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ASAI–FLACH CLASSES AND p -ADIC L-FUNCTIONS

GIADA GROSSI, DAVID LOEFFLER, AND SARAH LIVIA ZERBES

ABSTRACT. We prove a formula for the Bloch–Kato logarithm of the bottom class in the Asai–Flach Euler system associated to a quadratic Hilbert modular form. We show that this can be expressed as a value, outside the interpolation range, of the p -adic Asai L -function constructed in the prequel paper [GLZ23].

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1. INTRODUCTION

The theory of Euler systems – norm-compatible families of cohomology classes for global Galois representations – has played an important role in many discoveries in number theory. A key aspect of this theory is *explicit reciprocity laws*, which relate Euler systems to the special values of L -functions; this can be used as a criterion for the non-vanishing of Euler systems, a crucial input in applications to the Bloch–Kato conjecture and related problems.

One strategy for proving such explicit reciprocity laws, introduced in the seminal paper [BDP13] in the context of Heegner points, proceeds as follows. Supposing we already have an Euler system for some Galois representation V , we aim to carry out the following steps:

- (1) Construct a p -adic L -function, for some p -adic family of Galois representations specialising to V , whose values in a suitable interpolation range are critical values of the corresponding complex L -functions.
- (2) Relate values of this p -adic L -function at suitable points *outside* the range of interpolation to Bloch–Kato logarithms of Euler system classes.
- (3) Via analytic continuation, deduce a relation between Euler systems and p -adic L -values at points *inside* the interpolation range, where the p -adic L -function specialises to the complex L -value.

In the present paper we focus on the *Euler system of Asai–Flach classes*, introduced in [LLZ18], which is an Euler system for the Asai, or twisted tensor, Galois representation associated to a Hilbert modular form for a real quadratic field. In this setting, step (1) of the above strategy was carried out in the prequel paper [GLZ23]. The goal of the present paper is to carry out step (2): the main result (Theorem 7.7.6) expresses the logarithms of Euler system classes as non-critical values of the p -adic L -function constructed in *op.cit.*. The remaining step (3), with applications to the Bloch–Kato conjecture and the Iwasawa main conjecture for Asai L -functions, will be carried out in a third paper in the series, currently in preparation.

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This strategy was successfully carried out for the Euler system of Beilinson–Flach elements (attached to a Rankin–Selberg convolution of modular forms) in [BDR15, KLZ17]; and more recently in the GSp_4 setting in [LPSZ21, LZ20]. The Asai representation can be seen as a “twisted version” of the Rankin–Selberg convolution, and the construction of the Euler system classes is recognisably a direct generalisation of the Beilinson–Flach case; but the techniques used to construct p -adic L -functions and prove the reciprocity law for Rankin–Selberg convolutions do not straightforwardly generalise, since they rely on the decomposition of the Shimura variety for $\mathrm{GL}_2 \times \mathrm{GL}_2$ as a product of two factors, and new techniques are required. Hence the present paper follows a strategy rather closer to the GSp_4 case, relying on the use of *higher Hida theory*, which has been developed for Hilbert modular varieties at split primes by the first author [Gro21]; and its “overconvergent” version, *higher Coleman theory*. The higher Coleman theory we use is based on the general results of [BP20], with some slight modifications to allow p -adic interpolation at only one of the primes above p , for which we refer to [LZ22].

(We remark that a related problem is also considered in the paper [LSZ20], where Skinner and two of the present authors computed a pairing which also involves the Bloch–Kato logarithm of the Asai–Flach class, but projected into a different piece of the de Rham cohomology which can be computed using only conventional, rather than higher, Coleman theory. The earlier paper gives an explicitly computable formula in terms of p -adic Hilbert modular forms, but one which does not seem to have any straightforward relationship to p -adic L -values, in contrast to the formula of Theorem 7.7.6.)

2. CONVENTIONS

Throughout this paper F denotes a real quadratic field, and we fix an enumeration of the embeddings $F \hookrightarrow \mathbf{R}$ as σ_1, σ_2 .

2.1. Algebraic groups. As in [LLZ18], define $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$, and $H = \mathrm{GL}_2$, with $\iota : H \hookrightarrow G$ the natural embedding. We write B_H and B_G for the upper-triangular Borel subgroups.

Recall that if V is an algebraic representation of B_H , then V gives rise to a vector bundle on the Shimura variety Y_H (for any sufficiently small level group), endowed with an action of Hecke correspondences. There are two possible normalisations for this functor, and we normalise so that the defining 2-dimensional representation of H (restricted to B_H) maps to the relative de Rham *cohomology* sheaf of the universal elliptic curve $\mathcal{E} \rightarrow Y_H$ (rather than the relative homology, which is the other convention in use). There is an analogous construction for G as long as we restrict to representations trivial on the norm-one subgroup $\left\{ \begin{pmatrix} x & \\ & x \end{pmatrix} : N_{F/\mathbf{Q}}(x) = 1 \right\}$ of Z_G , cf. [LLZ18, §3.2c].

2.2. Algebraic representations and the Clebsch–Gordan map. For the group G , we shall always work with representations over fields containing F . Given two representations V, V' of GL_2 , we write $V \boxtimes V'$ for the tensor product of V, V' , endowed with a G -action as the tensor product of the actions via σ_1 on V and via σ_2 on V' .

Given integers $k_1, k_2 \geq 0$, and $0 \leq j \leq \min\{k_1, k_2\}$, let

$$V_G := \mathrm{Sym}^{k_1} W_G \boxtimes \mathrm{Sym}^{k_2} W_G \quad \text{and} \quad V_H = \mathrm{Sym}^t W_H, \quad t = k_1 + k_2 - 2j,$$

where W_H and W_G denote the defining 2-dimensional representations of GL_2/\mathbf{Q} and GL_2/F respectively.

The representation V_H has a canonical basis $(v^a w^{t-a})_{0 \leq a \leq t}$, where v, w are the two standard basis vectors of W_H ; and is equipped with a decreasing B_H -stable weight filtration $\mathrm{Fil}^n V_H = \langle v^a w^{t-a} : a \geq n \rangle$. Similarly, V_G is equipped with a bi-filtration (a decreasing filtration indexed by \mathbf{Z}^2) arising from the weight vector filtrations on each factor.

There is a non-zero morphism of H -representations (the *Clebsch–Gordan map*), unique up to scalars,

$$\mathrm{CG}^{[k_1, k_2, j]} : V_H \rightarrow V_G \otimes \det^{-j}.$$

See (7.1) below for explicit formulae. If we equip the one-dimensional representation \det^{-j} with a filtration concentrated in degree $-j$, then this map respects the filtrations, and hence induces a map on the graded pieces.

3. THE ASAI MOTIVE AND ITS REALISATIONS

3.1. Automorphic representations of G .

Definition 3.1.1. Let Π be the *unitary* cuspidal automorphic representation of $G(\mathbf{A})$ generated by a holomorphic Hilbert modular newform of weight $(k_1 + 2, k_2 + 2)$, for some integers $k_1, k_2 \geq 0$, and some level $\mathcal{N}_\Pi \trianglelefteq \mathcal{O}_F$.

We shall suppose throughout this work that the following conditions hold:

- Π is not of CM type,
- the central character of Π is trivial (and hence k_i are both even).

Both of these could almost certainly be relaxed with a little extra work, but we shall not pursue this here. The representation Π (and hence the integers (k_1, k_2)) will remain fixed throughout the remainder of this paper.

We define $a_{\mathfrak{n}}^{\circ}(\Pi)$, for each ideal $\mathfrak{n} \leq \mathcal{O}_F$, to be the Hecke eigenvalues of the new vector of Π , normalised in the analytic fashion, so that $|a_{\mathfrak{p}}^{\circ}(\Pi)| \leq 2$ for primes $\mathfrak{p} \nmid \mathcal{N}_{\Pi}$. For each prime $\mathfrak{p} \nmid \mathcal{N}_{\Pi}$ we let $\alpha_{\mathfrak{p}}^{\circ}$ and $\beta_{\mathfrak{p}}^{\circ}$ be the Satake parameters of $\Pi_{\mathfrak{p}}$, normalised so that $|\alpha_{\mathfrak{p}}^{\circ}| = |\beta_{\mathfrak{p}}^{\circ}| = 1$ and $\alpha_{\mathfrak{p}}^{\circ} + \beta_{\mathfrak{p}}^{\circ} = a_{\mathfrak{p}}^{\circ}(\Pi)$.

By a rationality theorem due to Shimura [Shi86, Proposition 1.2], there exists a number field $L \subset \mathbf{C}$ (depending on Π) such that the quantities $N(\mathfrak{n})^{1/2} a_{\mathfrak{n}}^{\circ}$ lie in L , for all $\mathfrak{n} \leq \mathcal{O}_F$. Extending L if necessary, we may assume that L also contains the images of the embeddings $\sigma_i : F \hookrightarrow \mathbf{R}$ (this is automatic if $k_1 \neq k_2$).

Definition 3.1.2. For $n \in \mathbf{Z}_{\geq 1}$, we define $a_n(\Pi)$ by

$$a_n(\Pi) = n^{(k_1+k_2+2)/2} a_n^{\circ}(\Pi).$$

One checks that $a_n(\Pi) \in \mathcal{O}_L$ for all n . (However, unless we make the stronger assumption that $k_1 = k_2 \pmod{4}$, we cannot extend this normalisation of the Hecke eigenvalues to include ideals \mathfrak{n} which do not come from \mathbf{Q} , without introducing infinitely many square roots to L .)

3.2. The Asai L -function. For each rational prime ℓ , let us set $\Pi_{\ell} = \bigotimes_{v|\ell} \Pi_v$, considered as a representation of $G(\mathbf{Q}_{\ell}) = \prod_{v|\ell} \mathrm{GL}_2(F_v)$. Associated to Π_{ℓ} we have an *Asai L -factor*¹, $L_{\mathrm{As}}(\Pi_{\ell}, s)$. We write $L_{\mathrm{As}}(\Pi, s) = \prod_{\ell \text{ prime}} L_{\mathrm{As}}(\Pi_{\ell}, s)$ (without any Archimedean factors) for the Asai L -function; note that this Dirichlet series has coefficients in L .

Since the Euler factors of L_{As} at primes dividing \mathcal{N}_{Π} are hard to describe explicitly, we shall also consider the “imprimitive” Asai L -function

$$L_{\mathrm{As}}^{\mathrm{imp}}(\Pi, s) = \zeta_{(N)}(2s) \sum_{n \geq 1} a_n^{\circ}(\Pi) n^{-s},$$

where the notation (N) signifies removing the Euler factors at the primes dividing $N = \mathcal{N}_{\Pi} \cap \mathbf{Z}$. Note that this Dirichlet series has coefficients in L . The following result is standard:

Lemma 3.2.1. *The function $L_{\mathrm{As}}^{\mathrm{imp}}(\Pi, s) / L_{\mathrm{As}}(\Pi, s)$ is a product of polynomials in ℓ^{-s} for $\ell \mid N$. All of its zeroes have real part in $\{0, -\frac{1}{2}, -1\}$. \square*

3.3. De Rham and coherent cohomology groups. We let $U_1(\mathcal{N}_{\Pi})$ denote the open compact subgroup $\{g \in G(\mathbf{A}_f) : g = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{N}_{\Pi}}\}$, so that the space of $U_1(\mathcal{N}_{\Pi})$ -invariants of Π_f (the new subspace of Π_f) is one-dimensional. We assume, for simplicity² that \mathcal{N}_{Π} does not divide 6 disc $_{F/\mathbf{Q}}$, so that $U_1(\mathcal{N}_{\Pi})$ is *sufficiently small* in the sense of [LLZ18, Definition 2.2.1]; thus the Hilbert modular variety Y_G of level $U_1(\mathcal{N}_{\Pi})$ is a smooth quasiprojective variety defined over \mathbf{Q} .

Definition 3.3.1. Let \mathcal{V}_G denote the vector bundle with connection on Y_G corresponding to the algebraic representation³ V_G ; and set

$$D_L^{\mathrm{As}}(\Pi) = H_{\mathrm{dR},c}^2(Y_G, \mathcal{V}_G)[\Pi_f],$$

which is 4-dimensional over L .

