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The excitation spectrum of a Bose gas with an impurity in the Gross–Pitaevskii regime

Jonas Lampart*, Arnaud Triay†

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We study a dilute system of N interacting bosons coupled to an impurity particle via a pair potential in the Gross–Pitaevskii regime. We derive an expansion of the ground state energy up to order one in the boson number, and show that the difference of excited eigenvalues to the ground state is given by the eigenvalues of the renormalized Bogoliubov–Fröhlich Hamiltonian in the limit $N \rightarrow \infty$.

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1 Introduction

At very low temperatures a gas of bosons forms a Bose–Einstein condensate (BEC), a quantum state in which a macroscopic fraction of the bosons occupy the same single-particle state. The particles outside the condensate state, often called excitations or phonons, play a crucial role in the thermodynamic properties of the BEC. In 1947, Bogoliubov developed a theory to explain the emergence of superfluidity in dilute Bose gases using Landau’s criterion of superfluidity [48]. In Bogoliubov’s theory [9], the excitations are described by a quantum field governed by a Hamiltonian that is quadratic in the creation and annihilation operators of the excitations. This allowed him to compute explicitly the ground state energy and the low-lying excitation spectrum of the system. The rigorous justification of Bogoliubov’s theory has now been studied in depth from a mathematical point of view, both from the spectral point of view [68, 31, 52, 71, 23, 5, 2, 62, 15, 28, 29, 12, 11, 16], and for the dynamics [66, 33, 34, 51, 3, 20, 8]. See also [58, 17, 26] for two-dimensional systems and [7, 37, 13, 38, 27, 18] for the positive temperature case.

The interaction of the Bose gas with an impurity particle will create additional excitations out of the condensate, which in turn act on the impurity. In the regime of large boson number, the Bogoliubov–Fröhlich Hamiltonian has been proposed as an effective model (cf. [35, 57, 46]) for the excitation spectrum of the system. In this Hamiltonian, the interaction between the impurity and the excitation field is given by a coupling term that is linear in the field operator. This model has already been mathematically justified in the mean-field limit [57, 46]. However, Bose–Einstein condensates, as they are prepared in experiments, are better modeled by the dilute

Gross–Pitaevskii limit, and the validity of the Bogoliubov–Fröhlich model has been questioned in the physics literature [21] (see Remark 1.7). The goal of this article is to study the ground state energy and spectrum of the Bose gas with an impurity in this limit and prove validity of the approximation by the Bogoliubov–Fröhlich Hamiltonian.

In the Gross–Pitaevskii regime, N bosons are spread out over a volume of diameter $L = N$ in three dimensions. The density is therefore of order N^{-2} . It is however more convenient to rescale the system in a volume of order one. This leads us to consider the Hamiltonian

$$H_N = -\Delta_x + \sum_{i=1}^N (-\Delta_{y_i}) + \sum_{1 \leq i < j \leq N} N^2 V(N(y_i - y_j)) + \sum_{i=1}^N N W(\sqrt{N}(x - y_i)). \quad (1.1)$$

For simplicity, we will consider a system with periodic boundary conditions. The position of the impurity will be denoted by $x \in \mathbb{T}^3$ and those of the N bosons by $(y_i)_{1 \leq i \leq N} \in \mathbb{T}^{3N}$, where $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ is the three-dimensional unit torus. We assume that the boson-boson potential V and boson-impurity potential W belong to $L^2(\mathbb{R}^3)$, have compact support and are non-negative. If N is sufficiently large, the scaled potentials $V_N(y) = N^2 V(Ny)$ and $W_N(x) = N W(\sqrt{N}x)$ have support inside the cube $[0, 1]^3$, therefore we may identify the support with a subset of the torus and replace the potentials by their periodic extension. The Hamiltonian H_N thus acts on the Hilbert space $\mathcal{H}_N = L^2(\mathbb{T}^3) \otimes L^2(\mathbb{T}^3)^{\otimes_{\text{sym}} N}$ with domain $H^2(\mathbb{T}^{3N+1}) \cap \mathcal{H}_N$. Heuristically, one should think that as $N \rightarrow \infty$, the boson-boson interaction behaves like $N^{-1} \delta(y_i - y_j)$, and the boson-impurity interaction like $N^{-1/2} \delta(x - y_i)$. This singular behavior leads to considerable difficulties in the analysis, even at leading order of the energy. For the case without an impurity, such an analysis has been carried out in the works [56, 53, 55, 54, 61, 6, 16].

The scaling for W is chosen so that the interaction between the impurity and the excitations above the condensate is of the same order as the kinetic energy of the Bogoliubov excitations, similarly to [57, 46], see Remark 1.3 below.

The starting point for our analysis will be to assume Bose-Einstein condensation of low-energy states, in the form of Condition 1.1 below. To state this, let \mathbf{a}_v denote the scattering length of a potential $0 \leq v \in L^1(\mathbb{R}^3)$, defined by

$$8\pi \mathbf{a}_v := \inf_{\varphi \in \dot{H}^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(2|\nabla \varphi(x)|^2 + v(x)|1 + \varphi(x)|^2 \right) dx. \quad (1.2)$$

Note the scaling property $\mathbf{a}_{\lambda^2 v(\lambda \cdot)} = \lambda^{-1} \mathbf{a}_v$.

Condition 1.1 (Bose–Einstein condensation). There exist $c, d > 0$ so that

$$H_N - 4\pi \mathbf{a}_V N \geq c \sum_{j=1}^N Q_j - d\sqrt{N},$$

where we Q_j denotes the projection onto the orthogonal of the constant function on \mathbb{T}^3 , acting on the j -th factor of $L^2(\mathbb{T}^3)^{\otimes_{\text{sym}} N}$.

As we are already assuming $W \geq 0$, the boson-impurity interaction can be dropped from H_N for a lower bound, which makes this a condition on V only. It is known to be satisfied if V is radial [6, 40], where it holds with a rate of order one instead of \sqrt{N} . For our use, the improved rate is of no importance since the impurity will change the energy at order \sqrt{N} in any case. Condition 1.1 is, however, expected to hold more generally. A discussion of the non-radial case can be found in [60].

Theorem 1.2. *Let $0 \leq V, W \in L^2(\mathbb{R}^3)$ be even and compactly supported. Assume moreover that Bose–Einstein condensation holds in the sense of Condition 1.1. Then as $N \rightarrow \infty$*

$$\inf \sigma(H_N) = 4\pi\mathfrak{a}_V N + 8\pi\mathfrak{a}_W \sqrt{N} - 32\pi(2\pi/3 - \sqrt{3})\mathfrak{a}_W^4 \log N + \mathcal{O}(1).$$

The first two terms in the expansion account for the direct boson-boson and the boson-impurity interaction, respectively. They are natural guesses, as two-body interactions are expected to be responsible for the leading order in the energy, due to the diluteness of the gas. The $\log N$ -term, on the other hand, is explained by an effective three-body interaction between the impurity and two bosonic excitations (see Remark 1.11). The emergence of such a term from the Bogoliubov–Fröhlich Hamiltonian with cutoff $\Lambda = N$ was proved in [45], after being observed in [36, 35].

Remark 1.3. The scaling for W is chosen so as to observe meaningful interactions of the impurity with the excitations. To explain this in more detail, recall that the Bogoliubov Hamiltonian represents the Hessian of the Gross–Pitaevskii functional [52]. In other words, without impurity, the second variation of the Gross–Pitaevskii energy describes the asymptotic excitation spectrum of H_N . The Gross–Pitaevskii energy with impurity is obtained by formally replacing the potentials in H_N by $8\pi\mathfrak{a}_V N^{-1}\delta(y_i - y_j)$, $8\pi\mathfrak{a}_W N^{-1/2}\delta(x - y_i)$ and calculating the quadratic form of H_N in an uncorrelated state $\psi \otimes u^{\otimes N}$. Doing so, we arrive at

$$\begin{aligned} \mathcal{E}_{\text{GP}}(\psi, u) &= \int |\nabla_x \psi|^2 + N \int |\nabla_y u|^2 + 4\pi\mathfrak{a}_V(N-1) \int |u(y)|^4 \\ &\quad + 8\pi\mathfrak{a}_W \sqrt{N} \int |u(x)|^2 |\psi(x)|^2. \end{aligned} \tag{1.3}$$

With no impurity, $\psi = 0$, the constant function $u = \mathbb{1}$ minimizes this functional. This function represents a condensate of N bosons, so finitely many excitations out of the condensate (as expected in the Gross–Pitaevskii regime [6]) correspond to a variation of $\int |u|^2$ of order N^{-1} , i.e., $u = \mathbb{1} + N^{-1/2}\delta u$. Inserting this into the

functional, we obtain

$$\begin{aligned} & \mathcal{E}_{\text{GP}}(\psi, \mathbb{1} + N^{-1/2}\delta u) - \mathcal{E}_{\text{GP}}(0, \mathbb{1}) - 8\pi\mathbf{a}_W\sqrt{N} \\ & \approx \int |\nabla_x \psi|^2 + \int (|\nabla_y \delta u|^2 + 8\pi\mathbf{a}_V |\delta u(y)|^2) \\ & \quad + 4\pi\mathbf{a}_V \int (\delta u(y)^2 + \overline{\delta u(y)^2}) + 8\pi\mathbf{a}_W \int (\delta u(y) + \overline{\delta u(y)}) |\psi(x)|^2, \end{aligned} \quad (1.4)$$

up to higher order terms. Here, the interaction between the impurity and the excitations represented by δu is of the same order as the interaction of the excitations among themselves due to our choice of interaction strength, which justifies the scaling for W .

Our second main result is that the low-lying spectrum of H_N is well approximated by the one of the renormalized Bogoliubov-Fröhlich Hamiltonian. This operator acts on

$$\mathcal{H}_+ = L^2(\mathbb{T}^3) \otimes \Gamma(\mathfrak{H}_+), \quad (1.5)$$

where $\Gamma(\mathfrak{H}_+)$ is the bosonic Fock space over $\mathfrak{H}_+ = \{\mathbb{1}\}^\perp$, the orthogonal to the (constant) condensate wave-function $\mathbb{1}_{\mathbb{T}^3}$. Formally, the Bogoliubov-Fröhlich Hamiltonian is given by

$$-\Delta_x + d\Gamma(\epsilon) + a(w_x) + a^*(w_x), \quad (1.6)$$

where ϵ is the Fourier multiplier with Bogoliubov's dispersion relation

$$\epsilon(p) = \sqrt{p^4 + 16\pi\mathbf{a}_V p^2}, \quad (1.7)$$

a^*, a are the bosonic creation and annihilation operators (see (2.11)) and $w_x(y) = w(x - y)$, where w is the distribution with Fourier coefficients

$$\widehat{w}(p) = 8\pi\mathbf{a}_W |p| \epsilon(p)^{-1/2}, \quad p \in 2\pi\mathbb{Z}^3 \setminus \{0\}. \quad (1.8)$$

However, the expression (1.6) is ill-defined due to the lack of decay of $\widehat{w}(p) \sim 8\pi\mathbf{a}_W$ for $p \rightarrow \infty$. For a rigorous definition, it should be replaced by the renormalized Bogoliubov-Fröhlich Hamiltonian on the torus, constructed as in [45].

Proposition 1.4. *Define w^Λ by its Fourier coefficients $\widehat{w}^\Lambda(k) = \widehat{w}(k)\mathbb{1}_{|k| \leq \Lambda}$ for $\Lambda \in \mathbb{R}_+$, and set*

$$H_{\text{BF}}^\Lambda = -\Delta_x + d\Gamma(\epsilon) + a(w_x^\Lambda) + a^*(w_x^\Lambda), \quad (1.9)$$

which defines a self-adjoint operator on $D(-\Delta_x + d\Gamma(\epsilon))$. There exists a family $(E_\Lambda)_{\Lambda \geq 0} \subset \mathbb{R}$ and a self-adjoint operator $(H_{\text{BF}}, D(H_{\text{BF}}))$ with compact resolvent so that

$$H_{\text{BF}}^\Lambda - E_\Lambda \xrightarrow{\Lambda \rightarrow \infty} H_{\text{BF}}$$

in norm resolvent sense.

We prove this proposition in Section 3, where we also give more details on H_{BF} . With the renormalized Bogoliubov-Fröhlich Hamiltonian, we can resolve the excitation spectrum of H_N for $N \rightarrow \infty$.

Theorem 1.5. *Let $0 \leq V, W \in L^2(\mathbb{R}^3)$ be even and compactly supported, and assume that Bose-Einstein condensation in the sense of Condition 1.1 holds. Let $(H_{\text{BF}}, D(H_{\text{BF}}))$ be the renormalized Bogoliubov-Fröhlich Hamiltonian of Proposition 1.4, and for $k \in \mathbb{N}_0$ let $e_k(H)$ denote the k -th min-max value of the operator H . Then as $N \rightarrow \infty$*

$$e_k(H_N) - e_0(H_N) = e_k(H_{\text{BF}}) - e_0(H_{\text{BF}}) + \mathcal{O}(N^{-1/80}).$$

Note that in Theorem 1.2 we determine the ground state energy $e_0(H_N) = \inf \sigma(H_N)$ with an error $\mathcal{O}(1)$ but in Theorem 1.5 we can express the elementary excitations $e_k(H_N) - e_0(H_N)$ up to a precision $o(1)$. From the proof of Theorem 1.5 one can extract an expression for $e_0(H_N)$ with this precision, but it is quite involved and not completely explicit, since the value of $e_0(H_{\text{BF}})$ is not known.

The proof of Theorem 1.5 also shows that the eigenfunctions of H_N can be expressed in terms of those of H_{BF} and two unitary transformations: the excitation map $U_X : \mathcal{H}_+ \rightarrow \mathcal{H}_N$ (see Section 4.1) and a Bogoliubov transformation U_B (see Section 4.5). This information is also sufficient to obtain an approximation of the unitary dynamics.

Corollary 1.6. *Assume that the hypothesis of Theorem 1.5 hold. Let U_X be the excitation map defined by (4.2) and U_B the Bogoliubov transformation whose generator is given by (4.164) with $N = \infty$. Then:*

- a) *For any $k \in \mathbb{N}$, the corresponding min-max values e_k and $\Psi_N \in \ker(H_N - e_k(H_N))$, the sequence $U_B^* U_X^* \Psi_N$ converges, up to a subsequence, to a limit $\Psi \in \ker(H_{\text{BF}} - e_k(H_{\text{BF}}))$.*
- b) *The corresponding spectral projections converge in norm: Let B be an open interval with $\sigma(H_{\text{BF}} - e_0(H_{\text{BF}})) \cap B = e_k(H_{\text{BF}}) - e_0(H_{\text{BF}})$, then*

$$\lim_{N \rightarrow \infty} U_X^* \mathbf{1}_B(H_N = e_k(H_N)) U_X = U_B \mathbf{1}(H_{\text{BF}} = e_k(H_{\text{BF}})) U_B^*.$$

- c) *For any sequence $\Psi_N \in \mathcal{H}_N$ so that $U_X^* \Psi_N$ has a limit $\Psi \in \mathcal{H}_+$,*

$$\lim_{N \rightarrow \infty} U_X^* e^{-it(H_N - e_0(H_N))} \Psi_N = U_B e^{-it(H_{\text{BF}} - e_0(H_{\text{BF}}))} U_B^* \Psi$$

in the norm of \mathcal{H}_+ .

Remark 1.7. There has been some debate in the physics literature concerning the applicability of the Bogoliubov-Fröhlich model. In [21], the authors performed an expansion of $e_0(H_N)$ in \mathfrak{a}_W and found an expression of order $\mathfrak{a}_W^3 \log \mathfrak{a}_W$. It is argued that this cannot arise from the Bogoliubov-Fröhlich Hamiltonian, whose (formal) perturbation series is even in \mathfrak{a}_W . In our scaling, the corresponding term in the perturbation series is of order $N^{-1/2} \log N$. It is related to the creation of pairs of Bogoliubov phonons by the impurity, which is suppressed by the fact that the boson-boson interaction is weaker than the boson-impurity interaction (see Section 4.5). Hence, its absence from our results is not in contradiction to [21], where no relation between the two interactions is assumed and the expansion is in \mathfrak{a}_W .

Our results show that, in the precise scaling we consider, the spectrum is correctly described by the Bogoliubov-Fröhlich Hamiltonian.

Remark 1.8. The first application of Bogoliubov's theory was to provide an explanation of superfluidity. Landau argued [48], based on conservation of energy and momentum, that an obstacle moving at velocity below the speed of sound relative to a quantum liquid could not excite phonons and would thus not create friction. This could be made more precise using the Bogoliubov-Fröhlich model, where the impurity serves to probe the superfluid behavior. A step in this direction was recently made [41], where the Bogoliubov-Fröhlich model in \mathbb{R}^3 is analyzed, and it is shown that a moving impurity does not experience friction when moving at velocities smaller than the speed of sound, which equals $4\sqrt{\pi\mathfrak{a}_V}$ in our derivation. Connecting our derivation of the Bogoliubov-Fröhlich Hamiltonian on the torus to the one on \mathbb{R}^3 (e.g., by taking the size of the torus to infinity, as is done in the physics literature [35], see also [23]) would thus provide a detailed explanation of this effect from a many-body quantum model. This has also been investigated in a semi-classical setting in [24, 30, 22, 50].

Remark 1.9. The prefactor of $\log N$ in Theorem 1.2 is due to our choice of the masses of both the bosons and the impurity being equal to one-half. For a general mass of the particle m and reduced mass $\mu^{-1} = m^{-1} + 2$, the prefactor is

$$c_m = -16\pi\mu^{-1}\mathfrak{a}_{\mu W}^4 \left(\frac{m}{\mu} \arcsin \left(\frac{\mu}{m} \right) - \sqrt{1 - \left(\frac{\mu}{m} \right)^2} \right), \quad (1.10)$$

as can be checked by keeping track of the constants in the calculations of Section 6.1 (see also [47]).

Remark 1.10. The scaling of our Hamiltonian can be expressed in the natural units of a dilute system (see also [47]). Consider a system of bosons with density ρ interacting among themselves through a potential V and with one impurity via W , in a box of side-length $\ell_{\text{GP}} = (\rho\mathfrak{a})^{-1/2}$ (the Gross-Pitaveskii length), where $\mathfrak{a} := \mathfrak{a}_V$

is the scattering length of V and where we also denote by \mathfrak{a}_W the one of W . The Hamiltonian for the total system is

$$H_{\text{GP}} = -\Delta_x + \sum_{i=1}^N (-\Delta_{y_i}) + \sum_{1 \leq i < j \leq N} V(y_i - y_j) + \sum_{1 \leq j \leq N} W(x - y_j), \quad (1.11)$$

We consider the dilute regime, in which the dimensionless parameter $\rho \mathfrak{a}^3$ is small and the polaronic coupling constant (compare [69, 35, 47], where definitions differ by factors of $\sqrt{8\pi}$)

$$\alpha = \mathfrak{a}_W^2 \ell_{\text{GP}} \rho = \frac{\mathfrak{a}_W^2}{\mathfrak{a} \ell_{\text{GP}}} \quad (1.12)$$

is of order one. The number of bosons is given by $N = \rho \ell_{\text{GP}}^3 = (\rho \mathfrak{a}^3)^{-1/2} = \ell_{\text{GP}} / \mathfrak{a} \gg 1$ and the number of excited bosons is of order one [6]. The coupling to the impurity satisfies the relation $\ell_{\text{GP}} / \mathfrak{a}_W = \sqrt{N/\alpha}$, so its range is still much shorter than ℓ_{GP} , though larger than \mathfrak{a} .

In order to normalize the interactions, we may write $V(y) = \mathfrak{a}^{-2} \tilde{V}(\mathfrak{a}^{-1}y)$ and $W(x) = \mathfrak{a}_W^{-2} \tilde{W}(\mathfrak{a}_W^{-1}x)$ with potentials \tilde{V} and \tilde{W} that have scattering length equal to one. Then, rescaling by ℓ_{GP} and multiplying with ℓ_{GP}^2 , we have $\ell_{\text{GP}}^2 V(\ell_{\text{GP}}x) = N^2 \tilde{V}(Nx)$, and $\ell_{\text{GP}}^2 W(\ell_{\text{GP}}x) = N/\alpha \tilde{W}(\sqrt{N/\alpha}x)$ which leads to a Hamiltonian of the form (1.1). The expansion of Theorem 1.2 can then be written as

$$\begin{aligned} & \frac{\inf \sigma(H_{\text{GP}})}{\ell_{\text{GP}}^3} \\ &= 4\pi \mathfrak{a} \rho^2 \left(1 + 2\sqrt{\alpha} (\rho \mathfrak{a}^3)^{1/4} - 8(2\pi/3 - \sqrt{3}) \alpha^2 \sqrt{\rho \mathfrak{a}^3} \log(\rho \mathfrak{a}^3) + \mathcal{O}(\sqrt{\rho \mathfrak{a}^3}) \right). \end{aligned}$$

In the case of different masses, the correct formula is obtained from (1.10) (see also [47]).

For comparison, note that the $\rho^{5/2} \log \rho$ term in this expansion is larger than the well-known Lee-Huang-Yang correction [49, 28, 2, 29, 26, 37], which is of the order $\rho^{5/2}$. It is however very similar in nature to the term proposed in [70, 67, 42] to capture the three-body effects in a Bose gas. The latter is expected to be of order $\rho^3 \log \rho$ and was very recently derived in the Gross-Pitaevskii limit [19].

Remark 1.11. The three-body nature of the $\log N$ -contribution to the energy can be understood by the following heuristics. Since the Bose gas is dilute, the effect of the boson-impurity interaction is dominated by simple two-body interactions. These lead to correlations between the impurity and a single boson since, due to the repulsive nature of W_N , these will try to avoid getting closer than distances of order $N^{-1/2}$. This is captured precisely by the solution φ to the minimization

problem (1.2) with $v = W$, and it is responsible for the scattering length \mathfrak{a}_W appearing in the coefficient of \sqrt{N} , instead of just the averaged potential $\int W$ as in the mean-field case. If we consider two bosons interacting with the impurity and make the ansatz $\Psi \approx \varphi(\sqrt{N}(x - y_1))\varphi(\sqrt{N}(x - y_2))\Phi(x, y_1, y_2)$ at short distances, then $H_2\Psi$ will contain a term with

$$v^{(3)}(x, y_1, y_2) := N\nabla\varphi(\sqrt{N}(x - y_1))\nabla\varphi(\sqrt{N}(x - y_2)), \quad (1.13)$$

coming from the action of $-\Delta_x$. This can be interpreted as an effective three-body interaction potential acting on Φ . Minimizing over Φ will make the scattering length of this three-body potential appear (cf. [59] and Equation (5.8), noting that this potential integrates to zero), which multiplied by the number of pairs of bosons $\sim N^2$ will give the term of order $\log N$ in Theorem 1.2. That this is of order $\log N$ can be easily checked. Note that for $1 \lesssim |y|$, $\varphi(y)$ behaves like $1/|y|$ and $\nabla\varphi(y) \sim 1/|y|^2$ (cf. [60, Theorem 6]). Then, passing to relative coordinates, the three-body scattering length, defined by a minimization problem analogous to (1.2), is approximately given by

$$\begin{aligned} & \int v^{(3)} - \langle v^{(3)}, (-(\nabla_{y_1} + \nabla_{y_2})^2 - \Delta_{y_1} - \Delta_{y_2} + v^{(3)})^{-1}v^{(3)} \rangle \\ & \sim -N^2 \int_{N^{-1/2} \lesssim |y_1|, |y_2|} \nabla\varphi(\sqrt{N}y_1)\nabla\varphi(\sqrt{N}y_2) \\ & \quad \times (-(\nabla_{y_1} + \nabla_{y_2})^2 - \Delta_{y_1} - \Delta_{y_2})^{-1}\nabla\varphi(\sqrt{N}y_1)\nabla\varphi(\sqrt{N}y_2) \\ & \sim -\frac{1}{N^2} \int_{N^{-1/2} \lesssim |y_1|, |y_2|} \frac{1}{|y_1|^2} \frac{1}{|y_2|^2} (-\Delta_{y_1} - \Delta_{y_2})^{-1} \frac{1}{|y_1|} \frac{1}{|y_2|} \sim \frac{\log N}{N^2}. \end{aligned}$$

The proof of Theorem 1.2 contains a rigorous implementation of these heuristic estimates where we do not use such an ansatz of product form, but rather a Weyl transformation which leads to similar expressions.

2 Preliminaries

2.1 Outline of the article

The following outline of the article may serve as a guide to the proofs of our main results. In the remainder of Section 2 we introduce some global conventions on notation, and summarize the key properties of the scattering length and the associated scattering solution, as well as its analogues on the torus. These results will be proved in Appendix A.

In Section 3, we discuss the renormalization of the Bogoliubov-Fröhlich Hamiltonian on the torus and prove Proposition 1.4. The proofs of the necessary technical

lemmas are postponed to Section 5, where a more general case with an additional excitation-impurity interaction is treated.

In Section 4, we write the Hamiltonian H_N in the excitation picture. We then apply several unitary transformations to extract the correct leading asymptotics of the energy, and, at the same time, to make the correct model parameters appear. The general scheme is to deal first with the large momenta, which are responsible for the bigger terms in the energy asymptotics, and then work downwards step-by-step. Specifically:

- 1) In Section 4.2 we apply a Bogoliubov transformation U_q , whose generator is quadratic in boson creation and annihilation operators. It acts on the large boson momenta and makes the scattering length \mathfrak{a}_V appear at the leading order of the energy.
- 2) In Section 4.3 we apply a Weyl-transformation U_W , which additionally depends on the impurity position x (similarly to the Gross transformation known from the literature on singular polaron models, like the Nelson model, cf. [63, 32]). This transformation also acts on large boson momenta and makes the scattering length \mathfrak{a}_W appear in the energy asymptotics.
- 3) In Section 4.4 we apply a transformation U_c to the boson Hamiltonian that has a generator which is cubic in creation and annihilation operators. This accounts for scattering processes that mix low and high energy bosons and makes the scattering length \mathfrak{a}_V appear in the boson Hamiltonian restricted to small energies.
- 4) In Section 4.5, we use Bogoliubov's transformation U_B , which transforms the remaining boson Hamiltonian into the simple expression $d\Gamma(\epsilon)$, with ϵ given by (1.7).
- 5) In Section 4.6 we apply another Weyl transformation U_G , which is precisely a variant of the Gross transform, which accounts for the two-particle impurity-excitation scattering at low energies.

The final result of Section 4 is Proposition 4.26, after which we can write the transformed Hamiltonian on low energy states as (cf. (5.1))

$$U^* H_N U \approx 4\pi\mathfrak{a}_V(N-1) + 8\pi\mathfrak{a}_W\sqrt{N} + e_N^{(U)} + \mathcal{H}_N^U$$

where \mathcal{H}_N^U is almost the Bogoliubov–Fröhlich Hamiltonian with an N -dependent cutoff Λ_N (up to a Gross transform), and $e_N^{(U)}$ is an energy contribution of order one.

In Section 5, we analyze \mathcal{H}_N^U and show that after removing the logarithmic divergence, it converges to the renormalized Bogoliubov–Fröhlich Hamiltonian constructed in Proposition 1.4. We essentially follow the proof of that proposition,

but with one key difference. The excitation Hamiltonian, and also \mathcal{H}_N^U , contains an additional term accounting for the interaction between two excitations and the impurity which is not present in the Bogoliubov–Fröhlich Hamiltonian. In our scaling regime, this term is responsible for a global shift of order one in the energy. However, the excitation spectrum is still correctly described by the Bogoliubov–Fröhlich Hamiltonian as stated in Theorem 1.5, since it only depends on the difference of eigenvalues.

In Section 6, we prove the main Theorems and Corollary 1.6.

2.2 Notation

We generally denote by $\widehat{f} \in \ell^2(2\pi\mathbb{Z}^3)$ the Fourier series of a function or distribution on \mathbb{T}^3 , with the normalization convention and inversion formula given by

$$\widehat{f}(k) = \int_{\mathbb{T}^3} e^{-ikx} f(x) dx, \quad f(x) = \sum_{2\pi\mathbb{Z}^3} e^{ikx} \widehat{f}(k). \quad (2.1)$$

For a symmetric operator A on a Hilbert space \mathcal{H} , we will denote by $D(A)$ its domain, and by $Q(A)$ the domain of the associated quadratic form $x \mapsto \langle x, Ax \rangle$.

For a function $f, g : \mathbb{N} \rightarrow \mathcal{C}$ where \mathcal{C} is a set with a partial order and admitting multiplication by positive real numbers (i.e., \mathbb{R}_+ or symmetric quadratic forms on \mathcal{H}), we denote

$$f \lesssim g \Leftrightarrow \exists C \in \mathbb{R}_+ : \forall N \in \mathbb{N} : f(N) \leq Cg(N). \quad (2.2)$$

For families of symmetric operators $A_N, B_N, N \in \mathbb{N}$ associated with quadratic forms on $Q(A), Q(B)$, independent of N , we denote by

$$A_N = \mathcal{O}(B_N) \quad (2.3)$$

the statement that $Q(B) \subset Q(A)$ and that there exists a constant C so that for all $\Psi \in Q(B)$ and $N \in \mathbb{N}$

$$|\langle \Psi, A_N \Psi \rangle| \leq C \langle \Psi, B_N \Psi \rangle. \quad (2.4)$$

Note that this relation is transitive and invariant under unitary transformations. Moreover, we have as a consequence of the Cauchy-Schwarz inequality that

$$A_N^* B_N + B_N^* A_N = \mathcal{O}(A_N^* A_N + B_N^* B_N), \quad (2.5)$$

when the left hand side is extended to the domain of the right.

For any function $h : \mathbb{T}^3 \rightarrow \mathbb{C}$, we will use the notation

$$h_x(y) = h(x - y). \quad (2.6)$$

For a Hilbert \mathfrak{H} denote by

$$\Gamma(\mathfrak{H}) := \mathbb{C} \otimes \bigoplus_{n=1}^{\infty} \mathfrak{H}^{\otimes_{\text{sym}} n} \quad (2.7)$$

the bosonic Fock space over \mathfrak{H} . In particular, $\Gamma(\mathfrak{H}_+)$ with $\mathfrak{H}_+ = \{\mathbf{1}\}^\perp \subset L^2(\mathbb{T}^3)$ (the orthogonal to the constant functions), is the Fock space of excitations. For $\Psi \in \Gamma(\mathfrak{H})$ we denote by $\Psi^{(n)} \in \mathfrak{H}^{\otimes_{\text{sym}} n}$ its n -particle component, and by

$$\mathcal{N}\Psi = \sum_{n=1}^{\infty} n\Psi^{(n)} \quad (2.8)$$

the number operator (with the canonical domain). In the case of $\Gamma(\mathfrak{H}_+)$ we denote the number operator by \mathcal{N}_+ .

For $f \in \mathfrak{H}$, we denote by $a^*(f)$, $a(f)$ the bosonic creation and annihilation operators on $\Gamma(\mathfrak{H})$, who satisfy the commutation relations

$$[a(f), a^*(g)] = \langle f, g \rangle, \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)]. \quad (2.9)$$

In the case $\mathfrak{H} = L^2(\mathbb{T}^3)$ we introduce the operators

$$a_p = a(e^{ip(\cdot)}), \quad a_p^* = a^*(e^{ip(\cdot)}), \quad (2.10)$$

and the operator-valued distributions a_y , a_y^* , $y \in \mathbb{T}^3$ as usual. For a distribution v on \mathbb{T}^3 and a quadratic form A on $L^2(\mathbb{T}^3)$ with Schwartz kernel $A(y, y')$, we also introduce the quadratic forms

$$a(v) = \int v(y)a_y, \quad a^*(v) = \int v(y)a_y^*, \quad d\Gamma(A) = \int A(y, y')a_y^*a_{y'} \quad (2.11)$$

on appropriate subspaces of $\Gamma(L^2(\mathbb{T}^3))$ (they are well defined at least on finite combinations of elements of $C^\infty(\mathbb{T}^{3n})$).

Any quadratic form on $\Gamma(L^2(\mathbb{T}^3))$ defines a quadratic form on $\Gamma(\mathfrak{H}_+)$ by restriction, which we denote by the same symbol. Note that this is not true for operators, where we would need to project the image back to $\Gamma(\mathfrak{H}_+)$, which is implicit for quadratic forms.

The excitation space with an impurity is

$$\mathcal{H}_+ = L^2(\mathbb{T}^3) \otimes \Gamma(\mathfrak{H}_+) = L^2(\mathbb{T}^3, \Gamma(\mathfrak{H}_+)). \quad (2.12)$$

We use the notations \mathcal{N}_+ , $d\Gamma(\epsilon)$, etc. for operators acting trivially on the first factor (i.e., we omit the tensor product with the identity). In this context, the variable x is usually reserved for the impurity, while the variables y_i describe the

bosons. To a function $x \mapsto A(x)$ on \mathbb{T}^3 whose values are quadratic forms with $Q(A_x) = D \subset \Gamma(\mathfrak{H}_+)$, we associate the quadratic form on \mathcal{H}_+

$$\begin{aligned} \langle \Psi, A\Psi \rangle &= \int \langle \Psi(x), A(x)\Psi(x) \rangle dx, \\ Q(A) &= \left\{ \Psi : \mathbb{T}^3 \rightarrow D \mid \int |\langle \Psi(x), A(x)\Psi(x) \rangle| < \infty \right\} \end{aligned} \tag{2.13}$$

by viewing \mathcal{H}_+ as $\Gamma(\mathfrak{H}_+)$ -valued functions on \mathbb{T}^3 .

Throughout the article we will explicitly write variables of integration or summation, their domains and the integration measures only when they are not obvious from the context.

2.3 The approximate scattering problems

By the scaling of the potentials, the interaction becomes increasingly short-ranged as $N \rightarrow \infty$, and thus the effective “interaction time” between two particles decreases to zero. For this reason, concepts from scattering theory of Schrödinger operators become relevant to the discussion, although we are working with periodic boundary conditions and the spectra of all operators are discrete.

The scattering length \mathfrak{a}_v of a potential $0 \leq v \in L^1(\mathbb{R}^3)$ is the first order in the low-energy expansion of the corresponding scattering amplitude [1]. In the expansion of the operator, it is accompanied by the generalized eigenfunction, also called scattering solution, at energy zero. This function $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ is asymptotic to one (i.e., e^{-ikx} with $k = 0$) and we may write it as $f = 1 + \varphi_{\mathbb{R}^3}$, where $\varphi_{\mathbb{R}^3} \in \dot{H}^1(\mathbb{R}^3)$ solves

$$-\Delta \varphi_{\mathbb{R}^3} + \frac{1}{2}v\varphi_{\mathbb{R}^3} = -\frac{1}{2}v. \tag{2.14}$$

This is also the Euler-Lagrange equation of the variational problem (1.2), and $\varphi_{\mathbb{R}^3}$ is the unique minimizer. The scattering length can thus be expressed as

$$8\pi\mathfrak{a}_v = \int v(1 + \varphi_{\mathbb{R}^3}). \tag{2.15}$$

As the support of the potentials V_N, W_N shrinks to a point, the boundary becomes less relevant and approximations of $\varphi_{\mathbb{R}^3}^3$ naturally appear in the analysis. Following [40], we will use solutions to the equation analogous to (2.14) on the torus, instead of truncations of $\varphi_{\mathbb{R}^3}$, and call these (approximate) scattering solutions. Their properties are discussed in detail in Appendix A. Here, we collect the information of practical relevance for our analysis. The scattering solution φ_B associated to the boson-boson interaction potential V_N on the torus is defined to be the unique solution in $\ell^2(2\pi\mathbb{Z}^3 \setminus \{0\})$ of

$$p^2 \widehat{\varphi}_B(p) + \frac{1}{2} \sum_{q \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \widehat{\varphi}_B(q) \widehat{V}_N(p - q) = -\frac{1}{2} \widehat{V}_N(p). \tag{2.16}$$

For $\alpha > 0$, we define its truncated version by the Fourier coefficients

$$\widehat{\tilde{\varphi}}_B(p) := \widehat{\varphi}_B(p) \mathbf{1}_{|p| > N^\alpha}. \quad (2.17)$$

In analogy with (2.15), the torus “scattering length” is then defined by

$$8\pi \tilde{\mathbf{a}}_{V_N} := \int_{\mathbb{T}^3} NV_N(y)(1 + \varphi_B(y)) = \widehat{V}(0) + \int NV_N \varphi_B. \quad (2.18)$$

Similarly, for the interaction potential between the bosons and the particle W_N , we define the scattering solution φ_I to be the unique solution in $\ell^2(2\pi\mathbb{Z}^3 \setminus \{0\})$ of

$$p^2 \tilde{\varphi}_I(p) + \frac{1}{2} \sum_{q \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \tilde{\varphi}_I(q) \widehat{W}_N(p - q) = -\frac{1}{2} \widehat{W}_N(p). \quad (2.19)$$

Its truncated version is denoted, for $\alpha > 0$, by

$$\widehat{\tilde{\varphi}}_I(p) = \widehat{\varphi}_I(p) \mathbf{1}_{|p| > N^\alpha}. \quad (2.20)$$

and the associated tours scattering length is defined by

$$8\pi \tilde{\mathbf{a}}_{W_N} = \int_{\mathbb{T}^3} \sqrt{N} W_N(y)(1 + \varphi_I(y)). \quad (2.21)$$

We gather in the following lemmas useful properties on $\varphi_B, \varphi_I, \tilde{\mathbf{a}}_{V_N}$ and $\tilde{\mathbf{a}}_{W_N}$.

Lemma 2.1 (Properties of φ_B). *We have for $0 \leq \alpha \leq 1$*

$$\begin{aligned} \|\tilde{\varphi}_B\|_2 &\lesssim N^{-1-\alpha/2}, & \|\tilde{\varphi}_B\|_\infty &\lesssim 1, \\ \|\widehat{\tilde{\varphi}}_B\|_\infty &\lesssim N^{-1-2\alpha}, & \|p^2 \widehat{\tilde{\varphi}}_B\|_\infty &\lesssim N^{-1}, \\ \||\nabla|^{1/2} \tilde{\varphi}_B\|_2 &\lesssim N^{-1} \sqrt{\log N}, & \|\tilde{\varphi}_B - \varphi_B\|_\infty &\lesssim N^{-1+\alpha}. \end{aligned}$$

Proof. This follows from Lemma A.1 in Appendix A, by taking $v = V_N$ □

Lemma 2.2 (Properties of φ_I). *We have for $0 \leq \alpha \leq 1$ and $0 < s \leq 1/2$*

$$\begin{aligned} \|\tilde{\varphi}_I\|_2 &\lesssim N^{-1/2-\alpha/2}, & \|\tilde{\varphi}_I\|_\infty &\lesssim 1, \\ \||\nabla|^{1/2+s} \tilde{\varphi}_I\|_2 &\lesssim N^{-1/2+s/2}, & \||\nabla|^{1/2} \tilde{\varphi}_I\|_2 &\lesssim N^{-1/2} \sqrt{\log N}, \\ \|p^2 \widehat{\tilde{\varphi}}_I\|_\infty &\lesssim N^{-1/2}, & \|\varphi_I - \tilde{\varphi}_I\|_\infty &\lesssim N^{-1/2+\alpha}. \end{aligned}$$

Proof. This follows from Lemma A.3 in Appendix A, by taking $v = W_N$. □

As we perform all our calculations on the torus, we need to control the difference of the quantities defined there to the real scattering lengths in \mathbb{R}^3 .

Lemma 2.3. *The differences between the scattering length on the torus and in \mathbb{R}^3 satisfy*

$$\begin{aligned} |\tilde{\mathbf{a}}_{V_N} - \mathbf{a}_V| &\lesssim N^{-1}, \\ |\tilde{\mathbf{a}}_{W_N} - \mathbf{a}_W| &\lesssim N^{-1/2}. \end{aligned}$$

Proof. This follows from Lemma A.4 in Appendix A, by taking respectively $v = V_N$ and $v = W_N$. \square

Remark 2.4. The error estimate above is in fact sharp. When V is radial it can be computed that [5]

$$\lim_{N \rightarrow \infty} 4\pi N (\tilde{\mathbf{a}}_{V_N} - \mathbf{a}_V) = 2 - \lim_{M \rightarrow \infty} \sum_{\substack{p \in \mathbb{Z}^3 \\ \|p\|_\infty \leq M}} \frac{\cos(|p|)}{|p|^2}.$$

3 Renormalization of the Bogoliubov-Fröhlich Hamiltonian

In this section we explain the renormalization of the Bogoliubov-Fröhlich Hamiltonian and prove Proposition 1.4. For the proof of the technical Lemmas, we will refer to Section 5, where the whole procedure is performed again, but on a slightly more general Hamiltonian. Although the proof is based on the ideas of [44, 45], we adapt it here for the use in the proof of Theorem 1.5.

The problem with the formal expression of the Hamiltonian (1.6) is that the interaction form factor doesn't decay for large momentum. Since the form factor is not square-integrable, the corresponding creation operator is not a densely defined operator on $\Gamma(\mathfrak{H}_+)$.

To approach this problem, we first consider the operator with ultra-violet cutoff $\Lambda \geq 0$,

$$H_{\text{BF}}^\Lambda = -\Delta_x + d\Gamma(\epsilon) + a(w_x^\Lambda) + a^*(w_x^\Lambda), \quad (3.1)$$

where

$$\hat{w}^\Lambda(p) = 8\pi \mathbf{a}_W |p| \epsilon(p)^{-1/2} \mathbb{1}_{|p| \leq \Lambda}. \quad (3.2)$$

This defines a self-adjoint operator on $\mathcal{H}_+ = L^2(\mathbb{T}^3) \otimes \Gamma(\mathfrak{H}_+)$. In our derivation of the Hamiltonian from H_N we will encounter a similar but somewhat more complicated Hamiltonian with Λ a function of the boson number N . It should be kept in mind that, as a consequence of this, the notation we introduce below is specific to this section.

We now perform a Weyl transformation on the Hamiltonian. This is a well-known procedure that is used for example in the renormalization of the Nelson

Hamiltonian [63, 32]. It will make the divergence of the Hamiltonian explicit, though in our case it will not yet suffice to arrive at an expression from which the cutoff may be removed. For $\kappa \geq 0$, which will be chosen sufficiently large later, we define $x \mapsto \widehat{f}_{\kappa,x}^\Lambda \in L^2(\mathbb{T}^3, \ell^2(2\pi\mathbb{Z}^3 \setminus \{0\}))$ by

$$\widehat{f}_{\kappa,x}^\Lambda(p) = -(-\Delta_x + \epsilon(p))^{-1} \widehat{w}_x^\Lambda(p) \mathbf{1}_{|p| > \kappa}, \quad (3.3)$$

and, denoting by $f_{\kappa,x}^\Lambda$ the inverse Fourier transform,

$$U_\kappa^\Lambda = \exp(a^*(f_{\kappa,x}^\Lambda) - a(f_{\kappa,x}^\Lambda)). \quad (3.4)$$

This transformation satisfies the identities [32, Sect.2]

$$(U_\kappa^\Lambda)^* a(g) U_\kappa^\Lambda = a(g) + \langle g, f_{\kappa,x}^\Lambda \rangle, \quad (3.5a)$$

$$(U_\kappa^\Lambda)^* i\nabla_x U_\kappa^\Lambda = i\nabla_x + a(i\nabla_x f_{\kappa,x}^\Lambda) + a^*(i\nabla_x f_{\kappa,x}^\Lambda). \quad (3.5b)$$

The choice of $f_{\kappa,x}^\Lambda$ leads to cancellations between terms in $(U_\kappa^\Lambda)^*(-\Delta_x + d\Gamma(\epsilon))U_\kappa^\Lambda$ and $a^*(w_x) + a(w_x)$, e.g., we have

$$(U_\kappa^\Lambda)^* d\Gamma(\epsilon) U_\kappa^\Lambda = d\Gamma(\epsilon) + a^*(\epsilon f_{\kappa,x}^\Lambda) + a(\epsilon f_{\kappa,x}^\Lambda) + \langle f_{\kappa,x}^\Lambda, \epsilon f_{\kappa,x}^\Lambda \rangle. \quad (3.6)$$

The terms coming from $-\Delta_x$ are similar but more involved, since taking the square of (3.5b) also yields new quadratic expressions in a^* , a (see [32, Sect.2]). Collecting all the terms and using $i\nabla_x f_{\kappa,x}^\Lambda(y) = -i\nabla f_{\kappa,x}^\Lambda(y)$, we obtain for $\Lambda \geq \kappa$

$$\begin{aligned} (U_\kappa^\Lambda)^* H_{\text{BF}}^\Lambda U_\kappa^\Lambda &= -\Delta_x + d\Gamma(\epsilon) + a^*(w_x^\kappa) + a(w_x^\kappa) \\ &\quad + a^*(i\nabla f_{\kappa,x}^\Lambda)^2 + a(i\nabla f_{\kappa,x}^\Lambda)^2 + 2a^*(i\nabla f_{\kappa,x}^\Lambda)a(i\nabla f_{\kappa,x}^\Lambda) \\ &\quad - 2a^*(i\nabla f_{\kappa,x}^\Lambda)i\nabla_x - 2i\nabla_x a(i\nabla f_{\kappa,x}^\Lambda) + \langle w_x, f_{\kappa,x}^\Lambda \rangle. \end{aligned} \quad (3.7)$$

The numbers

$$E_\Lambda^{(1)} = \langle w_x^\Lambda, f_{\kappa,x}^\Lambda \rangle = -(8\pi \mathbf{a}_W)^2 \sum_{p \in 2\pi\mathbb{Z}^3} \frac{p^2}{\epsilon(p)(p^2 + \epsilon(p))} \mathbf{1}_{\kappa \leq |p| \leq \Lambda} \sim -c\Lambda \quad (3.8)$$

diverge as $\Lambda \rightarrow \infty$, but the rest of the operator is now more regular than before. Indeed, the terms $a^*(i\nabla f_{\kappa,x}^\Lambda)a(i\nabla f_{\kappa,x}^\Lambda)$ and $a^*(i\nabla f_{\kappa,x}^\Lambda)i\nabla_x + \text{h.c.}$ now define quadratic forms on $Q(-\Delta_x + d\Gamma(\epsilon))$, even for $\Lambda = \infty$ (which was not the case before, as $\epsilon^{-1/2}w$ is not square-integrable). However, the term $a^*(i\nabla f_{\kappa,x}^\Lambda)^2 + \text{h.c.}$ is still singular. To deal with this, we employ an idea developed in [43, 44, 45]. We write

$$\begin{aligned} (U_\kappa^\Lambda)^* H_{\text{BF}}^\Lambda U_\kappa^\Lambda - E_\Lambda^{(1)} + 1 & \\ &= (1 - G_\Lambda^*)(-\Delta_x + d\Gamma(\epsilon) + 1)(1 - G_\Lambda) - G_\Lambda^*(-\Delta_x + d\Gamma(\epsilon) + 1)G_\Lambda + R_\Lambda, \end{aligned} \quad (3.9)$$

with

$$G_\Lambda = -(-\Delta_x + d\Gamma(\epsilon) + 1)^{-1} a^* (i\nabla f_{\kappa,x}^\Lambda)^2, \quad (3.10)$$

$$R_\Lambda = 2a^* (i\nabla f_{\kappa,x}^\Lambda) a (i\nabla f_{\kappa,x}^\Lambda) - (2i\nabla_x a (i\nabla f_{\kappa,x}^\Lambda) + a(w_x^\kappa) + \text{h.c.}). \quad (3.11)$$

Note that (3.9) is indeed just a simple identity, as expanding the first term and cancelling the term with $G_\Lambda^*(\cdot)G_\Lambda$ returns the expression (3.7). After rewriting the Hamiltonian in this way, the singularity is contained in the term $G_\Lambda^*(\cdot)G_\Lambda$ and can be extracted, as we will see. We will now analyze each term in (3.9) as $\Lambda \rightarrow \infty$.

The dressing factor G_Λ . We begin with the dressing factor G_Λ for which we have the following Lemma.

