



**HAL**  
open science

# A toy model provably featuring an arrow of time without past hypothesis

Pablo Arrighi, Gilles Dowek, Amélia Durbec

## ► To cite this version:

Pablo Arrighi, Gilles Dowek, Amélia Durbec. A toy model provably featuring an arrow of time without past hypothesis. RC 2024 - 16th International Conference on Reversible Computation, Jul 2024, Torun, Poland. pp.50-68, 10.1007/978-3-031-62076-8\_4. hal-04727052

**HAL Id: hal-04727052**

**<https://hal.science/hal-04727052v1>**

Submitted on 9 Oct 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A toy model provably featuring an arrow of time without past hypothesis

Pablo Arrighi<sup>1</sup>, Gilles Dowek<sup>1</sup>, and Amélia Durbec<sup>2</sup>

<sup>1</sup> Université Paris-Saclay, Inria, CNRS, LMF, 91190 Gif-sur-Yvette, France

<sup>2</sup> Université Paris-Saclay, CNRS, LISN, 91190 Gif-sur-Yvette, France

**Abstract.** The laws of Physics are time-reversible, making no qualitative distinction between the past and the future—yet we can only go towards the future. This apparent contradiction is known as the ‘arrow of time problem’. Its current resolution states that the future is the direction of increasing entropy. But entropy can only increase towards the future if it was low in the past, and past low entropy is a very strong assumption to make, because low entropy states are rather improbable, non-generic. Recent works from the Physics literature suggest, however, we may do away with this so-called ‘past hypothesis’, in the presence of reversible dynamical laws featuring expansion. We prove that this is the case, for a reversible causal graph dynamics-based toy model. It consists in graphs upon which particles circulate and interact according to local reversible rules. Some rules locally shrink or expand the graph. Generic states always expand; entropy always increases—thereby providing a local explanation for the arrow of time. This discrete setting allows us to deploy the full rigour of theoretical Computer Science proof techniques. These objects are also interesting from a Dynamical Systems point of view, as a simple generalisations of cellular automata exhibiting non-trivial behaviours.

**Keywords:** Models of computation · Cellular automata · Synchronous graph rewriting · Information theory · Invariants and termination

## 1 Introduction

*In short.* The main contribution of this paper is a first rigorous proof that an arrow of time typically emerges in some time-reversible dynamics, without the need for a past hypothesis. This problem pertains to a long foundations of Physics tradition. But the novelty of our approach is precisely the deployment of theoretical Computer Science models and techniques, in order to formalise it and solve it. I.e. the problem is first transposed to the world of models of computation (cellular automata, graph dynamics, term algebras) and information theory, and then addressed through the study of invariants and termination proofs. In the longer version of this paper, some natural variants of the model are also explored numerically.

*The problem.* Physics laws are time-reversible, i.e. time evolution can be inverted, making no qualitative distinction between the past and the future. Yet, we clearly experience the fact that we cannot go back to the past. This discrepancy is referred to as the ‘arrow of time problem’. It sets the requirement for an explanation of “how does the future versus past phenomenon that we witness in everyday life, arise from time-reversible dynamical laws alone”. I.e. the aim of the game is to resolve this apparent paradox by pinpointing specific physical quantities that can be used as ‘clocks’, and then identifying the future direction to be direction of increase of that quantity, thereby providing a time arrow.

Usually, these quantities are variants of the concept of entropy, and one shows that starting from a low entropy initial state, entropy typically increases even under a time-reversible law. Thus, in the conventional argument due to Boltzmann [14], “Time-reversible dynamical law + Past low entropy  $\Rightarrow$  Arrow of time”. This argument works even for systems of bounded size (e.g. a few of particles in a box). But it suffers three important criticisms: (i) By the Poincaré recurrence theorem, as we iterate the dynamics forward, the entropy typically increases. . . but then drops back, and increases, etc., as the dynamical system is necessarily (almost-)periodic. The criticism is usually dismissed by Physicists on the base that the recurrence period is beyond cosmological times. (ii) Starting from an initial low entropy state and iterating the dynamics forward, entropy typically increases (the ‘entropic clock’s arrow’ matches that of the external time-coordinate aka “dynamical clock”). However, had we applied the reversed dynamics instead, i.e. iterating the dynamics backward, entropy typically would have increased, too (the entropic clock’s arrow does not match the dynamical clock’s arrow) [20], as is quite often overlooked. (iii) The assumption that the initial state be of low entropy, also referred to as ‘the past hypothesis’ [1] is a very strong one. This is because generic state have maximal entropy. Hence, the presence of such an improbable state at dynamical clock time 0, remains a mystery, that again demands an explanation. We will review these points in Sec. 2.

