Tools for solving the div-curl problem with mixed boundary conditions in a polygonal domain

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Following [2], we continue to study the resolution of two-dimensional problems in non convex domains. In this previous paper, we considered several methods for solving Maxwell's equations (stationary or instationary) with a perfectly conducting boundary condition.

In the present report, we provide mathematical tools for solving the general div $-$ curl problem:
Find $\mathbf{u} \in \boldsymbol{L}^2(\Omega)$ *such that*

$$\begin{aligned} \operatorname{div} \mathbf{u} &= f \text{ in } \Omega, & (1)\\ \operatorname{curl} \mathbf{u} &= g \text{ in } \Omega, & (2)\\ \mathbf{u} \cdot \vec{\tau} &= \eta \text{ on } \Gamma_C, & (3)\\ \mathbf{u} \cdot \vec{\nu} &= \mu \text{ on } \Gamma_S. & (4) \end{aligned}$$

Here, Ω is a polygonal domain and its boundary Γ is made up of two parts Γ_C and Γ_S, and $\vec{\tau}$ and $\vec{\nu}$ are vectors, respectively tangent and normal to Γ. The solution $\mathbf{u}$ is a vector valued function which belongs to the Sobolev space $\mathbf{L}^2(\Omega) \stackrel{def}{=} L^2(\Omega)^2$, f and g are both in $L^2(\Omega)$, and η and μ are distributions defined on Γ_C and Γ_S respectively.

The report is organized as follows. Using the work of Grisvard concerning the Laplace problem [14], we determine an orthogonal decomposition of $\mathbf{L}^2(\Omega)$. Next, we deduce from this decomposition a splitting of the solution to the div $-$ curl problem into a regular part and a singular part. Then, we derive a method for the explicit computation of the singular part. Last, we obtain a way of computing numerically the regular part of the solution by using a standard finite element method. All these results will be mathematically justified.

Note that we use, without detail, a number of results proved in [2] for singularities which occur at reentrant corners in the presence of a (locally) uniform boundary condition, that is either (3) or (4). We also rely on Grisvard's theory [13]-[14].

Chapter 1

Notations and functional spaces

Let Ω be an open, connected and simply connected polygon in $\mathbb{R}^2$, and let Γ be its boundary. We assume that Γ is split up into two non empty subsets Γ_C and Γ_S. Γ_C and Γ_S can be disconnected, and we call $(\Gamma_C^k)_{0\leq k\leq K}$ (respectively $(\Gamma_S^k)_{0\leq k\leq K}$) the connected components of Γ_C (resp. Γ_S), and $\Gamma_C^* = \Gamma_C \setminus \Gamma_C^0$ (resp. $\Gamma_S^* = \Gamma_S \setminus \Gamma_S^0$).

Furthermore, it is possible to decompose each actual *geometrical* edge γ of the boundary Γ into $\gamma_C = \gamma \cap \Gamma_C$ and $\gamma_S = \gamma \cap \Gamma_S$, both of them possibly divided into several connected components. In other words, two cases might occur: either (for instance) γ_C is empty and to the *geometrical* edge γ corresponds a single *generalized* edge (itself), or both γ_C and γ_S are not empty and γ is made up of two or more *generalized* edges. From this local decomposition, one can then split up Γ, Γ_C and Γ_S into sets of *generalized* edges:

$$\begin{aligned}
\Gamma &= \bigcup_{j\in\mathcal{C}\cup\mathcal{S}} \overline{\Gamma}_j,\ \Gamma_j \cap \Gamma_k = \emptyset \text{ for } j \neq k \\
\overline{\Gamma}_C &= \bigcup_{j\in\mathcal{C}} \overline{\Gamma}_j, \\
\overline{\Gamma}_S &= \bigcup_{j\in\mathcal{S}} \overline{\Gamma}_j.
\end{aligned}$$

Two consecutive *generalized* edges Γ_j and Γ_{j+1} have an angle (in the interior of Ω) which is denoted by $\theta_j = \pi/\alpha_j$, where $\alpha_j > 1/2$ by definition. Finally, we call *generalized* corner the intersection $\{M_j\} = \overline{\Gamma}_j \cap \overline{\Gamma}_{j+1}$. In this report, we use either of the decompositions

$$\overline{\Gamma}_C = \bigcup_{0\leq k\leq K} \overline{\Gamma}_C^k = \bigcup_{j\in\mathcal{C}} \overline{\Gamma}_j.$$

Remark 1.1 Note that for all pairs of *generalized* edges imbedded in a common geometrical edge Γ, one has $\alpha_j = 1$. This artificial decomposition, which allows to take the change of the boundary condition on an edge γ in a way we hope to be elegant, will be justified at Remark 2.9. ■

In the following, the adjective *generalized* will be understood.

In $\mathbb{R}^2$, there exist a scalar curl from $\mathbb{R}^2$ to $\mathbb{R}$ and a vector **curl** operator from $\mathbb{R}$ to $\mathbb{R}^2$. We shall use boldface letters for functions, spaces of functions and operators with vector values subsequently. We define the well-known spaces:

$$\begin{aligned}\mathbf{H}(\mathrm{curl}\,,\Omega) &= \{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \mathrm{curl}\,\mathbf{v}=\partial_x v_y-\partial_y v_x\in L^2(\Omega)\},\\ \mathbf{H}(\mathrm{div}\,,\Omega) &= \{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \mathrm{div}\,\mathbf{v}=\partial_x v_x+\partial_y v_y\in L^2(\Omega)\},\ \text{and}\\ \mathbf{H}(\mathrm{div}\,0,\Omega) &= \{\mathbf{v}\in\mathbf{H}(\mathrm{div}\,,\Omega),\ \mathrm{div}\,\mathbf{v}=0\}.\end{aligned}$$

Now, let $\vec{\nu}=(\nu_x,\nu_y)^T$ be the unit outer normal at any point of the boundary (to the exception of corners), we denote by $\vec{\tau}=(\nu_y,-\nu_x)^T$ the associated tangent vector: the direct orthonormal reference is then $(M,\vec{\tau},\vec{\nu})$. On the boundary (cf. [13] (*Lemma 1.3.3.2. page 20*)), we introduce the space:

$$\begin{aligned}&H^{1/2}(\gamma)=\{\mu\in L^2(\gamma),|\mu|_{1/2,\gamma}<\infty\},\text{with}\\ &|\mu|_{1/2,\gamma}=\left[\int_\gamma\int_\gamma\frac{|\mu(\mathbf{x})-\mu(\mathbf{y})|^2}{||\mathbf{x}-\mathbf{y}||^2}\,d\sigma(\mathbf{x})d\sigma(\mathbf{y})\right]^{1/2}.\end{aligned}\tag{1.1}$$

where γ is any connected subset of the boundary Γ and $d\sigma$ a measure on it (*cf. [17], pages 14-15*). The associated norm is

$$||\mu||_{1/2,\gamma}=\left[||\mu||^2_{0,\gamma}+|\mu|^2_{1/2,\gamma}\right]^{1/2}.$$

Remark 1.2 It is possible to give an alternate definition of the fact that a function belongs to $H^{1/2}(\gamma)$ (and of (1.1)). Indeed, if γ is splitted up into $\overline{\gamma}=\bigcup_j\overline{\gamma}_j$, with $\{m_j\}=\overline{\gamma}_j\cap\overline{\gamma}_{j+1}$, one obtains by splitting up integrals that μ is in $H^{1/2}(\gamma)$ if and only if

$$\mu_{|\gamma_j}\in H^{1/2}(\gamma_j),\ \text{and}\ \mu_{|\gamma_j}\overset{1/2}{\equiv}\mu_{|\gamma_{j+1}}\ \text{at}\ m_j,$$

where $\mu_j\overset{1/2}{\equiv}\mu_{j+1}$ at m_j means

$$\begin{aligned}&\int_{\mathbf{x}_j\in\gamma_j}\int_{\mathbf{x}_{j+1}\in\gamma_{j+1}}\frac{|\mu_j(\mathbf{x}_j)-\mu_{j+1}(\mathbf{x}_{j+1})|^2}{||\mathbf{x}_j-\mathbf{x}_{j+1}||^2}\,d\sigma_j d\sigma_{j+1}<\infty,\ \text{or equivalently}\\ &\int_0^{\delta_j}|\mu_j(\mathbf{x}_j(-\sigma))-\mu_{j+1}(\mathbf{x}_{j+1}(\sigma))|^2\,\frac{d\sigma}{\sigma}<\infty,\ \text{for some non negative}\ \delta_j.\end{aligned}$$

The second characterization[1,2] comes from [14] (*page 17, following [9]*). ∎

In this report, we shall also often use the Sobolev space $\widetilde{H}^{1/2}(\gamma)$ and its dual $\widetilde{H}^{-1/2}(\gamma)$. Let $\{\mathbf{p}_1,\mathbf{p}_2\}$ denote the endpoints of γ and let $d_{\partial\gamma}(\mathbf{p})=\min(||\mathbf{p}-\mathbf{p}_1||,||\mathbf{p}-\mathbf{p}_2||)$ for any point $\mathbf{p}$. We define $\widetilde{H}^{1/2}(\gamma)$ by

$$\begin{aligned}&\widetilde{H}^{1/2}(\gamma)=\{f\in H^{1/2}(\gamma),\ f/\sqrt{d_{\partial\gamma}}\in L^2(\gamma)\},\\ &||\mu||_{\sim,1/2,\gamma}=\left[||\mu||^2_{1/2,\gamma}+||\mu/\sqrt{d_{\partial\gamma}}||^2_{0,\gamma}\right]^{1/2}.\end{aligned}$$

Remark 1.3 Note again (see footnote [2]) that we could have defined equivalently $d_{\partial\gamma}(\mathbf{p})$ by $\min(|\sigma(\mathbf{p})-\sigma(\mathbf{p}_1)|,|\sigma(\mathbf{p})-\sigma(\mathbf{p}_2)|)$. Indeed, the two quantities are equal if $\mathbf{p}$ is

[1] $\mathbf{x}_j(-\sigma)$ (resp. $\mathbf{x}_{j+1}(\sigma)$) is the point of γ_j (resp. γ_{j+1}) at distance σ of m_j.

[2] This characterization has two advantages and a drawback: first, it extends to the case of cracks; second, it yields the characterization of the Sobolev space $\widetilde{H}^{1/2}(\gamma)$ defined below; third, there is no clear (?) equivalent definition for γ a part of the boundary of a domain in $\mathbb{R}^3$.

close enough to either $\mathbf{p}_1$ or $\mathbf{p}_2$, and outside a vicinity of those two points, both definitions yield values which are bounded below by a non negative constant. ∎

One of the main properties of $\widetilde{H}^{1/2}(\gamma)$ is that the continuation of any of its elements by zero to Γ belongs to $H^{1/2}(\Gamma)$, which corresponds to one of the usual definitions of this space (cf. [14]).

Proposition 1.1 *Let μ be an element of $\widetilde{H}^{1/2}(\gamma)$. Then its continuation by zero to Γ belongs to $H^{1/2}(\Gamma)$. Conversely, if we let $\tilde{\mu}$ be an element of $H^{1/2}(\Gamma)$ such that $\tilde{\mu} = 0$ on $\Gamma \setminus \gamma$, then its restriction to γ is in $\widetilde{H}^{1/2}(\gamma)$. Furthermore, both the continuation and the restriction operators are continuous.*

Proof: Let us consider first an element μ of $\widetilde{H}^{1/2}(\gamma)$. By assumption,

$$\mu \in L^2(\gamma),\ \int_\gamma \int_\gamma \frac{|\mu(\mathbf{x}) - \mu(\mathbf{y})|^2}{||\mathbf{x}-\mathbf{y}||^2}\, d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) < \infty,\ \text{and}\ \int_\gamma \frac{|\mu(\mathbf{y})|^2}{d_{\partial\gamma}(\mathbf{y})} d\sigma(\mathbf{y}) < \infty.$$

Let $\tilde{\mu}$ be its continuation by zero to Γ. By definition, $\tilde{\mu} \in L^2(\Gamma)$. In addition, if we let $\gamma^* = \Gamma \setminus \gamma$,

$$\int_\Gamma \int_\Gamma \frac{|\tilde{\mu}(\mathbf{x}) - \tilde{\mu}(\mathbf{y})|^2}{||\mathbf{x}-\mathbf{y}||^2}\, d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) = |\mu|^2_{1/2,\gamma} + 2\int_\gamma |\mu(\mathbf{y})|^2 \int_{\gamma^*} \frac{d\sigma(\mathbf{x})}{||\mathbf{x}-\mathbf{y}||^2}\, d\sigma(\mathbf{y}).$$

Now, let us choose $R > 0$ in such a way that $\gamma^{**} = \{\mathbf{x} \in \gamma^*,\, d(\mathbf{x}, \gamma) \leq R\}$ is exactly made up of two segments originating from $\mathbf{p}_1$ and $\mathbf{p}_2$. Finally, let $\gamma^{***} = \gamma^* \setminus \gamma^{**}$. We are now in a position to use the estimate (i) in the next inequalities:

$$\begin{aligned}
\int_\gamma |\mu(\mathbf{y})|^2 \int_{\gamma^*} \frac{d\sigma(\mathbf{x})}{||\mathbf{x}-\mathbf{y}||^2}\, d\sigma(\mathbf{y}) &\leq \int_\gamma |\mu(\mathbf{y})|^2 \int_{\gamma^{**}} \frac{d\sigma(\mathbf{x})}{||\mathbf{x}-\mathbf{y}||^2}\, d\sigma(\mathbf{y}) + \frac{l(\gamma^{***})}{R^2} ||\mu||^2_{0,\gamma} \\
&\leq C_2 \int_\gamma \frac{|\mu(\mathbf{y})|^2}{d_{\partial\gamma}(\mathbf{y})} d\sigma(\mathbf{y}) + \frac{l(\gamma^{***})}{R^2} ||\mu||^2_{0,\gamma} \\
&\leq \max(C_2, \frac{l(\gamma^{***})}{R^2}) ||\mu||^2_{\sim,1/2,\gamma}.
\end{aligned}$$

Thus $\tilde{\mu}$ belongs to $H^{1/2}(\gamma)$. In addition, C_2 depends on γ^{**} and R, but not on μ, so the continuation operator is continuous.

Conversely, let $\tilde{\mu} \in H^{1/2}(\Gamma)$ such that $\tilde{\mu}_{|\gamma^*} = 0$. The regularity of $\tilde{\mu}$ amounts to

$$\tilde{\mu} \in L^2(\Gamma) \text{ and } \int_\Gamma \int_\Gamma \frac{|\tilde{\mu}(\mathbf{x}) - \tilde{\mu}(\mathbf{y})|^2}{||\mathbf{x}-\mathbf{y}||^2}\, d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) < \infty.$$

Let μ be its restriction to γ. Classically, $\mu \in H^{1/2}(\gamma)$. Also, if we let $C_1' = 1/C_1$ in (i),

$$\begin{aligned}
\int_\gamma \frac{|\mu(\mathbf{y})|^2}{d_{\partial\gamma}(\mathbf{y})} d\sigma(\mathbf{y}) &\leq C_1' \int_\gamma |\mu(\mathbf{y})|^2 \int_{\gamma^{**}} \frac{d\sigma(\mathbf{x})}{||\mathbf{x}-\mathbf{y}||^2} d\sigma(\mathbf{y}) \\
&\leq C_1' \int_{\gamma^{**}} \int_\gamma \frac{|\tilde{\mu}(\mathbf{x}) - \tilde{\mu}(\mathbf{y})|^2}{||\mathbf{x}-\mathbf{y}||^2} d\sigma(\mathbf{y}) d\sigma(\mathbf{x}) \\
&\leq C_1' \, |\tilde{\mu}|_{1/2,\Gamma}.
\end{aligned}$$

Therefore, μ is an element of $\widetilde{H}^{1/2}(\gamma)$. Again, the constant do not depend on $\tilde{\mu}$, which proves that the restriction operator is continuous. ∎

Remark 1.4 This result is not a straightforward consequence of the classical definition of the space $\widetilde{H}^{1/2}(\omega)$, where ω is an arbitrary subset of $\mathbb{R}^p$ $(p \geq 1)$, that is the space of all f defined on ω such that their continuation by zero outside ω belongs to $H^{1/2}(\mathbb{R}^p)$. Indeed, in the above proposition, the continuation occurs on a polygonal boundary. For a generalization to polyhedral domains, see [1] or [5]. ■

For a field $\mathbf{v}$, let v_τ and v_ν stand for the tangential and normal components of $\mathbf{v}$ on Γ, i. e. $v_\tau = \mathbf{v} \cdot \vec{\tau}$ and $v_\nu = \mathbf{v} \cdot \vec{\nu}$. Now, we are in a position to introduce

$$\mathcal{V} = \{\mathbf{v} \in \mathbf{H}(\mathrm{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}\,;\Omega),\ v_{\tau|\Gamma_C} = 0,\ v_{\nu|\Gamma_S} = 0\}.$$

The solution $\mathbf{u}$ to problem (1)-(4) precisely belongs to $\mathcal{V}$ when $\eta = \nu = 0$. We shall now investigate a number of properties related to $\mathcal{V}$.