Lemma 3.1 (Regularity and limit of G_Λ). *For all $0 \leq s < 1/2$, there exists a family $(C_\kappa)_{\kappa>0}$ with $\lim_{\kappa \rightarrow \infty} C_\kappa = 0$, so that for all $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$, $\kappa > 0$, $\Psi \in D(\mathcal{N}_+)$*

$$\|(-\Delta_x + d\Gamma(\epsilon))^s G_\Lambda \Psi\|_{\mathcal{H}_+} \leq C_\kappa \|(\mathcal{N}_+ + 1)^s \Psi\|_{\mathcal{H}_+}.$$

Moreover, for all $0 \leq s < 1/2$, there exists $(C_\Lambda)_{\Lambda>0}$ with $\lim_{\Lambda \rightarrow \infty} C_\Lambda = 0$, so that for all $\kappa, \Lambda > 0$, $\Psi \in D(\mathcal{N}_+)$,

$$\|(-\Delta_x + d\Gamma(\epsilon))^s (G_\Lambda - G_\infty) \Psi\|_{\mathcal{H}_+} \leq C_\Lambda \|\mathcal{N}_+^s \Psi\|_{\mathcal{H}_+}.$$

The proof follows the one of Lemma 5.2. The key point is that $\epsilon^{-1/2+s/2} k f_\kappa^\infty \lesssim |k|^s f_\kappa^\infty \in \ell^2$ for $s < 1/2$.

The dressed kinetic operator K_Λ . We can now discuss the behavior as $\Lambda \rightarrow \infty$ of the principal term,

$$K_\Lambda := (1 - G_\Lambda^*)(-\Delta_x + d\Gamma(\epsilon) + 1)(1 - G_\Lambda), \quad (3.12)$$

with,

$$D(K_\Lambda) = \{\Psi \in \mathcal{H}_+ : (1 - G_\Lambda)\Psi \in D(-\Delta_x + d\Gamma(\epsilon))\}. \quad (3.13)$$

Lemma 3.2 (Comparison estimates for K_Λ). *For κ sufficiently large and all $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$, K_Λ is self-adjoint with compact resolvent. Moreover, for $s < 1/2$ there is $C > 0$ such that for all $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$ and all $\Psi \in D(K_\Lambda)$*

$$\|(\mathcal{N}_+ + 1)^{1-s} (-\Delta_x + d\Gamma(\epsilon) + 1)^s \Psi\| \leq C \|K_\Lambda \Psi\|.$$

Proof. By Lemma 3.1, $1 - G_\Lambda$ is invertible for $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$ by Neumann series for κ large enough. In particular, $D(K_\Lambda)$ is dense. As $-\Delta_x + d\Gamma(\epsilon) + 1$ is invertible, the symmetric operator K_Λ is a product of three invertible operators, and thus self-adjoint. Since the resolvent of $-\Delta_x + d\Gamma(\epsilon) + 1$ is compact, so is that of K_Λ .

The bound follows by inserting $\pm G_\Lambda$ and using the properties of G_Λ . The details, and more general estimates, are given in Lemma 5.3. \square

Note that $D(K_\Lambda)$ coincides with $D(-\Delta + d\Gamma(\epsilon))$ for $\Lambda < \infty$, but from Lemma 3.1 for $\Lambda = \infty$ we only have $D(K_\infty) \subset D((-\Delta + d\Gamma(\epsilon))^s)$ for $s < 1/2$ (and in fact the intersection is trivial for $s \geq 1/2$, cf. [43]).

The singular term. We now consider the second term in (3.9), which contains the remaining singularity. We can observe that, denoting the Fock vacuum by \emptyset ,

$$\begin{aligned} E_\Lambda^{(2)} &:= -\langle \mathbf{1} \otimes \emptyset, G^*(-\Delta_x + d\Gamma(\epsilon) + 1)G\mathbf{1} \otimes \emptyset \rangle \\ &= -\langle \mathbf{1} \otimes \emptyset, a(i\nabla f_{\kappa,x}^\Lambda)^2(-\Delta_x + d\Gamma(\epsilon) + 1)^{-1}a^*(i\nabla f_{\kappa,x}^\Lambda)^2\mathbf{1} \otimes \emptyset \rangle \\ &= -2 \sum_{p,q \in 2\pi\mathbb{Z}^3} \frac{(p \cdot q)^2 |\widehat{f}_{\kappa,0}^\Lambda(p)|^2 |\widehat{f}_{\kappa,0}^\Lambda(q)|^2}{(p+q)^2 + \epsilon(p) + \epsilon(q) + 1} \sim -c \log \Lambda. \end{aligned} \quad (3.14)$$

Subtracting this divergent contribution, we define

$$T_\Lambda = -G^*(-\Delta_x + d\Gamma(\epsilon) + 1)G - E_\Lambda^{(2)}. \quad (3.15)$$

In order to exhibit the cancellation of the singularities, we move all creation operators to the left of the annihilation operators. To do so, we use that $i\nabla_x e^{-ikx} = e^{-ikx}(i\nabla_x + k)$ and $d\Gamma(\epsilon)a_k^* = a_k^*(d\Gamma(\epsilon) + \epsilon(k))$, which implies

$$\left(-\Delta_x + d\Gamma(\epsilon)\right)^{-1} e^{-ik \cdot x} a_k^* = e^{-ik \cdot x} a_k^* \left((i\nabla_x + k)^2 + \epsilon(k) + d\Gamma(\epsilon)\right)^{-1}. \quad (3.16)$$

This gives

$$T_\Lambda = \sum_{j=0}^2 \Theta_{\Lambda,j}, \quad (3.17)$$

where $\Theta_{\Lambda,j}$ contains the terms with j remaining creation/annihilation operators after reordering. These are given explicitly by

$$\Theta_{\Lambda,0} = -2 \sum_{p,q \in 2\pi\mathbb{Z}^3} \frac{(p \cdot q)^2 |\widehat{f}_{\kappa,0}^\Lambda(p)|^2 |\widehat{f}_{\kappa,0}^\Lambda(q)|^2}{(i\nabla_x + p + q)^2 + \epsilon(p) + \epsilon(q) + d\Gamma(\epsilon) + 1} - E_\Lambda^{(2)} \quad (3.18)$$

$$\Theta_{\Lambda,1} = 4 \sum_{p,q \in 2\pi\mathbb{Z}^3} e^{-ipx} a_p^* \theta_{\Lambda,1}(p, q) a_q e^{iqx} \quad (3.19)$$

$$\Theta_{\Lambda,2} = \sum_{p,q,k,\ell \in 2\pi\mathbb{Z}^3} e^{-ipx} e^{-iqx} a_p^* a_q^* \theta_{\Lambda,2}(p, q, k, \ell) a_k a_\ell e^{ikx} e^{i\ell x}, \quad (3.20)$$

with

$$\begin{aligned} &\theta_{\Lambda,1}(p, q) \\ &= - \sum_{k \in 2\pi\mathbb{Z}^3} \frac{\widehat{f}_{\kappa,0}^\Lambda(p)(p \cdot k)(q \cdot k) \widehat{f}_{\kappa,0}^\Lambda(k)^2 \widehat{f}_{\kappa,0}^\Lambda(q)}{(i\nabla_x + p + q + k)^2 + \epsilon(k) + \epsilon(p) + \epsilon(q) + d\Gamma(\epsilon) + 1}, \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} & \theta_{\Lambda,2}(p, q, k, \ell) \\ &= -\frac{\widehat{f}_{\kappa,0}^\Lambda(p)\widehat{f}_{\kappa,0}^\Lambda(q)(p \cdot q)(k \cdot \ell)\widehat{f}_{\kappa,0}^\Lambda(k)\widehat{f}_{\kappa,0}^\Lambda(\ell)}{(\mathrm{i}\nabla_x + p + q + k + \ell)^2 + \epsilon(p) + \epsilon(q) + \epsilon(k) + \epsilon(\ell) + \mathrm{d}\Gamma(\epsilon) + 1}. \end{aligned} \quad (3.22)$$

Note that the kernels $\theta_{\Lambda,1}, \theta_{\Lambda,2}$ as functions of the commuting operators $\mathrm{i}\nabla_x$ and $\mathrm{d}\Gamma(\epsilon)$, are well defined also for $\Lambda = \infty$, since in $\theta_{\Lambda,1}$ the sum over k converges. Moreover, writing the terms in $\Theta_{\Lambda,0}$ with a common denominator we obtain

$$\begin{aligned} \Theta_{\Lambda,0} &= \sum_{k, \ell \in 2\pi\mathbb{Z}^3} \left(\frac{(k \cdot \ell)^2 \widehat{f}_{\kappa,0}^\Lambda(k)^2 \widehat{f}_{\kappa,0}^\Lambda(\ell)^2}{(\mathrm{i}\nabla_x + k + \ell)^2 + \epsilon(k) + \epsilon(\ell) + \mathrm{d}\Gamma(\epsilon) + 1} \right. \\ & \quad \left. \times \frac{((\mathrm{i}\nabla_x + k + \ell)^2 + \mathrm{d}\Gamma(\epsilon) - (k + \ell)^2)}{(k + \ell)^2 + \epsilon(k) + \epsilon(\ell) + 1} \right), \end{aligned} \quad (3.23)$$

where the series also converges for $\Lambda = \infty$ to a well defined, although unbounded, function of $\mathrm{i}\nabla_x$ and $\mathrm{d}\Gamma(\epsilon)$. With these observations, we can make sense of T_∞ . The key properties of T_Λ are summarized in the following lemma. In particular, T_∞ defines an operator on $D(K_\infty)$ and $K_\infty + T_\infty$ is self-adjoint on $D(K_\infty)$.

Lemma 3.3. *(Estimates and convergence of T_Λ) Given $\delta > 0$, the following holds for κ sufficiently large. For all $\Lambda, \Lambda' \in \mathbb{R}_+ \cup \{\infty\}$, $\Psi \in D(K_{\Lambda'})$*

$$\|T_\Lambda \Psi\| \leq \delta \|K_{\Lambda'} \Psi\|.$$

Moreover, there exists $(C_\Lambda)_{\Lambda > 0}$ with $\lim_{\Lambda \rightarrow \infty} C_\Lambda = 0$ so that for all $\Lambda, \Lambda' \in \mathbb{R}_+ \cup \{\infty\}$, $\Psi \in D(K'_\Lambda)$

$$\|(T_\Lambda - T_\infty)\Psi\| \leq C_\Lambda \|K_{\Lambda'} \Psi\|.$$

The proof is similar to the one of Proposition 5.4 and is omitted here. The main idea is to estimate T_Λ , respectively the difference $T_\Lambda - T_\infty$, by operators of the form $\mathcal{N}_+^{1-s}(-\Delta_x + \mathrm{d}\Gamma(\epsilon))^s$, which are themselves controlled by $K_{\Lambda'}$ using Lemma 3.2.

The remainder term R_Λ . Since $p \mapsto p\widehat{f}_{\kappa,x}^\infty(p)$ is not square summable, the expression $a^*(-\mathrm{i}\nabla_{\kappa,x} f_{\kappa,x}^\Lambda)\mathrm{i}\nabla_x$ occurring in R_Λ only makes sense as a quadratic form for $\Lambda = \infty$.

Lemma 3.4. *Given $\delta > 0$, $|s| < 1/8$, the following holds for κ sufficiently large. There is $C > 0$, so that for all $\Lambda, \Lambda' \in \mathbb{R}_+ \cup \{\infty\}$ and $\Psi, \Phi \in D(K_{\Lambda'})$*

$$|\langle \Phi, R_\Lambda \Psi \rangle| \leq \delta \| (K_{\Lambda'}^{1/2-s} + C)\Phi \| \| (K_{\Lambda'}^{1/2+s} + C)\Psi \|.$$

Moreover, there exists $(C_\Lambda)_{\Lambda>0}$ with $\lim_{\Lambda\rightarrow\infty} C_\Lambda = 0$ so that for all $\Lambda, \Lambda' \in \mathbb{R}_+ \cup \{\infty\}$ and $\Psi \in Q(K_{\Lambda'})$

$$|\langle \Psi, (R_\Lambda - R_\infty)\Psi \rangle| \leq C_\Lambda \langle \Psi, K_{\Lambda'}\Psi \rangle.$$

The proof of this lemma is a combination of Lemma 5.12 and Lemma 3.2.

Proposition 3.5. *For κ sufficiently large, the quadratic form*

$$K_\infty + T_\infty + R_\infty$$

is associated with a unique self-adjoint operator with compact resolvent and domain contained in $Q(K_\infty) = (1 - G_\infty)^{-1}Q(-\Delta_x + d\Gamma(\epsilon))$. Moreover,

$$\lim_{\Lambda\rightarrow\infty} (K_\Lambda + T_\Lambda + R_\Lambda \pm i)^{-1} = (K_\infty + T_\infty + R_\infty \pm i)^{-1} \quad (3.24)$$

in the operator norm.

Proof. By Lemma 3.3 and the Kato-Rellich Theorem, $K_\Lambda + T_\Lambda$ is self-adjoint on $D(K_\Lambda)$ for κ sufficiently large. Lemma 3.4 shows that R_Λ is a perturbation of the quadratic form of $K_\Lambda + T_\Lambda$ with relative bound $\delta < 1$. This implies existence of the self-adjoint realization by the KLMN theorem, whose resolvent is compact since the resolvent of K_Λ is. Using the resolvent formula and the properties of T_Λ , R_Λ and K_Λ (in particular, the uniformity of the relative bounds), one shows norm-resolvent convergence by the same argument as in the proof of Proposition 5.7. \square

We can now conclude by proving the proposition on the renormalization of the Bogoliubov-Fröhlich Hamiltonian.

3.1 Proof of Proposition 1.4

Setting $E_\Lambda = E_\Lambda^{(1)} + E_\Lambda^{(2)}$, with the latter defined in (3.8), (3.14). Let κ be sufficiently large for Proposition 3.5 to hold. Then, with the notation of (3.9) and (3.15), we have

$$(U_\kappa^\Lambda)^* H_{\text{BF}}^\Lambda U_\kappa^\Lambda - E_\Lambda + 1 = K_\Lambda + T_\Lambda + R_\Lambda. \quad (3.25)$$

By Proposition 3.5, this converges to $K_\infty + T_\infty + R_\infty$ in norm resolvent sense as $\Lambda \rightarrow \infty$. It remains to deal with the convergence of the transformation U_κ . Note that $f_{\kappa,x}^\infty \in \ell^2(2\pi\mathbb{Z}^3)$, and thus (cf. [32, Lem.C.2]) for $\Psi \in Q(K_{\Lambda'})$, $\Lambda' \in \mathbb{R}_+ \cup \{\infty\}$,

$$\|(U_\kappa^\Lambda - U_\kappa^\infty)\Psi\| \leq 2\|f_\kappa^\Lambda - f_\kappa^\infty\| \|(\mathcal{N}_+ + 1)^{1/2}\Psi\| \xrightarrow{\Lambda\rightarrow\infty} 0. \quad (3.26)$$

Since $\|\mathcal{N}_+^{1/2}\Psi\| \lesssim \|K_\infty^{1/2}\Psi\| \lesssim \|(K_\infty + T_\infty + R_\infty)^{1/2}\Psi\|$ from Lemma 3.2 and Proposition 3.5 (inclusion of the domains), this proves that

$$\lim_{\Lambda \rightarrow \infty} (H_{\text{BF}}^\Lambda - E_\Lambda + 1 \pm i)^{-1} = U_\kappa^\infty (K_\infty + T_\infty + R_\infty + \pm i)^{-1} (U_\kappa^\infty)^* \quad (3.27)$$

in norm. We have thus proved Proposition 1.4 with the renormalized Bogoliubov–Fröhlich Hamiltonian given by

$$H_{\text{BF}} = U_\kappa^\infty (K_\infty + T_\infty + R_\infty - 1) (U_\kappa^\infty)^*. \quad (3.28)$$

The resolvent of this operator is compact by Proposition 3.5. \square

The proof above shows that

$$Q(H_{\text{BF}}) = U_\kappa^\infty Q(K_\infty) = U_\kappa^\infty (1 - G_\infty)^{-1} Q(-\Delta_x + d\Gamma(\epsilon)), \quad (3.29)$$

since R_∞ is only a form perturbation of K_∞ . More precise information on the operator domain is obtained using the slightly different formulation of [44, 45].

4 Transformation of the Hamiltonian

4.1 The excitation Hamiltonian

In a first step, we transfer the Hamiltonian

$$H_N = -\Delta_x + \sum_{i=1}^N (-\Delta_{y_i}) + \sum_{1 \leq i < j \leq N} V_N(y_i - y_j) + \sum_{i=1}^N W_N(x - y_i) \quad (4.1)$$

from the N -particle space \mathcal{H}_N to the excitation space \mathcal{H}_+ using the excitation map introduced in [52]. This map is given by

$$U_X : \mathcal{H}_+ \rightarrow \mathcal{H}_N$$

$$(U_X \Psi)(x, y_1, \dots, y_N) = \sum_{j=0}^N S_N \left(\mathbb{1}(y_{j+1}, \dots, y_N) \Psi^{(j)}(x, y_1, \dots, y_j) \right) \quad (4.2)$$

where $\mathbb{1}$ denotes the constant function and S_N the projection to symmetric functions in y_1, \dots, y_N . In words, U_X completes the function $\Psi^{(j)}(x, y_1, \dots, y_j)$ to a function on $\mathbb{T}^{3(N+1)}$ by multiplying with the constant function in the variables y_{j+1}, \dots, y_N and then symmetrizes the result. This defines a partial isometry, i.e., we have

$$U_X^* U_X = \mathbb{1}_{\mathcal{N}_+ \leq N}, \quad U_X U_X^* = \mathbb{1}_{\mathcal{H}_N}, \quad (4.3)$$

where \mathcal{N}_+ denotes the number operator on \mathcal{H}_+ .

The Bose–Einstein condensation condition that we will use in the proof of our main theorems ensures that there are at most \sqrt{N} excitations in a state of the bosons whose energy has the correct leading order for $N \rightarrow \infty$. Within the notations of second quantization, Condition 1.1 is equivalent to

$$H_N - 4\pi\mathfrak{a}_V N \geq cU_X \mathcal{N}_+ U_X^* - d\sqrt{N}. \quad (4.4)$$

We will now evaluate the action of U_X on H_N in detail.

Proposition 4.1. *In the sense of quadratic forms on $Q(-\Delta_x + d\Gamma(-\Delta)) \subset \mathcal{H}_+$, we have the identity*

$$U_X^* H_N U_X = \mathbf{1}_{\mathcal{N}_+ \leq N} \mathcal{H}_N \mathbf{1}_{\mathcal{N}_+ \leq N},$$

where

$$\mathcal{H}_N := \frac{1}{2}(N-1)\widehat{V}(0) + \sqrt{N}\widehat{W}(0) + \mathcal{H}_B + \mathcal{H}_I,$$

with

$$\begin{aligned} \mathcal{H}_B &= d\Gamma(-\Delta) + (N - \mathcal{N}_+)_+ d\Gamma(\widehat{V}_N(-i\nabla)) + \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \\ \mathcal{L}_0 &= -\frac{1}{2N}\widehat{V}(0)\mathcal{N}_+(\mathcal{N}_+ - 1)\mathbf{1}_{\mathcal{N}_+ \leq N}, \\ \mathcal{L}_2 &= \frac{1}{2}\sqrt{(1 - \mathcal{N}_+/N)_+}\sqrt{(1 - (\mathcal{N}_+ + 1)/N)_+} \int NV_N(y_1 - y_2)a_{y_1}a_{y_2} + \text{h.c.}, \\ \mathcal{L}_3 &= \frac{1}{2}\sqrt{(1 - \mathcal{N}_+/N)_+} \int \sqrt{N}V_N(y_1 - y_2)a_{y_1}^*a_{y_2}a_{y_1} + \text{h.c.}, \\ \mathcal{L}_4 &= \frac{1}{2} \int V_N(y_1 - y_2)a_{y_1}^*a_{y_2}^*a_{y_1}a_{y_2}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_I &= -\Delta_x + Q_1 + d\Gamma(W_{N,x}), \\ Q_1 &= \sqrt{(1 - \mathcal{N}_+/N)_+}a(\sqrt{N}W_{N,x}) + a^*(\sqrt{N}W_{N,x})\sqrt{(1 - \mathcal{N}_+/N)_+}. \end{aligned}$$

Proof. This follows by applying well-known identities, going back to [52], hence we only sketch the calculations. The kinetic operator is unchanged by U_X since the constant function has derivative zero, thus only the potentials are transformed in a non-trivial way. For the boson-boson interaction, we can write

$$\sum_{1 \leq i < j \leq N} V_N(y_i - y_j) = \frac{1}{2} \sum_{k,p,q \in 2\pi\mathbb{Z}^3} \widehat{V}_N(k)a_{p+k}^*a_q^*a_{q+k}a_p \quad (4.5)$$

by considering \mathcal{H}_N as a subspace of $L^2(\mathbb{T}^3) \otimes \Gamma(L^2(\mathbb{T}^3))$. With

$$U_X^* a_0^* a_0 U_X = (N - \mathcal{N}_+)_+, \quad \sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} U_X^* a_p^* a_p U_X = \mathcal{N}_+, \quad (4.6)$$

and

$$U_X^* a_p^* a_0 U_X = a_p^* \sqrt{(N - \mathcal{N}_+)_+}, \quad (4.7)$$

for $p \neq 0$, we can rewrite

$$\begin{aligned} \frac{1}{2} \sum_{k,p,q \in 2\pi\mathbb{Z}^3} \widehat{V}_N(k) U_X^* a_{p+k}^* a_q^* a_{q+k} a_p U_X &= \frac{1}{2N} \widehat{V}(0) (N - \mathcal{N}_+)_+ (N - \mathcal{N}_+ - 1)_+ \\ &+ (N - \mathcal{N}_+)_+ d\Gamma(\widehat{V}_N(-i\nabla) + N^{-1} \widehat{V}(0)) + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \mathbf{1}_{\mathcal{N}_+ \leq N}. \end{aligned} \quad (4.8)$$

With

$$\frac{\widehat{V}(0)}{2N} \{(N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1) + 2(N - \mathcal{N}_+) \mathcal{N}_+\} = \frac{1}{2} (N - 1) \widehat{V}(0) + \mathcal{L}_0 \quad (4.9)$$

this gives the claimed form of \mathcal{H}_B . For the boson-impurity interaction, we find similarly

$$\begin{aligned} U_X^* \sum_{i=1}^N W_N(x - y_j) U_X &= \sum_{k,p \in 2\pi\mathbb{Z}^3} \widehat{W}_N(k) e^{ikx} U_X^* a_p^* a_{p+k} U_X \\ &= \sqrt{N} \widehat{W}(0) + \sum_{k \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \widehat{W}_N(k) e^{ikx} \left(\sqrt{(N - \mathcal{N}_+)_+} a_k + a_{-k}^* \sqrt{(N - \mathcal{N}_+)_+} \right) \\ &+ \sum_{\substack{k,p \in 2\pi\mathbb{Z}^3 \setminus \{0\} \\ p+k \neq 0}} \widehat{W}_N(k) e^{ikx} a_p^* a_{p+k}, \end{aligned} \quad (4.10)$$

where we may identify $d\Gamma(W_{N,x})$ in the last line and Q_1 in the line before. This proves the identity. \square

4.2 Interactions of high-energy bosons: a first quadratic transformation

We start by implementing a transformation whose generator is quadratic in the creation and annihilation operators. This transformation makes explicit correlations among the bosons that are due to their interaction at high energies. More precisely it makes the correct leading order of the Bose gas $4\pi\mathbf{a}_V(N-1)$ appear and renormalizes the quadratic term \mathcal{L}_2 into a less singular version $\widetilde{\mathcal{L}}_2$. Variants of this transformation have been used in many works on the dilute Bose gas [68, 25, 71, 3, 14, 5, 39, 62]. Throughout all the paper, we will distinguish between high and low energies using

the cutoff N^α , where the parameter $0 < \alpha < 1/2$ will be chosen sufficiently small for our estimates to hold.

Let $\tilde{\varphi}_B$, defined in (2.17), be the truncated scattering solution at momentum N^α , $0 < \alpha \leq 1/2$. We set, with $Q(\mathcal{B}_q) = Q(\mathcal{N}_+)$,

$$\mathcal{B}_q = \frac{1}{2} \int_{\mathbb{T}^3} N \tilde{\varphi}_B(y_1 - y_2) a_{y_1}^* a_{y_2}^* dy_1 dy_2 - \text{h.c.} \quad (4.11)$$

Using the commutation relations (2.9) together with the identity $\mathcal{N}_+ = \int a_y^* a_y dy$ and the Cauchy-Schwarz inequality, we have for $\Psi \in D(\mathcal{N}_+^{1/2})$

$$\begin{aligned} \langle \Psi, [\mathcal{N}_+, \mathcal{B}_q] \Psi \rangle &= \text{Re} \int \langle \Psi, N \tilde{\varphi}_B(y_1 - y_2) a_{y_1}^* a_{y_2}^* \Psi \rangle dy_1 dy_2 \\ &= \text{Re} \int \langle (\mathcal{N}_+ + 1)^{-1/2} a_{y_2} a_{y_1} \Psi, N \tilde{\varphi}_B(y_1 - y_2) (\mathcal{N}_+ + 1)^{1/2} \Psi \rangle dy_1 dy_2 \\ &\leq \|N \tilde{\varphi}_B\|_2 \left(\int \|(\mathcal{N}_+ + 1)^{-1/2} a_{y_2} a_{y_1} \Psi\|^2 \right)^{1/2} \|\mathcal{N}_+^{1/2} \Psi\| \\ &\leq \|N \tilde{\varphi}_B\|_2 \|\mathcal{N}_+^{1/2} \Psi\|^2. \end{aligned} \quad (4.12)$$

As $\|N \tilde{\varphi}_B\|_2 \lesssim N^{-\alpha/2}$, see Lemma 2.1, this implies

$$[\mathcal{N}_+, \mathcal{B}_q] = \mathcal{O}(N^{-\alpha/2}(\mathcal{N}_+ + 1)). \quad (4.13)$$

Hence, by the commutator theorem [65, Thm.X.36], there exists an anti-self-adjoint operator, denoted also by \mathcal{B}_q , whose form is the above and for which $D(\mathcal{N}_+)$ is a core. We can thus define the unitary

$$U_q = e^{\mathcal{B}_q}, \quad (4.14)$$

and $U_q(t) = e^{t\mathcal{B}_q}$. The effect of this transformation on \mathcal{H}_N is summarized in the following proposition.

Proposition 4.2. *Let $0 \leq \alpha \leq 1/6$, and \mathbf{a}_V be the scattering length defined by (1.2). We have, in the sense of quadratic forms on \mathcal{H}_+ ,*

$$\begin{aligned} U_q^* \mathcal{H}_N U_q &= 4\pi \mathbf{a}_V (N - 1) + e_N^{(U_q)} + \sqrt{N} \widehat{W}(0) \\ &\quad + d\Gamma(-\Delta + N \widehat{V}_N(-i\nabla) + \widehat{V}(0) - 8\pi \mathbf{a}_V) + \widetilde{\mathcal{L}}_2 + \mathcal{L}_3^{U_q} + \mathcal{L}_4 \\ &\quad - \Delta_x + Q_1^{U_q} + d\Gamma(W_{N,x}) + \mathcal{E}^{(U_q)}, \end{aligned}$$

with

$$\begin{aligned}\tilde{\mathcal{L}}_2 &= \int \tilde{V}_N(x-y)a_x^*a_y^* + \text{h.c.}, & \widehat{\tilde{V}}_N(p) &= 4\pi\mathbf{a}_V\mathbf{1}_{0<|p|\leq N^\alpha}, \\ Q_1^{U_q} &= \sqrt{\left(1 - \frac{U_q^*\mathcal{N}_+U_q}{N}\right)_+} a(\sqrt{N}W_{N,x}) + \text{h.c.}, \\ \mathcal{L}_3^{U_q} &= \frac{1}{2} \int \sqrt{N}V_N(x-y)a_x^*a_ya_x + \text{h.c.},\end{aligned}$$

the scalar correction

$$e_N^{(U_q)} = 4\pi N(\tilde{\mathbf{a}}_{V_N} - \mathbf{a}_V) + \sum_{\substack{p \in 2\pi\mathbb{Z}^3 \\ |p| \leq N^\alpha}} \frac{(4\pi\mathbf{a}_V)^2}{p^2},$$

as well as the error bound

$$\pm \mathcal{E}^{(U_q)} \lesssim N^{-\alpha/2}(\mathcal{L}_4 + \mathcal{N}_+ + 1 + N^{-1/2}d\Gamma(W_{N,x})) + N^{\alpha/2-1}\mathcal{N}_+^2.$$

Remark 4.3. The term $e_N^{(U_q)}$ is a lower order energy contribution, where the first term is a boundary effect of order one, see Remark 2.4, and the second term is of order N^α . This divergence is due to the fact that we solely renormalized the quadratic part with momenta higher than N^α . It will later combine with a similar term coming from the diagonalization of the low momenta part of the Hamiltonian, yielding together a contribution of order one, which is the well-known Lee-Huang-Yang correction.

Proof of Proposition 4.2. The proof relies on several lemmas about the transformation of individual terms that we will prove later. We will be rather brief, as the treatment of the bosonic part of the Hamiltonian is standard, except for some estimates we need to adapt to our analysis.

The correlation energy that renormalizes $N\widehat{V}(0)$ into the torus scattering length $8\pi\tilde{\mathbf{a}}_{V_N}$ defined in (2.18) is hidden in $d\Gamma(-\Delta)$, \mathcal{L}_2 , and \mathcal{L}_4 . We can see this by combining Lemmas 4.7, 4.6, 4.5 below. These give

$$\begin{aligned}U_q^*(d\Gamma(-\Delta) + \mathcal{L}_2 + \mathcal{L}_4)U_q \\ = \frac{1}{2} \int N^2V_N\tilde{\varphi}_B^2 + \int N^2V_N\tilde{\varphi}_B\left(1 - \frac{1}{2N}\right) + N^2\langle -\Delta\tilde{\varphi}_B, \tilde{\varphi}_B \rangle\end{aligned}\tag{4.15a}$$

$$+ \int N\left(\frac{1}{2}V_N\tilde{\varphi}_B + \frac{1}{2}V_N - \Delta\tilde{\varphi}_B\right)(y_1 - y_2)a_{y_1}^*a_{y_2}^* + \text{h.c.}\tag{4.15b}$$

$$+ d\Gamma(-\Delta) - \mathcal{N}_+ \int NV_N\tilde{\varphi}_B + \mathcal{L}_4 + \mathcal{E},$$

with,

$$\pm \mathcal{E} \lesssim N^{-\alpha/2}(\mathcal{N}_+ + 1) + N^{-\alpha/2}\mathcal{L}_4 + N^{-1+\alpha/2}(\mathcal{N}_+ + 1)^2. \quad (4.16)$$

Adding to the scalar terms (4.15a) the contribution $\frac{1}{2}(N-1)\widehat{V}(0)$ from \mathcal{H}_N , we find using the definition of $\tilde{\mathbf{a}}_{V_N}$ in (2.18), and the scattering equation (2.16),

$$\begin{aligned} & \frac{N-1}{2} \int NV_N + \frac{1}{2} \int N^2 V_N \tilde{\varphi}_B^2 + \int N^2 V_N \tilde{\varphi}_B \left(1 - \frac{1}{2N}\right) + N^2 \langle -\Delta \tilde{\varphi}_B, \tilde{\varphi}_B \rangle \\ &= 4\pi \tilde{\mathbf{a}}_{V_N} (N-1) + \left(\frac{1}{2} - \frac{1}{2N}\right) \int N^2 V_N (\tilde{\varphi}_B - \varphi_B) \\ & \quad + N^2 \left\langle \tilde{\varphi}_B, -\Delta \tilde{\varphi}_B + \frac{1}{2} V_N \tilde{\varphi}_B + \frac{1}{2} V_N \right\rangle \\ &= 4\pi \tilde{\mathbf{a}}_{V_N} (N-1) + \left(\frac{1}{2} - \frac{1}{2N}\right) \int N^2 V_N (\tilde{\varphi}_B - \varphi_B) + \int N^2 V_N \tilde{\varphi}_B (\tilde{\varphi}_B - \varphi_B). \end{aligned} \quad (4.17)$$

Following the computation in the proof of [40, Proposition 4] and using that $\tilde{\mathbf{a}}_{V_N} = \mathbf{a}_V + \mathcal{O}(N^{-1})$ from Lemma 2.3 gives

$$\frac{1}{2} \int N^2 V_N (1 + \tilde{\varphi}_B) (\tilde{\varphi}_B - \varphi_B) = \sum_{|p| \leq N^\alpha} \frac{(4\pi \mathbf{a}_V)^2}{p^2} + \mathcal{O}(N^{2\alpha-1}). \quad (4.18)$$

The additional term with prefactor $1/(2N)$ is of order $N^{\alpha-1}$ by Lemma 2.1.

We now turn to the quadratic terms (4.15b) and show that they are approximated by $\tilde{\mathcal{L}}_2$. Again using the scattering equation (2.16), we have

$$\begin{aligned} & \int N \left(\frac{1}{2} V_N \tilde{\varphi}_B + \frac{1}{2} V_N - \Delta \tilde{\varphi}_B \right) (y_1 - y_2) a_{y_1}^* a_{y_2}^* \\ &= \frac{1}{2} \sum_{|p| \leq N^\alpha} N (\widehat{V}_N(p) + \widehat{V}_N \varphi_B(p)) a_p^* a_{-p}^* \end{aligned} \quad (4.19a)$$

$$+ \frac{1}{2} \int NV_N (\tilde{\varphi}_B - \varphi_B) (y_1 - y_2) a_{y_1}^* a_{y_2}^*. \quad (4.19b)$$

The first term (4.19a) may be approximated by $\tilde{\mathcal{L}}_2$. Indeed, noting that $\widehat{V}_N(0) + \widehat{V}_N \varphi_B(0) = 8\pi \tilde{\mathbf{a}}_{V_N}$, we have

$$\begin{aligned} |NV_N (\widehat{1 + \varphi_B})(p) - 8\pi \mathbf{a}_V| &\leq \int NV_N (1 + \varphi_B)(y) |e^{-ip \cdot y} - 1| \\ &\quad + |\tilde{\mathbf{a}}_{V_N} - \mathbf{a}_V| \lesssim |p| N^{-1}, \end{aligned} \quad (4.20)$$

where we used that $|y|V \in L^1(\mathbb{R}^3)$ and that $\|\varphi_B\|_\infty \lesssim 1$, from Lemma 2.1. Then, estimating the operator using the ℓ^2 -norm of its kernel,

$$\begin{aligned} ((4.19a) + \text{h.c.}) - \tilde{\mathcal{L}}_2 &= \frac{1}{2} \sum_{|p| \leq N^\alpha} \left(NV_N (\widehat{1 + \varphi_B})(p) - 8\pi \mathbf{a}_V \right) a_p^* a_{-p}^* + \text{h.c.} \\ &= \mathcal{O}(N^{5\alpha/2-1}(\mathcal{N}_+ + 1)). \end{aligned} \quad (4.21)$$

To estimate the second term (4.19b), we use the Cauchy-Schwarz inequality to get

$$\begin{aligned}
& \left| \left\langle \Psi, \int NV_N(\tilde{\varphi}_B - \varphi_B)(y_1 - y_2)a_{y_1}^* a_{y_2}^* \Psi \right\rangle \right| \\
& \leq \left(\int V_N(y_1 - y_2) \|a_{y_1} a_{y_2} \Psi\|^2 \right)^{1/2} N \|(\tilde{\varphi}_B - \varphi_B) V_N^{1/2}\|_2 \|\Psi\| \\
& \leq \delta \langle \Psi, \mathcal{L}_4 \Psi \rangle + (4\delta)^{-1} (N \|(\tilde{\varphi}_B - \varphi_B) V_N^{1/2}\|_2)^2 \|\Psi\|^2.
\end{aligned} \tag{4.22}$$

With Lemma 2.1, the choice $\delta = N^{-\alpha/2}$ and $\alpha \leq 1/6$ this gives

$$(4.19b) + \text{h.c.} = \mathcal{O}(N^{-\alpha/2} \mathcal{L}_4 + N^{-\alpha/2}). \tag{4.23}$$

Of the terms coming from $d\Gamma(-\Delta) + \mathcal{L}_2 + \mathcal{L}_4$, we still have the term $\mathcal{N}_+ \int NV_N \tilde{\varphi}_B$ that does not appear in the statement as such. It can be written as

$$\begin{aligned}
-\mathcal{N}_+ \int NV_N \tilde{\varphi}_B &= (\widehat{V}(0) - 8\pi \mathbf{a}_V) \mathcal{N}_+ \\
&+ \left(8\pi(\tilde{\mathbf{a}}_{V_N} - \mathbf{a}_V) + \int NV_N(\varphi_B - \tilde{\varphi}_B) \right) \mathcal{N}_+.
\end{aligned} \tag{4.24}$$

The first term now appears in the statement, and the second can be absorbed in the error by Lemma 2.1 and Lemma 2.3. For $d\Gamma(N\widehat{V}_N(-i\nabla))$, we easily find

$$U_q^* d\Gamma(N\widehat{V}_N(-i\nabla)) U_q^* = d\Gamma(N\widehat{V}_N(-i\nabla)) + \mathcal{O}(N^{-\alpha/2} \mathcal{N}_+), \tag{4.25}$$

using that $\|NV_N * N\tilde{\varphi}_B\|_2 \lesssim N^{-\alpha/2}$ by Lemma 2.1. Writing

$$(N - \mathcal{N}_+)_+ d\Gamma(\widehat{V}_N(-i\nabla)) = d\Gamma(N\widehat{V}_N(-i\nabla)) + \mathcal{O}(\mathcal{N}_+^2/N), \tag{4.26}$$

the previous estimates and Lemma 4.4 below show that $U_q^*(N - \mathcal{N}_+)_+ d\Gamma(\widehat{V}_N(-i\nabla)) U_q$ is equal to $d\Gamma(N\widehat{V}_N(-i\nabla))$ plus a term that can be absorbed in the error $\mathcal{E}^{(U_q)}$.

The transformation of $\mathcal{L}_0 = -(\mathcal{N}_+ + 1)\mathcal{N}_+/2N$ is absorbed in the error $\mathcal{E}_{\mathcal{B}_q}$ as well. The remaining terms are treated by applying Lemmas 4.9 and 4.8 below, after which the proof is complete. \square

Lemma 4.4. *For $\alpha > 0$ we have uniformly in $|s| \leq 1$*

$$U_q(s)^* \mathcal{N}_+ U_q(s) = \mathcal{N}_+ + \mathcal{O}(N^{-\alpha/2}(\mathcal{N}_+ + 1)),$$

and for $t \geq 0$

$$U_q(s)^* \mathcal{N}_+^t U_q(s) = \mathcal{O}(\mathcal{N}_+^t + 1).$$

Proof. The first claim follows directly from the bound (4.12) and Grönwall's inequality. For the generalization to arbitrary exponents see [14, Lem.3.1] for the case $t \in \mathbb{N}$, which implies the general case by operator monotonicity of $x \mapsto x^s$ for $0 \leq s \leq 1$. \square

Lemma 4.5. *In the sense of quadratic forms on \mathcal{H}_+ , we have for $\alpha > 0$*

$$U_q^* \mathcal{L}_4 U_q = \mathcal{L}_4 + \frac{1}{2} \int NV_N \tilde{\varphi}_B(y_1 - y_2) a_{y_1}^* a_{y_2}^* + \text{h.c.} + \frac{1}{2} \int N^2 V_N \tilde{\varphi}_B^2 + \mathcal{E}_{\mathcal{L}_4}^{(U_q)},$$

with

$$\pm \mathcal{E}_{\mathcal{L}_4}^{(U_q)} \lesssim N^{-\alpha/2} \mathcal{N}_+ + N^{-1} (\mathcal{N}_+ + 1)^2 + N^{-\alpha/2} \mathcal{L}_4.$$

Moreover, we have the a priori bound

$$U_q^*(s) \mathcal{L}_4 U_q(s) \lesssim \mathcal{L}_4 + N^{-1} (\mathcal{N}_+ + 1)^2 + N \quad (4.27)$$

uniformly in $s \in [0, 1]$.

Proof. From Duhamel's formula, we obtain

$$U_q^* \mathcal{L}_4 U_q = \mathcal{L}_4 + \int_0^1 U_q^*(s) [\mathcal{L}_4, \mathcal{B}_q] U_q(s) ds. \quad (4.28)$$

Calculating the commutator and putting terms in normal order, we find (using that V_N is even)

$$\begin{aligned} [\mathcal{L}_4, \mathcal{B}_q] &= \overbrace{\frac{1}{2} \int NV_N \tilde{\varphi}_B(y_1 - y_2) a_{y_1}^* a_{y_2}^* + \text{h.c.}}^{X_1} \\ &\quad + \underbrace{\int NV_N(y_1 - y_2) \tilde{\varphi}_B(y_3 - y_2) a_{y_1}^* a_{y_2}^* a_{y_3}^* a_{y_1} + \text{h.c.}}_{X_2}. \end{aligned} \quad (4.29)$$

Using the Cauchy-Schwarz inequality as in (4.22) gives

$$|\langle \Psi, X_1 \Psi \rangle| \leq \langle \Psi, \mathcal{L}_4 \Psi \rangle + N^2 \|\tilde{\varphi}_B V_N^{1/2}\|_2^2 \|\Psi\|^2 \lesssim \langle \Psi, \mathcal{L}_4 \Psi \rangle + N \|\Psi\|^2, \quad (4.30)$$

where we used Lemma 2.1 in the last step. With the same reasoning and another application of Cauchy-Schwarz, we obtain

$$\begin{aligned} &|\langle \Psi, X_2 \Psi \rangle| \\ &\leq \delta \langle \Psi, \mathcal{L}_4 \Psi \rangle + \frac{1}{\delta} \int N^2 V_N(y_1 - y_2) \left\| \int \tilde{\varphi}_B(y_3 - y_2) a_{y_3}^* a_{y_1} \Psi dy_3 \right\|^2 dy_1 dy_2 \\ &\leq \delta \langle \Psi, \mathcal{L}_4 \Psi \rangle + \delta^{-1} N^2 \|\tilde{\varphi}_B\|_2^2 \|V_N\|_1 \int \|a_{y_2} a_{y_1} \Psi\|^2 + \int \|a_{y_1} \Psi\|^2 \\ &\lesssim \delta \langle \Psi, \mathcal{L}_4 \Psi \rangle + \delta^{-1} N^{-1-\alpha} (\mathcal{N}_+ + 1)^2 \|\Psi\|. \end{aligned} \quad (4.31)$$

Here, we used the estimates of Lemma 2.1 and the bound for $g \in L^2(\mathbb{T}^3)$,

$$\begin{aligned}
& \int \left\| \int g(y_1 - y_2) a_{y_1}^* a_{y_2} \Psi dy_1 \right\|^2 dy_2 \\
&= \int g(y_1 - y_2) \overline{g(y_3 - y_2)} \langle \Psi, a_{y_2}^* a_{y_1} a_{y_3}^* a_{y_2} \Psi \rangle dy_1 dy_3 dy_2 \\
&= \int |g(y_1 - y_2)|^2 \langle \Psi, a_{y_2}^* a_{y_2} \Psi \rangle dy_1 dy_2 \\
&\quad + \int g(y_1 - y_2) \overline{g(y_3 - y_2)} \langle \Psi, a_{y_2}^* a_{y_3}^* a_{y_1} a_{y_2} \Psi \rangle dy_1 dy_3 dy_2 \\
&\leq \|g\|_2^2 \|(\mathcal{N}_+ + 1)\Psi\|^2, \tag{4.32}
\end{aligned}$$

where we used the Cauchy-Schwarz inequality to bound the second term, similarly as in (4.12). Now taking $\delta = 1$ in (4.31) and using Lemma 4.4 gives

$$U_q^*(s) \mathcal{L}_4 U_q(s) \lesssim \mathcal{L}_4 + N + N^{-1}(\mathcal{N}_+ + 1)^2 + \int_0^s U_q^*(\tau) \mathcal{L}_4 U_q(\tau) d\tau,$$

from which the a priori bound (4.27) on $U_q^*(s) \mathcal{L}_4 U_q(s)$ follows by Grönwall's inequality. To get the more precise statement for $s = 1$, we continue the Duhamel expansion to obtain

$$\begin{aligned}
U_q^* \mathcal{L}_4 U_q &= \mathcal{L}_4 + X_1 + \text{h.c.} + \int_0^1 \int_0^t U_q^*(s) ([X_1, \mathcal{B}_q] + \text{h.c.}) U_q(s) ds dt \\
&\quad + \int_0^1 U_q^*(s) (X_2 + \text{h.c.}) U_q(s) ds. \tag{4.33}
\end{aligned}$$

We may compute

$$[X_1, \mathcal{B}_q] = \frac{1}{2} \int N^2 V_N \tilde{\varphi}_B^2 + \int N V_N \tilde{\varphi}_B (y_1 - y) N \tilde{\varphi}_B (y_2 - y) a_{y_1}^* a_{y_2} dy dy_1 dy_2. \tag{4.34}$$

The first term contributes to the main term in the statement (with an additional factor 1/2 from the double integral). The second is estimated by $\|N V_N \tilde{\varphi}_B\|_1 \|N \tilde{\varphi}_B\|_2 \mathcal{N}_+$, so its contribution can be absorbed in the error by Lemma 4.4 and Lemma 2.1. In order to control the final contribution to (4.33), we write

$$\begin{aligned}
& U_q^*(s) X_2 U_q(s) \\
&= \int N V_N (y_1 - y_2) \tilde{\varphi}_B (y_3 - y_2) U_q^*(s) a_{y_1}^* a_{y_2}^* U_q(s) U_q^*(s) a_{y_3}^* a_{y_1} U_q(s) \\
&= \int N V_N (y_1 - y_2) \tilde{\varphi}_B (y_3 - y_2) a_{y_1}^* a_{y_2}^* U_q^*(s) a_{y_3}^* a_{y_1} U_q(s) \tag{4.35a}
\end{aligned}$$

$$\begin{aligned}
& + \int_0^s \int N V_N (y_1 - y_2) \tilde{\varphi}_B (y_3 - y_2) U_q^*(t) [a_{y_1}^* a_{y_2}^*, \mathcal{B}_q] U_q(t) \\
&\quad \times U_q^*(s) a_{y_3}^* a_{y_1} U_q(s) dt. \tag{4.35b}
\end{aligned}$$

With the reasoning of (4.31) and $\delta = N^{-\alpha/2}$, we find

$$|\langle \Psi, (4.35a)\Psi \rangle| \lesssim N^{-\alpha/2} \langle \Psi, \mathcal{L}_4 \Psi \rangle + N^{-1-\alpha/2} \|(\mathcal{N}_+ + 1)^2 U_q(s) \Psi\|^2. \quad (4.36)$$

To bound (4.35b), we use the identity

$$[a_{y_1}^* a_{y_2}^*, \mathcal{B}_q] = -N \tilde{\varphi}_B(y_1 - y_2) - N a_{y_1}^* a(\tilde{\varphi}_B(\cdot - y_2)) - N a_{y_2}^* a(\tilde{\varphi}_B(\cdot - y_1)), \quad (4.37)$$

to obtain

$$\begin{aligned} (4.35b) &= - \int_0^s \int N(V_N \tilde{\varphi}_B)(y_1 - y_2) U_q^*(s) a^*(N \tilde{\varphi}_B(\cdot - y_2)) a_{y_1} U_q(s) dt \\ &\quad - \int_0^s \int N V_N(y_1 - y_2) U_q^*(t) \left(a_{y_1}^* a(N \tilde{\varphi}_B(\cdot - y_2)) + a_{y_2}^* a(N \tilde{\varphi}_B(\cdot - y_1)) \right) \\ &\quad \times U_q(t) U_q^*(s) a^*(\tilde{\varphi}_B(\cdot - y_2)) a_{y_1} U_q(s) dt. \end{aligned} \quad (4.38)$$