*Comparison with related models.* In [16,17], Carroll and Chen try to fix the third criticism and provide the first plausible intuition why “Time-reversible dynamical law  $\Rightarrow$  Arrow of time”. Their key new ingredient is big bounce (big crunch then bang) then eternal expansion—as featured also in several early attempts to restore the particle-level charge/parity/time-reversal symmetry (CPT) at the scale of the universe [25,15,22]. Their discussion is left informal however, leaving plenty of room to discuss whether it really manages to do away with the past hypothesis [28,27,29]. In particular, they themselves raise the issue whether expansion mechanisms are actually compatible with reversibility.

Informally, in a big bouncing universe, matter gets compressed by the big crunch, and released by the big bang. Expansion happens so fast that matter then finds itself out of equilibrium, in a low entropy state. Matter then diffuses and entropy increases without ever reaching a maximum, as expansion is eternal. The entropic clock’s arrow thus matches that of the “size-of-the-universe clock”, which in turn matches that of the dynamical clock after the big bounce. This

explanation is compelling. Yet the following direct logical consequence is somewhat mind-bending: before the big bounce occur, the size-of-the-universe clock, and thus the entropic clock, have their arrow opposite to that of the dynamical clock. These twin entropic times facing each other have in fact been popularized as ‘Janus time’ by Barbour [10]. Still, as counter-intuitive as it may be, this is already happening in the conventional argument by Boltzmann: as pointed out by Golstein et al. [20], this is just the mentioned criticism (*ii*).

Barbour et al. [10,11] use the  $n$ -body problem, with (non-local) Newtonian classical gravity turned on, as an enlightening analogy of big-bounce-then-eternal-expansion. The question whether their model really does away with the past hypothesis has been argued in [30]. Indeed as the considered bodies travel on a pre-existing infinite space, the analogy blurs out the requirement of finite but unbounded configurations, which is needed to make the argument rigorous. Moreover, the entropic clock (quantity measuring the microscopic disorder) is replaced by a “shape complexity clock” (quantity measuring the macroscopic clumping) in these works. This is non-standard and arguable on the basis that in many situations there is no need for gravitational clumping in order to observe an arrow of time. The question whether expansion can be implemented as a local reversible mechanism also remains open in this strand of works [24].

The aim of this paper is to exhibit rigorously-defined local reversible dynamical laws (the local rule of reversible causal graph dynamics) for which we can prove that, for a rigorously-defined notion of entropy (alike that of perfect gas):

- Generic states always end up growing in size as we iterate the dynamics.
- Entropy always increases as size grows.

Thus we prove that an entropic clock direction emerges without the need to assume past low entropy. In other words the arrow of time is established from local reversible expansion mechanisms alone, doing away with the past hypothesis. Again, this works because size as a function of dynamical clock time is typically  $U$ -shaped. As generic states are somewhere on this  $U$ -curve, their size will end up growing, and their entropy will end up increasing. They will do so forever, as configurations are of finite but unbounded size. Of course generic states have, somewhere along the dynamical clock timeline, some states of smallest size and lowest entropy, which may be dubbed as “initial”. These particular states are non-generic, just like the minimum of any  $U$ -curve is non-generic. Because the  $U$ -curve is due to the dynamics alone, their existence is the result of dynamics alone.

Our toy model is set in  $1 + 1$  spacetime. It consists in circular graphs upon which particles move and interact when they are closeby. Moreover, local some patterns are interchanged, triggering shrinking or expansion of the circle. It is cast in the framework of reversible causal graph dynamics [8,6] and is inspired by the Hasslacher-Meyer model [23], for which there is numerical evidence of a  $U$ -shaped size curve, but no proof—this seems inherently hard to prove in fact [12]. There is no mention of the arrow of time nor entropy in their paper; moreover the sense in which it is reversible and causal is left informal and seems incompatible with quantum mechanics [3]. Instead, our toy model enjoys rigorous proofs of

$U$ -shaped size and entropy curves, as well as rigorous notions of reversibility [5] and causality (i.e. making sure that information propagates at a bounded speed with respect to graph distance), readily allowing for a quantum extension [4]. These results are provided in Sec. 3.

As far as we know, there are no other closely related models besides the above-mentioned ones. This may be because 1/ The fact that a time-reversible model always grows is somewhat counter-intuitive, 2/ let alone proving it—as shown by the efforts of [12]. 3/ Interdisciplinary work is not so common on this topic. In particular, the theoretical Computer Science analysis that we deploy is a new player. We believe it brings simplicity, clarity and rigour to a long-standing issue.

## 2 The conventional argument

*Entropy.* Entropy was defined by Boltzmann in the 1870s:

**Definition 1.** [13] *The entropy  $S$  of a macroscopic state is defined by :*

$$S = k \cdot \ln \Omega \tag{1}$$

with  $k$  the Boltzmann constant and  $\Omega$  is the number of microscopic configurations corresponding to the macroscopic state.

