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ASAI–FLACH CLASSES AND  $p$ -ADIC L-FUNCTIONS

GIADA GROSSI, DAVID LOEFFLER, AND SARAH LIVIA ZERBES

ABSTRACT. We prove a formula for the Bloch–Kato logarithm of the bottom class in the Asai–Flach Euler system associated to a quadratic Hilbert modular form. We show that this can be expressed as a value, outside the interpolation range, of the  $p$ -adic Asai  $L$ -function constructed in the prequel paper [GLZ23].

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## 1. INTRODUCTION

The theory of Euler systems – norm-compatible families of cohomology classes for global Galois representations – has played an important role in many discoveries in number theory. A key aspect of this theory is *explicit reciprocity laws*, which relate Euler systems to the special values of  $L$ -functions; this can be used as a criterion for the non-vanishing of Euler systems, a crucial input in applications to the Bloch–Kato conjecture and related problems.

One strategy for proving such explicit reciprocity laws, introduced in the seminal paper [BDP13] in the context of Heegner points, proceeds as follows. Supposing we already have an Euler system for some Galois representation  $V$ , we aim to carry out the following steps:

- (1) Construct a  $p$ -adic  $L$ -function, for some  $p$ -adic family of Galois representations specialising to  $V$ , whose values in a suitable interpolation range are critical values of the corresponding complex  $L$ -functions.
- (2) Relate values of this  $p$ -adic  $L$ -function at suitable points *outside* the range of interpolation to Bloch–Kato logarithms of Euler system classes.
- (3) Via analytic continuation, deduce a relation between Euler systems and  $p$ -adic  $L$ -values at points *inside* the interpolation range, where the  $p$ -adic  $L$ -function specialises to the complex  $L$ -value.

In the present paper we focus on the *Euler system of Asai–Flach classes*, introduced in [LLZ18], which is an Euler system for the Asai, or twisted tensor, Galois representation associated to a Hilbert modular form for a real quadratic field. In this setting, step (1) of the above strategy was carried out in the prequel paper [GLZ23]. The goal of the present paper is to carry out step (2): the main result (Theorem 7.7.6) expresses the logarithms of Euler system classes as non-critical values of the  $p$ -adic  $L$ -function constructed in *op.cit.*. The remaining step (3), with applications to the Bloch–Kato conjecture and the Iwasawa main conjecture for Asai  $L$ -functions, will be carried out in a third paper in the series, currently in preparation.

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This strategy was successfully carried out for the Euler system of Beilinson–Flach elements (attached to a Rankin–Selberg convolution of modular forms) in [BDR15, KLZ17]; and more recently in the  $\mathrm{GSp}_4$  setting in [LPSZ21, LZ20]. The Asai representation can be seen as a “twisted version” of the Rankin–Selberg convolution, and the construction of the Euler system classes is recognisably a direct generalisation of the Beilinson–Flach case; but the techniques used to construct  $p$ -adic  $L$ -functions and prove the reciprocity law for Rankin–Selberg convolutions do not straightforwardly generalise, since they rely on the decomposition of the Shimura variety for  $\mathrm{GL}_2 \times \mathrm{GL}_2$  as a product of two factors, and new techniques are required. Hence the present paper follows a strategy rather closer to the  $\mathrm{GSp}_4$  case, relying on the use of *higher Hida theory*, which has been developed for Hilbert modular varieties at split primes by the first author [Gro21]; and its “overconvergent” version, *higher Coleman theory*. The higher Coleman theory we use is based on the general results of [BP20], with some slight modifications to allow  $p$ -adic interpolation at only one of the primes above  $p$ , for which we refer to [LZ22].

(We remark that a related problem is also considered in the paper [LSZ20], where Skinner and two of the present authors computed a pairing which also involves the Bloch–Kato logarithm of the Asai–Flach class, but projected into a different piece of the de Rham cohomology which can be computed using only conventional, rather than higher, Coleman theory. The earlier paper gives an explicitly computable formula in terms of  $p$ -adic Hilbert modular forms, but one which does not seem to have any straightforward relationship to  $p$ -adic  $L$ -values, in contrast to the formula of Theorem 7.7.6.)

## 2. CONVENTIONS

Throughout this paper  $F$  denotes a real quadratic field, and we fix an enumeration of the embeddings  $F \hookrightarrow \mathbf{R}$  as  $\sigma_1, \sigma_2$ .

**2.1. Algebraic groups.** As in [LLZ18], define  $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$ , and  $H = \mathrm{GL}_2$ , with  $\iota : H \hookrightarrow G$  the natural embedding. We write  $B_H$  and  $B_G$  for the upper-triangular Borel subgroups.

Recall that if  $V$  is an algebraic representation of  $B_H$ , then  $V$  gives rise to a vector bundle on the Shimura variety  $Y_H$  (for any sufficiently small level group), endowed with an action of Hecke correspondences. There are two possible normalisations for this functor, and we normalise so that the defining 2-dimensional representation of  $H$  (restricted to  $B_H$ ) maps to the relative de Rham *cohomology* sheaf of the universal elliptic curve  $\mathcal{E} \rightarrow Y_H$  (rather than the relative homology, which is the other convention in use). There is an analogous construction for  $G$  as long as we restrict to representations trivial on the norm-one subgroup  $\left\{ \begin{pmatrix} x & \\ & x \end{pmatrix} : N_{F/\mathbf{Q}}(x) = 1 \right\}$  of  $Z_G$ , cf. [LLZ18, §3.2c].

**2.2. Algebraic representations and the Clebsch–Gordan map.** For the group  $G$ , we shall always work with representations over fields containing  $F$ . Given two representations  $V, V'$  of  $\mathrm{GL}_2$ , we write  $V \boxtimes V'$  for the tensor product of  $V, V'$ , endowed with a  $G$ -action as the tensor product of the actions via  $\sigma_1$  on  $V$  and via  $\sigma_2$  on  $V'$ .

Given integers  $k_1, k_2 \geq 0$ , and  $0 \leq j \leq \min\{k_1, k_2\}$ , let

$$V_G := \mathrm{Sym}^{k_1} W_G \boxtimes \mathrm{Sym}^{k_2} W_G \quad \text{and} \quad V_H = \mathrm{Sym}^t W_H, \quad t = k_1 + k_2 - 2j,$$

where  $W_H$  and  $W_G$  denote the defining 2-dimensional representations of  $\mathrm{GL}_2/\mathbf{Q}$  and  $\mathrm{GL}_2/F$  respectively.

The representation  $V_H$  has a canonical basis  $(v^a w^{t-a})_{0 \leq a \leq t}$ , where  $v, w$  are the two standard basis vectors of  $W_H$ ; and is equipped with a decreasing  $B_H$ -stable weight filtration  $\mathrm{Fil}^n V_H = \langle v^a w^{t-a} : a \geq n \rangle$ . Similarly,  $V_G$  is equipped with a bi-filtration (a decreasing filtration indexed by  $\mathbf{Z}^2$ ) arising from the weight vector filtrations on each factor.

There is a non-zero morphism of  $H$ -representations (the *Clebsch–Gordan map*), unique up to scalars,

$$\mathrm{CG}^{[k_1, k_2, j]} : V_H \rightarrow V_G \otimes \det^{-j}.$$

See (7.1) below for explicit formulae. If we equip the one-dimensional representation  $\det^{-j}$  with a filtration concentrated in degree  $-j$ , then this map respects the filtrations, and hence induces a map on the graded pieces.

## 3. THE ASAI MOTIVE AND ITS REALISATIONS

### 3.1. Automorphic representations of $G$ .

**Definition 3.1.1.** Let  $\Pi$  be the *unitary* cuspidal automorphic representation of  $G(\mathbf{A})$  generated by a holomorphic Hilbert modular newform of weight  $(k_1 + 2, k_2 + 2)$ , for some integers  $k_1, k_2 \geq 0$ , and some level  $\mathcal{N}_\Pi \trianglelefteq \mathcal{O}_F$ .

We shall suppose throughout this work that the following conditions hold:

- $\Pi$  is not of CM type,
- the central character of  $\Pi$  is trivial (and hence  $k_i$  are both even).

Both of these could almost certainly be relaxed with a little extra work, but we shall not pursue this here. The representation  $\Pi$  (and hence the integers  $(k_1, k_2)$ ) will remain fixed throughout the remainder of this paper.

We define  $a_{\mathfrak{n}}^{\circ}(\Pi)$ , for each ideal  $\mathfrak{n} \leq \mathcal{O}_F$ , to be the Hecke eigenvalues of the new vector of  $\Pi$ , normalised in the analytic fashion, so that  $|a_{\mathfrak{p}}^{\circ}(\Pi)| \leq 2$  for primes  $\mathfrak{p} \nmid \mathcal{N}_{\Pi}$ . For each prime  $\mathfrak{p} \nmid \mathcal{N}_{\Pi}$  we let  $\alpha_{\mathfrak{p}}^{\circ}$  and  $\beta_{\mathfrak{p}}^{\circ}$  be the Satake parameters of  $\Pi_{\mathfrak{p}}$ , normalised so that  $|\alpha_{\mathfrak{p}}^{\circ}| = |\beta_{\mathfrak{p}}^{\circ}| = 1$  and  $\alpha_{\mathfrak{p}}^{\circ} + \beta_{\mathfrak{p}}^{\circ} = a_{\mathfrak{p}}^{\circ}(\Pi)$ .

By a rationality theorem due to Shimura [Shi86, Proposition 1.2], there exists a number field  $L \subset \mathbf{C}$  (depending on  $\Pi$ ) such that the quantities  $N(\mathfrak{n})^{1/2} a_{\mathfrak{n}}^{\circ}$  lie in  $L$ , for all  $\mathfrak{n} \leq \mathcal{O}_F$ . Extending  $L$  if necessary, we may assume that  $L$  also contains the images of the embeddings  $\sigma_i : F \hookrightarrow \mathbf{R}$  (this is automatic if  $k_1 \neq k_2$ ).

**Definition 3.1.2.** For  $n \in \mathbf{Z}_{\geq 1}$ , we define  $a_n(\Pi)$  by

$$a_n(\Pi) = n^{(k_1+k_2+2)/2} a_n^{\circ}(\Pi).$$

One checks that  $a_n(\Pi) \in \mathcal{O}_L$  for all  $n$ . (However, unless we make the stronger assumption that  $k_1 = k_2 \pmod{4}$ , we cannot extend this normalisation of the Hecke eigenvalues to include ideals  $\mathfrak{n}$  which do not come from  $\mathbf{Q}$ , without introducing infinitely many square roots to  $L$ .)

**3.2. The Asai  $L$ -function.** For each rational prime  $\ell$ , let us set  $\Pi_{\ell} = \bigotimes_{v|\ell} \Pi_v$ , considered as a representation of  $G(\mathbf{Q}_{\ell}) = \prod_{v|\ell} \mathrm{GL}_2(F_v)$ . Associated to  $\Pi_{\ell}$  we have an *Asai  $L$ -factor*<sup>1</sup>,  $L_{\mathrm{As}}(\Pi_{\ell}, s)$ . We write  $L_{\mathrm{As}}(\Pi, s) = \prod_{\ell \text{ prime}} L_{\mathrm{As}}(\Pi_{\ell}, s)$  (without any Archimedean factors) for the Asai  $L$ -function; note that this Dirichlet series has coefficients in  $L$ .

Since the Euler factors of  $L_{\mathrm{As}}$  at primes dividing  $\mathcal{N}_{\Pi}$  are hard to describe explicitly, we shall also consider the “imprimitive” Asai  $L$ -function

$$L_{\mathrm{As}}^{\mathrm{imp}}(\Pi, s) = \zeta_{(N)}(2s) \sum_{n \geq 1} a_n^{\circ}(\Pi) n^{-s},$$

where the notation  $(N)$  signifies removing the Euler factors at the primes dividing  $N = \mathcal{N}_{\Pi} \cap \mathbf{Z}$ . Note that this Dirichlet series has coefficients in  $L$ . The following result is standard:

**Lemma 3.2.1.** *The function  $L_{\mathrm{As}}^{\mathrm{imp}}(\Pi, s) / L_{\mathrm{As}}(\Pi, s)$  is a product of polynomials in  $\ell^{-s}$  for  $\ell \mid N$ . All of its zeroes have real part in  $\{0, -\frac{1}{2}, -1\}$ .  $\square$*

**3.3. De Rham and coherent cohomology groups.** We let  $U_1(\mathcal{N}_{\Pi})$  denote the open compact subgroup  $\{g \in G(\mathbf{A}_f) : g = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{N}_{\Pi}}\}$ , so that the space of  $U_1(\mathcal{N}_{\Pi})$ -invariants of  $\Pi_f$  (the new subspace of  $\Pi_f$ ) is one-dimensional. We assume, for simplicity<sup>2</sup> that  $\mathcal{N}_{\Pi}$  does not divide 6 disc $_{F/\mathbf{Q}}$ , so that  $U_1(\mathcal{N}_{\Pi})$  is *sufficiently small* in the sense of [LLZ18, Definition 2.2.1]; thus the Hilbert modular variety  $Y_G$  of level  $U_1(\mathcal{N}_{\Pi})$  is a smooth quasiprojective variety defined over  $\mathbf{Q}$ .

