_2}{2} + \kappa_{\mathcal{U}_1} - \kappa_{\mathcal{U}_2} - \mathbf{j}, \frac{k_1 - k_2}{2} + \kappa_{\mathcal{U}_1} - \kappa_{\mathcal{U}_2} + \mathbf{j} \right) \in H^0(\mathfrak{X}_{H, \Gamma_0(p)}^m, \omega_H^\vee(\kappa_{\mathcal{U}_2} - \kappa_{\mathcal{U}_1})) \hat{\otimes} \mathcal{O}(W);$$

to lighten the notation, we denote it by  $\mathcal{E}_{\kappa_{\mathcal{U}_1} - \kappa_{\mathcal{U}_2}}(\mathbf{j})$ .

**Definition 4.4.1.** We define the  $p$ -adic Asai  $L$ -function by the pairing

$$\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})(\kappa_1, \kappa_2, \mathbf{j}) = \frac{1}{p} (\sqrt{D})^{-1 - (\kappa_1 + \kappa_2)/2 - \mathbf{j}} (-1)^{\mathbf{j}} \cdot \langle \mathcal{E}_{\kappa_{\mathcal{U}_1} - \kappa_{\mathcal{U}_2}}(\mathbf{j}), \iota_{\text{Iw}}^*(\underline{\eta}) \rangle \in \mathcal{O}(U \times W),$$

where  $U = \mathcal{U}_1 \times \mathcal{U}_2$ .

**4.5. Interpolation of critical values.** This function  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})$  satisfies the following interpolation property at  $\mathfrak{p}_1$ -dominant critical points:

**Theorem 4.5.1.** Let  $(P, Q) \in \tilde{\Sigma}_{\text{crit}}$ , with  $Q = j \in \mathbf{Z}$ . Then we have

$$\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})(P, j+1) = \frac{\Omega_p(\Pi[P])}{\Omega_\infty(\Pi[P])} \cdot \mathcal{E}(\Pi[P], \tau) \cdot \frac{\Gamma(j-1)\Gamma(j-k_2-2a_2)}{2^{k_1+2a_1-2} i^{1-k_2-2a_2} (-2\pi i)^{2j-k_2-2a_2-1}} \cdot \mathcal{L}_{\text{As}}^{\text{imp}}(\Pi[P], j+1),$$

where

$$\mathcal{E}(\Pi[P], s) = \left( 1 - \frac{p^{s-1}}{\alpha_{P,1}^\circ \alpha_{P,2}^\circ} \right) \left( 1 - \frac{p^{s-1}}{\alpha_{P,1}^\circ \beta_{P,2}^\circ} \right) \left( 1 - \frac{\beta_{P,1}^\circ \alpha_{P,2}^\circ}{p^s} \right) \left( 1 - \frac{\beta_{P,1}^\circ \beta_{P,2}^\circ}{p^s} \right)$$

$$\text{and } \tau = j - \frac{k_1 + 2a_1 + k_2 + 2a_2}{2}.$$

*Proof.* Since the cup-product defining  $\mathcal{L}_{p, \text{As}}(\underline{\Pi}, \underline{\eta})$  commutes with specialisation, and the specialisation of the Eisenstein series at this point is a classical modular form, the value at  $(P, j+1)$  can be written as a coherent-cohomology cup-product and hence as a zeta-integral. This is a very similar computation as in the proof of Theorem C in [GLZ23a]; the only difference is that we use a slightly different test vector in the local zeta-integral computation at  $p$ , since our local Whittaker function at  $\mathfrak{p}_2$  is now the normalised  $U_{\mathfrak{p}_2}$ -eigenfunction, rather than the spherical Whittaker function. However, this makes no difference to the zeta-integral computation (which only sees the value of the Whittaker function at 1).  $\square$