The de Rham cohomology can be computed via the coherent cohomology of a smooth toroidal compactification X_G of Y_G . We can then compute de Rham cohomology using the logarithmic de Rham complex $\mathcal{V}_G \otimes \Omega_{X_G}^{\bullet}(D)$, where $D = X_G - Y_G$ is the boundary divisor; and we can compute compactly-supported

¹This can be defined as the “common denominator” of a family of zeta-integrals, or alternatively via the local Langlands correspondence, but both definitions give the same L -factor. For ℓ split this is one of the defining properties of the local Langlands correspondence; for ℓ non-split, see [Mat10, Theorem 3.3].

²The excluded cases may be dealt with via the usual trick of introducing full level \mathfrak{h} structure, for an auxiliary ideal \mathfrak{h} , and then taking invariants under $\mathrm{GL}_2(\mathcal{O}/\mathfrak{h})$; we leave the details to the interested reader.

³More precisely, we need to twist by an appropriate character to make the action of norm-one units trivial, but the resulting vector bundle is independent of the choice up to a canonical isomorphism.

de Rham cohomology using the “minus-log” complex $\mathcal{V}_G \otimes \Omega_{X_G}^\bullet \langle -D \rangle$, where $\Omega^\bullet \langle -D \rangle := \Omega^\bullet(D)(-D)$. We denote the corresponding cohomologies by

$$\begin{aligned} R\Gamma_{\mathrm{dR}}(X_G, \mathcal{V}_G \langle D \rangle) &:= R\Gamma(X_G, \mathcal{V} \otimes \Omega_{X_G}^\bullet \langle D \rangle) \cong R\Gamma_{\mathrm{dR}}(Y_G, \mathcal{V}_G), \\ R\Gamma_{\mathrm{dR}}(X_G, \mathcal{V}_G \langle -D \rangle) &:= R\Gamma(X_G, \mathcal{V} \otimes \Omega_{X_G}^\bullet \langle -D \rangle) \cong R\Gamma_{\mathrm{dR},c}(Y_G, \mathcal{V}_G). \end{aligned}$$

Rather than explicitly working with the full de Rham complex, it is more convenient to work with the *dual BGG complex*

$$\mathrm{BGG}^\bullet = \left[\omega^{(-k_1, -k_2)} \longrightarrow \omega^{(-k_1, k_2+2)} \oplus \omega^{(k_1+2, -k_2)} \longrightarrow \omega^{(k_1+2, k_2+2)} \right]$$

and its compactly-supported analogue $\mathrm{BGG}^\bullet(-D)$, which are quasi-isomorphic to the logarithmic de Rham complexes $V \otimes \Omega^\bullet \langle D \rangle$ and $V \otimes \Omega^\bullet \langle -D \rangle$ respectively. (We shall give explicit formulae for the differentials of the BGG complex, and the quasi-isomorphism relating it to the de Rham complex, in Section 7.1 below, but we shall not need this just yet.)

The BGG complex is equipped with a natural decreasing \mathbf{Z}^2 -filtration $\mathrm{Fil}^{\bullet\bullet}$, which gives rise to a \mathbf{Z}^2 -filtration on its cohomology. We can (and do) normalise so that the nontrivial graded pieces are in bidegrees $\{(0, 0), (k_1 + 1, 0), (0, k_2 + 1), (k_1 + 1, k_2 + 1)\}$, and are given by

$$\begin{aligned} \mathrm{Gr}^{(k_1+1, k_2+1)} &= H^0 \left(X_G, \omega^{(k_1+1, k_2+1)}(-D) \right) [\Pi_{\mathfrak{f}}], \\ \mathrm{Gr}^{(k_1+1, 0)} &= H^1 \left(X_G, \omega^{(k_1+2, -k_2)}(-D) \right) [\Pi_{\mathfrak{f}}], & \mathrm{Gr}^{(0, k_2+1)} &= H^1 \left(X_G, \omega^{(-k_1, k_2+2)}(-D) \right) [\Pi_{\mathfrak{f}}], \\ \mathrm{Gr}^{(0, 0)} &= H^2 \left(X_G, \omega^{(-k_1, -k_2)}(-D) \right) [\Pi_{\mathfrak{f}}], \end{aligned}$$

where $\omega^{(r,s)}$ is the sheaf of Hilbert modular forms of weight (r, s) (so that $\omega^{(r,s)}(-D)$ is the subsheaf of cusp forms). The induced single filtration, with graded pieces in degrees $\{0, k_1 + 1, k_2 + 1, k_1 + k_2 + 2\}$, is the Hodge filtration.

Definition 3.3.2. We let ν be a basis of the 1-dimensional L -vector space $H^1(X_G, \omega^{(-k_1, k_2+2)}(-D))[\Pi_{\mathfrak{f}}]$.

Remark 3.3.3 (Occult periods). After base-extension to \mathbf{C} , these graded pieces have canonical bases arising from the comparison between sheaf cohomology and Dolbeault cohomology. The one which will interest us is $\mathrm{Gr}^{(0, k_2+1)}$, which is spanned by the differential form associated to the real-analytic Hilbert modular form $\mathcal{F}^{\mathrm{ah},1}$ (anti-holomorphic at the place σ_1 and holomorphic at σ_2) having the same Fourier–Whittaker coefficients as the holomorphic newform \mathcal{F} generating Π ; see Lemma 5.2.1 of [LLZ18] for further details. Harris’ *occult period* for Π at σ_1 (cf. [Har90]) is the ratio between ν and $\mathcal{F}^{\mathrm{ah},1}$, well-defined as an element of $\mathbf{C}^\times/L^\times$. \diamond

3.4. The Asai Galois representation. We now fix (for the remainder of this paper) a prime p , and a prime \mathfrak{P} of the coefficient field L above p .

Definition 3.4.1.

- (i) Let $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$ denote the four-dimensional Asai Galois representation associated to Π as in [LLZ18, Definition 4.4.2], defined as the $\Pi_{\mathfrak{f}}$ -eigenspace in the p -adic étale cohomology of the Hilbert modular variety $Y_{G,1}(\mathcal{N}_{\Pi}) \otimes \overline{\mathbf{Q}}$ (with coefficients in the étale local system of $L_{\mathfrak{P}}$ -vector spaces determined by (k_1, k_2)).
- (ii) Let $D_{\mathfrak{P}}^{\mathrm{As}}(\Pi) = L_{\mathfrak{P}} \otimes_L D_L^{\mathrm{As}}(\Pi)$, so that there is a canonical comparison isomorphism (compatible with the filtrations)

$$\mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p, V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)) \cong D_{\mathfrak{P}}^{\mathrm{As}}(\Pi).$$

Remark 3.4.2. More precisely, this is $M_{L_{\mathfrak{P}}}(\mathcal{F})$ in the notation of *op.cit.*, where \mathcal{F} is the normalised newform generating Π . \diamond

By results of Brylinski–Labesse and Nekovář recalled in *op.cit.*, the Galois representation $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$ can be characterised, up to isomorphism, as the unique semisimple Galois representation whose L -series is $L_{\mathrm{As}}(\Pi, s - \frac{k_1+k_2+2}{2})$. (However, since we want to consider Euler system classes for $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$, it is important to fix not only an abstract isomorphism class of Galois representations but a specific realisation of this isomorphism class in étale cohomology.) By Poincaré duality for the Hilbert modular surface, there is a canonical nondegenerate symmetric bilinear form

$$(3.1) \quad \lambda : \mathrm{Sym}^2(V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)) \rightarrow L_{\mathfrak{P}}(-2 - k_1 - k_2),$$

equivariant for the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. (The pairing is symmetric since $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$ appears in cohomology in even degree.)

Remark 3.4.3. By [LLZ18, Proposition 9.4.3], our hypotheses on Π imply that $V_{\mathfrak{P}}^{\text{As}}(\Pi)$ is either irreducible, or the direct sum of a character and a three-dimensional irreducible representation, with the latter case occurring if and only if Π is a twist of a base-change from GL_2/\mathbf{Q} . \diamond

3.5. The split-prime case. For the remainder of this paper, we suppose that p is split in F , and Π is unramified at the primes above p . Since the embeddings σ_i of F take values in $L \subset \mathbf{C}$, and we have fixed a prime $\mathfrak{P} \mid p$ in L , we can number the primes above p as $\mathfrak{p}_1, \mathfrak{p}_2$ such that $\sigma_i(\mathfrak{p}_i) \subset \mathfrak{P}$.

We define

$$a_{\mathfrak{p}_i}(\Pi) = p^{(k_i+1)/2} a_{\mathfrak{p}_i}^{\circ}(\Pi) \in L, \quad \text{so } a_p(\Pi) = a_{\mathfrak{p}_1}(\Pi) a_{\mathfrak{p}_2}(\Pi).$$

(Note this normalisation depends on the choice of prime $\mathfrak{P} \mid p$, and the $a_{\mathfrak{p}_i}$ are integral at \mathfrak{P} , but possibly not at other primes of L above p .) We likewise define

$$\alpha_i = p^{\frac{1+k_i}{2}} \alpha_{\mathfrak{p}_i}^{\circ}$$

and similarly β_i . Extending L if necessary, we may suppose that the α_i and β_i also lie in L .

Corollary 3.5.1. *The representation $V_{\mathfrak{P}}^{\text{As}}(\Pi)$ is crystalline at p , so $D_{\mathfrak{P}}(\Pi)$ is naturally a filtered φ -module; and the eigenvalues of φ on this module are the pairwise products $\{\alpha_1 \alpha_2, \dots, \beta_1 \beta_2\}$. \square*

Lemma 3.5.2. *Let $0 \leq j \leq \min(k_1, k_2)$ be an integer. Then p^j is not an eigenvalue of φ on $D_{\mathfrak{P}}^{\text{As}}(\Pi)$. Moreover, if $p^{(1+j)}$ is an eigenvalue of φ , then we must have $k_1 = k_2 = j$.*

Proof. This follows from the fact that the Satake parameters $\alpha_i^{\circ}, \beta_i^{\circ}$ have complex absolute value 1. \square

This implies that for the Galois representation $V_{\mathfrak{P}}^{\text{As}}(\Pi)^*(-j)$, the Bloch–Kato subspaces $H_e^1(\mathbf{Q}_p, -)$ and $H_f^1(\mathbf{Q}_p, -)$ agree; and these are also equal to H_g^1 , except possibly in the boundary case $k_1 = k_2 = j$ in which case H_g^1 can be strictly larger. Moreover, the inverse of the Bloch–Kato exponential map for $V_{\mathfrak{P}}^{\text{As}}(\Pi)^*(-j)$ is an isomorphism

$$(3.2) \quad \log : H_f^1(\mathbf{Q}_p, V_{\mathfrak{P}}^{\text{As}}(\Pi)^*(-j)) \xrightarrow{\cong} (\text{Fil}^{1+j} D_{\mathfrak{P}}^{\text{As}}(\Pi))^* = (\text{Fil}^1 D_{\mathfrak{P}}^{\text{As}}(\Pi))^*,$$

with both sides 3-dimensional over $L_{\mathfrak{P}}$.

3.6. Partial Frobenii. We may identify $D_{\mathfrak{P}}^{\text{As}}(\Pi)$, as a Frobenius module (forgetting the filtration), with the rigid cohomology of the special fibre of Y_G at p . This special fibre has two commuting endomorphisms, the **partial Frobenii** φ_1 and φ_2 at the primes \mathfrak{p}_i , whose composite is the Frobenius φ ; more precisely, φ_i corresponds to sending a Hilbert–Blumenthal abelian surface A to the quotient $A/(\ker(\varphi_A) \cap A[\mathfrak{p}_i])$. We refer to [TX16] or [Nek18] for detailed accounts of this construction.