Using the Cauchy-Schwarz inequality and the estimate (4.32), we have

$$\begin{aligned} |\langle \Psi, (4.35b)\Psi \rangle| &\lesssim \int_0^s N \|V_N \tilde{\varphi}_B\|_1 \|N \tilde{\varphi}_B\|_2 \|(\mathcal{N}_+ + 1)^{1/2} U_q(s) \Psi\|^2 dt \\ &\quad + \int_0^s \|V_N\|_1 \|N \tilde{\varphi}_B\|_2^2 \|(\mathcal{N}_+ + 1) U_q(t) \Psi\|^2 dt \\ &\quad + \int_0^s \|V_N\|_1 \|N \tilde{\varphi}_B\|_2^2 \|(\mathcal{N}_+ + 1) U_q(s) \Psi\|^2 dt \\ &\lesssim N^{-\alpha/2} \mathcal{N}_+ + N^{-1} (\mathcal{N}_+ + 1)^2, \end{aligned} \quad (4.39)$$

where we used that $U_q(s)^* \mathcal{N}_+^k U_q(s) = \mathcal{O}(\mathcal{N}_+^k + 1)$, see Lemma 4.4. This can be absorbed in $\mathcal{E}_{\mathcal{L}_4}^{(U_q)}$ and the proof is complete. \square

Lemma 4.6. *In the sense of quadratic forms on \mathcal{H}_+ , we have for $\alpha > 0$*

$$\begin{aligned} U_q^* \mathcal{L}_2 U_q &= \frac{1}{2} \int N V_N(y_1 - y_2) a_{y_1} a_{y_2} + \text{h.c.} \\ &\quad + \left(1 - \frac{\mathcal{N}_+}{N} - \frac{1}{2N} \right) \int N^2 V_N \tilde{\varphi}_B + \mathcal{E}_{\mathcal{L}_2}^{(U_q)}, \end{aligned}$$

with, for all $0 < \delta \leq N^{2\alpha}$,

$$\pm \mathcal{E}_{\mathcal{L}_2}^{(U_q)} \lesssim N^{-\alpha/2} (\mathcal{N}_+ + 1) + \delta \mathcal{L}_4 + \delta^{-1} N^{-1} (\mathcal{N}_+ + 1)^2.$$

Proof. Using that where $\mathcal{N}_+ \leq N$

$$\left| \sqrt{1 - \frac{\mathcal{N}_+ + 1}{N}} \sqrt{1 - \frac{\mathcal{N}_+}{N}} - \left(1 - \frac{\mathcal{N}_+}{N} - \frac{1}{2N} \right) \right| \lesssim \frac{\mathcal{N}_+ + 1}{N^2}, \quad (4.40)$$

we obtain with the argument of (4.22)

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{2} \left(1 - \frac{\mathcal{N}_+}{N} + \frac{1}{2N} \right) \int NV_N(y_1 - y_2)(a_{y_1}a_{y_2} + \text{h.c.}) + \mathcal{E}, \\ \pm \mathcal{E} &\lesssim N^{-2}\mathcal{L}_4 + N^{-1}(\mathcal{N}_+ + 1)^2.\end{aligned}\tag{4.41}$$

Using the a priori estimate of Lemma 4.5 and Lemma 4.4, the error term $U_q^* \mathcal{E} U_q$ can be absorbed in $\mathcal{E}_{\mathcal{L}_2}^{(U_q)}$. Denoting $\chi(\mathcal{N}_+) := \left(1 - \frac{\mathcal{N}_+}{N} - \frac{1}{2N} \right)$ and using Duhamel's formula, we obtain

$$\begin{aligned}&\frac{1}{2} U_q^* \left\{ \chi(\mathcal{N}_+) \int NV_N(y_1 - y_2) a_{y_1} a_{y_2} + \text{h.c.} \right\} U_q \\ &= \frac{1}{2} \chi(U_q^* \mathcal{N}_+ U_q) \int NV_N(y_1 - y_2) a_{y_1} a_{y_2} + \text{h.c.}\end{aligned}\tag{4.42a}$$

$$+ \frac{1}{2} \chi(U_q^* \mathcal{N}_+ U_q) \int_0^1 U_q(s)^* \left[\int NV_N(y_1 - y_2) a_{y_1} a_{y_2}, \mathcal{B}_q \right] U_q(s) ds + \text{h.c.}\tag{4.42b}$$

Let us start with the first term. Using the definition of $\chi(\mathcal{N}_+)$, we have by the argument of (4.22)

$$\pm \left((4.42a) - \frac{1}{2} \int NV_N(y_1 - y_2) a_{y_1} a_{y_2} + \text{h.c.} \right) \lesssim \delta \mathcal{L}_4 + \delta^{-1} N^{-1} (\mathcal{N}_+ + 1)^2,\tag{4.43}$$

which can be absorbed in $\mathcal{E}_{\mathcal{L}_2}^{(U_q)}$. Now for the commutator term, we have by (4.12)

$$\left[\int NV_N(y_1 - y_2) a_{y_1} a_{y_2}, \mathcal{B}_q \right] = \int N^2 V_N \tilde{\varphi}_B + \underbrace{d\Gamma(N \hat{V}_N(-i\nabla) N \hat{\tilde{\varphi}}_B(-i\nabla))}_{\lesssim \mathcal{N}_+ N^{-2\alpha}},\tag{4.44}$$

where we used Lemma 2.1. From Lemma 4.4, we get

$$\chi(U_q^* \mathcal{N}_+ U_q) = \chi(\mathcal{N}_+) + \mathcal{O}(N^{-1-\alpha/2}(\mathcal{N}_+ + 1)),\tag{4.45}$$

and together this yields

$$(4.42b) = \chi(\mathcal{N}_+) \int N^2 V_N \tilde{\varphi}_B + \mathcal{O}\left(N^{-\alpha/2}(\mathcal{N}_+ + 1) + N^{-1-2\alpha} \mathcal{N}_+^2\right).\tag{4.46}$$

This finishes the proof of Lemma 4.6. \square

Lemma 4.7. *In the sense of quadratic forms on \mathcal{H}_+ , we have for $\alpha > 0$*

$$\begin{aligned}U_q^* d\Gamma(-\Delta) U_q &= d\Gamma(-\Delta) + N \int (-\Delta \tilde{\varphi}_B)(y_1 - y_2) a_{y_1}^* a_{y_2}^* + \text{h.c.} \\ &\quad + N^2 \langle -\Delta \tilde{\varphi}_B, \tilde{\varphi}_B \rangle + \mathcal{O}(N^{-2\alpha} \mathcal{N}_+).\end{aligned}$$

Proof. From Duhamel's formula, we obtain

$$U_q^* d\Gamma(-\Delta)U_q = d\Gamma(-\Delta) + [d\Gamma(-\Delta), \mathcal{B}_q] \quad (4.47)$$

$$+ \int_0^1 \int_0^t U_q^*(s) [[d\Gamma(-\Delta), \mathcal{B}_q], \mathcal{B}_q] U_q(s) ds dt. \quad (4.48)$$

Moreover,

$$[d\Gamma(-\Delta), \mathcal{B}_q] = N \int (-\Delta \tilde{\varphi}_B)(y_1 - y_2) a_{y_1}^* a_{y_2}^* + \text{h.c.}, \quad (4.49)$$

and with Lemma 2.1

$$[[d\Gamma(-\Delta), \mathcal{B}_q], \mathcal{B}_q] = 2N^2 \langle -\Delta \tilde{\varphi}_B, \tilde{\varphi}_B \rangle + \underbrace{2 d\Gamma(N^2 \widehat{\tilde{\varphi}}_B(-i\nabla)(-\Delta \widehat{\tilde{\varphi}}_B(-i\nabla))}_{=\mathcal{O}(\mathcal{N}_+ N^{-2\alpha})}, \quad (4.50)$$

where we used that $\|\widehat{\tilde{\varphi}}_B\|_\infty \lesssim N^{-1-2\alpha}$ and $\|p^2 \widehat{\tilde{\varphi}}_B\|_\infty \lesssim N^{-1}$ from Lemma 2.1. Plugging this into (4.48) and using that $U_q^*(s) \mathcal{N}_+ U_q(s) = \mathcal{O}(\mathcal{N}_+ + 1)$ proves the claim (note that we obtain a factor 1/2 from the double integral in (4.48)). \square

Lemma 4.8. *As quadratic forms on \mathcal{H}_+ , we have for $\alpha > 0$*

$$U_q^* \mathcal{H}_1 U_q = -\Delta_x + \sqrt{\left(1 - \frac{U_q^* \mathcal{N}_+ U_q}{N}\right)_+} a(\sqrt{N} W_{N,x}) + \text{h.c.} + d\Gamma(W_{N,x}) + \mathcal{E}_1^{(U_q)}$$

where

$$\pm \mathcal{E}_1^{(U_q)} \lesssim N^{-\alpha/2} \left(N^{-1/2} d\Gamma(W_{N,x}) + (\mathcal{N}_+ + 1) \right).$$

Proof. First note that $U_q^*(-\Delta_x)U_q = -\Delta_x$, since U_q does not depend on the variable x . Let us now turn to Q_1 . From Duhamel's formula we obtain

$$U_q^* Q_1 U_q = \sqrt{\left(1 - \frac{U_q^* \mathcal{N}_+ U_q}{N}\right)_+} \left(a(\sqrt{N} W_{N,x}) + \int_0^1 U_q^*(s) [a(\sqrt{N} W_{N,x}), \mathcal{B}_q] U_q(s) ds \right) + \text{h.c.} \quad (4.51)$$

We have (using that V, φ_B are even)

$$\begin{aligned} [a(\sqrt{N} W_{N,x}), \mathcal{B}_q] &= \int \sqrt{N} W_N(x-y) [a_y, \mathcal{B}_q] dy \\ &= N^{3/2} a^*(\tilde{\varphi}_B * W_{N,x}) = \mathcal{O}\left(\mathcal{N}_+^{1/2} N^{3/2} \|\tilde{\varphi}_B\|_2 \|W_N\|_1\right). \end{aligned} \quad (4.52)$$

With $U_q^*(s)\mathcal{N}_+U_q(s) \lesssim (\mathcal{N}_+ + 1)$ and Lemma 2.1 we thus obtain

$$U_q^*Q_1U_q = \sqrt{\left(1 - \frac{U_q^*\mathcal{N}_+U_q}{N}\right)_+} a(\sqrt{N}W_{N,x}) + \text{h.c.} + \mathcal{O}(\mathcal{N}_+^{1/2}N^{-\alpha/2}). \quad (4.53)$$

Let us now turn to $d\Gamma(W_{N,x})$. Using the Duhamel formula again, we obtain for $t \in [-1, 1]$

$$U_q^*(t)d\Gamma(W_{N,x})U_q(t) = d\Gamma(W_{N,x}) + \int_0^t U_q^*(s)[d\Gamma(W_{N,x}), \mathcal{B}_q]U_q(s)ds. \quad (4.54)$$

Then, using the Cauchy-Schwarz inequality as in (4.22),

$$\begin{aligned} [d\Gamma(W_{N,x}), \mathcal{B}_q] &= N \int W_N(x - y_1)\tilde{\varphi}_B(y_1 - y_2)a_{y_1}^*a_{y_2}^* + \text{h.c.} \\ &= \mathcal{O}\left(N^{-\alpha/2-1/2}d\Gamma(W_{N,x}) + N^{-\alpha/2}(\mathcal{N}_+ + 1)\right). \end{aligned} \quad (4.55)$$

From (4.54), the above estimate, Lemma 4.4 and the Grönwall lemma, we first deduce the a priori estimate $U_q^*(t)d\Gamma(W_{N,x})U_q(t) \lesssim (d\Gamma(W_{N,x}) + \mathcal{N}_+ + 1)$, for all $t \in [-1, 1]$. Taking $t = 1$ in (4.54), we then obtain

$$U_q^*d\Gamma(W_{N,x})U_q = d\Gamma(W_{N,x}) + \mathcal{O}\left(N^{-\alpha/2-1/2}d\Gamma(W_{N,x}) + N^{-\alpha/2}(\mathcal{N}_+ + 1)\right). \quad (4.56)$$

This proves the claim. \square

Lemma 4.9. *For $\alpha > 0$, we have as quadratic forms on \mathcal{H}_+*

$$U_q^*\mathcal{L}_3U_q = \frac{1}{2} \int \sqrt{N}V_N(y_1 - y_2)a_{y_1}^*a_{y_2}a_{y_1} + \text{h.c.} + \mathcal{E}_{\mathcal{L}_3}^{(U_q)},$$

with, for all $N^{-\alpha/2} \leq \delta < 1$,

$$\pm \mathcal{E}_{\mathcal{L}_3}^{(U_q)} \lesssim N^{-\alpha/2}\mathcal{N}_+ + \delta^{-1}N^{-1}(\mathcal{N}_+ + 1)^2 + \delta\mathcal{L}_4.$$

Proof. Let us denote $\chi(\mathcal{N}_+) = \sqrt{\left(1 - \frac{\mathcal{N}_+}{N}\right)_+}$. From Duhamel's formula, we obtain

$$\begin{aligned} U_q^*\mathcal{L}_3U_q &= \chi(U_q^*\mathcal{N}_+U_q) \int \sqrt{N}V_N(y_1 - y_2)a_{y_1}^*a_{y_2}a_{y_1} + \text{h.c.} \\ &\quad + \chi(U_q^*\mathcal{N}_+U_q) \int_0^1 U_q^*(s) \int \sqrt{N}V_N(y_1 - y_2)[a_{y_1}^*a_{y_2}a_{y_1}, \mathcal{B}_q]U_q(s)ds + \text{h.c.} \end{aligned} \quad (4.57)$$

To obtain $\mathcal{L}_3^{(U_q)}$, we need to extract the main contribution from the first term. With $|\sqrt{1-x}-1| \leq x$ and $U_q^*\mathcal{N}_+U_q \lesssim (\mathcal{N}_+ + 1)$, we find

$$|1 - \chi(U_q^*\mathcal{N}_+U_q)| \lesssim N^{-1}(\mathcal{N}_+ + 1). \quad (4.58)$$

This allows us to bound for $\delta > 0$, as in (4.22),

$$\begin{aligned}
& \pm \left\{ \left(1 - \chi(U_q^* \mathcal{N}_+ U_q)\right) \int \sqrt{N} V_N(y_1 - y_2) a_{y_1}^* a_{y_2} a_{y_1} + \text{h.c.} \right\} \\
& \lesssim \delta \mathcal{L}_4 + \delta^{-1} \left(1 - \chi(U_q^* \mathcal{N}_+ U_q)\right)^2 \|V\|_1 \mathcal{N}_+ \\
& \lesssim \delta \mathcal{L}_4 + \delta^{-1} N^{-1} \mathcal{N}_+^2.
\end{aligned} \tag{4.59}$$

To bound the commutator terms, one proceeds as in the proof of Lemma 4.5. That is, one calculates the commutator, noticing that the terms linear in a^* , a vanish on \mathcal{H}_+ , and then inserts an additional factor $U_q^*(s)U_q(s)$ after a pair of creation/annihilation operators to apply Duhamel's formula once more. The details, with slightly different conventions, can be found in [40, Lem. 6]. \square

4.3 High-energy boson-impurity interactions: a Weyl transformation

As a next step we apply an x -dependent Weyl transformation to the Hamiltonian. Its role is very similar to the one of U_q used in the previous section. This transformation corrects the scalar $N\widehat{W}(0)$ from the excitation Hamiltonian to make the scattering length \mathfrak{a}_W appear. Additionally, it renormalizes the high momentum part of the interaction term Q_1 by transforming it into a less singular interaction.

Let $\tilde{\varphi}_I$ be the solution to the approximate impurity-boson scattering equation (2.19) cut-off at momentum N^α , for the same $\alpha \geq 0$ as in the previous section (cf. Equation (2.20)). Define the unitary

$$U_W(t) = \exp\left(t \int \sqrt{N} \tilde{\varphi}_I(x-y) a_y^* dy - \text{h.c.}\right), \quad U_W = U_W(1), \tag{4.60}$$

whose generator is clearly (essentially) anti-self-adjoint on $D(\mathcal{N}_+^{1/2})$. This transformation is very similar to the transformation U_κ^Λ from Section 2.3 with $\kappa = N^\alpha$, except that the definition of $\tilde{\varphi}_I$ also takes the interaction term $d\Gamma(W_{N,x})$ into account. Note that the generator is linear in a , a^* , but if we were to include creation and annihilation operators b^* , b , for the impurities, the kernel would be $\sqrt{N} \tilde{\varphi}_I(x-y) b_x^* b_x (a_y^* - a_y)$, which is cubic in total. The transformation satisfies the identities

$$U_W^* a_y U_W = a_y + \sqrt{N} \tilde{\varphi}_I(y-x), \tag{4.61}$$

and

$$\begin{aligned}
U_W^* \mathcal{N}_+ U_W &= \mathcal{N}_+ + (a(\sqrt{N} \tilde{\varphi}_{I,x}) + \text{h.c.}) + \|\sqrt{N} \tilde{\varphi}_I\|_2^2 \\
&= \mathcal{N}_+ + \mathcal{O}(N^{-\alpha/2}(\mathcal{N}_+ + 1)).
\end{aligned} \tag{4.62}$$

The result of transforming the Hamiltonian with U_W is given below.

Proposition 4.10. *Let $0 < \alpha \leq 1/6$, \mathbf{a}_W be the scattering length of W defined by (1.2), and recall the Hamiltonian $U_q^* \mathcal{H}_N U_q$ from Proposition 4.2. Then, we have as quadratic forms on \mathcal{H}_+*

$$U_W^* U_q^* \mathcal{H}_N U_q U_W = 4\pi \mathbf{a}_V (N-1) + 8\pi \mathbf{a}_W \sqrt{N} + e_N^{(U_W)} + \mathcal{H}_{\text{BB}}^{(U_W)} + \mathcal{H}_{\text{IB}}^{(U_W)} + \mathcal{E}^{(U_W)},$$

where

$$\mathcal{H}_{\text{BB}}^{(U_W)} = d\Gamma(-\Delta + 2\widehat{V}(0) - 8\pi \mathbf{a}_V) + \widetilde{\mathcal{L}}_2 + \mathcal{L}_3^{U_q} + \mathcal{L}_4,$$

and (with the notation $f_x(y) = f(x-y)$)

$$\begin{aligned} \mathcal{H}_{\text{IB}}^{(U_W)} &= -\Delta_x + \left(a^*(\widetilde{W}_{N,x}) + \text{h.c.} \right) + \left(a^*(\sqrt{N} i \nabla \widetilde{\varphi}_I)_x \right)^2 + \text{h.c.} + d\Gamma(W_{N,x}) \\ &\quad - 2a^*(\sqrt{N} i \nabla \widetilde{\varphi}_I)_x i \nabla_x + \text{h.c.} + 2a^*(\sqrt{N} i \nabla \widetilde{\varphi}_I)_x a(\sqrt{N} i \nabla \widetilde{\varphi}_I)_x, \end{aligned}$$

$$\widehat{W}_N(p) = 8\pi \mathbf{a}_W \mathbf{1}_{0 < |p| \leq N^\alpha},$$

$$e_N^{(U_W)} = 4\pi N(\widetilde{\mathbf{a}}_{V_N} - \mathbf{a}_V) + 8\pi \sqrt{N}(\widetilde{\mathbf{a}}_{W_N} - \mathbf{a}_W) + \sum_{\substack{p \in 2\pi\mathbb{Z}^3 \\ 0 < |p| \leq N^\alpha}} \frac{(4\pi \mathbf{a}_V)^2 + 2(4\pi \mathbf{a}_W)^2}{p^2},$$

and the error term $\mathcal{E}^{(U_W)}$ satisfies

$$\pm \mathcal{E}^{(U_W)} \lesssim N^{-\alpha/2} (1 + \mathcal{N}_+ + \mathcal{L}_4 + d\Gamma(W_{N,x})) + N^{-1+\alpha/2} \mathcal{N}_+^2 + N^{-1/4} d\Gamma(|i\nabla|).$$

Proof. Let us rewrite the result of Proposition 4.2 as

$$U_q^* \mathcal{H}_N U_q = 4\pi \mathbf{a}_V (N-1) + e_N^{(U_q)} + \mathcal{H}_{\text{IB}} + \mathcal{H}_{\text{BB}} + \mathcal{E}^{(U_q)}, \quad (4.63)$$

with

$$\begin{aligned} \mathcal{H}_{\text{IB}} &= \sqrt{N} \widehat{W}(0) - \Delta_x + d\Gamma(-\Delta) + Q_1^{U_q} + d\Gamma(W_{N,x}), \\ \mathcal{H}_{\text{BB}} &= d\Gamma(N \widehat{V}_N(-i\nabla) + \widehat{V}(0) - 8\pi \mathbf{a}_V) + \widetilde{\mathcal{L}}_2 + \mathcal{L}_3^{U_q} + \mathcal{L}_4. \end{aligned} \quad (4.64)$$

In Lemma 4.11 below, we prove that \mathcal{H}_{BB} remains essentially unchanged by the transformation U_W , except that we will be able to simplify $N \widehat{V}_N(-i\nabla)$ to $\widehat{V}(0)$. The relevant changes thus occur in the part of the Hamiltonian describing non-interacting bosons coupled to the impurity, that is in \mathcal{H}_{IB} . Let us therefore consider $U_W^* \mathcal{H}_{\text{IB}} U_W$.

Estimating $U_W^ Q_1^{U_q} U_W$.* The action of U_W on $Q_1^{U_q}$ is easily evaluated using the identity (4.61):

$$U_W^* Q_1^{U_q} U_W = \sqrt{\left(1 - \frac{U_W^* U_q^* \mathcal{N}_+ U_q U_W}{N} \right)_+} \left(a(\sqrt{N} W_{N,x}) + N \langle W_N, \widetilde{\varphi}_I \rangle \right) + \text{h.c.} \quad (4.65)$$

By Equation (4.62) and Lemma 4.4, we have for all Ψ

$$\begin{aligned} & \left\| \left(1 - \sqrt{\left(1 - \frac{U_W^* U_q^* \mathcal{N}_+ U_q U_W}{N} \right)_+} \right) \Psi \right\| \\ & \lesssim \|N^{-1} U_W^* U_q^* \mathcal{N}_+ U_q U_W \Psi\| \lesssim \|N^{-1} (\mathcal{N}_+ + 1) \Psi\|. \end{aligned} \quad (4.66)$$

By the Cauchy-Schwarz inequality,

$$2 \left| \langle \Phi, a(\sqrt{N} W_{N,x}) \Psi \rangle \right| \leq \underbrace{\left(\int W_N(x-y) \|a_y \Psi\|^2 \right)^{1/2}}_{= \langle \Psi, d\Gamma(W_{N,x}) \Psi \rangle} \underbrace{\left(\int N W_N \right)^{1/2}}_{= \sqrt{N} \widehat{W}(0)} \|\Phi\|. \quad (4.67)$$

Thus Lemma 2.2 with (2.5) and using that $N \langle W_N, \tilde{\varphi}_I \rangle = \mathcal{O}(N^{1/2})$ by Lemma 2.2 yield

$$\begin{aligned} U_W^* Q_1^{U_q} U_W &= 2N \operatorname{Re} \langle W_N, \tilde{\varphi}_I \rangle + (a(\sqrt{N} W_{N,x}) + \text{h.c.}) \\ &+ \mathcal{O}(N^{-1/2} d\Gamma(W_{N,x}) + N^{-1/2} (\mathcal{N}_+ + 1) + N^{-1} (\mathcal{N}_+ + 1)^2), \end{aligned} \quad (4.68)$$

where the last line can be absorbed in $\mathcal{E}(U_W)$.

Estimating $U_W^ (d\Gamma(-\Delta) + d\Gamma(W_{N,x})) U_W$.* With the shift property (4.61), we find

$$U_W^* d\Gamma(W_{N,x}) U_W = d\Gamma(W_{N,x}) + (a(\sqrt{N} (W_N \tilde{\varphi}_I)_x) + \text{h.c.}) + N \langle \tilde{\varphi}_I, W_N \tilde{\varphi}_I \rangle, \quad (4.69a)$$

$$U_W^* d\Gamma(-\Delta) U_W = d\Gamma(-\Delta) - (a(\sqrt{N} (\Delta \tilde{\varphi}_I)_x) + \text{h.c.}) - N \langle \tilde{\varphi}_I, \Delta \tilde{\varphi}_I \rangle. \quad (4.69b)$$

We keep these terms to combine them with those coming from Q_1 and $-\Delta_x$ later. We also note that, applying the Cauchy-Schwarz inequality as in (4.67), we have the a priori estimate

$$U_W^* d\Gamma(W_{N,x}) U_W \lesssim d\Gamma(W_{N,x}) + N \langle \tilde{\varphi}_I, W_N \tilde{\varphi}_I \rangle \lesssim d\Gamma(W_{N,x}) + \sqrt{N}. \quad (4.70)$$

Estimating $U_W^ \Delta_x U_W$.* As the generator of the transformation U_W depends on the position x of the particle, $-\Delta_x$ also transforms non-trivially. Using that $\nabla_x \tilde{\varphi}_{I,x} = -(\nabla \tilde{\varphi}_I)_x$, the shift property (4.61), and that $\tilde{\varphi}_I$ is real, we find

$$\begin{aligned} U_W^* (-i\nabla_x) U_W &= -i\nabla_x + \int_0^1 ds U_W^*(s) [-i\nabla_x, a^*(\sqrt{N} \tilde{\varphi}_{I,x}) - \text{h.c.}] U_W(s) \\ &= -i\nabla_x - (a^*(\sqrt{N} (-i\nabla \tilde{\varphi}_I)_x) + \text{h.c.}), \end{aligned} \quad (4.71)$$

and thus

$$\begin{aligned}
U_W^*(-\Delta_x)U_W &= \left(-i\nabla_x - (a^*(\sqrt{N}(-i\nabla\tilde{\varphi}_I)_x) + \text{h.c.}) \right)^2 \\
&= -\Delta - a^*(\sqrt{N}(i\nabla\tilde{\varphi}_I)_x)2i\nabla_x + \text{h.c.} \\
&\quad + 2a^*(\sqrt{N}(-i\nabla\tilde{\varphi}_I)_x)a(\sqrt{N}(-i\nabla\tilde{\varphi}_I)_x) + a^*(\sqrt{N}(-i\nabla\tilde{\varphi}_I)_x)^2 + \text{h.c.} \\
&\quad + a^*(\sqrt{N}(-\Delta\tilde{\varphi}_I)_x) + \text{h.c.} + N\langle\tilde{\varphi}_I, (-\Delta)\tilde{\varphi}_I\rangle. \tag{4.72}
\end{aligned}$$

Combining the scalar and linear terms. We now combine the scalars and terms linear in a, a^* from these equations, which add up in a non-trivial way. The remaining errors will be estimated at the end. Together with $\sqrt{N}\widehat{W}(0)$, the scalar terms from (4.68), (4.69a), (4.69b) add up to

$$\sqrt{N}\widehat{W}(0) + 2N\text{Re}\langle W_N, \tilde{\varphi}_I \rangle + N\langle (-2\Delta + W_N)\tilde{\varphi}_I, \tilde{\varphi}_I \rangle. \tag{4.73}$$

Similarly as in (4.18), using the scattering equation (2.19), the definition of $\tilde{\mathbf{a}}_{W_N}$ in (2.21) and that $\tilde{\mathbf{a}}_{W_N} = \mathbf{a}_W + \mathcal{O}(N^{-1/2})$ from Lemma 2.3, we obtain

$$\begin{aligned}
&\sqrt{N}8\pi\tilde{\mathbf{a}}_{W_N} + \langle W_N(1 + \tilde{\varphi}_I), \tilde{\varphi}_I - \varphi_I \rangle \\
&= \sqrt{N}8\pi\tilde{\mathbf{a}}_{W_N} + 2 \sum_{|p| < N^\alpha} \frac{(4\pi\mathbf{a}_W)^2}{p^2} + \mathcal{O}(N^{2\alpha-1/2}). \tag{4.74}
\end{aligned}$$

Gathering the terms linear in a, a^* from (4.68), (4.69a), (4.72) and (4.69b) yields

$$a\left(\sqrt{N}\underbrace{((-2\Delta + W_N)\tilde{\varphi}_I + W_N)_x}_{=:f}\right) + \text{h.c.} \tag{4.75}$$

Using the scattering equation (2.19) for φ_I , we see that the Fourier transform of the function f above equals

$$\begin{aligned}
\widehat{f}(p) &= -p^2\widehat{\varphi}_I(p)\mathbb{1}_{0 < |p| \leq N^\alpha} - W_N(\widehat{\varphi_I - \tilde{\varphi}_I})(p) \\
&= W_N(\widehat{1 + \varphi_I})(p)\mathbb{1}_{0 < |p| \leq N^\alpha} - W_N(\widehat{\varphi_I - \tilde{\varphi}_I})(p). \tag{4.76}
\end{aligned}$$

By the same reasoning as (4.20) and using that $\tilde{\mathbf{a}}_{W_N} = \mathbf{a}_W + \mathcal{O}(N^{-1/2})$ from Lemma 2.3, we have with $\alpha \leq 1/6$

$$\|(W_N(\widehat{1 + \varphi_I})(p) - 8\pi\mathbf{a}_W)\mathbb{1}_{|p| \leq N^\alpha}\|_2 \lesssim N^{5\alpha/2-1} + N^{3\alpha/2-1/2} \leq N^{-\alpha/2}. \tag{4.77}$$

Moreover, using the Cauchy-Schwarz inequality as in (4.67) and Lemma 2.2, we have

$$\pm(a(\sqrt{N}W_{N,x}(\varphi_I - \tilde{\varphi}_I)_x) + \text{h.c.}) \lesssim \delta d\Gamma(W_{N,x}) + \delta^{-1}N^{-1/2+2\alpha}. \tag{4.78}$$

Choosing $\delta = N^{-\alpha/2}$ and using that $\alpha < 1/4$, this yields

$$a(\sqrt{N}f_x) + \text{h.c} = a(\widetilde{W}_{N,x}) + \text{h.c} + \mathcal{O}(N^{-\alpha/2}(1 + \text{d}\Gamma(W_{N,x}))), \quad (4.79)$$

with the modified interaction \widetilde{W}_N . This proves that

$$\begin{aligned} U_W^* \mathcal{H}_{\text{IB}} U_W &= 8\pi \widetilde{\mathbf{a}}_{W_N} \sqrt{N} + 2 \sum_{|p| \leq N^\alpha} \frac{(4\pi \mathbf{a}_W)^2}{p^2} + \text{d}\Gamma(-\Delta) + \mathcal{H}_1^{(U_W)} \\ &\quad + \mathcal{O}(N^{-\alpha/2}(\text{d}\Gamma(W_{N,x}) + \mathcal{N}_+ + 1) + N^{-1}(\mathcal{N}_+ + 1)^2). \end{aligned} \quad (4.80)$$

Estimating $U_W^ \mathcal{E}^{(U_q)} U_W$.* It remains to track the behavior of the error terms $\mathcal{E}^{(U_q)}$ under U_W . Using the a priori estimates from Lemma 4.11 together with (4.70) (note that $\text{d}\Gamma(W_{N,x})$ appears with an additional factor $N^{-1/2}$ in the error $\mathcal{E}^{(U_q)}$), we obtain that $U_W^* \mathcal{E}^{(U_q)} U_W$ can be absorbed in $\mathcal{E}^{(U_W)}$. This completes the proof. \square

Lemma 4.11. *For $\alpha > 0$, we have as quadratic forms on \mathcal{H}_+*

$$U_W^* (\widetilde{\mathcal{L}}_2 + \text{d}\Gamma(N\widehat{V}_N(-i\nabla))) + \mathcal{L}_3^{U_q} + \mathcal{L}_4 U_W = \widetilde{\mathcal{L}}_2 + \widehat{V}(0)\mathcal{N}_+ + \mathcal{L}_3^{U_q} + \mathcal{L}_4 + \mathcal{E}_{\text{BB}}^{(U_W)},$$

with

$$\pm \mathcal{E}_{\text{BB}}^{(U_W)} \lesssim N^{-\alpha/2}(1 + \mathcal{N}_+ + \mathcal{L}_4) + N^{-1/3+\alpha/6} \text{d}\Gamma(|i\nabla|).$$

Moreover, we have for $t \geq 0$

$$U_W^* (\mathcal{N}_+ + 1)^t U_W \lesssim (\mathcal{N}_+ + 1)^t,$$

and

$$U_W^* \mathcal{L}_4 U_W \lesssim \mathcal{L}_4 + N^{-1/3+\alpha/6} \text{d}\Gamma(|i\nabla|) + N^{-\alpha/2}(\mathcal{N}_+ + 1).$$

Proof. Estimating $U_W^ (\mathcal{N}_+ + 1)^k U_W$.* For integer t , this bound follows directly from the identity (4.62) together with (2.5). The general case then follows by interpolation [65].

Estimating $U_W^ \widetilde{\mathcal{L}}_2 U_W$.* For the quadratic term $\widetilde{\mathcal{L}}_2$ we calculate using the shift property (4.61)

$$\begin{aligned} U_W^* \widetilde{\mathcal{L}}_2 U_W &= \widetilde{\mathcal{L}}_2 + 2 \int \sqrt{N} \widetilde{V}_N(y_1 - y_2) \widetilde{\varphi}_1(x - y_1) a_{y_2} + \text{h.c.} \\ &\quad + \underbrace{\int N \widetilde{V}_N(y_1 - y_2) \widetilde{\varphi}_1(y_1) \widetilde{\varphi}_1(y_2)}_{\lesssim \|\widehat{V}_N\|_\infty \|\sqrt{N} \widetilde{\varphi}_1\|_2^2 \lesssim N^{-\alpha}}. \end{aligned} \quad (4.81)$$

With $\|\tilde{V}_N * \sqrt{N}\tilde{\varphi}_I\|_2 \lesssim \|\widehat{\tilde{V}}_N\|_\infty \|\sqrt{N}\tilde{\varphi}_I\|_2 \lesssim N^{-\alpha/2}$ from Lemma 2.2, we thus find

$$U_W^* \tilde{\mathcal{L}}_2 U_W = \tilde{\mathcal{L}}_2 + \mathcal{O}(N^{-\alpha/2}(\mathcal{N}_+ + 1)). \quad (4.82)$$

Estimating $U_W^ d\Gamma(N\widehat{V}_N(-i\nabla))U_W$.* Similarly as above, we find

$$\begin{aligned} U_W^* d\Gamma(N\widehat{V}_N(-i\nabla))U_W &= d\Gamma(N\widehat{V}_N(-i\nabla)) + \left(a \underbrace{(N^{3/2}(\widehat{V}_N(-i\nabla)\tilde{\varphi}_I)_x)}_{\|\cdot\|_2 \lesssim N^{-\alpha/2}} \right) + \text{h.c.} \\ &\quad + \underbrace{N^2 \langle \tilde{\varphi}_I, \widehat{V}_N(-i\nabla)\tilde{\varphi}_I \rangle}_{\lesssim N^{-\alpha}}, \end{aligned} \quad (4.83)$$

and

$$\begin{aligned} d\Gamma(N\widehat{V}_N(-i\nabla)) &= \sum_{0 \neq p \in 2\pi\mathbb{Z}^3} \widehat{V}(p/N) a_p^* a_p = \widehat{V}(0) \mathcal{N}_+ + \underbrace{\sum_p \left(\widehat{V}(p/N) - \widehat{V}(0) \right) a_p^* a_p}_{\lesssim N^{-1} d\Gamma(|i\nabla|)}, \end{aligned} \quad (4.84)$$

where it is used that $|y|V \in L^1(\mathbb{R}^3)$, as in (4.20).

Estimating $U_W^ \tilde{\mathcal{L}}_4 U_W$.* For the quartic term we have

$$\begin{aligned} U_W^* \tilde{\mathcal{L}}_4 U_W - \mathcal{L}_4 &= \frac{1}{2} \int V_N(y_1 - y_2) U_W^* a_{y_1}^* a_{y_2}^* a_{y_1} a_{y_2} U_W - \mathcal{L}_4 \\ &= \int \sqrt{N} V_N(y_1 - y_2) \tilde{\varphi}_I(x - y_1) a_{y_1}^* a_{y_2}^* a_{y_1} + \text{h.c.} \end{aligned} \quad (4.85a)$$

$$+ \frac{1}{2} \int N V_N(y_1 - y_2) \tilde{\varphi}_I(x - y_1) \tilde{\varphi}_I(x - y_2) a_{y_1}^* a_{y_2}^* + \text{h.c.} \quad (4.85b)$$

$$+ \int N V_N(y_1 - y_2) \tilde{\varphi}_I(x - y_1) \tilde{\varphi}_I(x - y_2) a_{y_1}^* a_{y_2} \quad (4.85c)$$

$$+ d\Gamma(NV_N * |\tilde{\varphi}_{I,x}|^2) + \left(N^{3/2} a((V_N * |\tilde{\varphi}_{I,x}|^2) \tilde{\varphi}_{I,x}) + \text{h.c.} \right) \quad (4.85d)$$

$$+ \frac{1}{2} \int N^2 |\tilde{\varphi}_{I,x}(y_1)|^2 |\tilde{\varphi}_{I,x}(y_2)|^2 V_N(y_1 - y_2) \quad (4.85e)$$

To start with, the Hölder inequality yields for the scalar in the last line

$$0 \leq (4.85e) \lesssim \|NV_N\|_1 \|\sqrt{N}\tilde{\varphi}_I\|_2^2 \|\tilde{\varphi}_I\|_\infty^2 \lesssim N^{-\alpha}. \quad (4.86)$$

The second term in (4.85d) can easily be estimated using Lemma 2.2 by

$$\begin{aligned} &\pm \left(N^{3/2} a((V_N * |\tilde{\varphi}_{I,x}|^2) \tilde{\varphi}_{I,x}) + \text{h.c.} \right) \\ &\lesssim \|NV_N\|_1 \|\tilde{\varphi}_I\|_\infty^2 \|\sqrt{N}\tilde{\varphi}_I\|_2 (\mathcal{N}_+ + 1)^{1/2} \lesssim N^{-\alpha/2} (\mathcal{N}_+ + 1)^{1/2}. \end{aligned} \quad (4.87)$$

The first term in (4.85d) is non-negative and will also be used as a reference to bound other error terms later. By the Sobolev embedding of $H^{1/2}(\mathbb{T}^3)$ into L^3 , the norm of a multiplication operator from $H^{1/2}$ to $H^{-1/2}$ is controlled by the L^3 -norm of the multiplier. By Lemma 2.2 and Young's inequality we thus have the bound

$$\begin{aligned} |\langle \Psi, d\Gamma(NV_N * |\tilde{\varphi}_{1,x}|^2)\Psi \rangle| &\lesssim \|NV_N * |\tilde{\varphi}_1|^2\|_3 \langle \Psi, d\Gamma(|i\nabla|)\Psi \rangle \\ &\lesssim N^{-(1+\alpha)/3} \langle \Psi, d\Gamma(|i\nabla|)\Psi \rangle. \end{aligned} \quad (4.88)$$

We can now estimate the other error terms, starting with the cubic term (4.85a). Using the Cauchy-Schwarz inequality as in (4.22) we obtain

$$\pm(4.85a) \lesssim N^{-\alpha/2} \mathcal{L}_4 + N^{\alpha/2} d\Gamma(NV_N * |\tilde{\varphi}_{1,x}|^2). \quad (4.89)$$

For the quadratic term (4.85b), we have similarly by the Cauchy-Schwarz inequality

$$\pm(4.85b) \lesssim N^{-\alpha/2} \mathcal{L}_4 + N^{\alpha/2} (4.85e) \lesssim N^{-\alpha/2} (\mathcal{L}_4 + 1). \quad (4.90)$$

A similar reasoning gives

$$\pm(4.85c) \lesssim d\Gamma(NV_N * |\tilde{\varphi}_{1,x}|^2) \lesssim N^{-(1+\alpha)/3} d\Gamma(|i\nabla|). \quad (4.91)$$

We have thus obtained that

$$U_W^* \mathcal{L}_4 U_W = \mathcal{L}_4 + \mathcal{O}\left(N^{-\alpha/2} \mathcal{L}_4 + N^{-\alpha/2} (\mathcal{N}_+ + 1) + N^{-1/3+\alpha/6} d\Gamma(|i\nabla|)\right). \quad (4.92)$$

In particular, this proves the claim on $U_W^* \mathcal{L}_4 U_W$.

Estimating $U_W^ \mathcal{L}_3^{U_q} U_W$.* For the cubic term \mathcal{L}_3 we have

$$\begin{aligned} U_W^* \mathcal{L}_3^{U_q} U_W - \mathcal{L}_3^{U_q} &= \frac{1}{2} \int \sqrt{N} V_N(y_1 - y_2) U_W^* a_{y_1}^* a_{y_2}^* a_{y_1} U_W + \text{h.c.} - \mathcal{L}_3^{U_q} \\ &= d\Gamma(NV_N * \tilde{\varphi}_{1,x}) + \frac{1}{2} \int NV_N(y_1 - y_2) \tilde{\varphi}_{1,x}(y_1) a_{y_2}^* a_{y_1} + \text{h.c.} \end{aligned} \quad (4.93a)$$

$$+ \frac{1}{2} \int NV_N(y_1 - y_2) \tilde{\varphi}_1(x - y_1) a_{y_1}^* a_{y_2}^* + \text{h.c.} \quad (4.93b)$$

$$+ \frac{3}{2} a((N^{3/2} V_N * \tilde{\varphi}_{1,x}) \tilde{\varphi}_{1,x}) + \text{h.c.} \quad (4.93c)$$

$$+ \frac{1}{2} \int N^2 V_N(y_1 - y_2) \tilde{\varphi}_{1,x}(y_2) |\tilde{\varphi}_{1,x}(y_1)|^2. \quad (4.93d)$$

These terms are estimated by similar means as for \mathcal{L}_4 . Similarly to (4.88) and (4.91), we obtain

$$\pm(4.93a) \lesssim (\|NV_N * |\tilde{\varphi}_1|\|_3 + \|NV_N\|_1 \|\tilde{\varphi}_1\|_3) d\Gamma(|i\nabla|) \lesssim N^{-1/3-\alpha/3} d\Gamma(|i\nabla|). \quad (4.94)$$

For the quadratic terms in (4.93b), we have by the Cauchy-Schwarz inequality

$$\begin{aligned} \pm(4.93b) &\lesssim N^{-\alpha/2} \mathcal{L}_4 + N^{\alpha/2} N^2 \int V_N(y_1 - y_2) |\tilde{\varphi}_I(y_2)|^2 \\ &\lesssim N^{-\alpha/2} (\mathcal{L}_4 + 1). \end{aligned} \quad (4.95)$$

The linear terms in (4.93c) are estimated by

$$\pm(4.93c) \leq 2 \|(V_N * \tilde{\varphi}_I) \tilde{\varphi}_I\|_2 \mathcal{N}_+^{1/2} \lesssim N^{-\alpha/2} (\mathcal{N}_+ + 1). \quad (4.96)$$

Finally, the scalar term (4.93d) is bounded by the Hölder inequality and Lemma 2.2

$$|(4.93d)| \lesssim \|NV_N\|_1 \|\tilde{\varphi}_I\|_2 N \|\tilde{\varphi}_I\|_4^2 \lesssim N^{-\alpha}. \quad (4.97)$$

Together, these bounds show that

$$U_W^* \mathcal{L}_3^{U_q} U_W = \mathcal{L}_3^{U_q} + \mathcal{O}\left(N^{-\alpha/2} (\mathcal{L}_4 + \mathcal{N}_+ + 1) + N^{-1/3+\alpha/6} d\Gamma(|i\nabla|)\right). \quad (4.98)$$

This completes the proof of the Lemma. \square

4.4 Soft boson pairs: a cubic transformation

We now apply a transformation in the boson variables whose generator is cubic in creation and annihilation operators. This transformation eliminates the cubic term $\mathcal{L}_3^{U_q}$ while turning $d\Gamma(2\hat{V}(0) - 8\pi\mathbf{a}_V)$ into $d\Gamma(8\pi\mathbf{a}_V)$.

Such a renormalization has its roots in [71], where a cubic non-unitary transformation was used to derive the upper bound to the Lee-Huang-Yang formula. Later in [6, 5], a unitary version was implemented to prove optimal rate of condensation and derive the low-lying excitation spectrum in the Gross-Pitaevskii regime.

Its role is to account for the interactions of the so-called “soft pairs” of high energy modes, which, in contrast to hard pairs that annihilate to yield two particles in the condensate, only annihilate to give one particle in the condensate and one low energy mode. As before, we will distinguish between low and high energy modes with the cutoff N^α .

In [62], a cutoff in the number of excitations $\theta_M(\mathcal{N}_+) \simeq \mathbb{1}_{\mathcal{N}_+ \leq M}$ was introduced in the kernel of the cubic transformation. The idea is that if we know a priori a condensation rate on the low-lying eigenvectors $\langle \Psi, \mathcal{N}_+ \Psi \rangle \lesssim M_0 \ll N$, it is enough to renormalize the sectors with $\mathcal{N}_+ \leq M$ for $M \gg M_0$. The rate of condensation provided by Condition 1.1 means that we can work on sectors with $\mathcal{N}_+ \leq M$ for some $M \gg \sqrt{N}$.

Our implementation of the cubic transformation essentially follows [40], although an important difference is that we have to use $d\Gamma(|i\nabla|)$ instead of $d\Gamma(-\Delta)$ in the error

estimates, as the latter would interfere with the renormalization of the impurity-boson interaction explained in Section 3, because $Q(H_{\text{BF}}) \not\subseteq Q(d\Gamma(-\Delta))$.

Let $\tilde{\varphi}_{\text{B}}$ be the truncated scattering solution for the bosons (2.17), as in Section 4.2. Let $\theta : \mathbb{R}_+ \rightarrow [0, 1]$ be a smooth function satisfying $\theta(x) = 1$ for $|x| \leq 1/2$ and $\theta(x) = 0$ for $|x| \geq 1$, and set $\theta_M(x) = \theta(x/M)$. We define for $M \geq 1$

$$U_c(t) = e^{t\mathcal{B}_c}, \quad U_c = U_c(1), \quad (4.99)$$

with the generator

$$\mathcal{B}_c = \theta_M(\mathcal{N}_+) \mathcal{B}_c^+ - \mathcal{B}_c^- \theta_M(\mathcal{N}_+), \quad (4.100a)$$

$$\mathcal{B}_c^+ = \sum_{p,q \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \sqrt{N} \tilde{\varphi}_{\text{B}}(p) \mathbb{1}_{|q| \leq N^\alpha} a_{p+q}^* a_{-p}^* a_q, \quad \mathcal{B}_c^- = (\mathcal{B}_c^+)^*. \quad (4.100b)$$

Note that the sum is only over a finite set since $\tilde{\varphi}_{\text{B}}$ contains a cutoff. Due to the cutoff θ_M in \mathcal{N}_+ , \mathcal{B}_c is a bounded operator, since

$$\begin{aligned} |\langle \Phi, \mathcal{B}_c^- \Psi \rangle| &\leq \left(\sum_{p,q} N \tilde{\varphi}_{\text{B}}(p)^2 \|a_q(\mathcal{N}_+ + 1)^{-1/2} \Phi\|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{p,q} \|a_{p+q} a_{-p} (\mathcal{N}_+ + 2)^{1/2} \Psi\|^2 \right)^{1/2} \\ &\leq N^{-1/2} \|N \tilde{\varphi}_{\text{B}}\|_2 \|\Phi\| \|(\mathcal{N}_+ + 2)^{3/2} \Psi\|. \end{aligned} \quad (4.101)$$

Denoting by $\chi \in L^2(\mathbb{T}^3)$ the (even) function with Fourier coefficients $\hat{\chi}(p) = \mathbb{1}_{|p| \leq N^\alpha}$, we can also write

$$\mathcal{B}_c^+ = \sqrt{N} \int \chi(y_1 - y_3) \tilde{\varphi}_{\text{B}}(y_2 - y_3) a_{y_3}^* a_{y_2}^* a_{y_1}. \quad (4.102)$$

The main result of this section is the following proposition.