Comparing with the main theorem of [GLZ23a], which gives a 2-variable  $p$ -adic  $L$ -function varying analytically over the two-dimensional ‘‘slice’’  $\mathcal{U}_1 \times \{a_2\} \times \mathcal{W}$ , for any given  $a_2$  with  $k_2 + 2a_2 \geq 0$ , and noting that the intersection of  $\tilde{\Sigma}_{\text{crit}}$  with this set is Zariski-dense, we conclude that the restriction of  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})$  to this slice must coincide with the 2-variable  $p$ -adic  $L$ -function of *op.cit.*. In particular, its restriction to  $\{a_1\} \times \{a_2\} \times \mathcal{W}$ , for any  $P = (a_1, a_2) \in \Sigma_{\text{cl}}$  with  $k_1 + 2a_1 > k_2 + 2a_2 \geq 0$ , coincides with the 1-parameter  $p$ -adic  $L$ -function  $\mathcal{L}_{\text{As}, p}^{\text{imp}}(\Pi[P])$  from Theorem 2.5.1 for the specialisation  $\Pi[P]$ .

<sup>3</sup>c.f. §7.4 in [GLZ23a]

5.1. **Étale cohomology and Euler systems in families.** We recall some results due to Sheth [She24] on families of Galois representations associated to  $\underline{\Pi}$ , and families of Galois cohomology classes for these.

**Theorem 5.1.1** (Sheth). *For small enough  $\mathcal{U}$ , there exists a free rank 4 module  $\mathbf{V} = V(\underline{\Pi})$  over  $\mathcal{O}(\mathcal{U}_1 \times \mathcal{U}_2)$ , whose specialisation at each classical point  $P \in \Sigma_{\text{cl}}$  is canonically isomorphic to the Asai representation  $V_{p,\text{As}}(\Pi[P])$ .*

Let  $c$  be an integer coprime to  $6p\text{Nm}_{\mathbf{Q}}^F \mathfrak{N}$ ,  $a$  a generator of  $O_F/\mathbf{Z}$ , and suppose all prime factors of  $c$  lie in our set  $S$ .

**Theorem 5.1.2** (Sheth). *There exists a family of cohomology classes  $z_m(\underline{\Pi}, \mathbf{j}) \in H^1(\mathbf{Q}(\mu_m), \mathbf{V}^*(-\mathbf{j}))$  for all square-free integers coprime to  $pS$ , which has the following interpolation property at each point  $(P, j) \in \tilde{\Sigma}_{\text{geom}}$ :*

$${}_c z_m(\underline{\Pi}, \mathbf{j})(P, Q) = \frac{\left(1 - \frac{p^j \sigma_p}{\alpha_{P,1} \alpha_{P,2}}\right) \left(1 - \frac{\alpha_{P,1} \beta_{P,2}}{p^{1+j} \sigma_p}\right) \left(1 - \frac{\beta_{P,1} \alpha_{P,2}}{p^{1+j} \sigma_p}\right) \left(1 - \frac{\beta_{P,1} \beta_{P,2}}{p^{1+j} \sigma_p}\right)}{(-1)^j j! \binom{k_1+2a_1}{j} \binom{k_2+2a_2}{j}} \times {}_c z_m(\Pi[P], j).$$

Here  ${}_c z_m(\Pi[P], j)$  is the Asai–Flach class in  $H^1(\mathbf{Q}(\mu_m), V_{p,\text{As}}(\Pi[P])^*(-j))$ , as constructed in [LLZ18], and  $\sigma_p$  is the arithmetic Frobenius at  $p$ .