Let us define three spaces of stream functions:

$$\Phi = \{\phi \in H^1(\Omega),\ \Delta\phi \in L^2(\Omega),\ \frac{\partial \phi}{\partial \nu}_{|\Gamma_C} = 0,\ \phi_{|\Gamma_S^0} = 0,\ \frac{\partial \phi}{\partial \tau}_{|\Gamma_S^*} = 0\},$$
$$\Phi_0 = \{\phi \in \Phi,\ \phi_{|\Gamma_S} = 0\},\ \Phi_\Delta = \{\phi \in \Phi,\ \Delta\phi = 0\}.$$

Before we investigate the properties of these sets of functions, let us comment a little bit on the definition of Φ. First, for $\phi \in \Phi$, as $\phi \in H^1(\Omega)$, $\phi_{|\Gamma_S^0}$ belongs to $H^{1/2}(\Gamma_S^0)$. Furthermore, the relationship on Γ_S^* is equivalent to $\phi_{|\Gamma_S^k} = c_k$, with $(c_k)_{1\leq k\leq K} \in \mathbb{R}^K$, according to [11] (*Rmk 2.3, Chapter I, page 35*).

Now, let us prove that the relationship on the normal derivative is to be understood on $\widetilde{H}^{-1/2}(\Gamma_C^k)$, for $0 \leq k \leq K$. For that, we define the space

$$E(\Delta, L^2(\Omega)) = \{\phi \in H^1(\Omega),\ \Delta\phi \in L^2(\Omega)\},$$

a subset of which is Φ. We denote by $||\cdot||_E$ the associated (graph) norm.

Proposition 1.2 *Let $\phi \in E(\Delta, L^2(\Omega))$ and let γ be any connected subset of Γ. Then*

$$\frac{\partial \phi}{\partial \nu}_{|\gamma} \in \widetilde{H}^{-1/2}(\gamma).$$

In addition, given a decomposition of the boundary Γ into a set of connected subsets $(\gamma_l)_{1\leq l\leq L}$, we arrive at the integration by parts formula

$$\forall \phi \in H^1(\Omega) \text{ such that } \phi_{|\gamma_l} \in \widetilde{H}^{1/2}(\gamma_l),\ 1 \leq l \leq L, \text{ and}$$
$$\forall \phi' \in E(\Delta, L^2(\Omega)),$$
$$\int_\Omega (\mathbf{grad}\,\phi \cdot \mathbf{grad}\,\phi' + \phi\,\Delta\phi')\,d\mathbf{x} = \sum_{l=1}^{l=L} \left\langle \frac{\partial \phi'}{\partial \nu}_{|\gamma_l}, \phi_{|\gamma_l} \right\rangle_{\widetilde{H}^{-1/2}(\gamma_l), \widetilde{H}^{1/2}(\gamma_l)}. \quad (1.2)$$

Proof: Indeed, for all $\phi \in H^1(\Omega)$ and $\phi' \in H^2(\Omega)$,

$$\int_{\Gamma\setminus\gamma} \phi_{|\gamma} \frac{\partial \phi'}{\partial \nu}_{|\gamma} d\sigma + \int_\gamma \phi_{|\gamma} \frac{\partial \phi'}{\partial \nu}_{|\gamma} d\sigma = \int_\Omega (\mathbf{grad}\,\phi \cdot \mathbf{grad}\,\phi' + \phi\,\Delta\phi')\,d\mathbf{x}.$$

Now, take $\mu \in \widetilde{H}^{1/2}(\gamma)$ and denote $\tilde{\mu}$ is continuation by zero to Γ. Thanks to Proposition 1.1, there exists C_1 independent of μ such that $||\tilde{\mu}||_{1/2,\Gamma} \leq C_1\,||\mu||_{\sim,1/2,\gamma}$. Moreover, it is well known (cf. [17] for instance) that there exist a second constant C_2 independent of $\tilde{\mu}$

and $\phi \in H^1(\Omega)$ which satisfy $\phi_{|\Gamma} = \tilde{\mu}$, and $||\phi||_1 \leq C_2 \, ||\tilde{\mu}||_{1/2,\Gamma}$. Thus, there exists C_3 such that

$$\forall \mu \in \widetilde{H}^{1/2}(\gamma),\ \exists \phi \in H^1(\Omega),\ \phi_{|\gamma} = \mu,\ \phi_{|\Gamma\setminus\gamma} = 0 \text{ and } ||\phi||_1 \leq C_3 \, ||\mu||_{\sim,1/2,\gamma}.$$

We arrive at

$$\left| \int_\gamma \mu \, \frac{\partial \phi'}{\partial \nu}_{|\gamma} d\sigma \right| \leq C_3 ||\mu||_{\sim,1/2,\gamma} \, ||\phi'||_E. \tag{1.3}$$

The conclusion follows from [13] (*see Lemma 1.5.3.9. page 59*), where it is proved that $H^2(\Omega)$ is dense in $E(\Delta, L^2(\Omega))$.
Classically, if we let $(\phi'_n)_n$ be a sequence of elements of $H^2(\Omega)$ that converges to ϕ' and $E(\Delta, L^2(\Omega))$, we infer from (1.3) that $(\alpha_n)_n$, with

$$\alpha_n = \int_\gamma \mu \, \frac{\partial \phi'_n}{\partial \nu}_{|\gamma} d\sigma,$$

is a Cauchy sequence in $\mathbb{R}$. Therefore, it admits a limit α which is, by definition, equal to

$$\alpha = \int_\gamma \mu \, \frac{\partial \phi'}{\partial \nu}_{|\gamma} d\sigma.$$

Thus $\frac{\partial \phi'}{\partial \nu}_{|\gamma} \in \widetilde{H}^{-1/2}(\gamma)$.
Assume that we are given the decomposition $\Gamma = \cup_{1 \leq l \leq L} \overline{\gamma}_l$. In order to prove the integration by parts formula, we simply write, for $\phi \in H^1(\Omega)$ such that $\phi_{|\gamma_l} \in \widetilde{H}^{1/2}(\gamma_l)$, $1 \leq l \leq L$, and $\phi' \in H^2(\Omega)$

$$\int_\Omega (\mathbf{grad}\, \phi \cdot \mathbf{grad}\, \phi' + \phi \, \Delta \phi') \, d\mathbf{x} = \sum_{l=1}^{l=L} \int_{\gamma_l} \frac{\partial \phi'}{\partial \nu}_{|\gamma_l} \phi_{|\gamma_l} d\sigma.$$

Using again the density of $H^2(\Omega)$ in $E(\Delta, L^2(\Omega))$, together with (1.3), leads to (1.2). ■

Let γ and $(\gamma_l)_{1 \leq l \leq L}$ be defined as above.

Proposition 1.3 *Let* $\mathbf{v} \in \boldsymbol{H}(\mathrm{curl}\,;\Omega)$*, then* $v_{\tau|\gamma} \in \widetilde{H}^{-1/2}(\gamma)$*. In addition,*

$$\begin{aligned} &\forall \phi \in H^1(\Omega) \text{ such that } \phi_{|\gamma_l} \in \widetilde{H}^{1/2}(\gamma_l),\ 1 \leq l \leq L, \text{ and} \\ &\forall \mathbf{v} \in \boldsymbol{H}(\mathrm{curl}\,;\Omega), \\ &\int_\Omega (\mathbf{curl}\, \phi \cdot \mathbf{v} - \phi \, \mathrm{curl}\, \mathbf{v}) \, d\mathbf{x} = \sum_{l=1}^{l=L} \left\langle v_{\tau|\gamma_l}, \phi_{|\gamma_l} \right\rangle_{\widetilde{H}^{-1/2}(\gamma_l), \widetilde{H}^{1/2}(\gamma_l)}. \end{aligned} \tag{1.4}$$

Let $\mathbf{v} \in \boldsymbol{H}(\mathrm{div}\,;\Omega)$*, then* $v_{\nu|\gamma} \in \widetilde{H}^{-1/2}(\gamma)$*. In addition,*

$$\begin{aligned} &\forall \phi \in H^1(\Omega) \text{ such that } \phi_{|\gamma_l} \in \widetilde{H}^{1/2}(\gamma_l),\ 1 \leq l \leq L, \text{ and} \\ &\forall \mathbf{v} \in \boldsymbol{H}(\mathrm{div}\,;\Omega), \\ &\int_\Omega (\mathbf{grad}\, \phi \cdot \mathbf{v} + \phi \, \mathrm{div}\, \mathbf{v}) \, d\mathbf{x} = \sum_{l=1}^{l=L} \left\langle v_{\nu|\gamma_l}, \phi_{|\gamma_l} \right\rangle_{\widetilde{H}^{-1/2}(\gamma_l), \widetilde{H}^{1/2}(\gamma_l)}. \end{aligned} \tag{1.5}$$

Proof: Simply reproduce that of the previous Proposition, with the help of the density of $\mathbf{H}^1(\Omega)$ in either $\mathbf{H}(\mathrm{curl}\,;\Omega)$ or $\mathbf{H}(\mathrm{div}\,;\Omega)$. ■

Remark 1.5 In particular, this tells that there is no hope of finding a solution $\mathbf{u} \in \mathbf{L}^2(\Omega)$ to (1)-(4) (with f and g both in $L^2(\Omega)$) if either $\eta_{|\Gamma_C^k} \notin \widetilde{H}^{-1/2}(\Gamma_C^k)$ or $\mu_{|\Gamma_S^k} \notin \widetilde{H}^{-1/2}(\Gamma_S^k)$ for some k. ■

We have the

Proposition 1.4 *In* $\boldsymbol{L}^2(\Omega)$, $\mathbf{grad}\,\Phi_0 \perp \mathbf{grad}\,\Phi_\Delta$.

Proof: Let $\phi \in \Phi_0$ and $\phi' \in \Phi_\Delta$. We note first that as $\phi_{|\Gamma} \in H^{1/2}(\Gamma)$ and $\phi_{|\Gamma_S} = 0$, we know that $\phi_{|\Gamma_C^k} \in \widetilde{H}^{1/2}(\Gamma_C^k)$, for $0 \leq k \leq K$. Then, thanks to (1.2),

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\phi \cdot \mathbf{grad}\,\phi'\,d\mathbf{x} &= \int_\Omega (\mathbf{grad}\,\phi \cdot \mathbf{grad}\,\phi' + \phi\,\Delta\phi')\,d\mathbf{x} \\ &= \sum_k \left\langle \frac{\partial \phi'}{\partial \nu}_{|\Gamma_C^k}, \phi_{|\Gamma_C^k} \right\rangle + \sum_k \left\langle \frac{\partial \phi'}{\partial \nu}_{|\Gamma_S^k}, \phi_{|\Gamma_S^k} \right\rangle = 0.\end{aligned}$$

■

Now, in Φ, the norm

$$||\phi||_\Phi = \left[||\mathbf{grad}\,\phi||_0^2 + ||\Delta\phi||_0^2\right]^{1/2}$$

is equivalent to the graph norm $||\phi||_E = \left[||\phi||_1^2 + ||\Delta\phi||_0^2\right]^{1/2}$, due to the homogeneous boundary condition on Γ_S^0. Let $(\cdot,\cdot)_\Phi$ denote the associated scalar product. One has the

Corollary 1.1 $\Phi = \Phi_0 \overset{\perp_\Phi}{\oplus} \Phi_\Delta$.

Proof: The orthogonality stems from the previous Proposition.

Now, let $\phi \in \Phi$. By [14] (*Thm 2.1.1., page 38*), there exists a unique element ϕ_0 of Φ_0 such that $\Delta\phi_0 = \Delta\phi$ in Ω. The conclusion follows as $\phi - \phi_0$ clearly belongs to Φ_Δ. ■

As far as the regularity of elements of Φ is concerned, we have the

Theorem 1.1 Φ *is imbedded in* $\boldsymbol{H}^{5/4}(\Omega)$ *algebrically. More precisely, let*

$$\alpha = \min(\alpha_D, \alpha_N, \alpha_M/2), \ \textit{with}$$

$$\alpha_D = \min_{j \in J_D}(\alpha_j),\ \alpha_N = \min_{j \in J_N}(\alpha_j),\ \alpha_M = \min_{j \in J_M}(\alpha_j),$$

J_D (resp. J_N, J_M) being the set of indices such that there is a Dirichlet (resp. Neumann, mixed) boundary condition on the couple of edges (Γ_j, Γ_{j+1}).

Then, when $\alpha \notin \mathbb{N}$, $\Phi \subset H^{1+\alpha-\varepsilon}(\Omega)$ *(for any* $\varepsilon > 0$*) and* $\Phi \not\subset H^{1+\alpha}(\Omega)$.
When $\alpha \in \mathbb{N}$, $\Phi \subset H^{1+\alpha}(\Omega)$ *and* $\Phi \not\subset H^{1+\alpha+\varepsilon}(\Omega)$ *(for any* $\varepsilon > 0$*).*

Proof: Let $\phi \in \Phi$. Then, $\phi_{|\Gamma_S^k} = c_k$ for $0 \leq k \leq K$. If we let $g = \Delta\phi \in L^2(\Omega)$, ϕ is now the solution of

$$\phi \in H^1(\Omega),\ \Delta\phi = g,\ \phi_{|\Gamma_S^k} = c_k,\ \frac{\partial \phi}{\partial \nu}_{|\Gamma_C^k} = 0,\ \text{for } 0 \leq k \leq K. \tag{1.6}$$

In particular, the compatibility conditions (*[14], page 59*) are always satisfied: indeed, according to our notations, these are

$$c_j = c_{j+1} \text{ for } j \in J_D, \ \frac{\partial c_j}{\partial \nu_{j\pm 1}} = 0 \text{ for } j \in J_M.$$

From this point on, one can apply Grisvard's theory [14] directly. Indeed, one first splits the unique solution of (1.6) into a regular part (which belongs to $H^2(\Omega)$) and a singular part (*Rmks 2.4.5. and 2.4.6. pages 58-59 and Corollary 2.4.4. page 58*). Then, by inspection, it is easily seen that the singular part, an explicit expression of which is known near the corners, belongs to the aforementioned Sobolev spaces (*Thm 1.2.18. page 8*). To complete the proof, one has $\alpha > 1/4$ by definition. ■

Remark 1.6 The computations will be detailled in the Section 3, in the proof of Theorem 3.5. ■

In the following, we denote by $X \hookrightarrow Y$ the topological imbedding (*i. e. continuous*) of the Banach space X in the Banach space Y. Recall the

Theorem of the closed graph *Let X and Y be two Banach spaces, and let u be a linear mapping from X to Y. Then u is continuous if and only if its graph $\{(x, u(x)),\ x \in X\}$ is closed in $X \times Y$.*

Then we have

Corollary 1.2 *Let X, Y and Z be three Banach spaces such that*

$$X \hookrightarrow Z, \ Y \hookrightarrow Z \text{ and } X \subset Y.$$

Then $X \hookrightarrow Y$.

Proof: Let i (resp. i_x, i_Y) denote the imbedding of X into Y (resp. of X into Z, of Y into Z). Let us prove that $\{(x, i(x)),\ x \in X\}$ is closed in $X \times Y$.