Proposition 4.12. *Let $0 < \alpha \leq 1/8$ and $M = N^{1/2+\alpha/2}$. Let \mathcal{H}_N be given by Proposition 4.1 and recall its transformation by $U_q U_W$ given in Proposition 4.10. We have as quadratic forms on \mathcal{H}_+*

$$\begin{aligned} &U_c^* U_W^* U_q^* \mathcal{H}_N U_q U_W U_c \\ &= 4\pi \mathbf{a}_V (N - 1) + 8\pi \mathbf{a}_W \sqrt{N} + e_N^{(U_W)} \\ &\quad - \Delta_x + \left(a(\tilde{W}_{N,x}) + \text{h.c.} \right) + \left(a^*(\sqrt{N}(-i\nabla \tilde{\varphi}_{\text{I}})_x)^2 + \text{h.c.} \right) + d\Gamma(W_{N,x}) \\ &\quad - a^*(\sqrt{N}(i\nabla \tilde{\varphi}_{\text{I}})_x) 2i\nabla_x + \text{h.c.} + 2a^*(\sqrt{N}(i\nabla \tilde{\varphi}_{\text{I}})_x) a(\sqrt{N}(i\nabla \tilde{\varphi}_{\text{I}})_x) \\ &\quad + d\Gamma(-\Delta + 8\pi \mathbf{a}_V) + \tilde{\mathcal{L}}_2 + \mathcal{L}_4 + \mathcal{E}^{(U_c)}, \end{aligned}$$

where $\mathcal{E}^{(U_c)}$ satisfies

$$\begin{aligned} \pm \mathcal{E}^{(U_c)} &\lesssim N^{-\alpha/2} d\Gamma(W_{N,x}) + N^{-\alpha/4} \mathcal{L}_4 + N^{-1/2-\alpha/4} \mathcal{N}_+^2 \\ &\quad + N^{-\alpha/4} ((\log N) d\Gamma(|i\nabla|) + 1 + (\mathcal{N}_+ + 1)^{1/2} |\nabla_x|). \end{aligned}$$

We first give the proof of Proposition 4.12 using the commutator bounds and the a priori estimates of Lemmas 4.13, 4.14, 4.15, 4.16, 4.17, 4.18, 4.19, 4.20, 4.21 and 4.22 below, which we then prove in the remaining part of this section.

Proof of Proposition 4.12. From Proposition 4.10, we have (with the definitions given there)

$$\begin{aligned} U_c^* U_W^* U_q^* \mathcal{H}_N U_q U_W U_c &= 4\pi \mathbf{a}_{V_N} (N-1) + e_N^{(U_W)} \\ &\quad + U_c^* \mathcal{H}_{\text{BB}}^{(U_W)} U_c + U_c^* \mathcal{H}_{\text{IB}}^{(U_W)} U_c + U_c^* \mathcal{E}^{(U_W)} U_c. \end{aligned} \quad (4.103)$$

Let us estimate the three terms above separately.

Estimate of $U_c^ \mathcal{H}_{\text{BB}}^{(U_W)} U_c$.* Recall that

$$\mathcal{H}_{\text{BB}}^{(U_W)} = d\Gamma(-\Delta + 2\widehat{V}(0) - 8\pi \mathbf{a}_V) + \widetilde{\mathcal{L}}_2 + \mathcal{L}_3^{U_q} + \mathcal{L}_4.$$

We first deal with the term $d\Gamma(-\Delta) + \mathcal{L}_3^{U_q} + \mathcal{L}_4$. With

$$U_c^* \mathcal{L}_3^{U_q} U_c - \int_0^1 U_c^*(s) \mathcal{L}_3^{U_q} U_c(s) ds = \int_0^1 \int_s^1 U_c^*(t) [\mathcal{L}_3^{U_q}, \mathcal{B}_c] U_c(t) dt ds \quad (4.104)$$

and Duhamel's formula, we may write

$$\begin{aligned} &U_c^* \left(d\Gamma(-\Delta) + \mathcal{L}_3^{U_q} + \mathcal{L}_4 \right) U_c \\ &= d\Gamma(-\Delta) + \mathcal{L}_4 + \int_0^1 U_c^*(s) \left\{ [d\Gamma(-\Delta) + \mathcal{L}_4, \mathcal{B}_c] + \mathcal{L}_3^{U_q} \right\} U_c(s) ds \end{aligned} \quad (4.105a)$$

$$+ \int_0^1 \int_s^1 U_c^*(t) [\mathcal{L}_3^{U_q}, \mathcal{B}_c] U_c(t) dt ds. \quad (4.105b)$$

With the cancellation showed in Lemma 4.15 below, the error bounds from Lemmas 4.13, 4.14 and the conservation estimates of Lemmas 4.21 and 4.22 we find,

$$(4.105a) = d\Gamma(-\Delta) + \mathcal{L}_4 + \mathcal{E} \quad (4.106)$$

with, using that $M = N^{1/2+\alpha/2}$, $\alpha \leq 1/8$ and $d\Gamma(|i\nabla|) \geq \mathcal{N}_+$ on \mathcal{H}_+ ,

$$\pm \mathcal{E} \lesssim N^{-\alpha/4} (\mathcal{L}_4 + d\Gamma(|i\nabla|) + 1) + N^{-1/2-\alpha/4} \mathcal{N}_+^2. \quad (4.107)$$

Using Lemma 4.17 for the commutator with $\mathcal{L}_3^{U_q}$ yields with $M = N^{1/2+\alpha/2}$ (note that the double integral gives a factor of 1/2)

$$(4.105b) = \int_0^1 \int_s^1 U_c^*(t) (4d\Gamma(8\pi\mathbf{a}_V - \widehat{V}(0)) + \mathcal{E}_{[\mathcal{L}_3, \mathcal{B}_c]}) U_c(t) dt ds \quad (4.108)$$

$$= 2d\Gamma(8\pi\mathbf{a}_V - \widehat{V}(0)) + \mathcal{O}(N^{-\alpha/2}\mathcal{L}_4 + N^{-\alpha/4}(d\Gamma(|i\nabla|) + 1) + N^{-1/2-\alpha/2}\mathcal{N}_+^2),$$

by the conservation estimates of Lemmas 4.21 and 4.22.

Together with the estimate on the transformation of $\widetilde{\mathcal{L}}_2$ from Lemma 4.16, this shows that

$$U_c^* \mathcal{H}_{\text{BB}}^{(U_W)} U_c = d\Gamma(-\Delta + 8\pi\mathbf{a}_V) + \widetilde{\mathcal{L}}_2 + \mathcal{L}_4 + \mathcal{E}_{\text{B}}^{(U_c)}, \quad (4.109)$$

where $\mathcal{E}_{\text{B}}^{(U_c)}$ satisfies the error bounds claimed for $\mathcal{E}^{(U_c)}$.

Estimate of $U_c^ \mathcal{H}_{\text{IB}}^{(U_W)} U_c$.* Recall that

$$\mathcal{H}_{\text{IB}}^{(U_W)} = -\Delta_x + \left(a^*(\widetilde{W}_{N,x}) + \text{h.c.} \right) + \left(a^*(\sqrt{N}(-i\nabla\tilde{\varphi}_{\text{I}})_x)^2 + \text{h.c.} \right) + d\Gamma(W_{N,x})$$

$$- 2a^*(\sqrt{N}(i\nabla\tilde{\varphi}_{\text{I}})_x)i\nabla_x + \text{h.c.} + 2a^*(\sqrt{N}(i\nabla\tilde{\varphi}_{\text{I}})_x)a(\sqrt{N}(i\nabla\tilde{\varphi}_{\text{I}})_x), \quad (4.110)$$

and note that $U_c^* \Delta_x U_c = \Delta_x$ since \mathcal{B}_c does not depend on x . By the Duhamel formula, we then have

$$U_c^* \mathcal{H}_{\text{IB}}^{(U_W)} U_c = \mathcal{H}_{\text{IB}}^{(U_W)} + \int_0^1 U_c^*(t) [\mathcal{H}_{\text{I}}^{(U_W)} + \Delta_x, \mathcal{B}_c] U_c(t) dt \quad (4.111)$$

The commutators of individual terms are estimated in Lemmas 4.18, 4.19, 4.20 below. Combining these bounds with the conservation estimates of Lemmas 4.21 and 4.22, we obtain, setting $M = N^{1/2+\alpha/2}$ and using that $\alpha \leq 1/8$

$$\pm \int_0^1 U_c^*(t) [\mathcal{H}_{\text{I}}^{(U_W)} + \Delta_x, \mathcal{B}_c] U_c(t) dt \quad (4.112)$$

$$\lesssim N^{-\alpha/2} d\Gamma(W_{N,x}) + N^{-\alpha/4} (\log N d\Gamma(|i\nabla|) + (\mathcal{N}_+ + 1)^{1/2} |\nabla_x| + 1),$$

which can be absorbed in the final error term $\mathcal{E}^{(U_c)}$.

Estimate of $U_c^ \mathcal{E}^{(U_W)} U_c$.* Recalling the bound on the error $\mathcal{E}^{(U_W)}$ from Proposition 4.10, the expression $U_c^* \mathcal{E}^{(U_W)} U_c$ can be absorbed in the error $\mathcal{E}^{(U_c)}$ by Lemmas 4.21 and 4.22. \square

Commutator estimates: Boson Hamiltonian. Here, we treat the commutators of \mathcal{B}_c with the different terms of the boson-Hamiltonian. To remove the cutoff $\theta_M(\mathcal{N}_+)$ from the leading terms, we will make use of the inequality

$$|\theta_M(t-1) - \theta_M(t)| \lesssim M^{-1} \mathbb{1}_{t \leq M+1}, \quad (4.113)$$

which follows from Taylor's theorem.

Lemma 4.13. *For all $\alpha > 0$, $M \geq 1$ we have*

$$[\mathrm{d}\Gamma(-\Delta), \mathcal{B}_c] = - \sum_{\substack{p, q \in 2\pi\mathbb{Z}^3 \setminus \{0\} \\ |q| \leq N^\alpha}} \sqrt{N} (\widehat{V}_N(p) + \widehat{V}_N * \widehat{\varphi}_B(p)) a_{p+q}^* a_{-p}^* a_q + \text{h.c.} + \mathcal{E}_{[\mathrm{d}\Gamma(-\Delta), \mathcal{B}_c]},$$

where the error satisfies uniformly in M

$$\begin{aligned} \pm \mathcal{E}_{[\mathrm{d}\Gamma(-\Delta), \mathcal{B}_c]} &\lesssim N^{-\alpha/4} \mathcal{L}_4 + N^{\alpha/4} M^{-1} (\mathcal{N}_+ + 1)^2 \\ &\quad + \sqrt{M} N^{\alpha/2-1/2} \log N \mathrm{d}\Gamma(|i\nabla|) + \sqrt{M} N^{3\alpha/2-1/2} (N_+ + 1). \end{aligned}$$

Proof. A simple computation gives

$$\begin{aligned} \theta_M(\mathcal{N}_+) [\mathrm{d}\Gamma(-\Delta), \mathcal{B}_c^+] \\ = \theta_M \sum_{p, q} \sqrt{N} (2p^2 + 2pq) \widehat{\varphi}_B(p) \mathbb{1}_{|q| \leq N^\alpha} a_{p+q}^* a_{-p}^* a_q. \end{aligned} \quad (4.114)$$

By the scattering equation (2.16) (note that $\widehat{\varphi}_B$ has no cutoff)

$$2p^2 \widehat{\varphi}_B(p) = -(\widehat{V}_N(p) + \widehat{V}_N * \widehat{\varphi}_B(p)) \mathbb{1}_{|p| > N^\alpha}, \quad (4.115)$$

so we need to remove the cutoff $\mathbb{1}_{|p| > N^\alpha}$ and θ_M from the term with p^2 in order to obtain the leading term from the claim. Using the Cauchy-Schwarz inequality as in (4.101), we find

$$\begin{aligned} &\left| \left\langle \Psi, \theta_M(\mathcal{N}_+) \sum_{|p|, |q| \leq N^\alpha} \sqrt{N} (\widehat{V}_N(p) + \widehat{V}_N * \widehat{\varphi}_B(p)) a_{p+q}^* a_{-p}^* a_q \Psi \right\rangle \right| \\ &\lesssim \sqrt{N} \|(\widehat{V}_N + \widehat{V}_N * \widehat{\varphi}_B) \mathbb{1}_{|p| \leq N^\alpha}\|_2 \|\mathcal{N}_+^{1/2} \Psi\| \|\theta_M(\mathcal{N}_+) \mathcal{N}_+ \Psi\| \\ &\lesssim \sqrt{M} N^{-1/2+3\alpha/2} \|\mathcal{N}_+^{1/2} \Psi\|^2, \end{aligned} \quad (4.116)$$

which can be absorbed in the error and allows us to drop $\mathbb{1}_{|p| > N^\alpha}$. To remove θ_M , we first write

$$\begin{aligned} &\sum_{p, q} \sqrt{N} (\widehat{V}_N(p) + \widehat{V}_N * \widehat{\varphi}_B(p)) \mathbb{1}_{|q| \leq N^\alpha} a_{p+q}^* a_{-p}^* a_q \\ &= \int \chi(y_3 - y_1) \sqrt{N} V_N(1 + \varphi_B)(y_2 - y_1) a_{y_1}^* a_{y_2}^* a_{y_3}, \end{aligned} \quad (4.117)$$

where $\chi \in L^2(\mathbb{T}^3)$ denotes the function with Fourier coefficients $\widehat{\chi}(p) = \mathbb{1}_{|p| \leq N^\alpha}$. We then use that $1 - \theta_M(\mathcal{N}_+) \lesssim M^{-1} \mathcal{N}_+$ together with the Cauchy-Schwarz inequality

(as in (4.22), (4.31)), to obtain for $\delta > 0$

$$\begin{aligned}
& \pm \left((1 - \theta_M(\mathcal{N}_+)) \sum_{\substack{p,q \\ |q| \leq N^\alpha}} \sqrt{N} (\widehat{V}_N(p) + \widehat{V}_N * \widehat{\varphi}_B(p)) a_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right) \\
& \lesssim \delta \int V_N(1 + \varphi_B)(y_2 - y_1) a_{y_1}^* a_{y_2}^* a_{y_1} a_{y_2} + \delta^{-1} M^{-1} \mathcal{N}_+ \int \chi * \chi(y - y') a_{y'}^* a_y \\
& \lesssim \delta \mathcal{L}_4 + \delta^{-1} M^{-1} \mathcal{N}_+^2. \tag{4.118}
\end{aligned}$$

Choosing $\delta = N^{-\alpha/4}$ here gives the required bound on this term. It remains to bound the term coming from the pq -part in (4.114). This satisfies, for all $\delta > 0$,

$$\begin{aligned}
& \pm \left(2\theta_M(\mathcal{N}_+) \sum_{p,q} p \cdot q \sqrt{N} \widehat{\varphi}_B(p) \mathbb{1}_{|q| \leq N^\alpha} a_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right) \\
& \lesssim \delta \theta_M^2 \sum_{p,q \leq N^\alpha} |p||q| a_{p+q}^* a_{-p}^* a_{-p} a_{p+q} + \delta^{-1} N \sum_{p,q \leq N^\alpha} |p||q| |\widehat{\varphi}_B(p)|^2 a_q^* a_q \\
& \lesssim \delta N^\alpha \theta_M^2 d\Gamma(|i\nabla|) \mathcal{N}_+ + \delta^{-1} N \|\nabla\|^{1/2} \|\widehat{\varphi}_B\|_2^2 d\Gamma(|i\nabla|). \tag{4.119}
\end{aligned}$$

Using that $\mathcal{N}_+ \theta_M^2 \leq M$, Lemma 2.1, and choosing $\delta = M^{-1/2} N^{-\alpha/2-1/2} \log N$ then completes the proof. \square

Lemma 4.14. *For all $\alpha > 0$, $M \geq 1$ we have as quadratic forms on \mathcal{H}_+*

$$[\mathcal{L}_4, \mathcal{B}_c] = \sum_{\substack{p,q \in 2\pi\mathbb{Z}^3 \setminus \{0\} \\ |q| \leq N^\alpha}} \sqrt{N} \widehat{V}_N * \widehat{\varphi}_B(p) a_{p+q}^* a_{-p}^* a_q + \text{h.c.} + \mathcal{E}_{[\mathcal{L}_4, \mathcal{B}_c]}$$

where the error satisfies uniformly in M and $\delta > 0$

$$\pm \mathcal{E}_{[\mathcal{L}_4, \mathcal{B}_c]} \lesssim \delta \mathcal{L}_4 + \delta^{-1} (MN^{-1} + 1) MN^{-1+3\alpha} (\mathcal{N}_+ + 1) + M^{-1} \delta^{-1} \mathcal{N}_+^2.$$

Proof. Following the proof of [40, Lemma 12] (with a_0 replaced by \sqrt{N} and using θ_M to bound any powers of \mathcal{N}_+ in excess of one), we arrive at

$$\begin{aligned}
& \theta_M(\mathcal{N}_+) [\mathcal{L}_4, \mathcal{B}_c^+] \\
& = \theta_M(\mathcal{N}_+) \sum_{r,p,q} \sqrt{N} \widehat{V}_N(p-r) \widehat{\varphi}_B(r) \mathbb{1}_{|q| \leq N^\alpha} a_{p+q}^* a_{-p}^* a_q + \mathcal{E} \tag{4.120}
\end{aligned}$$

with

$$\pm \mathcal{E} + \text{h.c.} \lesssim \delta \mathcal{L}_4 + \delta^{-1} (MN^{-1} + 1) MN^{-1+3\alpha} (\mathcal{N}_+ + 1). \tag{4.121}$$

By the bound (4.118), the error due to the removal of θ_M from the main term can be absorbed in $\mathcal{E}_{[\mathcal{L}_4, \mathcal{B}_c]}$. \square

Lemma 4.15. *For all $\alpha > 0$, $M \geq 1$ and with the definitions from Lemma 4.13, Lemma 4.14, we have as quadratic forms on \mathcal{H}_+*

$$[\mathrm{d}\Gamma(-\Delta) + \mathcal{L}_4, \mathcal{B}_c] + \mathcal{L}_3^{U_q} = \mathcal{E}_{[\mathrm{d}\Gamma(-\Delta), \mathcal{B}_c]} + \mathcal{E}_{[\mathcal{L}_4, \mathcal{B}_c]} + \mathcal{O}(N^{-\alpha/2} (\mathcal{L}_4 + \mathrm{d}\Gamma(|i\nabla|))).$$

Proof. Combining Lemma 4.13, Lemma 4.14 and the Fourier representation of $\mathcal{L}_3^{U_q}$, we obtain

$$\begin{aligned} [\mathrm{d}\Gamma(-\Delta) + \mathcal{L}_4, \mathcal{B}_c] + \mathcal{L}_3^{U_q} &= \sum_{p,q} \sqrt{N} \widehat{V}_N(p) \mathbb{1}_{|q| > N^\alpha} a_{p+q}^* a_{-p}^* a_q + \text{h.c.} \\ &\quad + \mathcal{E}_{[\mathrm{d}\Gamma(-\Delta), \mathcal{B}_c]} + \mathcal{E}_{[\mathcal{L}_4, \mathcal{B}_c]}. \end{aligned} \quad (4.122)$$

We have by the reasoning of (4.118)

$$\sum_{p,q} \sqrt{N} \widehat{V}_N(p) \mathbb{1}_{|q| > N^\alpha} a_{p+q}^* a_{-p}^* a_q + \text{h.c.} \lesssim \delta \mathcal{L}_4 + \delta^{-1} \sum_{|p| > N^\alpha} a_p^* a_p. \quad (4.123)$$

Clearly, the last term is bounded by $N^{-\alpha} \mathrm{d}\Gamma(|i\nabla|)$, so choosing $\delta = N^{-\alpha/2}$ proves the claim. \square

Lemma 4.16. *For all $\alpha > 0$ and uniformly in $M \geq 1$ we have as quadratic forms on \mathcal{H}_+*

$$[\widetilde{\mathcal{L}}_2, \mathcal{B}_c] = \mathcal{O}(\sqrt{M} N^{\alpha-1/2} (\mathcal{N}_+ + 1)).$$

Proof. From the definition of $\widetilde{\mathcal{L}}_2$ in Proposition 4.2 and using that $a_p \mathcal{N}_+ = (\mathcal{N}_+ + 1) a_p$, we have

$$[\widetilde{\mathcal{L}}_2, \theta_M(\mathcal{N}_+)] = (\theta_M(\mathcal{N}_+) - \theta_M(\mathcal{N}_+ + 2)) \sum_{|p| \leq N^\alpha} 4\pi \mathbf{a}_V a_p a_p + \text{h.c.} \quad (4.124)$$

Using the bound (4.113) on the difference for θ_M together with the bound (4.101) on \mathcal{B}_c^- , gives

$$\begin{aligned} |\langle \Psi, [\widetilde{\mathcal{L}}_2, \theta_M(\mathcal{N}_+)] \mathcal{B}_c^+ \Psi \rangle| &\lesssim N^{\alpha-\frac{1}{2}} M^{-1} \|\mathcal{N}_+ \mathbb{1}_{\mathcal{N}_+ \leq 2M} \Psi\| \|(\mathcal{N}_+ + 1)^{3/2} \mathbb{1}_{\mathcal{N}_+ \leq 2M} \Psi\| \\ &\lesssim \sqrt{M} N^{\alpha-\frac{1}{2}} \|(\mathcal{N}_+ + 1)^{1/2} \Psi\|^2. \end{aligned} \quad (4.125)$$

Due to the cut-off $\mathbb{1}_{|p| > N^\alpha}$ in the definitions of $\widehat{\varphi}_B$ and \mathcal{B}_c^+ , we have

$$\begin{aligned} [\widetilde{\mathcal{L}}_2, \mathcal{B}_c^+] &= 8\pi \mathbf{a}_V \sum_{p,q} \sqrt{N} \widehat{\varphi}_B(p) \mathbb{1}_{|q| \leq N^\alpha} a_{-p}^* a_{p+q}^* a_q^* \\ &\quad - 8\pi \mathbf{a}_V \sum_{p,q} \sqrt{N} \widehat{\varphi}_B(p) \mathbb{1}_{|p+q| \leq N^\alpha} \mathbb{1}_{|q| \leq N^\alpha} a_{-p}^* a_{-(p+q)} a_q. \end{aligned} \quad (4.126)$$

With the reasoning of (4.116) we get from this

$$\begin{aligned} |\langle \Psi, \theta_M(\mathcal{N}_+) [\tilde{\mathcal{L}}_2, \mathcal{B}_c^+] \Psi \rangle| &\lesssim \sqrt{N} \|\widehat{\varphi}_B\|_\infty N^{3\alpha/2} \|\mathcal{N}_+^{1/2} \Psi\| \|1_{\mathcal{N}_+ \leq 2M} \mathcal{N}_+ \Psi\| \\ &\lesssim \sqrt{M} N^{-1/2-\alpha/2} \|\mathcal{N}_+^{1/2} \Psi\|^2. \end{aligned} \quad (4.127)$$

Together with the bound on the commutator with θ_M this proves the claim. \square

Lemma 4.17. *For all $\alpha > 0$, $M \geq 1$ we have as quadratic forms on \mathcal{H}_+*

$$[\mathcal{L}_3^{U_q}, \mathcal{B}_c] = 4(8\pi\mathbf{a}_V - \widehat{V}(0))\mathcal{N}_+ + \mathcal{E}_{[\mathcal{L}_3, \mathcal{B}_c]},$$

where the error satisfies uniformly in M

$$\pm \mathcal{E}_{[\mathcal{L}_3, \mathcal{B}_c]} \lesssim N^{-\alpha/2} \mathcal{L}_4 + N^{-\alpha} d\Gamma(|i\nabla|) + MN^{-1+5\alpha/2}(\mathcal{N}_+ + 1) + M^{-1} \mathcal{N}_+^2.$$

Proof. The proof follows the computations carried out in [40, Lem.15], with the only difference being the cut-off θ_M that does not entirely commute with $\mathcal{L}_3^{U_q}$, which we deal with in the same way as in [62]. First, write $\mathcal{L}_3^{U_q} = \mathcal{L}_3^+ + \mathcal{L}_3^-$ with

$$\mathcal{L}_3^+ = \frac{1}{2} \int \sqrt{N} V_N(y_1 - y_2) a_{y_1}^* a_{y_2}^* a_{y_1} \quad (4.128)$$

and $\mathcal{L}_3^- = (\mathcal{L}_3^+)^*$. With this, we find

$$\begin{aligned} [\mathcal{L}_3^{U_q}, \theta_M(\mathcal{N}_+)] \mathcal{B}_c^+ &= [\mathcal{L}_3^+, \theta_M(\mathcal{N}_+)] \mathcal{B}_c^+ \\ &\quad + (\theta_M(\mathcal{N}_+ + 1) - \theta_M(\mathcal{N}_+)) (\mathcal{B}_c^+ \mathcal{L}_3^- + [\mathcal{L}_3^-, \mathcal{B}_c^+]). \end{aligned} \quad (4.129)$$

Using the bounds (4.113) on θ_M , (4.101) on \mathcal{B}_c^\pm and the Cauchy-Schwarz inequality (as in [62, Lem.26]), gives

$$\begin{aligned} &(\mathcal{L}_3^+ [\theta_M(\mathcal{N}_+), \mathcal{B}_c^+] + (\theta_M(\mathcal{N}_+ + 1) - \theta_M(\mathcal{N}_+)) \mathcal{B}_c^+ \mathcal{L}_3^- + \text{h.c.}) \\ &\lesssim N^{-\alpha/2} \mathcal{L}_4 + MN^{-1-\alpha/2} (\mathcal{N}_+ + 1). \end{aligned} \quad (4.130)$$

In view of (4.129), we thus have

$$\begin{aligned} [\mathcal{L}_3^{U_q}, \mathcal{B}_c] &= \theta_M(\mathcal{N}_+) [\mathcal{L}_3^+, \mathcal{B}_c^+] + \theta_M(\mathcal{N}_+ + 1) [\mathcal{L}_3^-, \mathcal{B}_c^+] + \text{h.c.} \\ &\quad + \mathcal{O}(N^{-\alpha/2} \mathcal{L}_4 + MN^{-1-\alpha/2} (\mathcal{N}_+ + 1)). \end{aligned} \quad (4.131)$$

The remaining relevant terms correspond to those denoted (I) and (II) in [40, Eq.(77)]. Following the proof there yields (replacing a_0^\dagger, a_0 by \sqrt{N} and thus ignoring their commutators)

$$\theta_M(\mathcal{N}_+) [\mathcal{L}_3^+, \mathcal{B}_c^+] = \mathcal{O}(N^{-\alpha/2} \mathcal{L}_4 + MN^{-1+\alpha} (\mathcal{N}_+ + 1)). \quad (4.132)$$

Now following the proof from [40, Eq.(79)] on, we find (note that the term $(\text{II})_b$ in our case is just equal to $(\text{II})_{b_1}$ with $a_0^\dagger a_0$ is replaced by N)

$$\begin{aligned} \theta_M(\mathcal{N}_+ + 1)[\mathcal{L}_3^-, \mathcal{B}_c^+] &= \theta_M(\mathcal{N}_+ + 1) \sum_q N(\widehat{V}_N * \widehat{\varphi}_B(0) + \widehat{V}_N * \widehat{\varphi}_B(q)) \mathbf{1}_{|q| \leq N^\alpha} a_q^* a_q \\ &\quad + \mathcal{O}(N^{-\alpha/2} \mathcal{L}_4 + MN^{-1+5\alpha/2}(\mathcal{N}_+ + 1)). \end{aligned} \quad (4.133)$$

As in (4.20), we have

$$N|\widehat{V}_N * \widehat{\varphi}_B(q) - \widehat{V}_N * \widehat{\varphi}_B(0)| \lesssim |q|N^{-1}, \quad (4.134)$$

and together $\|\widehat{\varphi}_B - \varphi_B\|_\infty \lesssim N^{-1+\alpha}$ from Lemma 2.1, we obtain for $|q| \leq N^\alpha$

$$N|\widehat{V}_N * \widehat{\varphi}_B(q) - N\widehat{V}_N * \widehat{\varphi}_B(0)| \lesssim N^{-1+\alpha}. \quad (4.135)$$

Using this, we have

$$\begin{aligned} \theta_M(\mathcal{N}_+ + 1) \sum_q N\widehat{V}_N * \widehat{\varphi}_B(q) \mathbf{1}_{|q| \leq N^\alpha} a_q^* a_q \\ = \theta_M(\mathcal{N}_+ + 1) \sum_q N\widehat{V}_N * \varphi_B(0) \mathbf{1}_{|q| \leq N^\alpha} a_q^* a_q + \mathcal{O}(N^{-1+\alpha} \mathcal{N}_+). \end{aligned} \quad (4.136)$$

By definition (2.18), $N\widehat{V}_N * \widehat{\varphi}_B(0) = 8\pi\widetilde{\mathbf{a}}_{V_N} - \widehat{V}(0)$, which is bounded in N . To remove the cutoffs θ_M and $\mathbf{1}_{|q| \leq N^\alpha}$, we use that

$$\sum_q N\widehat{V}_N * \widehat{\varphi}_B(0) \mathbf{1}_{|q| > N^\alpha} a_q^* a_q = \mathcal{O}(N^{-\alpha} d\Gamma(|i\nabla|)), \quad (4.137)$$

and

$$(1 - \theta_M(\mathcal{N}_+ + 1)) \sum_q N\widehat{V}_N * \widehat{\varphi}_B(0) a_q^* a_q = \mathcal{O}(M^{-1} \mathcal{N}_+^2). \quad (4.138)$$

Since $\widetilde{\mathbf{a}}_{V_N} = \mathbf{a}_V + \mathcal{O}(N^{-1})$ by Lemma 2.3 this proves the claim. \square

Commutator estimates: Interaction Hamiltonian. We now prove bounds on the commutators of \mathcal{B}_c with the terms in the interaction Hamiltonian.

Lemma 4.18. *For all $\alpha > 0$ and uniformly in $M \geq 1$ we have as quadratic forms on \mathcal{H}_+*

$$[a^*(\widetilde{W}_{N,x}), \mathcal{B}_c] + \text{h.c.} = \mathcal{O}(N^{\alpha-1/2}(\mathcal{N}_+ + 1)).$$

Proof. Recall that $\widehat{W}_N(p) = 8\pi\mathbf{a}_W\mathbf{1}_{|p|<N^\alpha}$. Since

$$[a^*(\widetilde{W}_{N,x}), \theta_M(\mathcal{N}_+)] = (\theta_M(\mathcal{N}_+ - 1) - \theta_M(\mathcal{N}_+))a^*(\widetilde{W}_{N,x}) \quad (4.139)$$

and with (4.113) as well as (4.101) we find

$$\begin{aligned} |\langle \Psi, [a^*(\widetilde{W}_{N,x}), \theta_M]\mathcal{B}_c^+\Psi \rangle| &\lesssim \|\widetilde{W}_N\|_2 \|\mathcal{N}_+^{1/2}\Psi\| M^{-1} \|\mathbf{1}_{\mathcal{N}_+\leq 2M}\mathcal{B}_c^+\Psi\| \\ &\lesssim N^{3\alpha/2} M^{-1} \|\sqrt{N}\widetilde{\varphi}_B\|_2 \|\mathcal{N}_+^{1/2}\Psi\| \|\mathbf{1}_{\mathcal{N}_+\leq 2M}(\mathcal{N}_+ + 1)^{3/2}\Psi\| \\ &\lesssim N^{-1/2+\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\Psi\|^2. \end{aligned} \quad (4.140)$$

For the commutators with \mathcal{B}_c^\pm , we have

$$\begin{aligned} \theta_M[a^*(\widetilde{W}_{N,x}), \mathcal{B}_c^+] + \text{h.c.} &= \sum_{p,q} \sqrt{N}\widetilde{\varphi}_B(p)\theta_M a_{p+q}^* a_{-p}^* \widehat{W}_{N,x}(q) + \text{h.c.} \\ &= \mathcal{O}(N^{-1/2} \|N\widetilde{\varphi}_B\|_2 \|\widetilde{W}_{N,x}\|_2 (\mathcal{N}_+ + 1)) \\ &= \mathcal{O}(N^{\alpha-1/2} (\mathcal{N}_+ + 1)). \end{aligned} \quad (4.141)$$

A similar bound holds for $\theta_M[a^*(\widetilde{W}_{N,x}), \mathcal{B}_c^-]$. Together, these inequalities imply the claim. \square

Lemma 4.19. *We have as quadratic forms on \mathcal{H}_+ for all $\alpha > 0$ and uniformly in $M \geq 1$*

$$\pm [d\Gamma(W_{N,x}), \mathcal{B}_c] \lesssim (M/N)^{1/2} N^{-1/4+\alpha} (d\Gamma(W_{N,x}) + \mathcal{N}_+).$$

Proof. First, note that $d\Gamma(W_{N,x})$ commutes with $\theta_M(\mathcal{N}_+)$. Using the representation (4.102) of \mathcal{B}_c^+ , we compute

$$\begin{aligned} [d\Gamma(W_{N,x}), \mathcal{B}_c^+] & \quad (4.142) \\ &= \int \sqrt{N}\widetilde{\varphi}_B(y_2 - y_3) (W_{N,x}(y_3) + W_{N,x}(y_2) - W_{N,x}(y_1)) \chi(y_1 - y_3) a_{y_3}^* a_{y_2}^* a_{y_1}. \end{aligned}$$

Using the Cauchy Schwarz inequality similarly to (4.67), we get for any $\delta > 0$ the bound

$$\begin{aligned} &\pm \left(\int \sqrt{N}\widetilde{\varphi}_B(y_2 - y_3) \chi(y_1 - y_3) W_{N,x}(y_1) a_{y_3}^* a_{y_2}^* a_{y_1} + \text{h.c.} \right) \\ &\leq \delta d\Gamma(W_{N,x}) + \delta^{-1} N \|W_N\|_1 \|\chi\|_2^2 \|\widetilde{\varphi}_B\|_2^2 \mathcal{N}_+ (\mathcal{N}_+ - 1). \end{aligned} \quad (4.143)$$

The other terms in (4.142) satisfy the same bound with \mathcal{N}_+^2 replacing $\mathcal{N}_+(\mathcal{N}_+ - 1)$. With $N \|W_N\|_1 \|\chi\|_2^2 \|\widetilde{\varphi}_B\|_2^2 \lesssim N^{-3/2+2\alpha}$ and choosing $\delta = \sqrt{MN}^{-3/4+\alpha}$, this yields

$$\begin{aligned} &\pm (\theta_M(\mathcal{N}_+) [d\Gamma(W_{N,x}), \mathcal{B}_c^+] + \text{h.c.}) \\ &\lesssim (M/N)^{1/2} N^{-1/4+\alpha} (d\Gamma(W_{N,x}) + \mathcal{N}_+) \end{aligned} \quad (4.144)$$

and proves the claim. \square

Lemma 4.20. Denoting $g = \sqrt{N}i\nabla\tilde{\varphi}_I$, we have that for all $0 < \alpha \leq 1/6$ and uniformly in $M \geq 1$, as quadratic forms on \mathcal{H}_+ ,

$$\pm([a^*(g_x)^2, \mathcal{B}_c] + \text{h.c.}) \lesssim \sqrt{MN}^{-\frac{1}{4}-\frac{\alpha}{2}} (\log Nd\Gamma(|i\nabla|) + 1), \quad (4.145a)$$

$$\pm([a^*(g_x)a(g_x), \mathcal{B}_c] + \text{h.c.}) \lesssim \sqrt{MN}^{-\frac{1}{4}-\frac{\alpha}{2}} (\log Nd\Gamma(|i\nabla|) + 1), \quad (4.145b)$$

$$\pm([a^*(g_x)i\nabla_x, \mathcal{B}_c] + \text{h.c.}) \lesssim \sqrt{MN}^{-\frac{1}{4}-\frac{\alpha}{2}} \left(d\Gamma(|i\nabla|) + (\mathcal{N}_+ + 1)^{1/2} |i\nabla_x| + 1 \right). \quad (4.145c)$$

Proof. Note that $\|g\|_2 \lesssim N^{1/4}$ by Lemma 2.2. We prove successively the three inequalities.

Proof of (4.145a). We start by observing that $[a^*(g_x), \mathcal{B}_c^+] = 0$ due to the cut-off $\mathbb{1}_{|q| \leq N^\alpha}$ in the definition of \mathcal{B}_c^+ and $\mathbb{1}_{|q| \leq N^\alpha} \hat{g}(q) = 0$. For the commutator with \mathcal{B}_c^- we find

$$[\mathcal{B}_c^-, a^*(g_x)^2] = 2 \sum_{p,q} \sqrt{N} \widehat{\varphi}_B(p) \mathbb{1}_{|q| \leq N^\alpha} \hat{g}_x(p+q) \hat{g}_x(-p) a_q^* \quad (4.146a)$$

$$+ 2 \sum_{p,q} \sqrt{N} \widehat{\varphi}_B(p) \mathbb{1}_{|q| \leq N^\alpha} \hat{g}_x(p+q) a^*(g_x) a_q^* a_{-p} \quad (4.146b)$$

$$+ 2 \sum_{p,q} \sqrt{N} \widehat{\varphi}_B(p) \mathbb{1}_{|q| \leq N^\alpha} \hat{g}_x(-p) a^*(g_x) a_q^* a_{p+q}. \quad (4.146c)$$

With $\|\hat{g} * \widehat{\varphi}_B\|_2 \leq \|\hat{g}\|_2^2 \|\widehat{\varphi}_B\|_2 \lesssim N^{-1/2-\alpha/2}$ by Lemmas 2.1, 2.2, we find

$$(4.146a) = \mathcal{O}(N^{-\alpha/2} (\mathcal{N}_+ + 1)^{1/2}). \quad (4.147)$$

To bound the other terms, we will use the inequality

$$\|a(\nabla f)\Psi\| \leq \| |\nabla|^{1/2} f \|_2 \| (d\Gamma(|i\nabla|) + 1)^{1/2} \Psi \|. \quad (4.148)$$

For the first one, this yields with $\delta > 0$

$$\pm((4.146b) + \text{h.c.}) \lesssim \delta \| |\nabla|^{1/2} \tilde{\varphi}_I \|_2^2 (d\Gamma(|i\nabla|) + 1) + \delta^{-1} \|g\|_2^2 \|\sqrt{N} \tilde{\varphi}_B\|_2^2 (\mathcal{N}_+ + 1)^2. \quad (4.149)$$

For the second one, we have similarly

$$\pm((4.146c) + \text{h.c.}) \lesssim \delta \| |\nabla|^{1/2} \tilde{\varphi}_I \|_2^2 d\Gamma(|i\nabla|) + \delta^{-1} N^{3\alpha} \|\hat{g}\|_\infty^2 \|\sqrt{N} \tilde{\varphi}_B\|_2^2 (\mathcal{N}_+ + 1)^2. \quad (4.150)$$

Using Lemma 2.2 and $\alpha \leq 1/6$, combining with $\theta_M(\mathcal{N}_+)$ and then choosing $\delta = \sqrt{MN}^{-1/4-\alpha/2}$, we find

$$\begin{aligned} \pm([\mathcal{B}_c^-, a^*(g_x)^2] \theta_M(\mathcal{N}_+) + \text{h.c.}) \\ \lesssim \sqrt{MN}^{-1/4-\alpha/2} (\log Nd\Gamma(|i\nabla|) + \mathcal{N}_+ + 1). \end{aligned} \quad (4.151)$$

It remains to estimate

$$[a^*(g_x)^2, \theta_M(\mathcal{N}_+)]\mathcal{B}_c^+ + \text{h.c.} = (\theta_M(\mathcal{N}_+) - \theta_M(\mathcal{N}_+ + 2))a^*(g_x)^2\mathcal{B}_c^+ + \text{h.c.} \quad (4.152)$$

With the bound (4.101) on \mathcal{B}_c^- and (4.148) we obtain

$$\begin{aligned} & |\langle \Psi, [\mathcal{B}_c^-, a^*(g_x)^2]\theta_M(\mathcal{N}_+)\Psi \rangle| \\ & \lesssim \|g\|_2 \| |\nabla|^{1/2} \sqrt{N} \tilde{\varphi}_1 \|_2 \| \mathcal{N}_+^{1/2} \Psi \| M^{-1} \| \sqrt{N} \tilde{\varphi}_B \|_2 \| \mathbf{1}_{\mathcal{N}_+ \leq 2M} (\mathcal{N}_+ + 2)^2 \Psi \| \\ & \lesssim \sqrt{M} N^{-1/4 - \alpha/2} \| (\mathcal{N}_+ + 1)^{1/2} \Psi \|, \end{aligned} \quad (4.153)$$

by Lemmas 2.1, 2.2. This proves the claim on the commutator with $a^*(g_x)^2$.

Proof of (4.145b). The argument for the commutator with $a^*(g_x)a(g_x)$ is similar to the bounds on (4.146b), (4.146c), as all other terms vanish in this case.

Proof of (4.145c). Let us now deal with the term involving $a^*(g_x)i\nabla_x$. We have

$$\begin{aligned} [a^*(g_x)i\nabla_x, \mathcal{B}_c] &= [a^*(g_x)i\nabla_x, \theta_M(\mathcal{N}_+)\mathcal{B}_c^+ - \mathcal{B}_c^- \theta_M(\mathcal{N}_+)] \\ &= (\theta_M(\mathcal{N}_+ - 1) - \theta_M(\mathcal{N}_+))a^*(g_x)i\nabla_x \mathcal{B}_c^+ \end{aligned} \quad (4.154a)$$

$$+ (\theta_M(\mathcal{N}_+ + 1) - \theta_M(\mathcal{N}_+))a^*(g_x)i\nabla_x \mathcal{B}_c^- \quad (4.154b)$$

$$+ \theta_M(\mathcal{N}_+ + 1) \sum_{p, q \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \sqrt{N} \widehat{\varphi}_B(p) \mathbf{1}_{|q| \leq N^\alpha} \left(a_{-p}^* \widehat{g}_x(p+q) + a_{p+q}^* \widehat{g}_x(p) \right) a_q i\nabla_x. \quad (4.154c)$$

We used here that ∇_x commutes with \mathcal{B}_c , and that $[a^*(g_x)i\nabla_x, \mathcal{B}_c^+] = 0$ because of the cutoffs in the definitions of g and \mathcal{B}_c^+ . To obtain the desired bound, we need distribute ∇_x to the left and right argument of the quadratic form, since otherwise we would obtain a bound by $-\Delta_x$ and not $|\nabla_x|$. To do this, we use that

$$a^*(g_x)|\nabla_x| = |\nabla_x|^{1/2} a^*(g_x) |\nabla_x|^{1/2} + [a^*(g_x), |\nabla_x|^{1/2}] |\nabla_x|^{1/2}. \quad (4.155)$$

With $i\nabla_x e^{-ikx} = e^{-ikx}(i\nabla_x + k)$ and $||p - k|^{1/2} - |p|^{1/2}| \leq |k|^{1/2}$, we get that

$$\| [|\nabla_x|^{1/2}, a(g_x)] \Psi \| \lesssim \sum_k |k|^{1/2} |\widehat{g}(k)| \| a_k \Psi \| \lesssim \|g\|_2 \| d\Gamma(|i\nabla|)^{1/2} \Psi \|. \quad (4.156)$$

Using this and the estimate on \mathcal{B}_c^- and \mathcal{B}_c^+ (4.101), we obtain

$$\begin{aligned} & |\langle \Psi, (4.154a)\Psi \rangle| \\ & \lesssim M^{1/2} N^{-1/2} \| N \tilde{\varphi}_B \|_2 \|g\|_2 \| (\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi \|^2 \\ & \quad + M^{1/4} N^{-1/2} \| N \tilde{\varphi}_B \|_2 \|g\|_2 \| d\Gamma(|i\nabla|)^{1/2} \Psi \| \| (\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi \| \\ & \lesssim M^{1/2} N^{-1/4 - \alpha/2} \| (\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi \|^2 \\ & \quad + M^{1/4} N^{-1/4 - \alpha/2} \| d\Gamma(|i\nabla|)^{1/2} \Psi \| \| (\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi \|, \end{aligned}$$

where we used that $|(\theta_M(\mathcal{N}_+ - 1) - \theta_M(\mathcal{N}_+))| \lesssim M^{-1} \mathbb{1}_{\mathcal{N}_+ \leq 2M}$ and $\|N\tilde{\varphi}_B\|_2 \lesssim N^{-\alpha/2}$. Similarly, one shows that (4.154b) satisfies the same bound. For the third term, we commute again $|\nabla_x|^{1/2}$, and a similar reasoning leads to

$$\begin{aligned} |\langle \Psi, (4.154c)\Psi \rangle| &\lesssim M^{1/2} N^{-1/2} \|N\tilde{\varphi}_B\|_2 \|g\|_2 \|(\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi\|^2 \quad (4.157) \\ &\quad + M^{1/4} N^{-1/2} \|N\tilde{\varphi}_B\|_2 \|\mathbb{1}_{|\cdot| \leq N^\alpha}\|_2 \|\widehat{|\nabla|^{1/2}} g\|_\infty \\ &\quad \times \|(\mathcal{N}_+ + 1)^{1/2} \Psi\| \|(\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi\| \\ &\lesssim M^{1/2} N^{-1/4 - \alpha/2} \|(\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi\|^2 \\ &\quad + M^{1/4} N^{-1/2 + \alpha/2} \|(\mathcal{N}_+ + 1)^{1/2} \Psi\| \|(\mathcal{N}_+ + 1)^{1/4} |\nabla_x|^{1/2} \Psi\|, \end{aligned}$$

where we used that $\|\widehat{|\nabla|^{1/2}} g\|_\infty \lesssim \sqrt{N} \|p^{3/2} \widehat{\varphi}_I\|_\infty \lesssim \sqrt{N} \|p^2 \widehat{\varphi}_I\|_\infty^{3/4} \|\widehat{\varphi}_I\|_\infty^{1/4} \lesssim N^{-\alpha/2}$ from Lemma 2.2. The above can be absorbed in (4.145c) as $\mathcal{N}_+ \leq d\Gamma(|i\nabla|)$. This concludes the proof. \square

Conservation estimates. Here, we prove that the operators appearing in the bound on the error term remain of the same order when transformed with $U_c(t) = e^{t\mathcal{B}_c}$.

Lemma 4.21. *For all $s \geq 0$, we have uniformly in $M \geq 1$ and $t \in [-1, 1]$*

$$\begin{aligned} U_c(t)^* \mathcal{N}_+ U_c(t) &= \mathcal{N}_+ + \mathcal{O}(N^{-\alpha/2} (\mathcal{N}_+ + 1)), \\ U_c(t)^* (\mathcal{N}_+ + 1)^s U_c(t) &= \mathcal{O}((\mathcal{N}_+ + 1)^s), \end{aligned}$$

as quadratic forms on \mathcal{H}_+ .

Proof. This follows from a simple calculation of the commutator $[\mathcal{N}_+, \mathcal{B}_c]$ and Grönwall's Lemma, see [40, Lem.10], and interpolation for non-integer s . \square

Lemma 4.22. *Let $0 < \alpha \leq 1/6$, then we have as quadratic forms on \mathcal{H}_+*

$$U_c(t)^* d\Gamma(|i\nabla|) U_c(t) = \mathcal{O}(d\Gamma(|i\nabla|)), \quad (4.158a)$$

$$U_c(t)^* \mathcal{L}_4 U_c(t) = \mathcal{O}(\mathcal{L}_4 + \mathcal{N}_+ + M^{-1} \mathcal{N}_+^2 + 1), \quad (4.158b)$$

$$U_c(t)^* d\Gamma(W_{N,x}) U_c(t) = \mathcal{O}(d\Gamma(W_{N,x}) + \mathcal{N}_+), \quad (4.158c)$$

uniformly in $s \in [-1, 1]$ and $M \leq N^{1-3\alpha}$.