**Definition 3.3.1.** Let  $\mathcal{V}_G$  denote the vector bundle with connection on  $Y_G$  corresponding to the algebraic representation<sup>3</sup>  $V_G$ ; and set

$$D_L^{\mathrm{As}}(\Pi) = H_{\mathrm{dR},c}^2(Y_G, \mathcal{V}_G)[\Pi_f],$$

which is 4-dimensional over  $L$ .

The de Rham cohomology can be computed via the coherent cohomology of a smooth toroidal compactification  $X_G$  of  $Y_G$ . We can then compute de Rham cohomology using the logarithmic de Rham complex  $\mathcal{V}_G \otimes \Omega_{X_G}^{\bullet}(D)$ , where  $D = X_G - Y_G$  is the boundary divisor; and we can compute compactly-supported

<sup>1</sup>This can be defined as the “common denominator” of a family of zeta-integrals, or alternatively via the local Langlands correspondence, but both definitions give the same  $L$ -factor. For  $\ell$  split this is one of the defining properties of the local Langlands correspondence; for  $\ell$  non-split, see [Mat10, Theorem 3.3].

<sup>2</sup>The excluded cases may be dealt with via the usual trick of introducing full level  $\mathfrak{h}$  structure, for an auxiliary ideal  $\mathfrak{h}$ , and then taking invariants under  $\mathrm{GL}_2(\mathcal{O}/\mathfrak{h})$ ; we leave the details to the interested reader.

<sup>3</sup>More precisely, we need to twist by an appropriate character to make the action of norm-one units trivial, but the resulting vector bundle is independent of the choice up to a canonical isomorphism.

de Rham cohomology using the “minus-log” complex  $\mathcal{V}_G \otimes \Omega_{X_G}^\bullet \langle -D \rangle$ , where  $\Omega^\bullet \langle -D \rangle := \Omega^\bullet \langle D \rangle (-D)$ . We denote the corresponding cohomologies by

$$\begin{aligned} R\Gamma_{\mathrm{dR}}(X_G, \mathcal{V}_G \langle D \rangle) &:= R\Gamma(X_G, \mathcal{V} \otimes \Omega_{X_G}^\bullet \langle D \rangle) \cong R\Gamma_{\mathrm{dR}}(Y_G, \mathcal{V}_G), \\ R\Gamma_{\mathrm{dR}}(X_G, \mathcal{V}_G \langle -D \rangle) &:= R\Gamma(X_G, \mathcal{V} \otimes \Omega_{X_G}^\bullet \langle -D \rangle) \cong R\Gamma_{\mathrm{dR},c}(Y_G, \mathcal{V}_G). \end{aligned}$$

Rather than explicitly working with the full de Rham complex, it is more convenient to work with the *dual BGG complex*

$$\mathrm{BGG}^\bullet = \left[ \omega^{(-k_1, -k_2)} \longrightarrow \omega^{(-k_1, k_2+2)} \oplus \omega^{(k_1+2, -k_2)} \longrightarrow \omega^{(k_1+2, k_2+2)} \right]$$

and its compactly-supported analogue  $\mathrm{BGG}^\bullet(-D)$ , which are quasi-isomorphic to the logarithmic de Rham complexes  $V \otimes \Omega^\bullet \langle D \rangle$  and  $V \otimes \Omega^\bullet \langle -D \rangle$  respectively. (We shall give explicit formulae for the differentials of the BGG complex, and the quasi-isomorphism relating it to the de Rham complex, in Section 7.1 below, but we shall not need this just yet.)

The BGG complex is equipped with a natural decreasing  $\mathbf{Z}^2$ -filtration  $\mathrm{Fil}^{\bullet\bullet}$ , which gives rise to a  $\mathbf{Z}^2$ -filtration on its cohomology. We can (and do) normalise so that the nontrivial graded pieces are in bidegrees  $\{(0, 0), (k_1 + 1, 0), (0, k_2 + 1), (k_1 + 1, k_2 + 1)\}$ , and are given by

$$\begin{aligned} \mathrm{Gr}^{(k_1+1, k_2+1)} &= H^0 \left( X_G, \omega^{(k_1+1, k_2+1)}(-D) \right) [\Pi_{\mathfrak{f}}], \\ \mathrm{Gr}^{(k_1+1, 0)} &= H^1 \left( X_G, \omega^{(k_1+2, -k_2)}(-D) \right) [\Pi_{\mathfrak{f}}], & \mathrm{Gr}^{(0, k_2+1)} &= H^1 \left( X_G, \omega^{(-k_1, k_2+2)}(-D) \right) [\Pi_{\mathfrak{f}}], \\ \mathrm{Gr}^{(0, 0)} &= H^2 \left( X_G, \omega^{(-k_1, -k_2)}(-D) \right) [\Pi_{\mathfrak{f}}], \end{aligned}$$

where  $\omega^{(r,s)}$  is the sheaf of Hilbert modular forms of weight  $(r, s)$  (so that  $\omega^{(r,s)}(-D)$  is the subsheaf of cusp forms). The induced single filtration, with graded pieces in degrees  $\{0, k_1 + 1, k_2 + 1, k_1 + k_2 + 2\}$ , is the Hodge filtration.

**Definition 3.3.2.** We let  $\nu$  be a basis of the 1-dimensional  $L$ -vector space  $H^1(X_G, \omega^{(-k_1, k_2+2)})[\Pi_{\mathfrak{f}}]$ .

*Remark 3.3.3* (Occult periods). After base-extension to  $\mathbf{C}$ , these graded pieces have canonical bases arising from the comparison between sheaf cohomology and Dolbeault cohomology. The one which will interest us is  $\mathrm{Gr}^{(0, k_2+1)}$ , which is spanned by the differential form associated to the real-analytic Hilbert modular form  $\mathcal{F}^{\mathrm{ah},1}$  (anti-holomorphic at the place  $\sigma_1$  and holomorphic at  $\sigma_2$ ) having the same Fourier–Whittaker coefficients as the holomorphic newform  $\mathcal{F}$  generating  $\Pi$ ; see Lemma 5.2.1 of [LLZ18] for further details. Harris’ *occult period* for  $\Pi$  at  $\sigma_1$  (cf. [Har90]) is the ratio between  $\nu$  and  $\mathcal{F}^{\mathrm{ah},1}$ , well-defined as an element of  $\mathbf{C}^\times/L^\times$ .  $\diamond$

**3.4. The Asai Galois representation.** We now fix (for the remainder of this paper) a prime  $p$ , and a prime  $\mathfrak{P}$  of the coefficient field  $L$  above  $p$ .

**Definition 3.4.1.**

- (i) Let  $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$  denote the four-dimensional Asai Galois representation associated to  $\Pi$  as in [LLZ18, Definition 4.4.2], defined as the  $\Pi_{\mathfrak{f}}$ -eigenspace in the  $p$ -adic étale cohomology of the Hilbert modular variety  $Y_{G,1}(\mathcal{N}_{\Pi}) \otimes \overline{\mathbf{Q}}$  (with coefficients in the étale local system of  $L_{\mathfrak{P}}$ -vector spaces determined by  $(k_1, k_2)$ ).
- (ii) Let  $D_{\mathfrak{P}}^{\mathrm{As}}(\Pi) = L_{\mathfrak{P}} \otimes_L D_L^{\mathrm{As}}(\Pi)$ , so that there is a canonical comparison isomorphism (compatible with the filtrations)

$$\mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p, V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)) \cong D_{\mathfrak{P}}^{\mathrm{As}}(\Pi).$$

*Remark 3.4.2.* More precisely, this is  $M_{L_{\mathfrak{P}}}(\mathcal{F})$  in the notation of *op.cit.*, where  $\mathcal{F}$  is the normalised newform generating  $\Pi$ .  $\diamond$

By results of Brylinski–Labesse and Nekovář recalled in *op.cit.*, the Galois representation  $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$  can be characterised, up to isomorphism, as the unique semisimple Galois representation whose  $L$ -series is  $L_{\mathrm{As}}(\Pi, s - \frac{k_1+k_2+2}{2})$ . (However, since we want to consider Euler system classes for  $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$ , it is important to fix not only an abstract isomorphism class of Galois representations but a specific realisation of this isomorphism class in étale cohomology.) By Poincaré duality for the Hilbert modular surface, there is a canonical nondegenerate symmetric bilinear form

$$(3.1) \quad \lambda : \mathrm{Sym}^2(V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)) \rightarrow L_{\mathfrak{P}}(-2 - k_1 - k_2),$$

equivariant for the action of  $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ . (The pairing is symmetric since  $V_{\mathfrak{P}}^{\mathrm{As}}(\Pi)$  appears in cohomology in even degree.)

*Remark 3.4.3.* By [LLZ18, Proposition 9.4.3], our hypotheses on  $\Pi$  imply that  $V_{\mathfrak{P}}^{\text{As}}(\Pi)$  is either irreducible, or the direct sum of a character and a three-dimensional irreducible representation, with the latter case occurring if and only if  $\Pi$  is a twist of a base-change from  $\text{GL}_2/\mathbf{Q}$ .  $\diamond$

**3.5. The split-prime case.** For the remainder of this paper, we suppose that  $p$  is split in  $F$ , and  $\Pi$  is unramified at the primes above  $p$ . Since the embeddings  $\sigma_i$  of  $F$  take values in  $L \subset \mathbf{C}$ , and we have fixed a prime  $\mathfrak{P} \mid p$  in  $L$ , we can number the primes above  $p$  as  $\mathfrak{p}_1, \mathfrak{p}_2$  such that  $\sigma_i(\mathfrak{p}_i) \subset \mathfrak{P}$ .

We define

$$a_{\mathfrak{p}_i}(\Pi) = p^{(k_i+1)/2} a_{\mathfrak{p}_i}^{\circ}(\Pi) \in L, \quad \text{so } a_p(\Pi) = a_{\mathfrak{p}_1}(\Pi) a_{\mathfrak{p}_2}(\Pi).$$

(Note this normalisation depends on the choice of prime  $\mathfrak{P} \mid p$ , and the  $a_{\mathfrak{p}_i}$  are integral at  $\mathfrak{P}$ , but possibly not at other primes of  $L$  above  $p$ .) We likewise define

$$\alpha_i = p^{\frac{1+k_i}{2}} \alpha_{\mathfrak{p}_i}^{\circ}$$

and similarly  $\beta_i$ . Extending  $L$  if necessary, we may suppose that the  $\alpha_i$  and  $\beta_i$  also lie in  $L$ .

**Corollary 3.5.1.** *The representation  $V_{\mathfrak{P}}^{\text{As}}(\Pi)$  is crystalline at  $p$ , so  $D_{\mathfrak{P}}(\Pi)$  is naturally a filtered  $\varphi$ -module; and the eigenvalues of  $\varphi$  on this module are the pairwise products  $\{\alpha_1 \alpha_2, \dots, \beta_1 \beta_2\}$ .  $\square$*

**Lemma 3.5.2.** *Let  $0 \leq j \leq \min(k_1, k_2)$  be an integer. Then  $p^j$  is not an eigenvalue of  $\varphi$  on  $D_{\mathfrak{P}}^{\text{As}}(\Pi)$ . Moreover, if  $p^{(1+j)}$  is an eigenvalue of  $\varphi$ , then we must have  $k_1 = k_2 = j$ .*

*Proof.* This follows from the fact that the Satake parameters  $\alpha_i^{\circ}, \beta_i^{\circ}$  have complex absolute value 1.  $\square$

This implies that for the Galois representation  $V_{\mathfrak{P}}^{\text{As}}(\Pi)^*(-j)$ , the Bloch–Kato subspaces  $H_e^1(\mathbf{Q}_p, -)$  and  $H_f^1(\mathbf{Q}_p, -)$  agree; and these are also equal to  $H_g^1$ , except possibly in the boundary case  $k_1 = k_2 = j$  in which case  $H_g^1$  can be strictly larger. Moreover, the inverse of the Bloch–Kato exponential map for  $V_{\mathfrak{P}}^{\text{As}}(\Pi)^*(-j)$  is an isomorphism

$$(3.2) \quad \log : H_f^1(\mathbf{Q}_p, V_{\mathfrak{P}}^{\text{As}}(\Pi)^*(-j)) \xrightarrow{\cong} (\text{Fil}^{1+j} D_{\mathfrak{P}}^{\text{As}}(\Pi))^* = (\text{Fil}^1 D_{\mathfrak{P}}^{\text{As}}(\Pi))^*,$$

with both sides 3-dimensional over  $L_{\mathfrak{P}}$ .

**3.6. Partial Frobenii.** We may identify  $D_{\mathfrak{P}}^{\text{As}}(\Pi)$ , as a Frobenius module (forgetting the filtration), with the rigid cohomology of the special fibre of  $Y_G$  at  $p$ . This special fibre has two commuting endomorphisms, the **partial Frobenii**  $\varphi_1$  and  $\varphi_2$  at the primes  $\mathfrak{p}_i$ , whose composite is the Frobenius  $\varphi$ ; more precisely,  $\varphi_i$  corresponds to sending a Hilbert–Blumenthal abelian surface  $A$  to the quotient  $A/(\ker(\varphi_A) \cap A[\mathfrak{p}_i])$ . We refer to [TX16] or [Nek18] for detailed accounts of this construction.