The operators φ_i induce commuting linear operators on $D_{\mathfrak{P}}^{\text{As}}(\Pi)$, with $\varphi = \varphi_1 \varphi_2$; and it follows from the “partial Eichler–Shimura” comparison result proved in [Nek18] that for each i we have

$$(\varphi_i - \alpha_i)(\varphi_i - \beta_i) = 0 \quad \text{on } D_{\mathfrak{P}}^{\text{As}}(\Pi).$$

One checks easily that the partial Frobenii satisfy $\lambda(\varphi_i x, \varphi_i y) = p^{k_i+1} \lambda(x, y)$, where λ is the Poincaré duality form. This identifies the β_i -generalised eigenspace with the dual of that for α_i . Hence, if $\alpha_i \neq \beta_i$, the $\varphi_i = \alpha_i$ and $\varphi_i = \beta_i$ eigenspaces are both 2-dimensional, and each is isotropic with respect to λ .

Remark 3.6.1. We are using a slightly different normalisation of the partial Frobenii here from [LZ22]: the φ_i here is $p^{-t_i} \varphi_i$ in the notation of *op.cit.*, where (t_1, t_2) is an auxiliary choice of integers such that $w = k_i + 2t_i$ is independent of i . This reflects the fact that the α_i of *op.cit.* is $p^{(w+1)/2} \alpha_{\mathfrak{p}_i}^{\circ}$, while α_i here is $p^{(k_i+1)/2} \alpha_{\mathfrak{p}_i}^{\circ}$. The present normalisation is more convenient for comparison with higher Hida theory, since it matches the minimal integral normalisation of the $U'_{\mathfrak{p}_i}$ operators. \diamond

3.7. A lifting of ν . We write v_p for the valuation on $L_{\mathfrak{P}}$ normalized by $v_p(p) = 1$; and we fix an ordering of the Satake parameters at \mathfrak{p}_1 . Then we have $0 \leq v_p(\alpha_1) \leq k_1 + 1$.

Proposition 3.7.1. *Suppose that $\alpha_1 \neq \beta_1$, and $v_p(\alpha_1) < k_1$. Then the vector space*

$$D_{\mathfrak{P}}(\Pi)^{(\varphi_1 = \alpha_1)} \cap \text{Fil}^{(0, k_2+1)} D_{\mathfrak{P}}(\Pi)$$

is one-dimensional, and surjects onto $\text{Gr}^{(0, k_2+1)} D_{\mathfrak{P}}(\Pi)$.

Hence there exists a uniquely determined vector $\nu_{\text{dR}} \in D_{\mathfrak{P}}$ with the following properties:

- $\varphi_1 \nu_{\text{dR}} = \alpha_1 \cdot \nu_{\text{dR}}$;
- $\nu_{\text{dR}} \in \text{Fil}^{(0, k_2+1)} D_{\mathfrak{P}}$;

- the image of ν_{dR} in the graded piece

$$\text{Gr}^{(0, k_2+1)} D_{\mathfrak{F}} \cong L_{\mathfrak{F}} \otimes_L H^1 \left(X_G, \omega^{(-k_1, k_2+2)} \right) [\Pi_{\mathfrak{f}}]$$

coincides with the ν of Definition 3.3.2.

Proof. This is a special case of the first main theorem of [LZ22], which states that (over totally-real fields of any degree) the \mathbf{Z}^d -indexed filtration of $D_{\mathfrak{F}}^{\text{As}}(\Pi)$ has a canonical splitting given by intersecting with partial Frobenius eigenspaces. The assumption $v_p(\alpha_1) < k_1$ is the *strictly small slope* assumption of *op.cit.*. \square

Remark 3.7.2. Note that $\ker((\varphi - \alpha_1\alpha_2)(\varphi - \alpha_1\beta_2) : D_p \rightarrow D_p)$ contains $D_p^{(\varphi_1=\alpha_1)}$, but it may be larger; this always occurs if Π is a twist of a base-change from \mathbf{Q} (so that $\alpha_1/\beta_1 = \alpha_2/\beta_2$). \diamond

4. DEFINITION OF THE p -ADIC REGULATOR

4.1. Euler system classes. Let $0 \leq j \leq \min(k_1, k_2)$. We refer to [LLZ18, Definition 4.4.6] for the definition of the *étale Asai–Flach class*

$$\text{AF}_{\text{ét}}^{[\Pi, j]} \in H^1 \left(\mathbf{Z}[1/\Sigma], V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j) \right),$$

where Σ is the set of primes dividing $pN \text{disc}(F)$. (More precisely, this class is defined for any normalised eigenform \mathcal{F} , not necessarily new, and we are defining $\text{AF}_{\text{ét}}^{[\Pi, j]}$ as $\text{AF}_{\text{ét}}^{[\mathcal{F}, j]}$ where \mathcal{F} is the unique newform generating Π , consistently with our definition of the Galois representation $V_{\mathfrak{F}}^{\text{As}}(\Pi)$.)

Remark 4.1.1. In *op.cit.* we showed that $\text{AF}_{\text{ét}}^{[\Pi, j]}$ was the $m = 1$ case of a family of classes defined over $\mathbf{Q}(\mu_m)$ with norm-compatibility properties as m varies, but we shall not use this here. \diamond

Note 4.1.2. We briefly recall the definition of $\text{AF}_{\text{ét}}^{[\Pi, j]}$. Letting \mathcal{V}_G and \mathcal{V}_H denote the relative Chow motives (over L) associated to the representations V_G and V_H defined in Section 2.2, we have a pushforward map

$$\iota_*^{[j]} : H_{\text{mot}}^1(Y_{H,1}(N), \mathcal{V}_H(1+t)) \otimes_{\mathbf{Q}} L \longrightarrow H_{\text{mot}}^3(Y_{G,1}(\mathcal{N}_{\Pi}), \mathcal{V}_G(2+k_1+k_2-j)),$$

where $t = k_1 + k_2 - 2j$ as before. We have an analogous pushforward map in étale cohomology with $L_{\mathfrak{F}}$ -coefficients, and the two are compatible via the étale regulator map $r_{\text{ét}}$. Since the $\Pi_{\mathfrak{f}}$ -generalised eigenspace in the cohomology of $Y_{G,1}(\mathcal{N}_{\Pi})$ vanishes outside degree 2, the Hochschild–Serre spectral sequence gives a projection map from $H_{\text{ét}}^3(Y_{G,1}(\mathcal{N}_{\Pi}), \mathcal{V}_G(2+k_1+k_2-j))$ to $H^1(\mathbf{Z}[1/\Sigma], V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j))$. We can then define $\text{AF}_{\text{ét}}^{[\Pi, j]}$ as the image of the weight t Eisenstein class under this chain of maps. \diamond

4.2. Localisation at p and syntomic cohomology.

Proposition 4.2.1. *The localisation of $\text{AF}_{\text{ét}}^{[\Pi, j]}$ at p lies in the Bloch–Kato subspace*

$$H_{\mathfrak{f}}^1(\mathbf{Q}_p, V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j)) \subseteq H^1(\mathbf{Q}_p, V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j)).$$

Proof. This follows from the comparison between étale and syntomic cohomology, since the map from syntomic to étale cohomology factors through the Bloch–Kato exponential map; cf. [KLZ20, Proposition 5.4.1] in the analogous case of Beilinson–Flach elements. \square

Now fix a choice of α_1 satisfying the conditions of Proposition 3.7.1, and let $\nu_{\text{dR}} \in \text{Fil}^{(0, k_2+1)} D_{\mathfrak{F}}(\Pi)$ be the ensuing lifting of ν . Our goal will be to compute the pairing

$$(4.1) \quad \left\langle \nu_{\text{dR}}, \log \left(\text{loc}_p \text{AF}_{\text{ét}}^{[\Pi, j]} \right) \right\rangle_{D_{\mathfrak{F}}(\Pi)},$$

where \log is the Bloch–Kato logarithm (3.2), and $\langle -, - \rangle_{D_{\mathfrak{F}}(\Pi)}$ denotes the canonical pairing between $D_{\mathfrak{F}}^{\text{As}}(\Pi)$ and its dual.

Let \mathbb{Y}_G denote the canonical \mathbf{Z}_p -model of $Y_{G,1}(\mathcal{N}_{\Pi})$. This is a smooth \mathbf{Z}_p -scheme, and we may choose an arithmetic toroidal compactification \mathbb{X}_G such that $(\mathbb{Y}_G, \mathbb{X}_G)$ is a *smooth pair* over \mathbf{Z}_p in the sense of [KLZ20] (i.e. \mathbb{X}_G is smooth, and the cuspidal divisor $\mathbb{X}_G - \mathbb{Y}_G$ is a smooth normal-crossing divisor relative to $\text{Spec } \mathbf{Z}_p$). Thus Besser’s theory of *rigid syntomic* and *finite-polynomial* cohomology applies to \mathbb{Y}_G , and rigid-syntomic cohomology has a natural comparison to étale cohomology.

Notation 4.2.2. Let $P(T) = \left(1 - \frac{T}{\alpha_1\alpha_2}\right) \left(1 - \frac{T}{\alpha_1\beta_2}\right) \in L[T]$, and let $P_{1+j}(T) = P(p^{1+j}T)$.

Proposition 4.2.3. *There is a unique lift ν_{fp} of ν_{dR} to the space*

$$H_{\text{fp},c}^2(\mathbb{Y}_G, \mathcal{V}_G; 1+j, P_{1+j})[\Pi_f].$$

Remark 4.2.4. The above group is actually independent of j in the range $0 \leq j \leq \min(k_1, k_2)$: Besser's cohomology for twist r and polynomial Q is defined using the mapping fibre of $Q(p^{-r}\varphi)$, and we have $P_{1+j}(p^{-1-j}\varphi) = P(\varphi)$ for any j . However, different values of j will correspond to the étale cohomology of different twists of $V_{\mathfrak{p}}^{\text{As}}(\Pi)$. \diamond

Proof. Since Π is cuspidal, the Π_f -generalised eigenspace in de Rham (or, equivalently, rigid) cohomology vanishes outside degree 2. So the natural map

$$H_{\text{fp},c}^2(\mathbb{Y}_G, \mathcal{V}_G; 1+j, P) \rightarrow \text{Fil}^{(1+j)} H_{\text{dR},c}^2(\mathbb{Y}_G, \mathcal{V}_G)^{P(\varphi)=0}$$

is an isomorphism after localising at the Π_f -eigenspace. Since $P(\varphi)$ annihilates ν_{dR} the result follows. \square

The compatibility of étale and syntomic Abel–Jacobi maps for smooth pairs (cf. Proposition 5.4.1 of [KLZ20]) then implies that

$$(4.2) \quad \begin{aligned} \left\langle \nu_{\text{dR}}, \log \left(\text{loc}_p \text{AJ}_{\text{ét}}^{[\Pi, j]} \right) \right\rangle_{D_{\mathfrak{F}}(\Pi)} &= \left\langle \nu_{\text{dR}}, \log \circ \text{pr}_{\Pi}^{\text{As}} \circ \iota_*^{[j]} \left(\text{Eis}_{\text{ét}, N}^t \right) \right\rangle_{\text{dR}, Y_G} \\ &= \left\langle \nu_{\text{fp}}, \iota_*^{[j]} \left(\text{Eis}_{\text{syn}, N}^t \right) \right\rangle_{\text{fp}, \mathbb{Y}_G} \\ &= \left\langle \iota^{[j],*}(\nu_{\text{fp}}), \text{Eis}_{\text{syn}, N}^t \right\rangle_{\text{fp}, \mathbb{Y}_H}, \end{aligned}$$

where the last equality follows from the adjunction between pushforward and pullback. Note that

$$\iota^{[j],*}(\nu_{\text{fp}}) \in H_{\text{fp},c}^2(\mathbb{Y}_H, \mathcal{V}_H; 1+j, P_{1+j}),$$

and the coefficient module \mathcal{V}_H depends on j .

5. LIFTING ν TO THE \mathfrak{p}_1 -ORDINARY LOCUS

In this section, we shall lift the coherent class ν (and its cousins ν_{dR} and ν_{fp}) from the cohomology of the full variety Y_G , to cohomology groups associated to certain open subsets of the special fibre of Y_G .

5.1. Geometry of the Hilbert modular variety. Let $Y_{G,0}$ be the special fibre of \mathbb{Y}_G , which is a smooth \mathbb{F}_p -variety; and similarly for the compactification $X_{G,0}$.

Notation 5.1.1.