For that, let $(x_n, i(x_n))_n$ be a Cauchy sequence in $X \times Y$. Then $(x_n)_n$ converges to some x in X and $(i(x_n))_n$ converges to some y in Y. By assumption, this implies

$$i_X(x_n) \longrightarrow i_X(x) \text{ in } Z$$
$$i_Y \circ i(x_n) \longrightarrow i_Y(y) \text{ in } Z.$$

But $i_X(x') = i_Y \circ i(x')$, $\forall x' \in X$, and therefore $i_X(x) = i_Y \circ i(x) = i_Y(y)$. i_Y being injective, we finally get $y = i(x)$, which leads to the conclusion. ■

Corollary 1.3 *Φ is imbedded in $H^{5/4}(\Omega)$ topologically, i. e. $\Phi \hookrightarrow H^{5/4}(\Omega)$. Thus,*

$$\exists C(\Omega) > 0, \text{ such that } \forall \phi \in \Phi, ||\phi||_{5/4} \leq C(\Omega)\, ||\phi||_{\Phi}.$$

Proof: Indeed, by definition, Φ, $H^{5/4}(\Omega)$ and $H^1(\Omega)$ are three Banach spaces, and the following holds:

$$\Phi \hookrightarrow H^1(\Omega) \qquad H^{5/4}(\Omega) \hookrightarrow H^1(\Omega) \qquad \Phi \subset H^{5/4}(\Omega).$$

[by definition] [Sobolev imbedding Theorem] [Theorem 1.1]

■

Note that similar results can be obtained for

$$\Psi = \{\psi \in H^1(\Omega),\ \Delta\psi \in L^2(\Omega),\ \frac{\partial\psi}{\partial\nu}_{|\Gamma_S} = 0,\ \psi_{|\Gamma_C^0} = 0,\ \frac{\partial\psi}{\partial\tau}_{|\Gamma_C^*} = 0\},$$
$$\Psi_0 = \{\psi \in \Psi,\ \psi_{|\Gamma_C} = 0\},\ \Psi_\Delta = \{\psi \in \Psi,\ \Delta\psi = 0\}.$$

In particular, one has $\mathbf{grad}\,\phi \cdot \mathbf{grad}\,\psi = \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} = \mathbf{curl}\,\phi \cdot \mathbf{curl}\,\psi$ pointwise, so in $\mathbf{L}^2(\Omega)$,

$$\mathbf{curl}\,\Phi_0 \perp \mathbf{curl}\,\Phi_\Delta \text{ and } \mathbf{curl}\,\Psi_0 \perp \mathbf{curl}\,\Psi_\Delta.$$

With these definitions and results, we can now prove the

Theorem 1.2 *In $\boldsymbol{L}^2(\Omega)$, one has*

$$\mathcal{V} = \mathbf{curl}\,\Phi_0 \overset{\perp}{\oplus} \mathbf{curl}\,\Phi_\Delta \overset{\perp}{\oplus} \mathbf{grad}\,\Psi_0.$$

Proof: First, notice that both $\mathbf{curl}\,\Phi$ and $\mathbf{grad}\,\Psi_0$ are subsets on $\mathcal{V}$. Next, following [10], for any element $\mathbf{v}$ of $\mathcal{V}$, we define

$$\psi_0 \in H^1(\Omega),\ \text{such that } \Delta\psi_0 = \operatorname{div}\mathbf{v} \text{ in } \Omega,\ \psi_{0|\Gamma_C} = 0,\ \frac{\partial\psi_0}{\partial\nu}_{|\Gamma_S} = 0.$$

From [14], we know that there exists one and only one solution ψ_0. By construction, $\psi_0 \in \Psi_0$. Now $\mathbf{w} = \mathbf{v} - \mathbf{grad}\,\psi_0$ is a divergence-free element of $\mathcal{V}$. Also:

$$\langle \mathbf{w}\cdot\vec{\nu}, 1\rangle = \int_\Omega \operatorname{div}\mathbf{w}\, d\mathbf{x} = \int_\Omega (\operatorname{div}\mathbf{v} - \Delta\psi_0)\, d\mathbf{x} = 0.$$

Then, as Ω is simply connected, there exists one and only one (*see Thm 3.1, Chapter 1 of [11]*) ϕ of $H^1(\Omega)$ such that $\mathbf{w} = \mathbf{curl}\,\phi$. In other words, as $\Delta \equiv -\mathrm{curl}\,\mathbf{curl}$ and $\partial\phi/\partial\nu = (\mathbf{curl}\,\phi)\cdot\vec{\tau}$, $\partial\phi/\partial\tau = -(\mathbf{curl}\,\phi)\cdot\vec{\nu}$, ϕ satisfies the following:

$$\phi \in H^1(\Omega),\ \text{such that } \Delta\phi = -\mathrm{curl}\,\mathbf{w} \text{ in } \Omega,\ \frac{\partial\phi}{\partial\nu}_{|\Gamma_C} = 0,\ \frac{\partial\phi}{\partial\tau}_{|\Gamma_S} = 0,$$

that is $\phi \in \Phi$. We already know that ϕ can be written as $\phi_0 + \phi_\Delta$, with $(\phi_0, \phi_\Delta) \in \Phi_0 \times \Phi_\Delta$, so there remains to check the orthogonality of $\mathbf{curl}\,\Phi$ and $\mathbf{grad}\,\Psi_0$: as a matter of fact, if we let $\phi \in \Phi$ and $\psi_0 \in \Psi_0$, $\psi_{0|\Gamma_S^k}$ belongs to $\widetilde{H}^{1/2}(\Gamma_S^k)$, for $1 \le k \le K$. Thus (1.5) finally yields

$$\int_\Omega \mathbf{curl}\,\phi \cdot \mathbf{grad}\,\psi_0\, d\mathbf{x} = 0.$$

■

Corollary 1.4 *Algebrically, $\mathcal{V} \subset \boldsymbol{H}^{1/4}(\Omega)$. More precisely, when $\alpha \notin \mathbb{N}$, $\mathcal{V} \subset \boldsymbol{H}^{\alpha-\varepsilon}(\Omega)$ (for any $\varepsilon > 0$) and $\mathcal{V} \not\subset \boldsymbol{H}^{\alpha}(\Omega)$, whereas when $\alpha \in \mathbb{N}$, $\mathcal{V} \subset \boldsymbol{H}^{\alpha}(\Omega)$ and $\mathcal{V} \not\subset \boldsymbol{H}^{\alpha+\varepsilon}(\Omega)$ (for any $\varepsilon > 0$).*

Proof: This is a straightforward consequence of the previous Theorem. ■

Remark 1.7 (For $\alpha \notin \mathbb{N}$) By taking $\phi_S \in \Phi_0$ such that $\phi_S \notin H^{1+\alpha}(\Omega)$, one gets a function $\mathbf{curl}\,\phi_S$ which does not belong to $\mathbf{H}^\alpha(\Omega)$. ■

Corollary 1.5 $\mathcal{V} \hookrightarrow \boldsymbol{H}^{1/4}(\Omega)$. *As a consequence, $\mathcal{V}$ is compactly imbedded in $\boldsymbol{L}^2(\Omega)$.*

Proof: It suffices to remark that

$$\mathcal{V} \hookrightarrow \mathbf{L}^2(\Omega),\ \mathbf{H}^{1/4}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega),\ \text{and}\ \mathcal{V} \subset \mathbf{H}^{1/4}(\Omega)$$

to prove the first part. Now, as $\mathbf{H}^{1/4}(\Omega)$ is compactly imbedded into $\mathbf{L}^2(\Omega)$ by the Sobolev imbedding Theorem, the second part follows. ■

Remark 1.8 In a two-dimensional polygonal domain, this corresponds to the generalization of Weber's imbedding result [18] to the case of mixed bounday conditions. ■

Now, for a given $\mathbf{v}$ of $\mathcal{V}$, we are in a position to characterize $(\phi_0, \phi_\Delta, \psi_0)$ such that

$$\mathbf{v} = \mathbf{curl}\,\phi_0 + \mathbf{curl}\,\phi_\Delta + \mathbf{grad}\,\psi_0, (\phi_0, \phi_\Delta, \psi_0) \in \Phi_0 \times \Phi_\Delta \times \Psi_0. \tag{1.7}$$

Before we give the characterization, let us notice that, from its definition, it is clear that Φ_Δ is a finite dimensional space, a basis of which is the set of functions $(\chi_l)_{1\le l\le K}$ defined by

$$\chi_l \in H^1(\Omega),\ \text{such that}\ \Delta\chi_l = 0\ \text{in}\ \Omega,\ \chi_{l|\Gamma_S^k} = \delta_{kl},\ \frac{\partial \chi_l}{\partial \nu}_{|\Gamma_C} = 0.$$

Indeed, it is clear that the linear combinations of the $(\chi_l)_l$ span Φ_Δ and, if $\chi = \sum_l \beta_l\chi_l = 0$, $\chi_{|\Gamma_S^l} = \beta_l = 0$. We thus define the capacitance matrix $A = (A_{kl})_{1\le k,l\le K}$

$$A_{kl} = \int_\Omega \mathbf{grad}\,\chi_k \cdot \mathbf{grad}\,\chi_l\, d\mathbf{x}.$$

A is a symmetric positive definite matrix, as $(\mathbf{y}, A\mathbf{y}) = ||\mathbf{grad}\,(\sum_k y_k\chi_k)||_0^2$.

One has the

Theorem 1.3 *Let $\mathbf{v} \in \mathcal{V}$, then the triplet which satisfies (1.7) is characterized by*

$$\begin{cases} \Delta\phi_0 = -\mathrm{curl}\,\mathbf{v} \\ \phi_\Delta = \sum_k c_k\chi_k,\ \textit{where}\ \sum_l A_{kl}c_l = \int_\Omega \mathbf{v}\cdot\mathbf{curl}\,\chi_k\, d\mathbf{x} \\ \Delta\psi_0 = \mathrm{div}\,\mathbf{v}. \end{cases}$$

Proof: This is already known for ϕ_0 and ψ_0. Concerning ϕ_Δ, written as the linear combination above (with $\phi_{\Delta|\Gamma_S^k} = c_k$), one obtains by orthogonality

$$\int_\Omega \mathbf{v}\cdot\mathbf{curl}\,\chi_k\, d\mathbf{x} = \int_\Omega \mathbf{curl}\,\phi_\Delta \cdot \mathbf{curl}\,\chi_k\, d\mathbf{x} = \sum_l A_{kl}c_l.$$

■

Remark 1.9 The identity $\mathbf{grad}\,\Psi_\Delta = \mathbf{curl}\,\Phi_\Delta$ holds (see Annex B). ■

Remark 1.10 In all the proofs given above, the need for two distinct partitions of the boundary Γ is now clear. Indeed, the first one,

$$\Gamma_S = \bigcup_{0\le k\le K} \Gamma_S^k,\ \Gamma_C = \bigcup_{0\le k\le K} \Gamma_C^k,$$

which is based on the connected components of Γ_S and Γ_C, is used in order to characterize the space Φ_Δ. Whereas the second one,

$$\Gamma = \bigcup_{j\in\mathcal{C}\cup\mathcal{S}} \overline{\Gamma}_j,\ \overline{\Gamma}_C = \bigcup_{j\in\mathcal{C}} \overline{\Gamma}_j,\ \overline{\Gamma}_S = \bigcup_{j\in\mathcal{S}} \overline{\Gamma}_j,$$

is needed to determine the regularity of functions that are considered throughout this report. ■

Corollary 1.6 *In $\mathcal{V}$, $||\cdot||_{\mathcal{V}}$:*

$$\mathbf{v} \mapsto \left[||\mathrm{curl}\,\mathbf{v}||_0^2 + ||\mathrm{div}\,\mathbf{v}||_0^2 + \sum_{k=1}^{K} R_k(\mathbf{v})^2 \right]^{1/2},$$
$$\textit{with } R_k(\mathbf{v}) = \int_\Omega \mathbf{v}\cdot\mathbf{curl}\,\chi_k\,d\mathbf{x},$$

is a norm which is equivalent to the canonical norm of $\mathcal{V}$, i. e. $||\mathbf{v}||_{0,div,curl}$.

Proof: We know that any $\mathbf{v}$ of $\mathcal{V}$ can be written as in (1.7), i. e.

$$\mathbf{v} = \mathbf{curl}\,\phi_0 + \mathbf{curl}\,\phi_\Delta + \mathbf{grad}\,\psi_0.$$

Also, by orthogonality, one has

$$||\mathbf{v}||_0^2 = ||\mathbf{curl}\,\phi_0||_0^2 + ||\mathbf{curl}\,\phi_\Delta||_0^2 + ||\mathbf{grad}\,\psi_0||_0^2.$$

In addition, recall that ϕ_0 and ψ_0 are solutions of two variational problems, according to [14] (*Thm 2.1.1. page 38*). The problems to be solved are respectively

$$\int_\Omega \mathbf{grad}\,\phi_0\cdot\mathbf{grad}\,\phi\,d\mathbf{x} = \int_\Omega \phi\,\mathrm{curl}\,\mathbf{v}\,d\mathbf{x},\quad \forall\phi\in\{\phi\in H^1(\Omega),\ \phi_{|\Gamma_S}=0\},$$
$$\int_\Omega \mathbf{grad}\,\psi_0\cdot\mathbf{grad}\,\psi\,d\mathbf{x} = -\int_\Omega \psi\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x},\quad \forall\psi\in\{\psi\in H^1(\Omega),\ \psi_{|\Gamma_C}=0\}.$$

Therefore, there exists a constant C_1 independent of $\mathbf{v}$ such that $||\mathbf{curl}\,\phi_0||_0 \le C_1\,||\mathrm{curl}\,\mathbf{v}||_0$ and $||\mathbf{grad}\,\psi_0||_0 \le C_1\,||\mathrm{div}\,\mathbf{v}||_0$.

Concerning the remaining part, let $\vec{c}$ (resp. $\vec{R}$) denote the vector with components c_k (resp. $R_k(\mathbf{v})$). One gets,

$$\begin{aligned} ||\mathbf{curl}\,\phi_\Delta||_0^2 &= \int_\Omega |\mathbf{grad}\,\phi_\Delta|^2\,d\mathbf{x} = \sum_{k,l} c_k\,c_l \int_\Omega \mathbf{grad}\,\chi_k\cdot\mathbf{grad}\,\chi_l\,d\mathbf{x} \\ &= (\vec{c},A\vec{c}) = (A^{-1}\vec{R},\vec{R}) \le \frac{1}{\lambda_{min}}||\vec{R}||^2 = \frac{1}{\lambda_{min}}\sum_{k_1}^{K} R_k(\mathbf{v})^2. \end{aligned}$$

Thus, we arrive at

$$||\mathbf{v}||_0 \le C_2\,||\mathbf{v}||_{\mathcal{V}},\ \text{with } C_2 = \max(C_1, \frac{1}{\sqrt{\lambda_{min}}}).$$

The conclusion follows. ■

Remark 1.11 An alternate proof can be derived by using the fact that $\mathcal{V}$ is compactly imbedded into $\mathbf{L}^2(\Omega)$ (by contradiction). In the same way, one can prove that, in Φ, $||\cdot||_\Phi$: $\phi \to \left[||\Delta\phi||_0^2 + \sum_{k=1}^{K}(\phi_{|\Gamma_S^k})^2\right]^{1/2}$ is equivalent to the graph norm $||\cdot||_E$. ■

Chapter 2

Homogeneous boundary conditions

Given f and g elements of $L^2(\Omega)$ and $\vec{c} \in \mathbb{R}^K$, we consider the following problem, called (P_{SS}):
Find $\mathbf{u} \in \boldsymbol{H}(\mathrm{curl}\,,\Omega) \cap \boldsymbol{H}(\mathrm{div}\,,\Omega)$ *such that*

$$\begin{aligned} \mathrm{curl}\,\mathbf{u} &= f \text{ in } \Omega, && (2.1)\\ \mathrm{div}\,\mathbf{u} &= g \text{ in } \Omega, && (2.2)\\ \mathbf{u}\cdot\vec{\tau} &= 0 \text{ on } \Gamma_C, && (2.3)\\ \mathbf{u}\cdot\vec{\nu} &= 0 \text{ on } \Gamma_S, && (2.4)\\ R_k(\mathbf{u}) &= c_k \text{ for } 1 \le k \le K. && (2.5) \end{aligned}$$

Proposition 2.1 *Take f, g and $\vec{c}$ as above. Then problem (P_{SS}) admits a unique solution $\mathbf{u} \in \boldsymbol{H}(\mathrm{curl}\,,\Omega) \cap \boldsymbol{H}(\mathrm{div}\,,\Omega)$.*

Proof: This is a straightforward application of Theorem 1.3. ∎

The vector $\vec{c}$ can be taken to be equal to zero, as this amounts to adding some $\mathbf{curl}\,\phi_\Delta/\mathbf{grad}\,\psi_\Delta$. In the sequel, we shall actually consider that (2.5) holds with $c_k = 0$, for $1 \le k \le K$, i. e.

$$R_k(\mathbf{u}) = 0 \text{ for } 1 \le k \le K.$$

2.1 A variational formulation of (P_{SS}) via a stream function when $f = 0$

Thanks to Theorem 1.3, $\mathbf{u}$ is a solution to (P_{SS}) with $f = 0$ if and only if $\mathbf{u}$ can be written in the form $\mathbf{u} = \mathbf{grad}\,\psi_0$, where ψ_0 is the solution to the next problem, labelled (P_ψ):
Find $\psi_0 \in H^1(\Omega)$ *such that*

$$\begin{aligned} \Delta\psi_0 &= g \text{ in } \Omega, && (2.6)\\ \psi_0 &= 0 \text{ on } \Gamma_C, && (2.7)\\ \frac{\partial\psi_0}{\partial\nu} &= 0 \text{ on } \Gamma_S. && (2.8) \end{aligned}$$

According to [14] (*pages 36-39*), (P_ψ) is equivalent to the variational formulation
Find $\psi_0 \in \Psi_- = \{\psi \in H^1(\Omega),\ \psi_{|\Gamma_C} = 0\}$ *such that*

$$\int_\Omega \mathbf{grad}\,\psi_0 \cdot \mathbf{grad}\,\psi\, d\mathbf{x} = -\int_\Omega g\,\psi\, d\mathbf{x}, \quad \forall \psi \in \Psi_-.$$

2.2 A variational formulation of (P_{SS}) via a stream function when $g = 0$

In this case, $\mathbf{u}$ satisfies to (2.1)-(2.5) if and only if $\mathbf{u}$ can be written in the form $\mathbf{u} = \mathbf{curl}\,\phi_0$ with ϕ_0 the solution to (P_ϕ):
Find $\phi_0 \in H^1(\Omega)$ *such that*

$$\begin{aligned} \Delta\phi_0 &= -f \text{ in } \Omega, && (2.9)\\ \phi_0 &= 0 \text{ on } \Gamma_S, && (2.10)\\ \frac{\partial\phi_0}{\partial\nu} &= 0 \text{ on } \Gamma_C. && (2.11) \end{aligned}$$

From [14] again, we infer this time that (P_ϕ) is equivalent to the variational formulation
Find $\phi_0 \in \Phi_- = \{\phi \in H^1(\Omega),\ \phi_{|\Gamma_S} = 0\}$ *such that*

$$\int_\Omega \mathbf{grad}\,\phi_0 \cdot \mathbf{grad}\,\phi\, d\mathbf{x} = \int_\Omega f\,\phi\, d\mathbf{x}, \quad \forall\phi \in \Phi_-.$$