The operators  $\varphi_i$  induce commuting linear operators on  $D_{\mathfrak{P}}^{\text{As}}(\Pi)$ , with  $\varphi = \varphi_1 \varphi_2$ ; and it follows from the “partial Eichler–Shimura” comparison result proved in [Nek18] that for each  $i$  we have

$$(\varphi_i - \alpha_i)(\varphi_i - \beta_i) = 0 \quad \text{on } D_{\mathfrak{P}}^{\text{As}}(\Pi).$$

One checks easily that the partial Frobenii satisfy  $\lambda(\varphi_i x, \varphi_i y) = p^{k_i+1} \lambda(x, y)$ , where  $\lambda$  is the Poincaré duality form. This identifies the  $\beta_i$ -generalised eigenspace with the dual of that for  $\alpha_i$ . Hence, if  $\alpha_i \neq \beta_i$ , the  $\varphi_i = \alpha_i$  and  $\varphi_i = \beta_i$  eigenspaces are both 2-dimensional, and each is isotropic with respect to  $\lambda$ .

*Remark 3.6.1.* We are using a slightly different normalisation of the partial Frobenii here from [LZ22]: the  $\varphi_i$  here is  $p^{-t_i} \varphi_i$  in the notation of *op.cit.*, where  $(t_1, t_2)$  is an auxiliary choice of integers such that  $w = k_i + 2t_i$  is independent of  $i$ . This reflects the fact that the  $\alpha_i$  of *op.cit.* is  $p^{(w+1)/2} \alpha_{\mathfrak{p}_i}^{\circ}$ , while  $\alpha_i$  here is  $p^{(k_i+1)/2} \alpha_{\mathfrak{p}_i}^{\circ}$ . The present normalisation is more convenient for comparison with higher Hida theory, since it matches the minimal integral normalisation of the  $U'_{\mathfrak{p}_i}$  operators.  $\diamond$

**3.7. A lifting of  $\nu$ .** We write  $v_p$  for the valuation on  $L_{\mathfrak{P}}$  normalized by  $v_p(p) = 1$ ; and we fix an ordering of the Satake parameters at  $\mathfrak{p}_1$ . Then we have  $0 \leq v_p(\alpha_1) \leq k_1 + 1$ .

**Proposition 3.7.1.** *Suppose that  $\alpha_1 \neq \beta_1$ , and  $v_p(\alpha_1) < k_1$ . Then the vector space*

$$D_{\mathfrak{P}}(\Pi)^{(\varphi_1 = \alpha_1)} \cap \text{Fil}^{(0, k_2+1)} D_{\mathfrak{P}}(\Pi)$$

*is one-dimensional, and surjects onto  $\text{Gr}^{(0, k_2+1)} D_{\mathfrak{P}}(\Pi)$ .*

*Hence there exists a uniquely determined vector  $\nu_{\text{dR}} \in D_{\mathfrak{P}}$  with the following properties:*

- $\varphi_1 \nu_{\text{dR}} = \alpha_1 \cdot \nu_{\text{dR}}$ ;
- $\nu_{\text{dR}} \in \text{Fil}^{(0, k_2+1)} D_{\mathfrak{P}}$ ;

- the image of  $\nu_{\text{dR}}$  in the graded piece

$$\text{Gr}^{(0, k_2+1)} D_{\mathfrak{F}} \cong L_{\mathfrak{F}} \otimes_L H^1 \left( X_G, \omega^{(-k_1, k_2+2)} \right) [\Pi_{\mathfrak{f}}]$$

coincides with the  $\nu$  of Definition 3.3.2.

*Proof.* This is a special case of the first main theorem of [LZ22], which states that (over totally-real fields of any degree) the  $\mathbf{Z}^d$ -indexed filtration of  $D_{\mathfrak{F}}^{\text{As}}(\Pi)$  has a canonical splitting given by intersecting with partial Frobenius eigenspaces. The assumption  $v_p(\alpha_1) < k_1$  is the *strictly small slope* assumption of *op.cit.*  $\square$

*Remark 3.7.2.* Note that  $\ker((\varphi - \alpha_1\alpha_2)(\varphi - \alpha_1\beta_2) : D_p \rightarrow D_p)$  contains  $D_p^{(\varphi_1=\alpha_1)}$ , but it may be larger; this always occurs if  $\Pi$  is a twist of a base-change from  $\mathbf{Q}$  (so that  $\alpha_1/\beta_1 = \alpha_2/\beta_2$ ).  $\diamond$

#### 4. DEFINITION OF THE $p$ -ADIC REGULATOR

**4.1. Euler system classes.** Let  $0 \leq j \leq \min(k_1, k_2)$ . We refer to [LLZ18, Definition 4.4.6] for the definition of the *étale Asai–Flach class*

$$\text{AF}_{\text{ét}}^{[\Pi, j]} \in H^1 \left( \mathbf{Z}[1/\Sigma], V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j) \right),$$

where  $\Sigma$  is the set of primes dividing  $pN \text{disc}(F)$ . (More precisely, this class is defined for any normalised eigenform  $\mathcal{F}$ , not necessarily new, and we are defining  $\text{AF}_{\text{ét}}^{[\Pi, j]}$  as  $\text{AF}_{\text{ét}}^{[\mathcal{F}, j]}$  where  $\mathcal{F}$  is the unique newform generating  $\Pi$ , consistently with our definition of the Galois representation  $V_{\mathfrak{F}}^{\text{As}}(\Pi)$ .)

*Remark 4.1.1.* In *op.cit.* we showed that  $\text{AF}_{\text{ét}}^{[\Pi, j]}$  was the  $m = 1$  case of a family of classes defined over  $\mathbf{Q}(\mu_m)$  with norm-compatibility properties as  $m$  varies, but we shall not use this here.  $\diamond$

*Note 4.1.2.* We briefly recall the definition of  $\text{AF}_{\text{ét}}^{[\Pi, j]}$ . Letting  $\mathcal{V}_G$  and  $\mathcal{V}_H$  denote the relative Chow motives (over  $L$ ) associated to the representations  $V_G$  and  $V_H$  defined in Section 2.2, we have a pushforward map

$$\iota_*^{[j]} : H_{\text{mot}}^1(Y_{H,1}(N), \mathcal{V}_H(1+t)) \otimes_{\mathbf{Q}} L \longrightarrow H_{\text{mot}}^3(Y_{G,1}(\mathcal{N}_{\Pi}), \mathcal{V}_G(2+k_1+k_2-j)),$$

where  $t = k_1 + k_2 - 2j$  as before. We have an analogous pushforward map in étale cohomology with  $L_{\mathfrak{F}}$ -coefficients, and the two are compatible via the étale regulator map  $r_{\text{ét}}$ . Since the  $\Pi_{\mathfrak{f}}$ -generalised eigenspace in the cohomology of  $Y_{G,1}(\mathcal{N}_{\Pi})$  vanishes outside degree 2, the Hochschild–Serre spectral sequence gives a projection map from  $H_{\text{ét}}^3(Y_{G,1}(\mathcal{N}_{\Pi}), \mathcal{V}_G(2+k_1+k_2-j))$  to  $H^1(\mathbf{Z}[1/\Sigma], V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j))$ . We can then define  $\text{AF}_{\text{ét}}^{[\Pi, j]}$  as the image of the weight  $t$  Eisenstein class under this chain of maps.  $\diamond$

#### 4.2. Localisation at $p$ and syntomic cohomology.

**Proposition 4.2.1.** *The localisation of  $\text{AF}_{\text{ét}}^{[\Pi, j]}$  at  $p$  lies in the Bloch–Kato subspace*

$$H_{\mathfrak{f}}^1(\mathbf{Q}_p, V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j)) \subseteq H^1(\mathbf{Q}_p, V_{\mathfrak{F}}^{\text{As}}(\Pi)^*(-j)).$$

*Proof.* This follows from the comparison between étale and syntomic cohomology, since the map from syntomic to étale cohomology factors through the Bloch–Kato exponential map; cf. [KLZ20, Proposition 5.4.1] in the analogous case of Beilinson–Flach elements.  $\square$

Now fix a choice of  $\alpha_1$  satisfying the conditions of Proposition 3.7.1, and let  $\nu_{\text{dR}} \in \text{Fil}^{(0, k_2+1)} D_{\mathfrak{F}}(\Pi)$  be the ensuing lifting of  $\nu$ . Our goal will be to compute the pairing

$$(4.1) \quad \left\langle \nu_{\text{dR}}, \log \left( \text{loc}_p \text{AF}_{\text{ét}}^{[\Pi, j]} \right) \right\rangle_{D_{\mathfrak{F}}(\Pi)},$$

where  $\log$  is the Bloch–Kato logarithm (3.2), and  $\langle -, - \rangle_{D_{\mathfrak{F}}(\Pi)}$  denotes the canonical pairing between  $D_{\mathfrak{F}}^{\text{As}}(\Pi)$  and its dual.

Let  $\mathbb{Y}_G$  denote the canonical  $\mathbf{Z}_p$ -model of  $Y_{G,1}(\mathcal{N}_{\Pi})$ . This is a smooth  $\mathbf{Z}_p$ -scheme, and we may choose an arithmetic toroidal compactification  $\mathbb{X}_G$  such that  $(\mathbb{Y}_G, \mathbb{X}_G)$  is a *smooth pair* over  $\mathbf{Z}_p$  in the sense of [KLZ20] (i.e.  $\mathbb{X}_G$  is smooth, and the cuspidal divisor  $\mathbb{X}_G - \mathbb{Y}_G$  is a smooth normal-crossing divisor relative to  $\text{Spec } \mathbf{Z}_p$ ). Thus Besser’s theory of *rigid syntomic* and *finite-polynomial* cohomology applies to  $\mathbb{Y}_G$ , and rigid-syntomic cohomology has a natural comparison to étale cohomology.

**Notation 4.2.2.** Let  $P(T) = \left(1 - \frac{T}{\alpha_1\alpha_2}\right) \left(1 - \frac{T}{\alpha_1\beta_2}\right) \in L[T]$ , and let  $P_{1+j}(T) = P(p^{1+j}T)$ .

**Proposition 4.2.3.** *There is a unique lift  $\nu_{\text{fp}}$  of  $\nu_{\text{dR}}$  to the space*

$$H_{\text{fp},c}^2(\mathbb{Y}_G, \mathcal{V}_G; 1+j, P_{1+j})[\Pi_f].$$

*Remark 4.2.4.* The above group is actually independent of  $j$  in the range  $0 \leq j \leq \min(k_1, k_2)$ : Besser's cohomology for twist  $r$  and polynomial  $Q$  is defined using the mapping fibre of  $Q(p^{-r}\varphi)$ , and we have  $P_{1+j}(p^{-1-j}\varphi) = P(\varphi)$  for any  $j$ . However, different values of  $j$  will correspond to the étale cohomology of different twists of  $V_{\mathfrak{p}}^{\text{As}}(\Pi)$ .  $\diamond$

*Proof.* Since  $\Pi$  is cuspidal, the  $\Pi_f$ -generalised eigenspace in de Rham (or, equivalently, rigid) cohomology vanishes outside degree 2. So the natural map

$$H_{\text{fp},c}^2(\mathbb{Y}_G, \mathcal{V}_G; 1+j, P) \rightarrow \text{Fil}^{(1+j)} H_{\text{dR},c}^2(\mathbb{Y}_G, \mathcal{V}_G)^{P(\varphi)=0}$$

is an isomorphism after localising at the  $\Pi_f$ -eigenspace. Since  $P(\varphi)$  annihilates  $\nu_{\text{dR}}$  the result follows.  $\square$

The compatibility of étale and syntomic Abel–Jacobi maps for smooth pairs (cf. Proposition 5.4.1 of [KLZ20]) then implies that

$$(4.2) \quad \begin{aligned} \left\langle \nu_{\text{dR}}, \log \left( \text{loc}_p \text{AJ}_{\text{ét}}^{[\Pi, j]} \right) \right\rangle_{D_{\mathfrak{F}}(\Pi)} &= \left\langle \nu_{\text{dR}}, \log \circ \text{pr}_{\Pi}^{\text{As}} \circ \iota_*^{[j]} \left( \text{Eis}_{\text{ét}, N}^t \right) \right\rangle_{\text{dR}, Y_G} \\ &= \left\langle \nu_{\text{fp}}, \iota_*^{[j]} \left( \text{Eis}_{\text{syn}, N}^t \right) \right\rangle_{\text{fp}, \mathbb{Y}_G} \\ &= \left\langle \iota^{[j],*}(\nu_{\text{fp}}), \text{Eis}_{\text{syn}, N}^t \right\rangle_{\text{fp}, \mathbb{Y}_H}, \end{aligned}$$

where the last equality follows from the adjunction between pushforward and pullback. Note that

$$\iota^{[j],*}(\nu_{\text{fp}}) \in H_{\text{fp},c}^2(\mathbb{Y}_H, \mathcal{V}_H; 1+j, P_{1+j}),$$

and the coefficient module  $\mathcal{V}_H$  depends on  $j$ .

## 5. LIFTING $\nu$ TO THE $\mathfrak{p}_1$ -ORDINARY LOCUS

In this section, we shall lift the coherent class  $\nu$  (and its cousins  $\nu_{\text{dR}}$  and  $\nu_{\text{fp}}$ ) from the cohomology of the full variety  $Y_G$ , to cohomology groups associated to certain open subsets of the special fibre of  $Y_G$ .