5.2. **Localisation at  $p$ .** By comparison with the *standard* representation in families, we see that (after possibly shrinking  $\mathcal{U}$ ) there are free rank 2 subrepresentations  $\mathcal{F}_i^+ \mathbf{V}$  whose specialisations at all points  $P \in \Sigma_{\text{cl}}$  of *non-parallel weight* interpolate the subrepresentations  $\mathcal{F}_i^+ V_P$  described above.<sup>4</sup>

**Proposition 5.2.1.** *The localisation of  ${}_c z_m(\underline{\Pi}, \mathbf{j})$  at  $p$  takes values in the rank 3 subrepresentation  $\mathcal{F}_1^+ \mathbf{V}^* + \mathcal{F}_2^+ \mathbf{V}^*$ .*

*Proof.* The analogous result for the specialisations at classical points is Corollary 9.2.3 of [LLZ18], and the result for the family  $\underline{\Pi}$  follows from this by interpolation.  $\square$

We now consider the image of  ${}_c z_m(\underline{\Pi}, \mathbf{j})$  in the rank 1 quotient  $\mathbf{M}_{\mathfrak{p}_1} = (\mathcal{F}_1^+ \mathbf{V}^* + \mathcal{F}_2^+ \mathbf{V}^*)/\mathcal{F}_1^+ \mathbf{V}^*$ . Note that  $\mathbf{M}_{\mathfrak{p}_1}(-k_2 - 2\kappa_2 - 1)$  is unramified, so we can define a Perrin–Riou regulator for  $\mathbf{M}_{\mathfrak{p}_1}$  by pulling back the regulator for its unramified twist (compare [KLZ17, Theorem 8.2.8]), giving a map of  $\mathcal{O}(\mathcal{U} \times \mathcal{W})$ -modules

$$\mathcal{L}_{\text{PR}} : H^1(\mathbf{Q}_p, \mathbf{M}_{\mathfrak{p}_1}(-\mathbf{j})) \rightarrow \mathbf{D}_{\text{cris}}(\mathbf{M}_{\mathfrak{p}_1}(-\mathbf{j})).$$

Comparing the interpolating property of Perrin–Riou’s map with the preceding theorem (as in the Rankin–Selberg case, see [KLZ17, proof of Theorem 10.2.2]), we obtain:

**Proposition 5.2.2.** *For  $(P, Q) \in \tilde{\Sigma}_{\text{geom}}$  we have*

$$\mathcal{L}_{\text{PR}}({}_c z_1(\underline{\Pi}, \mathbf{j})) \bmod (P, Q) = \frac{(-1)^{k_2-j}}{(k_2 + 2a_2 - j)!} \cdot \frac{\left(1 - \frac{p^j}{\alpha_{P,1} \beta_{P,2}}\right)}{\left(1 - \frac{\alpha_{P,1} \beta_{P,2}}{p^{j+1}}\right)} \cdot \log({}_c z_1(\Pi[P], j)).$$

The following theorem is the main result from [GLZ23b], reformulated in terms of the Eichler–Shimura isomorphism  $\text{ES}_{\Pi[P], w_1}$  (see (2.2)):

**Theorem 5.2.3.** *For each  $(P, Q) \in \tilde{\Sigma}_{\text{geom}}$ , say  $P = (a_1, a_2)$  and  $Q = j$ , such that  $\alpha_{1,P} \beta_{2,P} \neq \beta_{1,P} \alpha_{2,P}$ , we have*

$$\langle \mathcal{L}_{\text{PR}}({}_c z_1(\underline{\Pi}[P], j)), \text{ES}_{\Pi[P], w_1}(\eta_P) \rangle = (c^2 - c^{-t} \chi(c)^{-1}) \cdot \sqrt{D}^{j+1} (-1)^{(k_1-k_2)/2+j+1} N^t \cdot \mathcal{L}_{p,\text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})(P, j+1).$$

Here,  $t = k_1 + k_2 + 2(a_1 + a_2) - 2j$ , and  $\chi$  is the restriction to  $\mathbf{Q}$  of the nebentype character of  $\underline{\Pi}$ .

<sup>4</sup>More generally, this applies to any specialisation with  $\alpha_{P,1} \beta_{P,2} \neq \beta_{P,1} \alpha_{P,2}$ , which is automatic in non-parallel weights. If these two quantities are equal, then the subrepresentations  $\mathcal{F}_1^+ V_P$  and  $\mathcal{F}_2^+ V_P$  are isomorphic to each other as  $G_{\mathbf{Q}_p}$ -representations, and cannot be distinguished without using the plectic structures; so it is not immediately clear which of these subrepresentations is the one given by specialising the (uniquely defined) “family” subrepresentations  $\mathcal{F}_i^+ \mathbf{V}$ .