2.3 Another formulation of (P_{SS})

Proposition 2.2 *An equivalent re-formulation of* (P_{SS}) *is problem* (P_{CC})
Find $\mathbf{u} \in \mathbf{H}(\mathrm{curl}\,,\Omega) \cap \mathbf{H}(\mathrm{div}\,,\Omega)$ such that

$$\begin{aligned} \mathbf{curl}\,(\mathrm{curl}\,\mathbf{u}) &= \mathbf{curl}\,f \ \textit{in}\ \Omega, && (2.12)\\ \mathbf{grad}\,(\mathrm{div}\,\mathbf{u}) &= \mathbf{grad}\,g \ \textit{in}\ \Omega, && (2.13)\\ \mathbf{u}\cdot\vec{\tau} &= 0 \ \textit{on}\ \Gamma_C, && (2.14)\\ \mathbf{u}\cdot\vec{\nu} &= 0 \ \textit{on}\ \Gamma_S, && (2.15)\\ R_k(\mathbf{u}) &= 0 \ \textit{for}\ 1 \le k \le K, && (2.16)\\ (\mathrm{curl}\,\mathbf{u} - f) &= 0 \ \textit{on}\ \gamma,\ \textit{meas}(\gamma) > 0, && (2.17)\\ (\mathrm{div}\,\mathbf{u} - g) &= 0 \ \textit{on}\ \gamma',\ \textit{meas}(\gamma') > 0. && (2.18) \end{aligned}$$

Proof: It is clear that if $\mathbf{u}$ is a solution to (P_{SS}), then it is also a solution to (P_{CC}). Conversely, if $\mathbf{u}'$ is a solution to (P_{CC}), then (2.12) implies $\mathrm{curl}\,(\mathbf{u} - \mathbf{u}') = cst$ in Ω (Ω is connected). As $f = \mathrm{curl}\,\mathbf{u}$ in Ω, (2.17) yields $\mathrm{curl}\,(\mathbf{u} - \mathbf{u}')_{|\gamma} = 0$, and therefore $cst = 0$. In the same way, (2.13) and (2.18) lead to $\mathrm{div}\,\mathbf{u}' = \mathrm{div}\,\mathbf{u}$. Thus $\mathbf{u} = \mathbf{u}'$. ■

2.4 A second variational formulation of (P_{SS})

Here, we choose to "break the symmetry" in curl-div. For that, let us introduce the set $X_C = \{\mathbf{v} \in \mathbf{H}(\mathrm{curl}\,,\Omega),\ \mathbf{v}\cdot\tau_{|\Gamma_C} = 0\}$. By assumptions (2.1) and (2.3), $\mathbf{u}$ belongs to X_C. In addition, (2.1) implies

$$\int_\Omega \mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v}\, d\mathbf{x} = \int_\Omega f\,\mathrm{curl}\,\mathbf{v}\, d\mathbf{x}, \quad \forall\mathbf{v} \in X_C.$$

Clearly, (2.5) yields (see Annex B)

$$\int_\Omega \mathbf{u}\cdot\mathbf{grad}\,\psi_\Delta\, d\mathbf{x} = 0, \quad \forall\psi_\Delta \in \Psi_\Delta.$$

Let $M_C = \Psi_-$ (see above). We now proceed to show that (2.2) and (2.4) lead to $\int_\Omega \mathbf{u} \cdot \mathbf{grad}\, \psi \, d\mathbf{x} = -\int_\Omega g\, \psi \, d\mathbf{x}$, for all ψ in M_C. Grisvard [14] (*page 28*) states that $M_C^m = M_C \cap H^m(\Omega)$ is dense in M_C for all $m > 1$. Therefore, if we choose a smooth function $\psi \in M_C^m$, we can integrate $\int_\Omega \psi \,\mathrm{div}\, \mathbf{u}\, d\mathbf{x}$ by parts (recall that $\mathbf{u} \in \mathbf{H}^{1/4}(\Omega)$, and so $u_{\nu|\gamma} \in H^{-1/4}(\gamma)$). Then

$$\int_\Omega g\, \psi \, d\mathbf{x} = -\int_\Omega \mathbf{u} \cdot \mathbf{grad}\, \psi \, d\mathbf{x} + \sum_k \left\{ \langle u_\nu, \psi \rangle_{\Gamma_S^k} + \langle u_\nu, \psi \rangle_{\Gamma_C^k} \right\}.$$

Thanks to (2.4) and $\psi \in M_C^m$, the bracketed terms vanish and so $\int_\Omega \mathbf{u} \cdot \mathbf{grad}\, \psi \, d\mathbf{x} = -\int_\Omega g\, \psi \, d\mathbf{x}$. By density, this holds for all $\psi \in M_C$.

Thus, we come to yet another problem, called (P_{INT}): *Find* $\mathbf{u}' \in X_C$ *such that*

$$\int_\Omega \mathrm{curl}\, \mathbf{u}' \,\mathrm{curl}\, \mathbf{v}\, d\mathbf{x} = \int_\Omega f\, \mathrm{curl}\, \mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v} \in X_C \tag{2.19}$$

$$\int_\Omega \mathbf{u}' \cdot \mathbf{grad}\, (\psi + \psi_\Delta)\, d\mathbf{x} = -\int_\Omega g\, \psi \, d\mathbf{x}, \quad \forall (\psi, \psi_\Delta) \in M_C \times \Psi_\Delta. \tag{2.20}$$

By construction, the solution $\mathbf{u}$ to (P_{SS}) is a solution to (P_{INT}). There is a reciprocal assertion.

Proposition 2.3 *Let* $\mathbf{u}'$ *be a solution to* (P_{INT}). *Then* $\mathbf{u}' = \mathbf{u}$.

Proof: Consider (2.20), for $\psi = 0$: this yields $R_k(\mathbf{u}') = 0$ for $1 \le k \le K$.

Next, let $\psi_\Delta = 0$ and $\psi \in \mathcal{D}(\Omega)$: then $\mathrm{div}\, \mathbf{u}' = g$ in Ω.

In addition, for all $\mu \in \widetilde{H}^{1/2}(\Gamma_S^k)$, there exists $\psi \in M_C$ such that $\psi_{|\Gamma_S^k} = \mu$ (extend μ by zero to Γ). Then we readily come to $u_\nu = 0$ on Γ_S.

Now, (2.19) can be rewritten $\int_\Omega h\, \mathrm{curl}\, \mathbf{v}\, d\mathbf{x} = 0$, $\forall \mathbf{v} \in X_C$, with $h = (\mathrm{curl}\, \mathbf{u}' - f) \in L^2(\Omega)$. Thus $\mathbf{curl}\, h = 0$ in the sense of distributions, which implies in turn that $h \in H^1(\Omega)$ and, as a consequence, since Ω is connected, it is equal to a constant.

To conclude, if we let $\mathbf{v} \in X_C$ such that $\int_\Omega \mathrm{curl}\, \mathbf{v}\, d\mathbf{x} \neq 0$, we find $h = 0$. Thus $\mathbf{u}'$ is the solution to (P_{SS}). ∎

Then, let us introduce a saddle-point formulation of (P_{INT}), i. e. the problem (P_{SP}): *Find* $(\mathbf{u}', \underline{\psi}, \underline{\psi}_\Delta) \in X_C \times M_C \times \Psi_\Delta$ *such that*

$$a(\mathbf{u}', \mathbf{v}) + b(\mathbf{v}, (\underline{\psi}, \underline{\psi}_\Delta)) = \int_\Omega f\, \mathrm{curl}\, \mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v} \in X_C \tag{2.21}$$

$$b(\mathbf{u}', (\psi, \psi_\Delta)) = -\int_\Omega g\, \psi \, d\mathbf{x}, \quad \forall (\psi, \psi_\Delta) \in M_C \times \Psi_\Delta. \tag{2.22}$$

Here, $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathrm{curl}\, \mathbf{u}\, \mathrm{curl}\, \mathbf{v}\, d\mathbf{x}$ and $b(\mathbf{v}, (\psi, \psi_\Delta)) = \int_\Omega \mathbf{v} \cdot \mathbf{grad}\, (\psi + \psi_\Delta)\, d\mathbf{x}$ are bilinear continuous forms on $X_C \times X_C$ and $X_C \times (M_C \times \Psi_\Delta)$ respectively.

We now establish the well-posedeness of (P_{SP}).

Proposition 2.4 *There exists one and only one solution* $(\mathbf{u}', \underline{\psi}, \underline{\psi}_\Delta) \in X_C \times M_C \times \Psi_\Delta$ *to* (P_{SP}). *In addition,* $\underline{\psi} = \underline{\psi}_\Delta = 0$ *and* $\mathbf{u}' = \mathbf{u}$.

Proof: In order to prove that there exists a unique solution to this saddle-point problem, we have to prove (cf. [11]) that

(i) a is K-elliptic, where K is the kernel of b.

(ii) An inf-sup condition holds.

(i) By definition,

$$\begin{aligned} K &= \{\mathbf{v} \in X_C,\ b(\mathbf{v}, (\psi, \psi_\Delta)) = 0,\ \forall (\psi, \psi_\Delta) \in M_C \times \Psi_\Delta\} \\ &= \{\mathbf{v} \in \mathbf{H}(\mathrm{curl}\,, \Omega),\ \mathrm{div}\,\mathbf{v} = 0,\ v_{\tau|\Gamma_C} = 0,\ v_{\nu|\Gamma_S} = 0,\ \vec{R}(\mathbf{v}) = 0\} \subset \mathcal{V}. \end{aligned}$$

Then, thanks to Corollary 1.6, a is K-elliptic.

(ii) The inf-sup condition is satisfied with a unit constant. Indeed, if we let $(\psi, \psi_\Delta) \in M_C \times \Psi_\Delta$, then $\mathbf{grad}\,(\psi + \psi_\Delta)$ belongs to X_C and, as a consequence:

$$\sup_{\mathbf{v} \in X_C} \frac{|b(\mathbf{v}, (\psi, \psi_\Delta))|}{||\mathbf{v}||_{0,\mathrm{curl}}\ ||(\psi, \psi_\Delta)||} \geq \frac{|b(\mathbf{grad}\,(\psi + \psi_\Delta), (\psi, \psi_\Delta))|}{||(\psi, \psi_\Delta)||^2} = 1,$$

where $||(\psi, \psi_\Delta)|| = \{||\mathbf{grad}\,\psi||_0^2 + ||\mathbf{grad}\,\psi_\Delta||_0^2\}^{1/2}$.

N. B. $\displaystyle\int_\Omega \mathbf{grad}\,\psi \cdot \mathbf{grad}\,\psi_\Delta\, d\mathbf{x} = 0$ by orthogonality.

To conclude, if we let $\mathbf{v} = \mathbf{grad}\,\underline{\psi} + \mathbf{grad}\,\underline{\psi}_\Delta$ in (2.21), we come to $||(\underline{\psi}, \underline{\psi}_\Delta)|| = 0$ and then $\mathbf{u}'$ is the solution to (P_{INT}), i. e. $\mathbf{u}' = \mathbf{u}$. ■

Chapter 3

Orthogonal decompositions and singularities

Let $V = \mathcal{V} \cap \mathbf{H}(\mathrm{div}\, 0, \Omega)$ and $V_0 = \{\mathbf{v} \in V,\ R_k(\mathbf{v}) = 0,\ 1 \leq k \leq K\}$. Note that, in V_0, $||\mathbf{v}||_{\mathcal{V}}$ reduces to $||\mathrm{curl}\, \mathbf{v}||_0$. In a similar manner, in Φ_0, $||\phi||_\Phi$ is simply $||\Delta\phi||_0$. In the following, the notations $V_{(0)}$ and $\Phi_{(0)}$ correspond either to the situation V and Φ or to V_0 and Φ_0. By applying standard results, one obtains the

Theorem 3.1 *(i) Δ is an isomorphism from Φ_0 onto $L^2(\Omega)$.*

(ii) **curl** *is an isomorphism from $\Phi_{(0)}$ onto $V_{(0)}$.*

(iii) curl *is an isomorphism from V_0 onto $L^2(\Omega)$.*

Proof: (i) stems directly from [14], Chapter II.

(ii) is a consequence of [11] (*page 37*).

(iii) is due to the identity $-\mathrm{curl}\ \mathbf{curl} = \Delta$ and points (i) and (ii). ■

A direct consequence of Corollary 1.4 and Theorem 1.1 is that V_0 and Φ_0 are imbedded into $\mathbf{H}^1(\Omega)$ and $H^2(\Omega)$ respectively if and only if the value of the parameter α is larger than or equal to one. *In particular, this excludes a change of the boundary condition on a geometrical edge.* In the general case, we introduce the **regularized** spaces:

$$V_{0,R} = V_0 \cap \mathbf{H}^1(\Omega) \text{ and } \Phi_{0,R} = \Phi_0 \cap H^2(\Omega).$$

Remark 3.1 Clearly, it is possible to proceed in exactly the same way for $W = \{\mathbf{v} \in \mathcal{V},\ \mathrm{curl}\, \mathbf{v} = 0\}$ and $W_0 = \{\mathbf{v} \in W,\ R_k(\mathbf{v}) = 0,\ 1 \leq k \leq K\}$ by replacing Φ_0 (resp. **curl**, curl) by Ψ_0 (resp. **grad**, div). In this case, on simply uses the identity div **grad** $= \Delta$. ■

3.1 Bilinear forms

Let $\mathcal{V}_R = \mathcal{V} \cap \mathbf{H}^1(\Omega)$ be a **regularized** space.

In order to obtain interesting results, it is worthwhile to prove a density result *a la Costabel-Dauge-Nicaise* [8]. Let

$$\mathbf{D} = \{\mathbf{v} \in \mathcal{D}(\Omega)^2,\ v_{\tau|\Gamma_C} = 0,\ v_{\nu|\Gamma_S} = 0,\ \mathbf{v} = 0 \text{ in a neighborhood of the corners } (M_j)_j\}.$$

Proposition 3.1 ***D*** *is dense in* $\mathcal{V}_R$.

Proof: We follow here the proof which can be found in [8]. Given $\mathbf{u} \in \mathcal{V}_R$ and $\varepsilon > 0$, we shall build a function $\mathbf{u}^{***}$ of $\mathbf{D}$ such that $||\mathbf{u} - \mathbf{u}^{***}||_1 \leq \varepsilon$.

Let χ be a truncation function, equal to one in the vicinity of zero and to zero in the vicinity of ∞, let r_j denote the distance from the corner M_j and set $\Omega_j^c = B(M_j, R_j) \cap \Omega$ (for some positive R_j).

Step 1: Let us apply Lemma 2.4 (*page 9*) of [8] to get $n \in \mathbb{N}$ big enough and $\alpha > 0$ small enough such that the function

$$\mathbf{u}^j = r_j^\alpha \, (1 - \chi(nr_j))\mathbf{u}$$

of $\mathbf{H}^1(\Omega_j^c)$ is such that $||\mathbf{u}-\mathbf{u}^j||_{1,\Omega_j^c}$ is small (with respect to ε). By construction, $\mathbf{u}^j$ is equal to zero in a neighborhood of the corner M_j and, in addition, $\mathbf{u}^*_{\tau|\Gamma_C \cap \partial\Omega_j^c} = \mathbf{u}^*_{\nu|\Gamma_S \cap \partial\Omega_j^c} = 0$. It is then possible to build $\mathbf{u}^* \in \mathcal{V}_R$ (locally equal to $\mathbf{u}^j$ around M_j) such that

$$||\mathbf{u} - \mathbf{u}^*||_1 \leq \varepsilon/4;$$
$$\mathbf{u}^* = 0 \text{ in a neighborhood } \mathcal{V}_0 \text{ of the corners.}$$

Step 2: Let $\mathcal{V}_0'$ be another neighborhood of the corners, $\overline{\mathcal{V}_0'} \subset \mathcal{V}_0$. Then, $\mathbf{u}^*$ is regularized by convolution: let $(\chi_m)_m$ be a sequence of mollifiers and set $\mathbf{u}^{**} = \mathbf{u}^* * \chi_m$ for m large enough with the properties:

a. $\mathbf{u}^{**} = 0$ in $\mathcal{V}_0'$.

b. $||\mathbf{u}^* - \mathbf{u}^{**}||_1 \leq \varepsilon/4$.

c. $||\mathbf{u}^{**} \cdot \nu||_{1/2,\Gamma_S} \leq \varepsilon/4\, C_\nu$ and $||\mathbf{u}^{**} \cdot \tau||_{1/2,\Gamma_C} \leq \varepsilon/4\, C_\tau$.

Properties a. and b. are consequences of the classical theory of convolution [4] (*Proposition IV.18 page 68 for a. and Corollary IX.8 page 162, applied to a domain of class* $\mathcal{C}^1$ *"equivalent" to* $\Omega \setminus \mathcal{V}_0'$*, for b.*). What about c.? Here, C_ν (resp. C_τ) is equal to the norm of the lifting operator L_{ν,Γ_S} (resp. L_{τ,Γ_C}) of the normal (resp. tangential) trace of functions of $H^{1/2}(\Gamma_S)$ (resp. $H^{1/2}(\Gamma_C)$) into $\mathbf{H}^1(\Omega)$ which are *equal to zero on* $\mathcal{V}_0' \cap \Gamma$. (ouf !)
Both C_ν and C_τ thus depend on ε (via $\mathcal{V}_0'$). But, like ε, they are *given*. Therefore, as far as the traces are continuous with respect to the $\mathbf{H}^1$ norm of $\mathbf{u}^* - \mathbf{u}^{**}$, c. can be obtained through b..