Proof. We start with the inequality (4.158a) and compute

$$[d\Gamma(|i\nabla|), \mathcal{B}_c^+] = \sum_{p,q} \sqrt{N} \widehat{\varphi}_B(p) \mathbb{1}_{|q| \leq N^\alpha} (|p+q| + |p| - |q|) a_{p+q}^* a_{-p}^* a_q. \quad (4.159)$$

Using that $||p+q|-|q|| \leq |p|$, we obtain by the Cauchy-Schwarz inequality, for $\delta > 0$

$$\begin{aligned} & \pm \theta_M (\mathcal{N}_+) [\mathrm{d}\Gamma(|\nabla|), \mathcal{B}_c^+] + \text{h.c.} \\ & \lesssim \delta \theta_M^2 \sum_{p,q} |p| a_{p+q}^* a_p^* a_p a_{p+q} + \delta^{-1} \sum_{p,q} N |p| \widehat{\varphi}_B(p)^2 \mathbb{1}_{|q| \leq N^\alpha} a_q^* a_q \\ & \lesssim \delta M \mathrm{d}\Gamma(|\nabla|) + \delta^{-1} N^{-1} \log N \mathcal{N}_+, \end{aligned} \quad (4.160)$$

since $||\nabla|^{1/2} \widehat{\varphi}_B||_2^2 \lesssim N^{-2} \log N$ by Lemma 2.1. Taking $\delta = \sqrt{N/(M \log N)}$ this gives

$$\pm [\mathrm{d}\Gamma(|\nabla|), \mathcal{B}_c] \lesssim \sqrt{M \log N/N} (\mathcal{N}_+ + \mathrm{d}\Gamma(|\nabla|)). \quad (4.161)$$

Let $f(t) = U_c(t)^* \mathrm{d}\Gamma(|\nabla|) U_c(t)$. Then, as $\sqrt{M \log N/N} \lesssim 1$ by assumption, we obtain with $\mathcal{N}_+ \leq \mathrm{d}\Gamma(|\nabla|)$ on \mathcal{H}_+ .

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = U_c(t)^* [\mathrm{d}\Gamma(|\nabla|), \mathcal{B}_c^+] U_c(t) \lesssim f(t). \quad (4.162)$$

The bound (4.158a) now follows from Grönwall's inequality.

To prove the inequality (4.158b), we use the calculation of the commutator $[\mathcal{L}_4, \mathcal{B}_c]$ from Lemma 4.14. The principal term satisfies, using the Cauchy-Schwarz inequality as in (4.118),

$$\sum_{\substack{p,q \in 2\pi\mathbb{Z}^3 \setminus \{0\} \\ |q| \leq N^\alpha}} \widehat{V}_N * \widehat{\varphi}_B(p) a_{p+q}^* a_{-p}^* a_q + \text{h.c.} = \mathcal{O}(\mathcal{L}_4 + \mathcal{N}_+). \quad (4.163)$$

The difference of the principal term and the full commutator is bounded by using Lemma 4.14 with $\delta = 1$ and the constraint $M \leq N^{1-3\alpha}$. The claim then follows from Lemma 4.21 and Grönwall's inequality, as above.

The inequality (4.158c) follows from Lemma 4.19 and Grönwall's inequality. \square

4.5 Low-energy bosons: Bogoliubov's transformation

In this section we apply the well known Bogoliubov transformation U_B (defined by its generator in (4.164)) that diagonalizes the boson Hamiltonian for momenta below N^α . This goes back to the original work of Bogoliubov [9], and has been employed in all subsequent works on the topic. In addition to the standard computations, that we will omit, we will also need to transform the boson-impurity interaction terms. The article [21] found a term in the expansion of the ground state energy H_N in powers of the scattering lengths that does not come from the Bogoliubov–Fröhlich model. This arises from interaction terms that create or annihilate two excitations from the condensate, which are generated by acting on $\mathrm{d}\Gamma(W_{N,x})$ with U_B . In Lemma 4.24 below, we show that these terms are small relative to $\mathrm{d}\Gamma(W_{N,x})$. Since the latter

contributes to the energy at order one, this shows that the contribution of these interaction terms is small in N , in our setting.

The generator of the transformation is

$$\mathcal{B}_B = \frac{1}{8} \sum_{\substack{p \in 2\pi\mathbb{Z}^3 \setminus \{0\} \\ |p| \leq N^\alpha}} \log \left(1 + \frac{16\pi\mathbf{a}_V}{p^2} \right) (a_p a_{-p} - \text{h.c.}). \quad (4.164)$$

The action of the unitary $U_B = e^{\mathcal{B}_B}$ on the creation and annihilation operators can be expressed by the explicit formulas

$$U_B^* a^*(f) U_B = a^*(Cf) + a(\overline{Sf}), \quad U_B^* a(f) U_B = a(Cf) + a^*(\overline{Sf}), \quad (4.165)$$

where C, S are the Fourier multipliers with coefficients

$$C(p) = \frac{1}{2} \left(\frac{|p|}{(p^4 + 16\pi\mathbf{a}_V p^2)^{1/4}} + \frac{(p^4 + 16\pi\mathbf{a}_V p^2)^{1/4}}{|p|} \right) \mathbb{1}_{|p| \leq N^\alpha} + \mathbb{1}_{|p| > N^\alpha} \quad (4.166a)$$

$$S(p) = \frac{1}{2} \left(\frac{|p|}{(p^4 + 16\pi\mathbf{a}_V p^2)^{1/4}} - \frac{(p^4 + 16\pi\mathbf{a}_V p^2)^{1/4}}{|p|} \right) \mathbb{1}_{|p| \leq N^\alpha}. \quad (4.166b)$$

Notice that $C(p) = \cosh(-\frac{1}{4} \log(1 + 16\pi\mathbf{a}_V/p^2))$, and $S(p) = \sinh(-\frac{1}{4} \log(1 + 16\pi\mathbf{a}_V/p^2))$.

The main result of this section is the following proposition, which states that for momenta below N^α the Hamiltonian acts almost like the Bogoliubov-Fröhlich Hamiltonian (1.9) with cutoff $\Lambda = N^\alpha$, except for the continued presence of the positive terms $d\Gamma(W_{N,x})$ and \mathcal{L}_4 .

Proposition 4.23. *Let $0 < \alpha \leq 1/8$ and $U_c^* U_W^* U_q^* \mathcal{H}_N U_q U_W U_c$ be given by Proposition 4.12. Then as quadratic forms on \mathcal{H}_+*

$$\begin{aligned} & U_B^* U_c^* U_W^* U_q^* \mathcal{H}_N U_q U_W U_c U_B \\ &= 4\pi\mathbf{a}_V(N-1) + 8\pi\mathbf{a}_W \sqrt{N} + e_N^{(U_B)} \\ &+ d\Gamma(\epsilon) - \Delta_x + a(w_{N,x}) + a^*(w_{N,x}) + \mathcal{H}_{\text{Int},>} + d\Gamma(W_{N,x}) + \mathcal{L}_4 + \mathcal{E}^{(U_B)}, \end{aligned}$$

with

$$\widehat{w}_{N,x}(p) = \widehat{w}_x(p) \mathbb{1}_{0 < |p| < N^\alpha}, \quad \widehat{w}_x(p) = \frac{8\pi\mathbf{a}_W |p| e^{-ipx}}{(p^4 + 16\pi\mathbf{a}_V p^2)^{1/4}},$$

the high momentum interaction

$$\begin{aligned} \mathcal{H}_{\text{Int},>} &= \left(a^*(\sqrt{N}(-i\nabla\tilde{\varphi}_I)_x)^2 + \text{h.c.} \right) + 2a^*(\sqrt{N}(i\nabla\tilde{\varphi}_I)_x) a(\sqrt{N}(i\nabla\tilde{\varphi}_I)_x) \\ &- 2a^*(\sqrt{N}(i\nabla\tilde{\varphi}_I)_x) i\nabla_x + \text{h.c.}, \end{aligned}$$

the scalar,

$$e_N^{(U_B)} = 4\pi N(\tilde{\mathbf{a}}_{V_N} - \mathbf{a}_V) + 8\pi\sqrt{N}(\tilde{\mathbf{a}}_{W_N} - \mathbf{a}_W) \\ + \sum_{\substack{0 \neq p \in 2\pi\mathbb{Z}^3 \\ |p| \leq N^\alpha}} \left(\frac{1}{2}\sqrt{p^4 + 16\pi\mathbf{a}_V p^2} - \frac{1}{2}p^2 - 4\pi\mathbf{a}_V + \frac{(4\pi\mathbf{a}_V)^2}{p^2} + \frac{2(4\pi\mathbf{a}_W)^2}{p^2} \right),$$

and where the error satisfies

$$\pm \mathcal{E}^{(U_B)} \lesssim N^{-\alpha/2} d\Gamma(W_{N,x}) + N^{-\alpha/4} \mathcal{L}_4 + N^{-1/2-\alpha/4} \mathcal{N}_+^2 \\ + N^{-\alpha/4} ((\log N) d\Gamma(|i\nabla|) + (\log N)^2 + (\mathcal{N}_+ + 1)^{1/2} |\nabla_x|).$$

As usual, we will first prove Proposition 4.23 assuming known the bounds given by Lemmas 4.24 and 4.26, which we will discuss in the remainder of this section.

Proof of Proposition 4.23. First, note that $-\Delta_x$ and $\mathcal{H}_{\text{Int},>}$ (which already appears in Proposition 4.12) are left unchanged by U_B since the generator does not depend on x and acts only on momenta $|p| \leq N^\alpha$. By standard calculations (see e.g. [40]) using the relations (4.165), one finds that

$$U_B^*(d\Gamma(-\Delta + 8\pi\mathbf{a}_V) + \tilde{\mathcal{L}}_2)U_B \\ = \sum_p \left(\sqrt{p^4 + 16\pi\mathbf{a}_V p^2} \mathbb{1}_{|p| \leq N^\alpha} + (p^2 + 8\pi\mathbf{a}_V) \mathbb{1}_{|p| > N^\alpha} \right) a_p^* a_p \\ + \sum_{|p| \leq N^\alpha} \left(\frac{1}{2}\sqrt{p^4 + 16\pi\mathbf{a}_V p^2} - \frac{1}{2}p^2 - 4\pi\mathbf{a}_V \right). \quad (4.167)$$

The second term contributes to $e_N^{(U_B)}$ and the first equals $d\Gamma(\epsilon) + \mathcal{O}(N^{-2\alpha}\mathcal{N}_+)$ using that

$$|\epsilon(p) - (p^2 + 8\pi\mathbf{a}_V)| \mathbb{1}_{|p| > N^\alpha} \lesssim N^{-2\alpha}. \quad (4.168)$$

For the interaction terms on momenta below N^α , one finds with (4.165)

$$U_B^*(a(\tilde{W}_{N,x}) + a^*(\tilde{W}_{N,x}))U_B = \sum_p (C(p) + S(p)) e^{-ipx} \widehat{W}_N(p) (a_p^* + a_{-p}) \\ = \sum_{|p| \leq N^\alpha} \frac{e^{-ipx} |p| 8\pi\mathbf{a}_W}{(p^4 + 16\pi\mathbf{a}_V p^2)^{1/4}} (a_p^* + a_{-p}), \quad (4.169)$$

which gives $w_{N,x}$.

The terms $d\Gamma(W_{N,x})$ and \mathcal{L}_4 are preserved by U_B up to errors that can be absorbed in $\mathcal{E}^{(U_B)}$ by Lemmas 4.24, 4.25 below. Moreover, Lemma 4.25 also shows that

$U_B^* \mathcal{E}^{(U_c)} U_B$ can also be absorbed in the error, where in the case of the error term $(\mathcal{N}_+ + 1)^{1/2} |\nabla_x|$ we additionally make use of the fact that ∇_x commutes with both U_B , and \mathcal{N}_+ , which gives

$$\begin{aligned} U_B^* (\mathcal{N}_+ + 1)^{1/2} |\nabla_x| U_B &= |\nabla_x|^{1/2} U_B^* (\mathcal{N}_+ + 1)^{1/2} U_B |\nabla_x|^{1/2} \\ &= \mathcal{O}((\mathcal{N}_+ + 1)^{1/2} |\nabla_x|), \end{aligned} \quad (4.170)$$

by Lemma 4.25. This proves the claim. \square

Lemma 4.24. *We have for $\alpha > 0$ as quadratic forms on \mathcal{H}_+*

$$U_B^* d\Gamma(W_{N,x}) U_B = d\Gamma(W_{N,x}) + \mathcal{O}(N^{-1/4} d\Gamma(W_{N,x}) + N^{-1/4} (\mathcal{N}_+ + 1)).$$

Proof. Writing $d\Gamma(W_{N,x})$ as

$$d\Gamma(W_{N,x}) = \sum_{p,q} e^{-ix(p-q)} \widehat{W}_N(p-q) a_p^* a_q \quad (4.171)$$

we find from (4.165)

$$\begin{aligned} U_B^* d\Gamma(W_{N,x}) U_B &= \sum_{p,q} (C(p)C(q) + S(p)S(q)) e^{-ix(p-q)} \widehat{W}_N(p-q) a_p^* a_q \\ &\quad + \sum_{p,q} C(p)S(q) e^{-ix(p-q)} \widehat{W}_N(p-q) a_p^* a_q^* + \text{h.c.} \\ &\quad + \sum_p S(p)^2 \widehat{W}_N(0). \end{aligned} \quad (4.172)$$

We have

$$S(p)^2 = \frac{(1 - \sqrt{1 + 16\pi\mathbf{a}_V/p^2})^2}{4\sqrt{1 + 16\pi\mathbf{a}_V/p^2}} \mathbf{1}_{|p| \leq N^\alpha} \leq \frac{(4\pi\mathbf{a}_V)^2}{p^4} \in \ell^1(2\pi\mathbb{Z}^3). \quad (4.173)$$

From this the last term in (4.172) is a $\mathcal{O}(N^{-1/2})$. To bound the $a^* a^*$ term, we use that $C(p) = 1 + H(p)$ with $H(p) = \sqrt{S(p)^2 + 1} - 1$. By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\pm \sum_{p,q} S(q) e^{-ix(p-q)} \widehat{W}_N(p-q) a_p^* a_q^* + \text{h.c.} = \pm \int W_{N,x}(y) \check{S}(y-y') a_y^* a_{y'}^* + \text{h.c.} \\ &\lesssim \delta d\Gamma(W_{N,x}) + \delta^{-1} \int W_{N,x}(y) \check{S}(y-y_1) \check{S}(y-y_2) a_{y_1} a_{y_2}^* \\ &\lesssim N^{-1/4} d\Gamma(W_{N,x}) + N^{1/4} \|W_N\|_1 \|S\|_2^2 (\mathcal{N}_+ + 1), \end{aligned} \quad (4.174)$$

where we took $\delta = N^{-1/4}$ and which can be absorbed it in the error term. The remaining term is estimated as

$$\begin{aligned} & \pm \sum_{p,q} H(p)S(q)e^{-ix(p-q)}\widehat{W}_N(p-q)a_p^*a_q^* + \text{h.c.} \\ & \lesssim \|\widehat{W}_N\|_\infty \|S\|_2 \|H\|_2 (\mathcal{N}_+ + 1) \lesssim N^{-1/2}(\mathcal{N}_+ + 1). \end{aligned} \quad (4.175)$$

From (4.173) we also immediately obtain

$$\sum_{p,q} S(p)S(q)e^{-ix(p-q)}\widehat{W}_N(p-q)a_p^*a_q = \mathcal{O}(N^{-1/2}\mathcal{N}_+), \quad (4.176)$$

and $\sum S(p)^2\widehat{W}(0) \lesssim N^{-1/2}$. Writing again $C(p) = 1 + H(p)$ and arguing as above gives

$$\begin{aligned} & \sum_{p,q} C(p)C(q)e^{-ix(p-q)}\widehat{W}_N(p-q)a_p^*a_q \\ & = d\Gamma(W_{N,x}) + \mathcal{O}(N^{-1/4}d\Gamma(W_{N,x}) + N^{-1/4}(\mathcal{N}_+ + 1)). \end{aligned} \quad (4.177)$$

This proves the claim. \square

Lemma 4.25. *For all $t \geq 0$ and $0 \leq \alpha < 1/2$ we have as quadratic forms on \mathcal{H}_+*

$$\begin{aligned} U_B^* \mathcal{N}_+^t U_B &= \mathcal{O}(\mathcal{N}_+^t + 1), \\ U_B \mathcal{N}_+^t U_B^* &= \mathcal{O}(\mathcal{N}_+^t + 1), \\ U_B^* \mathcal{L}_4 U_B &= \mathcal{L}_4 + \mathcal{O}(N^{-1+2\alpha} + N^{-1}\mathcal{N}_+^2), \\ U_B^* d\Gamma(|i\nabla|) U_B &= \mathcal{O}(d\Gamma(|i\nabla|) + \log N). \end{aligned}$$

Proof. The first three estimates follow from the same arguments as in [40, Lem.18] (first, for integer t , and then for all t by interpolation). The final inequality follows from the fact that $\| |p|^{1/2} S \|_2 \lesssim \log N$ and $|p| S(p)^2 \lesssim 1$, $C(p) \lesssim 1$, by an argument similar to Lemma 4.24. \square

4.6 Excitation-impurity scattering: a second Weyl transformation

In this section we apply another Weyl transformation, similar to the one from Section 4.3. However, it acts now on small momenta and the form factor of the transformation is obtained from an equation involving the Bogoliubov dispersion relation ϵ , instead of simply $-\Delta$, and the interaction w_N instead of the potential W_N . The transformation is closely related to the first step of the renormalization in Section 3.

Let $\kappa > 0$, to be chosen later, and let $\alpha > 0$ be the parameter of the previous sections. We define the function $f \in L^2(\mathbb{T}^3)$ by

$$\widehat{f}_x(p) = -\frac{\widehat{w}_N(p)e^{ipx}}{p^2 + \epsilon(p)} \mathbf{1}_{\kappa < |p|} = \frac{-8\pi\mathbf{a}_W |p| e^{ipx} \mathbf{1}_{\kappa < |p| \leq N^\alpha}}{(p^2 + \sqrt{p^4 + 16\pi\mathbf{a}_V p^2})(p^4 + 16\pi\mathbf{a}_V p^2)^{1/4}}. \quad (4.178)$$

Note that this corresponds to f_κ^Λ with $\Lambda = N^\alpha$ in (3.3). One easily checks that $\|f_x\|_2 \lesssim 1$, uniformly in κ and N . We then define the following unitary transformation, which is similar to the Gross transformation used in the renormalization of some polaron models [63, 32],

$$U_G(t) = \exp(t(a^*(f_x) - a(f_x))), \quad U_G = U_G(1). \quad (4.179)$$

After this transformation, the Hamiltonian from Proposition 4.23 resembles closely the operator (3.7) appearing in the renormalization procedure.

Proposition 4.26. *Let $\kappa > 0$, $0 < \alpha \leq 1/10$ and $M = N^{1/2+\alpha/2}$, denote $U = U_G U_q U_W U_c U_B$ and recall the transformation of \mathcal{H}_N from Proposition 4.23. Then, for $N^\alpha > \kappa$*

$$\begin{aligned} U^* \mathcal{H}_N U &= 4\pi \mathbf{a}_V (N-1) + 8\pi \mathbf{a}_W \sqrt{N} + e_N^{(U)} \\ &\quad - \Delta_x + d\Gamma(\epsilon) + a(w_x^\kappa) + a^*(w_x^\kappa) + a^*(i\nabla v_{N,x})^2 + a(i\nabla v_{N,x})^2 \\ &\quad + 2a^*(i\nabla v_{N,x})a(i\nabla v_{N,x}) + 2a^*(i\nabla_x v_{N,x})i\nabla_x + 2i\nabla_x a(i\nabla_x v_{N,x}) \\ &\quad + d\Gamma(W_{N,x}) + \mathcal{L}_4 + \mathcal{E}^{(U)}, \end{aligned} \quad (4.180)$$

where

$$\begin{aligned} \widehat{w}_x^\kappa(p) &= 8\pi a(W) |p| \epsilon(p)^{-1/2} e^{ipx} \mathbb{1}_{0 < |p| \leq \kappa}, \\ \widehat{v}_{N,x} &= \begin{cases} \frac{8\pi \mathbf{a}_W}{p^2 + \epsilon(p)} \frac{|p|}{\sqrt{\epsilon(p)}} e^{ipx} & \kappa < |p| \leq N^\alpha, \\ \sqrt{N} \widehat{\varphi}_1(p) e^{ipx} & N^\alpha < |p|, \end{cases} \end{aligned}$$

the scalar term has the value

$$\begin{aligned} e_N^{(U)} &= 4\pi N (\widetilde{\mathbf{a}}_{V_N} - \mathbf{a}_V) + 8\pi \sqrt{N} (\widetilde{\mathbf{a}}_{W_N} - \mathbf{a}_W) \\ &\quad + \frac{1}{2} \sum_{\substack{0 \neq p \in 2\pi\mathbb{Z}^3 \\ |p| \leq N^\alpha}} \left(\sqrt{p^4 + 16\pi \mathbf{a}_V p^2} - p^2 - 8\pi \mathbf{a}_V + 2 \frac{(4\pi \mathbf{a}_V)^2}{p^2} \right) \\ &\quad + (8\pi \mathbf{a}_W)^2 \sum_{\substack{0 \neq p \in 2\pi\mathbb{Z}^3 \\ |p| < N^\alpha}} \left(\frac{1}{2p^2} - \frac{p^2 \mathbb{1}_{|p| > \kappa}}{(p^2 + \epsilon(p))\epsilon(p)} \right), \end{aligned}$$

and the error term satisfies

$$\begin{aligned} \pm \mathcal{E}^{(U)} &\lesssim N^{-\alpha/2} d\Gamma(W_{N,x}) + N^{-\alpha/4} \mathcal{L}_4 + N^{-1/2-\alpha/4} \mathcal{N}_+^2 \\ &\quad + N^{-\alpha/4} ((\log N) d\Gamma(|i\nabla|) + (\log N)^2 + (\mathcal{N}_+ + 1)^{1/2} |\nabla_x|). \end{aligned} \quad (4.181)$$

Proof. Recall from Proposition 4.23 that

$$U^* \mathcal{H}_N U = 4\pi \mathbf{a}_V (N-1) + 8\pi \mathbf{a}_W \sqrt{N} + e_N^{(U_B)} + d\Gamma(\epsilon) - \Delta_x + a(w_{N,x}) + a^*(w_{N,x}) + \mathcal{H}_{\text{Int},>} + d\Gamma(W_{N,x}) + \mathcal{L}_4 + \mathcal{E}^{(U_B)}. \quad (4.182)$$

We have the identities (compare (3.5a), (3.5b) and (4.71))

$$U_G^* a_y U_G = a_y + f_x(y), \quad (4.183a)$$

$$U_G^*(i\nabla_x)U_G = i\nabla_x + a^*(i\nabla_x f_x) + a(i\nabla_x f_x), \quad (4.183b)$$

which gives with $\nabla_x f_x = -(\nabla f)_x$

$$\begin{aligned} & U_G^* (-\Delta_x + d\Gamma(\epsilon) + a(w_{N,x}) + a^*(w_{N,x})) U_G \\ &= -\Delta_x + d\Gamma(\epsilon) + a(w_x^\kappa) + a^*(w_x^\kappa) + a(i\nabla f_x)^2 + a^*(i\nabla f_x)^2 \\ &\quad + 2a(i\nabla f_x)a^*(i\nabla f_x) - 2a^*(i\nabla f_x)i\nabla_x - 2i\nabla_x a(i\nabla f_x) \\ &\quad - \sum_{\kappa < |p| \leq N^\alpha} \frac{\widehat{w}(p)^2}{(p^2 + \epsilon(p))}. \end{aligned} \quad (4.184)$$

We now pass to $\mathcal{H}_{\text{Int},>}$, which we recall is defined in Proposition 4.23. Note that the transformation U_G^* leaves invariant the terms involving momenta greater than N^α . Therefore, all the terms in $\mathcal{H}_{\text{Int},>}$ remain unchanged except the ones containing the gradient ∇_x . This gives, with $\nabla_x f_x = -(\nabla f)_x$

$$U_G^* \mathcal{H}_{\text{Int},>} U_G = \mathcal{H}_{\text{Int},>} + 2a^*(\sqrt{N}i\nabla \tilde{\varphi}_{I,x})(a(i\nabla f_x) + a^*(i\nabla f_x)) + \text{h.c.} \quad (4.185)$$

Introducing $v_{N,x} = f_x + \sqrt{N}\tilde{\varphi}_{I,x}$ and noting that the additional terms in (4.185) complete the square with (4.184), we find

$$\begin{aligned} (4.184) + (4.185) &= -\Delta_x + d\Gamma(\epsilon) + a(w_x^\kappa) + a^*(w_x^\kappa) - \sum_{\kappa < |p| \leq N^\alpha} \frac{\widehat{w}(p)^2}{(p^2 + \epsilon(p))} \\ &\quad + a^*(i\nabla v_{N,x})^2 + a(i\nabla v_{N,x})^2 + 2a^*(i\nabla v_{N,x})a(i\nabla v_{N,x}) \\ &\quad - 2a^*(i\nabla v_{N,x})i\nabla - 2i\nabla a(i\nabla v_{N,x}). \end{aligned} \quad (4.186)$$

It remains to consider $d\Gamma(W_{N,x})$, \mathcal{L}_4 and $\mathcal{E}^{(U_B)}$. With the shift property (4.183a), we obtain

$$U_G^* d\Gamma(W_{N,x}) U_G = d\Gamma(W_{N,x}) + a^*((W_N f)_x) + a((W_N f)_x) + \langle f, W_N f \rangle. \quad (4.187)$$

Next, one easily checks that

$$\langle f, W_N f \rangle \lesssim \|W_N\|_1 \|f\|_\infty^2 \lesssim N^{-1/2+2\alpha}, \quad (4.188)$$

and, using the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} a^*((W_N f)_x) + a((W_N f)_x) &\lesssim N^{-\alpha/2} d\Gamma(W_{N,x}) + N^{\alpha/2} \langle f, W_N f \rangle \\ &\lesssim N^{-\alpha/2} d\Gamma(W_{N,x}) + N^{-\alpha/2}, \end{aligned} \quad (4.189)$$

for $\alpha \leq 1/6$. This gives

$$U_G^* d\Gamma(W_{N,x}) U_G = d\Gamma(W_{N,x}) + \mathcal{O}(N^{-\alpha/2} (d\Gamma(W_{N,x}) + 1)). \quad (4.190)$$

We now estimate the transformation of the quartic term \mathcal{L}_4 , which gives

$$U_G^* \mathcal{L}_4 U_G - \mathcal{L}_4 = \int V_N(y_1 - y_2) f_x(y_1) a_{y_1}^* a_{y_2}^* a_{y_2} + \text{h.c.} \quad (4.191a)$$

$$+ \frac{1}{2} \int V_N(y_1 - y_2) f_x(y_1) f_x(y_2) a_{y_1}^* a_{y_2}^* + \text{h.c.} \quad (4.191b)$$

$$+ \int V_N(y_1 - y_2) f_x(y_1) f_x(y_2) a_{y_1}^* a_{y_2} \quad (4.191c)$$

$$+ d\Gamma(V_N * f_x^2) + \left(a((V_N * f_x^2) f_x) + \text{h.c.} \right) \quad (4.191d)$$

$$+ \frac{1}{2} \int f_x(y_1)^2 f_x(y_2)^2 V_N(y_1 - y_2). \quad (4.191e)$$

Since $\|f_x\|_\infty \lesssim \|\widehat{f}_x\|_1 \lesssim N^\alpha$ and $\|f_x\|_2 \lesssim 1$ we have

$$(4.191e) \lesssim \|V_N\|_1 \|f_x\|_\infty^2 \|f_x\|_2^2 \lesssim N^{-1+2\alpha}. \quad (4.192)$$

Similarly, we obtain

$$(4.191d) = \mathcal{O}(N^{-1+2\alpha}(\mathcal{N}_+ + 1)). \quad (4.193)$$

Using the Cauchy-Schwarz inequality as in (4.22) and (4.59) yields

$$(4.191b) = \mathcal{O}(N^{-\alpha/2} \mathcal{L}_4 + N^{-1+5\alpha/2}) \quad (4.194a)$$

$$(4.191a) = \mathcal{O}(N^{-\alpha/2} \mathcal{L}_4 + N^{\alpha/2} d\Gamma(V_N * f_x^2)) = \mathcal{O}(N^{-\alpha/2} \mathcal{L}_4 + N^{-1+5\alpha/2} \mathcal{N}_+). \quad (4.194b)$$

These estimates, together with $\alpha \leq 1/10$, show that

$$U_G^* \mathcal{L}_4 U_G = \mathcal{L}_4 + \mathcal{O}(N^{-\alpha/2}(\mathcal{L}_4 + \mathcal{N}_+ + 1)). \quad (4.195)$$

Finally, we estimate the action of U_G on $\mathcal{E}^{(U_B)}$. Similarly as in (4.189), $\| |p|^{1/2} \widehat{f} \|_2^2 \lesssim \log N$ yields

$$U_G^* d\Gamma(|i\nabla|) U_G = \mathcal{O}(d\Gamma(|i\nabla|) + \log N) \quad (4.196)$$

With $\|f\|_2 \lesssim 1$, one easily obtains

$$U_G^*(t)\mathcal{N}_+^k U_G(t) = \mathcal{O}(\mathcal{N}_+^k + 1) \quad (4.197)$$

for $k \in \mathbb{N}$. Moreover,

$$\begin{aligned} U_G^*(\mathcal{N}_+ + 1)^{1/2} |\nabla_x| U_G & \quad (4.198) \\ &= (\mathcal{N}_+ + 1)^{1/2} |\nabla_x| + \int_0^1 U_G(t)^* \left[(\mathcal{N}_+ + 1)^{1/2} |\nabla_x|, a^*(f_x) \right] + \text{h.c.} U_G(t). \end{aligned}$$

Let us estimate the commutator

$$\begin{aligned} & [(\mathcal{N}_+ + 1)^{1/2} |\nabla_x|, a^*(f_x)] \\ &= (\mathcal{N}_+ + 1)^{1/2} [|\nabla_x|, a^*(f_x)] + [(\mathcal{N}_+ + 1)^{1/2}, a^*(f_x)] |\nabla_x|. \end{aligned} \quad (4.199)$$

To estimate the first term, we use that $i\nabla_x e^{-ikx} = e^{-ikx} (i\nabla_x + k)$ and the triangle inequality $\|i\nabla_x + p\| - \|i\nabla_x\| \leq |p|$ for all p , to obtain that

$$\begin{aligned} |\langle \Psi, [i\nabla_x|, a^*(f_x)] \Psi \rangle| &= \left| \langle \Psi, \sum_p (|i\nabla_x| - |i\nabla_x - p|) \widehat{f}_x(p) a_p^* \Psi \rangle \right| \\ &\leq \|d\Gamma(|i\nabla|)^{1/2} \Psi\| \left(\sum_p \left\| \frac{(|i\nabla_x| - |i\nabla_x - ip|)^2 |\widehat{f}_x(p)|^2}{|p|} \Psi \right\|^2 \right)^{1/2} \\ &\lesssim \sqrt{\log N} \|d\Gamma(|i\nabla|)^{1/2} \Psi\| \|\Psi\|, \end{aligned} \quad (4.200)$$

where we used that $\| |p|^{1/2} \widehat{f} \|_2^2 \lesssim \log N$. Hence, the first term in (4.199) is bounded by

$$(\mathcal{N}_+ + 1)^{1/2} [|\nabla_x|, a^*(f_x)] + \text{h.c.} = \mathcal{O}((d\Gamma(|i\nabla|) + \log N)(\mathcal{N}_+ + 1)). \quad (4.201)$$

To bound the second term in (4.199), we write

$$a^*(f_x) |\nabla_x| = |\nabla_x|^{1/2} a^*(f_x) |\nabla_x|^{1/2} - [|\nabla_x|^{1/2}, a^*(f_x)] |\nabla_x|^{1/2} \quad (4.202)$$

and use again that $i\nabla_x e^{-ikx} = e^{-ikx} (i\nabla_x + k)$ and that $\| |i\nabla_x + p|^{1/2} - |i\nabla_x|^{1/2} \| \leq |p|^{1/2}$ from which we obtain

$$\begin{aligned} [(\mathcal{N}_+ + 1)^{1/2}, a^*(f_x)] |\nabla_x| &= ((\mathcal{N}_+ + 1)^{1/2} - \mathcal{N}_+^{1/2}) a^*(f_x) |\nabla_x| \\ &= \mathcal{O}(|\nabla_x| + \log N), \end{aligned} \quad (4.203)$$

where we used that $\| |p|^{1/2} \widehat{f} \|_2^2 \lesssim \log N$ and that $(\mathcal{N}_+ + 1)^{1/2} - \mathcal{N}_+^{1/2} \leq (\mathcal{N}_+ + 1)^{-1/2}$.

It thus follows from (4.196), (4.197) and Grönwall's inequality that

$$U_G^*(\mathcal{N}_+ + 1)^{1/2} |\nabla_x| U_G = \mathcal{O} \left((\mathcal{N}_+ + 1)^{1/2} |\nabla_x| + d\Gamma(|i\nabla|) + \log N(\mathcal{N}_+ + 1) \right). \quad (4.204)$$

This completes the proof of the proposition as $\mathcal{N}_+ \leq d\Gamma(|i\nabla|)$ on \mathcal{H}_+ . \square

5 Convergence of the transformed Hamiltonian

In the section, we study the main part of the Hamiltonian appearing at the end of the previous section in Proposition 4.26. The remaining singular contribution, which will give rise to the $\log N$ -term in the energy, is the interaction term $a^*(i\nabla_x v_{N,x})^2 + \text{h.c.}$ from (4.180) (compare Section 3). For technical reasons, we will cut this term off at particle numbers $\mathcal{N}_+ > M = N^{1-\alpha}$ for the same $\alpha > 0$ as before.

We can rephrase the result of Proposition 4.26 as

$$U^* \mathcal{H}_N U = \mathcal{H}_{\text{Sing}} + R_N + 4\pi \mathbf{a}_V (N-1) + 8\pi \mathbf{a}_W \sqrt{N} + e_N^{(U)} + \mathcal{L}_4 + \mathcal{E}^{(U)} + \mathcal{O}(N^{-3/2+2\alpha}(\mathcal{N}_+ + 1)^3). \quad (5.1)$$

Here,

$$\begin{aligned} \mathcal{H}_{\text{Sing}} &= -\Delta_x + d\Gamma(\epsilon) + d\Gamma(W_{N,x}) \\ &\quad + \mathbf{1}_{\mathcal{N}_+ \leq N^{1-\alpha}} a^*(i\nabla v_{N,x})^2 + a(i\nabla v_{N,x})^2 \mathbf{1}_{\mathcal{N}_+ \leq N^{1-\alpha}} \end{aligned} \quad (5.2)$$

generalizes the singular Hamiltonian from Section 3, and

$$\begin{aligned} R_N &= 2a^*(i\nabla v_{N,x})a(i\nabla v_{N,x}) - 2a^*(i\nabla v_{N,x})i\nabla_x - 2i\nabla_x a(i\nabla v_{N,x}) \\ &\quad + a^*(w_x^\kappa) + a(w_x^\kappa), \end{aligned} \quad (5.3)$$

collects the remaining terms from the transformed operator of Proposition 4.26, except \mathcal{L}_4 , the constants, and the error terms, and we recall that for $\alpha > 0$

$$\begin{aligned} \widehat{w}_x^\kappa(p) &= 8\pi \mathbf{a}_W |p| \epsilon(p)^{-1/2} e^{ipx} \mathbf{1}_{0 < |p| \leq \kappa} \\ \widehat{v}_{N,x}(p) &= \begin{cases} \frac{8\pi \mathbf{a}_W}{p^2 + \epsilon(p)} \frac{|p|}{\sqrt{\epsilon(p)}} e^{ipx} & \kappa < |p| \leq N^\alpha \\ \sqrt{N} \widehat{\varphi}_1(p) e^{ipx} & N^\alpha < |p|. \end{cases} \end{aligned} \quad (5.4)$$

Note that, since $\sqrt{N} \widehat{\varphi}_1(p) \lesssim |p|^{-2}$ by Lemma 2.2,

$$\widehat{v}_N(p) \lesssim |p|^{-2}, \quad (5.5)$$

and that \widehat{v}_∞ is also a well-defined function. Also by Lemma 2.2 with $s = 1/2$, we have $\|\nabla v_{N,x}\|_2 \lesssim N^{1/4}$, so, by the argument of (4.12),

$$\mathbf{1}_{\mathcal{N}_+ > M} a^*(i\nabla_x v_{N,x})^2 + \text{h.c.} = \mathcal{O}(M^{-2} \sqrt{N} (\mathcal{N}_+ + 1)^3). \quad (5.6)$$

Cutting off the interaction in (5.2) thus indeed gives rise to errors of the order stated in (5.1).

For later use, we introduce the notation for the relevant part of the transformed Hamiltonian

$$\mathcal{H}_N^U := \mathcal{H}_{\text{Sing}} + R_N. \quad (5.7)$$

Without the term $d\Gamma(W_{N,x})$ in $\mathcal{H}_{\text{Sing}}$, one should think of \mathcal{H}_N^U as the Bogoliubov-Fröhlich Hamiltonian H_{BF}^Λ defined in (3.1) for $\Lambda = N^\alpha$ dressed with the appropriate Weyl transformation (compare with the expression given in (3.7)). This will be discussed in detail in Section 5.1.

We will then show in Section 5.2 that the extra term $d\Gamma(W_{N,x})$ only shifts the limiting operator by a constant of order one. Indeed, the divergence in $\mathcal{H}_{\text{sing}}$ is expressed by the scalar (compare (3.14))

$$E_{N,W} := -2 \left\langle \nabla v_N \otimes \nabla v_N, \left((\nabla_{y_1} + \nabla_{y_2})^2 + \epsilon(i\nabla_{y_1}) + \epsilon(i\nabla_{y_2}) + W_N(y_1) + W_N(y_2) + 1 \right)^{-1} \nabla v_N \otimes \nabla v_N \right\rangle, \quad (5.8)$$

which differs from the same quantity with W_N omitted from the resolvent at order one, as shown in the proof of Theorem 1.2. The results of this Section are summarized by:

Proposition 5.1. *Recall the unitary operator U_κ^∞ defined in (3.4). For κ large enough and $0 < \alpha \leq 1/2$ we have*

$$\lim_{N \rightarrow \infty} (\mathcal{H}_N^U - E_{N,W}) = (U_\kappa^\infty)^* H_{\text{BF}} U_\kappa^\infty$$

in norm resolvent sense.

Proof of Proposition 5.1. In Proposition 3.5, we defined a self-adjoint operator associated to the quadratic form $K_\infty + T_\infty + R_\infty$, whose domain is contained in $Q(K_\infty) = (1 - G_\infty)^{-1}Q(-\Delta_x + d\Gamma(\epsilon))$. In (3.28) we proved that

$$H_{\text{BF}} = U_\kappa^\infty (K_\infty + T_\infty + R_\infty - 1) (U_\kappa^\infty)^*. \quad (5.9)$$

On the other side of the equation, we will show in Proposition 5.7 that $\mathcal{H}_N^U - E_{N,W}$ converges to $K_\infty + T_\infty + R_\infty - 1$ in norm resolvent sense. Together, these prove the claim. \square

5.1 Analysis of the simplified singular Hamiltonian

Let us denote

$$\mathcal{H}_{\text{Sing},0} = -\Delta_x + d\Gamma(\epsilon) + a^*(i\nabla v_{N,x})^2 + a(i\nabla v_{N,x})^2. \quad (5.10)$$

Note that here we do not cut off the interaction at $\mathcal{N}_+ \leq N^{1-\alpha}$, as this is unnecessary when the interaction term $d\Gamma(W_{N,x})$ is absent, and such a cutoff is not used in Section 3. Following the procedure from Section 3, we introduce

$$G_N := -(-\Delta_x + d\Gamma(\epsilon) + 1)^{-1} a^* (i\nabla v_{N,x})^2, \quad (5.11)$$

and rewrite

$$\mathcal{H}_{\text{Sing},0} = K_N + T_N + E_N - 1 \quad (5.12)$$

with

$$\begin{aligned} K_N &:= (1 - G_N^*)(-\Delta_x + d\Gamma(\epsilon) + 1)(1 - G_N), \\ T_N &:= -a(i\nabla v_{N,x})^2(-\Delta_x + d\Gamma(\epsilon) + 1)^{-1} a^* (i\nabla v_{N,x})^2 - E_N \end{aligned} \quad (5.13)$$

and

$$E_N := -2 \left\langle \nabla v_N \otimes \nabla v_N, \left((\nabla_{y_1} + \nabla_{y_2})^2 + \epsilon(i\nabla_{y_1}) + \epsilon(i\nabla_{y_2}) + 1 \right)^{-1} \nabla v_N \otimes \nabla v_N \right\rangle. \quad (5.14)$$

We also use the definitions of $G_\infty, K_\infty, T_\infty$ from Section 3.

We now show regularity estimates for G_N , and its convergence rate to G_∞ . We then use these to infer comparison estimates between the dressed K_N and the bare Hamiltonian $-\Delta_x + d\Gamma(\epsilon)$, as well as its convergence rate towards K_∞ .

Lemma 5.2 (Properties of G_N). *For all $0 \leq s < 1/2$ there exists $(C_\kappa)_{\kappa>0}$, C with $\lim_{\kappa \rightarrow \infty} C_\kappa = 0$ so that for all $\kappa > 0$, $N \in \mathbb{N} \cup \{\infty\}$*

$$\|G_N^*(-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{-s}\| \leq C_\kappa, \quad (5.15a)$$

and for all $\alpha > 0$

$$\|(G_N^* - G_\infty^*)(-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{-s}\| \leq CN^{(2s-1)\alpha}. \quad (5.15b)$$

Moreover, for $1/2 \leq s \leq 1$, $\epsilon > 0$ there is C so that for all $\kappa > 0$, $N \in \mathbb{N}$

$$\|G_N^*(-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{-s}\| \leq CN^{(s-\frac{1}{2})+\epsilon}. \quad (5.15c)$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &a^* (i\nabla v_{N,x})^2 a (i\nabla v_{N,x})^2 \\ &= \sum_{p,q,k,r} (k \cdot r)(p \cdot q) \widehat{v}_{N,x}(k) \widehat{v}_{N,x}(r) \widehat{v}_{N,x}(p) \widehat{v}_{N,x}(q) a_k^* a_r^* a_p a_q \\ &\leq \sum_{p,q,k,r} (p \cdot q)^2 \frac{|\widehat{v}_N(p)|^2}{\epsilon(p)^{1-s}} \frac{|\widehat{v}_N(q)|^2}{\epsilon(q)^{1-s}} \epsilon(k)^{1-s} \epsilon(r)^{1-s} a_k^* a_r^* a_r a_k \leq \left\| \left| k \right| \frac{\widehat{v}_N}{\epsilon^{\frac{1-s}{2}}} \right\|_2^4 d\Gamma(\epsilon^{1-s})^2. \end{aligned} \quad (5.16)$$

for all $s \geq 0$, $N \in \mathbb{N} \cup \{\infty\}$ for which the right hand side is finite. Using Hölder's inequality on each particle sector $\sum_{j=1}^k \epsilon_j^{1-s} \leq k^s \left(\sum_{j=1}^k \epsilon_j\right)^{1-s}$, we obtain

$$a^*(i\nabla v_{N,x})^2 a(i\nabla v_{N,x})^2 \leq \left\| \left| k \frac{\widehat{v}_N}{\epsilon^{\frac{1-s}{2}}} \right\|_2^4 (-\Delta_x + d\Gamma(\epsilon))^{2(1-s)} \mathcal{N}_+^{2s}, \quad (5.17)$$

since $0 \leq -\Delta_x$ and $d\Gamma(\epsilon)$ commute. Using (5.5), we have for $0 \leq s < 1/2$ that

$$C_\kappa := \left\| \left| k \frac{\widehat{v}_N}{\epsilon^{\frac{1-s}{2}}} \right\|_2^2 \lesssim \sum_{|k| > \kappa} \frac{1}{|k|^2 \epsilon(k)^{1-s}} \lesssim \kappa^{2s-1}. \quad (5.18)$$

Thus, denoting $\mathbb{H}_0 := -\Delta_x + d\Gamma(\epsilon) + 1$,

$$(\mathcal{N}_+ + 1)^{-s} \mathbb{H}_0^s G_N G_N^* \mathbb{H}_0^s (\mathcal{N}_+ + 1)^{-s} \leq C_\kappa^2, \quad (5.19)$$

which proves (5.15a).

To prove the statement for the difference (5.15b), note that $v_N(k) - v_\infty(k)$ is supported on the set where $|k| > N^\alpha$, so this follows from the same argument where now $\kappa = N^\alpha$.

By the same reasoning, we have for $s \geq 1/2$ and $N < \infty$

$$(\mathcal{N}_+ + 1)^{-s} \mathbb{H}_0^s G_N G_N^* \mathbb{H}_0^s (\mathcal{N}_+ + 1)^{-s} \lesssim C_N^2, \quad (5.20)$$

with, using Lemma 2.2,

$$C_N = \left\| \left| k \frac{\widehat{v}_N}{\epsilon^{\frac{1-s}{2}}} \right\|_2^2 \lesssim \sum_{|k| \leq N^\alpha} \frac{1}{|k|^{4-2s}} + N \|\nabla|^s \widehat{\varphi}_1\|_2^2 \lesssim N^{(s-\frac{1}{2})_+ + \varepsilon}. \quad (5.21)$$

□

Lemma 5.3 (Properties of K_N). *For κ sufficiently large, the following hold. The operators K_N with domain*

$$D(K_N) := (1 - G_N)^{-1} D(-\Delta_x + d\Gamma(\epsilon)),$$

are self-adjoint for any $N \in \mathbb{N} \cup \{\infty\}$. Moreover, for $0 \leq s \leq 1$ there is $C > 0$, such that for all $N \in \mathbb{N} \cup \{\infty\}$ we have

$$\|(-\Delta_x + d\Gamma(\epsilon))^s (1 - G_N) K_N^{-s}\| \leq C, \quad (5.22a)$$

$$\|\mathcal{N}_+^s K_N^{-s}\| \leq C. \quad (5.22b)$$

Furthermore, K_N converges to K_∞ in norm resolvent sense, and for $0 \leq s, \alpha < 1/2$, there exists $C > 0$ so that for all $N \in \mathbb{N}$

$$\left\| K_{N,0}^s (K_{N,0}^{-1} - K_\infty^{-1}) K_\infty^s \right\| \leq C N^{(2s-1)\alpha}. \quad (5.22c)$$

Proof. The operator $\mathbb{H}_0 = -\Delta + d\Gamma(\epsilon) + 1$ is invertible and, for κ sufficiently large, so is $1 - G_N$ (by Lemma 5.2), for all $N \in \mathbb{N} \cup \{\infty\}$. From this, one easily checks that K_N is invertible on its domain. Since it is manifestly symmetric, K_N must be self-adjoint.

We now prove the estimate (5.22a). We will argue by interpolation, so first let $s = 1$. By Lemma 5.2, for $N \in \mathbb{N} \cup \{\infty\}$, inserting $(1 - G_N^*)$ and its inverse yields

$$\|(-\Delta_x + d\Gamma(\epsilon) + 1)(1 - G_N)K_N^{-1}\| = \|(1 - G_N^*)^{-1}\| \leq (1 - C_\kappa)^{-1}. \quad (5.23)$$

For $s = 0$, we simply have, also by Lemma 5.2,

$$\|(-\Delta_x + d\Gamma(\epsilon) + 1)^0(1 - G_N)K_N^0\| \leq 1 + C_\kappa. \quad (5.24)$$

The estimate (5.22a) now follows by complex interpolation (cf. [65, Theorem IX.20]). To prove the bound (5.22b) on \mathcal{N}_+ , we use that $\mathcal{N}_+G_N = G_N(\mathcal{N}_+ + 2)$. Adding and subtracting G_N , we obtain

$$\|\mathcal{N}_+^s K_N^{-s}\| \leq \|\mathcal{N}_+^s(1 - G_N)K_N^{-s}\| + C_\kappa \|(\mathcal{N}_+ + 2)^s K_N^{-s}\|, \quad (5.25)$$

where we bounded G_N using Lemma 5.2. As $x \mapsto x^s$ is sub-additive for $s \leq 1$ this implies the claim (5.22b) if κ is sufficiently large for $C_\kappa < 1$ to hold.