### 5.3. Explicit reciprocity laws and meromorphic Eichler–Shimura.

**Definition 5.3.1.** Denote by  $\Sigma_{\text{cl}}^{\sharp} \subset \Sigma_{\text{cl}}$  the set of  $P$  with the following property: there exists a  $j$  such that  $(P, j) \in \widetilde{\Sigma}_{\text{geom}}$  and  $\mathcal{L}_p^{\text{imp}}(\Pi[P], \eta_P; j+1) \neq 0$ . (We call these good classical points.)

**Lemma 5.3.2.** The set  $\Sigma_{\text{cl}}^{\sharp}$  is Zariski-dense in  $\mathcal{U}$ .

*Proof.* Consider a function  $f$  vanishing at all these points. Then  $f \cdot \mathcal{L}_p^{\text{imp}}(\underline{\Pi}, \underline{\eta})$  must vanish on  $\widetilde{\Sigma}_{\text{geom}}$ . As  $\widetilde{\Sigma}_{\text{geom}}$  is Zariski-dense, it is identically 0 on  $\mathcal{U} \times \mathcal{W}$ . But  $\mathcal{L}_p^{\text{imp}}(\underline{\Pi}, \underline{\eta})$  is not a zero-divisor, since every component of  $\mathcal{U} \times \mathcal{W}$  contains a point where  $\mathcal{L}_p(\underline{\Pi}, \underline{\eta})$  interpolates a non-vanishing complex  $L$ -value. Thus  $f = 0$ .  $\square$

Write  $\mathcal{Q}(\mathcal{U})$  for the fraction field of  $\mathcal{O}(\mathcal{U})$ ; and let us write  ${}_c\mathcal{L}_{p, \text{As}}(\underline{\Pi})$  for the  $p$ -adic  $L$ -function multiplied by the  $c$ -factor  $(c^2 - \dots)$  (which clearly interpolates to an element of  $\mathcal{O}(\mathcal{U} \times \mathcal{W})$ ).

#### Theorem 5.3.3.

(1) There exists an isomorphism of  $\mathcal{Q}(U)$ -modules

$$\text{ES}_{\underline{\Pi}}^1 : \mathcal{S}^1(\underline{\Pi}) \otimes_{\mathcal{O}(U)} \mathcal{Q}(U) \cong \mathbf{D}_{\text{cris}}(\mathbf{M}_{\mathfrak{p}_1}) \otimes_{\mathcal{O}(U)} \mathcal{Q}(U),$$

depending only on  $\underline{\Pi}$ , which satisfies the following property: for all  $P \in \Sigma_{\text{cl}}^{\sharp}$ , the specialisation of the morphism  $\text{ES}_{\underline{\Pi}}^1$  at  $P$  is well-defined and coincides with the Eichler–Shimura isomorphism  $\text{ES}_{\Pi[P]}^1$ .

(2) Extending  $\text{ES}_{\underline{\Pi}}^1$  to an isomorphism of  $\mathcal{O}(U \times \mathcal{W})$ -modules, we have an explicit reciprocity law

$$\left\langle \mathcal{L}_{\text{PR}}(z_1(\underline{\pi}, \mathbf{j})), \text{ES}_{\underline{\Pi}}^1(\underline{\eta}) \right\rangle = {}_c\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})(1 + \mathbf{j}).$$