Step 3: Classically [17], the lifting preserves the smoothness of the trace. A smooth trace yields a smooth lifting. Thus, the nonzero boundary condition on $\mathbf{u}^{**}$ is corrected by taking:

$$\mathbf{u}^{***} = \mathbf{u}^{**} - L_{\tau,\Gamma_C}(u_\tau^{**}) - L_{\nu,\Gamma_S}(u_\nu^{**}).$$

Note that L_{τ,Γ_C} are two *independent* liftings thanks to the zero boundary condition in the neighborhood of each corner;

$$||\mathbf{u}^{**} - \mathbf{u}^{***}||_1 \leq \varepsilon/2 \text{ by construction.}$$

As a conclusion, $\mathbf{u}^{***} \in \mathbf{D}$ and $||\mathbf{u}^{**} - \mathbf{u}||_1 \leq \varepsilon$. ■

In $\mathcal{V}_R$, after Costabel [7] who proved the result in three-dimensional polyhedral domains (in the case when $\Gamma_S = \emptyset$ or $\Gamma_C = \emptyset$), the following result holds

Theorem 3.2 $\forall(\mathbf{u},\mathbf{v}) \in \mathcal{V}_R^2$*:*

$$\int_\Omega \mathbf{grad}\,\mathbf{u} \,:\, \mathbf{grad}\,\mathbf{v}\; d\mathbf{x} = \int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\; d\mathbf{x} + \int_\Omega \operatorname{curl}\mathbf{u}\operatorname{curl}\mathbf{v}\; d\mathbf{x}.$$

Before proving the Theorem proper, let us consider the case of smooth functions which belong to $\mathcal{D}(\overline{\Omega})^2$.

Proposition 3.2 *Let* $(\mathbf{u},\mathbf{v}) \in \mathcal{D}(\overline{\Omega})^{2^2}$*, then*

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\mathbf{u} \,:\, \mathbf{grad}\,\mathbf{v}\; d\mathbf{x} &= \int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\; d\mathbf{x} + \int_\Omega \operatorname{curl}\mathbf{u}\operatorname{curl}\mathbf{v}\; d\mathbf{x} \\ &+ \sum_{j\in\mathcal{C}\cup\mathcal{S}} \int_{\Gamma_j} \left\{ \frac{\partial u_\nu}{\partial\tau}\, v_\tau - \frac{\partial u_\tau}{\partial\tau}\, v_\nu \right\} d\sigma,\end{aligned}$$

where $(\Gamma_j)_{j\in\mathcal{C}\cup\mathcal{S}}$ *is the set of generalized edges and* (u_τ, u_ν) *are the components of* $\mathbf{u}$ *in the local reference* (O,τ_j,ν_j) *(i.e. for each* Γ_j*).*

Proof: First, let us recall that, given two 2×2 matrices A and B, one has

$$A \,:\, B = \sum_{i,j=1}^{2} a_{ij} b_{ij}.$$

As a consequence, integrating by parts, one obtains on the one hand

$$\int_\Omega \mathbf{grad}\,\mathbf{u} \,:\, \mathbf{grad}\,\mathbf{v}\; d\mathbf{x} = -\int_\Omega (\Delta u_1\, v_1 + \Delta u_2\, v_2)\; d\mathbf{x} + \int_\Gamma (\frac{\partial u_1}{\partial\nu}\, v_1 + \frac{\partial u_2}{\partial\nu}\, v_2)\, d\sigma,$$

and, on the other hand,

$$\begin{aligned}\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\; d\mathbf{x} &= -\int_\Omega (\mathbf{grad}\operatorname{div}\mathbf{u})\cdot\mathbf{v}\; d\mathbf{x} + \int_\Gamma \operatorname{div}\mathbf{u}\, v_\nu\, d\sigma, \\ \int_\Omega \operatorname{curl}\mathbf{u}\operatorname{curl}\mathbf{v}\; d\mathbf{x} &= \int_\Omega (\mathbf{curl}\operatorname{curl}\mathbf{u})\cdot\mathbf{v}\; d\mathbf{x} - \int_\Gamma \operatorname{curl}\mathbf{u}\, v_\tau\, d\sigma.\end{aligned}$$

In other respects

$$-(\Delta u_1\, v_1 + \Delta u_2\, v_2) = (\mathbf{curl}\operatorname{curl}\mathbf{u})\cdot\mathbf{v} - (\mathbf{grad}\operatorname{div}\mathbf{u})\cdot\mathbf{v},$$

thus one arrives readily at

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\mathbf{u} \,:\, \mathbf{grad}\,\mathbf{v}\; d\mathbf{x} &= \int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\; d\mathbf{x} + \int_\Omega \operatorname{curl}\mathbf{u}\operatorname{curl}\mathbf{v}\; d\mathbf{x} \\ &+ \int_\Gamma (\frac{\partial u_1}{\partial\nu}\, v_1 + \frac{\partial u_2}{\partial\nu}\, v_2)\, d\sigma - \int_\Gamma \operatorname{div}\mathbf{u}\, v_\nu\, d\sigma + \int_\Gamma \operatorname{curl}\mathbf{u}\, v_\tau\, d\sigma.\end{aligned}$$

From this point on, it is possible to split up each integral over Γ into a sum of integrals over the Γ_j: on each edge, the vectors τ and ν are constant, and as a consequence one can change the reference from $(O,\vec{e}_1,\vec{e}_2)$ to (O,τ_j,ν_j) in order to obtain

$$\begin{aligned}\frac{\partial u_1}{\partial\nu}\, v_1 + \frac{\partial u_2}{\partial\nu}\, v_2 &= \frac{\partial u_\tau}{\partial\nu}\, v_\tau + \frac{\partial u_\nu}{\partial\nu}\, v_\nu, \\ \operatorname{curl}\mathbf{u} &= \frac{\partial u_\nu}{\partial\tau} - \frac{\partial u_\tau}{\partial\nu}, \\ \operatorname{div}\mathbf{u} &= \frac{\partial u_\tau}{\partial\tau} + \frac{\partial u_\nu}{\partial\nu}.\end{aligned}$$

This finally leads to the result. ■

Proof of Theorem 3.2: Let $\mathbf{v} \in \mathbf{D}$ (and $\mathbf{u} \in \mathcal{D}(\overline{\Omega})^2$). Then

$$\sum_{j\in\mathcal{C}\cup\mathcal{S}} \int_{\Gamma_j} \left\{ \frac{\partial u_\nu}{\partial \tau}\, v_\tau - \frac{\partial u_\tau}{\partial \tau}\, v_\nu \right\} d\sigma = \sum_{j\in\mathcal{S}} \int_{\Gamma_j} \frac{\partial u_\nu}{\partial \tau}\, v_\tau \, d\sigma - \sum_{j\in\mathcal{C}} \int_{\Gamma_j} \frac{\partial u_\tau}{\partial \tau}\, v_\nu \, d\sigma.$$

Now, if $\mathbf{u} \in \mathbf{D}$, then $u_{\nu|\Gamma_S} = 0$, which implies in turn $\partial u_\nu/\partial\tau_{|\Gamma_S} = 0$; in the same way, $\partial u_\tau/\partial\tau_{|\Gamma_C} = 0$. Thus the above sum is equal to 0 for any $(\mathbf{u}, \mathbf{v}) \in \mathbf{D}^2$. The conclusion follows thanks to the density of $\mathbf{D}$ in $\mathcal{V}_R$. ■

For an alternate proof (without the help of the density result), see Annex C.

Remark 3.2 The results of [7] (derivation of the formula in the case of smooth functions) and [8] (density of locally zero smooth functions) allow to prove the same identity in a polyhedron Ω with a Lipschitz boundary in the case of mixed boundary conditions, $\mathbf{u} \times \mathbf{n}_{|\Gamma_C} = 0$, $\mathbf{u} \cdot \mathbf{n}_{|\Gamma_S} = 0$, where Γ_C and Γ_S are sets of faces. ■

3.2 Orthogonal decompositions and splitting of solutions

Recall that we introduced the regularized subspaces of V_0 and Φ_0 which we called $V_{0,R}$ and $\Phi_{0,R}$. Let us prove that

Proposition 3.3 *$V_{0,R}$ and $\Phi_{0,R}$ are closed subspaces of V_0 and Φ_0 respectively.*

Proof: Thanks to Theorem 3.2, for all $\mathbf{v}$ in $V_{0,R}$, one has the identity $||\mathbf{v}||_{0,\mathrm{curl},\mathrm{div}} = ||\mathbf{v}||_1$. The above result is then a straightforward consequence of the completeness of $\mathbf{H}^1(\Omega)$ for its canonical norm and of Corollary 1.6.

Concerning $\Phi_{0,R}$, according to Point (ii) of Theorem 3.1, it suffices to note that $\mathbf{curl}^{-1}(V_{0,R}) = \Phi_{0,R}$. ■

As a consequence, we may write

$$V_0 = V_{0,R} \overset{\perp}{\oplus} V_{0,S} \text{ and } \Phi_0 = \Phi_{0,R} \overset{\perp}{\oplus} \Phi_{0,S}.$$

In what follows, our aim will be to characterize the singular part of the solution $\mathbf{u}$ to (2.1)-(2.5) when $g = 0$ and $\vec{c} = 0$: $\mathbf{u} = \mathbf{u}_R + \mathbf{u}_S$ with $\mathbf{u}_R \in V_{0,R}$ and $\mathbf{u}_S \in V_{0,S}$. Note that it is equivalent to split up ϕ_0 the solution to (2.9)-(2.11) into $\phi_0 = \phi_{0,R} + \phi_{0,S}$, with $\phi_{0,R} \in \Phi_{0,R}$ and $\phi_{0,S} \in \Phi_{0,S}$. For that, we rely on Theorem 3.1 and Grisvard's orthogonal decomposition of $L^2(\Omega)$, stated below:

$$L^2(\Omega) = \Delta\Phi_{0,R} \overset{\perp}{\oplus} N.$$

According to [14] (*Thm 2.3.3. and Lemma 2.3.2. page 47*), an element of N is characterized by

Theorem 3.3 *An element p of $L^2(\Omega)$ belongs to N if and only if*

$$\Delta p = 0 \text{ in } \Omega, \quad p_{|\Gamma_j} = 0,\ j \in \mathcal{S}, \quad \frac{\partial p}{\partial \nu}_{|\Gamma_j} = 0,\ j \in \mathcal{C}.^1$$

[1] In addition, there are some compatibility conditions in the neighborhood of corners $(M_j)_{j\in J_N}$ only (no condition for mixed corners, cf. [16]).

We infer, from Theorem 3.1, the

Proposition 3.4

$$\begin{aligned} V_{0,S} &= \mathrm{curl}^{-1} N. \\ \Phi_{0,S} &= \Delta^{-1} N. \end{aligned}$$

In this way, we are able to characterize those elements of both $V_{0,S}$ and $\Phi_{0,S}$. Note that, by construction, they satisfy to some orthogonality properties with respect to the scalar products associated to $||\cdot||_{\mathcal{V}}$ and $||\cdot||_{\Phi}$: they are orthogonal to the smooth fields of $V_{0,R}$ and to the smooth potentials of $\Phi_{0,R}$ respectively. In the next subsection, we shall derive, following [2], an algorithm to compute the elements of both $V_{0,S}$ and $\Phi_{0,S}$.

3.3 Regularity of the singular part

We begin by some brief highlights of the theory developed by Grisvard. Then we reproduce the framework developed in [2]. In short, Grisvard's approach[2] to singular solutions of elliptic problems articulates along the following lines:

Lemma 3.1 *Let $v \in N$, then $v \in \mathcal{C}^\infty(\overline{\Omega} \setminus \mathcal{O})$ where $\mathcal{O}$ is any neighborhood of the set of corners $(M_j)_j$.*

Proof (Main steps): We know that v is harmonic in Ω, i. e. $\Delta v = 0$, which yields that v is locally $\mathcal{C}^\infty$ (cf. for instance [12] (*page 79*)).
On the boundary, away from corners, let $\overline{v}$ be an extension of v across the boundary, by odd or even reflection, depending on the boundary condition imposed on v. Then one proves that $\Delta\overline{v} = 0$ in an open set which covers (locally) both sides of the boundary. (see also Annex D). ■

There are several consequences to this Lemma: Let $\Omega_j^c \stackrel{def}{=} B(M_j, R_j) \cap \Omega = \{(r,\theta) \in]0, R_j[\times]0, \pi/\alpha_j[\} \cap \Omega$, (r,θ) being the polar coordinates with center M_j and $x - axis$ Γ_j (note that we drop the suscript j for the polar coordinates).

[1] When the polar coordinates (r,θ) are used in a neighborhood of M_j, the boundary condition can be written **meaningfully** as $v = 0$ or $\partial v/\partial\theta = 0$ for $r > 0$.

[2] In Ω_j^c, $\Delta v = 0$ is equivalent to $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial\theta^2} = 0$, for $0 < r < R$, $0 < \theta < \pi/\alpha_j$. Now, it is clear from the Lemma that:

$$\theta \mapsto v(r\,\mathrm{e}^{\mathrm{i}\theta}) \text{ belongs to } H^2(]0, \pi/\alpha_j[), \text{ for } 0 < r < R.$$

In other words, this function is in the domain D_j of the operator (in variable θ) Λ_j: $\varphi \mapsto -\varphi''$ with *ad hoc* boundary conditions.
Thus one gets:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}\Lambda_j v = 0,\ 0 < r < R.$$

[3] Let $(\varphi_{j,m})_{m\geq 0}$ denote the eigenvectors of Λ_j, with eigenvalues $\lambda_{j,m}^2$, $\lambda_{j,m} \geq 0$. The following holds:

[2] There are other relevant approaches, such as the use of the Mellin transform [3].

Proposition 3.5 *v can be written in the form $\sum_{m\geq 0} v_{j,m}(r)\,\varphi_{j,m}(\theta)$, with*

$$\begin{aligned} v_{j,m}(r) &= \alpha_{j,m}\, r^{\lambda_{j,m}} + \beta_{j,m}\, r^{-\lambda_{j,m}},\ \textit{for } 0 \leq \lambda_{j,m} < 1, \\ &= \alpha_{j,m}\, r^{\lambda_{j,m}},\ \textit{for } 1 \leq \lambda_{j,m}. \end{aligned}$$

Proof (Main steps): The "separation of variables" which corresponds exactly, here, to an expansion of $r \mapsto v(r\,\mathrm{e}^{\mathrm{i}\theta})$ in the basis $(\varphi_{j,m})_{m\geq 0}$, is justified by Point [2] and the fact that $v \in \mathcal{C}^\infty(]0, R[, D_j)$.
It is the property $v \in L^2(\Omega_j^c)$ which limits the value of $v_{j,m}(r)$ to $\alpha_{j,m}\, r^{\lambda_{j,m}}$, for $1 \leq \lambda_{j,m}$. When $\lambda_{j,m} = 0$, there is a $\log(r)$ term which vanishes thanks to the orthogonality conditions mentioned footnote 1. ■

From that point on, Grisvard proves:

Lemma 3.2 *For each j and $\lambda_{j,m} \in]0,1[$, there exists $\sigma_{j,m} \in N$ such that*

$$\sigma_{j,m} - \eta_j(r) r^{-\lambda_{j,m}} \varphi_{j,m}(\theta) \in H^1(\Omega),$$

where η_j is a cut-off function.

Proof (Main steps): Given $u_{j,m} = \eta_j(r) r^{-\lambda_{j,m}} \varphi_{j,m}(\theta)$, let $f_{j,m} = \Delta u_{j,m}$. By construction $f_{j,m}$ is in $L^2(\Omega)$.
There exists one and only one element of $H^1(\Omega)$, called $w_{j,m}$, the variational solution to $\Delta w_{j,m} = f_{j,m}$ with the relevant boundary conditions.
By a direct computation, it is easily checked "by hand" that $\sigma_{j,m} = u_{j,m} - w_{j,m}$ is in N. ■

This finally leads to

Theorem 3.4 $\dim N = \sum_j \mathrm{Card}\ \{\lambda_{j,m},\ 0 < \lambda_{j,m} < 1\}$.

Using the above results, assuming that $\alpha < 1$ (cf. Theorem 1.1), we are now in the position to prove the

Theorem 3.5 *An element $\mathbf{u}$ of $V_0 \setminus V_{0,R}$ belongs to $\boldsymbol{H}^{\alpha-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ and does not belong to $\boldsymbol{H}^{\alpha}(\Omega)$.*

Proof: $\mathbf{u}$ can be decomposed into $\mathbf{u}_R + \mathbf{u}_S$, with $\mathbf{u}_S \neq 0$. As $\mathbf{u}_R$ is at least of regularity H^1 for both its components, there remains to study $\mathbf{u}_S \in V_{0,S}$. From Proposition 3.4, this amounts to studying elements of N. Let $p = \mathrm{curl}\,\mathbf{u}_S \in N$.