In this definition the word ‘macroscopic state’ refers to a set of values for the macroscopic properties of the system, such as its temperature, pressure, volume or density. Given a certain macroscopic state, a ‘statistical ensemble’ is a way to assign a probability distribution to the set of microscopic states that correspond to the macroscopic state. It is often reasonable to assume that the probability distribution be uniform (aka a ‘microcanonical ensemble’). The probability  $p_i$  of a microscopic state  $x_i$  is then  $p_i = \frac{1}{\Omega}$ . The connection between the Shannon entropy [26] of this probability distribution, and the Boltzmann entropy of the macrostate, is then obvious:

$$S = -k \sum_{i=0}^{\Omega-1} p_i \ln(p_i) = -k \sum_{i=0}^{\Omega-1} \frac{1}{\Omega} \ln\left(\frac{1}{\Omega}\right) = k \cdot \ln \Omega \tag{2}$$

For a dynamical system over the state space  $X$ , the microscopic states are simply the configurations  $x_i \in X$  of the system. We formalize the macroscopic states as equivalence classes on  $X$ . The entropy function associates, to each microscopic state, the entropy of its macrostate.

**Definition 2.** *Consider  $X$  a set and  $\equiv$  an equivalence relation on  $X$ . We define the entropy function  $S : X \rightarrow \mathbb{R}$  associated with  $\equiv$  as:*

$$S(x) = \ln(|[x]|) \tag{3}$$

with  $[x]$  the  $\equiv$ -equivalence class of  $x$  and  $|[x]|$  is its size.

*The case of bounded size dynamical systems.* The second principle of thermodynamics states that “entropy increases in time”. However, even for closed, bounded size dynamical systems, the situation is not so obvious:

**Remark 1** *Let  $X$  be a finite state space and  $f : X \rightarrow X$  a bijection. For any entropy function  $S$ , and for any configuration  $x \in X$ , the sequence  $(S(f^n(x)))_{n \in \mathbb{N}}$  is periodic because the sequence  $f^n(x)_{n \in \mathbb{N}}$  is periodic. This implies that the sequence of entropy variations  $(S(f^{n+1}(x)) - S(f^n(x)))_{n \in \mathbb{N}}$  is itself periodic. If  $(S(f^n(x)))_{n \in \mathbb{N}}$  is not constant, then these entropy variations can be negative.*

How can we justify, then, that when we dilute a drop of dye in a sealed glass of water, entropy seems to just rise, unambiguously indicating an arrow of time? Besides the fact that in practice the glass is not quite a closed system, and hence undergoes a not quite time-reversible system, several other assumptions are implicit in this emblematic experiment.

First, the duration of such an experiment likely to be far too short to observe periodicity. To give an order of magnitude, for a molar volume of 22.4L of perfect gas, the number of microstates  $\Omega$  is  $\sim 10^{5 \times 10^{24}}$ . Intuitively, the period ought to be of that same order of magnitude, as expressed in units of Planck time (minimal observable time). However, it is estimated that only  $10^{60}$  Planck times have elapsed since the Big Bang.

Second, the drop of dye would likely have diluted just as well if it had undergone the time-symmetrized versions of Physics laws instead. In other words, entropy typically does increase when we start from a low entropy initial configuration... but it does so in both directions of the dynamical clock [20].

Third, we must realise that the experiment starts off at a rather improbable time: the first  $10^{60}$  ticks of dynamical clock time represent a negligible fraction of the  $\sim 10^{5 \times 10^{24}}$  entropy recurrence period. Had current time been picked up at random within the period, there would be no reason to expect it to be a time of increase of entropy, rather than of decrease. Another way to say this is that the experiment starts off from an improbable configuration. Indeed in any generic configuration the dye is diluted already; entropy is almost maximal already; and the entropy variation is zero on average, independently of  $t$  the number of steps between the two observations:

$$\sum_{x \in \Sigma} (S(f^t(x)) - S(x)) = \sum_{x \in \Sigma} S(x) - \sum_{x \in \Sigma} S(x) = 0.$$

In order to witness an arrow of time, we must start from a low entropy configuration, but in practice the equivalence relation and therefore the entropy function are chosen so that low-entropy configurations are non-generic.

*Past hypothesis.* So, to this day, phenomena such as the dilution of the drop of dye in a glass of water, and the increasing entropy therein, are paradigmatic of current understanding of the arrow of time problem... and yet, a careful inspection of the assumptions underlying the conventional argument shows that it only displaces the problem. The question “Why do we observe an arrow of time” has become “Why was the Universe originally of low entropy?”. In fact

the conventional argument requires that the entropy at the Big Bang be so low that an arrow of time is still observable  $\sim 13.7$  billion years later—making it a very strongly non-generic configuration. This strong assumption of a low entropy initial configuration is referred to as the ‘past hypothesis’ [1], and was criticised right from its birth, on account of this unlikelihood [14]. Luckily, more recent accounts of the arrow of time suggest we could do without it [16,17,10,9]. The key ingredient is expansion.