We now show the convergence by proving the bound (5.22c). With the resolvent formula, we have

$$\begin{aligned} K_N^s(K_N^{-1} - K_\infty^{-1})K_\infty^s &= K_N^{-1+s}(K_\infty - K_N)K_\infty^{-1+s} \\ &= K_N^{s-1}\left((G_N^* - G_\infty^*)\mathbb{H}_0(1 - G_\infty) + (1 - G_N^*)\mathbb{H}_0(G_N - G_\infty)\right)K_\infty^{s-1}. \end{aligned} \quad (5.26)$$

Thus (5.15b) and (5.22a) imply that for $0 \leq s < 1/2$

$$\begin{aligned} &\left\|K_N^s(K_N^{-1} - K_\infty^{-1})K_\infty^s\right\| \\ &\lesssim \|K_N^{-1+s}(G_N^* - G_\infty^*)\mathbb{H}_0^s\| \|\mathbb{H}_0^{1-s}(1 - G_\infty)K_\infty^{-1+s}\| \\ &\quad + \|K_N^{-1+s}(1 - G_N^*)\mathbb{H}_0^{1-s}\| \|\mathbb{H}_0^s(G_N - G_\infty)K_\infty^{-1+s}\| \\ &\lesssim N^{(2s-1)\alpha}(\|(\mathcal{N}_+ + 2)^s K_N^{-1+s}\| + \|(\mathcal{N}_+ + 2)^s K_\infty^{-1+s}\|), \end{aligned} \quad (5.27)$$

where we used that $G_N^*\mathcal{N}_+ = (\mathcal{N}_+ + 2)G_N^*$. In view of (5.22b) this proves (5.22c). \square

We now discuss the operator T_N . Rewriting $\mathcal{H}_{\text{sing},0}$ as in (5.12) extracts the divergent scalar contribution E_N , and T_∞ is well defined as the limit of T_N . To make this explicit, we follow Section 3. Using the commutation rules

$$i\nabla_x e^{-ikx} = e^{-ikx}(i\nabla_x + k), \quad d\Gamma(\epsilon)a_k^* = a_k^*(d\Gamma(\epsilon) + \epsilon(k)) \quad (5.28)$$

we write T_N as a sum of terms in “normal order”,

$$T_N = \Theta_{N,0} + \Theta_{N,1} + \Theta_{N,2}, \quad (5.29)$$

with

$$\Theta_{N,0} = -2 \sum_{\substack{k, \ell \in 2\pi\mathbb{Z}^3 \\ |k|, |\ell| > \kappa}} \frac{(k \cdot \ell)^2 \widehat{v}_N(k)^2 \widehat{v}_N(\ell)^2}{(i\nabla_x - k - \ell)^2 + \epsilon(k) + \epsilon(\ell) + d\Gamma(\epsilon) + 1} - E_N \quad (5.30a)$$

$$\Theta_{N,1} = 4 \sum_{\substack{k, \ell \in 2\pi\mathbb{Z}^3 \\ |k|, |\ell| > \kappa}} e^{-ikx} a_k^* \theta_{N,1}(k, \ell) a_\ell e^{i\ell x} \quad (5.30b)$$

$$\Theta_{N,2} = \sum_{\substack{k_1, k_2 \in 2\pi\mathbb{Z}^3 \\ |k_1|, |k_2| > \kappa}} \sum_{\substack{\ell_1, \ell_2 \in 2\pi\mathbb{Z}^3 \\ |\ell_1|, |\ell_2| > \kappa}} e^{-ik_1 x} e^{-ik_2 x} a_{k_1}^* a_{k_2}^* \theta_{N,2}(k_1, k_2, \ell_1, \ell_2) a_{\ell_1} a_{\ell_2} e^{-i\ell_1 x} e^{i\ell_2 x}, \quad (5.30c)$$

where the kernels are defined as

$$\theta_{N,1}(k, \ell) = - \sum_{\substack{\xi \in 2\pi\mathbb{Z}^3 \\ |\xi| > \kappa}} \frac{(k \cdot \xi)(\ell \cdot \xi) \widehat{v}_N(k) \widehat{v}_N(\ell) \widehat{v}_N(\xi)^2}{(i\nabla_x + k - \ell - \xi)^2 + \epsilon(k) + \epsilon(\ell) + \epsilon(\xi) + d\Gamma(\epsilon) + 1} \quad (5.31)$$

and

$$\begin{aligned} \theta_{N,2}(k_1, k_2, \ell_1, \ell_2) & \quad (5.32) \\ &= - \frac{(k_1 \cdot k_2) \widehat{v}_N(k_1) \widehat{v}_N(k_2) (\ell_1 \cdot \ell_2) \widehat{v}_N(\ell_1) \widehat{v}_N(\ell_2)}{(i\nabla_x + k_1 + k_2 - \ell_1 - \ell_2)^2 + \epsilon(k_1) + \epsilon(k_2) + \epsilon(\ell_1) + \epsilon(\ell_2) + d\Gamma(\epsilon) + 1}. \end{aligned}$$

Observe that the series $\theta_{N,j}$ converge as functions of $i\nabla_x$ and $d\Gamma(\epsilon)$. As we will see, the operators $\Theta_{N,j}$ are bounded relatively to K_N . Because of this, T_∞ is well defined on $D(K_N)$. This entails:

Proposition 5.4. *For κ sufficiently large, $0 < \alpha \leq 1/2$, $N, N' \in \mathbb{N} \cup \{\infty\}$ and all $\Psi \in D(K_{N'})$*

$$\|T_N \Psi\| \leq C_\kappa \|K_{N'} \Psi\|$$

where $\lim_{\kappa \rightarrow \infty} C_\kappa = 0$. Moreover, for all $0 < \varepsilon < 1/4$ there is $C > 0$ so that for $N, N' \in \mathbb{N} \cup \{\infty\}$, $0 < \alpha \leq 1/2$

$$\|(T_N - T_\infty) K_{N'}^{-1}\| \leq C N^{-\alpha\varepsilon}.$$

In particular, for $\kappa > 0$ large enough, the operator $K_\infty + T_\infty$ is self-adjoint on

$$D(K_\infty) := (1 - G_\infty)^{-1} D(-\Delta_x + d\Gamma(\epsilon)).$$

and $\mathcal{H}_{\text{sing},0} - E_N$ converges to $K_\infty + T_\infty$ in norm resolvent sense.

Proof of Proposition 5.4. We split the proof of the bounds on $T_N = \Theta_{N,0} + \Theta_{N,1} + \Theta_{N,2}$ into two parts, one on uniform bounds for $\Theta_{N,j}$, and one on the differences $\Theta_{N,j} - \Theta_{\infty,j}$.

Lemma 5.5. *Let $0 < s < 1/2$. There is C_κ with $\lim_{\kappa \rightarrow \infty} C_\kappa = 0$ so that for all $\kappa > 0$, $N \in \mathbb{N} \cup \{\infty\}$ and $\Psi \in D(\mathcal{N}_+^{1-s}(-\Delta_x + d\Gamma(\epsilon)^s))$*

$$\begin{aligned}\|\Theta_{N,1}\Psi\| &\leq C_\kappa \|\mathcal{N}_+^{1/2-s} d\Gamma(\epsilon)^s \Psi\| \\ \|\Theta_{N,2}\Psi\| &\leq C_\kappa \|\mathcal{N}_+^{1-s} d\Gamma(\epsilon)^s \Psi\|.\end{aligned}$$

Moreover, for $\varepsilon > 0$ there is C so that for all $N \in \mathbb{N}$ and $\kappa > 0$

$$\|\Theta_{N,0}\Psi\| \leq C\kappa^{-\varepsilon} \|(-\Delta_x + d\Gamma(\epsilon))^\varepsilon \Psi\|.$$

Proof. We start by proving the estimate on $\Theta_{N,1}$. First note that using (5.5) we have the point-wise bound

$$|\theta_{N,1}(k, \ell)| \lesssim \frac{1}{|k||\ell|(\epsilon(k) + \epsilon(\ell) + d\Gamma(\epsilon))^{1/2}}. \quad (5.33)$$

Using this and the Cauchy-Schwarz inequality, we have that for any $\Phi \in \mathcal{H}_+$, $\Psi \in D(d\Gamma(\epsilon)^s \mathcal{N}_+^{1-s})$ and $\delta > 0$, denoting $t = s + 1/2 \in (1/2, 1)$,

$$\begin{aligned}\langle \Phi, \Theta_{N,1}\Psi \rangle &= \sum_{|k|, |\ell| > \kappa} \left\langle \epsilon(\ell)^{-(1+t)/2} \epsilon(k)^{t/2} a_k \Phi, \epsilon(\ell)^{(t+1)/2} \epsilon(k)^{-t/2} \theta_{N,1}(k, \ell) a_\ell \Psi \right\rangle \\ &\lesssim \delta \sum_{|k|, |\ell| > \kappa} \epsilon(\ell)^{-(1+t)} \epsilon(k)^t \left\| \frac{1}{(\epsilon(k) + \epsilon(\ell) + d\Gamma(\epsilon))^{t/2}} a_k \mathcal{N}_+^{-(1-t)/2} \Phi \right\|^2 \\ &\quad + \frac{1}{\delta} \sum_{|k|, |\ell| > \kappa} \epsilon(k)^{-(1+t)} \epsilon(\ell)^t \left\| \frac{1}{(\epsilon(k) + \epsilon(\ell) + d\Gamma(\epsilon))^{1/2-t/2}} a_\ell \mathcal{N}_+^{(1-t)/2} \Psi \right\|^2 \\ &\lesssim C_\kappa \left(\delta \|d\Gamma(\epsilon^t)^{1/2} d\Gamma(\epsilon)^{-t/2} \mathcal{N}_+^{-(1-t)/2} \Phi\|^2 \right. \\ &\quad \left. + \delta^{-1} \|d\Gamma(\epsilon^t)^{1/2} d\Gamma(\epsilon)^{-(1-t)/2} \mathcal{N}_+^{(1-t)/2} \Psi\|^2 \right) \\ &\lesssim C_\kappa \left(\delta \|\Phi\|^2 + \delta^{-1} \|d\Gamma(\epsilon)^{t-1/2} \mathcal{N}_+^{1-t} \Psi\|^2 \right) \\ &= C_\kappa \left(\delta \|\Phi\|^2 + \delta^{-1} \|d\Gamma(\epsilon)^s \mathcal{N}_+^{1/2-s} \Psi\|^2 \right),\end{aligned} \quad (5.34)$$

where we used that $d\Gamma(\epsilon^t) \leq d\Gamma(\epsilon)^t \mathcal{N}_+^{1-t}$ and denoted

$$C_\kappa = \sum_{|k| > \kappa} \epsilon(k)^{-(1+t)} \xrightarrow{\kappa \rightarrow \infty} 0. \quad (5.35)$$

Optimizing over δ yields the claim.

We now turn to the proof of the bound on $\Theta_{N,2}$. With the same reasoning as (5.33) we find the bound

$$|\theta_{N,2}(k_1, k_2, \ell_1, \ell_2)| \lesssim \frac{1}{|k_1||k_2||\ell_1||\ell_2|(\epsilon(k_1) + \epsilon(k_2) + \epsilon(\ell_1) + \epsilon(\ell_2) + d\Gamma(\epsilon))}. \quad (5.36)$$

Similarly as above, let $\Phi \in \mathcal{H}_+$, $\Psi \in \mathcal{D}(\mathcal{N}_+^{1-2s} d\Gamma(\epsilon)^{2s})$, $\delta > 0$ using the Cauchy-Schwarz inequality and denoting again $t = (1+s)/2 \in (1/2, 3/4)$, we obtain

$$\begin{aligned} & \langle \Phi, \Theta_{N,2} \Psi \rangle \\ & \lesssim \delta \sum_{\substack{|k_1|, |k_2| > \kappa \\ |\ell_1|, |\ell_2| > \kappa}} \epsilon(\ell_1)^{-(1+t)} \epsilon(\ell_2)^{-(1+t)} \epsilon(k_1)^t \epsilon(k_2)^t \\ & \quad \times \|(\epsilon(k_1) + \epsilon(k_2) + \epsilon(\ell_1) + \epsilon(\ell_2) + d\Gamma(\epsilon))^{-t} a_{k_1} a_{k_2} \mathcal{N}_+^{-2(1-t)} \Phi\|^2 \\ & + \frac{1}{\delta} \sum_{\substack{|k_1|, |k_2| > \kappa \\ |\ell_1|, |\ell_2| > \kappa}} \epsilon(k_1)^{-(1+t)} \epsilon(k_2)^{-(1+t)} \epsilon(\ell_1)^t \epsilon(\ell_2)^t \\ & \quad \times \|(\epsilon(k_1) + \epsilon(k_2) + \epsilon(\ell_1) + \epsilon(\ell_2) + d\Gamma(\epsilon))^{-1+t} a_{k_1} a_{k_2} \mathcal{N}_+^{2(1-t)} \Psi\|^2 \\ & \lesssim C_\kappa^2 (\delta \|d\Gamma(\epsilon^t) d\Gamma(\epsilon)^{-t} \mathcal{N}_+^{-(1-t)} \Phi\|^2 + \delta^{-1} \|d\Gamma(\epsilon^t) d\Gamma(\epsilon)^{-(1-t)} \mathcal{N}_+^{1-t} \Psi\|^2) \\ & \lesssim C_\kappa^2 (\delta \|\Phi\|^2 + \delta^{-1} \|d\Gamma(\epsilon)^s \mathcal{N}_+^{1-s} \Psi\|^2). \end{aligned} \quad (5.37)$$

Optimizing over δ proves the desired bound.

Finally, we prove the bound on $\Theta_{N,0}$. Since $\Theta_{N,0}$ is a function of the commuting operators $i\nabla_x$ and $d\Gamma(\epsilon)$, it is sufficient to prove a point-wise bound on this function, by the functional calculus. First, rewrite $\Theta_{N,0}$ as

$$\begin{aligned} \Theta_{N,0} = \sum_{k, \ell \in 2\pi\mathbb{Z}^3} & \left(\frac{2(k \cdot \ell)^2 \widehat{v}_N(k)^2 \widehat{v}_N(\ell)^2}{(i\nabla_x + k + \ell)^2 + \epsilon(k) + \epsilon(\ell) + d\Gamma(\epsilon) + 1} \right. \\ & \left. \times \frac{(i\nabla_x + k + \ell)^2 + d\Gamma(\epsilon) - (k + \ell)^2}{(k + \ell)^2 + \epsilon(k) + \epsilon(\ell) + 1} \right). \end{aligned} \quad (5.38)$$

Using that $|\widehat{v}_N(k)| \lesssim k^{-2}$, we have for all $\varepsilon > 0$,

$$\begin{aligned}
|\Theta_{N,0}| &\lesssim \sum_{|k|,|\ell|>\kappa} \frac{1}{|k|^2|\ell|^2(\epsilon(k) + \epsilon(\ell))} \frac{(\mathbf{i}\nabla_x)^2 + |\mathbf{i}\nabla_x||k| + \mathbf{d}\Gamma(\epsilon)}{((\mathbf{i}\nabla_x + k + \ell)^2 + \epsilon(k) + \epsilon(\ell) + \mathbf{d}\Gamma(\epsilon))} \\
&\lesssim (\mathbf{i}\nabla_x)^2 \sum_{|k|,|\ell|>\kappa} \frac{1}{|k|^3|\ell|^3} \frac{1}{(\mathbf{i}\nabla_x + k + \ell)^2 + |k|^2 + |\ell|^2} \\
&\quad + |\mathbf{i}\nabla_x| \sum_{|k|,|\ell|>\kappa} \frac{1}{|k|^3|\ell|^3} \frac{|k|}{(\mathbf{i}\nabla_x + k + \ell)^2 + |k|^2 + |\ell|^2} \\
&\quad + \mathbf{d}\Gamma(\epsilon) \sum_{|k|,|\ell|>\kappa} \frac{1}{|k|^3|\ell|^3} \frac{1}{\mathbf{d}\Gamma(\epsilon) + |k|^2 + |\ell|^2}. \tag{5.39}
\end{aligned}$$

Dropping $\mathbf{d}\Gamma(\epsilon)$, $(\mathbf{i}\nabla_x + k + \ell)^2$ from a power $(1 - \varepsilon)$ of the denominator and $|k|^2 + |\ell|^2$ from the remaining power ε , and using that that

$$\frac{1}{(\mathbf{i}\nabla_x + k + \ell)^2 + |k|^2 + |\ell|^2} \lesssim \frac{1}{(\mathbf{i}\nabla_x)^2}. \tag{5.40}$$

we obtain

$$|\Theta_{N,0}| \lesssim \kappa^{-\varepsilon} ((\mathbf{i}\nabla_x)^2 + \mathbf{d}\Gamma(\epsilon))^\varepsilon. \tag{5.41}$$

This proves the claim. \square

Lemma 5.6. *For $0 < \alpha, s < 1/2$, $0 < \varepsilon < s$ there is C so that for all $N \in \mathbb{N}$, $\kappa > 0$ and $\Psi \in D(\mathcal{N}_+^{1-s}(-\Delta_x + \mathbf{d}\Gamma(\epsilon))^s)$*

$$\begin{aligned}
\|(\Theta_{N,1} - \Theta_{\infty,1})\Psi\| &\leq CN^{-\alpha\varepsilon} \|\mathcal{N}_+^{\frac{1}{2} + \frac{\varepsilon}{2} - s} \mathbf{d}\Gamma(\epsilon)^s \Psi\|, \\
\|(\Theta_{N,2} - \Theta_{\infty,2})\Psi\| &\leq CN^{-\alpha\varepsilon} \|\mathcal{N}_+^{1-s} \mathbf{d}\Gamma(\epsilon)^s \Psi\|.
\end{aligned}$$

Moreover, for all $0 < \varepsilon \leq 2$ there is $C > 0$ so that for $N \in \mathbb{N}$, $\kappa > 0$

$$\|(\Theta_{N,0} - \Theta_{\infty,0})\Psi\| \leq CN^{-\alpha\varepsilon} \|(-\Delta_x + \mathbf{d}\Gamma(\epsilon))^\varepsilon \Psi\|.$$

Proof. To estimate the convergence rate as $N \rightarrow \infty$, we use (5.4), (5.5) in the form

$$|\widehat{v}_N(p) - \widehat{v}_\infty(p)| \lesssim \mathbf{1}_{|p|>N^\alpha} |p|^{-2} \leq N^{-\alpha\varepsilon} |p|^{-2+\varepsilon}. \tag{5.42}$$

From this it follows that

$$|\theta_{N,1}(k, \ell) - \theta_{\infty,1}(k, \ell)| \lesssim \frac{N^{-\alpha\varepsilon}}{|k|^{1-\varepsilon} |\ell|^{1-\varepsilon} (\epsilon(k) + \epsilon(\ell) + \mathbf{d}\Gamma(\epsilon))^{1/2-\varepsilon/2}} \tag{5.43}$$

and

$$\begin{aligned} & |\theta_{N,2} - \theta_{\infty,2}|(k_1, k_2, \ell_1, \ell_2) \\ & \lesssim \frac{N^{-\alpha\varepsilon}}{|k_1|^{1-\varepsilon}|k_2|^{1-\varepsilon}|\ell_1|^{1-\varepsilon}|\ell_2|^{1-\varepsilon}(\varepsilon(k_1) + \varepsilon(k_2) + \varepsilon(\ell_1) + \varepsilon(\ell_2) + d\Gamma(\varepsilon))}. \end{aligned} \quad (5.44)$$

The rest of the proof follows as in the proof of Lemma 5.5. We only give the details for $\Theta_{N,1}$. Choosing $t = s + \frac{1}{2} + \frac{\varepsilon}{2}$, from the condition on s and ε , we have $1 - \varepsilon + t > 3/2$, and therefore

$$\begin{aligned} & N^{\alpha\varepsilon} |\langle \Phi, (\Theta_{N,1} - \Theta_{\infty,1})\Psi \rangle| \\ & \lesssim \delta \sum_{|k|, |\ell| > \kappa} \varepsilon(\ell)^{-(1-\varepsilon+t)} \varepsilon(k)^{t-\varepsilon} \left\| \frac{(\mathcal{N}_+ + 1)^{-(1-t+\varepsilon)/2}}{(\varepsilon(k) + \varepsilon(\ell) + d\Gamma(\varepsilon))^{t/2-\varepsilon/2}} a_k \Phi \right\|^2 \\ & \quad + \frac{1}{\delta} \sum_{|k|, |\ell| > \kappa} \varepsilon(k)^{-(1-\varepsilon+t)} \varepsilon(\ell)^{t-\varepsilon} \left\| \frac{(\mathcal{N}_+ + 1)^{(1-t+\varepsilon)/2}}{(\varepsilon(k) + \varepsilon(\ell) + d\Gamma(\varepsilon))^{1/2-t/2}} a_\ell \Psi \right\|^2 \\ & \lesssim C_\kappa \left(\delta \|\Phi\|^2 + \delta^{-1} \|d\Gamma(\varepsilon)^s \mathcal{N}_+^{1/2-s+\varepsilon/2} \Psi\|^2 \right). \end{aligned} \quad (5.45)$$

As before, optimizing over δ gives the desired estimate. \square

We conclude this section with the proof of Proposition 5.4.

Proof of Proposition 5.4. Recall that $T_N = \Theta_{N,0} + \Theta_{N,1} + \Theta_{N,2}$, so in view of Lemmas 5.5 and 5.6, we only need to bound the operators $\mathcal{N}_+^{1/2-s} d\Gamma(\varepsilon)^s$, $\mathcal{N}_+^{1-2s} d\Gamma(\varepsilon)^{2s}$ and $(-\Delta_x + d\Gamma(\varepsilon))^s$, for $0 < s < 1/4$, relative to $K_{N'}$. The worst case is $\mathcal{N}_+^{1-2s} d\Gamma(\varepsilon)^{2s}$, for which we have

$$\begin{aligned} \|\mathcal{N}_+^{1-2s} d\Gamma(\varepsilon)^{2s} \Psi\| & \leq \|\mathcal{N}_+^{1-2s} d\Gamma(\varepsilon)^{2s} (1 - G_N) \Psi\| + \|\mathcal{N}_+^{1-2s} d\Gamma(\varepsilon)^{2s} G_N \Psi\| \\ & \lesssim \|K_{N'} \Psi\| + \|(\mathcal{N}_+ + 2) \Psi\| \lesssim \|(K_{N'} + 2) \Psi\|, \end{aligned} \quad (5.46)$$

by applying first (5.22a) and (5.15a), and then (5.22b).

This implies self-adjointness of $K_\infty + T_\infty$ by the Kato-Rellich theorem. Resolvent convergence follows from the convergence of T_N and the uniform relative bound. A detailed argument is provided in the more general case of Proposition 5.7. \square

5.2 Analysis of the full Hamiltonian

We now turn to the full Hamiltonian \mathcal{H}_N^U , starting with the singular part $\mathcal{H}_{\text{Sing}}$ (which includes $d\Gamma(W_{N,x})$). In analogy with the previous section and Section 3, we introduce

$$G_{N,W} = -(-\Delta_x + d\Gamma(\varepsilon) + d\Gamma(W_{N,x}) + 1)^{-1} \mathbf{1}_{\mathcal{N}_+ \leq N^{1-\alpha}} a^*(i\nabla v_{N,x})^2, \quad (5.47)$$

where now the particle number is cut off at $M = N^{1-\alpha}$ as in (5.2). Then, we can write

$$\mathcal{H}_{\text{Sing}} = K_{N,W} + T_{N,W} + E_{N,W} - 1, \quad (5.48)$$

with $E_{N,W}$ defined in (5.8) and

$$K_{N,W} := (1 - G_{N,W}^*)(-\Delta_x + d\Gamma(\epsilon) + d\Gamma(W_{N,x}) + 1)(1 - G_{N,W}), \quad (5.49)$$

$$\begin{aligned} T_{N,W} &:= -\mathbf{1}_{\mathcal{N}_+ \leq N^{1-\alpha}} a(i\nabla v_{N,x})^2 (-\Delta_x + d\Gamma(\epsilon) + d\Gamma(W_{N,x}) + 1)^{-1} a^*(i\nabla v_{N,x})^2 \\ &\quad - E_{N,W}. \end{aligned} \quad (5.50)$$

In this section we prove that $\mathcal{H}_{\text{Sing}} + R_N - E_{N,W} + 1$ converges to the self-adjoint operator $K_\infty + T_\infty + R_\infty$ in norm resolvent sense, where R_∞ is defined by setting $N = \infty$ in R_N , as in Section 3.

Proposition 5.7. *For κ sufficiently large there exists C so that for all $0 < \alpha \leq 1/2$, $\epsilon > 0$, and $N \in \mathbb{N}$ we have*

$$\|(\mathcal{H}_N^U - E_{N,W} + 1 \pm i)^{-1} - (K_\infty + T_\infty + R_\infty \pm i)^{-1}\| \leq CN^{-(\frac{1}{4}-\epsilon)\alpha}.$$

Moreover, for $0 \leq s < 5/8$ there exists C so that for all $N \in \mathbb{N}$

$$\|K_{N,W}^s (\mathcal{H}_N^U - E_{N,W} \pm i)^{-1}\| \leq C.$$

In order to prove this, we proceed similarly as in the previous section and first establish bounds on $G_{N,W}$, $K_{N,W}$, as well as their differences to G_N , K_N .

Lemma 5.8 (Properties of $G_{N,W}$). *For $0 \leq s \leq 5/8$ there exists $(C_\kappa)_{\kappa>0}$, C with $\lim_{\kappa \rightarrow \infty} C_\kappa = 0$ so that for all $\kappa, \alpha > 0$, and sufficiently large $N \in \mathbb{N}$*

$$\|G_{N,W}^*(-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{-s}\| \leq C_\kappa, \quad s < 1/2, \quad (5.51a)$$

$$\|G_{N,W}^*(-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{-s}\| \leq CN^{s-\frac{1}{2}+\epsilon}, \quad s \geq 1/2. \quad (5.51b)$$

Moreover, for $0 \leq s \leq 1/2$ there is C so that for all $\kappa, \alpha > 0$, $N \in \mathbb{N}$,

$$\|(G_{N,W}^* - G_N^*)(-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{-\frac{1}{2}}\| \leq CN^{(-\frac{1}{4}+(s-\frac{1}{4})_+)(1-\alpha)} \quad (5.51c)$$

Proof. Denote $\mathbb{H}_0 = -\Delta_x + d\Gamma(\epsilon) + 1$, $\mathbb{W} = d\Gamma(W_{N,x})$. From the proof of Lemma 5.2, specifically (5.17), using that $\mathbb{W} \geq 0$ we obtain the a priori bound on $G_{N,W}$,

$$\begin{aligned} &(\mathcal{N}_+ + 1)^{-1/2} \mathbb{H}_0^{1/2} G_{N,W} G_{N,W}^* \mathbb{H}_0^{1/2} (\mathcal{N}_+ + 1)^{-1/2} \\ &\lesssim C_N^2 \mathbb{H}_0^{1/2} (\mathbb{H}_0 + \mathbb{W})^{-1} \mathbb{H}_0 (\mathbb{H}_0 + \mathbb{W})^{-1} \mathbb{H}_0^{1/2} \lesssim C_N^2, \end{aligned} \quad (5.52)$$

with the constant

$$C_N = \|\epsilon^{-1/4}|k|\hat{v}_N\|_2^2 \lesssim \sum_{|k| \leq N^\alpha} \frac{1}{|k|^3} + N\|\nabla|^{1/2}\widehat{\varphi}_I\|_2^2 \lesssim \log N, \quad (5.53)$$

by Lemma 2.2. This proves that $\|G_{N,W}^*\mathbb{H}_0^{1/2}(\mathcal{N}_+ + 1)^{-1/2}\| \lesssim \log N$. Now, by the resolvent formula,

$$\begin{aligned} G_{N,W}^* - G_N^* \mathbb{1}_{\mathcal{N}_+ \leq N^{1-\alpha}} &= -a(i\nabla v_{N,x})^2 \mathbb{1}_{\mathcal{N}_+ \leq N^{1-\alpha}} ((\mathbb{H}_0 + \mathbb{W})^{-1} - \mathbb{H}_0^{-1}) \\ &= -G_{N,W}^* \mathbb{W} \mathbb{H}_0^{-1}. \end{aligned} \quad (5.54)$$

As we already have bounds on $\|G_N^* \mathbb{H}_0^s (\mathcal{N}_+ + 1)^{-s} \mathbb{1}_{\mathcal{N}_+ \leq N^{1-\alpha}}\| \leq \|G_N^* \mathbb{H}_0^s (\mathcal{N}_+ + 1)^{-s}\|$ from Lemma 5.2, we will prove the bounds on $G_{N,W}$ by estimating the right hand side. With (5.52) we have

$$\|G_{N,W}^* \mathbb{W} \mathbb{H}_0^{-1+s} (\mathcal{N}_+ + 1)^{-s}\| \lesssim \log N \|\mathbb{H}_0^{-1/2} \mathbb{W} \mathbb{H}_0^{s-1} (\mathcal{N}_+ + 1)^{1/2-s} \mathbb{1}_{\mathcal{N}_+ \leq N^{1-\alpha}}\|. \quad (5.55)$$

To bound this, we use that the Sobolev inequality on the torus gives

$$W_N \lesssim \|W_N\|_{3/(2t)} (-\Delta + 1)^t, \quad (5.56)$$

for $t \in (0, 3/2)$, from which we obtain that

$$d\Gamma(W_{N,x}) \lesssim N^{1-t} d\Gamma(\epsilon^t) \lesssim N^{1-t} \mathcal{N}_+^{(1-t)_+} d\Gamma(\epsilon)^t \quad (5.57)$$

uniformly in x . Note we assume that $W \in L^1 \cap L^2$, which corresponds to $t \in [3/4, 3/2]$. Consider first the case $s \leq 1/4$. Applying the inequality once for $t = 1$ and once for $t = 3/2 - 2\epsilon$, for $0 < \epsilon < 3/8$ yields

$$\|\mathbb{H}_0^{-1/2} \mathbb{W} \mathbb{H}_0^{s-1} (\mathcal{N}_+ + 1)^{1/2-s} \mathbb{1}_{\mathcal{N}_+ \leq N^{1-\alpha}}\| \lesssim N^{-\alpha(1/4-\epsilon)}. \quad (5.58)$$

Similarly, for $1/4 \leq s < 1/2$, we use $t = 2(1-s) > 1$ and find

$$\|\mathbb{H}_0^{-1/2} \mathbb{W} \mathbb{H}_0^{s-1} (\mathcal{N}_+ + 1)^{1/2-s} \mathbb{1}_{\mathcal{N}_+ \leq N^{1-\alpha}}\| \lesssim N^{-\alpha(1/2-s)}. \quad (5.59)$$

Choosing N sufficiently large then proves the claim (5.51a). The inequality (5.51b), where $s \geq 1/2$, follows from the same the reasoning, but here we do not need to use the cutoff $\mathbb{1}_{\mathcal{N}_+ \leq N^{1-\alpha}}$ or choose N large (the restriction $s \leq 5/8$ ensures that only the norms $\|W_N\|_p$ with $1 \leq p \leq 2$ intervene). To prove the final bound (5.51c), we use again the resolvent formula to obtain

$$\begin{aligned} &\|(G_{N,W}^* - G_N^*) \mathbb{H}_0^s (\mathcal{N}_+ + 1)^{-\frac{1}{2}}\| \\ &\leq \|G_N^* \mathbb{H}_0^s (\mathcal{N}_+ + 1)^{-\frac{1}{2}} \mathbb{1}_{\mathcal{N}_+ > N^{1-\alpha}}\| + \|G_{N,W}^* \mathbb{W} \mathbb{H}_0^{-1+s} (\mathcal{N}_+ + 1)^{-\frac{1}{2}}\|. \end{aligned} \quad (5.60)$$

The first term is bounded by $N^{(1-\alpha)(s-1/2)}$ by Lemma 5.2, and the second by $N^{-\frac{1}{4}+(s-\frac{1}{4})+\varepsilon}$, as before (without having to use the cutoff on \mathcal{N}_+ due to the larger power of $(\mathcal{N}_+ + 1)^{-1}$). Both of these quantities are smaller than the claimed one for an appropriate choice of ε (depending on α, s). This proves the claim. \square

Lemma 5.9 (Properties of $K_{N,W}$). *For κ sufficiently large and $0 < \alpha \leq 1/2$ the following holds. The operators $K_{N,W}$ are self-adjoint and converge to K_∞ in norm-resolvent sense as $N \rightarrow \infty$.*

Moreover, for $0 \leq s \leq 1$ there is $C > 0$, such that for large enough $N \in \mathbb{N}$

$$\|(-\Delta_x + d\Gamma(\epsilon) + d\Gamma(W_{N,x}))^s(1 - G_{N,W})K_{N,W}^{-s}\| \leq C, \quad (5.61a)$$

$$\|\mathcal{N}_+^s K_{N,W}^{-s}\| \leq C, \quad (5.61b)$$

and for $0 \leq s < 1/2$, $\varepsilon > 0$ there exists $C > 0$ so that for $N \in \mathbb{N}$ sufficiently large

$$\|K_{N,W}^s(K_{N,W}^{-1} - K_N^{-1})K_N^s\| \leq CN^{(-\frac{1}{4}+(s-\frac{1}{4})_+)(1-\alpha)} \quad (5.61c)$$

Proof. In view of the bounds on $G_{N,W}$ from Lemma 5.8, self-adjointness and the estimates (5.61a), (5.61b) follow from exactly the same argument as their analogues for K_N in Lemma 5.3. Convergence to K_∞ will follow once we have established (5.61c), since we already know that K_N converges to K_∞ .

It remains to prove (5.61c). Similarly as above, we have with $\mathbb{W} = d\Gamma(W_{N,x})$, $\mathbb{H}_0 = -\Delta_x + d\Gamma(\epsilon) + 1$,

$$\begin{aligned} K_{N,W}^s(K_{N,W}^{-1} - K_N^{-1})K_N^s &= K_{N,W}^{s-1}(K_N - K_{N,W})K_N^{s-1} \\ &= K_{N,W}^{s-1}((1 - G_{N,W}^*)(\mathbb{H}_0 + \mathbb{W})(G_N - G_{N,W}))K_N^{s-1} \end{aligned} \quad (5.62a)$$

$$+ K_{N,W}^{s-1}((1 - G_{N,W}^*)\mathbb{W}(1 - G_N))K_N^{s-1} \quad (5.62b)$$

$$+ K_{N,W}^{s-1}((G_N^* - G_{N,W}^*)\mathbb{H}_0(1 - G_N))K_N^{s-1}. \quad (5.62c)$$

To bound (5.62a), we use that $\|(\mathbb{H}_0 + \mathbb{W})^s \mathbb{H}_0^{-s}\| \lesssim 1$ (which holds by (5.57) and for $s = 1/2$ by operator monotonicity or interpolation) to obtain

$$\begin{aligned} \|(5.62a)\| &\stackrel{(5.61a)}{\lesssim} \|(\mathbb{H}_0 + \mathbb{W})^s(G_N - G_{N,W})K_N^{s-1}\| \lesssim \|\mathbb{H}_0^s(G_N - G_{N,W})K_N^{s-1}\| \\ &\stackrel{(5.51c)}{\lesssim} N^{(-\frac{1}{4}+(s-\frac{1}{4})_+)(1-\alpha)} \|(\mathcal{N}_+ + 3)^{\frac{1}{2}} K_N^{s-1}\|. \end{aligned} \quad (5.63)$$

The final quantity is bounded by (5.22b), which proves the required estimate. The bound on (5.62c) is obtained in the same way.

The middle term (5.62b) is bounded using (5.22a), (5.61a) and (5.57), to obtain for $\varepsilon > 0$

$$\begin{aligned} \|(5.62b)\| &\lesssim \|K_{N,W}^{s-1}(1 - G_{N,W}^*)(\mathbb{H}_0 + \mathbb{W})^{1/2}\| \|(\mathbb{H}_0 + \mathbb{W})^{-1/2}\mathbb{H}_0^{1/2}\| \\ &\quad \times \|d\Gamma(\varepsilon)^{-1/2}\mathbb{W}d\Gamma(\varepsilon)^{s-1}\| \|d\Gamma(\varepsilon)^{1-s}(1 - G_N)K_{N,0}^{s-1}\| \\ &\lesssim N^{-\frac{1}{4}+(s-\frac{1}{4})_++\varepsilon}. \end{aligned} \quad (5.64)$$

This concludes the proof of (5.61c), and the proposition. \square

Corollary 5.10 (Comparison estimates for $K_{N,W}$). *For all $0 \leq s < 1/2$, and $s \leq t \leq 1$ we have for κ, N large enough*

$$\|\mathcal{N}_+^{t-s}(-\Delta_x + d\Gamma(\varepsilon))^s K_{N,W}^{-t}\| \lesssim 1.$$

Moreover, for $0 < s, \varepsilon \leq 1/8$

$$\|(-\Delta_x + d\Gamma(\varepsilon))^{1/2+s} K_{N,W}^{-1/2-s}\| \lesssim N^{s+\varepsilon}. \quad (5.65)$$

Proof. Let us denote $\mathbb{H}_0 = -\Delta_x + d\Gamma(\varepsilon) + 1$ and $\mathbb{W} = d\Gamma(W_{N,x})$. Using Lemma 5.8, we obtain with $\mathcal{N}_+ G_{N,0} = G_{N,0}(\mathcal{N}_+ + 2)$

$$\begin{aligned} &\|\mathcal{N}_+^{t-s}\mathbb{H}_0^s K_{N,W}^{-t}\| \\ &\lesssim \|\mathcal{N}_+^{t-s}(\mathbb{H}_0 + \mathbb{W} + 1)^s(1 - G_{N,W})K_{N,W}^{-t}\| + \|\mathcal{N}_+^{t-s}(\mathbb{H}_0 + \mathbb{W})^s G_{N,W} K_{N,W}^{-t}\| \\ &\lesssim \|(\mathbb{H}_0 + \mathbb{W} + 1)^t(1 - G_{N,W})K_{N,W}^{-t}\| + C_\kappa \|(\mathcal{N}_+ + 2)^t K_{N,W}^{-t}\|. \end{aligned} \quad (5.66)$$

We also used that $\mathbb{W} \geq 0$ and that $s < 1/2$. The first bound thus follows from (5.61a) and (5.61b). To prove the second bound, the same calculation using (5.51b) yields

$$\begin{aligned} &\|\mathbb{H}_0^{1/2+s} K_{N,W}^{-1/2-s}\| \\ &\lesssim \|\mathbb{H}_0^{1/2+s}(1 - G_{N,W})K_{N,W}^{-1/2-s}\| + CN^{s+\varepsilon} \|(\mathcal{N}_+ + 1)^{1/2+s} K_{N,W}^{-1/2-s}\|. \end{aligned} \quad (5.67)$$

The second term on the right hand side is bounded using (5.61b). For the first term, we use (5.61a) to obtain

$$\begin{aligned} &\|\mathbb{H}_0^{1/2+s}(1 - G_{N,W})K_{N,W}^{-1/2-s}\| \lesssim \|\mathbb{H}_0^{1/2+s}(\mathbb{H}_0 + \mathbb{W})^{-1/2-s}\| \\ &\lesssim \|\mathbb{H}_0^{-1/2+s}(\mathbb{H}_0 + \mathbb{W})^{1/2-s}\| + \|\mathbb{H}_0^{s-1/2}\mathbb{W}(\mathbb{H}_0 + \mathbb{W})^{-1/2-s}\| \\ &\lesssim 1 + \|W_N\|_{3/2}^{1/2} \|W_N\|_{3/(2-4s)}^{1/2} \lesssim N^s, \end{aligned} \quad (5.68)$$

where we used again that $\mathbb{H}_0^t(\mathbb{H}_0 + \mathbb{W})^{-t}$ is uniformly bounded for $-1/2 \leq t \leq 1/2$, and (5.57). The restriction $s \leq 1/8$ means that $3/(2-4s) \leq 2$, so the corresponding norm of W_N is finite by hypothesis. \square

Lemma 5.11. For all $0 < s < 1/4$ there are $C, (C_\kappa)_{\kappa>0}$ with $C_\kappa \rightarrow 0$ as $\kappa \rightarrow \infty$ such that for all $\kappa > 0$, and sufficiently large $N \in \mathbb{N}$, it holds that for all $\Psi \in D(\mathcal{N}_+^{1/2-s/2}(-\Delta_x + d\Gamma(\epsilon))^s)$

$$|\langle \Psi, T_{N,W} \Psi \rangle| \leq C_\kappa \|(-\Delta_x + d\Gamma(\epsilon))^s \mathcal{N}_+^{1/2-s} \Psi\|^2$$

and

$$|\langle \Psi, (T_{N,W} - T_N) \Psi \rangle| \leq CN^{-(1-\alpha)s} \|\mathcal{N}_+^{(1-s)/2} (-\Delta_x + d\Gamma(\epsilon))^s \Psi\|^2.$$

It is important to note that this lemma together with Lemma 5.6 implies that $\lim_{N \rightarrow \infty} \langle \Psi, T_{N,W} \Psi \rangle = \langle \Psi, T_\infty \Psi \rangle$, which is crucial for the fact that $d\Gamma(W_{N,x})$ does not change the excitation spectrum. It only contributes a scalar term of order one which is $\lim_{N \rightarrow \infty} (E_{N,W} - E_N)$. The fact that $d\Gamma(W_{N,x})$ does contribute at order one in this way makes the proof of Lemma 5.11 rather subtle. This proof is given in Section 5.2.1 below.

Lemma 5.12. For $3/8 < s \leq 1/2$, $|t| \leq 1/2 - s$ and $\delta > 0$, there exists κ_0 , so that for all $\kappa \geq \kappa_0$ there is C with

$$\begin{aligned} |\langle \Phi, R_N \Psi \rangle| &\leq \delta \left\| \left\{ (-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{1/2-s+t} + C \right\} \Phi \right\| \\ &\quad \times \left\| \left\{ (-\Delta_x + d\Gamma(\epsilon))^s (\mathcal{N}_+ + 1)^{1/2-s-t} + C \right\} \Psi \right\|. \end{aligned} \quad (5.69a)$$

for all $N \in \mathbb{N} \cup \{\infty\}$, $\Phi, \Psi \in D((-\Delta_x + d\Gamma(\epsilon))^s \mathcal{N}_+^{1/2+t})$. Moreover, for $3/8 < s < 1/2$ there is C so that for all $N \in \mathbb{N}$, $\kappa > 0$

$$\pm(R_N - R_\infty) \leq CN^{-(4s-3/2)\alpha} (-\Delta_x + d\Gamma(\epsilon))^{2s} (\mathcal{N}_+ + 1)^{1-2s}. \quad (5.69b)$$

The proof of this Lemma is given in Section 5.2.2 below. These estimates allow us to define a self-adjoint realization of $K_{N,W} + T_{N',W} + R_{N'}$.

Lemma 5.13. For $\kappa > 0$, $N' \in \mathbb{N} \cup \{\infty\}$ sufficiently large and all $N \in \mathbb{N} \cup \{\infty\}$ the quadratic form

$$\langle \Psi, (K_{N,W} + T_{N',W}) \Psi \rangle + \langle \Psi, R_{N'} \Psi \rangle$$

defines a unique self-adjoint operator $K_{N,W} + T_{N',W} + R_{N'}$ with domain contained in $Q(K_{N,W})$.

Moreover, for $3/8 < t < 5/8$ there exists a constant C so that for all $N \in \mathbb{N} \cup \{\infty\}$ and $N' \in \mathbb{N} \cup \{\infty\}$ sufficiently large

$$\|K_{N,W}^t (K_{N,W} + T_{N',W} + R_{N'} + i)^{-1} K_{N,W}^{1-t}\| \leq C.$$

Proof. Recall that by Corollary 5.10 we can bound uniformly the quadratic forms $(-\Delta + d\Gamma(\epsilon))^s \mathcal{N}_+^{1-s}$ by $K_{N,W}$ for all $0 \leq s < 1$. Therefore, it follows from Lemmas 5.11 and 5.12 that $T_{N',W}$ and $R_{N'}$ are form bounded by $K_{N,W}$, uniformly in N, N' . The same lemmas show that for κ and N' large enough the relative constant can be chosen smaller than one. This allows us to apply the KLMN theorem [65, Theorem X.17]: the quadratic form $K_{N,W} + T_{N',W} + R_{N'}$ then defines a unique self-adjoint operator with domain contained in $Q(K_{N,W})$. This also implies the resolvent bound for $t = 1/2$. Moreover, Lemmas 5.12 and 5.11 imply that

$$\begin{aligned} \|(K_{N,W} + C)^{t-1} R_{N'} (K_{N,W} + C)^{-t}\| &< 1 \\ \|(K_{N,W} + C)^{t-1} T_{N',W} (K_{N,W} + C)^{-t}\| &< 1 \end{aligned} \quad (5.70)$$

for $3/8 < t < 5/8$ and κ, C, N' sufficiently large. This implies the bound in the general case by application of the Kato-Rellich Theorem to $K_{N,W}$ and $(K_{N,W} + C)^{t-1} R_{N'} (K_{N,W} + C)^{1-t}$, $(K_{N,W} + C)^{t-1} T_{N',W} (K_{N,W} + C)^{1-t}$. \square

We can now prove Proposition 5.7.