*Proof.* Let  $j : \mathcal{S}_{\underline{\Pi}}^1 \rightarrow \mathbf{M}_{\mathfrak{p}_1}$  be some isomorphism. Since  $\mathcal{L}_p(\underline{\Pi}, \underline{\eta})$  is not identically zero, we can consider the ratio

$$R = \frac{1}{{}_c\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})(1 + \mathbf{j})} \cdot \left\langle \mathcal{L}_{\text{PR}}(z_1(\underline{\pi}, \mathbf{j})), j(\underline{\eta}) \right\rangle \in \mathcal{O}(U \times \mathcal{W}).$$

Let  $(P, Q)$  be an geometric point such that  $\alpha_{P,1}\beta_{P,2} \neq \beta_{P,1}\alpha_{P,2}$  and  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})(P, Q) \neq 0$ . Then it follows from Theorem 5.2.3 that  $R$  is regular at  $(P, Q)$ , and its value is equal to  $j_P/\text{ES}_{\Pi_P}^1$ . In particular, this ratio depends only on  $P$ , and is independent of  $Q$ . From the Zariski-density of the set of such  $(P, Q)$ , it follows that the meromorphic function  $R$  on  $\mathcal{U} \times \mathcal{W}$  is actually independent of the  $\mathcal{W}$  variable.

Hence  $R \in \mathcal{Q}(U)^\times$ , regular at all points  $P \in \Sigma_{\text{cl}}^{\sharp}$ , and coinciding at each such point with  $j_P/\text{ES}_{\Pi_P}^1$ . It follows that if we define  $\text{ES}_{\underline{\Pi}}^1 = R^{-1}j$ , then  $\text{ES}_{\underline{\Pi}}^1$  is regular at all points in  $\Sigma_{\text{cl}}^{\sharp}$  and coincides at such points with  $\text{ES}_{\Pi[P]}^1$ . By Lemma 5.3.2, this interpolating property uniquely determines  $\text{ES}_{\underline{\Pi}}^1$ , and the reciprocity law holds by construction.  $\square$

It follows, in particular, that if  $(0, 0) \in \Sigma_{\text{cl}}^{\sharp}$  (i.e.  $\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})$  is non-zero for some  $j$  in the geometric range), and  $\alpha_1\beta_2 \neq \beta_1\alpha_2$ , then we have an equality in  $\Lambda(\Gamma)$

$$\left\langle \mathcal{L}_{\text{PR}}(z_1(\underline{\Pi}, \mathbf{j})), \text{ES}_{\underline{\Pi}}^1(\underline{\eta}) \right\rangle = {}_c\mathcal{L}_{p, \text{As}}^{\text{imp}}(\underline{\Pi}, \underline{\eta})$$

for any basis  $\eta$  of  $\mathcal{S}_{w_1}(\underline{\Pi})$ . However, if  $(0, 0) \notin \Sigma_{\text{cl}}^{\sharp}$  then we cannot rule out the possibility that  $\mathcal{L}_{\text{PR}}(z_1(\underline{\Pi}, \mathbf{j}))$  is identically 0.

## 6. LEADING-TERM ARGUMENTS AND APPLICATIONS

**6.1. Assumptions.** Let  $\Pi$  be an automorphic representation of  $G(\mathbf{A})$  as in Definition 2.2.1. We impose the following assumptions on  $\Pi$ :

#### Assumptions 6.1.1.

- (1)  $\Pi$  is ordinary at  $p$ ;
- (2) the local Euler factor  $P_p(\Pi, X)$  does not vanish at  $p^{-j}$  for any  $j \in \mathbf{Z}$ ;
- (3)  $\Pi$  satisfies the “big image” condition (BI) of [LLZ18, §9.4], as do all its twists  $\Pi \otimes \psi$  for Dirichlet characters  $\psi$  of  $p$ -power order and conductor coprime to some finite set of primes  $S$ .