From the short excerpts of Grisvard's theory above, we know in particular that p is smooth away from the corners. If we use the standard results of Annex D, we deduce that $\phi = \Delta^{-1} p$ $(\in \Phi_{0,S})$ is also smooth away from the corners: $\phi_{0,S} \in \mathcal{C}^\infty(\overline{\Omega} \setminus \mathcal{O})$, where $\mathcal{O}$ is any neighborhood of the corners. As $\mathbf{u}_S = \mathrm{curl}\,\phi_S$, it is then clear that it is also smooth away from the corners. The regularity of this function is therefore determined by its behavior at the corners.

Let J_M^D (resp. J_M^N) be the subset of J_M such that there is a Dirichlet (resp. Neumann) boundary condition on Γ_j in the definition of N. Then, with the help of the (local) polar coordinates, (r, θ), we find by inspection that S_j^0, the restriction of p to Ω_j^c, takes one of the following forms

$$S_j^0 = \sum_{n\geq n_0} A_{n,j}\, r^{n\alpha_j}\ \sin(n\alpha_j\theta),\ \text{if } j \in J_D,$$

$$
\begin{aligned}
&= \sum_{n\geq n_0} A_{n,j}\, r^{n\alpha_j}\, \cos(n\alpha_j\theta),\ \text{if } j\in J_N,\\
&= \sum_{n\geq n_0} A_{n,j}\, r^{(n+1/2)\alpha_j}\, \sin((n+1/2)\alpha_j\theta),\ \text{if } j\in J_M^D,\\
&= \sum_{n\geq n_0} A_{n,j}\, r^{(n+1/2)\alpha_j}\, \cos((n+1/2)\alpha_j\theta),\ \text{if } j\in J_M^N.
\end{aligned}
$$

N. B. These expansions of variables are justified by Proposition 3.5.

The index of the first term, n_0, depends on the values of the angle π/α_j. As the trigonometric functions are mutually orthogonal, the fact that S_j^0 belongs to $L^2(\Omega_j^c)$ yields:

[1] $j\in J_D$ or $j\in J_N$: $n_0=0$ if $\alpha_j\geq 1$, $n_0=-1$ if $1>\alpha_j>1/2$.

$j\in J_M$: $n_0=0$ if $\alpha_j\geq 2$, $n_0=-1$ if $2>\alpha_j\geq 3/2$, $n_0=-2$ if $3/2>\alpha_j>1/2$.

[2] $\displaystyle\sum_{n\geq n_0}\frac{A_{n,j}^2}{n}{R_j}^{2n\alpha_j}$ is bounded, where R_j is the radius of Ω_j^c.

Now, with Theorem 3.4, we conclude that, if $j\in J_D\cup J_N$, the corner contributes for one additional element of a basis of N if and only if $\alpha_j<1$; if $j\in J_M$, it contributes for a single element if and only if $2>\alpha_j\geq 2/3$, and for two elements if and only if $2/3>\alpha_j>1/2$.[3] [4] This means that there is no contribution as soon as $\theta_j\leq\pi$ (when $j\in J_D\cup J_N$) and $\theta_j\leq\pi/2$ ($j\in J_M$). Also, based on this labeling of a basis of N, it is clear that S_j^0 is determined by $A_{-1,j}$ when the summation begins at $n_0=-1$, and by $(A_{-2,j},A_{-1,j})$ when the value of n_0 is -2.

Remark 3.3 Evidently, it is possible to give the following alternate characterization, with the complex variable $z_j=r\mathrm{e}^{\mathrm{i}\theta}$ and $f(z)=\sum_{n\geq n_0}A_{n,j}\,z^n$:

$$
\begin{aligned}
S_j^0(z_j) &= \mathcal{I}m(f(z_j^{\alpha_j})),\ \text{if } j\in J_D \text{ or } \mathcal{R}e(f(z_j^{\alpha_j})),\ \text{if } j\in J_N\\
&= \mathcal{I}m(z_j^{\alpha_j/2}\, f(z_j^{\alpha_j})),\ \text{if } j\in J_M^D \text{ or } \mathcal{R}e(z_j^{\alpha_j/2}\, f(z_j^{\alpha_j})),\ \text{if } j\in J_M^N.
\end{aligned}
$$

■

Thanks to the result of Annex D, as both $\phi=-\Delta^{-1}p$ and $\mathbf{u}_S=-\mathbf{curl}\,(\Delta^{-1}p)$ are smooth up to the boundary, the same tools (i. e. separation of variables and expansion) can be applied to derive the expressions of $S_j^2=\phi_{|\Omega_j^c}$ and $\mathbf{S}_j^1=\mathbf{u}_{S|\Omega_j^c}$. In both cases, the solution is made up of two parts: a part which corresponds to the homogeneous problem, and a particular solution. As before, in order to truncate the summation in a correct manner, we use the properties $\mathbf{S}_j^1\in\mathbf{L}^2(\Omega_j^c)$ and $S_j^2\in H^1(\Omega_j^c)$. This yields:

$$
\mathbf{S}_j^1 = \sum_{n\geq n_0} A_{n,j}\, r^{n\alpha_j+1}\begin{pmatrix} -\dfrac{n\alpha_j}{4n\alpha_j+4}\cos(n\alpha_j\theta)\\ \dfrac{n\alpha_j+2}{4n\alpha_j+4}\sin(n\alpha_j\theta)\end{pmatrix} + \sum_{l\geq l_0} B_{l,j}\, r^{l\alpha_j-1}\begin{pmatrix}\cos(l\alpha_j\theta)\\ -\sin(l\alpha_j\theta)\end{pmatrix},\ [J_D],
$$

[3] $\lambda_{j,m}^2=(m\,\alpha_j)^2$, if $j\in J_D\cup J_N$. $0<\lambda_{j,m}<1$ only for $m=1$, $\alpha_j<1$.
[4] $\lambda_{j,m}^2=((m+1/2)\,\alpha_j)^2$, if $j\in J_M$. $0<\lambda_{j,m}<1$ only for $m=1$, $\alpha_j<2$, and for $m\in\{1,2\}$, $\alpha_j<2/3$.

$$
\begin{aligned}
&= \sum_{n\geq n_0} A_{n,j}\, r^{n\alpha_j+1} \begin{pmatrix} \dfrac{n\alpha_j}{4n\alpha_j+4}\sin(n\alpha_j\theta) \\ \dfrac{n\alpha_j+2}{4n\alpha_j+4}\cos(n\alpha_j\theta) \end{pmatrix} + \sum_{l\geq l_0} B_{l,j}\, r^{l\alpha_j-1} \begin{pmatrix} \sin(l\alpha_j\theta) \\ \cos(l\alpha_j\theta) \end{pmatrix}, \ [J_N], \\
&= \sum_{n\geq n_0} A_{n,j}\, r^{(n+1/2)\alpha_j+1} \begin{pmatrix} -\dfrac{(n+1/2)\alpha_j}{4(n+1/2)\alpha_j+4}\cos((n+1/2)\alpha_j\theta) \\ \dfrac{(n+1/2)\alpha_j+2}{4(n+1/2)\alpha_j+4}\sin((n+1/2)\alpha_j\theta) \end{pmatrix} \\
&+ \sum_{l\geq l_0} B_{l,j}\, r^{(l+1/2)\alpha_j-1} \begin{pmatrix} \cos((l+1/2)\alpha_j\theta) \\ -\sin((l+1/2)\alpha_j\theta) \end{pmatrix}, \ [J_M^D], \\
&= \sum_{n\geq n_0} A_{n,j}\, r^{(n+1/2)\alpha_j+1} \begin{pmatrix} \dfrac{(n+1/2)\alpha_j}{4(n+1/2)\alpha_j+4}\sin((n+1/2)\alpha_j\theta) \\ \dfrac{(n+1/2)\alpha_j+2}{4(n+1/2)\alpha_j+4}\cos((n+1/2)\alpha_j\theta) \end{pmatrix} \\
&+ \sum_{l\geq l_0} B_{l,j}\, r^{(l+1/2)\alpha_j-1} \begin{pmatrix} \sin((l+1/2)\alpha_j\theta) \\ \cos((l+1/2)\alpha_j\theta) \end{pmatrix}, \ [J_M^N].
\end{aligned}
$$

As $\mathbf{S}_j^1$ is in $\mathbf{L}^2(\Omega_j^c)$, we find that $l_0 = 1$ in the case when $j \in J_D \cup J_N$, whereas l_0 takes the value 0 when $j \in J_M$, regardless of the value of α_j. Then,

$$
\begin{aligned}
S_j^2 &= -\sum_{n\geq n_0} \frac{A_{n,j}}{4n\alpha_j+4}\, r^{n\alpha_j+2}\sin(n\alpha_j\theta) + \sum_{l\geq l_0} \frac{B_{l,j}}{l\alpha_j}\, r^{l\alpha_j}\sin(l\alpha_j\theta), \ [J_D], \\
&= -\sum_{n\geq n_0} \frac{A_{n,j}}{4n\alpha_j+4}\, r^{n\alpha_j+2}\cos(n\alpha_j\theta) - \sum_{l\geq l_0} \frac{B_{l,j}}{l\alpha_j}\, r^{l\alpha_j}\cos(l\alpha_j\theta), \ [J_N], \\
&= -\sum_{n\geq n_0} \frac{A_{n,j}}{4(n+1/2)\alpha_j+4}\, r^{(n+1/2)\alpha_j+2}\sin((n+1/2)\alpha_j\theta) \\
&+ \sum_{l\geq l_0} \frac{B_{l,j}}{(l+1/2)\alpha_j}\, r^{(l+1/2)\alpha_j}\sin((l+1/2)\alpha_j\theta), \ [J_M^D], \\
&= -\sum_{n\geq n_0} \frac{A_{n,j}}{4(n+1/2)\alpha_j+4}\, r^{(n+1/2)\alpha_j+2}\cos((n+1/2)\alpha_j\theta) \\
&- \sum_{l\geq l_0} \frac{B_{l,j}}{(l+1/2)\alpha_j}\, r^{(l+1/2)\alpha_j}\cos((l+1/2)\alpha_j\theta), \ [J_M^N].
\end{aligned}
$$

N. B. The coefficients $(B_l)_l$ are identical in the expressions of $\mathbf{S}_j^1$ and S_j^2, as they are related by the relationship $\mathbf{S}_j^1 = \operatorname{curl} S_j^2$.

Then, the regularity of $\mathbf{u}_S$ (resp. p, ϕ) is given by that of its local expansion (**first** term of the **second** summation, in l), and the conclusion follows. ∎

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Chapter 4

Annex

In the Annex, we give some results which are either standard or technical. By standard, we mean that they can be found in advanced textbooks. By technical, we mean that the computations are straightforward but sometimes a little bit too long to fit in the core of the report itself.

A Distances

We consider the following situation, with edges Γ_1 and Γ_2 of length L_1 and L_2.

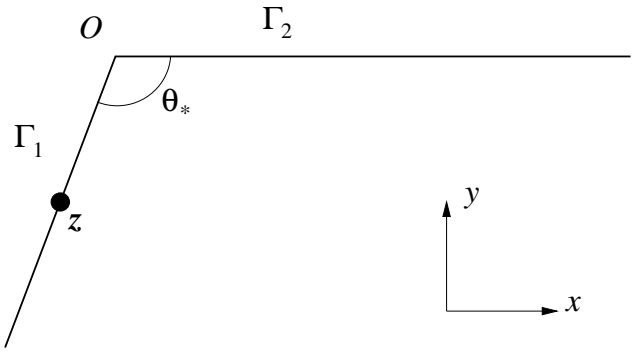

Figure 4.1: Notations.

We prove that there exist non negative constants C_1 and C_2 which depend only on the geometry such that

$$\frac{C_1}{||\mathbf{z}||} \leq \int_{\mathbf{t}\in\Gamma_2} \frac{d\sigma(\mathbf{t})}{||\mathbf{z}-\mathbf{t}||^2} \leq \frac{C_2}{||\mathbf{z}||}, \quad \forall \mathbf{z} \in \Gamma_1. \tag{i}$$

Remark 4.1 This is a case similar to the one considered for instance in [13] (*(1.3.2.12)*), with a difference due to the piecewise smoothness of the boundary Γ. ■

By symmetry, we can restrict θ_* to $]0, \pi]$. Let

$$\mathbf{z} = (z_x, z_y)^T \text{ and } I(\mathbf{z}) = \int_{\mathbf{t}\in\Gamma_2} \frac{d\sigma(\mathbf{t})}{||\mathbf{z}-\mathbf{t}||^2}.$$

Let us focuse on two particular cases:

[a] $\theta_* = \pi$:
z_x spans $]-L_1, 0[$ and $z_y = 0$ (this corresponds to a smooth boundary).

$$\begin{aligned} I(\mathbf{z}) &= \int_0^{L_2} \frac{dt}{(t-z_x)^2} = \left[-\frac{1}{t-z_x}\right]_0^{L_2} = -\frac{1}{L_2 - z_x} - \frac{1}{z_x} = -\frac{1}{L_2 + ||\mathbf{z}||} + \frac{1}{||\mathbf{z}||} \\ &= \frac{L_2}{||\mathbf{z}||(L_2 + ||\mathbf{z}||)}. \end{aligned}$$

Then (i) holds for $C_1 = \dfrac{L_2}{L_1 + L_2}$ and $C_2 = 1$.

[b] $\theta_* = \pi/2$:
$z_x = 0$ and z_y spans $]-L_1, 0[$.

$$I(\mathbf{z}) = \int_0^{L_2} \frac{dt}{t^2 + {z_y}^2} = \frac{1}{|z_y|} \left[\arctan(\frac{t}{|z_y|})\right]_0^{L_2} = \frac{1}{||\mathbf{z}||} \arctan(\frac{L_2}{||\mathbf{z}||}).$$

As $x \mapsto \arctan(x)$ is an increasing function in x, (i) holds for $C_1 = \arctan(\dfrac{L_2}{L_1})$ and $C_2 = \pi/2$.

Let us finally consider the general case:

[c] $\theta_* \in]0, \pi/2[\cup]\pi/2, \pi[$:
we have $z_y = -\alpha z_x$, with $\alpha = \tan(\theta_*)$, and $z_y \in]z_y^{\min}, 0[$, with $z_y^{\min} = -\dfrac{L_1}{\sqrt{1 + 1/\alpha^2}}$.

$$\begin{aligned} I(\mathbf{z}) &= \int_0^{L_2} \frac{dt}{(t-z_x)^2 + {z_y}^2} = \int_{-z_x}^{L_2 - z_x} \frac{dt}{t^2 + {z_y}^2} = \frac{1}{z_y} \left[\arctan(\frac{t}{z_y})\right]_{-z_x}^{L_2 - z_x} \\ &= \frac{1}{|z_y|} \left(\arctan(\frac{1}{\alpha}) - \arctan(\frac{L_2}{z_y} + \frac{1}{\alpha})\right). \end{aligned}$$

When z_y spans $]z_y^{\min}, 0[$, $z_y \mapsto -\arctan(L_2/z_y + 1/\alpha)$ is and increasing function in z_y. Therefore, if we let

$$\begin{aligned} \tilde{C}_1 &= \arctan(\frac{1}{\alpha}) - \arctan(\frac{1}{\alpha} - \frac{L_2}{L_1}\sqrt{1 + 1/\alpha^2}), \\ \tilde{C}_2 &= \arctan(\frac{1}{\alpha}) + \frac{\pi}{2}, \end{aligned}$$

both constants are non negative and, in addition,

$$\frac{\tilde{C}_1}{|z_y|} < I(\mathbf{z}) < \frac{\tilde{C}_2}{|z_y|}, \quad \forall z_y \in]z_y^{\min}, 0[.$$

As $\dfrac{1}{|z_y|} = \sqrt{1 + 1/\alpha^2}\dfrac{1}{||\mathbf{z}||}$, (i) holds with $C_i = \sqrt{1 + 1/\alpha^2}\,\tilde{C}_i$, $i = 1, 2$.

B Identification of spaces of stream functions

Let us consider the spaces $\mathbf{grad}\,\Psi_\Delta$ and $\mathbf{curl}\,\Phi_\Delta$. Ψ_Δ is a finite dimensional space, a basis of which is the set of functions $(\eta_k)_{1\leq k\leq K}$ defined by

$$\eta_k \in H^1(\Omega),\ \text{such that } \Delta\eta_k = 0 \text{ in } \Omega,\ \eta_{k|\Gamma_C^l} = \delta_{kl},\ \frac{\partial \eta_k}{\partial \nu}_{|\Gamma_S} = 0.$$

By construction, $\mathbf{w} = \mathbf{grad}\,\eta_k$ belongs to $\mathcal{V}$. Thanks to Theorem 1.3, $\mathbf{w} \in \mathbf{curl}\,\Phi_\Delta$. Therefore, $\mathbf{grad}\,\Psi_\Delta \subset \mathbf{curl}\,\Phi_\Delta$ and, as these vector spaces are both of dimension K:

$$\mathbf{grad}\,\Psi_\Delta = \mathbf{curl}\,\Phi_\Delta.$$

Then, one the one hand

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\eta_k \cdot \mathbf{curl}\,\chi_l\,d\mathbf{x} &= \left\langle \frac{\partial \eta_k}{\partial \tau}, \chi_l \right\rangle \\ &= \sum_{m=0}^{K} \left\langle \frac{\partial \eta_k}{\partial \tau}, \chi_l \right\rangle_{\Gamma_S^m} + \sum_{m=0}^{K} \left\langle \frac{\partial \eta_k}{\partial \tau}, \chi_l \right\rangle_{\Gamma_C^m} \\ &= \left\langle \frac{\partial \eta_k}{\partial \tau}, 1 \right\rangle_{\Gamma_S^l}.\end{aligned}$$

N. B. The splitting of duality brackets is fully justified, as $\mathbf{grad}\,\eta_k \in \mathbf{H}^{1/4}(\Omega)$. The same argument carries below, as $\mathbf{curl}\,\chi_l \in \mathbf{H}^{1/4}(\Omega)$.