Proof of Proposition 5.7. Recall that $\mathcal{H}_N^U - E_{N,W} + 1 = K_{N,W} + T_{N,W} + R_N$, hence the resolvent identity gives

$$\begin{aligned} &(\mathcal{H}_N^U - E_{N,W} + 1 \pm i)^{-1} - (K_\infty + T_\infty + R_\infty \pm i)^{-1} \\ &= (K_{N,W} + T_{N,W} + R_N \pm i)^{-1} (T_\infty + R_\infty - T_{N,W} - R_N) \\ &\quad \times (K_{N,W} + T_\infty + R_\infty \pm i)^{-1} \end{aligned} \quad (5.71a)$$

$$\begin{aligned} &+ (K_{N,W} + T_\infty + R_\infty \pm i)^{-1} (K_N - K_{N,W}) \\ &\quad \times (K_N + T_\infty + R_\infty \pm i)^{-1} \end{aligned} \quad (5.71b)$$

$$\begin{aligned} &+ (K_N + T_\infty + R_\infty \pm i)^{-1} (K_\infty - K_N) \\ &\quad \times (K_\infty + T_\infty + R_\infty \pm i)^{-1}. \end{aligned} \quad (5.71c)$$

Using the bound from Lemma 5.13, the resolvents can absorb factors of $K_{N,W}^t$ for $t < 5/8$. Therefore we will obtain the convergence from the estimates on $T_{N,W} - T_N$ in Lemma 5.11, on $T_N - T_\infty$ in Lemma 5.6 and on $R_N - R_\infty$ in Lemma 5.12, together with the comparison estimates between $(-\Delta_x + d\Gamma(\epsilon))^s \mathcal{N}_+^{t-s}$ and $K_{N,W}^{-t}$ in Corollary 5.10. Explicitly, we obtain from Lemma 5.11, Lemma 5.6 with $s = 1/8 - \epsilon$ and (5.69b) with $s = 1/2 - \epsilon$, for $0 < \epsilon < 1/8$,

$$\|(5.71a)\| \lesssim N^{(-1/8+\epsilon)(1-\alpha)} + N^{-(1/2-4\epsilon)\alpha}. \quad (5.72)$$

From (5.61c) with $s = 3/8 + \epsilon$ for any $0 < \epsilon < 1/8$,

$$\|(5.71b)\| \lesssim \|K_{N,W}^s (K_{N,0}^{-1} - K_{N,W}^{-1}) K_{N,0}^s\| \lesssim N^{(-1/8+\epsilon)(1-\alpha)}. \quad (5.73)$$

and from (5.22c) with $s = 3/8 + \varepsilon$, for any $0 < \varepsilon < 1/8$

$$\|(5.71c)\| \lesssim \|K_{N,0}^s(K_{N,0}^{-1} - K_\infty^{-1})K_\infty^s\| \lesssim N^{-(1/4-2\varepsilon)\alpha}. \quad (5.74)$$

Gathering the above estimates proves the claimed convergence, and inserting that $M^{-1/8+\varepsilon} \lesssim N^{-(1/4-2\varepsilon)\alpha}$ for $M = N^{1-\alpha}$ and $\alpha \leq 1/2$ yields the stated rate. The relative bound for $K_{N,W}^s$ is a direct corollary to Lemma 5.13. \square

5.2.1 Proof of Lemma 5.11

We again use the notation $\mathbb{H}_0 = -\Delta_x + d\Gamma(\varepsilon) + 1$, $\mathbb{W} = d\Gamma(W_{N,x})$. We start by proving the bound on the difference $T_{N,W} - T_N$, which will also imply the bound on $T_{N,W}$ by using Lemma 5.5. We first introduce a cutoff at $\mathbb{1}_{\mathcal{N}_+ \leq M}$ into T_N , to match that of $T_{N,W}$. Since T_N commutes with \mathcal{N}_+ , Lemma 5.5 gives us for $0 < s < 1/4$,

$$\begin{aligned} |\langle \Psi, T_N \mathbb{1}_{\mathcal{N}_+ > M} \Psi \rangle| &\leq M^{-s} |\langle \Psi, T_N \mathcal{N}_+^s \Psi \rangle| \\ &\lesssim N^{-(1-\alpha)s} \|\mathcal{N}_+^{(1+s)/2} \Psi\| \|\mathcal{N}_+^{(1-s)/2} \mathbb{H}_0^s \Psi\|. \end{aligned} \quad (5.75)$$

It thus remains to consider the difference $T_N \mathbb{1}_{\mathcal{N}_+ \leq M} - T_{N,W}$. Recall that $T_{N,W}$ is given by (5.50). To evaluate the difference with $T_{N,0}$, we decompose $T_{N,W}$ similarly to (5.29), in the form

$$T_{N,W} = \Theta_{N,W,0} + \Theta_{N,W,1} + \Theta_{N,W,2}. \quad (5.76)$$

Since $T_{N,W}$ commutes with \mathcal{N}_+ it is characterized by its restriction to the n -boson sectors $\mathcal{H}_+^{(n)}$ for $n \leq M$. Denote by $(\tau_x f)(y) = f(x - y)$ the translation by x on $L^2(\mathbb{T}_x^3) \otimes L^2(\mathbb{T}_y^3) = L^2(\mathbb{T}_x^3, L^2(\mathbb{T}_y^3))$ and set $\mathcal{T}_x = \tau_x \otimes \tau_x \otimes \mathbb{1}$, which satisfies

$$\mathcal{T}_x^*(-\Delta_x)\mathcal{T}_x = |\nabla_x - \nabla_{y_1} - \nabla_{y_2}|^2. \quad (5.77)$$

Moreover, denote

$$\mathbb{H}^{(2)} = -(\nabla_{y_1} + \nabla_{y_2})^2 + \varepsilon(i\nabla_{y_1}) + \varepsilon(i\nabla_{y_2}) + W_N(y_1) + W_N(y_2) + 1. \quad (5.78)$$

For $\Psi^{(n)} \in \mathcal{H}_+^{(n)} \subset L^2(\mathbb{T}_x^3, L^2(\mathbb{T}^3)^{\otimes sn})$ we then set

$$\langle \Psi^{(n)}, \Theta_{N,W,0} \Psi^{(n)} \rangle \quad (5.79a)$$

$$= -2 \langle \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)}, \mathcal{T}_x^*(\mathbb{H}_0 + \mathbb{W})^{-1} \mathcal{T}_x - (\mathbb{H}^{(2)})^{-1} \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)} \rangle,$$

$$\langle \Psi^{(n)}, \Theta_{N,W,1} \Psi^{(n)} \rangle \quad (5.79b)$$

$$= -4n \langle \nabla v_{N,x} \otimes \nabla v_{N,x} \otimes \Psi^{(n)}, (\mathbb{H}_0 + \mathbb{W})^{-1} \nabla v_{N,x} \otimes \Psi^{(n)} \otimes \nabla v_{N,x} \rangle,$$

$$\langle \Psi^{(n)}, \Theta_{N,W,2} \Psi^{(n)} \rangle \quad (5.79c)$$

$$= -n(n-1) \langle \nabla v_{N,x} \otimes \nabla v_{N,x} \otimes \Psi^{(n)}, (\mathbb{H}_0 + \mathbb{W})^{-1} \Psi^{(n)} \otimes \nabla v_{N,x} \otimes \nabla v_{N,x} \rangle.$$

Note that the tensor products here do not involve symmetrization. The coordinate x of the particle is always integrated over at the end, while the tensor products concern the variables y_1, \dots, y_{n+2} . A simple computation shows that these operators sum to $T_{N,W}$. The operators $\Theta_{N,0}, \Theta_{N,1}, \Theta_{N,2}$ without $d\Gamma(W_{N,x})$ can of course be expressed in the same way, with the difference that \mathbb{W} is set to zero.

When calculating the difference of these operators, the resolvent formula yields expressions involving $W_{N,x}(y_i)$ for $i = 1, \dots, n+2$. The key is to use the regularity of $\Psi^{(n)}$ in either y_i or x . We need to take care, however, since not all of the resolvents commute with derivatives in these directions.

We start by simplifying $\Theta_{N,W,0}$ by replacing $\mathbb{H}_0 + \mathbb{W}$ with $\mathbb{H}_0 + W_{N,x}(y_1) + W_{N,x}(y_2)$, which no longer contains the potentials in y_3, \dots, y_{n+2} and thus commutes with derivatives in these directions. To estimate the difference, we will use that

$$\begin{aligned} & \|(\epsilon^{1/4} \otimes \epsilon^{1/4} \otimes \mathbf{1})(\mathbb{H}_0 + \mathbb{W})^{-1/2}\| \\ & \leq \|(\epsilon^{1/4} \otimes \mathbf{1})\mathbb{H}_0^{-1/4}\|^2 \|\mathbb{H}_0^{1/2}(\mathbb{H}_0 + \mathbb{W})^{-1/2}\| \lesssim 1. \end{aligned} \quad (5.80)$$

We thus take $0 < s < 1/4$, and bound

$$\begin{aligned} & \left| \left\langle \nabla v_{N,x} \otimes \nabla v_{N,x} \otimes \Psi^{(n)}, \right. \right. \\ & \quad \left. \left. (\mathbb{H}_0 + \mathbb{W})^{-1} - (\mathbb{H}_0 + W_{N,x}(y_1) + W_{N,x}(y_2))^{-1} \nabla v_{N,x} \otimes \nabla v_{N,x} \otimes \Psi^{(n)} \right\rangle \right| \\ & = \left| \left\langle \nabla v_{N,x} \otimes \nabla v_{N,x} \otimes \Psi^{(n)} (\mathbb{H}_0 + \mathbb{W})^{-1}, \right. \right. \\ & \quad \left. \left. \left(\sum_{j=3}^{n+2} W_{N,x}(y_j) \right) (\mathbb{H}_0 + W_{N,x}(y_1) + W_{N,x}(y_2))^{-1} \nabla v_{N,x} \otimes \nabla v_{N,x} \otimes \Psi^{(n)} \right\rangle \right| \\ & \leq \|\epsilon^{-1/4} \nabla v_N\|^4 \|\mathbb{H}_0^{-1/2} \mathbb{W} \mathbb{H}_0^{-1/2} d\Gamma(\epsilon)^{-s}\| \|\Psi^{(n)}\| \|d\Gamma(\epsilon)^s \Psi^{(n)}\| \\ & \lesssim (\log N)^2 N^{-s} \|\Psi^{(n)}\| \|d\Gamma(\epsilon)^s \Psi^{(n)}\|, \end{aligned} \quad (5.81)$$

since $\|\epsilon^{-1/4} \nabla v_N\|^2 \lesssim \log N$ (see the definition of v_N in (5.4) and Lemma 2.2), and \mathbb{W} is bounded using (5.57). Another application of the resolvent identity leads to

$$\begin{aligned} \langle \Psi^{(n)}, \Theta_{N,W,0} \Psi^{(n)} \rangle & = 2 \left\langle \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)} (\mathcal{T}_x^* \mathbb{H}_0 \mathcal{T}_x + W_N(y_1) + W_N(y_2))^{-1}, \right. \\ & \quad \left. \left(-\Delta_x + 2\nabla_x(\nabla_{y_1} + \nabla_{y_2}) + \sum_{j=3}^{n+2} \epsilon(i\nabla_{y_j}) \right) (\mathbb{H}^{(2)})^{-1} \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)} \right\rangle \\ & \quad + \mathcal{O}(N^{-s} (\log N)^2 \|\Psi\| \|d\Gamma(\epsilon)^s \Psi\|). \end{aligned} \quad (5.82)$$

To arrive at the quadratic form of $\Theta_{N,0}$, we have to set W_N to zero in the first resolvent and $(H^{(2)})^{-1}$. Consider for example the term arising from expanding

$(\mathcal{T}_x^* \mathbb{H}_0 \mathcal{T}_x + W_N(y_1) + W_N(y_2))^{-1}$. Using that $\mathbb{H}^{(2)}$ commutes with derivatives in x and y_3, \dots, y_{n+2} , it is bounded by

$$\begin{aligned}
& \left| \left\langle \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)} (\mathcal{T}_x^* \mathbb{H}_0 \mathcal{T}_x + W_N(y_1) + W_N(y_2))^{-1} W_N(y_1) (\mathcal{T}_x^* \mathbb{H}_0 \mathcal{T}_x)^{-1}, \right. \right. \\
& \quad \left. \left(-\Delta_x + 2\nabla_x (\nabla_{y_1} + \nabla_{y_2}) + \sum_{j=3}^{n+2} \epsilon (i\nabla_{y_j}) \right) (\mathbb{H}^{(2)})^{-1} \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)} \right\rangle \Big| \\
& \leq \|\mathbb{H}_0^{-1/2} (\nabla v_N)^{\otimes 2}\| \|(\mathbb{H}^{(2)})^{-1/2} (\nabla v_N)^{\otimes 2}\| \|\mathbb{H}_0^{-1/2} W_{N,x}(y_1) \mathbb{H}_0^{-s} \epsilon (i\nabla_{y_1})^{-1/2}\| \\
& \quad \times \left\{ \left(\|\mathbb{H}_0^{-1+s} \mathcal{T}_x^* \Delta_x^{1-s} \mathcal{T}_x\| + \|\mathbb{H}_0^{-1+s} \mathcal{T}_x^* |\nabla_x|^{1-2s} |\nabla_{y_1}| \mathcal{T}_x\| \right) \|(-\Delta_x)^s \Psi^{(n)}\| \|\Psi^{(n)}\| \right. \\
& \quad \left. + \|\mathbb{H}_0^{-1+s} d\Gamma(\epsilon)^{1-s}\| \| (d\Gamma(\epsilon))^s \Psi^{(n)} \| \|\Psi^{(n)}\| \right\} \\
& \leq \|\epsilon^{-1/4} \nabla v_N\|^4 \|\epsilon^{-1/2} W_N \epsilon^{-1/2-s}\| \|\mathbb{H}_0^s \Psi^{(n)}\| \|\Psi^{(n)}\| \\
& \lesssim (\log N)^2 N^{-s} \|\mathbb{H}_0^s \Psi^{(n)}\| \|\Psi^{(n)}\|, \tag{5.83}
\end{aligned}$$

where we applied (5.56). The error terms coming from replacing $\mathbb{H}^{(2)}$ by $-(\nabla_{y_1} + \nabla_{y_2})^2 + \epsilon(i\nabla_{y_1}) + \epsilon(i\nabla_{y_2})$ are bounded in a similar way, which yields

$$\|\langle \Psi^{(n)}, (\Theta_{N,W,0} - \Theta_{N,0}) \Psi^{(n)} \rangle\| \lesssim (\log N)^2 N^{-s} \|\mathbb{H}_0^s \Psi^{(n)}\| \|\Psi\| \tag{5.84}$$

for $0 < s < 1/4$ and proves the claim for $\Theta_{N,W,0}$.

To bound the difference of $\Theta_{N,W,1}$ and $\Theta_{N,1}$ we use that on $\mathcal{H}_+^{(n+2)}$

$$\begin{aligned}
\mathbb{H}_0^{-1} - (\mathbb{H}_0 + \mathbb{W})^{-1} &= \left(\mathbb{H}_0 + \sum_{j=1}^{n+1} W_{N,x}(y_j) \right)^{-1} W_{N,x}(y_{n+2}) (\mathbb{H}_0 + \mathbb{W})^{-1} \\
&+ \left(\mathbb{H}_0 + \sum_{j=1}^{n+1} W_{N,x}(y_j) \right)^{-1} \sum_{j=2}^{n+1} W_{N,x}(y_j) \left(\mathbb{H}_0 + W_{N,x}(y_1) \right)^{-1} \\
&+ \left(\mathbb{H}_0 + W_{N,x}(y_1) \right)^{-1} W_{N,x}(y_1) \mathbb{H}_0^{-1}. \tag{5.85}
\end{aligned}$$

With this decomposition, we can use the regularity of Ψ and the fact that the derivative in the relevant variable commutes with the resolvent on the right to estimate the middle term of each summand. To bound the factor n in the definition of $\Theta_{N,W,1}$, we also need to use the symmetry of $\Psi^{(n)}$. We proceed in the following way, denoting by $S_{a,b}$, for $1 \leq a < b \leq n+2$, the projection onto functions symmetric under

permutations of y_a, \dots, y_b , we have for all $\Phi \in L^2(\mathbb{T}^{3(n+2)})$

$$\begin{aligned}
& \|\epsilon^{1/4}(\mathrm{i}\nabla_{y_1})\epsilon^{1/4}(\mathrm{i}\nabla_{y_{n+2}})\mathrm{d}\Gamma(\epsilon)^{-1/2}S_{3,n+2}\Phi\|^2 \\
&= n^{-1}\langle S_{3,n+2}\Phi, \mathrm{d}\Gamma(\epsilon)^{-1}\epsilon(\mathrm{i}\nabla_{y_1})^{1/2}\sum_{j=3}^{n+2}\epsilon(\mathrm{i}\nabla_{y_j})^{1/2}S_{3,n+2}\Phi \rangle \\
&\leq n^{-1/2}\langle S_{3,n+2}\Phi, \mathrm{d}\Gamma(\epsilon)^{-1/2}\epsilon(\mathrm{i}\nabla_{y_1})^{1/2}S_{3,n+2}\Phi \rangle \\
&\lesssim n^{-1/2}\|\Phi\|^2.
\end{aligned} \tag{5.86}$$

Then, from the decomposition (5.85) and the estimate above we obtain, using the symmetry of $\Psi^{(n)}$ and again (5.57), that for $0 \leq s < 1/4$

$$\begin{aligned}
& |\langle \Psi^{(n)}, (\Theta_{N,W,1} - \Theta_{N,1})\Psi^{(n)} \rangle| \\
&\lesssim N^{-s}n\|S_{2,n+1}\mathrm{d}\Gamma(\epsilon)^{-1/2}\nabla v_N \otimes \nabla v_N \otimes \epsilon(\mathrm{i}\nabla_{y_{n+2}})^s\Psi^{(n)}\| \\
&\quad \times \|S_{3,n+2}\mathrm{d}\Gamma(\epsilon)^{-1/2}\nabla v_N \otimes \Psi^{(n)} \otimes \nabla v_N\| \\
&\quad + N^{-s}n\|S_{2,n+1}\mathrm{d}\Gamma(\epsilon)^{-1/2}\nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)}\| \\
&\quad \times \|S_{3,n+2}\mathrm{d}\Gamma(\epsilon)^{-1/2}\nabla v_N \otimes \mathrm{d}\Gamma(\epsilon)^s\Psi^{(n)} \otimes \nabla v_N\| \\
&\quad + N^{-s}n\|S_{2,n+1}\mathrm{d}\Gamma(\epsilon)^{-1/2}\nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)}\| \\
&\quad \times \|S_{3,n+2}\mathrm{d}\Gamma(\epsilon)^{-1/2}\nabla v_N \otimes \epsilon(\mathrm{i}\nabla_{y_1})^s\Psi^{(n)} \otimes \nabla v_N\| \\
&\lesssim N^{-s}n^{1/2}\|\epsilon^{-1/4}\nabla v_N\|_2^4\|\Psi^{(n)}\| \left(\|\epsilon(\mathrm{i}\nabla_1)^s\Psi^{(n)}\| + \|\mathrm{d}\Gamma(\epsilon^s)\Psi^{(n)}\| \right) \\
&\lesssim (\log N)^2 N^{-s}\|\mathcal{N}_+^{1/4}\Psi^{(n)}\| \|\mathcal{N}_+^{1/4}\mathrm{d}\Gamma(\epsilon)^s\Psi^{(n)}\|.
\end{aligned} \tag{5.87}$$

For the difference of $\Theta_{N,W,2}$ and $\Theta_{N,2}$ we proceed in a similar way, writing on $\mathcal{H}_+^{(n+2)}$

$$\begin{aligned}
\mathbb{H}_0^{-1} - (\mathbb{H}_0 + \mathbb{W})^{-1} &= \mathbb{H}_0^{-1} \sum_{j=3}^{n+2} W_{N,x}(y_j) \left(\mathbb{H}_0 + \sum_{j=3}^{n+2} W_{N,x}(y_j) \right)^{-1} \\
&\quad + (\mathbb{H}_0 + \mathbb{W})^{-1} \left(W_{N,x}(y_1) + W_{N,x}(y_2) \right) \left(\mathbb{H}_0 + \sum_{j=3}^{n+2} W_{N,x}(y_j) \right)^{-1}.
\end{aligned} \tag{5.88}$$

From this we obtain

$$\begin{aligned}
& \|\langle \Psi^{(n)}, (\Theta_{N,W,2} - \Theta_{N,2}) \Psi^{(n)} \rangle\| \\
&= n(n-1) \left| \left\langle S_{1,n} (\tau_x \nabla v_N \otimes \tau_x \nabla v_N \otimes \Psi^{(n)}), \left(\mathbb{H}_0^{-1} - (\mathbb{H}_0 + \mathbb{W})^{-1} \right) \right. \right. \\
&\quad \left. \left. S_{3,n+2} (\Psi^{(n)} \otimes \tau_x \nabla v_N \otimes \tau_x \nabla v_N) \right\rangle \right| \\
&\lesssim N^{-s} n(n-1) \|\mathrm{d}\Gamma(\epsilon)^{-1/2} S_{3,n+2} (\epsilon (i\nabla_{y_1})^s + \epsilon (i\nabla_{y_2})^s) \Psi^{(n)} \otimes \nabla v_N \otimes \nabla v_N\| \\
&\quad \times \|\mathrm{d}\Gamma(\epsilon)^{-1/2} S_{1,n} \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)}\| \tag{5.89a}
\end{aligned}$$

$$\begin{aligned}
&+ N^{-s} n(n-1) \|\mathrm{d}\Gamma(\epsilon)^{-1/2} S_{1,n} \nabla v_N \otimes \nabla v_N \otimes (\mathrm{d}\Gamma(\epsilon)^s \Psi^{(n)})\| \\
&\quad \times \|\mathrm{d}\Gamma(\epsilon)^{-1/2} S_{3,n+2} \Psi^{(n)} \otimes \nabla v_N \otimes \nabla v_N\|, \tag{5.89b}
\end{aligned}$$

where we used the bound (5.56). We now use the symmetry to absorb the factors in n . We have for $\Phi \in L^2(\mathbb{T}^{3(n+2)})$

$$\begin{aligned}
& \|\epsilon^{1/4} (i\nabla_{y_1}) \epsilon^{1/4} (i\nabla_{y_2}) \mathrm{d}\Gamma(\epsilon)^{-1/2} S_{1,n} \Phi\|^2 \\
&= \langle S_{1,n} \Phi, \mathrm{d}\Gamma(\epsilon)^{-1} \epsilon (i\nabla_{y_1})^{1/2} \epsilon (i\nabla_{y_2})^{1/2} S_{1,n} \Phi \rangle \\
&= \frac{1}{n(n-1)} \left\langle S_{1,n} \Phi, \left(\sum_{\ell=1}^{n+2} \epsilon (i\nabla_{y_\ell}) \right)^{-1} \sum_{i \neq j=1}^n \epsilon (i\nabla_{y_i})^{1/2} \epsilon (i\nabla_{y_j})^{1/2} S_{1,n} \Phi \right\rangle \\
&\leq \frac{1}{\sqrt{n(n-1)}} \|\Phi\|^2, \tag{5.90}
\end{aligned}$$

where in the final step we used the Cauchy-Schwarz inequality on the double sum. We can now bound (5.89a) and (5.89b). Consider the first factor in (5.89a). We have

$$\begin{aligned}
& \|\mathrm{d}\Gamma(\epsilon)^{-1/2} S_{1,n} \nabla v_N \otimes \nabla v_N \otimes \Psi^{(n)}\| \\
&\leq \|\epsilon^{-1/4} \nabla v_N\|^2 \|\Psi^{(n)}\| \|\epsilon^{1/4} (i\nabla_{y_1}) \epsilon^{1/4} (i\nabla_{y_2}) \mathrm{d}\Gamma(\epsilon)^{-1/2} S_{1,n} \mathbf{1}_{L^2(\mathbb{T}^{3(n+2)})}\| \\
&\leq \frac{\log N}{(n(n-1))^{1/4}} \|\Psi^{(n)}\|. \tag{5.91}
\end{aligned}$$

Similarly, the second factor in (5.89a) is bounded by

$$\begin{aligned}
& \|\mathrm{d}\Gamma(\epsilon)^{-1/2} S_{3,n+2} (\epsilon (i\nabla_{y_1})^s + \epsilon (i\nabla_{y_2})^s) \Psi^{(n)} \otimes \nabla v_N \otimes \nabla v_N\| \\
&\lesssim \|\epsilon^{1/4} (i\nabla_{y_{n+1}}) \epsilon^{1/4} (i\nabla_{y_{n+2}}) \mathrm{d}\Gamma(\epsilon)^{-1/2} S_{3,n+2} \mathbf{1}_{L^2(\mathbb{R}^{3(n+2)})}\| \|\epsilon (i\nabla_{y_1})^s \Psi^{(n)}\| \\
&\lesssim \frac{\log N}{(n(n-1))^{1/4}} \|\mathrm{d}\Gamma(\epsilon)^s \mathcal{N}_+^{-s} \Psi^{(n)}\|. \tag{5.92}
\end{aligned}$$

From this we obtain that

$$\begin{aligned}
(5.89a) &\lesssim N^{-s} (\log N)^2 \sqrt{n(n-1)} \|\Psi^{(n)}\| \|\mathrm{d}\Gamma(\epsilon)^s \mathcal{N}_+^{-s} \Psi^{(n)}\| \\
&\lesssim N^{-s} (\log N)^2 \|\mathcal{N}_+^{1/2} \Psi^{(n)}\| \|\mathcal{N}_+^{1/2-s} \mathrm{d}\Gamma(\epsilon)^s \Psi^{(n)}\|. \tag{5.93}
\end{aligned}$$

The same computation leads to

$$(5.89b) \lesssim N^{-s}(\log N)^2 \|\mathcal{N}_+^{(1+s)/2} \Psi^{(n)}\| \|\mathcal{N}_+^{(1-s)/2} d\Gamma(\epsilon)^s \Psi^{(n)}\|. \quad (5.94)$$

We thus obtain

$$\begin{aligned} & \|\langle \Psi^{(n)}, (\Theta_{N,W,2} - \Theta_{N,2}) \Psi^{(n)} \rangle\| \\ & \lesssim N^{-s}(\log N)^2 \|\mathcal{N}_+^{(1+s)/2} \Psi^{(n)}\| \|\mathcal{N}_+^{(1-s)/2} d\Gamma(\epsilon)^s \Psi^{(n)}\|. \end{aligned} \quad (5.95)$$

Collecting the estimates (5.84), (5.87) and (5.95) and the bound on the cutoff (5.75) shows the bound on the difference $T_N - T_{N,W}$.

To prove the estimate of $T_{N,W}$, simply combine the bounds (5.84), (5.87) and (5.95) on $\Theta_{N,W,j}$ with those on $\Theta_{N,j}$ from Lemma 5.5 for $n \leq M$. Moreover, in (5.95) bound n^s by M^s , which yields

$$\|\langle \Psi^{(n)}, (\Theta_{N,W,2} - \Theta_{N,2}) \Psi^{(n)} \rangle\| \lesssim (M/N)^s (\log N)^2 \|\mathcal{N}_+^{1/2} \Psi^{(n)}\| \|\mathcal{N}_+^{1/2-s} d\Gamma(\epsilon)^s \Psi^{(n)}\|. \quad (5.96)$$

This implies the claim when N is chosen large enough. \square

5.2.2 Proof of Lemma 5.12

Recall that

$$R_N = 2a^*(i\nabla v_{N,x})a(i\nabla v_{N,x}) - 2a^*(i\nabla v_{N,x})i\nabla_x + \text{h.c.} + a(w_x^\kappa) + a^*(w_x^\kappa).$$

We will only give the proof of the estimate of R_N in (5.69a), as the one for the difference $R_N - R_\infty$ in (5.69b) follows from the same reasoning, replacing v_N by $v_N(k) - v_\infty$, which using (5.5) satisfies

$$\|\epsilon^{1/2-2s}|k|\widehat{v}_N(k) - \widehat{v}_\infty(k)\|_2 \lesssim \|\mathbf{1}_{|k|>N^\alpha}|k|^{-4s}\|_2 \lesssim N^{-\alpha(4s-3/2)}, \quad (5.97)$$

for $3/8 < s \leq 1/2$. We now prove (5.69a) by estimating each term.

The term $a^(w_x^\kappa)$.* This term is bounded using that $w^\kappa \in L^2(\mathbb{T}^3)$ (see the definition (5.4)). We obtain that for all $\delta > 0$ using Young's inequality

$$\begin{aligned} |\langle \Phi, a(w_x^\kappa) \Psi \rangle| & \leq \|w^\kappa\|^2 \|(\mathcal{N}_+ + 1)^{1/4} \Phi\| \|(\mathcal{N}_+ + 1)^{1/4} \Psi\| \\ & \leq \|(\delta \mathcal{N}_+^{3/8} + C_\delta) \Phi\| \|(\delta \mathcal{N}_+^{3/8} + C_\delta) \Psi\|. \end{aligned} \quad (5.98)$$

For the other terms in R_N , we make the following observation. Each term X in R_N satisfies that $\mathcal{N}X = X(\mathcal{N} + k)$ for some $k \in \mathbb{Z}$, therefore to conclude the proof of the desired bound (5.69a) it is enough to prove estimates of the form

$$|\langle \Phi, X \Psi \rangle| \lesssim \delta \|((-\Delta_x + d\Gamma(\epsilon))^s \mathcal{N}_+^a \Phi)\| \|((-\Delta_x + d\Gamma(\epsilon))^s \mathcal{N}_+^b \Psi)\|. \quad (5.99)$$

for some $0 \leq a, b$ such that $a + b \leq 1 - 2s$.

The term $a^*(i\nabla v_{N,x})i\nabla_x$. For this term, we decompose

$$|\langle \Phi, i\nabla_x a(\nabla v_{N,x})\Psi \rangle| \leq \| |\nabla_x|^{2s}\Phi \| \| |\nabla_x|^{1-2s}a(\nabla v_{N,x})\Psi \|. \quad (5.100)$$

The first factor above is bounded by $\| ((-\Delta_x + d\Gamma(\epsilon))^s \Phi) \|$. To bound the second factor, we commute the gradient to the left,

$$|\nabla_x|^{1-2s}a(\nabla v_{N,x}) = a(\nabla v_{N,x})|\nabla_x|^{1-2s} + [|\nabla_x|^{1-2s}, a(\nabla v_{N,x})]. \quad (5.101)$$

For $a > 1/4$, we have from (5.4) that $\epsilon^{-a}\nabla v_N \in L^2$ and that

$$C_\kappa = \sup_{N \geq 1} \|\epsilon^{-a}\nabla v_N\|_2 \rightarrow 0, \text{ as } \kappa \rightarrow \infty. \quad (5.102)$$

Using the Cauchy-Schwarz inequality, we can therefore bound the contribution coming from the first term in (5.101) by

$$\begin{aligned} \|a(\nabla v_{N,x})|\nabla_x|^{1-2s}\Psi\| &\leq \|\epsilon^{-a}\nabla v_N\|_2 \|d\Gamma(\epsilon^{2a})^{1/2}|\nabla_x|^{1-2s}\Psi\| \\ &\leq C_\kappa \|(-\Delta_x + d\Gamma(\epsilon))^{a+1/2-s}\mathcal{N}_+^{1/2-a}\Psi\| \\ &\leq C_\kappa \|(-\Delta_x + d\Gamma(\epsilon))^s\mathcal{N}_+^{1-2s}\Psi\|, \end{aligned} \quad (5.103)$$

where we took $a = 2s - 1/2 > 1/4$ for $s > 3/8$. To bound the term coming from the commutator in (5.101), we use that $i\nabla_x e^{-ikx} = e^{-ikx}(i\nabla_x + k)$ and that $\| |p-k|^{1-2s} - |p|^{1-2s} \| \lesssim |k|^{1-2s}$, to obtain

$$\begin{aligned} \| [|\nabla_x|^{1-2s}, a(\nabla v_{N,x})]\Psi \| &\lesssim \| a(|\nabla|^{2-2s}v_{N,x})\Psi \| \\ &\leq \|\epsilon^{-s}|\nabla|^{2-2s}v_N\|_2 \|d\Gamma(\epsilon^{2s})^{1/2}\Psi\|, \\ &\leq \|\epsilon^{-s}|\nabla|^{2-2s}v_N\|_2 \|d\Gamma(\epsilon)^s\mathcal{N}_+^{1/2-s}\Psi\|, \end{aligned} \quad (5.104)$$

where the norm tends to zero for $\kappa \rightarrow \infty$ since $|k|^{2-4s}\widehat{v}_N(k) \lesssim |k|^{-4s}\mathbf{1}_{|k|>\kappa}$. This proves that for all $\delta > 0$,

$$|\langle \Phi, i\nabla_x a(\nabla v_{N,x})\Psi \rangle| \leq \delta \| (-\Delta_x + d\Gamma(\epsilon))^s \Phi \| \| (-\Delta_x + d\Gamma(\epsilon))^s \mathcal{N}_+^{1-2s} \Psi \|, \quad (5.105)$$

for κ sufficiently large.

The term $a^*(i\nabla v_{N,x})a(i\nabla v_{N,x})$. Finally, we finish the bound (5.69a) on R_N by estimating the term with a creation and an annihilation operator. Similarly as above, we have

$$\langle \Phi, a^*(i\nabla v_{N,x})a(i\nabla v_{N,x})\Psi \rangle \leq \|\epsilon^{-s}\nabla v_N\|_2^2 \|d\Gamma(\epsilon)^s\mathcal{N}_+^{1/2-s}\Phi\| \|d\Gamma(\epsilon)^s\mathcal{N}_+^{1/2-s}\Psi\|. \quad (5.106)$$

Using (5.102) we conclude the proof of the desired bound (5.69a).

6 Proof of the main results

In this section we prove Theorem 1.2, Theorem 1.5, and Corollary 1.6. We start by recalling the results that form the basis of the proofs, and proving precise upper and lower bounds on the spectrum of H_N as preliminary lemmas.

In Section 4 we computed

$$U_X^* H_N U_X = \mathbb{1}_{\mathcal{N}_+ \leq N} \mathcal{H}_N \mathbb{1}_{\mathcal{N}_+ \leq N}, \quad (6.1)$$

with U_X the excitation map (4.2). We then showed in Proposition 4.26 and (5.1)

$$U^* \mathcal{H}_N U = 4\pi \mathfrak{a}_V (N-1) + 8\pi \mathfrak{a}_W \sqrt{N} + e_N^{(U)} + \mathcal{H}_N^U + \mathcal{L}_4 + \mathcal{E} \quad (6.2)$$

where U is a unitary operator on \mathcal{H}_+ . Here, the notation \mathcal{H}_N^U is defined in (5.7), \mathcal{L}_4 is the quartic term defined in Proposition 4.1, $e_N^{(U)}$ is the scalar of order one defined in Proposition 4.26, and the error $\mathcal{E} = \mathcal{E}^{(U)} + \mathcal{O}(N^{-3/2+2\alpha}(\mathcal{N}_+ + 1)^3)$ satisfies

$$\begin{aligned} \pm \mathcal{E} \lesssim & N^{-\alpha/2} d\Gamma(W_{N,x}) + N^{-\alpha/4} \mathcal{L}_4 + N^{-1/2-\alpha/4} \mathcal{N}_+^2 + N^{-3/2+2\alpha} \mathcal{N}_+^3 \\ & + N^{-\alpha/4} (\log N d\Gamma(|i\nabla|) + (\log N)^2 + (\mathcal{N}_+ + 1)^{1/2} |\nabla_x|). \end{aligned} \quad (6.3)$$

for any $\alpha \leq 1/10$.

Moreover, we proved the following conservation estimates under the transformation U .

Lemma 6.1. *Let $U = U_q U_W U_c U_B U_G$ be the product of the unitaries from Section 4. Under the assumptions of Proposition 4.26, for any $t \geq 0$ we have*

$$U^* \mathcal{N}_+^t U = \mathcal{O}(\mathcal{N}_+^t + 1), \quad U \mathcal{N}_+^t U^* = \mathcal{O}(\mathcal{N}_+^t + 1),$$

and it holds

$$\begin{aligned} U^* \mathcal{L}_4 U &= \mathcal{O}(\mathcal{N}_+ + N^{-1/2-\alpha/2} \mathcal{N}_+^2 + \mathcal{L}_4 + N^{-1/2} \log N d\Gamma(|i\nabla|) + N), \\ U^* d\Gamma(W_{N,x}) U &= \mathcal{O}(d\Gamma(W_{N,x}) + \mathcal{N}_+ + \sqrt{N}). \end{aligned}$$

Proof. This follows from the various bounds in Section 4. Specifically: for \mathcal{N}_+ , Lemmas 4.4, 4.11, 4.21, 4.25 and Equation (4.197); for \mathcal{L}_4 Lemmas 4.5, 4.11, 4.22, 4.25 and Equations (4.195), (4.196); For $d\Gamma(W_{N,x})$ Equations (4.56), (4.70), Lemmas 4.22, 4.24 and Equation (4.190). \square

In Proposition 5.1 we proved that $\mathcal{H}_N^U - E_{N,W}$ converges to an operator that is unitarily equivalent to H_{BF} in norm resolvent sense, where $E_{N,W}$ is given by (5.8) and contains the terms of order $\log N$. Moreover, Proposition 5.7 and Corollary 5.10 imply that for $0 \leq s < 1$

$$\mathcal{N}_+^{1-s} (-\Delta_x + d\Gamma(\epsilon))^s \lesssim \mathcal{H}_N^U - E_{N,W} + C, \quad (6.4)$$

for some $C > 0$, independent of N .

Lemma 6.2 (Upper bound). *Assume the hypothesis of Theorem 1.2 and let*

$$\lambda_N := 4\pi\mathbf{a}_V(N-1) + 8\pi\mathbf{a}_W\sqrt{N} + e^{(U)} + E_{N,W}.$$

Then for any $k \in \mathbb{N}$

$$e_k(H_N) - \lambda_N \leq e_k(H_{\text{BF}}) + \mathcal{O}(N^{-1/80}).$$

Proof. By Proposition 5.1, $\mathcal{H}_N^U - E_{N,W}$ converges to $(U_\kappa^\infty)H_{\text{BF}}(U_\kappa^\infty)^*$ in norm resolvent sense, so the min-max values also converge. Moreover, the spectral projections associated to an interval around $e_k(H_{\text{BF}})$ that contains no other eigenvalues of H_{BF} also converge, cf. [64, Thm.VII.23]. In particular, their ranks are equal for N sufficiently large. To deduce the rate of convergence, consider such an interval and write for all eigenvalues in this interval $e_j(\mathcal{H}_N^U - E_{N,W}) = r_j^{-1} - i$, where r_j is the corresponding eigenvalue of $(\mathcal{H}_N^U - E_{N,W} + i)^{-1}$. The eigenvalues r_j all lie in the complement of a disc in \mathbb{C} , where the function $z \mapsto z^{-1} - i$ is Lipschitz. It is thus sufficient to bound $|r_j - (e_k(H_{\text{BF}}) + i)^{-1}|$, which is bounded by the norm-difference of the resolvents (cf. [4, Theorem 6.3.3], which applies to compact operators by the same proof). Proposition 5.7 thus gives us the rate of convergence

$$|e_k(H_{\text{BF}}) - e_k(\mathcal{H}_N^U) - E_{N,W}| \lesssim N^{(-\frac{1}{4} + \varepsilon)\alpha}, \quad (6.5)$$

for $0 \leq \alpha \leq 1/2$ and $\varepsilon > 0$, where we will take $\alpha = 1/10$, the largest value permitted by Proposition 4.26. It is thus sufficient to prove the upper bound using $e_k(\mathcal{H}_N^U) - E_{N,W}$. Let Y_k denote the span of the first $k+1$ eigenfunction of \mathcal{H}_N^U . Our test function for H_N is of the form

$$\Psi = U_X U \chi_M \Phi$$

with $\Phi \in Y_k$ normalized and χ_M a cutoff on $\mathcal{N}_+ \leq M = N^{\alpha/9}$. More precisely, let $\chi, \eta : \mathbb{R} \rightarrow [0, 1]$ be two smooth functions with $\chi^2 + \eta^2 = 1$ and $\chi(r) = 1$ for $r \leq 1/2$, $\chi(r) = 0$ for $r \geq 1$, and define $\eta_M = \eta(\mathcal{N}_+/M)$, $\chi_M = \chi(\mathcal{N}_+/M)$.

Normalizing Ψ . We first show that $\Psi = U_X U \chi_M \Phi$ is almost normalized. Recall that U_X is a partial isometry, hence we first bound, using that $U^* \mathcal{N}_+ U \lesssim \mathcal{N}_+$ by Lemma 6.1, and (6.4),

$$\begin{aligned} \|U \chi_M \Phi\|^2 - \|\Psi\|^2 &= \langle U \chi_M \Phi, (1 - U_X^* U_X) U \chi_M \Phi \rangle = \langle U \chi_M \Phi, \mathbf{1}_{\mathcal{N}_+ > N} U \chi_M \Phi \rangle \\ &\leq N^{-1} \langle \chi_M \Phi, U^* \mathcal{N}_+ U \chi_M \Phi \rangle \\ &\lesssim N^{-1} \langle \Phi, (\mathcal{H}_N^U - E_{N,W} + C) \Phi \rangle. \end{aligned} \quad (6.6)$$

Then, we find that for $\delta > 0$ small enough

$$\begin{aligned} 1 - \|U \chi_M \Phi\|^2 &= \langle \Phi, \eta_M^2 \Phi \rangle \leq M^{-5/4 + \delta} \langle \Phi, \mathcal{N}_+^{5/4 - \delta} \Phi \rangle \\ &\lesssim N^{-\alpha/8} \langle \Phi, (\mathcal{H}_N^U - E_{N,W} + C)^2 \Phi \rangle \lesssim N^{-\alpha/8}, \end{aligned} \quad (6.7)$$

where we used the second bound of Proposition 5.7 together with (5.61b), and chose $M = N^{\alpha/9}$. We also used that the $k + 1$ first eigenvalues of $\mathcal{H}_N^U - E_{N,W}$ are uniformly bounded in N because of the resolvent convergence we discussed before. In particular, the map $U_X U_{\chi_M} : Y_k \rightarrow \mathcal{H}_N$ is injective for large enough N . Thus we have for the min-max value

$$\begin{aligned}
e_k(H_N) &= \min_{\substack{X \subset \mathcal{H}_N \\ \dim X = k+1}} \max_{\Psi \in X} \frac{\langle \Psi, H_N \Psi \rangle}{\|\Psi\|^2} \\
&\leq \max_{\substack{\Phi \in Y_k \\ \|\Phi\|=1}} \langle \chi_M \Phi, U^* U_X^* H_N U_X U_{\chi_M} \Phi \rangle (1 + \mathcal{O}(N^{-\alpha/8})) \\
&\leq \max_{\substack{\Phi \in Y_k \\ \|\Phi\|=1}} \langle \Phi, \chi_M U^* \mathbf{1}_{N_+ \leq N} \mathcal{H}_N \mathbf{1}_{N_+ \leq N} U_{\chi_M} \Phi \rangle (1 + \mathcal{O}(N^{-\alpha/8})).
\end{aligned} \tag{6.8}$$

Removing $\mathbf{1}_{N_+ \leq N}$. Since $\mathbf{1}_{N \leq N_+} \mathcal{H}_N \mathbf{1}_{N > N_+} = 0$ and $\mathbf{1}_{N > N_+} \mathcal{H}_N \mathbf{1}_{N > N_+} \geq 0$, we can remove the projections for an upper bound. We thus have with (6.2)

$$e_k(H_N) - \lambda_N \leq \max_{\substack{\Phi \in Y_k \\ \|\Phi\|=1}} \langle \chi_M \Phi, (\mathcal{H}_N^U - E_{N,W} + \mathcal{L}_4 + \mathcal{E}) \chi_M \Phi \rangle (1 + \mathcal{O}(N^{-\alpha/8})). \tag{6.9}$$

Estimating \mathcal{E} . We can now bound the error term \mathcal{E} which satisfies (6.3). We proceed term by term. The powers of the number operator are bounded by $\chi_M \mathcal{N}_+^k \leq M^{k-1} \chi_M \mathcal{N}_+$, which is then bounded as in (6.6). Moreover, by (6.4) we have that

$$\langle \chi_M \Phi, (d\Gamma(|i\nabla|) + \mathcal{N}_+^{1/2} |\nabla_x|) \chi_M \Phi \rangle \lesssim \langle \chi_M \Phi, \mathcal{N}_+^{1/2} (-\Delta_x + d\Gamma(\epsilon))^{1/2} \chi_M \Phi \rangle \tag{6.10}$$

is of order one since χ_M commutes with the operator in the middle. For $t = 1 - \alpha/4 \leq 1$, we have (compare with (5.57))

$$d\Gamma(W_{N,x}) \lesssim d\Gamma((-\Delta)^t) \|W_N\|_{3/2t} \lesssim \mathcal{N}_+^{1-t} d\Gamma(\epsilon)^t \|W_N\|_{3/2t}, \tag{6.11}$$

and using the bound (6.4), we obtain for some $C > 0$,

$$N^{-\alpha/2} d\Gamma(W_{N,x}) \leq C N^{-\alpha/4} (\mathcal{H}_N^U - E_{N,W} + C). \tag{6.12}$$

Hence from (6.3), the above estimates give

$$|\langle \chi_M \Phi, \mathcal{E} \chi_M \Phi \rangle| \lesssim N^{-\alpha/4} \log N \langle \chi_M \Phi, (\mathcal{H}_N^U - E_{N,W} + \log N) \chi_M \Phi \rangle. \tag{6.13}$$

Estimating \mathcal{L}_4 . For the remaining term \mathcal{L}_4 , we use the analogue of the bound (5.56) on W_N to obtain for $1 \leq t < 3/2$

$$\mathcal{L}_4 \leq \|V_N\|_{3/2t} d\Gamma((-\Delta + 1)^t) \mathcal{N}_+ \lesssim N^{2-2t} d\Gamma(-\Delta + 1)^t \mathcal{N}_+. \tag{6.14}$$

Here, we cannot use $t < 1$ since \mathcal{L}_4 does not come with a small constant. However, \mathcal{L}_4 comes from V_N which scales with N , while the singularities of \mathcal{H}_N^U come from W_N which only scales with \sqrt{N} , so by using $t > 1$ we gain more decay from V_N than we lose from \mathcal{H}_N^U and this term will still be small. To see this, we set $t = 1 + \alpha/4$ and first use the cutoff at M , which gives

$$\langle \chi_M \Phi, \mathcal{L}_4 \chi_M \Phi \rangle \lesssim N^{-\alpha/2} M \langle \chi_M \Phi, d\Gamma(-\Delta)^{1+\alpha/4} \chi_M \Phi \rangle. \quad (6.15)$$

As $d\Gamma(-\Delta)$ commutes with \mathcal{N}_+ and thus χ_M , we can now apply the bound (5.65) from Corollary 5.10 with the second bound from Proposition 5.7, with $M = N^{\alpha/9}$, to obtain

$$\begin{aligned} \langle \chi_M \Phi, \mathcal{L}_4 \chi_M \Phi \rangle &\lesssim N^{-\alpha/2} M \|d\Gamma(-\Delta)^{1/2+\alpha/8} K_{N,W}^{-1/2-\alpha/8}\|^2 \\ &\quad \times \|K_{N,W}^{1/2+\alpha/8} (\mathcal{H}_N^U - E_{N,W} + i)^{-1}\|^2 \\ &\lesssim N^{-\alpha/4+2\varepsilon} M \lesssim N^{-\alpha/8}, \end{aligned} \quad (6.16)$$

for $\varepsilon > 0$ small enough. Putting these bounds together, the inequality (6.9) yields

$$e_k(H_N) - \lambda_N \leq \max_{\substack{\Phi \in Y_k \\ \|\Phi\|=1}} \langle \chi_M \Phi, (\mathcal{H}_N^U - E_{N,W}) \chi_M \Phi \rangle (1 + \mathcal{O}(N^{-\alpha/8})). \quad (6.17)$$

Removing χ_M . It remains to prove that the quadratic form on the right gives approximately $\langle \Phi, (\mathcal{H}_N^U - E_{N,W}) \Phi \rangle$. We use the IMS localization formula [52, Proposition 6.1]

$$\mathcal{H}_N^U = \chi_M \mathcal{H}_N^U \chi_M + \eta_M \mathcal{H}_N^U \eta_M + \frac{1}{2}([\chi_M, [\chi_M, \mathcal{H}_N^U]] + [\eta_M, [\eta_M, \mathcal{H}_N^U]]). \quad (6.18)$$

Using that the smallest eigenvalue of $\mathcal{H}_N^U - E_{N,W}$ is approximated by $e_0(H_{\text{BF}})$ from (6.5), we find for the term localized on large particle numbers

$$\langle \Phi, \eta_M (\mathcal{H}_N^U - E_{N,W}) \eta_M \Phi \rangle \geq e_0(H_{\text{BF}}) \|\eta_M \Phi\|^2 + \mathcal{O}(N^{-\alpha/8}) = \mathcal{O}(N^{-\alpha/8}), \quad (6.19)$$

where we used the approximate normalization (6.7). Collecting from the explicit expression of \mathcal{H}_N^U in (5.7) the terms that do not commute with \mathcal{N}_+ , we see that

$$\begin{aligned} [\chi_M, [\chi_M, \mathcal{H}_N^U]] &= [\chi_M, [\chi_M, a(i\nabla v_{N,x})^2 + 2i\nabla_x a(i\nabla v_{N,x}) + a(w^\kappa) + \text{h.c.}]] \\ &= (2i\nabla_x a(i\nabla v_{N,x}) + a(w^\kappa)) (\chi_M(\mathcal{N}_+ + 1) - \chi_M(\mathcal{N}_+))^2 + \text{h.c.} \\ &\quad + a(i\nabla v_{N,x})^2 (\chi_M(\mathcal{N}_+ + 2) - \chi_M(\mathcal{N}_+))^2 + \text{h.c.} \end{aligned} \quad (6.20)$$

With the bounds of Lemma 5.12 on the first line and (5.17) (with $t = 1/2 + \varepsilon$) together with (6.4) for the last line, this gives

$$\begin{aligned} |\langle \Phi, [\chi_M, [\chi_M, \mathcal{H}_N^U]] \Phi \rangle| &\leq CN^\varepsilon \langle \Phi, (\mathcal{H}_N^U - E_{N,W} + C) \Phi \rangle \sup_{x \geq 0} (\chi(\frac{x+1}{M}) - \chi(\frac{x}{M}))^2 \\ &\lesssim M^{-2} N^\varepsilon \lesssim N^{-\alpha/8}. \end{aligned} \quad (6.21)$$

The same reasoning applies to the commutator with η_M , and we thus obtain

$$\langle \chi_M \Phi, (\mathcal{H}_N^U - E_{N,W}) \chi_M \Phi \rangle \leq \langle \Phi, (\mathcal{H}_N^U - E_{N,W}) \Phi \rangle + \mathcal{O}(N^{-\alpha/8}). \quad (6.22)$$

In view of (6.17) and with the choice $\alpha = 1/10$ this proves the claim. \square

Lemma 6.3 (Lower bound). *Assume the hypothesis of Theorem 1.2 and let*

$$\lambda_N := 4\pi\mathbf{a}_V(N-1) + 8\pi\mathbf{a}_W\sqrt{N} + e^{(U)} + E_{N,W}.$$

Then for any $k \in \mathbb{N}$

$$e_k(H_N) - \lambda_N \geq e_k(H_{\text{BF}}) + \mathcal{O}(N^{-1/80}).$$

Proof. Localizing in \mathcal{N}_+ . In order to control the error terms with $\mathcal{N}_+^2, \mathcal{N}_+^3$ in \mathcal{E} from (6.3), we need to localize on a subspace of \mathcal{H}_+ with $\mathcal{N}_+ \leq M$. Let χ_M, η_M be as in Lemma 6.2, with the choice of $M = N^{1/2+\alpha/8}$ as the cutoff scale. We now apply the localization formula (6.18) to $\mathcal{H}_N^{\leq N} := U_X^* H_N U_X$. The term localized by χ_M will be analyzed further, and the other terms need to be estimated as errors. This is where the condensation hypothesis, Condition 1.1 enters into play. It translates to (compare with (4.4))

$$\mathcal{H}_N^{\leq N} \geq (4\pi\mathbf{a}_V N + c\mathcal{N}_+ - d\sqrt{N}) \mathbb{1}_{\mathcal{N}_+ \leq N}. \quad (6.23)$$

For the choice $M = N^{1/2+\alpha/8}$, using that $\eta_M^2 \leq M^{-1} \mathcal{N}_+ \eta_M^2$, we can reconstruct the energy λ_N on the sector of large particle numbers using the gap, that is

$$\eta_M \mathcal{H}_N^{\leq N} \eta_M \geq \lambda_N \eta_M^2 \mathbb{1}_{\mathcal{N}_+ \leq N} \quad (6.24)$$

for N sufficiently large.