- For $i = 1, 2$, denote by $X_{G,0}^{i-\text{ss}} \subset X_{G,0}$ the \mathfrak{p}_i -supersingular locus (the vanishing locus of the partial Hasse invariant, as constructed in [TX16, §3.2]).
- Let $X_{G,0}^{i-\text{ord}}$ be the complement of $X_{G,0}^{i-\text{ss}}$, and $X_{G,0}^{\text{ord}} = X_{G,0}^{1-\text{ord}} \cap X_{G,0}^{2-\text{ord}}$.

We write $Y_{G,0}^{i-\text{ord}}$ etc for the intersection of these subvarieties with $Y_{G,0} \subset X_{G,0}$.

The following results on the geometry of the supersingular loci are well-known (see e.g. [TX16]):

Lemma 5.1.2. *For $i = 1, 2$, $X_{G,0}^{i-\text{ss}}$ is a smooth codimension 1 closed subscheme of $X_{G,0}$, disjoint from the toroidal boundary; and $X_{G,0}^{1-\text{ss}} \cap X_{G,0}^{2-\text{ss}}$ is a smooth closed subvariety of codimension 2 (i.e. a finite disjoint union of points).*

Remark 5.1.3. The preimage of either $X_{G,0}^{1-\text{ord}}$ or $X_{G,0}^{2-\text{ord}}$ under the finite map $\iota : X_{H,0} \rightarrow X_{G,0}$ is the ordinary locus $X_{H,0}^{\text{ord}}$. \diamond

Proposition 5.1.4. *The extension-by-0 map*

$$R\Gamma_{\text{rig},c}(Y_{G,0}^{1-\text{ord}}, \mathcal{V}_G) \longrightarrow R\Gamma_{\text{rig},c}(Y_{G,0}, \mathcal{V}_G) \cong R\Gamma_{\text{dR},c}(Y_G, \mathcal{V}_G)$$

is a quasi-isomorphism on the Π generalised eigenspace for the prime-to- Np Hecke operators.

Proof. This is a special case of Proposition 4.3 of [LZ22]. \square

Notation 5.1.5. Write $\nu_{\text{dR}}^{(1-\text{ord})} \in H_{\text{rig},c}^2(Y_{G,0}^{1-\text{ord}}, \mathcal{V}_G)$ for the preimage of ν_{dR} under the isomorphism in Proposition 5.1.4.

5.2. Coherent cohomology of the 1-ordinary locus. We write \mathcal{X}_G for the dagger space associated to X_G/\mathbf{Q}_p ; and we denote the tubes in \mathcal{X}_G of the various subvarieties of $X_{G,0}$ considered above by the corresponding superscripts on \mathcal{X}_G , so $\mathcal{X}_G^{i\text{-ord}}$ is the tube of $X_{G,0}^{i\text{-ord}}$ in \mathcal{X}_G etc.

Theorem 5.2.1. *Suppose $v_p(\alpha_1) < k$, and $\alpha_1 \neq \beta_1$. Then there exists a unique class*

$$\nu^{1\text{-ord}} \in H_c^1\left(\mathcal{X}_G^{(1\text{-ord})}, \omega^{(-k_1, k_2+2)}(-D)\right)$$

satisfying the following properties:

- (1) it is a φ_1 -eigenvector with eigenvalue α_1 ;
- (2) its image in $H^1(\mathcal{X}_G, \omega^{(-k_1, k_2+2)}(-D))$ is ν .

Proof. This is an instance of Proposition 5.2 of [LZ22]. \square

Remark 5.2.2. Observe that we have chosen a lifting of the de Rham class ν_{dR} to $X_{G,0}^{(1\text{-ord})}$ characterised by information about the action of Hecke operators *away* from p ; and, separately, we have lifted the coherent class ν to $\mathcal{X}_G^{(1\text{-ord})}$ using information about the action of the Frobenius φ_1 at \mathfrak{p}_1 . So it is not obvious how these liftings are related, and our next task is to find a way to reconcile the two, which we will carry out in Section 5.4 – see Proposition 5.4.1 below. \diamond

5.3. Comparison with a higher-level class. We now compare the class $\nu^{1\text{-ord}}$ with algebraic coherent classes at level \mathfrak{p}_1 . If $X_G(\mathfrak{p}_1)$ denotes the Shimura variety of level $U_1(\mathcal{N}_\Pi) \cap \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}) \bmod \mathfrak{p}_1\}$, then the special fibre $X_G(\mathfrak{p}_1)_0$ has a stratification with three strata,

$$X_G(\mathfrak{p}_1)_0 = X_G(\mathfrak{p}_1)_0^m \cup X_G(\mathfrak{p}_1)_0^{\text{ét}} \cup X_G(\mathfrak{p}_1)_0^\alpha,$$

on which the level structure is multiplicative, étale, or α_p respectively; and this gives a corresponding decomposition of the dagger space $\mathcal{X}_G(\mathfrak{p}_1)$, which restricts to an isomorphism of dagger spaces

$$\mathcal{X}_G(\mathfrak{p}_1)^m \xrightarrow{\cong} \mathcal{X}_G^{1\text{-ord}}.$$

(The inverse map is given by the “canonical subgroup” construction.) From the functoriality of pushforward maps we have a commutative square of cohomology groups (all with coefficients in $\omega^{-k_1, k_2+2}(-D)$)

$$\begin{array}{ccc} H_c^1(\mathcal{X}_G(\mathfrak{p}_1)^m) & \longrightarrow & H^1(\mathcal{X}_G(\mathfrak{p}_1)) \\ \parallel & & \downarrow \pi_{\mathfrak{p}_1,*} \\ H_c^1(\mathcal{X}_G^{1\text{-ord}}) & \longrightarrow & H^1(\mathcal{X}_G) \end{array}$$

where the horizontal maps are extension-by-zero (i.e. pushforwards along open embeddings) and the right-hand vertical map is pushforward along the natural degeneracy map $\pi_{\mathfrak{p}_1} : \mathcal{X}_G(\mathfrak{p}_1) \rightarrow \mathcal{X}_G$. By [LZ22, Lemma 5.7], our Frobenius lift φ_1 on $H_c^1(\mathcal{X}_G^{1\text{-ord}})$ corresponds to the Hecke operator $U'_{\mathfrak{p}_1}$ at level $\mathcal{X}_G(\mathfrak{p}_1)$. Hence we have the following compatibility:

Proposition 5.3.1. *The image of $\nu^{1\text{-ord}}$ in $H^1(\mathcal{X}_G(\mathfrak{p}_1))$ is the unique class which lies in the $\Pi_{\mathfrak{f}}$ -eigenspace away from p , is a $(U'_{\mathfrak{p}_1} = \alpha_1)$ -eigenvector, and maps to ν under the trace map. \square*

We briefly compare this with the choice of basis vector used in [GLZ23, §6–7], since this will be needed for our final formula. In §7.3 of *op.cit.* we define a $U'_{\mathfrak{p}_1} = \alpha_1$ eigenvector $\check{\nu}_{\Pi, \alpha} \in H^1(\mathcal{X}_G(\mathfrak{p}_1))$, depending on a choice of basis $W_{\mathfrak{f}}^{(p)}$ of the Whittaker model away from $p\infty$. If we choose this basis to be the normalised new-vector, then by construction we have

$$\check{\nu}_{\Pi, \alpha} = \left(1 - \frac{\beta_1}{U'_{\mathfrak{p}_1}}\right) \pi_{\mathfrak{p}_1}^*(\nu).$$

Since $\check{\nu}_{\Pi, \alpha}$ and the image of $\nu^{1\text{-ord}}$ lie in the same one-dimensional space, we may compare them by computing their images in $H^1(\mathcal{X}_G)$. An elementary computation shows that the map

$$\pi_{\mathfrak{p}_1,*} \circ \left(1 - \frac{\beta_1}{U'_{\mathfrak{p}_1}}\right) \circ \pi_{\mathfrak{p}_1}^*$$

acts on the Π -eigenspace as multiplication by $p(1 - \frac{\beta_1}{p\alpha_1})$ (which is not zero, since α_1/β_1 has complex absolute value 1); so we have

$$\check{\nu}_{\Pi, \alpha} = p\left(1 - \frac{\beta_1}{p\alpha_1}\right) \cdot \text{image}(\nu^{1\text{-ord}}).$$

5.4. **The de Rham spectral sequence for $\mathcal{X}_G^{1\text{-ord}}$.** Since $(\mathbb{X}_G, \mathbb{Y}_G)$ is a smooth pair, and our coefficient system \mathcal{V}_G extends to a vector bundle on \mathcal{X}_G whose connection has log poles along the boundary divisor D , we can compute rigid cohomology of $Y_{G,0}$ using the analytification of the BGG complex on \mathcal{X}_G (just as we did for de Rham cohomology above). By taking the mapping fibre of the restriction map we obtain the same result for compactly-supported cohomology of $Y_{G,0}^{1\text{-ord}}$; that is, we have

$$R\Gamma_{\text{rig},c}(Y_{G,0}^{1\text{-ord}}, \mathcal{V}) \cong R\Gamma_{\text{dR},c}(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle) \cong R\Gamma_c(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^\bullet(-D)).$$

This gives rise to a first-quadrant spectral sequence converging to $H_{\text{rig},c}^*(Y_{G,0}^{1\text{-ord}}, \mathcal{V}_G)$, whose E_1^{mn} terms are $H_c^n(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^m(-D))$.

We denote by $\widetilde{R}\Gamma_{\text{dR},c}(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle)$ the cohomology of the truncated complex

$$\tau_{\geq 1} \text{BGG}^\bullet(-D) = \left[0 \longrightarrow \omega^{(-k_1, k_2+2)} \oplus \omega^{(k_1+2, -k_2)} \longrightarrow \omega^{(k_1+2, k_2+2)} \right](-D),$$

which is quasi-isomorphic to the filtered de Rham complex $\text{Fil}^{1+j} \text{BGG}^\bullet(-D)$, for any j in our range. Since \mathcal{X}_G is connected and non-compact, $H_c^0(\mathcal{X}_G^{1\text{-ord}}, -)$ is zero for all locally-free sheaves, and so we obtain an isomorphism

$$\alpha_{\text{dR}}^{(1\text{-ord})} : H_c^1(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^1(-D))^{\nabla=0} \cong \widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle).$$

Moreover, the inclusion of the subcomplex $\tau_{\geq 1} \text{BGG}$ into the full BGG complex gives a commutative square of maps

$$\begin{array}{ccc} \widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle) & \longrightarrow & \text{Fil}^1 H_{\text{dR}}^2(X_G, \mathcal{V}\langle -D \rangle)_{L_{\mathfrak{p}}} \\ \downarrow & & \downarrow \\ H_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle) & \longrightarrow & H_{\text{dR}}^2(X_G, \mathcal{V}\langle -D \rangle)_{L_{\mathfrak{p}}}, \end{array}$$

in which the top horizontal arrow is compatible, via $\alpha_{\text{dR}}^{(1\text{-ord})}$, with the natural map

$$H_c^1(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^1(-D)) \rightarrow H_c^1(\mathcal{X}_G, \text{BGG}^1(-D)) = \text{Fil}^1 H_{\text{dR}}^2 / \text{Fil}^{k_1+k_2+2}.$$

Since the partial Frobenius φ_1 lifts to $\mathcal{X}_G^{1\text{-ord}}$, there is an action of φ_1 on both of the spaces in the left-hand column, compatible with the action on $H_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle, 1+j)$ given by comparison with the rigid cohomology of $X_{G,0}$.

Proposition 5.4.1. *If $\nu^{1\text{-ord}}$ is as in Theorem 5.2.1, then the class $(\nu^{1\text{-ord}}, 0)$ in*

$$H_c^1(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^1(-D)) = H_c^1(\mathcal{X}_G^{1\text{-ord}}, \omega^{-k_1, k_2+2}(-D)) \oplus H_c^1(\mathcal{X}_G^{1\text{-ord}}, \omega^{k_1+2, -k_2}(-D))$$

is in the kernel of ∇ , and hence defines a class in $\widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle)$. The image of this class in $H_{\text{rig},c}^2(Y_{G,0}^{1\text{-ord}}, \mathcal{V}_G)$, under the left vertical map of the above diagram, is $\nu_{\text{dR}}^{(1\text{-ord})}$.