**5.1. Geometry of the Hilbert modular variety.** Let  $Y_{G,0}$  be the special fibre of  $\mathbb{Y}_G$ , which is a smooth  $\mathbb{F}_p$ -variety; and similarly for the compactification  $X_{G,0}$ .

**Notation 5.1.1.**

- For  $i = 1, 2$ , denote by  $X_{G,0}^{i-\text{ss}} \subset X_{G,0}$  the  $\mathfrak{p}_i$ -supersingular locus (the vanishing locus of the partial Hasse invariant, as constructed in [TX16, §3.2]).
- Let  $X_{G,0}^{i-\text{ord}}$  be the complement of  $X_{G,0}^{i-\text{ss}}$ , and  $X_{G,0}^{\text{ord}} = X_{G,0}^{1-\text{ord}} \cap X_{G,0}^{2-\text{ord}}$ .

We write  $Y_{G,0}^{i-\text{ord}}$  etc for the intersection of these subvarieties with  $Y_{G,0} \subset X_{G,0}$ .

The following results on the geometry of the supersingular loci are well-known (see e.g. [TX16]):

**Lemma 5.1.2.** *For  $i = 1, 2$ ,  $X_{G,0}^{i-\text{ss}}$  is a smooth codimension 1 closed subscheme of  $X_{G,0}$ , disjoint from the toroidal boundary; and  $X_{G,0}^{1-\text{ss}} \cap X_{G,0}^{2-\text{ss}}$  is a smooth closed subvariety of codimension 2 (i.e. a finite disjoint union of points).*

*Remark 5.1.3.* The preimage of either  $X_{G,0}^{1-\text{ord}}$  or  $X_{G,0}^{2-\text{ord}}$  under the finite map  $\iota : X_{H,0} \rightarrow X_{G,0}$  is the ordinary locus  $X_{H,0}^{\text{ord}}$ .  $\diamond$

**Proposition 5.1.4.** *The extension-by-0 map*

$$R\Gamma_{\text{rig},c}(Y_{G,0}^{1-\text{ord}}, \mathcal{V}_G) \longrightarrow R\Gamma_{\text{rig},c}(Y_{G,0}, \mathcal{V}_G) \cong R\Gamma_{\text{dR},c}(Y_G, \mathcal{V}_G)$$

*is a quasi-isomorphism on the  $\Pi$  generalised eigenspace for the prime-to- $Np$  Hecke operators.*

*Proof.* This is a special case of Proposition 4.3 of [LZ22].  $\square$

**Notation 5.1.5.** Write  $\nu_{\text{dR}}^{(1-\text{ord})} \in H_{\text{rig},c}^2(Y_{G,0}^{1-\text{ord}}, \mathcal{V}_G)$  for the preimage of  $\nu_{\text{dR}}$  under the isomorphism in Proposition 5.1.4.



**5.2. Coherent cohomology of the 1-ordinary locus.** We write  $\mathcal{X}_G$  for the dagger space associated to  $X_G/\mathbf{Q}_p$ ; and we denote the tubes in  $\mathcal{X}_G$  of the various subvarieties of  $X_{G,0}$  considered above by the corresponding superscripts on  $\mathcal{X}_G$ , so  $\mathcal{X}_G^{i\text{-ord}}$  is the tube of  $X_{G,0}^{i\text{-ord}}$  in  $\mathcal{X}_G$  etc.

**Theorem 5.2.1.** *Suppose  $v_p(\alpha_1) < k$ , and  $\alpha_1 \neq \beta_1$ . Then there exists a unique class*

$$\nu^{1\text{-ord}} \in H_c^1\left(\mathcal{X}_G^{(1\text{-ord})}, \omega^{(-k_1, k_2+2)}(-D)\right)$$

satisfying the following properties:

- (1) it is a  $\varphi_1$ -eigenvector with eigenvalue  $\alpha_1$ ;
- (2) its image in  $H^1(\mathcal{X}_G, \omega^{(-k_1, k_2+2)}(-D))$  is  $\nu$ .

*Proof.* This is an instance of Proposition 5.2 of [LZ22].  $\square$

*Remark 5.2.2.* Observe that we have chosen a lifting of the de Rham class  $\nu_{\text{dR}}$  to  $X_{G,0}^{(1\text{-ord})}$  characterised by information about the action of Hecke operators *away* from  $p$ ; and, separately, we have lifted the coherent class  $\nu$  to  $\mathcal{X}_G^{(1\text{-ord})}$  using information about the action of the Frobenius  $\varphi_1$  at  $\mathfrak{p}_1$ . So it is not obvious how these liftings are related, and our next task is to find a way to reconcile the two, which we will carry out in Section 5.4 – see Proposition 5.4.1 below.  $\diamond$

**5.3. Comparison with a higher-level class.** We now compare the class  $\nu^{1\text{-ord}}$  with algebraic coherent classes at level  $\mathfrak{p}_1$ . If  $X_G(\mathfrak{p}_1)$  denotes the Shimura variety of level  $U_1(\mathcal{N}_\Pi) \cap \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}) \bmod \mathfrak{p}_1\}$ , then the special fibre  $X_G(\mathfrak{p}_1)_0$  has a stratification with three strata,

$$X_G(\mathfrak{p}_1)_0 = X_G(\mathfrak{p}_1)_0^m \cup X_G(\mathfrak{p}_1)_0^{\text{ét}} \cup X_G(\mathfrak{p}_1)_0^\alpha,$$

on which the level structure is multiplicative, étale, or  $\alpha_p$  respectively; and this gives a corresponding decomposition of the dagger space  $\mathcal{X}_G(\mathfrak{p}_1)$ , which restricts to an isomorphism of dagger spaces

$$\mathcal{X}_G(\mathfrak{p}_1)^m \xrightarrow{\cong} \mathcal{X}_G^{1\text{-ord}}.$$

(The inverse map is given by the “canonical subgroup” construction.) From the functoriality of pushforward maps we have a commutative square of cohomology groups (all with coefficients in  $\omega^{-k_1, k_2+2}(-D)$ )

$$\begin{array}{ccc} H_c^1(\mathcal{X}_G(\mathfrak{p}_1)^m) & \longrightarrow & H^1(\mathcal{X}_G(\mathfrak{p}_1)) \\ \parallel & & \downarrow \pi_{\mathfrak{p}_1,*} \\ H_c^1(\mathcal{X}_G^{1\text{-ord}}) & \longrightarrow & H^1(\mathcal{X}_G) \end{array}$$

where the horizontal maps are extension-by-zero (i.e. pushforwards along open embeddings) and the right-hand vertical map is pushforward along the natural degeneracy map  $\pi_{\mathfrak{p}_1} : \mathcal{X}_G(\mathfrak{p}_1) \rightarrow \mathcal{X}_G$ . By [LZ22, Lemma 5.7], our Frobenius lift  $\varphi_1$  on  $H_c^1(\mathcal{X}_G^{1\text{-ord}})$  corresponds to the Hecke operator  $U'_{\mathfrak{p}_1}$  at level  $\mathcal{X}_G(\mathfrak{p}_1)$ . Hence we have the following compatibility:

**Proposition 5.3.1.** *The image of  $\nu^{1\text{-ord}}$  in  $H^1(\mathcal{X}_G(\mathfrak{p}_1))$  is the unique class which lies in the  $\Pi_{\mathfrak{f}}$ -eigenspace away from  $p$ , is a  $(U'_{\mathfrak{p}_1} = \alpha_1)$ -eigenvector, and maps to  $\nu$  under the trace map.  $\square$*

We briefly compare this with the choice of basis vector used in [GLZ23, §6–7], since this will be needed for our final formula. In §7.3 of *op.cit.* we define a  $U'_{\mathfrak{p}_1} = \alpha_1$  eigenvector  $\check{\nu}_{\Pi, \alpha} \in H^1(\mathcal{X}_G(\mathfrak{p}_1))$ , depending on a choice of basis  $W_{\mathfrak{f}}^{(p)}$  of the Whittaker model away from  $p\infty$ . If we choose this basis to be the normalised new-vector, then by construction we have

$$\check{\nu}_{\Pi, \alpha} = \left(1 - \frac{\beta_1}{U'_{\mathfrak{p}_1}}\right) \pi_{\mathfrak{p}_1}^*(\nu).$$

Since  $\check{\nu}_{\Pi, \alpha}$  and the image of  $\nu^{1\text{-ord}}$  lie in the same one-dimensional space, we may compare them by computing their images in  $H^1(\mathcal{X}_G)$ . An elementary computation shows that the map

$$\pi_{\mathfrak{p}_1,*} \circ \left(1 - \frac{\beta_1}{U'_{\mathfrak{p}_1}}\right) \circ \pi_{\mathfrak{p}_1}^*$$

acts on the  $\Pi$ -eigenspace as multiplication by  $p(1 - \frac{\beta_1}{p\alpha_1})$  (which is not zero, since  $\alpha_1/\beta_1$  has complex absolute value 1); so we have

$$\check{\nu}_{\Pi, \alpha} = p\left(1 - \frac{\beta_1}{p\alpha_1}\right) \cdot \text{image}(\nu^{1\text{-ord}}).$$

5.4. **The de Rham spectral sequence for  $\mathcal{X}_G^{1\text{-ord}}$ .** Since  $(\mathbb{X}_G, \mathbb{Y}_G)$  is a smooth pair, and our coefficient system  $\mathcal{V}_G$  extends to a vector bundle on  $\mathcal{X}_G$  whose connection has log poles along the boundary divisor  $D$ , we can compute rigid cohomology of  $Y_{G,0}$  using the analytification of the BGG complex on  $\mathcal{X}_G$  (just as we did for de Rham cohomology above). By taking the mapping fibre of the restriction map we obtain the same result for compactly-supported cohomology of  $Y_{G,0}^{1\text{-ord}}$ ; that is, we have

$$R\Gamma_{\text{rig},c}(Y_{G,0}^{1\text{-ord}}, \mathcal{V}) \cong R\Gamma_{\text{dR},c}(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle) \cong R\Gamma_c(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^\bullet(-D)).$$

This gives rise to a first-quadrant spectral sequence converging to  $H_{\text{rig},c}^*(Y_{G,0}^{1\text{-ord}}, \mathcal{V}_G)$ , whose  $E_1^{mn}$  terms are  $H_c^n(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^m(-D))$ .

We denote by  $\widetilde{R}\Gamma_{\text{dR},c}(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle)$  the cohomology of the truncated complex

$$\tau_{\geq 1} \text{BGG}^\bullet(-D) = \left[ 0 \longrightarrow \omega^{(-k_1, k_2+2)} \oplus \omega^{(k_1+2, -k_2)} \longrightarrow \omega^{(k_1+2, k_2+2)} \right](-D),$$

which is quasi-isomorphic to the filtered de Rham complex  $\text{Fil}^{1+j} \text{BGG}^\bullet(-D)$ , for any  $j$  in our range. Since  $\mathcal{X}_G$  is connected and non-compact,  $H_c^0(\mathcal{X}_G^{1\text{-ord}}, -)$  is zero for all locally-free sheaves, and so we obtain an isomorphism

$$\alpha_{\text{dR}}^{(1\text{-ord})} : H_c^1(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^1(-D))^{\nabla=0} \cong \widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle).$$

Moreover, the inclusion of the subcomplex  $\tau_{\geq 1} \text{BGG}$  into the full BGG complex gives a commutative square of maps

$$\begin{array}{ccc} \widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle) & \longrightarrow & \text{Fil}^1 H_{\text{dR}}^2(X_G, \mathcal{V}\langle -D \rangle)_{L_{\mathfrak{p}}} \\ \downarrow & & \downarrow \\ H_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle) & \longrightarrow & H_{\text{dR}}^2(X_G, \mathcal{V}\langle -D \rangle)_{L_{\mathfrak{p}}}, \end{array}$$

in which the top horizontal arrow is compatible, via  $\alpha_{\text{dR}}^{(1\text{-ord})}$ , with the natural map

$$H_c^1(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^1(-D)) \rightarrow H_c^1(\mathcal{X}_G, \text{BGG}^1(-D)) = \text{Fil}^1 H_{\text{dR}}^2 / \text{Fil}^{k_1+k_2+2}.$$

Since the partial Frobenius  $\varphi_1$  lifts to  $\mathcal{X}_G^{1\text{-ord}}$ , there is an action of  $\varphi_1$  on both of the spaces in the left-hand column, compatible with the action on  $H_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle, 1+j)$  given by comparison with the rigid cohomology of  $X_{G,0}$ .