On the other hand

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\eta_k \cdot \mathbf{curl}\,\chi_l\,d\mathbf{x} &= -\left\langle \frac{\partial \chi_l}{\partial \tau}, \eta_k \right\rangle \\ &= -\sum_{m=0}^{K} \left\langle \frac{\partial \chi_l}{\partial \tau}, \eta_k \right\rangle_{\Gamma_S^m} - \sum_{m=0}^{K} \left\langle \frac{\partial \chi_l}{\partial \tau}, \eta_k \right\rangle_{\Gamma_C^m} \\ &= -\left\langle \frac{\partial \chi_l}{\partial \tau}, 1 \right\rangle_{\Gamma_C^k}.\end{aligned}$$

As a consequence

$$\left\langle \frac{\partial \eta_k}{\partial \tau}, 1 \right\rangle_{\Gamma_S^l} + \left\langle \frac{\partial \chi_l}{\partial \tau}, 1 \right\rangle_{\Gamma_C^k} = 0,\ 1 \leq l, k \leq K.$$

C An alternate proof

Theorem 4.1 *$\forall (\mathbf{u}, \mathbf{v}) \in \mathcal{V}_R^2$:*

$$\int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\ d\mathbf{x} = \int_\Omega \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\ d\mathbf{x} + \int_\Omega \operatorname{curl}\mathbf{u} \operatorname{curl}\mathbf{v}\ d\mathbf{x}.$$

Proof: First, thanks to Proposition 3.2, we already know that for all $(\mathbf{u}, \mathbf{v}) \in \mathcal{D}(\overline{\Omega})^{2^2}$, one has

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\ d\mathbf{x} &= \int_\Omega \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\ d\mathbf{x} + \int_\Omega \operatorname{curl}\mathbf{u} \operatorname{curl}\mathbf{v}\ d\mathbf{x} \\ &+ \sum_{j\in\mathcal{C}\cup\mathcal{S}} \int_{\Gamma_j} \left\{ \frac{\partial u_\nu}{\partial \tau}\, v_\tau - \frac{\partial u_\tau}{\partial \tau}\, v_\nu \right\} d\sigma,\end{aligned}$$

where $(\Gamma_j)_{j\in\mathcal{C}\cup\mathcal{S}}$ is the set of generalized edges and (u_τ, u_ν) are the components of $\mathbf{u}$ in the local reference (O, τ_j, ν_j) (i.e. for each Γ_j).

In the second place, with the help of the density of $\mathcal{D}(\overline{\Omega})^2$ in $\mathbf{H}^1(\Omega)$, for all $(\mathbf{u}, \mathbf{v}) \in \mathcal{D}(\overline{\Omega})^2 \times \mathbf{H}^1(\Omega)$, we arrive at

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\; d\mathbf{x} &= \int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\; d\mathbf{x} + \int_\Omega \operatorname{curl}\mathbf{u}\operatorname{curl}\mathbf{v}\; d\mathbf{x} \\ &+ \sum_{j\in\mathcal{C}\cup\mathcal{S}} \int_{\Gamma_j} \left\{ \frac{\partial u_\nu}{\partial\tau}\, v_\tau - \frac{\partial u_\tau}{\partial\tau}\, v_\nu \right\} d\sigma. \qquad \text{(ii)}\end{aligned}$$

Unfortunately, it is not possible to perform the same operation twice without precautions. Indeed, as $u_{\tau|\Gamma_j}$ and $u_{\nu|\Gamma_j}$ are elements of $H^{1/2}(\Gamma_j)$, we know, from [15] (*1st volume, Prop 12.1, page 94*), that

$$\frac{\partial u_\tau}{\partial\tau}_{|\Gamma_j} \text{ and } \frac{\partial u_\nu}{\partial\tau}_{|\Gamma_j} \text{ then belong to } \widetilde{H}^{-1/2}(\Gamma_j),$$

the application $\dfrac{\partial\cdot}{\partial\tau}_{|\Gamma_j}$ being continuous from $H^{1/2}(\Gamma_j)$ into $\widetilde{H}^{-1/2}(\Gamma_j)$.

Remark 4.2 The alternate idea which consists in considering integral over the connected components of Γ_C and Γ_S does not work either, as in this case the quantities u_τ and u_ν are not well-defined around the generalized corners. ■

Therefore, passing to the limit in (ii) requires that $v_{\tau|\Gamma_j}$ and $v_{\nu|\Gamma_j}$ belong to $\widetilde{H}^{1/2}(\Gamma_j)$. This is precisely what we shall show for elements $\mathbf{v}$ of $\mathcal{V}_R$. First, for all $(\mathbf{u}, \mathbf{v}) \in \mathcal{D}(\overline{\Omega})^2 \times \mathcal{V}_R$, (ii) reduces to

$$\begin{aligned}\int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\; d\mathbf{x} &= \int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\; d\mathbf{x} + \int_\Omega \operatorname{curl}\mathbf{u}\operatorname{curl}\mathbf{v}\; d\mathbf{x} \\ &+ \sum_{j\in\mathcal{S}} \int_{\Gamma_j} \frac{\partial u_\nu}{\partial\tau}\, v_\tau\, d\sigma - \sum_{j\in\mathcal{C}} \int_{\Gamma_j} \frac{\partial u_\tau}{\partial\tau}\, v_\nu\, d\sigma. \qquad \text{(iii)}\end{aligned}$$

Second, recall that $\{M_j\} = \overline{\Gamma}_j \cap \overline{\Gamma}_{j+1}$ and let

$$\begin{aligned}\widetilde{H}^{1/2}_*(\Gamma_j) &= \{\mu \in H^{1/2}(\Gamma_j),\ \int_{\Gamma_j} \frac{|\mu(\mathbf{z})|^2}{||\mathbf{z} - M_j||}\, d\sigma(\mathbf{z}) < \infty\}, \\ \widetilde{H}^{1/2}_*(\Gamma_{j+1}) &= \{\mu \in H^{1/2}(\Gamma_{j+1}),\ \int_{\Gamma_{j+1}} \frac{|\mu(\mathbf{z})|^2}{||\mathbf{z} - M_j||}\, d\sigma(\mathbf{z}) < \infty\}.\end{aligned}$$

Third, on the couple of edges (Γ_j, Γ_{j+1}), either a Dirichlet ($j \in J_D$), a Neumann ($j \in J_N$) or a mixed ($j \in J_M$) boundary condition is imposed. Then, for an element $\mathbf{v}$ of $\mathcal{V}_R$, one can prove the following:

Proposition 4.1 *(i)* $\forall j \in J_D$, $v_{\nu|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$ *and* $v_{\nu|\Gamma_{j+1}} \in \widetilde{H}^{1/2}_*(\Gamma_{j+1})$.

(ii) $\forall j \in J_N$, $v_{\tau|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$ *and* $v_{\tau|\Gamma_{j+1}} \in \widetilde{H}^{1/2}_*(\Gamma_{j+1})$.

(iii) $\forall j \in J_M$ *such that* $\theta_j \not\equiv \pi/2\ [\pi]$,

- *If* $\Gamma_j \subset \Gamma_S$ *and* $\Gamma_{j+1} \subset \Gamma_C$*:* $v_{\tau|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$ *and* $v_{\nu|\Gamma_{j+1}} \in \widetilde{H}^{1/2}_*(\Gamma_{j+1})$.

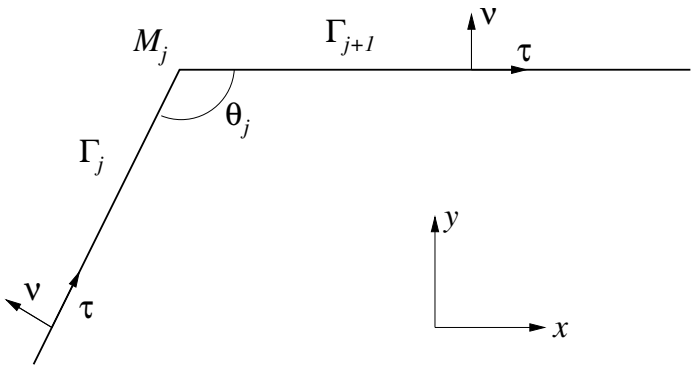

Figure 4.2: Around the corner M_j.

– *If* $\Gamma_j \subset \Gamma_C$ *and* $\Gamma_{j+1} \subset \Gamma_S$*:* $v_{\nu|\Gamma_j} \in \widetilde{H}_*^{1/2}(\Gamma_j)$ *and* $v_{\tau|\Gamma_{j+1}} \in \widetilde{H}_*^{1/2}(\Gamma_{j+1})$.

Proof: In the sequel, (O, x, y) is the absolute reference. Assume we have the following situation, with Γ_{j+1} parallel to the x-axis:

Case (i): by definition, $v_{\tau|\Gamma_j} = 0$ and $v_{x|\Gamma_{j+1}} = v_{\tau|\Gamma_{j+1}} = 0$. We shall prove that $v_{y|\Gamma_{j+1}} = v_{\nu|\Gamma_{j+1}}$ belongs to $\widetilde{H}_*^{1/2}(\Gamma_{j+1})$ (one proves $v_{\nu|\Gamma_j} \in \widetilde{H}_*^{1/2}(\Gamma_j)$ in a similar manner).

From that point on, the proof can be carried out in three steps:

(a_1) $v_{x|\Gamma_j} \in \widetilde{H}_*^{1/2}(\Gamma_j)$: as $\mathbf{v} \in \mathbf{H}^{1/2}(\Gamma)$, $v_x \in H^{1/2}(\Gamma)$, which implies in turn $v_{x|\Gamma_j} \in H^{1/2}(\Gamma_j)$ and

$$|v_x|_{1/2,\Gamma}^2 = \int_\Gamma \int_\Gamma \frac{|v_x(\mathbf{z}) - v_x(\mathbf{t})|^2}{||\mathbf{z} - \mathbf{t}||^2}\, d\sigma(\mathbf{z}) d\sigma(\mathbf{t}) < \infty.$$

Now, $v_{x|\Gamma_{j+1}} = 0$ and thanks to (i), we arrive at ($C_1' = 1/C_1$)

$$\int_{\Gamma_j} \frac{|v_x(\mathbf{z})|^2}{||\mathbf{z} - M_j||}\, d\sigma(\mathbf{z}) \le C_1' \, |v_x|_{1/2,\Gamma}^2 < \infty.$$

This proves (a_1).

(b_1) $v_{y|\Gamma_j} \in \widetilde{H}_*^{1/2}(\Gamma_j)$: by assumption, $v_{\tau|\Gamma_j} = 0$ or, equivalently,

$$(-\cos\theta_j \, v_x + \sin\theta_j \, v_y)_{|\Gamma_j} = 0, \text{ with } \sin\theta_j \neq 0.$$

(b_1) thus follows from (a_1).

(c_1) $v_{y|\Gamma_{j+1}} \in \widetilde{H}_*^{1/2}(\Gamma_{j+1})$: indeed, $v_y \in H^{1/2}(\Gamma)$ and, from Annex A,

$$\begin{aligned}
\int_{\Gamma_{j+1}} \frac{|v_y(\mathbf{z})|^2}{||\mathbf{z} - M_j||}\, d\sigma(\mathbf{z}) &\le C_1' \int_{\mathbf{z}\in\Gamma_{j+1}} |v_y(\mathbf{z})|^2 \, d\sigma(\mathbf{z}) \int_{\mathbf{t}\in\Gamma_j} \frac{d\sigma(\mathbf{t})}{||\mathbf{z} - \mathbf{t}||^2} \\
&\le C_1' \int_{\mathbf{z}\in\Gamma_{j+1}} \int_{\mathbf{t}\in\Gamma_j} \left\{ \frac{|v_y(\mathbf{z}) - v_y(\mathbf{t})|^2}{||\mathbf{z} - \mathbf{t}||^2} + \frac{|v_y(\mathbf{t})|^2}{||\mathbf{z} - \mathbf{t}||^2} \right\} d\sigma(\mathbf{z}) d\sigma(\mathbf{t}) \\
&\le C_1' \left\{ |v_y|_{1/2,\Gamma}^2 + C_2 \int_{\Gamma_j} \frac{|v_y(\mathbf{t})|^2}{||\mathbf{t} - M_j||}\, d\sigma(\mathbf{t}) \right\}.
\end{aligned}$$

Both terms are bounded, thanks to point (b_1). In other words, $v_{y|\Gamma_{j+1}} \in \widetilde{H}_*^{1/2}(\Gamma_{j+1})$.

Case (ii): by definition, $v_{\nu|\Gamma_j} = 0$ and $v_{y|\Gamma_{j+1}} = 0$. We sketch the proof of the fact that $v_{x|\Gamma_{j+1}}$ actually belongs to $\widetilde{H}^{1/2}_*(\Gamma_{j+1})$.

(a_2) $v_{y|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$: this is the transposition of (a_1) with v_y *in lieu of* v_x.

(b_2) $v_{x|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$: by hypothesis, $v_{\nu|\Gamma_j} = 0$. This can rewritten as:

$$(-\sin\theta_j\, v_x - \cos\theta_j\, v_y)_{|\Gamma_j} = 0, \text{ with } \sin\theta_j \neq 0.$$

(b_2) then follows from (a_2).

(c_2) $v_{x|\Gamma_{j+1}} \in \widetilde{H}^{1/2}_*(\Gamma_{j+1})$: this is ($c_1$) with v_x *in lieu of* v_y.

Case (iii): Assume (for instance) that $v_{\nu|\Gamma_j} = 0$ and $v_{x|\Gamma_{j+1}} = 0$, with $\cos\theta_j \neq 0$.

(a_3) $v_{x|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$: same proof as in ($a_1$).

(b_3) $v_{y|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$: as $v_{\nu|\Gamma_j} = 0$,

$$(-\sin\theta_j\, v_x - \cos\theta_j\, v_y)_{|\Gamma_j} = 0, \text{ with } \cos\theta_j \neq 0.$$

(b_3) follows from (a_3).

(c_3) $v_{y|\Gamma_{j+1}} \in \widetilde{H}^{1/2}_*(\Gamma_{j+1})$: same reasoning as in ($c_1$).

Therefore, $v_{y|\Gamma_{j+1}} \in \widetilde{H}^{1/2}_*(\Gamma_{j+1})$. In the same way, one gets that $v_{x|\Gamma_{j+1}} \in \widetilde{H}^{1/2}_*(\Gamma_{j+1})$. This completes the proof. ∎

Assume that, for all $j \in J_M$, $\theta_j \not\equiv \pi/2\ [\pi]$. Then $v_{\nu|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$ and $v_{\tau|\Gamma_j} \in \widetilde{H}^{1/2}_*(\Gamma_j)$ for all the edges, thanks to the Proposition. Now, let $\mathbf{u} \in \mathcal{V}_R$: by density, there exists a sequence $(\mathbf{u}^k)_k$ of element of $\mathcal{D}(\overline{\Omega})^2$ which converges to $\mathbf{u}$ in $\mathbf{H}^1(\Omega)$. Then according to the continuity of $\mu \mapsto \dfrac{\partial \mu}{\partial \tau}_{|\Gamma_j}$ from $H^{1/2}(\Gamma_j)$ into $\widetilde{H}^{-1/2}(\Gamma_j)$, one has

$$\frac{\partial u^k_\nu}{\partial \tau}_{|\Gamma_j} \to \frac{\partial u_\nu}{\partial \tau}_{|\Gamma_j} = 0 \text{ in } \widetilde{H}^{-1/2}(\Gamma_j), \qquad \forall j \in \mathcal{S}$$
$$\frac{\partial u^k_\tau}{\partial \tau}_{|\Gamma_j} \to \frac{\partial u_\tau}{\partial \tau}_{|\Gamma_j} = 0 \text{ in } \widetilde{H}^{-1/2}(\Gamma_j), \qquad \forall j \in \mathcal{S}.$$

Thus passing to the limit in (iii) is allowed and this leads to the conclusion.

Last, in the case when $\theta_j \equiv \pi/2\ [\pi]$ for some $j \in J_M$, then there is no additional difficulty, although the results (case (iii)) of the previous Proposition don't hold any more. Indeed, if one assumes that $(j, j+1) \in \mathcal{S}\times\mathcal{C}$, then let $t = \tau_j$, $n = \nu_j$ and $(\mathbf{u}, \mathbf{v}) \in \mathcal{D}(\overline{\Omega})^2 \times \mathcal{V}_R$.