As noted in *op.cit.*, if  $\Pi$  is non-CM, and there is at least one prime which ramifies in  $F$  and is coprime to  $\mathcal{N}_{\Pi}$ , then  $\Pi$  satisfies condition (BI) for a positive-density set of primes  $\mathfrak{P}$  of  $L$  (and this can be upgraded to “all but finitely many primes” if  $L = \mathbf{Q}$ ). The argument easily generalises to show that the twists  $\Pi \otimes \psi$  have big image as well, after discarding a finite set of possible primes from the conductor

of  $\psi$  (the primes which ramify in  $F'/\mathbf{Q}$ , where  $F'$  is a finite abelian extension of  $F$  determined by the inner twists of  $\Pi$ , see the proof of Theorem 9.4.6 of *op.cit.*).

**6.2. Leading terms.** Let  $\mathcal{U}' = \mathcal{U}_1 \cap \mathcal{U}_2$ , embedded diagonally in  $\mathcal{U}_1 \times \mathcal{U}_2$ . By restricting  $\underline{\Pi}$ , we obtain a one-parameter family of Hilbert modular forms with weight  $(k_1 + 2a, k_2 + 2a; w)$  for varying  $a \in \mathcal{U}'$ . Note that the restriction of  $\mathcal{L}_{p, \text{As}}(\underline{\Pi}, \underline{\eta})$  to  $\mathcal{U}'$  is well-defined, and non-vanishing at 0.

Let  $T$  be a uniformiser at  $0 \in \mathcal{U}'$ . From the meromorphic reciprocity law and the non-vanishing of the  $p$ -adic  $L$ -function modulo  $T$ , we deduce that the  $T$ -adic valuation of  $\mathcal{L}_{\text{PR}}(z_1(\underline{\pi}, \mathbf{j}))|_{\mathcal{U}' \times \mathcal{W}}$  is equal to the order of the pole of  $\text{ES}_{\underline{\Pi}}^1(\underline{\eta})$ . Let  $c \in \mathbf{Z}_{\geq 0}$  be this common value.

Repeating the construction with the Eisenstein series replaced by a Dirichlet-character twist (but the same  $\underline{\Pi}$  and  $\underline{\eta}$ ), we conclude that in fact  $\mathcal{L}_{\text{PR}}(z_m(\underline{\pi}, \mathbf{j}))|_{\mathcal{U}' \times \mathcal{W}}$  is divisible by  $T^c$ , for all  $m$ . So not only the  $m = 1$  class of the Euler system, but in fact all of the classes, have vanishing projection to the 1-dimensional local subquotient at  $p$  used to define the regulator map. So specialising at  $T = 0$  gives an Euler system whose local condition is “too strong”: it has core rank 0 in the sense of [MR04]. Arguing as in [LZ20], we conclude that in fact the specialisation at  $T = 0$  must vanish identically, for all  $m$ ; and we can therefore divide out factors of  $T$  repeatedly, and compensate by removing powers of  $T$  from the denominator of  $\text{ES}_{\underline{\Pi}}^1$ , until we have  $c = 0$ .

**Corollary 6.2.1.** *There exists an Euler system  $(\hat{\mathbf{z}}_m)$  for  $V_{p, \text{As}}(\Pi)^*$  such that the image of  $\hat{\mathbf{z}}_1$  under the Perrin-Riou regulator is a scalar multiple of  ${}_c\mathcal{L}_{p, \text{As}}(\Pi, \eta)(1 + \mathbf{j})$ .*

*Proof.* The above construction gives an Euler system mapping to a multiple of the imprimitive  $L$ -series  ${}_c\mathcal{L}_{p, \text{As}}^{\text{imp}}(\Pi, \eta)(1 + \mathbf{j})$ . However, both the Euler system and the  $p$ -adic  $L$ -function depend on a choice of test data at the bad primes for  $\Pi$ , and both have the same equivariance property when the test data changes. Since the space of linear forms with this equivariance property is 1-dimensional, we obtain the reciprocity law for any choice of test data, including the (inexplicit) test data which give the optimal  $p$ -adic  $L$ -function.  $\square$

**6.3. Arithmetic applications.** Let  $T \subset V_{p, \text{As}}(\Pi)$  be a  $G_{\mathbf{Q}}$ -invariant lattice, and define  $A = T \otimes_{\mathbf{Q}_p} \mathbf{Q}_p$ . Let  $\mathcal{F}_1^+ A$  be the corank 2  $R$ -submodule of  $A$  dual to the kernel  $\mathcal{F}_1^+ V_{p, \text{As}}(\Pi)^* \rightarrow M_{\mathfrak{p}_1}$ .