Proof. We first show that $\nu^{1\text{-ord}}$ is in the kernel of ∇ . This follows from the fact that it has strictly small slope for φ_1 : the slopes of φ_1 on $\omega^{(k_1+2, k_2+2)}$ are all at least k_1+1 , and the operator ∇ commutes with the Frobenius, so it must be zero on all Frobenius eigenspaces of slope smaller than k_1+1 .

This shows that $\nu^{1\text{-ord}}$ has a well-defined image in $H_{\text{rig},c}^2(Y_{G,0}^{1\text{-ord}}, \mathcal{V}_G)$. Let us temporarily write $\hat{\nu}_{\text{dR}}^{(1\text{-ord})}$ for this image; our goal is to show that it coincides with $\nu_{\text{dR}}^{1\text{-ord}}$. Since the latter is characterised as the unique lifting of ν_{dR} compatible with Hecke operators away from $p\mathcal{N}$, it suffices to show that $\hat{\nu}_{\text{dR}}^{(1\text{-ord})}$ lies in the correct Hecke eigenspace, and that it maps to ν_{dR} in $H_{\text{dR},c}^2(Y_G, \mathcal{V}_G)$.

It follows readily from the construction of $\nu^{1\text{-ord}}$ that it lies in the Π -eigenspace for the Hecke operators away from \mathfrak{p}_1 (including $T_{\mathfrak{p}_2}$), since these operators commute with φ_1 . So $\hat{\nu}_{\text{dR}}^{(1\text{-ord})}$ has the correct Hecke action. Moreover, its image in $H_{\text{dR},c}^2(\mathcal{Y}_G, \mathcal{V}_G)$ is in the $\varphi_1 = \alpha_1$ eigenspace (because $\nu^{1\text{-ord}}$ is); and it lies in $\text{Fil}^{(0, 1+k_2)}$, and maps to ν in $\text{Gr}^{(0, 1+k_2)}$, so it must be equal to ν_{dR} . \square

5.5. FP-cohomology of the \mathfrak{p}_1 -ordinary locus. We now consider a modified form of Besser’s finite-polynomial cohomology, namely *Gros fp-cohomology*, for the \mathfrak{p}_1 -ordinary locus (with compact supports). This cohomology, denoted by $\widetilde{R}\Gamma_{\text{fp},c}(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$ can be defined as the mapping fibre of the map

$$\widetilde{R}\Gamma_{\text{dR},c}(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle) \xrightarrow{P_{1+j}(p^{-1-j}\varphi)} R\Gamma_{\text{rig},c}(X_{G,0}^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle).$$

(As before, this is in fact independent of j in the stated range, despite the notations.) Note that although φ_1 lifts to $\mathcal{X}_G^{1-\text{ord}}$, the full Frobenius φ does not; so although $R\Gamma_{\text{rig},c}(X_{G,0}^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle)$ is isomorphic to de Rham cohomology of $\mathcal{X}_G^{1-\text{ord}}$, the action of Frobenius (given by the functoriality of rigid cohomology) cannot be ‘seen’ via this description.

For proper schemes such as \mathbb{X}_G , there is no difference between Gros fp-cohomology and the usual fp-cohomology, so there is an extension-by-zero map

$$\widetilde{R}\Gamma_{\text{fp},c}(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) \rightarrow R\Gamma_{\text{fp},c}(\mathbb{X}_G, \mathcal{V}_G\langle -D \rangle; 1+j, P_{1+j}).$$

Proposition 5.5.1. *There exists a class*

$$\tilde{\nu}_{\text{fp}}^{1-\text{ord}} \in \widetilde{H}_{\text{fp},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$$

with the following properties:

- Its image in $H_{\text{fp},c}^2(\mathbb{X}_G, \mathcal{V}_G\langle -D \rangle; 1+j, P_{1+j})$ is the ν_{fp} of Proposition 4.2.3.
- Its image in $\widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle)$ is the class $(\nu^{1-\text{ord}}, 0)$ of Proposition 5.4.1.

Remark 5.5.2. The reader may be relieved to hear that the class $\tilde{\nu}_{\text{fp}}^{1-\text{ord}}$ is “the ultimate among liftings of ν ”: all other variants of ν will be images of this one. \diamond

Proof. From the mapping-fibre definition of Gros fp-cohomology we have a long exact sequence

$$\cdots \rightarrow H_{\text{rig},c}^1(X_{G,0}^{1-\text{ord}}, \mathcal{V}\langle -D \rangle) \rightarrow \widetilde{H}_{\text{fp},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) \rightarrow \widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle) \rightarrow \cdots,$$

in which the boundary map is $P(\varphi) \circ \iota$. Moreover, this is compatible under extension-by-zero with the corresponding sequence for fp-cohomology of \mathbb{X}_G .

We have seen that the image of $(\nu^{1-\text{ord}}, 0)$ under ι is the class $\nu_{\text{dR}}^{1-\text{ord}}$, which is annihilated by $P(\varphi)$. Hence it lifts to $\widetilde{H}_{\text{fp},c}^2$. This lift is unique up to the image of an element of $H_{\text{rig},c}^1(X_{G,0}^{1-\text{ord}}, \mathcal{V}\langle -D \rangle)$; but from Proposition 5.1.4 it follows that this group has trivial Π_f -eigenspace for the prime-to- p Hecke operators, so there is a unique Hecke-equivariant lifting of $(\nu^{1-\text{ord}}, 0)$ to Gros-fp cohomology. The image of this class under extension-by-0 is therefore a Hecke-equivariant lifting of ν_{dR} to fp-cohomology of \mathbb{X}_G , so it must be ν_{fp} . \square

It is easily checked that $\iota^{-1}(\mathcal{X}_G^{\text{ord}}) = \mathcal{X}_H^{\text{ord}}$, so we have the following:

Corollary 5.5.3. *The pairing (4.1) is equal to $\langle \iota^{[j],*}(\tilde{\nu}_{\text{fp}}^{1-\text{ord}}), \widetilde{\text{Eis}}_{\text{syn}}^{t,\text{ord}} \rangle$, where $\widetilde{\text{Eis}}_{\text{syn}}^{t,\text{ord}}$ is the image of $\text{Eis}_{\text{syn}}^t$ in the Gros fp-cohomology of $\mathcal{X}_H^{\text{ord}}$. \square*

Here we define $\iota^{[j],*}$ for classes in Gros fp-cohomology using the quasi-isomorphism from the BGG complex to the full de Rham complex. We shall give explicit formulae in Section 7.1 below, but first we need to give an explicit form for $\tilde{\nu}_{\text{fp}}^{1-\text{ord}}$, which can only be done after restricting to $\mathcal{X}_G^{\text{ord}} \subset \mathcal{X}_G^{1-\text{ord}}$.

6. RESTRICTING TO THE FULLY-ORDINARY LOCUS

6.1. Cohomology with partial compact support. We now consider the cohomology of the fully ordinary locus $X_{G,0}^{\text{ord}}$. Since the complement $X_{G,0} - X_{G,0}^{\text{ord}}$ is the disjoint union of a closed subvariety $X_{G,0}^{(1-\text{ss})}$ and the open subvariety $X_{G,0}^{(2-\text{ss})} \cap X_{G,0}^{(1-\text{ord})}$, we can apply the formalism of [LZ20, §13] to define “cohomology of $\mathcal{X}_G^{\text{ord}}$ with compact support towards $\mathcal{X}_G^{(1-\text{ss})}$ ” (with coefficients in any abelian sheaf on \mathcal{X}_G). We write this as $R\Gamma_{c1}(\mathcal{X}_G, -)$. By construction, this comes equipped with a restriction map

$$R\Gamma_c(\mathcal{X}_G^{1-\text{ord}}, -) \rightarrow R\Gamma_{c1}(\mathcal{X}_G^{\text{ord}}, -),$$

which fits into a triangle whose third term is the compactly-supported cohomology of $\mathcal{X}_G^{1-\text{ord}} \cap \mathcal{X}_G^{2-\text{ss}}$. In particular, by the same argument as Proposition 5.1.4, we have isomorphisms

$$R\Gamma_{\text{rig},c1}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle) [\Pi_f] \longleftarrow R\Gamma_{\text{rig},c}(\mathcal{X}_G^{1-\text{ord}}, \mathcal{V}\langle -D \rangle) [\Pi_f] \longrightarrow R\Gamma_{\text{rig},c}(\mathcal{X}_G, \mathcal{V}\langle -D \rangle) [\Pi_f].$$

The advantage of working with $\mathcal{X}_G^{\text{ord}}$ is that both φ_1 and φ_2 have liftings.

Notation 6.1.1. Write $\nu^{\text{ord}} \in H_{c1}^1(\mathcal{X}_G^{\text{ord}}, \omega^{(-k_1, k_2+2)}(-D))$ for the image of $\nu^{1-\text{ord}}$ under the above restriction map.

Note 6.1.2. Over the ordinary locus, we have commuting liftings of both φ_1 and φ_2 , which both act on $R\Gamma_{c1}(\mathcal{X}_G^{\text{ord}}, \omega^{(\dots)})$; and the operator $T_{\mathfrak{p}_2}$ decomposes as $T_{\mathfrak{p}_2} = U_{\mathfrak{p}_2} + \varphi_2$, with $U_{\mathfrak{p}_2} \circ \varphi_2 = p^{k_2+1}\langle \mathfrak{p}_2 \rangle$. \diamond

Corollary 6.1.3. The class $P(\varphi) \cdot \nu^{\text{ord}}$ lies in the kernel of the Hecke operator $U_{\mathfrak{p}_2}$.

Proof. We have $\varphi = \varphi_1\varphi_2$. The result follows easily from Theorem 5.2.1 and Note 6.1.2, using the fact that $\nu^{(1-\text{ord})}$ is a $T_{\mathfrak{p}_2}$ -eigenvector. \square

6.2. The Poznan spectral sequence. We now recall a spectral sequence (introduced in [LZ20]) relating Gros fp-cohomology to coherent cohomology. Here Gros fp-cohomology is defined in the same way as for the \mathfrak{p}_1 -ordinary locus above, but now with $c1$ -support.

Definition 6.2.1. We define groups $\mathcal{C}_{\text{fp}, c1}^{m, n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$, for $m, n \geq 0$, by

$$\mathcal{C}_{\text{fp}, c1}^{m, n}(\dots) = H_{c1}^n(\mathcal{X}_G^{\text{ord}}, (\tau_{\geq 1} \text{BGG})^m(-D)) \oplus H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \text{BGG}^{m-1}(-D));$$

and we define differentials $\mathcal{C}_{\text{fp}, c1}^{m, n}(\dots) \rightarrow \mathcal{C}_{\text{fp}, c1}^{m+1, n}(\dots)$ by

$$(x, y) \mapsto (\nabla x, P(\varphi/p^n)\iota(x) - \nabla y),$$

where ι is the inclusion of $\text{Fil}^{1+j} \text{BGG}^\bullet$ into BGG^\bullet , and ∇ the differential of the BGG complex.

Note that $\mathcal{C}_{\text{fp}, c1}^{m, n}$ is zero for $m \leq 0$ (this is obvious for $m \leq -1$, and holds for $m = 0$ since $\text{Fil}^{(1+j)} \text{BGG}^0 = 0$). It is also zero for $n \leq 0$, since H_{c1}^0 vanishes for locally-free sheaves.

Proposition 6.2.2. There is a first-quadrant spectral sequence, the Poznań spectral sequence, with

$${}^{\text{Pz}}E_1^{mn} = \mathcal{C}_{\text{fp}, c1}^{m, n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}),$$

and the differentials on the E_1 page given by the formula above. This spectral sequence abuts to the Gros fp-cohomology $\tilde{H}_{\text{fp}, c1}^{m+n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$.

Definition 6.2.3. We define a coherent fp-pair (of degree (m, n) , twist $1+j$ and $c1$ -support) to be an element of the kernel of the differential $\mathcal{C}_{\text{fp}, c1}^{m, n} \rightarrow \mathcal{C}_{\text{fp}, c1}^{m+1, n}$; we write the group of these as $\mathcal{Z}_{\text{fp}, c1}^{m, n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$.