### 3 Arrow of time without past hypothesis

In this section we prove via a toy model that an entropic arrow of time can originate from expansion, and that this expansion can be implemented locally and reversibly. In this model Remark 1 does not apply because, although each configuration is finite, the state space itself is infinite, as configurations can grow. That Remark 1 can be circumvented in an infinite state space is not surprising by itself: think of  $\mathbb{N}$  the set of integers for instance. Each number can be written with a finite number of digits, but the set itself is of course infinite, and it is easy to define a non-periodic bijection on the set:

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ n + 2 & \text{if } n \text{ is even} \\ n - 2 & \text{else} \end{cases} \quad (4)$$

What is much less obvious and harder to prove is the existence of a time-reversible, causal, homogeneous, ultimately expanding dynamics on generic configurations. We do not know of another such model.

#### 3.1 State space

The states of the model are circular graphs of 1 dimension. These circles are of unbounded but finite size, i.e. the line is not allowed. The vertices are equipped with ports  $a$  and  $b$ , and edges go from port to port, each being used exactly once, as in Fig. 1.

Each vertex carries an internal state, amongst four possible states: ‘containing no particle’, ‘containing a particle moving along port  $a$ ’, ‘containing a particle moving along port  $b$ ’, ‘containing two particles’. Notice that this set of internal states is the

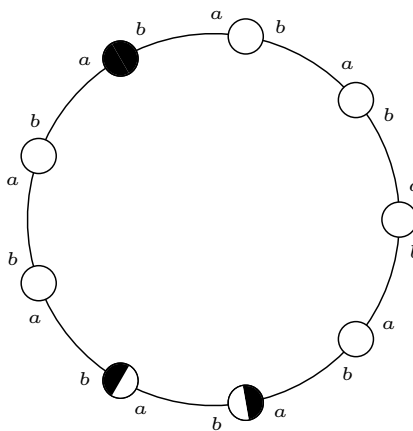


Fig. 1: A configuration. Each full half-disks represents the presence of an (undistinguishable) particle that is about to hop along the corresponding port.

same as that used to model electrons in gas-on-grid methods [21], or in quantum walks to represent the spin of a fermionic particle such as the electron [2]. Later we will generalize this by associating, to each port of each node, not just one information bit, but two or three.

There is one subtlety: vertices are named, and these names form a little algebra. This is so that a vertex  $u$  may be able to split into  $u.l$  and  $u.r$ , and later merge back into  $u.l \vee u.r$ , and that this be in fact the same as just  $u$ . There is no escaping this formalism in order to achieve both reversibility and local vertex creation/destruction [6], particularly if one wants to preserve causality in the quantum regime [3]. In order to remain self-contained, the full definition of these named graphs is provided in Appendix A.

We denote by  $\mathcal{C}_n$  the set of circular graphs with  $2n$  information bits per vertex ( $n$  per port). For any vertex  $x$ , we denote by  $p_i(x)$  the value of the  $i$ -th bit of the port  $p$  of vertex  $x$ .

### 3.2 The toy model

Our main model ( $\sqrt{\tau}I$ ) is inspired by the Hasslacher-Meyer dynamics [23]. It acts on the set  $\mathcal{C}_1$  and consists in composing two steps:  $I$ , then  $\sqrt{\tau}$ . Each of them is a reversible causal graph dynamics, thus so is their composition—the reader is left to referred to [8,6] for further theoretical aspects about reversible causal graph dynamics, including general definitions. Thus the whole dynamics is a time-reversible. It consists in a composition of steps which, taken individually, are time-symmetric [19]:

**Definition 3.** *A causal graph dynamics  $f$ , it is said to be time-symmetric if and only if there exists a causal graph dynamics  $T$  such that  $T^2 = Id$  and  $TfT = f^{-1}$ .*

*Step  $\sqrt{\tau}$ .* Let  $\tau$  be the operation that moves all particles along their corresponding port. The operation  $\sqrt{\tau}$  is such  $\sqrt{\tau} \circ \sqrt{\tau} = \tau$ : it moves particles by half an edge instead, see Fig. 2a. One way of thinking about this operation is as inverting the roles of edges and nodes in the sense of taking the dual graph. Notice that alternatively, we could have used a (renaming-equivalent but less symmetrical) operation that moves only those particles associated to port  $b$ , but by a whole edge.