We can now harvest some applications to the Iwasawa Main Conjecture, following [LLZ18, §9].

**Definition 6.3.1.** *Define the Selmer group*

$$\text{Sel}^{(\mathfrak{p}_1)}(\mathbf{Q}(\mu_{p^\infty}), A) = \left\{ \begin{array}{l} x \in H^1(\mathbf{Q}(\mu_{p^\infty}), A) : \text{loc}_\ell(x) = 0 \ \forall \ell \neq p \\ \text{loc}_p(x) \in \text{image } H^1(\mathbf{Q}_p(\mu_{p^\infty}), \mathcal{F}_1^+ A) \end{array} \right\},$$

and write  $X^{(\mathfrak{p}_1)}(\mathbf{Q}(\mu_{p^\infty}), A)$  for its Pontryagin dual.

Let  $\Gamma = \text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ , and let  $\Lambda = R \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma]]$ , where  $R$  is the ring of integers of  $L_{\mathfrak{F}}$ .

**Theorem 6.3.2.** *Assume that Assumptions 6.1.1 are satisfied. There exists an integer  $r$  such that*

$$\text{char}_\Lambda \left( X^{(\mathfrak{p}_1)}(\mathbf{Q}(\mu_{p^\infty}), A) \right) \Big| p^r \cdot \mathcal{L}_{p, \text{As}}(\Pi, 1 + \mathbf{j}).$$

*Proof.* It is immediate from Theorem 9.5.3 in *op.cit.* and Theorem 5.3.3 that the characteristic ideal of the Selmer group divides  $p^r {}_c\mathcal{L}_{p, \text{As}}(\Pi, \eta)(1 + \mathbf{j})$ , for some  $r$ . It remains to get rid of the factor  $(c^2 - c^{2\mathbf{j} - k_1 - k_2} \chi(c)^{-1})$  relating the  $p$ -adic  $L$ -functions with and without the  $c$ -subscript.

If the Dirichlet character  $\chi$  is non-trivial, then we can choose  $c$  such that the  $c$ -factor is a unit in the Iwasawa algebra. If  $\chi = 1$ , then we use the compatibility of the Iwasawa main conjecture with functional equations. The functional equation interchanges  $\mathbf{j}$  with  $k_1 + k_2 + 1 - \mathbf{j}$ , so the  $c$ -factor  $c^2 - c^{2\mathbf{j} - k_1 - k_2}$  is coprime to its image under this involution.  $\square$

*Remark 6.3.3.* The arbitrary power of  $p$  is inevitable since  $\mathcal{L}_{p, \text{As}}(\Pi)$  is itself only well-defined up to an arbitrary scalar factor. It should be possible to nail down a canonical normalisation up to  $p$ -adic units using integrality of equivariant  $p$ -adic  $L$ -functions as in [LZ20], but we have not attempted this here for reasons of space.  $\diamond$

By descent from  $\mathbf{Q}(\mu_{p^\infty})$  to  $\mathbf{Q}$ , we also obtain cases of the Bloch–Kato conjecture for critical values of the  $L$ -series.