Thus an fp-pair is a pair of elements

$$x \in H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \mathcal{F}i^{1+j} \text{BGG}^m(-D)), \quad y \in H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \mathcal{F}i^{1+j} \text{BGG}^{m-1}(-D))$$

which satisfy

$$(6.1) \quad \nabla(x) = 0 \quad \text{and} \quad \nabla(y) = P(p^{-1-j}\varphi)\iota(x).$$

Note 6.2.4. Given x , the equation (6.1) does not determine the element y uniquely: it is determined up to an element of $H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \text{BGG}^{m-1}(-D))^{\nabla=0}$. \diamond

Proposition 6.2.5. For $0 \leq j \leq \min\{k_1, k_2\}$, the spectral sequence gives rise to an isomorphism

$$\alpha_{G, \text{fp}} : \mathcal{Z}_{\text{fp}, c1}^{1, 1}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) \xrightarrow{\cong} \tilde{H}_{\text{fp}, c1}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}).$$

Proof. Since the E_1 page of the spectral sequence is zero for $m \leq 0$, the term ${}^{\text{Pz}}E_2^{1, 1}$ is the kernel of the differential on ${}^{\text{Pz}}E_1^{1, 1}$, which is the group of fp-pairs. Since this is the only nonzero term with $m+n=2$, and the incoming and outgoing differentials at ${}^{\text{Pz}}E_r^{1, 1}$ are trivially zero for all $r \geq 1$, we conclude that ${}^{\text{Pz}}E_2^{1, 1}$ coincides with \tilde{H}^2 of the abutment, as required. \square

Corollary 6.2.6. Every cohomology class in $\tilde{H}_{\text{fp}, c1}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$ can be uniquely represented by a coherent fp-pair of degree $(1, 1)$.

Proof. Since the E_1 terms of the spectral sequence are supported in the region $m, n \geq 1$, there are no other terms of total degree 2 except $(m, n) = (1, 1)$; and clearly $E_2^{(1, 1)} = E_\infty^{(1, 1)}$ since the differentials on the E_2 page and beyond land outside this region. \square

An exactly analogous argument shows that for the truncated de Rham cohomology groups $\tilde{H}_{\text{dR}, c1}^i$ (the hypercohomology of $\tau_{\geq 1} \text{BGG}^\bullet(-D)$), we have an isomorphism

$$H_{c1}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^1(-D)) \xrightarrow{\alpha_{G, \text{rig}}} \tilde{H}_{\text{dR}, c1}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle, 1+j).$$

Lemma 6.2.7. *Let $0 \leq j \leq \min\{k_1, k_2\}$. We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}_{\text{fp}, \text{cl}}^{1,1}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}(-D); 1+j, P_{1+j}) & \xrightarrow{\alpha_{G, \text{fp}}} & \tilde{H}_{\text{fp}, \text{cl}}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}(-D); 1+j, P_{1+j}) \\ \downarrow & & \downarrow \\ H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^1(-D)) & \xrightarrow{\alpha_{G, \text{rig}}} & \tilde{H}_{\text{dR}, \text{cl}}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}(-D), 1+j) \end{array}$$

where the vertical arrows are the natural projection maps.

Proof. Clear from the constructions. \square

6.3. Construction of a coherent fp-pair.

Notation 6.3.1. *Define*

$$\tilde{\nu}_{\text{fp}}^{(\text{ord})} \in \tilde{H}_{\text{fp}, \text{cl}}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}(-D); 1+j, P_{1+j})$$

to be the restriction of the class $\tilde{\nu}_{\text{fp}}^{(1-\text{ord})}$ constructed above.

Corollary 6.3.2. *There exists a uniquely determined class*

$$\xi \in H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \omega^{(-k_1, -k_2)}(-D)),$$

which is independent of j , such that $(\nu^{(\text{ord})}, \xi)$ forms an fp-pair representing the class $\tilde{\nu}_{\text{fp}}^{(\text{ord})}$, and such that ξ lies in the Π -eigenspace for the Hecke operators away from pN .

Proof. The existence of ξ is immediate from Corollary 6.2.6 and Lemma 6.2.7; the independence from j is clear by construction.

Now, if ξ' is another element such that $(\nu^{(\text{ord})}, \xi')$ also represents $\tilde{\nu}_{\text{fp}}^{(\text{ord})}$, then

$$\xi - \xi' \in H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^0(-D))^{\nabla=0} \cong H_{\text{rig}, \text{cl}}^1(X_{G,0}, \mathcal{V}(-D)).$$

As we have seen, the Π_f -eigenspace in this cohomology is zero, so there is a unique choice of ξ which is Hecke-equivariant. \square

Lemma 6.3.3. *The element ξ has the following properties:*

$$(6.2) \quad \varphi_1 \cdot \xi = \alpha_1 \xi \quad \text{and} \quad U_{p_2} \cdot \xi = 0.$$

Proof. We first show the latter statement. We deduce from Corollary 6.1.3 that

$$U_{p_2} \cdot \xi \in H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^0(-D))^{\nabla=0}[\Pi'_f].$$

But as observed above, this space is zero. The former statement follows analogously, by considering the element $(\varphi_1 - \alpha_1) \cdot \xi$. \square

We now lift these classes from the BGG complex to the full de Rham complex:

Definition 6.3.4.

(1) Write $\dot{\xi}$ for the image of ξ in $H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \mathcal{V}(-D))$.

(2) Similarly, for $0 \leq j \leq \min\{k_1, k_2\}$ write $\dot{\nu}_j^{(\text{ord})}$ for the image of $\nu^{(\text{ord})}$ in

$$H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \mathcal{F}il^j \mathcal{V} \otimes \Omega^1(-D)).$$

Lemma 6.3.5. *The element $\dot{\xi}$ also satisfies*

$$(6.3) \quad \varphi_1 \cdot \dot{\xi} = \alpha_1 \dot{\xi} \quad \text{and} \quad U_{p_2} \cdot \dot{\xi} = 0.$$

The following result will be very important for the regulator evaluation:

Proposition 6.3.6. *We have*

$$U_p \circ \iota^*(\dot{\xi}) = 0.$$

Proof. As maps $H^\bullet(\mathcal{X}_G^{\text{ord}}) \rightarrow H^\bullet(\mathcal{X}_H^{\text{ord}})$, we have the following identity:

$$(6.4) \quad U_p \circ \iota^* \circ \varphi_{p_1} = \iota^* \circ (\langle p_1 \rangle U_{p_2}).$$

Since $\dot{\xi}$ is an eigenvector of φ_{p_1} with non-zero eigenvalue, and it is in the kernel of φ_{p_2} , the Proposition follows. \square

7.1. Relating the BGG and de Rham complexes. We need to recall some formulae relating the de Rham and BGG complexes for GL_2 . Let $k \in \mathbf{Z}_{\geq 0}$. The BGG complex for weight $k \geq 0$ is given by $[\omega^{-k} \xrightarrow{\Theta} \omega^{k+2}]$, where Θ is a differential operator given in terms of q -expansions by

$$\Theta = \frac{(-1)^k}{k!} \theta^{k+1}, \quad \theta = q \frac{d}{dq}.$$

(Cf. [TX16, Remark 2.17] for example.) The map $\mathrm{BGG}^\bullet \rightarrow \mathrm{DR}^\bullet$ is given as follows.

- In degree 1, it is given by the tensor product of the natural inclusion $\omega^k = \mathrm{Fil}^k(\mathrm{Sym}^k \mathcal{W}_H) \hookrightarrow \mathcal{W}_H$ and the Kodaira–Spencer isomorphism $\omega^2 \cong \Omega^1(D)$.
- In degree 0, a section s of ω^{-k} is mapped to the unique section of $\mathrm{Sym}^k \mathcal{W}_H$ whose image in $\mathrm{Sym}^k \mathcal{W}_H / \mathrm{Fil}^0 \cong \omega^{-k}$ is s , and whose image under the differential ∇ lands in $\omega^k \otimes \Omega^1(D)$.

We now recall (and somewhat reformulate) some results from §4 of [KLZ20] giving a completely explicit description of these maps. We write $\mathcal{X}_H^{\mathrm{ord}}$ for the ordinary locus *as a classical rigid space* (not a dagger space), i.e. neglecting overconvergence. Passing to the Igusa tower $\mathcal{I}\mathcal{G}_H$ (the canonical \mathbf{Z}_p^\times -covering of $\mathcal{X}_H^{\mathrm{ord}}$ parametrising ordinary elliptic curves with a trivialization of their formal group), we obtain a canonical section v of ω , corresponding to the invariant differential form $\frac{dT}{T}$ on the Tate curve $\mathbf{G}_m/q^\mathbf{Z}$; note that $\varphi^*(v) = pv$. If we let w be the unique section such that $\nabla(v) = u \otimes \xi$, where ξ the local basis of $\Omega^1_{\mathcal{X}_H^{\mathrm{ord}}}(D)$ corresponding to $\frac{dq}{q}$, then v and w are a basis of sections of \mathcal{W}_H over the Igusa tower, with w spanning the unit-root subspace (we have $\varphi^*(w) = w$) and $\nabla w = 0$. Hence we obtain a basis of sections $(v^a w^{k-a})_{0 \leq a \leq k}$ of $\mathrm{Sym}^k \mathcal{W}_H$ in which the actions of ∇ and φ are completely explicit.

In these coordinates, the map from ω^{-k} to $\mathrm{Sym}^k \mathcal{W}_H$ sends a section \mathcal{G} of ω^{-k} to the section

$$\sum_{i=0}^k \frac{(-1)^i}{i!} \theta^i(\mathcal{G}) \cdot v^i w^{k-i}.$$

One verifies easily that the image of this under ∇ is $\theta^{k+1}(\mathcal{G})v^k \otimes \xi$, as expected. (Note that it is not *a priori* obvious that such a sum is overconvergent if \mathcal{G} is, since neither θ nor the local bases $v^i w^{k-i}$ have any meaning outside the ordinary locus.)

7.2. The Eisenstein class as a coherent fp-pair. Recall that the GL_2 Eisenstein class $\mathrm{Eis}_{\mathrm{syn},N}^t$ lies in $H_{\mathrm{syn}}^1(\mathbb{Y}_H, \mathcal{V}_H(1+t))$, where \mathbb{Y}_H is the \mathbf{Z}_p -model of the modular curve $Y_1(N)$, and \mathcal{V}_H is the sheaf corresponding to the t -th symmetric power of the standard representation.

If we restrict to the open subscheme $\mathbb{Y}_H^{\mathrm{ord}}$ given by removing the supersingular points of the special fibre (so $\mathbb{Y}_H^{\mathrm{ord}}$ has the same generic fibre as \mathbb{Y}_H), and work with Gros syntomic cohomology, then we have a convenient explicit description: a class $x \in \widetilde{H}_{\mathrm{syn}}^1(\mathcal{X}_1(N)^{\mathrm{ord}}, \mathcal{V}_H, 1+t)$ is given by a pair (x_0, x_1) , where

$$x_0 \in H^0(\mathcal{X}_1(N)^{\mathrm{ord}}, \mathcal{V}), \quad x_1 \in H^0(\mathcal{X}_1(N)^{\mathrm{ord}}, \mathrm{Fil}^t \mathcal{V} \otimes \Omega^1(D)), \quad \nabla x_0 = (1 - p^{-1-t} \varphi) x_1.$$

The sheaf $\mathrm{Fil}^t \mathcal{V} \otimes \Omega^1(D)$ is simply ω^{t+2} , so x_1 is an overconvergent p -adic modular form of weight $t+2$; via this description, φ acts on overconvergent forms as $p^{t+1} \langle p \rangle V_p$, where $\langle p \rangle$ is the diamond operator for $p \bmod N$.

We let $\widetilde{\mathrm{Eis}}_{\mathrm{syn},N}^{t,\mathrm{ord}}$ be the image of $\mathrm{Eis}_{\mathrm{syn},N}^t$ in this group; we shall now write down an explicit representing pair. The following is a reformulation of Theorem 4.5.7 of [KLZ20]; the formulations differ because we are using symmetric powers here rather than symmetric tensors (the basis vector $v^{[r,s]}$ of *op.cit.* corresponds to $\frac{v^s w^r}{r!s!}$ in our present notation) and because we use a slightly different notation for Eisenstein series following [Kat04].