**Definition 4** ( $\sqrt{\tau}$ ). *Step  $\sqrt{\tau}$  is defined for any graph  $X \in \mathcal{C}_n$  as follows:*

- $V(\sqrt{\tau}(X)) = \{u.r \vee v.l \mid \{u : b, v : a\} \in E(X)\}$
- $E(\sqrt{\tau}(X)) = \{\{x' : b, y' : a\} \mid u, x, y \in V(X) \text{ et } x' = u.r \vee x.l \text{ et } y' = y.r \vee u.l\}$



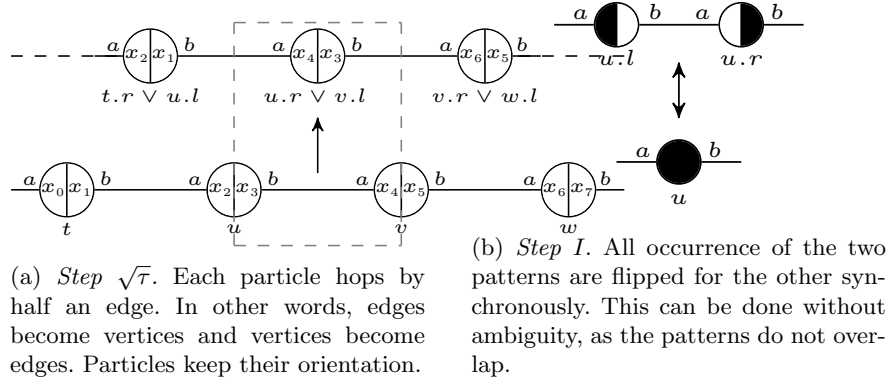


Fig. 2: Rules of the toy model.

- $\forall x' \in V(\sqrt{\tau}(X))$ , and for all  $i \in [1, n]$ ,  $a_i(x) = a_i(u)$  and  $b_i(x) = b_i(v)$ , where  $u$  and  $v$  are the vertices of  $X$  such that  $x'.l = u.r$  and  $x'.r = v.l$  respectively.

This step is both time-reversible and time-symmetrical, since

$$\sqrt{\tau}^{-1} = T\sqrt{\tau}T \quad (5)$$

with  $T$  the function exchanging the left and right information bits of each vertex.

*Step I*. *Step I* consists in splitting a vertex into two when it holds two particles, or conversely merging two vertices holding a pair of back-to-back particles, as in Fig. 2b. Thus, a vertex  $u$  such as  $a_1(u) = b_1(u) = 1$  will produce two vertices  $u.l$  and  $u.r$  with  $a_1(u.l) = b_1(u.r) = 1$  and  $b_1(u.l) = a_1(u.r) = 0$ . Conversely, two vertices  $u$  and  $v$ , with  $a_1(u) = b_1(v) = 1$  and  $b_1(u) = a_1(v) = 0$  will merge into a vertex  $u \vee v$  such as  $a_1(u \vee v) = b_1(u \vee v) = 1$ .

This step is obviously time-reversible and time-symmetric, since it is involutive:  $I^2 = Id$ .

*Spacetime diagram*. The evolution of a configuration under toy model  $\sqrt{\tau}I$  can be represented in the form of a spacetime diagrams, e.g. Fig. 3 represents  $\sqrt{\tau}I$ . In these diagrams, the spatial dimension is represented horizontally, and dynamical clock time is represented vertically downwards. Each vertex is represented by a cell, separated from its neighbours with a vertical black line. Its internal state of a cell is captured by its colour. The cells are depicted in variable-sizes, allowing each split/merge to be done “on the spot”.

### 3.3 Size increases

The first observation that can be made from Fig. 3 is that the dynamics  $\sqrt{\tau}I$ , although time-reversible, grows the size of the graph. It does not grow from the borders, there are no borders, it just expands locally. The numerics in Fig. 4a suggest this is typical. We will now prove that this happens for generic initial