**Proposition 5.4.1.** *If  $\nu^{1\text{-ord}}$  is as in Theorem 5.2.1, then the class  $(\nu^{1\text{-ord}}, 0)$  in*

$$H_c^1(\mathcal{X}_G^{1\text{-ord}}, \text{BGG}^1(-D)) = H_c^1(\mathcal{X}_G^{1\text{-ord}}, \omega^{-k_1, k_2+2}(-D)) \oplus H_c^1(\mathcal{X}_G^{1\text{-ord}}, \omega^{k_1+2, -k_2}(-D))$$

*is in the kernel of  $\nabla$ , and hence defines a class in  $\widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{1\text{-ord}}, \mathcal{V}\langle -D \rangle)$ . The image of this class in  $H_{\text{rig},c}^2(Y_{G,0}^{1\text{-ord}}, \mathcal{V}_G)$ , under the left vertical map of the above diagram, is  $\nu_{\text{dR}}^{(1\text{-ord})}$ .*

*Proof.* We first show that  $\nu^{1\text{-ord}}$  is in the kernel of  $\nabla$ . This follows from the fact that it has strictly small slope for  $\varphi_1$ : the slopes of  $\varphi_1$  on  $\omega^{(k_1+2, k_2+2)}$  are all at least  $k_1+1$ , and the operator  $\nabla$  commutes with the Frobenius, so it must be zero on all Frobenius eigenspaces of slope smaller than  $k_1+1$ .

This shows that  $\nu^{1\text{-ord}}$  has a well-defined image in  $H_{\text{rig},c}^2(Y_{G,0}^{1\text{-ord}}, \mathcal{V}_G)$ . Let us temporarily write  $\hat{\nu}_{\text{dR}}^{(1\text{-ord})}$  for this image; our goal is to show that it coincides with  $\nu_{\text{dR}}^{1\text{-ord}}$ . Since the latter is characterised as the unique lifting of  $\nu_{\text{dR}}$  compatible with Hecke operators away from  $p\mathcal{N}$ , it suffices to show that  $\hat{\nu}_{\text{dR}}^{(1\text{-ord})}$  lies in the correct Hecke eigenspace, and that it maps to  $\nu_{\text{dR}}$  in  $H_{\text{dR},c}^2(Y_G, \mathcal{V}_G)$ .

It follows readily from the construction of  $\nu^{1\text{-ord}}$  that it lies in the  $\Pi$ -eigenspace for the Hecke operators away from  $\mathfrak{p}_1$  (including  $T_{\mathfrak{p}_2}$ ), since these operators commute with  $\varphi_1$ . So  $\hat{\nu}_{\text{dR}}^{(1\text{-ord})}$  has the correct Hecke action. Moreover, its image in  $H_{\text{dR},c}^2(\mathcal{Y}_G, \mathcal{V}_G)$  is in the  $\varphi_1 = \alpha_1$  eigenspace (because  $\nu^{1\text{-ord}}$  is); and it lies in  $\text{Fil}^{(0, 1+k_2)}$ , and maps to  $\nu$  in  $\text{Gr}^{(0, 1+k_2)}$ , so it must be equal to  $\nu_{\text{dR}}$ .  $\square$

**5.5. FP-cohomology of the  $\mathfrak{p}_1$ -ordinary locus.** We now consider a modified form of Besser’s finite-polynomial cohomology, namely *Gros fp-cohomology*, for the  $\mathfrak{p}_1$ -ordinary locus (with compact supports). This cohomology, denoted by  $\widetilde{R}\Gamma_{\text{fp},c}(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$  can be defined as the mapping fibre of the map

$$\widetilde{R}\Gamma_{\text{dR},c}(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle) \xrightarrow{P_{1+j}(p^{-1-j}\varphi)} R\Gamma_{\text{rig},c}(X_{G,0}^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle).$$

(As before, this is in fact independent of  $j$  in the stated range, despite the notations.) Note that although  $\varphi_1$  lifts to  $\mathcal{X}_G^{1-\text{ord}}$ , the full Frobenius  $\varphi$  does not; so although  $R\Gamma_{\text{rig},c}(X_{G,0}^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle)$  is isomorphic to de Rham cohomology of  $\mathcal{X}_G^{1-\text{ord}}$ , the action of Frobenius (given by the functoriality of rigid cohomology) cannot be ‘seen’ via this description.

For proper schemes such as  $\mathbb{X}_G$ , there is no difference between Gros fp-cohomology and the usual fp-cohomology, so there is an extension-by-zero map

$$\widetilde{R}\Gamma_{\text{fp},c}(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) \rightarrow R\Gamma_{\text{fp},c}(\mathbb{X}_G, \mathcal{V}_G\langle -D \rangle; 1+j, P_{1+j}).$$

**Proposition 5.5.1.** *There exists a class*

$$\tilde{\nu}_{\text{fp}}^{1-\text{ord}} \in \widetilde{H}_{\text{fp},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$$

with the following properties:

- Its image in  $H_{\text{fp},c}^2(\mathbb{X}_G, \mathcal{V}_G\langle -D \rangle; 1+j, P_{1+j})$  is the  $\nu_{\text{fp}}$  of Proposition 4.2.3.
- Its image in  $\widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle)$  is the class  $(\nu^{1-\text{ord}}, 0)$  of Proposition 5.4.1.

*Remark 5.5.2.* The reader may be relieved to hear that the class  $\tilde{\nu}_{\text{fp}}^{1-\text{ord}}$  is “the ultimate among liftings of  $\nu$ ”: all other variants of  $\nu$  will be images of this one.  $\diamond$

*Proof.* From the mapping-fibre definition of Gros fp-cohomology we have a long exact sequence

$$\cdots \rightarrow H_{\text{rig},c}^1(X_{G,0}^{1-\text{ord}}, \mathcal{V}\langle -D \rangle) \rightarrow \widetilde{H}_{\text{fp},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) \rightarrow \widetilde{H}_{\text{dR},c}^2(\mathcal{X}_G^{(1-\text{ord})}, \mathcal{V}\langle -D \rangle) \rightarrow \cdots,$$

in which the boundary map is  $P(\varphi) \circ \iota$ . Moreover, this is compatible under extension-by-zero with the corresponding sequence for fp-cohomology of  $\mathbb{X}_G$ .

We have seen that the image of  $(\nu^{1-\text{ord}}, 0)$  under  $\iota$  is the class  $\nu_{\text{dR}}^{1-\text{ord}}$ , which is annihilated by  $P(\varphi)$ . Hence it lifts to  $\widetilde{H}_{\text{fp},c}^2$ . This lift is unique up to the image of an element of  $H_{\text{rig},c}^1(X_{G,0}^{1-\text{ord}}, \mathcal{V}\langle -D \rangle)$ ; but from Proposition 5.1.4 it follows that this group has trivial  $\Pi_f$ -eigenspace for the prime-to- $p$  Hecke operators, so there is a unique Hecke-equivariant lifting of  $(\nu^{1-\text{ord}}, 0)$  to Gros-fp cohomology. The image of this class under extension-by-0 is therefore a Hecke-equivariant lifting of  $\nu_{\text{dR}}$  to fp-cohomology of  $\mathbb{X}_G$ , so it must be  $\nu_{\text{fp}}$ .  $\square$

It is easily checked that  $\iota^{-1}(\mathcal{X}_G^{\text{ord}}) = \mathcal{X}_H^{\text{ord}}$ , so we have the following:

**Corollary 5.5.3.** *The pairing (4.1) is equal to  $\langle \iota^{[j],*}(\tilde{\nu}_{\text{fp}}^{1-\text{ord}}), \widetilde{\text{Eis}}_{\text{syn}}^{t,\text{ord}} \rangle$ , where  $\widetilde{\text{Eis}}_{\text{syn}}^{t,\text{ord}}$  is the image of  $\text{Eis}_{\text{syn}}^t$  in the Gros fp-cohomology of  $\mathcal{X}_H^{\text{ord}}$ .  $\square$*

Here we define  $\iota^{[j],*}$  for classes in Gros fp-cohomology using the quasi-isomorphism from the BGG complex to the full de Rham complex. We shall give explicit formulae in Section 7.1 below, but first we need to give an explicit form for  $\tilde{\nu}_{\text{fp}}^{1-\text{ord}}$ , which can only be done after restricting to  $\mathcal{X}_G^{\text{ord}} \subset \mathcal{X}_G^{1-\text{ord}}$ .

## 6. RESTRICTING TO THE FULLY-ORDINARY LOCUS

**6.1. Cohomology with partial compact support.** We now consider the cohomology of the fully ordinary locus  $X_{G,0}^{\text{ord}}$ . Since the complement  $X_{G,0} - X_{G,0}^{\text{ord}}$  is the disjoint union of a closed subvariety  $X_{G,0}^{(1-\text{ss})}$  and the open subvariety  $X_{G,0}^{(2-\text{ss})} \cap X_{G,0}^{(1-\text{ord})}$ , we can apply the formalism of [LZ20, §13] to define “cohomology of  $\mathcal{X}_G^{\text{ord}}$  with compact support towards  $\mathcal{X}_G^{(1-\text{ss})}$ ” (with coefficients in any abelian sheaf on  $\mathcal{X}_G$ ). We write this as  $R\Gamma_{c1}(\mathcal{X}_G, -)$ . By construction, this comes equipped with a restriction map

$$R\Gamma_c(\mathcal{X}_G^{1-\text{ord}}, -) \rightarrow R\Gamma_{c1}(\mathcal{X}_G^{\text{ord}}, -),$$

which fits into a triangle whose third term is the compactly-supported cohomology of  $\mathcal{X}_G^{1-\text{ord}} \cap \mathcal{X}_G^{2-\text{ss}}$ . In particular, by the same argument as Proposition 5.1.4, we have isomorphisms

$$R\Gamma_{\text{rig},c1}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle) [\Pi_f] \longleftarrow R\Gamma_{\text{rig},c}(\mathcal{X}_G^{1-\text{ord}}, \mathcal{V}\langle -D \rangle) [\Pi_f] \longrightarrow R\Gamma_{\text{rig},c}(\mathcal{X}_G, \mathcal{V}\langle -D \rangle) [\Pi_f].$$

The advantage of working with  $\mathcal{X}_G^{\text{ord}}$  is that both  $\varphi_1$  and  $\varphi_2$  have liftings.

**Notation 6.1.1.** Write  $\nu^{\text{ord}} \in H_{c1}^1(\mathcal{X}_G^{\text{ord}}, \omega^{(-k_1, k_2+2)}(-D))$  for the image of  $\nu^{1-\text{ord}}$  under the above restriction map.

*Note 6.1.2.* Over the ordinary locus, we have commuting liftings of both  $\varphi_1$  and  $\varphi_2$ , which both act on  $R\Gamma_{c1}(\mathcal{X}_G^{\text{ord}}, \omega^{(\dots)})$ ; and the operator  $T_{\mathfrak{p}_2}$  decomposes as  $T_{\mathfrak{p}_2} = U_{\mathfrak{p}_2} + \varphi_2$ , with  $U_{\mathfrak{p}_2} \circ \varphi_2 = p^{k_2+1}\langle \mathfrak{p}_2 \rangle$ .  $\diamond$

**Corollary 6.1.3.** The class  $P(\varphi) \cdot \nu^{\text{ord}}$  lies in the kernel of the Hecke operator  $U_{\mathfrak{p}_2}$ .

*Proof.* We have  $\varphi = \varphi_1\varphi_2$ . The result follows easily from Theorem 5.2.1 and Note 6.1.2, using the fact that  $\nu^{(1-\text{ord})}$  is a  $T_{\mathfrak{p}_2}$ -eigenvector.  $\square$

**6.2. The Poznan spectral sequence.** We now recall a spectral sequence (introduced in [LZ20]) relating Gros fp-cohomology to coherent cohomology. Here Gros fp-cohomology is defined in the same way as for the  $\mathfrak{p}_1$ -ordinary locus above, but now with  $c1$ -support.

**Definition 6.2.1.** We define groups  $\mathcal{C}_{\text{fp}, c1}^{m, n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$ , for  $m, n \geq 0$ , by

$$\mathcal{C}_{\text{fp}, c1}^{m, n}(\dots) = H_{c1}^n(\mathcal{X}_G^{\text{ord}}, (\tau_{\geq 1} \text{BGG})^m(-D)) \oplus H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \text{BGG}^{m-1}(-D));$$

and we define differentials  $\mathcal{C}_{\text{fp}, c1}^{m, n}(\dots) \rightarrow \mathcal{C}_{\text{fp}, c1}^{m+1, n}(\dots)$  by

$$(x, y) \mapsto (\nabla x, P(\varphi/p^n)\iota(x) - \nabla y),$$

where  $\iota$  is the inclusion of  $\text{Fil}^{1+j} \text{BGG}^\bullet$  into  $\text{BGG}^\bullet$ , and  $\nabla$  the differential of the BGG complex.

Note that  $\mathcal{C}_{\text{fp}, c1}^{m, n}$  is zero for  $m \leq 0$  (this is obvious for  $m \leq -1$ , and holds for  $m = 0$  since  $\text{Fil}^{(1+j)} \text{BGG}^0 = 0$ ). It is also zero for  $n \leq 0$ , since  $H_{c1}^0$  vanishes for locally-free sheaves.

**Proposition 6.2.2.** There is a first-quadrant spectral sequence, the Poznań spectral sequence, with

$${}^{\text{Pz}}E_1^{mn} = \mathcal{C}_{\text{fp}, c1}^{m, n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}),$$

and the differentials on the  $E_1$  page given by the formula above. This spectral sequence abuts to the Gros fp-cohomology  $\tilde{H}_{\text{fp}, c1}^{m+n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$ .

**Definition 6.2.3.** We define a coherent fp-pair (of degree  $(m, n)$ , twist  $1+j$  and  $c1$ -support) to be an element of the kernel of the differential  $\mathcal{C}_{\text{fp}, c1}^{m, n} \rightarrow \mathcal{C}_{\text{fp}, c1}^{m+1, n}$ ; we write the group of these as  $\mathcal{Z}_{\text{fp}, c1}^{m, n}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$ .