Proposition 7.2.1. *The Eisenstein class $\widetilde{\mathrm{Eis}}_{\mathrm{syn},N}^{t,\mathrm{ord}}$ is represented by the pair of sections $(\epsilon_0^t, \epsilon_1^t)$ whose restrictions to $\mathcal{X}_H^{\mathrm{ord}}$ are*

$$\begin{aligned} \epsilon_0 &= -N^k \sum_{u=0}^t \frac{(-1)^{t-u}}{u!} \theta^u \left(E_{0,1/N}^{-t,\mathrm{ord}} \right) \cdot v^{t-u} w^u, \\ \epsilon_1 &= -\frac{N^t}{t!} F_{0,1/N}^{t+2} \cdot v^t \otimes \xi. \end{aligned}$$

Here $F_{0,1/N}^{t+2}$ is the algebraic Eisenstein series with q -expansion

$$\zeta(-1-t) + \sum_{n>0} q^n \sum_{d|n} \binom{n}{d}^{t+1} (\zeta_N^d + (-1)^t \zeta_N^{-d}),$$

and $E_{0,1/N}^{-t,\text{ord}}$ is an ordinary p -adic Eisenstein series of weight $-t$, satisfying $\theta^{t+1}(E_{0,1/N}^{-t,\text{ord}}) = (1 - \langle p \rangle V_p) F_{0,1/N}^{t+2}$. \square

Remark 7.2.2. Note that the individual terms $\theta^u(E_{0,1/N}^{-t,\text{ord}})$ are p -adic modular forms, but they are not overconvergent (unless $u = 0$). Nonetheless, ϵ_0 is an overconvergent section of \mathcal{V}_H ; the non-overconvergence arises because the sections v, w we are using to trivialize \mathcal{V}_H over the ordinary locus are not themselves overconvergent. \diamond

7.3. Explicit formulae for the Clebsch–Gordan map. The map $\text{CG}^{[j]}$ is given explicitly by the following formula (c.f. [KLZ20, Prop. 5.1.2])⁴: for $0 \leq s \leq t$, we have

$$(7.1) \quad \iota^{[j]}(v^s w^{t-s}) = \sum_{\substack{0 \leq r \leq k_1 - j \\ 0 \leq r' \leq k_2 - j \\ r + r' = s}} \sum_{i=0}^j (-1)^i \frac{s!(t-s)!}{r!(r')!(k_1-r-j)!(k_2-r'-j)!i!(j-i)!} v^{r+i} w^{k_1-r-i} \boxtimes v^{r'+j-i} w^{k_2-r'-j+i} \otimes e_{-j}.$$

Lemma 7.3.1. For given values of k_1, k_2, j , the image of $\iota^{[j]}(v^s w^{t-s})$ in the line spanned by the basis vector $v^{k_1} \boxtimes w^{k_2} \otimes e_{-j}$ is zero for all s except $s = k_1 - j$, in which case it is equal to $\frac{(-1)^j}{j!} \cdot v^{k_1} \boxtimes w^{k_2} \otimes e_{-j}$.

Proof. This is a superficially modified version of Proposition 5.1.2 of [KLZ20]. \square

Remark 7.3.2. In particular, this shows that the Clebsch–Gordan map is not in general defined integrally (i.e. does not respect the lattice given by the \mathbf{Z} -span of the basis vectors). However, the coefficient $j!$ is the worst possible – one can check that $j! \iota^{[j]}$ is integral. \diamond

7.4. Reduction to a pairing in coherent cohomology. Recall that we want to evaluate the pairing

$$(7.2) \quad \left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn},N}^t \right\rangle.$$

Now we have

$$(7.3) \quad \begin{aligned} (7.2) &= \left\langle (\iota^{[j]})^* \left(\nu_{\text{fp}}^{(1-\text{ord})} \right), \text{Eis}_{\text{syn},N}^{t,\text{ord}} \right\rangle \\ &= \left\langle (\iota^{[j]})^* \left(\tilde{\nu}_{\text{fp}}^{(1-\text{ord})} \right), \widetilde{\text{Eis}}_{\text{syn},N}^{t,\text{ord}} \right\rangle \\ &= \left\langle (\iota^{[j]})^* \left(\tilde{\nu}_{\text{fp}}^{(\text{ord})} \right), \widetilde{\text{Eis}}_{\text{syn},N}^{t,\text{ord}} \right\rangle \\ &= \left\langle (\iota^{[j]})^* \left(\dot{\nu}_j^{(\text{ord})}, \dot{\xi} \right), (\epsilon_1, \epsilon_0) \right\rangle. \end{aligned}$$

Here, (7.3) takes values in the group

$$\tilde{H}_{\text{fp},c}^2(\mathcal{X}_H^{\text{ord}}, \mathbf{Q}_p \langle -\mathcal{D}_H \rangle, 2; P_j) \cong^{\text{tr}} \mathbf{Q}_p.$$

We evaluate (7.3) using the formalism of cup products in fp-cohomology (cf. [Bes00]). We have

$$P_j(st) = a(t, s)P_j(t) + b(t, s)(1-s)$$

with $a(t, s) = s$ and

$$b(t, s) = \frac{P_j(st) - sP_j(t)}{1-s} = 1 - b \cdot t^2 s, \quad b = \frac{p^{2j+2}}{\alpha_1^2 \alpha_2 \beta_2} = \frac{\beta_1}{\alpha_1} \cdot \frac{1}{p^{(k_1+k_2-2j)}}.$$

Hence

$$\begin{aligned} &P_j(p^{-1}) \times \left\langle (\iota^{[j]})^* \left(\dot{\nu}_j^{(\text{ord})}, \dot{\xi} \right), (\epsilon_1^t, \epsilon_0^t) \right\rangle \\ &= (\iota^{[j]})^*(\dot{\xi}) \cup \varphi_H^*(\epsilon_1^t) + \left(1 - \frac{\beta_1 \cdot (\varphi_H^{*,2} \otimes \varphi_H^*)}{p^{k_1+k_2-2j} \langle p \rangle \alpha_1} \right) \left((\iota^{[j]})^*(\dot{\nu}_j^{(\text{ord})}) \cup \epsilon_0^t \right). \end{aligned}$$

⁴The factorials appear slightly different from *op.cit.* since we are here working with symmetric powers v^m rather than symmetric tensors $v^{[m]}$. This is also the reason for the presence of $t!$ in the formula for the Eisenstein series.

Lemma 7.4.1. *We have $(\iota^{[j]})^*(\dot{\xi}) \cup \varphi_H^*(\epsilon_0^t) = 0$.*

Proof. Observe that U_p acts on the top-degree rigid cohomology as multiplication by a power of p . But we also have

$$U_p \left((\iota^{[j]})^*(\dot{\xi}) \cup \varphi_H^*(\epsilon_0^t) \right) = U_p \left((\iota^{[j]})^*(\dot{\xi}) \right) \cup \epsilon_0^t,$$

which is equal to zero by Proposition 6.3.6. The Proposition follows. \square

Proposition 7.4.2. *Equation (7.3) is equal to*

$$\frac{\left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right)} \left\langle (\iota^{[j]})^*(\dot{\nu}_j^{(\text{ord})}), \epsilon_0^t \right\rangle.$$

Proof. Since any p -depleted form will pair to 0 with a form that's in a direct sum of finite-slope φ -eigenspaces, we only care about the ϵ_0 term modulo exact forms. We also only care about its projection to the $\langle p \rangle = 1$ eigenspace, because we are pairing it with a class in the Π -eigenspace and Π has trivial central character.

On the space $H_{\text{rig},c}^2(\mathcal{X}_H^{\text{ord}}(-\mathcal{D}), \mathbf{Q}_p(2))$, the Frobenius acts as p^{-1} , and $\langle p \rangle$ acts trivially; so we can replace $\frac{\beta_1 \cdot (\varphi_H^{*,2} \otimes \varphi_H^*)}{p^{k_1+k_2-2j}\langle p \rangle \alpha_1}$ with $\frac{\beta_1 \cdot (1 \otimes \varphi_H^{-1})}{p^{k_1+k_2-2j+2}\alpha_1}$.

The operator φ^{-1} makes sense modulo p -depleted forms, and in this quotient we have

$$\varphi^{-1} \cdot \epsilon_0^m = p^{m+1} \langle p \rangle^{-1} \epsilon_0^m.$$

So we are done. \square

Remark 7.4.3. Note that $\dot{\nu}_j^{(\text{ord})}$ makes sense over the \mathfrak{p}_1 -ordinary locus $\mathcal{X}_G^{1-\text{ord}}$; the antiderivative $\dot{\xi}$ is only defined over the fully ordinary locus $\mathcal{X}_G^{\text{ord}}$, but this term has disappeared from our formula. So we can also interpret the pairing of Proposition 7.4.2 as a cup-product in the cohomology of $\mathcal{X}_G^{1-\text{ord}}$, which will allow us to compare with the construction of p -adic L -functions from [GLZ23]. \diamond

7.5. A partial unit root splitting. We saw above that over $\mathcal{X}_H^{\text{ord}}$ (the ordinary locus as a classical rigid space) the Hodge filtration of \mathcal{V}_H has a canonical splitting given by the unit-root subspace. For X_G we have the more refined structure of a \mathbf{Z}^2 -filtration, and we can ask for splittings of either factor. We state below the case of interest.

Proposition 7.5.1. *Over $\mathcal{X}_G^{1-\text{ord}}$, the natural inclusion map $\omega^{(k_1, -k_2)} \hookrightarrow \mathcal{V}_G / \text{Fil}^{k_1+1}$ admits a canonical splitting.*

Proof. We have a projection map from \mathcal{V}_G onto $\mathcal{V}_G / \text{Fil}^{(0,1)} \mathcal{V}_G \cong \text{Sym}^{k_1} \mathcal{W}_G \boxtimes \omega^{-k_2}$. So it suffices to show that the filtration on $\text{Sym}^{k_1} \mathcal{W}_G$ is splittable over the 1-ordinary locus; but this is clear, since a complement is provided by the unit-root subspace for the operator φ_1 . \square

7.6. Comparison of pushforward maps. To begin with, we observe that there exist pushforward maps in coherent cohomology:

Proposition 7.6.1. *There exists a pushforward map on coherent cohomology*

$$\iota_{\text{coh},*} : H^0(\mathcal{X}_H^{\text{ord}}, \omega^{k_1-k_2}) \longrightarrow H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}).$$

Proof. Clear since the conormal sheaf of the embedding ι is isomorphic to ω^2 . \square

We want to compare this map with the pushforward maps $\iota_*^{[j]}$ on the cohomology of the de Rham sheaves.

Notation 7.6.2. *We use the notation of Section 2.2, and we write*

- $\mathcal{V}_G = \text{Sym}^{[k_1, k_2]} \mathcal{H}(\mathcal{A})_{\mathbf{Q}_p}$;
- $\mathcal{V}_H = \text{Sym}^t \mathcal{H}(\mathcal{E})_{\mathbf{Q}_p}$ for $0 \leq j \leq \min\{k_1, k_2\}$.

for vector bundles attached to V_H and V_G , respectively.