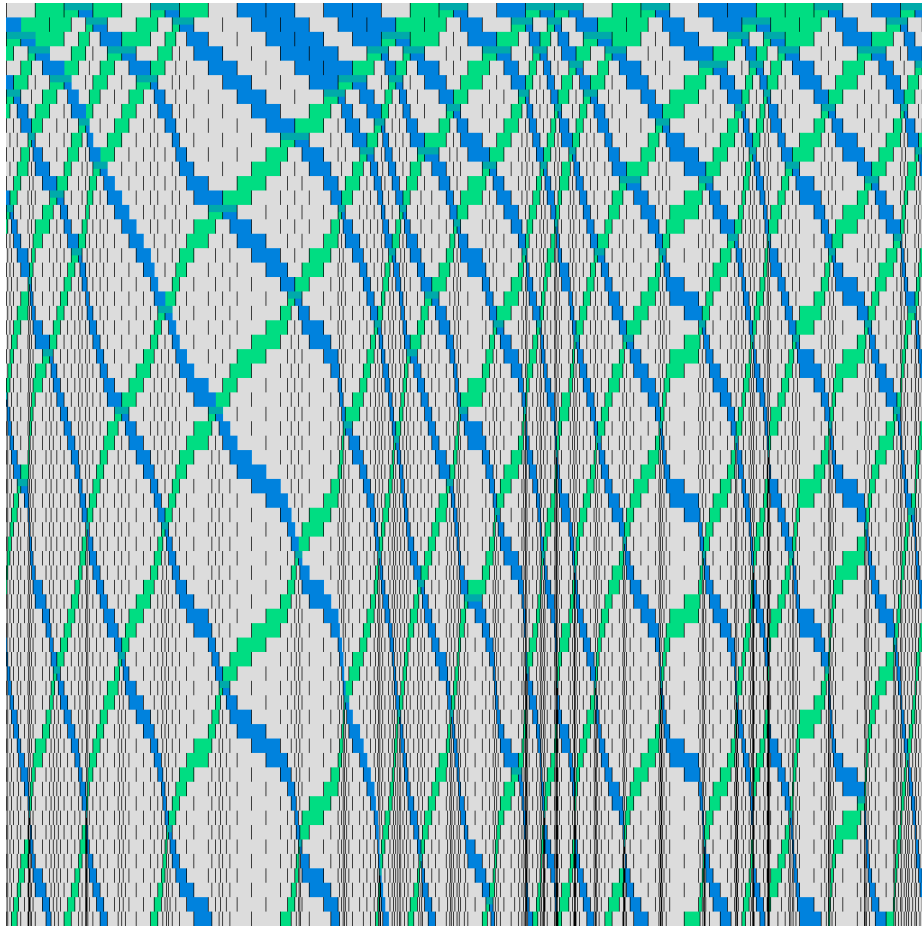


Fig. 3: *Spacetime diagram of dynamics  $\sqrt{\tau}I$ .* Dynamical clock time flows towards the bottom. Particles corresponding moving along port  $a$  (resp.  $b$ ) are represented green (resp. blue). Observe how space keeps on expanding, making this spacetime diagram look like a curtain.

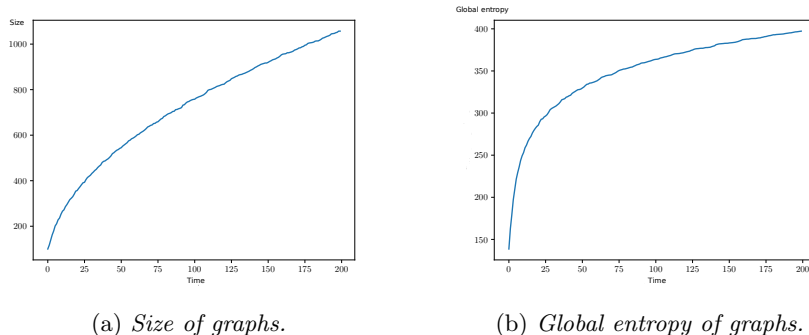


Fig. 4: Typical size and entropy curves for dynamics  $\sqrt{\tau}I$ . The horizontal axis represents the number of steps of the dynamics, aka dynamical clock time. The initial configuration is drawn uniformly at random amongst all graphs of size 100.

states. In fact, we will prove the stronger result that graphs always end up growing, and that the growth is strict as soon as they contain at least one particle of each type.

Intuitively this is due to the fact that vertex merger only occurs in the presence of a pattern which is unstable:

**Lemma 1 (Merger Instability).** . . *Let be a circular graph  $X \in \mathcal{C}_1$ . Given a pair  $u$  and  $v$  of adjacent vertices of  $X$ , these are said to belong to a merger pattern if and only if  $a_1(u) = b_1(v) = 1$  and  $b_1(u) = a_1(v) = 0$ . For any  $u, v$  forming a merger pattern in  $X$ , there are two vertices  $u.r, v.l \in V((\sqrt{\tau}I)^{-1}(X))$  such that  $u.r$  and  $v.l$  form a merger pattern.*

*Proof.* By inspection of Fig. 5, which represents the pre-image of a merger pattern.

Let us fix notations before we state the expansion theorem. first; in what follows we will write  $(u_n)$  to designate the sequence  $(u_n)_{n \in \mathbb{N}}$  in the absence of ambiguity and use:

**Definition 5.** *Let there be two sequences  $(u_n)$  and  $(v_n)$ . We say that  $(u_n)$  is of the order of  $(v_n)$ , and write  $(u_n) = \Theta((v_n))$  if there exist positive numbers  $a, b \in \mathbb{R}$  and  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $a.v_n \leq u_n \leq b.v_n$ .*

We have:

**Theorem 1 (Expansion).** *For any  $X \in \mathcal{C}_1$  containing at least one particle of each type, let  $u_n = |V((\sqrt{\tau}I)^n(X))|$ . We have  $(u_n) = \Theta((\sqrt{n}))$ .*

*Proof.* (Outline). As proved in Lem. 1 dynamics  $\sqrt{\tau}I$  cannot create a merger pattern as they are stable by  $(\sqrt{\tau}I)^{-1}$ . This entails that any interference disrupting a merger pattern will permanently destroy it. We prove, by means of

a strictly decreasing measure, that such interference will occur if and only if the graph contains at least two particles going in opposite direction. Lastly we quantify the growth rate once all merger patterns have been removed from the graph, placing bounds corresponding to the best and worst case scenarios. The proof technique is thus akin to a program termination and complexity analysis proof. It is technical, we shift it to Appendix B for readability.