Thus an fp-pair is a pair of elements

$$x \in H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \mathcal{F}i^{1+j} \text{BGG}^m(-D)), \quad y \in H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \mathcal{F}i^{1+j} \text{BGG}^{m-1}(-D))$$

which satisfy

$$(6.1) \quad \nabla(x) = 0 \quad \text{and} \quad \nabla(y) = P(p^{-1-j}\varphi)\iota(x).$$

*Note 6.2.4.* Given  $x$ , the equation (6.1) does not determine the element  $y$  uniquely: it is determined up to an element of  $H_{c1}^n(\mathcal{X}_G^{\text{ord}}, \text{BGG}^{m-1}(-D))^{\nabla=0}$ .  $\diamond$

**Proposition 6.2.5.** For  $0 \leq j \leq \min\{k_1, k_2\}$ , the spectral sequence gives rise to an isomorphism

$$\alpha_{G, \text{fp}} : \mathcal{Z}_{\text{fp}, c1}^{1, 1}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) \xrightarrow{\cong} \tilde{H}_{\text{fp}, c1}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}).$$

*Proof.* Since the  $E_1$  page of the spectral sequence is zero for  $m \leq 0$ , the term  ${}^{\text{Pz}}E_2^{1, 1}$  is the kernel of the differential on  ${}^{\text{Pz}}E_1^{1, 1}$ , which is the group of fp-pairs. Since this is the only nonzero term with  $m+n=2$ , and the incoming and outgoing differentials at  ${}^{\text{Pz}}E_r^{1, 1}$  are trivially zero for all  $r \geq 1$ , we conclude that  ${}^{\text{Pz}}E_2^{1, 1}$  coincides with  $\tilde{H}^2$  of the abutment, as required.  $\square$

**Corollary 6.2.6.** Every cohomology class in  $\tilde{H}_{\text{fp}, c1}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$  can be uniquely represented by a coherent fp-pair of degree  $(1, 1)$ .

*Proof.* Since the  $E_1$  terms of the spectral sequence are supported in the region  $m, n \geq 1$ , there are no other terms of total degree 2 except  $(m, n) = (1, 1)$ ; and clearly  $E_2^{(1, 1)} = E_\infty^{(1, 1)}$  since the differentials on the  $E_2$  page and beyond land outside this region.  $\square$

An exactly analogous argument shows that for the truncated de Rham cohomology groups  $\tilde{H}_{\text{dR}, c1}^i$  (the hypercohomology of  $\tau_{\geq 1} \text{BGG}^\bullet(-D)$ ), we have an isomorphism

$$H_{c1}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^1(-D)) \xrightarrow{\alpha_{G, \text{rig}}} \tilde{H}_{\text{dR}, c1}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle, 1+j).$$

**Lemma 6.2.7.** *Let  $0 \leq j \leq \min\{k_1, k_2\}$ . We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}_{\text{fp}, \text{cl}}^{1,1}(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) & \xrightarrow{\alpha_{G, \text{fp}}} & \tilde{H}_{\text{fp}, \text{cl}}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j}) \\ \downarrow & & \downarrow \\ H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^1(-D)) & \xrightarrow{\alpha_{G, \text{rig}}} & \tilde{H}_{\text{dR}, \text{cl}}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle, 1+j) \end{array}$$

where the vertical arrows are the natural projection maps.

*Proof.* Clear from the constructions.  $\square$

### 6.3. Construction of a coherent fp-pair.

**Notation 6.3.1.** *Define*

$$\tilde{\nu}_{\text{fp}}^{(\text{ord})} \in \tilde{H}_{\text{fp}, \text{cl}}^2(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle; 1+j, P_{1+j})$$

to be the restriction of the class  $\tilde{\nu}_{\text{fp}}^{(1-\text{ord})}$  constructed above.

**Corollary 6.3.2.** *There exists a uniquely determined class*

$$\xi \in H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \omega^{(-k_1, -k_2)}(-D)),$$

which is independent of  $j$ , such that  $(\nu^{(\text{ord})}, \xi)$  forms an fp-pair representing the class  $\tilde{\nu}_{\text{fp}}^{(\text{ord})}$ , and such that  $\xi$  lies in the  $\Pi$ -eigenspace for the Hecke operators away from  $pN$ .

*Proof.* The existence of  $\xi$  is immediate from Corollary 6.2.6 and Lemma 6.2.7; the independence from  $j$  is clear by construction.

Now, if  $\xi'$  is another element such that  $(\nu^{(\text{ord})}, \xi')$  also represents  $\tilde{\nu}_{\text{fp}}^{(\text{ord})}$ , then

$$\xi - \xi' \in H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^0(-D))^{\nabla=0} \cong H_{\text{rig}, \text{cl}}^1(X_{G,0}, \mathcal{V}\langle -D \rangle).$$

As we have seen, the  $\Pi_f$ -eigenspace in this cohomology is zero, so there is a unique choice of  $\xi$  which is Hecke-equivariant.  $\square$

**Lemma 6.3.3.** *The element  $\xi$  has the following properties:*

$$(6.2) \quad \varphi_1 \cdot \xi = \alpha_1 \xi \quad \text{and} \quad U_{p_2} \cdot \xi = 0.$$

*Proof.* We first show the latter statement. We deduce from Corollary 6.1.3 that

$$U_{p_2} \cdot \xi \in H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \text{BGG}^0(-D))^{\nabla=0}[\Pi'_f].$$

But as observed above, this space is zero. The former statement follows analogously, by considering the element  $(\varphi_1 - \alpha_1) \cdot \xi$ .  $\square$

We now lift these classes from the BGG complex to the full de Rham complex:

**Definition 6.3.4.**

- (1) Write  $\dot{\xi}$  for the image of  $\xi$  in  $H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \mathcal{V}\langle -D \rangle)$ .
- (2) Similarly, for  $0 \leq j \leq \min\{k_1, k_2\}$  write  $\dot{\nu}_j^{(\text{ord})}$  for the image of  $\nu^{(\text{ord})}$  in

$$H_{\text{cl}}^1(\mathcal{X}_G^{\text{ord}}, \mathcal{F}il^j \mathcal{V} \otimes \Omega^1\langle -D \rangle).$$

**Lemma 6.3.5.** *The element  $\dot{\xi}$  also satisfies*

$$(6.3) \quad \varphi_1 \cdot \dot{\xi} = \alpha_1 \dot{\xi} \quad \text{and} \quad U_{p_2} \cdot \dot{\xi} = 0.$$

The following result will be very important for the regulator evaluation:

**Proposition 6.3.6.** *We have*

$$U_p \circ \iota^*(\dot{\xi}) = 0.$$

*Proof.* As maps  $H^\bullet(\mathcal{X}_G^{\text{ord}}) \rightarrow H^\bullet(\mathcal{X}_H^{\text{ord}})$ , we have the following identity:

$$(6.4) \quad U_p \circ \iota^* \circ \varphi_{p_1} = \iota^* \circ (\langle p_1 \rangle U_{p_2}).$$

Since  $\dot{\xi}$  is an eigenvector of  $\varphi_{p_1}$  with non-zero eigenvalue, and it is in the kernel of  $\varphi_{p_2}$ , the Proposition follows.  $\square$

**7.1. Relating the BGG and de Rham complexes.** We need to recall some formulae relating the de Rham and BGG complexes for  $\mathrm{GL}_2$ . Let  $k \in \mathbf{Z}_{\geq 0}$ . The BGG complex for weight  $k \geq 0$  is given by  $[\omega^{-k} \xrightarrow{\Theta} \omega^{k+2}]$ , where  $\Theta$  is a differential operator given in terms of  $q$ -expansions by

$$\Theta = \frac{(-1)^k}{k!} \theta^{k+1}, \quad \theta = q \frac{d}{dq}.$$

(Cf. [TX16, Remark 2.17] for example.) The map  $\mathrm{BGG}^\bullet \rightarrow \mathrm{DR}^\bullet$  is given as follows.

- In degree 1, it is given by the tensor product of the natural inclusion  $\omega^k = \mathrm{Fil}^k(\mathrm{Sym}^k \mathcal{W}_H) \hookrightarrow \mathcal{W}_H$  and the Kodaira–Spencer isomorphism  $\omega^2 \cong \Omega^1(D)$ .
- In degree 0, a section  $s$  of  $\omega^{-k}$  is mapped to the unique section of  $\mathrm{Sym}^k \mathcal{W}_H$  whose image in  $\mathrm{Sym}^k \mathcal{W}_H / \mathrm{Fil}^0 \cong \omega^{-k}$  is  $s$ , and whose image under the differential  $\nabla$  lands in  $\omega^k \otimes \Omega^1(D)$ .

We now recall (and somewhat reformulate) some results from §4 of [KLZ20] giving a completely explicit description of these maps. We write  $\mathcal{X}_H^{\mathrm{ord}}$  for the ordinary locus *as a classical rigid space* (not a dagger space), i.e. neglecting overconvergence. Passing to the Igusa tower  $\mathcal{IG}_H$  (the canonical  $\mathbf{Z}_p^\times$ -covering of  $\mathcal{X}_H^{\mathrm{ord}}$  parametrising ordinary elliptic curves with a trivialization of their formal group), we obtain a canonical section  $v$  of  $\omega$ , corresponding to the invariant differential form  $\frac{dT}{T}$  on the Tate curve  $\mathbf{G}_m/q^\mathbf{Z}$ ; note that  $\varphi^*(v) = pv$ . If we let  $w$  be the unique section such that  $\nabla(v) = u \otimes \xi$ , where  $\xi$  the local basis of  $\Omega^1_{\mathcal{X}_H^{\mathrm{ord}}}(D)$  corresponding to  $\frac{dq}{q}$ , then  $v$  and  $w$  are a basis of sections of  $\mathcal{W}_H$  over the Igusa tower, with  $w$  spanning the unit-root subspace (we have  $\varphi^*(w) = w$ ) and  $\nabla w = 0$ . Hence we obtain a basis of sections  $(v^a w^{k-a})_{0 \leq a \leq k}$  of  $\mathrm{Sym}^k \mathcal{W}_H$  in which the actions of  $\nabla$  and  $\varphi$  are completely explicit.

In these coordinates, the map from  $\omega^{-k}$  to  $\mathrm{Sym}^k \mathcal{W}_H$  sends a section  $\mathcal{G}$  of  $\omega^{-k}$  to the section

$$\sum_{i=0}^k \frac{(-1)^i}{i!} \theta^i(\mathcal{G}) \cdot v^i w^{k-i}.$$

One verifies easily that the image of this under  $\nabla$  is  $\theta^{k+1}(\mathcal{G})v^k \otimes \xi$ , as expected. (Note that it is not *a priori* obvious that such a sum is overconvergent if  $\mathcal{G}$  is, since neither  $\theta$  nor the local bases  $v^i w^{k-i}$  have any meaning outside the ordinary locus.)

**7.2. The Eisenstein class as a coherent fp-pair.** Recall that the  $\mathrm{GL}_2$  Eisenstein class  $\mathrm{Eis}_{\mathrm{syn},N}^t$  lies in  $H_{\mathrm{syn}}^1(\mathbb{Y}_H, \mathcal{V}_H(1+t))$ , where  $\mathbb{Y}_H$  is the  $\mathbf{Z}_p$ -model of the modular curve  $Y_1(N)$ , and  $\mathcal{V}_H$  is the sheaf corresponding to the  $t$ -th symmetric power of the standard representation.

If we restrict to the open subscheme  $\mathbb{Y}_H^{\mathrm{ord}}$  given by removing the supersingular points of the special fibre (so  $\mathbb{Y}_H^{\mathrm{ord}}$  has the same generic fibre as  $\mathbb{Y}_H$ ), and work with Gros syntomic cohomology, then we have a convenient explicit description: a class  $x \in \widetilde{H}_{\mathrm{syn}}^1(\mathcal{X}_1(N)^{\mathrm{ord}}, \mathcal{V}_H, 1+t)$  is given by a pair  $(x_0, x_1)$ , where

$$x_0 \in H^0(\mathcal{X}_1(N)^{\mathrm{ord}}, \mathcal{V}), \quad x_1 \in H^0(\mathcal{X}_1(N)^{\mathrm{ord}}, \mathrm{Fil}^t \mathcal{V} \otimes \Omega^1(D)), \quad \nabla x_0 = (1-p^{-1-t}\varphi)x_1.$$

The sheaf  $\mathrm{Fil}^t \mathcal{V} \otimes \Omega^1(D)$  is simply  $\omega^{t+2}$ , so  $x_1$  is an overconvergent  $p$ -adic modular form of weight  $t+2$ ; via this description,  $\varphi$  acts on overconvergent forms as  $p^{t+1}\langle p \rangle V_p$ , where  $\langle p \rangle$  is the diamond operator for  $p \bmod N$ .