Proposition 7.6.3. *Pushforward along $\iota^{[j]}$ induces a commutative diagram*

$$\begin{array}{ccc}
H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) & \xrightarrow{\iota_*^{[j]}} & H^1(\mathcal{X}_G^{1-\text{ord}}, V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1(D)) \\
\uparrow & & \uparrow \\
H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H) & \longrightarrow & H^1(\mathcal{X}_G^{1-\text{ord}}, \text{Gr}^{k_1} V_G \otimes \Omega^1(D)) \\
& \searrow \frac{1}{j!} \iota_{\text{coh},*} & \downarrow \\
& & H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}).
\end{array}$$

Proof. For the commutativity of the top square, we note that the Clebsch–Gordan map is compatible with filtrations and hence induces a commutative diagram of B_H -representations

$$\begin{array}{ccc}
V_H/\text{Fil}^{k_1-j} V_H & \longrightarrow & V_G/\text{Fil}^{k_1} V_G \\
\uparrow & & \uparrow \\
\text{Gr}^{k_1-j} V_H & \longrightarrow & \text{Gr}^{k_1} V_G.
\end{array}$$

The identification of the diagonal map as $\frac{1}{j!} \iota_{\text{coh},*}$ arises similarly, using the explicit formula (7.1). \square

Proposition 7.6.4. *The diagram in Proposition 7.6.3 is compatible with the unit root splitting*

$$u_H : H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) \longrightarrow H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H)$$

and the partial unit root splitting

$$u_G : H^1(\mathcal{X}_G^{1-\text{ord}}, V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1) \longrightarrow H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}) :$$

we have a commutative diagram

$$\begin{array}{ccc}
H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) & \xrightarrow{\iota_*^{[j]}} & H^1(\mathcal{X}_G^{1-\text{ord}}, V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1) \\
\uparrow & & \uparrow \\
H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H) & \longrightarrow & H^1(\mathcal{X}_G^{1-\text{ord}}, \text{Gr}^{k_1} V_G \otimes \Omega^1) \\
& \searrow \frac{1}{j!} \iota_{\text{coh},*} & \downarrow \\
& & H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2})
\end{array}$$

$\xrightarrow{u_H}$ (left dashed arrow) $\xrightarrow{u_G}$ (right dashed arrow)

Proof. Over $\mathcal{X}_H^{\text{ord}}$, we have a canonical splitting of the Hodge filtration of \mathcal{W}_H , as above. To prove the proposition, it is sufficient to check that the unit root splittings induce a commutative diagram

$$\begin{array}{ccc}
H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) & \rightarrow & H^0(\mathcal{X}_H^{\text{ord}}, \iota^*(V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1)) \\
\uparrow & & \uparrow \\
H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H) & \longrightarrow & H^0(\mathcal{X}_H^{\text{ord}}, \iota^*(\text{Gr}^{k_1} V_G \otimes \Omega^1)) \\
& \searrow & \downarrow \\
& & H^0(\mathcal{X}_H^{\text{ord}}, \iota^*(\omega^{(k_1+2, -k_2)}))
\end{array}$$

$\xrightarrow{u_H}$ (left dashed arrow) $\xrightarrow{u_G}$ (right dashed arrow)

which boils down to an explicit computation with the Clebsch–Gordan map using equation (7.1). \square

It follows from the classicity theorem of higher Hida theory proved in [GLZ23] that the natural restriction map (forgetting overconvergence)

$$H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}) \rightarrow H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2})$$

is an isomorphism on the ordinary (slope 0) eigenspace for $U_{\mathfrak{p}_1}$. If Π is ordinary at \mathfrak{p}_1 (and α_1 is, necessarily, the unit root) then pairing with ν^{ord} factors through this eigenspace, so we obtain the following:

Corollary 7.6.5. *Assume Π is ordinary at \mathfrak{p}_1 . Then the linear functional on $H^0(\mathcal{X}_H^{\text{ord}}, V_H / \text{Fil}^{k_1-j+1} V_H)$ given by pairing with $(\iota^{[j],*})(\nu^{\text{ord}})$ factors through the unit root splitting into $: H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H)$, and it is given by*

$$\frac{k_1! k_2!}{j!} \cdot \left\langle \nu^{(1-\text{ord})}, \iota_{\text{coh},*}(-) \right\rangle_{\mathcal{X}_G^{1-\text{ord}}}. \quad \square$$

Remark 7.6.6. The factors $k_i!$ arise from the pairing of the basis vectors v^{k_i} and w^{k_i} of $\text{Sym}^{k_i} W_G$. \diamond

7.7. Relation to p -adic L -functions. We use Corollary 7.6.5 in order to relate the formula in Proposition 7.4.2 to values of p -adic L -functions. We assume henceforth that Π is ordinary at \mathfrak{p}_1 .

Proposition 7.7.1. *The pairing (7.2) is given by*

$$\begin{aligned} \left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn},N}^t \right\rangle = \\ \frac{N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} \left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right)} \binom{k_1}{j} k_2! \cdot \left(\nu^{(\text{ord})} \cup \iota_{\text{coh},*}(\theta^{(k_1-j)} E_{0,1/N}^{-t,\text{ord}}) \right). \end{aligned}$$

Compare Corollary 6.5.7 in [KLZ20].

Proof. We deduce from Proposition 7.2.1 that the image of $\epsilon_0^{k_1+k_2-2j}$ under projection to $H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} \mathcal{V}_H)$ is given by $-N^k (-1)^{k_2-j} \frac{(t)!}{(k_1-j)!} \theta^{k_1-j} (E_{0,1/N}^{-t,\text{ord}})$. Combining this with Proposition 7.4.2 and Corollary 7.6.5 gives the result. \square

Note 7.7.2. By adjunction, we can write the formula in Proposition 7.7.1 as

$$\begin{aligned} \left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn},N}^t \right\rangle = \\ \frac{N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} \left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right)} \binom{k_1}{j} k_2! \cdot \left(\iota_{\text{coh}}^*(\nu^{(\text{ord})}) \cup \theta^{(k_1-j)} E_{0,1/N}^{-t,\text{ord}} \right). \end{aligned}$$

\diamond

In order to relate this formula to a (non-critical) value of a p -adic L -function, we need to replace

$$\theta^{(k_1-j)} E_{0,1/N}^{-t,\text{ord}} = E_{0,1/N}^{j-k_2,\text{ord}}$$

by its p -depletion

$$E_{0,1/N}^{j-k_2,[p]} = (1 - \varphi \circ U_p) E_{0,1/N}^{j-k_2,\text{ord}}.$$

We adapt the argument from [KLZ20, §6.5]. Let $V_{\mathfrak{p}_2} = p^{-1-k_2} \langle \mathfrak{p}_2 \rangle^{-1} \varphi_2$, so $V_{\mathfrak{p}_2}$ is a right inverse of $U_{\mathfrak{p}_2}$. Let λ, μ be constants such that

$$U_{\mathfrak{p}_2}(\nu^{\text{ord}}) = \lambda \nu^{\text{ord}} - \mu V_{\mathfrak{p}_2}(\nu^{\text{ord}})$$

(explicitly, we have $\lambda = a_{\mathfrak{p}_2}(\mathcal{F})$ and $\mu = p^{1+k_2}$), and let $\gamma = p^{k_1-j}$. Using an analogue of Lemma 6.5.8 from *op.cit.*, we deduce the following result:

Lemma 7.7.3. *We have*

$$\iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]} = (1 - \lambda \gamma p^{-k_1} \beta_{\mathfrak{p}_1} \cdot V_p + \mu \gamma^2 (p^{-k_1} \beta_{\mathfrak{p}_1})^2 V_p^2) \left(\iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} \right).$$

Proof. Arguing as in §6.5 in *op.cit.*, we see that

$$\begin{aligned} \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]} = & (1 - \lambda \gamma p^{-k_1} \beta_{\mathfrak{p}_1} \cdot V_p + \mu \gamma^2 (p^{-k_1} \beta_{\mathfrak{p}_1})^2 V_p^2) \left(\iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} \right) \\ & + (1 - V_p U_p) \left(\mu \varphi^2 \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} - \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \varphi \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} \right). \end{aligned}$$

But the operator $1 - V_p U_p$ acts as the zero map, which proves the result. \square

We deduce the following formula:

Proposition 7.7.4. *We have*

$$\left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn}, N}^t \right\rangle = N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} \times \frac{\left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right) \left(1 - \frac{\beta_1\alpha_2}{p^{j+1}}\right) \left(1 - \frac{\beta_1\beta_2}{p^{j+1}}\right)} \binom{k_1}{j} k_2! \cdot \left(\iota_{\text{coh}}^*(\nu^{(\text{ord})}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]}\right).$$

We can also express this in terms of the class $\check{\nu}_{\Pi, \alpha}$ on the higher-level variety $\mathcal{X}_G(\mathfrak{p}_1)$ described above. If $\pi_{\mathfrak{p}_1}$ denotes the degeneracy map $\mathcal{X}_G(\mathfrak{p}_1) \rightarrow \mathcal{X}_G$, then we have $\text{pr}_{\mathfrak{p}_1, *}(\check{\nu}_{\Pi, \alpha}) = p \left(1 - \frac{\beta_1}{p\alpha_1}\right) \nu^{(\text{ord})}$; so the formula in Proposition 7.7.4 is equivalent to

$$\left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn}, N}^t \right\rangle = N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} p^{-1} \times \frac{1}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right) \left(1 - \frac{\beta_1\alpha_2}{p^{j+1}}\right) \left(1 - \frac{\beta_1\beta_2}{p^{j+1}}\right)} \binom{k_1}{j} k_2! \cdot \left(\iota_{\text{coh}}^* \circ \text{pr}_{\mathfrak{p}_1, *} \left(\nu_{1w(\mathfrak{p}_1)}^{(\text{ord})}\right) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]}\right).$$

We now recall the definition of the (imprimitive) p -adic Asai L -function in [GLZ23, Def. 7.3.1]. For a p -adic character $\sigma : \mathbf{Z}_p^\times \rightarrow \mathbf{C}_p^\times$, we have

$$(7.4) \quad L_{p, \text{Asai}}^{\text{imp}}(\Pi, \sigma) = \frac{1}{p} (\sqrt{D})^{-1-(k_1+k_2)/2-\sigma} (-1)^\sigma \left\langle \check{\nu}_{\Pi, \alpha}, \iota_{\text{coh}, *}^{(p)}(\mathcal{E}_{k_1-k_2}(\sigma)) \right\rangle,$$

where

- $X_H(p)$ is the modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$;
- $\iota^{(p)} : X_H(p) \rightarrow X_G(\mathfrak{p}_1)$ is the natural embedding induced by ι ;
- $\mathcal{E}_{k_1-k_2}$ is a p -adic family of p -depleted Eisenstein series, of constant weight $k_1 - k_2$, defined in [GLZ23, §7.2], with the Schwartz function $\Phi^{(p)}$ chosen as in §7.3.2 of *op.cit.*; at $\sigma = -\frac{t}{2}$, this specialises to $\theta^{k_1-j} \left(E_{0,1/N}^{-t,[p]}\right)$.

Lemma 7.7.5. *The following diagram is cartesian:*

$$\begin{array}{ccc} X_H(p) & \xrightarrow{\iota^{(p)}} & X_G(\mathfrak{p}_1) \\ \downarrow \pi_p & & \downarrow \pi_{\mathfrak{p}_1} \\ X_{H, \text{sph}} & \xrightarrow{\iota} & X_{G, \text{sph}} \end{array}$$

$$\text{so } \pi_{\mathfrak{p}_1}^* \circ \iota_* = \iota_*^{(p)} \circ \pi_p^*.$$

Comparing the previous proposition with (7.4), we hence deduce the main theorem of this paper:

Theorem 7.7.6. *Let $0 \leq j \leq \min\{k_1, k_2\}$, and let $t = k_1 + k_2 - 2j \geq 0$. Then we have*

$$\left\langle \nu_{\text{dR}}, \log \left(\text{loc}_p \text{AF}_{\text{ét}}^{[\Pi, j]} \right) \right\rangle = \frac{(\sqrt{D})^{j+1} (-1)^{(k_1-k_2)/2+1} N^t \binom{k_1}{j} k_2!}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right) \left(1 - \frac{\beta_1\alpha_2}{p^{j+1}}\right) \left(1 - \frac{\beta_1\beta_2}{p^{j+1}}\right)} \cdot L_{p, \text{Asai}}^{\text{imp}} \left(\Pi, -\frac{t}{2} \right).$$

Remark 7.7.7. Note that in [GLZ23] we also defined a primitive p -adic L -function, interpolating the critical values of the primitive Asai L -function (with the optimal Euler factors at ramified primes). One can likewise modify the construction of the Asai–Flach class by incorporating more general test data away from p (see [Gro20]), and prove a refinement of Theorem 7.7.6, relating Asai–Flach classes with appropriate “optimal” test data to non-critical values of the primitive p -adic Asai L -function. We leave the details to the interested reader. \diamond

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