[a] If $\theta_j = \pi/2$:

$$\begin{aligned}\int_{\Gamma_j} \frac{\partial u_\nu}{\partial \tau}\, v_\tau\, d\sigma - \int_{\Gamma_{j+1}} \frac{\partial u_\tau}{\partial \tau}\, v_\nu\, d\sigma &= \int_{\Gamma_j} \frac{\partial u_n}{\partial \tau}\, v_t\, d\sigma - \int_{\Gamma_{j+1}} \frac{\partial(-u_n)}{\partial \tau}\, v_t\, d\sigma \\ &= \int_{\Gamma_j \cup \Gamma_{j+1}} \frac{\partial u_n}{\partial \tau}\, v_t\, d\sigma.\end{aligned}$$

[b] If $\theta_j = 3\pi/2$:

$$\begin{aligned}\int_{\Gamma_j} \frac{\partial u_\nu}{\partial \tau}\, v_\tau \, d\sigma - \int_{\Gamma_{j+1}} \frac{\partial u_\tau}{\partial \tau}\, v_\nu \, d\sigma &= \int_{\Gamma_j} \frac{\partial u_n}{\partial \tau}\, v_t \, d\sigma - \int_{\Gamma_{j+1}} \frac{\partial u_n}{\partial \tau}\, (-v_t) \, d\sigma \\ &= \int_{\Gamma_j \cup \Gamma_{j+1}} \frac{\partial u_n}{\partial \tau}\, v_t \, d\sigma.\end{aligned}$$

In other words, the mixed corner "vanishes", and it allows to consider the two edges as a single one, $\Gamma_{j,j+1}$. The proof is then carried out using this new edge as, with the help of Proposition 1.3 ($\mathbf{grad}\, u_n \in \mathbf{H}(\mathrm{curl}\,, \Omega)$), we get $\dfrac{\partial u_n}{\partial \tau} \in \widetilde{H}^{-1/2}(\Gamma_{j,j+1})$. ■

Remark 4.3 If, for all the indices j, j is in J_M and $\theta_j \equiv \pi/2\ [\pi]$, then the sum of all the boundary terms becomes

$$\sum_{j \in \mathcal{S}} \int_{\Gamma_j} \frac{\partial u_\nu}{\partial \tau}\, v_\tau \, d\sigma - \sum_{j \in \mathcal{C}} \int_{\Gamma_j} \frac{\partial u_\tau}{\partial \tau}\, v_\nu \, d\sigma = \int_{\Gamma} \frac{\partial u_n}{\partial \tau}\, v_t \, d\sigma.$$

The problem then disappears altogether as now $\dfrac{\partial u_n}{\partial \tau}_{\,|\Gamma} \in H^{-1/2}(\Gamma)$ and $v_t \in H^{1/2}(\Gamma)$ directly! ■

NB: a straighforward consequence of the proof of Proposition 4.1 is that there exist constants $C > 0$ such that

(i) $\forall j \in J_D$, $\forall \mathbf{v} \in \mathcal{V}_R$,

$$||v_\nu||_{1/2,*,\Gamma_j} < C\, ||\mathbf{v}||_{1/2,\Gamma} \text{ and } ||v_\nu||_{1/2,*,\Gamma_{j+1}} < C\, ||\mathbf{v}||_{1/2,\Gamma}.$$

(ii) $\forall j \in J_N$, $\forall \mathbf{v} \in \mathcal{V}_R$,

$$||v_\tau||_{1/2,*,\Gamma_j} < C\, ||\mathbf{v}||_{1/2,\Gamma} \text{ and } ||v_\tau||_{1/2,*,\Gamma_{j+1}} < C\, ||\mathbf{v}||_{1/2,\Gamma}.$$

(iiia) $\forall j \in J_M$ s. t. $\theta_j \not\equiv \pi/2\ [\pi]$, $\Gamma_j \subset \Gamma_S$ and $\Gamma_{j+1} \subset \Gamma_C$,

$$||v_\tau||_{1/2,*,\Gamma_j} < C\, ||\mathbf{v}||_{1/2,\Gamma} \text{ and } ||v_\nu||_{1/2,*,\Gamma_{j+1}} < C\, ||\mathbf{v}||_{1/2,\Gamma}.$$

(iiib) $\forall j \in J_M$ s. t. $\theta_j \not\equiv \pi/2\ [\pi]$, $\Gamma_j \subset \Gamma_C$ and $\Gamma_{j+1} \subset \Gamma_S$,

$$||v_\nu||_{1/2,*,\Gamma_j} < C\, ||\mathbf{v}||_{1/2,\Gamma} \text{ and } ||v_\tau||_{1/2,*,\Gamma_{j+1}} < C\, ||\mathbf{v}||_{1/2,\Gamma}.$$

D Regularity of the solution to a Laplace problem

Let $p \in N$.

Proposition 4.2 *The solution ϕ to the problem:*
Find $\phi \in H^1(\Omega)$ such that

$$\begin{aligned}\Delta\phi &= p \text{ in } \Omega, \\ \phi &= 0 \text{ on } \Gamma_S, \\ \frac{\partial \phi}{\partial \nu} &= 0 \text{ on } \Gamma_C,\end{aligned}$$

satisfies to $\phi \in \mathcal{C}^\infty(\overline{\Omega} \setminus \mathcal{O})$, where $\mathcal{O}$ is any neighborhood of the corners $(M_j)_j$.

Proof:[1] [1] ϕ is smooth inside Ω:

for that, given a neighborhood $\mathcal{O}_\varepsilon = \cup_j B(M_j, \varepsilon)$, $\varepsilon > 0$, let $\eta_\varepsilon \in \mathcal{D}(\mathbb{R}^2)$ with the properties:

$$\begin{aligned} \eta_\varepsilon &= 1 \text{ in } \Omega_\varepsilon = \{x \in \Omega,\ d(x, \Gamma) > \varepsilon\}, \\ &= 0 \text{ outside } \Omega_{\varepsilon/2}. \end{aligned}$$

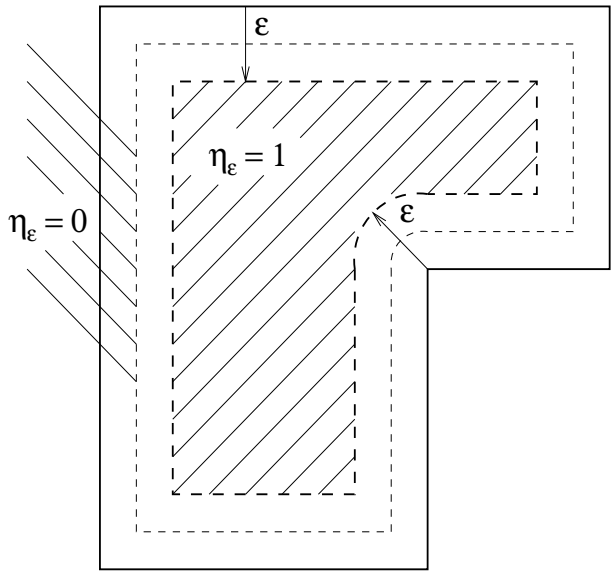

Figure 4.3: Ω_ε and η_ε.

As a consequence, since $p \in \mathcal{C}^\infty(\overline{\Omega} \setminus \mathcal{O}_{\varepsilon/2})$, $p_\varepsilon = p\,\eta_\varepsilon$ actually belongs to $\mathcal{D}(\mathbb{R}^2)$.

Now, let ϕ_ε be the solution to $\Delta\phi_\varepsilon = p_\varepsilon$ in $\mathbb{R}^2$. If we call E the (elementary) solution to $\Delta E = \delta_0$, we know that, in polar coordinates, $E = 1/r \in L^1_{loc}(\mathbb{R}^2)$ and $\phi_\varepsilon = E * p_\varepsilon$. Thanks to [4] (*page 69*), $\phi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^2)$. Then, $\Delta(\phi - \phi_\varepsilon) = p\,(1 - \eta_\varepsilon) = 0$ in Ω_ε. In other words, $(\phi - \phi_\varepsilon)$ is harmonic in Ω_ε and therefore of class $\mathcal{C}^\infty$ (cf. [12] (*page 79*)).

[2] $\phi \in \mathcal{C}^\infty(\overline{\Omega} \setminus \mathcal{O})$;

Let us consider a given edge, Γ_j, of the boundary. We define local cartesian (orthonormal) coordinates (x_1, x_2), such that $\vec{e}_1$ is parallel to Γ_j, and Γ_j is imbedded in $x_2 = 0$. Let us call $R =]a, b[\times]0, c[$ a rectangle imbedded in Ω and $I =]a, b[\times\{0\}$. By assumption, we know that $p \in \mathcal{C}^\infty(R)$. Let $\mathcal{R} =]a, b[\times] - c, c[$.

(a) Assume that $p_{|\Gamma_j} = 0$, that is $j \in \mathcal{S}$. Then, according to the local expansion of p in polar coordinates, we obtain immediately:

$$\forall \alpha_1, \forall \alpha_2 \text{ **even**}, \frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}} p(x_1, 0) = 0. \tag{iv}$$

Indeed, as p is "summable" in the variables (r, θ) in the angular sector $]0, R_j/2[\times]0, \pi/\alpha_j[$, we can proceed by differentation under the sign Σ.

[1] for an alternate proof, see [4], pages 181-188.

Then, we define the **odd** reflection of p, called $\overline{p}$, in $\mathcal{R}$, by:

$$\begin{aligned}\overline{p}(x_1,x_2) &= p(x_1,x_2) \text{ if } x_2>0,\\ &= -p(x_1,-x_2) \text{ if } x_2<0.\end{aligned}$$

Let us prove that $\overline{p}\in\mathcal{C}^\infty(\mathcal{R})$. We already know that $\overline{p}\in\mathcal{C}^\infty(]a,b[\times\{]-c,0[\cup]0,c[)$. There remains to check that, for all $(\alpha_1,\alpha_2)\in\mathbb{N}^2$ and for all $x_1\in]a,b[$:

$$\lim_{x_2^+\to 0^+}\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}\overline{p}(x_1,x_2^+)=\lim_{x_2^-\to 0^-}\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}\overline{p}(x_1,x_2^-). \tag{v}$$

Clearly, the left-hand side of (v) is equal to

$$\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}p(x_1,0).$$

Concerning the right-hand side, let us prove by induction that, for all the elements $(x_1,x_2^-)\in]a,b[\times]-c,0[$, one has:

$$\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}\overline{p}(x_1,x_2^-)=(-1)^{\alpha_2+1}\,\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}p(x_1,-x_2^-). \tag{vi}$$

Indeed, it is true for $\alpha_1=0$ and $\alpha_2=0$. Assume it is true for $(0,\alpha_2)$. Then:

$$\begin{aligned}\frac{\partial^{\alpha_2+1}}{\partial x_2^{\alpha_2+1}}\overline{p}(x_1,x_2^-) &= \lim_{h\to 0}\frac{1}{h}\left\{\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\overline{p}(x_1,x_2^-+h)-\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\overline{p}(x_1,x_2^-)\right\}\\ &= (-1)^{\alpha_2+1}\lim_{h\to 0}\frac{1}{h}\left\{\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}p(x_1,-x_2^--h)-\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}p(x_1,-x_2^-)\right\}\\ &= (-1)^{\alpha_2+2}\frac{\partial^{\alpha_2+1}}{\partial x_2^{\alpha_2+1}}p(x_1,-x_2^-).\end{aligned}$$

Therefore, it is true for all $(0,\alpha_2)$. For a fixed α_2, we now proceed by induction on α_1. If this is true for (α_1,α_2), then:

$$\begin{aligned}\frac{\partial^{(\alpha_1+1,\alpha_2)}}{\partial x_1^{\alpha_1+1}\,\partial x_2^{\alpha_2}}\overline{p}(x_1,x_2^-) &= \lim_{h\to 0}\frac{1}{h}\left\{\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}\overline{p}(x_1+h,x_2^-)-\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}\overline{p}(x_1,x_2^-)\right\}\\ &= (-1)^{\alpha_2+1}\lim_{h\to 0}\frac{1}{h}\left\{\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}p(x_1+h,-x_2^-)-\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}p(x_1,-x_2^-)\right\}\\ &= (-1)^{\alpha_2+1}\frac{\partial^{(\alpha_1+1,\alpha_2)}}{\partial x_1^{\alpha_1+1}\,\partial x_2^{\alpha_2}}p(x_1,-x_2^-),\ \text{QED.}\end{aligned}$$

Thanks to (vi), (v) holds for (α_1,α_2), such that α_2 is odd. Thanks to (iv) and (vi), the result is still valid for (α_1,α_2), such that α_2 is even.

Therefore, $\overline{p}$ belongs to $\mathcal{C}^\infty(\mathcal{R})$.

N. B. Note that an alternate proof can be also obtained by using what follows.

Let us prove now that $\overline{\phi}$, the odd reflection of ϕ, is also an element of $\mathcal{C}^\infty(\mathcal{R})$. For that, given $\phi' \in \mathcal{D}(\overline{R})$, let $\overline{\phi}'$ be the odd reflection of ϕ'. Finally, let $\varphi \in \mathcal{D}(\mathcal{R})$. With a double integration by parts, we get:

$$\begin{aligned} < \Delta\overline{\phi}', \varphi > = < \overline{\phi}', \Delta\varphi > &= \int_{\mathcal{R}} \overline{\phi}' \, \Delta\varphi \, dx_1 \, dx_2 \\ &= \int_R \overline{\phi}' \, \Delta\varphi \, dx_1 \, dx_2 + \int_{R'} \overline{\phi}' \, \Delta\varphi \, dx_1 \, dx_2 \\ &= -\int_{R\cup R'} \mathbf{grad}\, \overline{\phi}' \cdot \mathbf{grad}\, \varphi \, dx_1 \, dx_2 + 2\int_I \phi'(x_1,0) \, \frac{\partial\varphi}{\partial\nu}(x_1,0) \, dx_1 \\ &= \int_{R\cup R'} \Delta\overline{\phi}' \, \varphi \, dx_1 \, dx_2 + 2\int_I \phi'(x_1,0) \, \frac{\partial\varphi}{\partial\nu}(x_1,0) \, dx_1. \end{aligned}$$

For the first integration by parts, we used the property $\lim_{x_2^- \to 0^-} \overline{\phi}'(x_1, x_2^-) = -\phi'(x_1, 0)$, whereas, for the second one, we used $\lim_{x_2^- \to 0^-} \partial\overline{\phi}'/\partial\nu'(x_1, x_2^-) = -\partial\phi'/\partial\nu(x_1, 0)$ (with the help of (vi) for $\alpha_1 = 0$ and $\alpha_2 = 1$).

By density (cf. [13]), as ϕ is by definition an element of $E(\Delta, R)$, we finally obtain that, for all $\varphi \in \mathcal{D}(\mathcal{R})$:

$$< \Delta\overline{\phi}, \varphi > = \int_{R\cup R'} \Delta\overline{\phi} \, \varphi \, dx_1 \, dx_2 = \int_{R\cup R'} \overline{p} \, \varphi \, dx_1 \, dx_2.$$

Here, the second equality is derived from (vi), with (α_1, α_2) in $\{(0,2),(2,0)\}$. In other words, we arrive at $\Delta\overline{\phi} = \overline{p}$ in $\mathcal{R}$. As $\overline{p} \in \mathcal{C}^\infty(\mathcal{R})$, Point [1] shows that $\overline{\phi}$ in its turn also belongs to $\mathcal{C}^\infty(\mathcal{R})$, and therefore $\phi \in \mathcal{C}^\infty(]a, b[\times[0, c[)$. This proves the Proposition in Case [a].

[b] On the other hand, if we suppose that $\partial p/\partial\nu_{|\Gamma_j} = 0$, that is $j \in \mathcal{C}$, we obtain successively:

$$\forall\alpha_1, \forall\alpha_2 \ \mathbf{odd}, \frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1} \, \partial x_2^{\alpha_2}} p(x_1, 0) = 0. \tag{vii}$$

We define the **even** reflection of p in $\mathcal{R}$ by

$$\begin{aligned} \overline{\overline{p}}(x_1, x_2) &= p(x_1, x_2) \text{ if } x_2 > 0, \\ &= p(x_1, -x_2) \text{ if } x_2 < 0. \end{aligned}$$

(v) holds with $\overline{\overline{p}}$ thanks to (vii) and to the property hereunder.
For all $(x_1, x_2^-) \in]a, b[\times]-c, 0[$, one has:

$$\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1} \, \partial x_2^{\alpha_2}} \overline{\overline{p}}(x_1, x_2^-) = (-1)^{\alpha_2} \, \frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1} \, \partial x_2^{\alpha_2}} p(x_1, -x_2^-).$$

Finally, if we call $\overline{\overline{\phi}}$ the even reflection of ϕ, we readily reach, for all $\varphi \in \mathcal{D}(\mathcal{R})$:

$$< \Delta\overline{\overline{\phi}}, \varphi > = \int_{R\cup R'} \overline{\overline{p}} \, \varphi \, dx_1 \, dx_2.$$

Point [1] shows again that $\overline{\overline{\phi}} \in \mathcal{C}^\infty(\mathcal{R})$, and the Proposition is therefore proved in the general case. ■