We let  $\widetilde{\mathrm{Eis}}_{\mathrm{syn},N}^{t,\mathrm{ord}}$  be the image of  $\mathrm{Eis}_{\mathrm{syn},N}^t$  in this group; we shall now write down an explicit representing pair. The following is a reformulation of Theorem 4.5.7 of [KLZ20]; the formulations differ because we are using symmetric powers here rather than symmetric tensors (the basis vector  $v^{[r,s]}$  of *op.cit.* corresponds to  $\frac{v^s w^r}{r!s!}$  in our present notation) and because we use a slightly different notation for Eisenstein series following [Kat04].

**Proposition 7.2.1.** *The Eisenstein class  $\widetilde{\mathrm{Eis}}_{\mathrm{syn},N}^{t,\mathrm{ord}}$  is represented by the pair of sections  $(\epsilon_0^t, \epsilon_1^t)$  whose restrictions to  $\mathcal{X}_H^{\mathrm{ord}}$  are*

$$\begin{aligned} \epsilon_0 &= -N^k \sum_{u=0}^t \frac{(-1)^{t-u}}{u!} \theta^u \left( E_{0,1/N}^{-t,\mathrm{ord}} \right) \cdot v^{t-u} w^u, \\ \epsilon_1 &= -\frac{N^t}{t!} F_{0,1/N}^{t+2} \cdot v^t \otimes \xi. \end{aligned}$$

Here  $F_{0,1/N}^{t+2}$  is the algebraic Eisenstein series with  $q$ -expansion

$$\zeta(-1-t) + \sum_{n>0} q^n \sum_{d|n} \binom{n}{d}^{t+1} (\zeta_N^d + (-1)^t \zeta_N^{-d}),$$

and  $E_{0,1/N}^{-t, \text{ord}}$  is an ordinary  $p$ -adic Eisenstein series of weight  $-t$ , satisfying  $\theta^{t+1}(E_{0,1/N}^{-t, \text{ord}}) = (1 - \langle p \rangle V_p) F_{0,1/N}^{t+2}$ .  $\square$

**Remark 7.2.2.** Note that the individual terms  $\theta^u(E_{0,1/N}^{-t, \text{ord}})$  are  $p$ -adic modular forms, but they are not overconvergent (unless  $u = 0$ ). Nonetheless,  $\epsilon_0$  is an overconvergent section of  $\mathcal{V}_H$ ; the non-overconvergence arises because the sections  $v, w$  we are using to trivialize  $\mathcal{V}_H$  over the ordinary locus are not themselves overconvergent.  $\diamond$

**7.3. Explicit formulae for the Clebsch–Gordan map.** The map  $\text{CG}^{[j]}$  is given explicitly by the following formula (c.f. [KLZ20, Prop. 5.1.2])<sup>4</sup>: for  $0 \leq s \leq t$ , we have

$$(7.1) \quad \iota^{[j]}(v^s w^{t-s}) = \sum_{\substack{0 \leq r \leq k_1 - j \\ 0 \leq r' \leq k_2 - j \\ r + r' = s}} \sum_{i=0}^j (-1)^i \frac{s!(t-s)!}{r!(r')!(k_1-r-j)!(k_2-r'-j)!i!(j-i)!} v^{r+i} w^{k_1-r-i} \boxtimes v^{r'+j-i} w^{k_2-r'-j+i} \otimes e_{-j}.$$

**Lemma 7.3.1.** For given values of  $k_1, k_2, j$ , the image of  $\iota^{[j]}(v^s w^{t-s})$  in the line spanned by the basis vector  $v^{k_1} \boxtimes w^{k_2} \otimes e_{-j}$  is zero for all  $s$  except  $s = k_1 - j$ , in which case it is equal to  $\frac{(-1)^j}{j!} \cdot v^{k_1} \boxtimes w^{k_2} \otimes e_{-j}$ .

*Proof.* This is a superficially modified version of Proposition 5.1.2 of [KLZ20].  $\square$

**Remark 7.3.2.** In particular, this shows that the Clebsch–Gordan map is not in general defined integrally (i.e. does not respect the lattice given by the  $\mathbf{Z}$ -span of the basis vectors). However, the coefficient  $j!$  is the worst possible – one can check that  $j! \iota^{[j]}$  is integral.  $\diamond$

**7.4. Reduction to a pairing in coherent cohomology.** Recall that we want to evaluate the pairing

$$(7.2) \quad \left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn}, N}^t \right\rangle.$$

Now we have

$$(7.3) \quad \begin{aligned} (7.2) &= \left\langle (\iota^{[j]})^* \left( \nu_{\text{fp}}^{(1-\text{ord})} \right), \text{Eis}_{\text{syn}, N}^{t, \text{ord}} \right\rangle \\ &= \left\langle (\iota^{[j]})^* \left( \tilde{\nu}_{\text{fp}}^{(1-\text{ord})} \right), \widetilde{\text{Eis}}_{\text{syn}, N}^{t, \text{ord}} \right\rangle \\ &= \left\langle (\iota^{[j]})^* \left( \tilde{\nu}_{\text{fp}}^{(\text{ord})} \right), \widetilde{\text{Eis}}_{\text{syn}, N}^{t, \text{ord}} \right\rangle \\ &= \left\langle (\iota^{[j]})^* \left( \dot{\nu}_j^{(\text{ord})}, \dot{\xi} \right), (\epsilon_1, \epsilon_0) \right\rangle. \end{aligned}$$

Here, (7.3) takes values in the group

$$\tilde{H}_{\text{fp}, c}^2(\mathcal{X}_H^{\text{ord}}, \mathbf{Q}_p \langle -\mathcal{D}_H \rangle, 2; P_j) \cong^{\text{tr}} \mathbf{Q}_p.$$

We evaluate (7.3) using the formalism of cup products in fp-cohomology (cf. [Bes00]). We have

$$P_j(st) = a(t, s)P_j(t) + b(t, s)(1-s)$$

with  $a(t, s) = s$  and

$$b(t, s) = \frac{P_j(st) - sP_j(t)}{1-s} = 1 - b \cdot t^2 s, \quad b = \frac{p^{2j+2}}{\alpha_1^2 \alpha_2 \beta_2} = \frac{\beta_1}{\alpha_1} \cdot \frac{1}{p^{(k_1+k_2-2j)}}.$$

Hence

$$\begin{aligned} &P_j(p^{-1}) \times \left\langle (\iota^{[j]})^* \left( \dot{\nu}_j^{(\text{ord})}, \dot{\xi} \right), (\epsilon_1^t, \epsilon_0^t) \right\rangle \\ &= (\iota^{[j]})^*(\dot{\xi}) \cup \varphi_H^*(\epsilon_1^t) + \left( 1 - \frac{\beta_1 \cdot (\varphi_H^{*,2} \otimes \varphi_H^*)}{p^{k_1+k_2-2j} \langle p \rangle \alpha_1} \right) \left( (\iota^{[j]})^*(\dot{\nu}_j^{(\text{ord})}) \cup \epsilon_0^t \right). \end{aligned}$$

<sup>4</sup>The factorials appear slightly different from *op.cit.* since we are here working with symmetric powers  $v^m$  rather than symmetric tensors  $v^{[m]}$ . This is also the reason for the presence of  $t!$  in the formula for the Eisenstein series.

**Lemma 7.4.1.** *We have  $(\iota^{[j]})^*(\dot{\xi}) \cup \varphi_H^*(\epsilon_0^t) = 0$ .*

*Proof.* Observe that  $U_p$  acts on the top-degree rigid cohomology as multiplication by a power of  $p$ . But we also have

$$U_p \left( (\iota^{[j]})^*(\dot{\xi}) \cup \varphi_H^*(\epsilon_0^t) \right) = U_p \left( (\iota^{[j]})^*(\dot{\xi}) \right) \cup \epsilon_0^t,$$

which is equal to zero by Proposition 6.3.6. The Proposition follows.  $\square$

**Proposition 7.4.2.** *Equation (7.3) is equal to*

$$\frac{\left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right)} \left\langle (\iota^{[j]})^*(\dot{\nu}_j^{(\text{ord})}), \epsilon_0^t \right\rangle.$$

*Proof.* Since any  $p$ -depleted form will pair to 0 with a form that's in a direct sum of finite-slope  $\varphi$ -eigenspaces, we only care about the  $\epsilon_0$  term modulo exact forms. We also only care about its projection to the  $\langle p \rangle = 1$  eigenspace, because we are pairing it with a class in the  $\Pi$ -eigenspace and  $\Pi$  has trivial central character.

On the space  $H_{\text{rig},c}^2(\mathcal{X}_H^{\text{ord}}(-\mathcal{D}), \mathbf{Q}_p(2))$ , the Frobenius acts as  $p^{-1}$ , and  $\langle p \rangle$  acts trivially; so we can replace  $\frac{\beta_1 \cdot (\varphi_H^{*,2} \otimes \varphi_H^*)}{p^{k_1+k_2-2j}\langle p \rangle \alpha_1}$  with  $\frac{\beta_1 \cdot (1 \otimes \varphi_H^{-1})}{p^{k_1+k_2-2j+2}\alpha_1}$ .

The operator  $\varphi^{-1}$  makes sense modulo  $p$ -depleted forms, and in this quotient we have

$$\varphi^{-1} \cdot \epsilon_0^m = p^{m+1} \langle p \rangle^{-1} \epsilon_0^m.$$

So we are done.  $\square$

*Remark 7.4.3.* Note that  $\dot{\nu}_j^{(\text{ord})}$  makes sense over the  $\mathfrak{p}_1$ -ordinary locus  $\mathcal{X}_G^{1-\text{ord}}$ ; the antiderivative  $\dot{\xi}$  is only defined over the fully ordinary locus  $\mathcal{X}_G^{\text{ord}}$ , but this term has disappeared from our formula. So we can also interpret the pairing of Proposition 7.4.2 as a cup-product in the cohomology of  $\mathcal{X}_G^{1-\text{ord}}$ , which will allow us to compare with the construction of  $p$ -adic  $L$ -functions from [GLZ23].  $\diamond$

**7.5. A partial unit root splitting.** We saw above that over  $\mathcal{X}_H^{\text{ord}}$  (the ordinary locus as a classical rigid space) the Hodge filtration of  $\mathcal{V}_H$  has a canonical splitting given by the unit-root subspace. For  $X_G$  we have the more refined structure of a  $\mathbf{Z}^2$ -filtration, and we can ask for splittings of either factor. We state below the case of interest.

**Proposition 7.5.1.** *Over  $\mathcal{X}_G^{1-\text{ord}}$ , the natural inclusion map  $\omega^{(k_1, -k_2)} \hookrightarrow \mathcal{V}_G / \text{Fil}^{k_1+1}$  admits a canonical splitting.*

*Proof.* We have a projection map from  $\mathcal{V}_G$  onto  $\mathcal{V}_G / \text{Fil}^{(0,1)} \mathcal{V}_G \cong \text{Sym}^{k_1} \mathcal{W}_G \boxtimes \omega^{-k_2}$ . So it suffices to show that the filtration on  $\text{Sym}^{k_1} \mathcal{W}_G$  is splittable over the 1-ordinary locus; but this is clear, since a complement is provided by the unit-root subspace for the operator  $\varphi_1$ .  $\square$

**7.6. Comparison of pushforward maps.** To begin with, we observe that there exist pushforward maps in coherent cohomology:

**Proposition 7.6.1.** *There exists a pushforward map on coherent cohomology*

$$\iota_{\text{coh},*} : H^0(\mathcal{X}_H^{\text{ord}}, \omega^{k_1-k_2}) \longrightarrow H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}).$$

*Proof.* Clear since the conormal sheaf of the embedding  $\iota$  is isomorphic to  $\omega^2$ .  $\square$

We want to compare this map with the pushforward maps  $\iota_*^{[j]}$  on the cohomology of the de Rham sheaves.

**Notation 7.6.2.** *We use the notation of Section 2.2, and we write*

- $\mathcal{V}_G = \text{Sym}^{[k_1, k_2]} \mathcal{H}(\mathcal{A})_{\mathbf{Q}_p}$ ;
- $\mathcal{V}_H = \text{Sym}^t \mathcal{H}(\mathcal{E})_{\mathbf{Q}_p}$  for  $0 \leq j \leq \min\{k_1, k_2\}$ .

for vector bundles attached to  $V_H$  and  $V_G$ , respectively.