As can be seen in Fig. 4a, for a randomly chosen configuration, the asymptotic regime is reached quickly and is quite stable.

### 3.4 Entropy increases with size

It turns out that growth in the size of the graph implies growth of entropy, for a natural notion of entropy.

Indeed from this point on, we will focus on the entropy function associated with the following equivalence class: two configurations are considered equivalent if and only if they have the same size and the same number of particles. This entropy function can be seen as analogous to the one used in the study of perfect gases.

This is because the entropy of a perfect gas configuration is generally associated with the macroscopic properties of pressure, volume, temperature and number of particles. As these four variables are related by the perfect gas law, three are independent. Moreover, in our toy model the speed of the particles is constant, which makes it unnecessary to consider the temperature. We therefore have to consider just two variables amongst: the number of particles; the size of the graph (analogous to the volume); and the density of particles (analogous to the pressure).

We ignore the names of the vertices when counting the microstates corresponding to the macrostates, i.e. when counting the number of graphs of a given size and having a given number of particles. We denote  $\binom{n}{p}$  the binomial coefficient  $p$  among  $n$ , i.e.  $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ .

**Definition 6.** *The entropy function  $S'$  is defined by  $S'(X) = \log\left(\binom{2|V(X)|}{p}\right)$  where  $p$  is the number of particles in  $X$ .*

In the case of  $\sqrt{\tau}I$  we have proven in Th. 1 that  $V(X)$  grows as a square root. Because this rule preserves the number of particles we automatically obtains the growth of entropy. But we can be more precise:

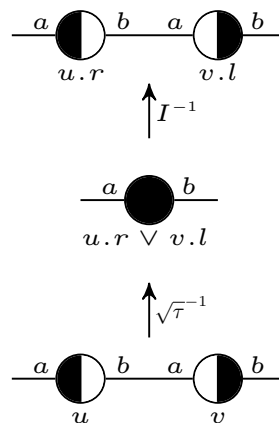


Fig. 5: *Instability of merger patterns under  $\sqrt{\tau}I$ .*

**Corollary 1** For any  $X \in \mathcal{C}_1$  containing at least one particle of each type, let  $e_n = S'(\sqrt{\tau}I)^n(X)$ . We have  $(e_n) = \Theta(\log(n))$ .

*Proof.* Thanks to Th. 1, we have that  $(S'(\sqrt{\tau}I)^n(X))_{n \in \mathbb{N}} = \Theta(\log(\binom{\Theta(\sqrt{n})}{p}))$ . Using the bounds  $(\frac{a}{b})^b \leq \binom{a}{b} \leq e^b (\frac{a}{b})^b$  [18], and noting that  $p$  is constant, we obtain :

$$(S'(\sqrt{\tau}I)^n(X))_{n \in \mathbb{N}} = \Theta(p \log(\frac{\Theta(\sqrt{n})}{p})) = \Theta(\log(n^{1/2})) = \Theta(\log(n))$$

Note that this corollary would apply equally well to any dynamics where the number of particles remains constant and the size of the graph grows polynomially (not necessarily as a square root).

The asymptotic regime of the size function was reached as soon as all the merger patterns were destroyed, cf. Fig. 4a. The same happens with global entropy, cf. Fig. 4b.

If we are only interested in whether entropy grows, without seeking to characterise its asymptotic behaviour, we can state a more general theorem, relating it to size growth. Indeed, under the assumption that the particles do not fill the whole space, nor disappear completely, then any dynamics that increase the size of the graph will also increase the entropy:

**Theorem 2 (Entropy increases with size).** For all  $X \in \mathcal{C}_u$  and  $f : \mathcal{C}_u \rightarrow \mathcal{C}_u$  such that :

- $\lim_{n \rightarrow +\infty} |V(f^n(X))| = +\infty$
- $\exists m \in \mathbb{N}$  such that  $\forall n \geq m, 1 \leq p_n \leq 2u \times |V(f^n(X))| - 1$  where  $p_n$  is the number of particles in the step in  $f^n(X)$ .

We have that  $\lim_{n \rightarrow +\infty} S'(f^n(X)) = +\infty$ .

*Proof.* Thanks to the second condition, we have for all  $n \geq m$  :

$$S'(f^n(X)) = \log\left(\binom{n}{p}_{2|V(f^n(X))|}\right) \geq \log\left(\binom{2|V(f^n(X))|}{1}\right) = \log(2|V(f^n(X))|)$$

As  $\log(2|V(f^n(X))|)$  tends to  $+\infty$  when  $n$  tends to  $+\infty$ , this is also the case for  $S'(f^n(X))$ .