**Theorem 6.3.4.** *Assume that the Assumptions 6.1.1 are satisfied. Let  $j$  be a critical value. Then the Bloch–Kato Selmer group*

$$H_f^1(\mathbf{Q}, V_{p, \text{As}}(\Pi)^*(-j))$$

*is zero.*

*Proof.* This follows from the previous theorem via the same arguments as in §11.7 of [KLZ17] in the Rankin–Selberg case, using the fact that the critical  $L$ -values are automatically non-zero.  $\square$

#### REFERENCES

- [AIS15] F. ANDREATTA, A. IOVITA, and G. STEVENS, *Overconvergent Eichler–Shimura isomorphisms*, J. Inst. Math. Jussieu **14** (2015), no. 2, 221–274. MR 3315057  $\uparrow$  2
- [Gro21] G. GROSSI, *Higher Hida theory for Hilbert modular varieties in the totally split case*, preprint, 2021, arXiv:2106.05666.  $\uparrow$  7, 8
- [GLZ23a] G. GROSSI, D. LOEFFLER, and S.L. ZERBES,  *$p$ -adic Asai  $L$ -functions for quadratic Hilbert eigenforms*, preprint, 2023, arXiv:2307.07004.  $\uparrow$  2, 4, 6, 7, 8, 9
- [GLZ23b] ———, *Asai–Flach classes and  $p$ -adic  $L$ -functions*, 2023, arXiv:2309.07536.  $\uparrow$  2, 5, 10
- [KLZ17] G. KINGS, D. LOEFFLER, and S.L. ZERBES, *Rankin–Eisenstein classes and explicit reciprocity laws*, Cambridge J. Math. **5** (2017), no. 1, 1–122. MR 3637653  $\uparrow$  1, 10, 13
- [LLZ18] A. LEI, D. LOEFFLER, and S.L. ZERBES, *Euler systems for Hilbert modular surfaces*, Forum Math. Sigma **6** (2018), no. e23. MR 3876763  $\uparrow$  2, 4, 5, 10, 11, 12
- [LZ20] D. LOEFFLER and S.L. ZERBES, *Iwasawa theory for quadratic Hilbert modular forms*, preprint, 2020, arXiv:2006.14491.  $\uparrow$  1, 2, 12
- [LZ23] ———, *Plectic structures in  $p$ -adic de Rham cohomology*, J. Number Theory (2023), published online 2023, print version to appear.  $\uparrow$  5
- [MR04] B. MAZUR and K. RUBIN, *Kolyvagin systems*, Mem. Amer. Math. Soc. **168** (2004), no. 799. MR 2031496  $\uparrow$  12
- [NS16] J. NEKOVÁŘ and A.J. SCHOLL, *Introduction to plectic cohomology*, Advances in the theory of automorphic forms and their  $L$ -functions, Contemp. Math., vol. 664, Amer. Math. Soc., Providence, RI, 2016, pp. 321–337. MR 3502988  $\uparrow$  5
- [Oht00] M. OHTA, *Ordinary  $p$ -adic étale cohomology groups attached to towers of elliptic modular curves. II*, Math. Ann. **318** (2000), no. 3, 557–583. MR 1800769  $\uparrow$  2
- [She24] A. SHETH, *The Asai–Flach Euler system in  $p$ -adic families*, in preparation, 2024.  $\uparrow$  2, 10
- [Shi86] G. SHIMURA, *On a class of nearly holomorphic automorphic forms*, Ann. of Math. (2) **123** (1986), no. 2, 347–406. MR 835767  $\uparrow$  3

(G. Grossi) CNRS, INSTITUT GALILÉE, UNIVERSITÉ SORBONNE PARIS NORD, 93430 VILLETANEUSE, FRANCE  
*Email address:* grossi@math.univ-paris13.fr  
*ORCID:* 0000-0002-0883-4494

(D. Loeffler) FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIDISTANCE SUISSE, SCHINERSTRASSE 18, 3900 BRIG, SWITZERLAND  
*Email address:* david.loeffler@unidistance.ch  
*ORCID:* 0000-0001-9069-1877

(S.L. Zerbes) DEPARTMENT OF MATHEMATICS, ETH ZÜRICH, RÄMISTRASSE 101, 8092 ZÜRICH, SWITZERLAND  
*Email address:* sarah.zerbes@math.ethz.ch  
*ORCID:* 0000-0001-8650-9622