**Proposition 7.6.3.** *Pushforward along  $\iota^{[j]}$  induces a commutative diagram*

$$\begin{array}{ccc}
H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) & \xrightarrow{\iota_*^{[j]}} & H^1(\mathcal{X}_G^{1-\text{ord}}, V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1(D)) \\
\uparrow & & \uparrow \\
H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H) & \longrightarrow & H^1(\mathcal{X}_G^{1-\text{ord}}, \text{Gr}^{k_1} V_G \otimes \Omega^1(D)) \\
& \searrow \frac{1}{j!} \iota_{\text{coh},*} & \downarrow \\
& & H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}).
\end{array}$$

*Proof.* For the commutativity of the top square, we note that the Clebsch–Gordan map is compatible with filtrations and hence induces a commutative diagram of  $B_H$ -representations

$$\begin{array}{ccc}
V_H/\text{Fil}^{k_1-j} V_H & \longrightarrow & V_G/\text{Fil}^{k_1} V_G \\
\uparrow & & \uparrow \\
\text{Gr}^{k_1-j} V_H & \longrightarrow & \text{Gr}^{k_1} V_G.
\end{array}$$

The identification of the diagonal map as  $\frac{1}{j!} \iota_{\text{coh},*}$  arises similarly, using the explicit formula (7.1).  $\square$

**Proposition 7.6.4.** *The diagram in Proposition 7.6.3 is compatible with the unit root splitting*

$$u_H : H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) \longrightarrow H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H)$$

and the partial unit root splitting

$$u_G : H^1(\mathcal{X}_G^{1-\text{ord}}, V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1) \longrightarrow H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}) :$$

we have a commutative diagram

$$\begin{array}{ccc}
H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) & \xrightarrow{\iota_*^{[j]}} & H^1(\mathcal{X}_G^{1-\text{ord}}, V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1) \\
\uparrow & & \uparrow \\
H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H) & \longrightarrow & H^1(\mathcal{X}_G^{1-\text{ord}}, \text{Gr}^{k_1} V_G \otimes \Omega^1) \\
& \searrow \frac{1}{j!} \iota_{\text{coh},*} & \downarrow \\
& & H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2})
\end{array}$$

$\xrightarrow{u_H} \quad \xrightarrow{u_G}$

*Proof.* Over  $\mathcal{X}_H^{\text{ord}}$ , we have a canonical splitting of the Hodge filtration of  $\mathcal{W}_H$ , as above. To prove the proposition, it is sufficient to check that the unit root splittings induce a commutative diagram

$$\begin{array}{ccc}
H^0(\mathcal{X}_H^{\text{ord}}, V_H/\text{Fil}^{k_1-j+1} V_H) & \rightarrow & H^0(\mathcal{X}_H^{\text{ord}}, \iota^*(V_G/\text{Fil}^{k_1+1} V_G \otimes \Omega^1)) \\
\uparrow & & \uparrow \\
H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H) & \longrightarrow & H^0(\mathcal{X}_H^{\text{ord}}, \iota^*(\text{Gr}^{k_1} V_G \otimes \Omega^1)) \\
& \searrow & \downarrow \\
& & H^0(\mathcal{X}_H^{\text{ord}}, \iota^*(\omega^{(k_1+2, -k_2)}))
\end{array}$$

$\xrightarrow{u_H} \quad \xrightarrow{u_G}$

which boils down to an explicit computation with the Clebsch–Gordan map using equation (7.1).  $\square$

It follows from the classicity theorem of higher Hida theory proved in [GLZ23] that the natural restriction map (forgetting overconvergence)

$$H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2}) \rightarrow H^1(\mathcal{X}_G^{1-\text{ord}}, \omega^{k_1+2, -k_2})$$

is an isomorphism on the ordinary (slope 0) eigenspace for  $U_{\mathfrak{p}_1}$ . If  $\Pi$  is ordinary at  $\mathfrak{p}_1$  (and  $\alpha_1$  is, necessarily, the unit root) then pairing with  $\nu^{\text{ord}}$  factors through this eigenspace, so we obtain the following:

**Corollary 7.6.5.** *Assume  $\Pi$  is ordinary at  $\mathfrak{p}_1$ . Then the linear functional on  $H^0(\mathcal{X}_H^{\text{ord}}, V_H / \text{Fil}^{k_1-j+1} V_H)$  given by pairing with  $(\iota^{[j],*})(\nu^{\text{ord}})$  factors through the unit root splitting into  $: H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} V_H)$ , and it is given by*

$$\frac{k_1! k_2!}{j!} \cdot \left\langle \nu^{(1-\text{ord})}, \iota_{\text{coh},*}(-) \right\rangle_{\mathcal{X}_G^{1-\text{ord}}}. \quad \square$$

*Remark 7.6.6.* The factors  $k_i!$  arise from the pairing of the basis vectors  $v^{k_i}$  and  $w^{k_i}$  of  $\text{Sym}^{k_i} W_G$ .  $\diamond$

**7.7. Relation to  $p$ -adic  $L$ -functions.** We use Corollary 7.6.5 in order to relate the formula in Proposition 7.4.2 to values of  $p$ -adic  $L$ -functions. We assume henceforth that  $\Pi$  is ordinary at  $\mathfrak{p}_1$ .

**Proposition 7.7.1.** *The pairing (7.2) is given by*

$$\begin{aligned} \left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn},N}^t \right\rangle = \\ \frac{N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} \left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right)} \binom{k_1}{j} k_2! \cdot \left( \nu^{(\text{ord})} \cup \iota_{\text{coh},*}(\theta^{(k_1-j)} E_{0,1/N}^{-t,\text{ord}}) \right). \end{aligned}$$

Compare Corollary 6.5.7 in [KLZ20].

*Proof.* We deduce from Proposition 7.2.1 that the image of  $\epsilon_0^{k_1+k_2-2j}$  under projection to  $H^0(\mathcal{X}_H^{\text{ord}}, \text{Gr}^{k_1-j} \mathcal{V}_H)$  is given by  $-N^k (-1)^{k_2-j} \frac{(t)!}{(k_1-j)!} \theta^{k_1-j} (E_{0,1/N}^{-t,\text{ord}})$ . Combining this with Proposition 7.4.2 and Corollary 7.6.5 gives the result.  $\square$

*Note 7.7.2.* By adjunction, we can write the formula in Proposition 7.7.1 as

$$\begin{aligned} \left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn},N}^t \right\rangle = \\ \frac{N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} \left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right)} \binom{k_1}{j} k_2! \cdot \left( \iota_{\text{coh}}^*(\nu^{(\text{ord})}) \cup \theta^{(k_1-j)} E_{0,1/N}^{-t,\text{ord}} \right). \end{aligned}$$

$\diamond$

In order to relate this formula to a (non-critical) value of a  $p$ -adic  $L$ -function, we need to replace

$$\theta^{(k_1-j)} E_{0,1/N}^{-t,\text{ord}} = E_{0,1/N}^{j-k_2,\text{ord}}$$

by its  $p$ -depletion

$$E_{0,1/N}^{j-k_2,[p]} = (1 - \varphi \circ U_p) E_{0,1/N}^{j-k_2,\text{ord}}.$$

We adapt the argument from [KLZ20, §6.5]. Let  $V_{\mathfrak{p}_2} = p^{-1-k_2} \langle \mathfrak{p}_2 \rangle^{-1} \varphi_2$ , so  $V_{\mathfrak{p}_2}$  is a right inverse of  $U_{\mathfrak{p}_2}$ . Let  $\lambda, \mu$  be constants such that

$$U_{\mathfrak{p}_2}(\nu^{\text{ord}}) = \lambda \nu^{\text{ord}} - \mu V_{\mathfrak{p}_2}(\nu^{\text{ord}})$$

(explicitly, we have  $\lambda = a_{\mathfrak{p}_2}(\mathcal{F})$  and  $\mu = p^{1+k_2}$ ), and let  $\gamma = p^{k_1-j}$ . Using an analogue of Lemma 6.5.8 from *op.cit.*, we deduce the following result:

**Lemma 7.7.3.** *We have*

$$\iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]} = (1 - \lambda \gamma p^{-k_1} \beta_{\mathfrak{p}_1} \cdot V_p + \mu \gamma^2 (p^{-k_1} \beta_{\mathfrak{p}_1})^2 V_p^2) \left( \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} \right).$$

*Proof.* Arguing as in §6.5 in *op.cit.*, we see that

$$\begin{aligned} \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]} = & (1 - \lambda \gamma p^{-k_1} \beta_{\mathfrak{p}_1} \cdot V_p + \mu \gamma^2 (p^{-k_1} \beta_{\mathfrak{p}_1})^2 V_p^2) \left( \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} \right) \\ & + (1 - V_p U_p) \left( \mu \varphi^2 \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} - \iota_{\text{coh}}^*(\nu^{\text{ord}}) \cup \varphi \theta^{k_1-j} E_{0,1/N}^{-t,\text{ord}} \right). \end{aligned}$$

But the operator  $1 - V_p U_p$  acts as the zero map, which proves the result.  $\square$

We deduce the following formula:

**Proposition 7.7.4.** *We have*

$$\left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn}, N}^t \right\rangle = N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} \times \frac{\left(1 - \frac{\beta_1}{p\alpha_1}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right) \left(1 - \frac{\beta_1\alpha_2}{p^{j+1}}\right) \left(1 - \frac{\beta_1\beta_2}{p^{j+1}}\right)} \binom{k_1}{j} k_2! \cdot \left(\iota_{\text{coh}}^*(\nu^{(\text{ord})}) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]}\right).$$

We can also express this in terms of the class  $\check{\nu}_{\Pi, \alpha}$  on the higher-level variety  $\mathcal{X}_G(\mathfrak{p}_1)$  described above. If  $\pi_{\mathfrak{p}_1}$  denotes the degeneracy map  $\mathcal{X}_G(\mathfrak{p}_1) \rightarrow \mathcal{X}_G$ , then we have  $\text{pr}_{\mathfrak{p}_1, *}(\check{\nu}_{\Pi, \alpha}) = p \left(1 - \frac{\beta_1}{p\alpha_1}\right) \nu^{(\text{ord})}$ ; so the formula in Proposition 7.7.4 is equivalent to

$$\left\langle (\iota^{[j]})^*(\nu_{\text{fp}}), \text{Eis}_{\text{syn}, N}^t \right\rangle = N^{(k_1+k_2-2j)} (-1)^{k_2-j+1} p^{-1} \times \frac{1}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right) \left(1 - \frac{\beta_1\alpha_2}{p^{j+1}}\right) \left(1 - \frac{\beta_1\beta_2}{p^{j+1}}\right)} \binom{k_1}{j} k_2! \cdot \left(\iota_{\text{coh}}^* \circ \text{pr}_{\mathfrak{p}_1, *} \left(\nu_{1w(\mathfrak{p}_1)}^{(\text{ord})}\right) \cup \theta^{k_1-j} E_{0,1/N}^{-t,[p]}\right).$$

We now recall the definition of the (imprimitive)  $p$ -adic Asai  $L$ -function in [GLZ23, Def. 7.3.1]. For a  $p$ -adic character  $\sigma : \mathbf{Z}_p^\times \rightarrow \mathbf{C}_p^\times$ , we have

$$(7.4) \quad L_{p, \text{Asai}}^{\text{imp}}(\Pi, \sigma) = \frac{1}{p} (\sqrt{D})^{-1-(k_1+k_2)/2-\sigma} (-1)^\sigma \left\langle \check{\nu}_{\Pi, \alpha}, \iota_{\text{coh}, *}^{(p)}(\mathcal{E}_{k_1-k_2}(\sigma)) \right\rangle,$$

where

- $X_H(p)$  is the modular curve of level  $\Gamma_1(N) \cap \Gamma_0(p)$ ;
- $\iota^{(p)} : X_H(p) \rightarrow X_G(\mathfrak{p}_1)$  is the natural embedding induced by  $\iota$ ;
- $\mathcal{E}_{k_1-k_2}$  is a  $p$ -adic family of  $p$ -depleted Eisenstein series, of constant weight  $k_1 - k_2$ , defined in [GLZ23, §7.2], with the Schwartz function  $\Phi^{(p)}$  chosen as in §7.3.2 of *op.cit.*; at  $\sigma = -\frac{t}{2}$ , this specialises to  $\theta^{k_1-j} \left(E_{0,1/N}^{-t,[p]}\right)$ .

**Lemma 7.7.5.** *The following diagram is cartesian:*

$$\begin{array}{ccc} X_H(p) & \xrightarrow{\iota^{(p)}} & X_G(\mathfrak{p}_1) \\ \downarrow \pi_p & & \downarrow \pi_{\mathfrak{p}_1} \\ X_{H, \text{sph}} & \xrightarrow{\iota} & X_{G, \text{sph}} \end{array}$$

$$\text{so } \pi_{\mathfrak{p}_1}^* \circ \iota_* = \iota_*^{(p)} \circ \pi_p^*.$$

Comparing the previous proposition with (7.4), we hence deduce the main theorem of this paper:

**Theorem 7.7.6.** *Let  $0 \leq j \leq \min\{k_1, k_2\}$ , and let  $t = k_1 + k_2 - 2j \geq 0$ . Then we have*

$$\left\langle \nu_{\text{dR}}, \log \left( \text{loc}_p \text{AF}_{\text{ét}}^{[\Pi, j]} \right) \right\rangle = \frac{(\sqrt{D})^{j+1} (-1)^{(k_1-k_2)/2+1} N^t \binom{k_1}{j} k_2!}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right) \left(1 - \frac{p^j}{\alpha_1\beta_2}\right) \left(1 - \frac{\beta_1\alpha_2}{p^{j+1}}\right) \left(1 - \frac{\beta_1\beta_2}{p^{j+1}}\right)} \cdot L_{p, \text{Asai}}^{\text{imp}} \left( \Pi, -\frac{t}{2} \right).$$

*Remark 7.7.7.* Note that in [GLZ23] we also defined a primitive  $p$ -adic  $L$ -function, interpolating the critical values of the primitive Asai  $L$ -function (with the optimal Euler factors at ramified primes). One can likewise modify the construction of the Asai–Flach class by incorporating more general test data away from  $p$  (see [Gro20]), and prove a refinement of Theorem 7.7.6, relating Asai–Flach classes with appropriate “optimal” test data to non-critical values of the primitive  $p$ -adic Asai  $L$ -function. We leave the details to the interested reader.  $\diamond$

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