To control the commutators, we collect the terms that do not commute with \mathcal{N}_+ from Proposition 4.1, which gives

$$[\eta_M, [\eta_M, \mathcal{H}_N^{\leq N}]] = [\eta_M, [\eta_M, (\mathcal{L}_2 + \mathcal{L}_3 + Q_1)]] \quad (6.25)$$

Arguing as in (6.21) and using the bounds $\mathcal{L}_2 + \mathcal{L}_3 \lesssim \mathcal{L}_4 + N$, $Q_1 \lesssim d\Gamma(W_{N,x}) + \sqrt{N}$, which follow from the Cauchy-Schwarz inequality (cf. (4.30)), we can bound the localization errors by

$$\begin{aligned} [\eta_M, [\eta_M, \mathcal{H}_N^{\leq N}]] + [\chi_M, [\chi_M, \mathcal{H}_N^{\leq N}]] &\lesssim M^{-2}(\mathcal{L}_4 + d\Gamma(W_{N,x}) + N) \mathbb{1}_{M/2 \leq \mathcal{N}_+ \leq M} \\ &\lesssim N^{-1-\alpha/4}(\mathcal{H}_N^{\leq N} + N). \end{aligned} \quad (6.26)$$

Here we used that $\mathbb{1}_{\mathcal{N}_+ \leq N}(\mathcal{L}_4 + d\Gamma(W_{N,x})) \lesssim \mathcal{H}_N^{\leq N} + N$ which follows again from the Cauchy-Schwarz inequality applied to the terms composing \mathcal{H}_N .

For the term localized at $\mathcal{N}_+ \leq M = N^{1/2+\alpha/8}$, first note that $\chi_M \mathbb{1}_{\mathcal{N}_+ \leq N} = \chi_M$ for N large enough, so we can drop the projections. Making use of the unitary to rewrite this term as in (6.2) yields together with (6.24)

$$\mathcal{H}_N^{\leq N} \geq \lambda_N \mathbb{1}_{\mathcal{N}_+ \leq N} + \chi_M U (\mathcal{H}_N^U - E_{N,W} + \mathcal{L}_4 + \mathcal{E}) U^* \chi_M. \quad (6.27)$$

Estimating \mathcal{E} . We can now control the error term $\mathcal{L}_4 + \mathcal{E}$. Recall the estimate (6.3). As $\mathcal{L}_4 \geq 0$, we can simply drop $\mathcal{L}_4 + \mathcal{O}(N^{-\alpha/4} \mathcal{L}_4) \geq 0$ for large N . To bound \mathcal{N}_+^2 , \mathcal{N}_+^3 , we use that by Lemma 6.1

$$\chi_M U \mathcal{N}_+^k U^* \chi_M \lesssim \chi_M (\mathcal{N}_+ + 1)^k \chi_M \lesssim M^{k-1} \chi_M (\mathcal{N}_+ + 1) \chi_M. \quad (6.28)$$

We have thus obtained

$$\begin{aligned} \chi_M U (\mathcal{L}_4 + \mathcal{E}) U^* \chi_M &\geq -\chi_M U \left(N^{-\alpha/2} d\Gamma(W_{N,x}) + (N^{-\alpha/8} + N^{-1/2+9\alpha/4}) \mathcal{N}_+ \right. \\ &\quad \left. + N^{-\alpha/4} (\log N d\Gamma(|i\nabla|) + (\log N)^2 + (\mathcal{N}_+ + 1)^{1/2} |\nabla_x|) \right) U^* \chi_M. \end{aligned} \quad (6.29)$$

These remaining error terms have to be estimated using $\mathcal{H}_N^U - E_{N,W}$. The term $d\Gamma(W_{N,x})$ is bounded as in (6.12). With $d\Gamma(|i\nabla|) \leq \mathcal{N}_+^{1/2} d\Gamma(\epsilon)^{1/2}$ and (6.4) we also get

$$\begin{aligned} N^{-\alpha/8} \mathcal{N}_+ + N^{-\alpha/4} (\log N d\Gamma(|i\nabla|) + (\log N)^2 + (\mathcal{N}_+ + 1)^{1/2} |\nabla_x|) \\ \lesssim N^{-\alpha/8} (\mathcal{H}_N^U - E_{N,W} + C). \end{aligned} \quad (6.30)$$

We have thus shown that

$$U_X^* H_N U_X \geq \lambda_N \mathbb{1}_{\mathcal{N}_+ \leq N} + (1 - CN^{-\alpha/8}) \chi_M U (\mathcal{H}_N^U - E_{N,W}) U^* \chi_M - CN^{-\alpha/8}. \quad (6.31)$$

Removing the localization. We are now ready to bring the proof of the lower bound to conclusion. Let $X_k \subset L^2(\mathbb{T}^{3(N+1)})$ denote the span of the first $k+1$ eigenvectors of H_N and let $\Psi \in X_k$ with norm equal to one. We first need to show that $\chi_M U_X^* \Psi \approx U_X^* \Psi$ for large N . We have with $M = N^{1/2+\alpha/8}$ as before, using the a priori bound (6.23) and the upper bound from Lemma 6.2,

$$\begin{aligned} \|(1 - \chi_M) U_X^* \Psi\|^2 &\leq 2M^{-1} \langle (1 - \chi_M) U_X^* \Psi, \mathcal{N}_+ (1 - \chi_M) U_X^* \Psi \rangle \\ &\lesssim M^{-1} \langle \Psi, (H_N - 4\pi\alpha_V(N-1)) \Psi \rangle + \mathcal{O}(N^{-\alpha/8}) \\ &= \mathcal{O}(N^{-\alpha/8}). \end{aligned} \quad (6.32)$$

In particular, $\chi_M U_X^* : X_k \rightarrow \mathcal{H}_+$ is injective for sufficiently large N . For the min-max values this implies, coming from (6.31),

$$\begin{aligned} e_k(H_N) - \lambda_N &\geq \max_{\substack{\Psi \in X_k \\ \|\Psi\|=1}} (1 - CN^{-\alpha/8}) \langle \chi_M U_X^* \Psi, U(\mathcal{H}_N^U - E_{N,W}) U^* \chi_M U_X^* \Psi \rangle \\ &\quad + \mathcal{O}(N^{-\alpha/8}) \\ &\geq e_k(\mathcal{H}_N^U - E_{N,W}) + \mathcal{O}(N^{-\alpha/8}). \end{aligned} \quad (6.33)$$

Taking $\alpha = 1/10$, the largest value permitted by Proposition 4.26, proves the claim. \square

6.1 Proof of Theorem 1.2

Proof of Theorem 1.2.

The upper and lower bounds proved in the lemmas above imply that

$$\inf \sigma(H_N) = \lambda_N + \mathcal{O}(1). \quad (6.34)$$

The proof now simply consists in analyzing

$$\lambda_N := 4\pi\mathfrak{a}_V(N-1) + 8\pi\mathfrak{a}_W\sqrt{N} + e^{(U)} + E_{N,W}, \quad (6.35)$$

where $e^{(U)}$ is given in Proposition 4.26 and $E_{N,W}$ by (5.8). Using Lemma 2.3 and straightforward estimates on the sums, we obtain that $e^{(U)}$ is of order one. Hence, it remains to extract the logarithmic contribution from $E_{N,W}$. Let us write

$$\begin{aligned} E_{N,W} &= -2 \left\langle \nabla v_N \otimes \nabla v_N, (\mathbb{H}^{(2)})^{-1} \nabla v_N \otimes \nabla v_N \right\rangle, \\ \mathbb{H}^{(2)} &= -(\nabla_{y_1} + \nabla_{y_2})^2 + \epsilon(i\nabla_{y_1}) + \epsilon(i\nabla_{y_2}) + W_N(y_1) + W_N(y_2) + 1. \end{aligned} \quad (6.36)$$

Recall that for large momenta, which are the relevant ones for the divergence, v_N is essentially given by the scattering solution, $v_N = \sqrt{N}\tilde{\varphi}_1$ for $N > N^\alpha$ (cf. (5.4)). The term of order $\log N$ is already contained in $E_N := E_{N,0}$, which is obtained by replacing $\mathbb{H}^{(2)}$ with

$$\mathbb{H}_0^{(2)} = -(\nabla_{y_1} + \nabla_{y_2})^2 + \epsilon(i\nabla_{y_1}) + \epsilon(i\nabla_{y_2}) + 1. \quad (6.37)$$

We have

$$\begin{aligned} E_{N,W} - E_N &= 4 \left\langle \nabla v_N \otimes \nabla v_N, (\mathbb{H}^{(2)})^{-1} W_N(y_1) (\mathbb{H}_0^{(2)})^{-1} \nabla v_N \otimes \nabla v_N \right\rangle \\ &= 4 \left\langle \nabla v_N \otimes \nabla v_N, (\mathbb{H}_0^{(2)})^{-1} W_N(y_1) (\mathbb{H}_0^{(2)})^{-1} \nabla v_N \otimes \nabla v_N \right\rangle \end{aligned} \quad (6.38a)$$

$$\begin{aligned} &- 4 \left\langle \nabla v_N \otimes \nabla v_N, (\mathbb{H}_0^{(2)})^{-1} W_N(y_1) (\mathbb{H}^{(2)})^{-1} \right. \\ &\quad \left. \times (W_N(y_1) + W_N(y_2)) (\mathbb{H}_0^{(2)})^{-1} \nabla v_N \otimes \nabla v_N \right\rangle. \end{aligned} \quad (6.38b)$$

Using that $\|W_N^{1/2}(\mathbb{H}^{(2)})^{-1/2}\|$ is uniformly bounded, we can bound the expression of the last two lines by

$$|(6.38b)| \lesssim \|W_N(y_1)^{1/2}(\mathbb{H}_0^{(2)})^{-1}\nabla v_N \otimes \nabla v_N\|^2 = \frac{1}{2}(6.38a). \quad (6.39)$$

This is now bounded using (5.56) and Lemma 2.2 with $0 < s < 1/4$

$$(6.38a) \leq \underbrace{\|\epsilon^{-1/4+s/2}\nabla v_N\|^4}_{\substack{\text{Lemma 2.2} \\ \lesssim N^{2s}}} \underbrace{\|\epsilon^{-1/2-s}W_N\epsilon^{-1/2-s}\|}_{\substack{(5.56) \\ \lesssim N^{-2s}}} \lesssim 1. \quad (6.40)$$

We have thus reduced the problem to the analysis of

$$E_N = -2 \sum_{p,q \in 2\pi\mathbb{Z}^3} \frac{(p \cdot q)^2 |\widehat{v}_N(p)|^2 |\widehat{v}_N(q)|^2}{((p+q)^2 + \epsilon(p) + \epsilon(q) + 1)}. \quad (6.41)$$

We may cut-off any of the sums at scale \sqrt{N} at the expense of an error of order one, since, with $p^2\widehat{v}(p) = \mathcal{O}(1)$ from (5.5) and $\|\nabla v_N\|_2 = \mathcal{O}(N^{1/4})$ from Lemma 2.2,

$$\sum_{|p| > \sqrt{N}} \frac{(p \cdot q)^2 |\widehat{v}_N(p)|^2 |\widehat{v}_N(q)|^2}{((p+q)^2 + \epsilon(p) + \epsilon(q) + 1)} \leq \|\nabla v_N\|_2^2 \sum_{|p| > \sqrt{N}} \frac{1}{p^2 \epsilon(p)} \lesssim 1. \quad (6.42)$$

To compute the sum, we can replace $\epsilon(p)$ by p^2 , at the price of another error of order one. We also need to deal with \widehat{v}_N which we recall is defined in (5.4). We can replace $\widehat{v}_N \mathbb{1}_{\kappa < |p| \leq N^\alpha}$ by $4\pi \mathbf{a}_W$ in (6.41) up to an error of order one. For the large momentum part $\widehat{v}_N \mathbb{1}_{|p| > N^\alpha} = \sqrt{N} \widehat{\varphi}_I(p) e^{ipx}$, we use the scattering equation (2.19) and the definition of $\widetilde{\mathbf{a}}_{W_N}$ in (2.21), to obtain

$$\begin{aligned} \left| \sqrt{N} \widehat{\varphi}_I(p) - \frac{4\pi \widetilde{\mathbf{a}}_{W_N}}{p^2} \right| &\lesssim \frac{1}{2} \left| \int \sqrt{N} W_N(1 + \varphi_I) \frac{(e^{ip \cdot x} - 1)}{p^2} \right| \\ &\lesssim \|\varphi_I\|_\infty \int N^{3/2} W(\sqrt{N}x) \frac{|p \cdot x|}{p^2} \lesssim \frac{1}{|p| \sqrt{N}}. \end{aligned} \quad (6.43)$$

Therefore, using the cutoff in the sum and also that $\widetilde{\mathbf{a}}_{W_N} = \mathbf{a}_W + \mathcal{O}(N^{-1/2})$, we can replace entirely \widehat{v}_N by $4\pi \mathbf{a}_W / p^2$ in (6.41) up to an error of order one. Next, we may also replace the sum by the corresponding integral, as the difference of the summand at two different points decays more quickly. Thus,

$$E_N = -\frac{2}{(2\pi)^6} \int_{|p|, |q| \leq \sqrt{N}} \frac{(4\pi \mathbf{a}_W)^4 (p \cdot q)^2 dq dp}{p^4 q^4 ((p+q)^2 + p^2 + q^2 + 1)} + \mathcal{O}(1). \quad (6.44)$$

This integral can now be evaluated up to order $\log N$. Using spherical coordinates and Gaussian integrals, we obtain

$$\begin{aligned} \int \frac{(p \cdot q)^2 dq}{(p^2 q^4 ((p+q)^2 + p^2 + q^2))} &= 2\pi \int_{-1}^1 \int_0^\infty \frac{s^2 dr ds}{(2r^2 + 2p^2 + 2|p|rs)} \\ &= \pi \int_{-1}^1 \int_0^\infty \int_0^\infty s^2 e^{-t(r^2 + p^2 + |p|rs)} dt dr ds \\ &= \frac{\pi^2}{2|p|} \int_{-1}^1 \frac{s^2 ds}{\sqrt{1 - s^2/4}} = \frac{\pi^2}{|p|} \left(4 \arcsin\left(\frac{1}{2}\right) - 2\sqrt{\frac{3}{4}} \right). \end{aligned} \quad (6.45)$$

Integrating over p , this yields

$$E_N = -32\pi \alpha_W^4 \left(\frac{2\pi}{3} - \sqrt{3} \right) \log N + \mathcal{O}(1). \quad (6.46)$$

This completes the proof of Theorem 1.2. \square

6.2 Proof of Theorem 1.5 and Corollary 1.6

Proof of Theorem 1.5.

This follows immediately by combining the upper bound of Lemma 6.2 with the lower bound of Lemma 6.3 and cancelling all the scalar contributions to $e_k(H_N)$, $e_0(H_N)$. \square

Proof of Corollary 1.6.

a) Convergence of eigenfunctions up to subsequences follows immediately from the convergence of spectral projections of part b). Indeed, denote $e_0 := e_0(H_{\text{BF}})$ and let B be an interval containing $e_k(H_{\text{BF}}) - e_0$, but no different eigenvalues of $H_{\text{BF}} - e_0$. Then, for N sufficiently large, $e_k(H_N) - e_0(H_N) \in B$ and $U_{\text{B}}^* U_X^* \Psi_N = U_{\text{B}}^* U_X^* \mathbf{1}_B(H_N - e_0(H_N)) \Psi_N$. With b) the difference to $\mathbf{1}_B(H_{\text{BF}} - e_0) U_{\text{B}}^* U_X^* \Psi_N$ tends to zero. The latter sequence has a convergent subsequence in the finite-dimensional range of $\mathbf{1}_B(H_{\text{BF}} - e_0) = \mathbf{1}(H_{\text{BF}} = e_k(H_{\text{BF}}))$.

b) Recall from (6.2) that $U_X^* H_N U_X = \mathbf{1}_{\mathcal{N}_+ \leq N} \mathcal{H}_N \mathbf{1}_{\mathcal{N}_+ \leq N}$, with

$$U^* \mathcal{H}_N U = \mathcal{H}_N^U - E_{N,W} + \lambda_N + \mathcal{L}_4 + \mathcal{E}.$$

We already know from the norm-resolvent convergence of Proposition 5.1 that

$$\lim_{N \rightarrow \infty} \mathbf{1}_B(\mathcal{H}_N^U - E_{N,W} - e_0) = (U_\kappa^\infty)^* \mathbf{1}_B(H_{\text{BF}} - e_0) U_\kappa^\infty. \quad (6.47)$$

We thus need to take care of the interaction term \mathcal{L}_4 , the error term \mathcal{E} and the convergence of the unitary $U = U_q U_W U_c U_B U_G$, which depends on N via the generators of its factors defined in Section 4.

Convergence of U . Concerning the unitary, we have

$$\lim_{N \rightarrow \infty} U \mathbf{1}_B(\mathcal{H}_N^U - E_{N,W} - e_0) U^* = \lim_{N \rightarrow \infty} U_B U_\kappa^\infty \mathbf{1}_B(\mathcal{H}_N^U - E_{N,W} - e_0) (U_B U_\kappa^\infty)^*. \quad (6.48)$$

The reason is that for the transformations U_q , U_W and U_c acting on large momenta,

$$\begin{aligned} 0 &= \lim_{N \rightarrow \infty} (U_q - 1)(\mathcal{N}_+ + 1)^{-1/2} = \lim_{N \rightarrow \infty} (U_c - 1)(\mathcal{N}_+ + 1)^{-1/2} \\ &= \lim_{N \rightarrow \infty} (U_W - 1)(\mathcal{N}_+ + 1)^{-1/2}, \end{aligned} \quad (6.49)$$

which follows easily using Duhamel's formula as in Sections 4.2, 4.4 in the first two cases and a simple bound on the generator (compare (3.26)) in the last case. By the same reasoning, the transformations U_G and U_B that act on small momenta converge to their limits, defined by setting $N = \infty$ in the generators, when regularised by $(\mathcal{N}_+ + 1)^{-1/2}$ (for $N = \infty$, U_G equals U_κ^∞ as defined in (3.4)). Since all the unitaries are uniformly bounded on $Q(\mathcal{N}_+)$ (cf. Lemma 6.1) and $(\mathcal{N}_+ + 1)^{1/2}$ is uniformly bounded on the range of the spectral projection by (6.4) this proves (6.48).

Convergence of the spectral projections. It remains to prove that

$$\lim_{N \rightarrow \infty} \left(\mathbf{1}_B(\mathcal{H}_N^U - E_{N,W} - e_0) - U^* U_X^* \mathbf{1}_B(H_N - e_0(H_N)) U_X U \right) = 0. \quad (6.50)$$

We will argue by induction on the number k of the min-max value. For any k let B_k be an interval containing $e_k(H_{\text{BF}}) - e_0$, but no other part of the spectrum of $H_{\text{BF}} - e_0$, as in the hypothesis. Define δ_k by $\sup B_k = e_k(H_{\text{BF}}) - e_0 + \delta_k$, and then for $j \in \mathbb{N}_0$

$$\begin{aligned} \mathbb{P}_k &= U^* U_X^* \mathbf{1}(H_N \leq e_k(H_N) + \delta_k) U_X U = 1 - \mathbb{Q}_k, \\ \tilde{\mathbb{P}}_j &= \mathbf{1}(\mathcal{H}_N^U \leq e_k(\mathcal{H}_N^U) + \delta_k) = 1 - \tilde{\mathbb{Q}}_k. \end{aligned} \quad (6.51)$$

Then we have for N sufficiently large

$$\begin{aligned} U^* U_X^* \mathbf{1}_{B_k}(H_N - e_0(H_N)) U_X U &= \mathbb{P}_k - \mathbb{P}_{k-1}, \\ \mathbf{1}_{B_k}(\mathcal{H}_N^U - E_{N,W} - e_0) &= \tilde{\mathbb{P}}_k - \tilde{\mathbb{P}}_{k-1}, \end{aligned} \quad (6.52)$$

with the convention that $\mathbb{P}_{-1} = \tilde{\mathbb{P}}_{-1} = 0$. Therefore, it is enough to prove that $\mathbb{P}_k - \tilde{\mathbb{P}}_k = \tilde{\mathbb{Q}}_k \mathbb{P}_k - \tilde{\mathbb{P}}_k \mathbb{Q}_k$ converges to zero for every k . In fact, we will prove that $\tilde{\mathbb{Q}}_k \mathbb{P}_k$ and $\tilde{\mathbb{P}}_k \mathbb{Q}_k$ converge to zero. Let us now assume the statement that has been proved up to $k-1$ for some $k \geq 0$ (it is obvious for $k < 0$, as $\mathbb{P}_{-1} = 0$). In Lemma 6.2,

we showed that

$$\begin{aligned}
& \operatorname{Tr}\left((\mathcal{H}_N^U - E_{N,W} - e_0)\tilde{\mathbb{P}}_k\right) \\
& \stackrel{(6.22)}{\geq} \operatorname{Tr}\left(\tilde{\mathbb{P}}_k\chi_M(\mathcal{H}_N^U - E_{N,W} - e_0)\chi_M\tilde{\mathbb{P}}_k\right) + \mathcal{O}(N^{-1/80}) \\
& \geq \operatorname{Tr}\left(\tilde{\mathbb{P}}_k\chi_M(U^*U_X H_N U_X U - \lambda_N - e_0)\chi_M\tilde{\mathbb{P}}_k\right) + \mathcal{O}(N^{-1/80}) \\
& \geq \operatorname{Tr}\left(\tilde{\mathbb{P}}_k\chi_M(U^*U_X H_N U_X U - e_0(H_N))\mathbb{P}_k\chi_M\tilde{\mathbb{P}}_k\right) \\
& \quad + (e_{k+1}(H_N) - e_0(H_N))\operatorname{Tr}\left(\tilde{\mathbb{P}}_k\chi_M\mathbb{Q}_k\chi_M\tilde{\mathbb{P}}_k\right) + \mathcal{O}(N^{-1/80}), \tag{6.53}
\end{aligned}$$

where we specifically used (6.9) together with (6.16) and (6.13) for the second inequality to get rid of \mathcal{L}_4 and \mathcal{E} , as well as the lower bound on $e_0(H_N)$ from Lemma 6.3. We may remove the localization χ_M in the two terms in (6.53) thanks to the presence of \mathbb{P}_k and because

$$\|\tilde{\mathbb{P}}_j(1 - \chi_M)\|^2 \stackrel{(6.7)}{=} \mathcal{O}(N^{-1/80}). \tag{6.54}$$

Let d_k be the dimension of $\operatorname{ran} \mathbb{1}_{B_k}(\mathcal{H}_N^U - E_{N,W} - e_0)$, which equals the multiplicity of $e_k(H_{\text{BF}})$ by (6.47) for N large enough. By the induction hypothesis, which implies in particular that $\mathbb{P}_{k-1}\tilde{\mathbb{P}}_k - \mathbb{P}_{k-1} \rightarrow 0$, we then have by Theorem 1.5,

$$\lim_{N \rightarrow \infty} \operatorname{Tr}\left(\tilde{\mathbb{P}}_k\chi_M(U^*U_X H_N U_X U - e_0(H_N))\mathbb{P}_{k-1}\chi_M\tilde{\mathbb{P}}_k\right) = \sum_{j=0}^{k-1} d_j(e_j(H_{\text{BF}}) - e_0). \tag{6.55}$$

Moreover, by (6.5) we also have

$$\lim_{N \rightarrow \infty} \operatorname{Tr}\left((\mathcal{H}_N^U - E_{N,W} - e_0)\tilde{\mathbb{P}}_k\right) = \sum_{j=0}^k d_j(e_j(H_{\text{BF}}) - e_0). \tag{6.56}$$

We can thus take the limit in (6.53), whence the terms with $j < k$ cancel, giving

$$\begin{aligned}
d_k(e_k(H_{\text{BF}}) - e_0) & \geq \limsup_{N \rightarrow \infty} \left((e_k(H_{\text{BF}}) - e_0)\operatorname{Tr}((\tilde{\mathbb{P}}_k - \tilde{\mathbb{P}}_{k-1})\mathbb{P}_k) \right. \\
& \quad \left. + (e_{k+1}(H_{\text{BF}}) - e_0)\operatorname{Tr}((\tilde{\mathbb{P}}_k - \tilde{\mathbb{P}}_{k-1})\mathbb{Q}_k) \right). \tag{6.57}
\end{aligned}$$

As $\mathbb{P}_k + \mathbb{Q}_k = 1$ and $d_k = \operatorname{Tr}(\tilde{\mathbb{P}}_k - \tilde{\mathbb{P}}_{k-1})$ for large N , dividing by d_k and eliminating e_0 expresses $e_k(H_{\text{BF}})$ as a convex combination of itself and $e_{k+1}(H_{\text{BF}})$. But since $e_k(H_{\text{BF}}) < e_{k+1}(H_{\text{BF}})$ this implies,

$$0 = \lim_{N \rightarrow \infty} \operatorname{Tr}((\tilde{\mathbb{P}}_k - \tilde{\mathbb{P}}_{k-1})\mathbb{Q}_k) = \lim_{N \rightarrow \infty} \operatorname{Tr}(\tilde{\mathbb{P}}_k\mathbb{Q}_k) \geq \lim_{N \rightarrow \infty} \|\tilde{\mathbb{P}}_k\mathbb{Q}_k\|, \tag{6.58}$$

where we used the induction hypothesis once more. The argument for the convergence of $\mathbb{P}_k \tilde{\mathbb{Q}}_k$ is essentially the same, using the inequalities proved in Lemma 6.3.

c) This follows from b) by an approximation argument. To be precise, let $\varepsilon > 0$ and let N be large enough for $\|U_X^* \Psi_N - \Psi\| < \varepsilon$ to hold. Choose k so that

$$\|\mathbb{1}(H_{\text{BF}} > e_k(H_{\text{BF}}) + \delta) U_{\text{B}}^* \Psi\| < \varepsilon, \quad (6.59)$$

where δ is such that part b) applies to $B_\delta(e_k(H_{\text{BF}})H_{\text{BF}})$. By the convergence of spectral projections proved there, we have

$$\|\mathbb{1}(H_N > e_k(H_N) + \delta) U_X \Psi\| < 2\varepsilon, \quad (6.60)$$

for sufficiently large N . Consequently, using unitarity of the group,

$$\begin{aligned} & \left\| U_X^* e^{-it(H_N - e_0(H_N))} \Psi_N - U_{\text{B}} e^{-it(H_{\text{BF}} - e_0(H_{\text{BF}}))} U_{\text{B}}^* \Psi \right\| \\ & \leq \left\| \sum_{j=0}^k \left(e^{-it(e_j(H_N) - e_0(H_N))} U_X^* \mathbb{1}(H_N = e_j(H_N)) U_X \Psi \right. \right. \\ & \quad \left. \left. - e^{-it(e_j(H_{\text{BF}}) - e_0(H_{\text{BF}}))} U_{\text{B}} \mathbb{1}(H_N = e_j(H_N)) U_{\text{B}}^* \Psi \right) \right\| + 4\varepsilon, \end{aligned} \quad (6.61)$$

which is less than 5ε for sufficiently large N by the convergence of the eigenvalue differences from Theorem 1.5 and the spectral projections from part b). \square

A The scattering problem on the torus

A.1 Properties of the scattering solution

Let $0 \leq v \in L^2(\mathbb{T}^3)$ be even and let us define

$$\varphi = -\frac{1}{2} \frac{1}{P^\perp(-\Delta + \frac{1}{2}v)P^\perp} P^\perp v \in P^\perp L^2(\mathbb{T}^3), \quad (\text{A.1})$$

where P^\perp is the orthogonal projection to the complement of the constant $\mathbb{1}$, i.e, functions that integrate to zero. The Fourier coefficients of φ then solve the scattering equation (2.16) with potential v ,

$$p^2 \widehat{\varphi}(p) + \frac{1}{2} \sum_{q \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \widehat{\varphi}(q) \widehat{v}(p - q) = -\frac{1}{2} \widehat{v}(p) \quad (\text{A.2})$$

Lemma A.1 (Regular estimates). *There is $C > 0$ so that for $0 \leq v \in L^2(\mathbb{T}^3)$*

$$\|\varphi\|_2 \leq C \|v\|_2^{2/3} \|v\|_1^{4/3}, \quad \|\nabla \varphi\|_2 \leq C \|v\|_2^{1/3} \|v\|_1^{2/3}, \quad (\text{A.3a})$$

$$|\widehat{\varphi}(p)| \leq \frac{C}{p^2} \|v\|_2^{2/3} \|v\|_1^{4/3}, \quad p \in 2\pi\mathbb{Z}^3 \setminus \{0\}, \quad (\text{A.3b})$$

$$\|\widehat{\varphi}\|_1 \leq C \|v\|_1^{1/3} \|v\|_2^{2/3} \left(1 + (1 + \|v\|_2^2) \|v\|_1\right). \quad (\text{A.3c})$$

Remark A.2. Taking $v_n(x) = n^2 v(nx)$ for $n \geq 1$, that is $\widehat{v}_n(p) = n^{-1} \widehat{v}(p/n)$, we obtain

$$\begin{aligned} \|v_n\|_2^{2/3} \|v_n\|_1^{4/3} &= n^{-1} \|v\|_2^{2/3} \|v\|_1^{4/3}, \\ \|v_n\|_1^{1/3} \|v_n\|_2^{2/3} &= \|v\|_1^{1/3} \|v\|_2^{2/3}, \\ (1 + \|v_n\|_2^2) \|v_n\|_1 &= (n^{-1} + \|v\|_2^2) \|v\|_1. \end{aligned}$$

Proof. Since $v \geq 0$ is in L^2 , we have $\varphi \in L^2(\mathbb{T}^3)$. Writing the scalar product of $-\Delta\varphi$ with φ in Fourier space, we obtain with $v \geq 0$

$$\|p\widehat{\varphi}\|_2^2 \leq \|p\widehat{\varphi}\|_2^2 + \frac{1}{2} \|\sqrt{v}\varphi\|_2^2 = -\frac{1}{2} \langle \widehat{v}, \widehat{\varphi} \rangle \leq \left(\sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \frac{|\widehat{v}(p)|^2}{p^2} \right)^{1/2} \|p\widehat{\varphi}\|_2. \quad (\text{A.4})$$

We deduce that $\varphi \in H^1(\mathbb{T}^3)$ and $\|p\widehat{\varphi}\|_2 \leq \| |p|^{-1} \widehat{v} \|_2$. Let us now bound, for any $A \geq 1$,

$$\begin{aligned} \| |p|^{-1} \widehat{v} \|_2^2 &= \sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \frac{|\widehat{v}(p)|^2}{p^2} = \sum_{|p| > A} \frac{|\widehat{v}(p)|^2}{p^2} + \sum_{|p| \leq A} \frac{|\widehat{v}(p)|^2}{p^2} \\ &\leq A^{-2} \|\widehat{v}\|_2^2 + CA \|\widehat{v}\|_\infty^2 \leq C \|v\|_2^{2/3} \|v\|_1^{4/3}, \end{aligned} \quad (\text{A.5})$$

where we took $A = \|v\|_2^{2/3} \|v\|_1^{-2/3} \geq 1$, used that $\|\widehat{v}\|_\infty \leq \|v\|_1 \leq \|v\|_2$. This proves the second estimate in (A.3a). The same computation gives also $\| |p|^{-2} * |\widehat{v}|^2 \|_\infty \leq C \|v\|_2^{2/3} \|v\|_1^{4/3}$,

$$\|\widehat{v} * \widehat{\varphi}\|_\infty \leq \sup_{q \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \left(\sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \frac{|\widehat{v}(p-q)|^2}{p^2} \right)^{1/2} \|p\widehat{\varphi}\|_2 \leq C \|v\|_2^{2/3} \|v\|_1^{4/3}, \quad (\text{A.6})$$

From this and the scattering equation (A.2), we obtain the point-wise bound on $\widehat{\varphi}$ (A.3b) as well as the first estimate in (A.3a). Multiplying (A.2) by $|p|^{-2}$ and summing over p , we also obtain

$$2\|\widehat{\varphi}\|_1 \leq \| |p|^{-2} \widehat{v} \|_1 + \| |p|^{-2} \widehat{v} * \widehat{\varphi} \|_1. \quad (\text{A.7})$$

The first term is bounded as follows, picking $A = (\|v\|_2/\|v\|_1)^{2/3} \geq 1$, we obtain

$$\| |p|^{-2} \widehat{v} \|_1 \leq \|\mathbf{1}_{|p| \leq A} |p|^{-2}\|_1 \|\widehat{v}\|_\infty + \|\mathbf{1}_{|p| > A} |p|^{-2}\|_2 \|v\|_2 \leq C \|v\|_1^{1/3} \|v\|_2^{2/3}. \quad (\text{A.8})$$

By iterating the equation (A.1) and using that $\||p|^{-2}\widehat{v} * h\|_{6p/(p+6)+\varepsilon} \leq C_\varepsilon \|h\|_p$ for all $p \geq 6/5$, $\varepsilon > 0$, we can estimate the second term in (A.7) and already obtain that $\|\widehat{\varphi}\|_1 < \infty$. To obtain a quantitative estimate, we decompose the second term for $A \geq 1$ as

$$\begin{aligned} \||p|^{-2}\widehat{v} * \widehat{\varphi}\|_1 &\leq \||p|^{-2}\mathbf{1}_{|p|\leq A}\widehat{v} * \widehat{\varphi}\|_1 + \||p|^{-2}\mathbf{1}_{|p|>A}\widehat{v} * \widehat{\varphi}\|_1 \\ &\leq \||p|^{-2}\mathbf{1}_{|p|\leq A}\|_1 \|\widehat{v} * \widehat{\varphi}\|_\infty + \||p|^{-2}\mathbf{1}_{|p|>A}\|_2 \|\widehat{v} * \widehat{\varphi}\|_2 \\ &\leq C \left(A \|v\|_2^{2/3} \|v\|_1^{4/3} + A^{-1/2} \|\widehat{v}\|_2 \|\widehat{\varphi}\|_1 \right). \end{aligned} \quad (\text{A.9})$$

For $A = C^2 \|v\|_2^2 + 1 > C^2 \|v\|_2^2$ this yields

$$\begin{aligned} \|\widehat{\varphi}\|_1 &\lesssim (1 - A^{-1/2} \|v\|_2)^{-1} \left(\|v\|_1^{1/3} \|v\|_2^{2/3} + A \|v\|_2^{2/3} \|v\|_1^{4/3} \right) \\ &\lesssim \|v\|_1^{1/3} \|v\|_2^{2/3} \left(1 + (1 + \|v\|_2^2) \|v\|_1 \right), \end{aligned} \quad (\text{A.10})$$

which concludes the proof. \square

We now consider, for $m \geq 1$, the truncated scattering solution

$$\widehat{\varphi}_m(p) = \mathbf{1}_{|p|>m} \widehat{\varphi}(p). \quad (\text{A.11})$$

Lemma A.3 (Estimates on φ_m). *There is $C > 0$ so that for $0 \leq v \in L^2(\mathbb{T}^3)$ and all $m \geq 1$, we have*

$$\|\varphi_m\|_2 \leq C m^{-1/2} \|v\|_2^{2/3} \|v\|_1^{4/3}, \quad \|\nabla \varphi_m\|_2 \leq C \|v\|_2^{1/3} \|v\|_1^{2/3}, \quad (\text{A.12a})$$

$$\|\varphi - \varphi_m\|_\infty \leq C m \|v\|_2^{2/3} \|v\|_1^{4/3} \quad (\text{A.12b})$$

Moreover, for $0 \leq s < 1$ there is $C > 0$ so that if $s < 1/2$

$$\||p|^s \widehat{\varphi}_m\|_2 \leq C m^{-(1/2-s)} \|v\|_2^{2/3} \|v\|_1^{4/3} \quad (\text{A.12c})$$

$$\||p|^{1/2} \widehat{\varphi}\|_2 \leq C \|v\|_2^{2/3} \|v\|_1^{4/3} \sqrt{|\log \|v\|_2| + 1}, \quad (\text{A.12d})$$

and for $1/2 < s \leq 1$

$$\||p|^s \widehat{\varphi}_m\|_2 \leq C m^{-(1-s)/2} \|v\|_2^{2-s/3} \|v\|_1^{2(2-s)/3}.$$

Proof. The estimates (A.12a), (A.12b) and (A.12c) follow easily from (A.3b) and (A.3a). The estimate for $1/2 < s < 1$ follows by Hölder's inequality $\||p|^s \widehat{\varphi}_m\|_2 \leq \||p| \widehat{\varphi}_m\|_2^s \|\widehat{\varphi}_m\|_2^{1-s}$ together with the two bounds in (A.12a).

It remains to prove the case $s = 1/2$. We use the equation (A.2) to obtain

$$2\||p|^{1/2} \widehat{\varphi}\|_2 \leq \||p|^{-3/2} \widehat{v}\|_2 + \||p|^{-3/2} \widehat{v} * \widehat{\varphi}\|_2. \quad (\text{A.13})$$

We bound the first term as follows, for any $A \geq 1$, we have

$$\begin{aligned} \||p|^{-3/2}\widehat{v}\|_2^2 &\leq \|\mathbf{1}_{|p|\leq A}|p|^{-3/2}\widehat{v}\|_2^2 + \|\mathbf{1}_{|p|>A}|p|^{-3/2}\widehat{v}\|_2^2 \\ &\leq C \log A \|\widehat{v}\|_\infty^2 + A^{-3} \|\widehat{v}\|_2^2 \\ &\leq C \|\widehat{v}\|_\infty^2 (\log(\|\widehat{v}\|_2/\|\widehat{v}\|_\infty) + 1) \end{aligned} \quad (\text{A.14})$$

where we chose $A = \|\widehat{v}\|_2^{2/3}/\|\widehat{v}\|_\infty^{2/3} \geq 1$. Similarly, we have

$$\begin{aligned} \||p|^{-3/2}\widehat{v} * \widehat{\varphi}\|_2 &\leq C \|\widehat{v} * \widehat{\varphi}\|_\infty \sqrt{\log(\|\widehat{v} * \widehat{\varphi}\|_2/\|\widehat{v} * \widehat{\varphi}\|_\infty) + 1} \\ &\leq C \|v\|_2^{2/3} \|v\|_1^{4/3} \sqrt{|\log \|v\|_2| + 1}. \end{aligned} \quad (\text{A.15})$$

where we used the bounds on $\|\widehat{v} * \widehat{\varphi}\|_\infty \leq \|v\|_2^{2/3} \|v\|_1^{4/3} \leq \|v\|_2^2$ from (A.6) and

$$\|\widehat{v} * \widehat{\varphi}\|_2 \leq C \|v\|_2 \left(\||p|^{-2}\widehat{v}\|_1 + (1 + \|v\|_2^2) \|v\|_2^{1/3} \|v\|_1^{2/3} \right) \leq C (\|v\|_2 + 1)^4. \quad (\text{A.16})$$

Finally combing the above estimates, we obtain

$$\||p|^{1/2}\widehat{\varphi}\|_2 \leq C \|v\|_2^{2/3} \|v\|_1^{4/3} \sqrt{|\log \|v\|_2| + 1}. \quad (\text{A.17})$$

□

A.2 Comparison between torus and free space scattering lengths

For a generic potential $0 \leq v \in L^2(\mathbb{T}^3)$, we define the torus scattering length by minimization problem

$$4\pi \mathbf{a}_{\mathbb{T}^3}(v) = \inf_{\substack{\varphi \in H^1(\mathbb{T}^3) \\ \int \varphi = 0}} \left\{ \int_{\mathbb{T}^3} |\nabla \varphi|^2 + \frac{1}{2} \int_{\mathbb{T}^3} v |1 + \varphi|^2 \right\} \quad (\text{A.18})$$

By the Lax-Milgram Theorem this problem has a unique minimizer $\varphi_{\mathbb{T}^3}$, which satisfies the equation (A.2). We may extend v beyond the unit cube by zero, to obtain a potential $0 \leq v \in L^1(\mathbb{R}^3)$. Denote by $\mathbf{a}_{\mathbb{R}^3}(v)$ the scattering length on the full space defined in (1.2) of this potential.

Lemma A.4. *Let $0 \leq v \in L^2(\mathbb{T}^3)$. There exists $C > 0$ so that, denoting $v_n(x) = n^2 v(nx)$, we have for all $n \geq 1$*

$$|n \mathbf{a}_{\mathbb{T}^3}(v_n) - \mathbf{a}_{\mathbb{R}^3}(v)| \leq C n^{-1}.$$

Proof. Let $\varphi_{\mathbb{R}^3}$ be the solution to (2.14), then from [60, Theorem 6] we know that

$$-\frac{1}{|x|+1} \lesssim \varphi_{\mathbb{R}^3}(x) \leq 0, \quad |\nabla \varphi_{\mathbb{R}^3}(x)| \lesssim \frac{1}{|x|^2+1} \quad (\text{A.19})$$

for all $x \in \mathbb{R}^3$. Let $0 \leq \chi \in C^\infty(\mathbb{R}^3)$ such that $\chi(x) = 1$ for $|x| \leq 1/3$ and $\chi(x) = 0$ for $|x| \geq 1/2$. We set $\varphi_n(x) = \chi(x)\varphi_{\mathbb{R}^3}(nx) + c_n$, with c_n chosen so that the integral of φ_n over the unit cube equals zero. Then φ_n satisfies for $n \geq 2$

$$-\Delta \varphi_n + \frac{1}{2}v_n(1 + \varphi_n) = (-\Delta \chi)\varphi(n\cdot) - 2n\nabla \chi \cdot \nabla \varphi(n\cdot) + \frac{1}{2}v_n c_n \quad (\text{A.20})$$

since $\text{supp } v_n \subset \{|x| \leq 1/3\} \subset \{\chi \equiv 1\}$ for $n \geq 2$. Note that

$$|c_n| \leq \int |\chi \varphi_{\mathbb{R}^3}(n\cdot)| \lesssim \int_{[-1/2, 1/2]^3} \frac{1}{n|x|+1} \lesssim n^{-1}, \quad (\text{A.21})$$

so, since $\varphi_{\mathbb{R}^3}$ is bounded and $\int v_n = n^{-1} \int v$,

$$\int v_n(|1 + \varphi_n|^2 - |1 + \chi \varphi_{\mathbb{R}^3}(n\cdot)|^2) = \mathcal{O}(n^{-2}). \quad (\text{A.22})$$

Then, using the variational characterization of $4\pi\mathbf{a}_{\mathbb{T}^3}(v_n)$ and (2.15), we obtain

$$\begin{aligned} 4\pi\mathbf{a}_{\mathbb{T}^3}(v_n) &\leq \int_{\mathbb{T}^3} |\nabla \varphi_n|^2 + \frac{1}{2} \int_{\mathbb{T}^3} v_n |1 + \varphi_n|^2 \\ &= 4\pi\mathbf{a}_{\mathbb{R}^3}(v_n) + \int_{\mathbb{R}^3} ((-\Delta \chi)\varphi_{\mathbb{R}^3}(n\cdot) - 2n\nabla \chi \cdot \nabla \varphi_{\mathbb{R}^3}(n\cdot)) \varphi_n + \mathcal{O}(n^{-2}) \\ &\leq 4\pi\mathbf{a}_{\mathbb{R}^3}(v_n) + C \int_{[-1/2, 1/2]^3} \left(\frac{1}{(n|x|+1)^2} + \frac{n}{(n|x|+1)^3} \right) + \mathcal{O}(n^{-2}) \\ &\leq 4\pi\mathbf{a}_{\mathbb{R}^3}(v)n^{-1} + \mathcal{O}(n^{-2}), \end{aligned} \quad (\text{A.23})$$

where we used that $\text{supp } \chi \subset [-1/2, 1/2]^3$ and the exact scaling relation $n\mathbf{a}_{\mathbb{R}^3}(v_n) = \mathbf{a}_{\mathbb{R}^3}(v)$.

Reciprocally, take this time $\varphi_n(x) = \chi(x)\varphi_{\mathbb{T}^3}^{(n)}(x)$ where

$$\varphi_{\mathbb{T}^3}^{(n)} = -\frac{1}{2} \frac{1}{P^\perp(-\Delta + \frac{1}{2}v_n)P^\perp} P^\perp v_n \in P^\perp \in L^2(\mathbb{T}^3) \cong L^2([-1/2, 1/2]^3). \quad (\text{A.24})$$

The same computation as above now leads to, since $\varphi_{\mathbb{T}^3}$ is real,

$$\begin{aligned} 4\pi\mathbf{a}_{\mathbb{R}^3}(v_n) - 4\pi\mathbf{a}_{\mathbb{T}^3}(v_n) &\leq \int_{\mathbb{R}^3} \left((-\Delta \chi)\varphi_{\mathbb{T}^3}^{(n)} - 2\nabla \chi \cdot \nabla \varphi_{\mathbb{T}^3}^{(n)} \right) \chi \varphi_{\mathbb{T}^3}^{(n)} \\ &= \int_{\mathbb{R}^3} |\nabla \chi|^2 |\varphi_{\mathbb{T}^3}^{(n)}|^2 \\ &\leq C \|\varphi_{\mathbb{T}^3}^{(n)}\|_2^2 = \mathcal{O}(n^{-2}), \end{aligned} \quad (\text{A.25})$$

where for the last estimate we used Lemma A.1 together with Remark A.2. \square

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Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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