### 3.5 Recovering an arrow of time

With Th. 1 and Cor. 1, we have proven that an entropic arrow of time emerges in some time-reversible, causal, homogeneous laws (namely the  $\sqrt{\tau}I$  toy model), without relying on the past hypothesis. More precisely, we have proven that starting from generic configurations, entropy ultimately grows as we iterate the dynamics. Intuitively, after a finite period of dynamical clock time, the entropic

clock's arrow aligns with that of the dynamical clock. This solves criticism (iii) of the conventional argument. Notice how, ultimately, this resolution boils down to the fact that configurations are of finite but unbounded size. In this context, assuming that the universe "starts small" is reasonable, because for any configuration, there are many more larger configurations than smaller ones. The same happens with entropy: any starting value is small within the set of positive real numbers. In that sense past low entropy is no longer unreasonable, it is unavoidable.

An immediate consequence is that the toy model is not periodic, i.e. there is no recurrence time: this solves criticism (i) in a way more satisfactory manner than arguing that "there is a recurrence time but it is typically too big to be observed". Let us look at criticism (ii).

Since the system  $\sqrt{\tau}I$  is time-reversible, one can naturally ask what happens if one tries to "go back in time", i.e. how a generic graph evolves when one applies the dynamics  $(\sqrt{\tau}I)^{-1} = I^{-1}\sqrt{\tau}^{-1}$ . Numerics suggest it also increases the size of the graph, but at a different rate, see Fig. 6a. We can prove it:

**Theorem 3.** *For any  $X \in \mathcal{C}_1$  containing at least one particle of each type, the sequence  $(|V((\sqrt{\tau}I)^{-n}(X))|)_{n \in \mathbb{N}}$  is of the order of  $n$ .*

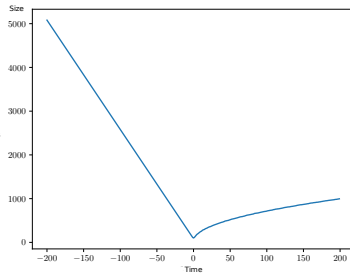
*Proof.* In the absence of patterns  $\textcircled{\small\bullet} \text{---} \textcircled{\small\bullet}$ , the size of the graph decreases strictly each time two particles meet. By conservation of momentum, the particles will continue to cross each other. Since the graph cannot decrease continuously a pattern  $\textcircled{\small\bullet} \text{---} \textcircled{\small\bullet}$  will inevitably form. As can be seen in the proof of Lem. 1, the pattern  $\textcircled{\small\bullet} \text{---} \textcircled{\small\bullet}$  is stable by  $I^{-1}\sqrt{\tau}^{-1}$ , and cannot be crossed by other particles. This implies that once such a pattern is present, any pair of particles not belonging to such a pattern can only collide once. When all these collisions have occurred, each application of  $I^{-1}\sqrt{\tau}^{-1}$  increases the size of the graph by the number of patterns present. We can bound by  $\min(n_a, n_b)$  the number of such patterns in  $X$  and its successors by  $I^{-1}\sqrt{\tau}^{-1}$ . Thus there exists  $m \in \mathbb{N}$  such that for all  $n \geq m$  we have :

$$|V((\sqrt{\tau}I)^{-m}(X))| + n \leq |V((\sqrt{\tau}I)^{-n}(X))| \leq |V((\sqrt{\tau}I)^{-m}(X))| + n \times \min(n_a, n_b)$$

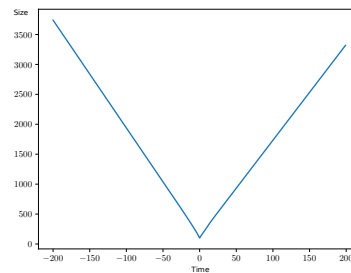
By the same proof scheme as for corollary 1, we obtain the entropy growth for  $I^{-1}\sqrt{\tau}^{-1}$ :

**Corollary 2** *For any  $X \in \mathcal{C}_1$  containing at least one particle of each type, let  $e_n = S'(\sqrt{\tau}I)^{-n}(X)$  the sequence  $(e_n)$  is of the order of  $\log(n)$ .*

Thus, a variation of criticism (ii) still holds, as entropy increases in both directions from a 'source'. But here, since the model is not periodic, there is a single such source: a possibility discussed under the name of 'Janus point' [16,17,10,9]. To move away from this region of minimal entropy, whether by iterating the forward dynamics or its reverse, means augmenting entropy and hence 'going towards the future'. The important point is the way in which this variation relates to the resolution of criticism (iii): it is no longer necessary to assume that



(a) Typical curve of size of the graphs under  $(\sqrt{\tau}I)^{-1}$  and  $\sqrt{\tau}I$ . The initial condition is drawn uniformly at random amongst graphs of size 100. We run the dynamics backwards and forward to explore negative and positive dynamical clock times.



(b) Typical curve of size of the graphs under  $\sqrt{\tau}I_2$ , a time-symmetrised version of  $\sqrt{\tau}I$ . The initial condition is drawn uniformly at random amongst graphs of size 100, thus particle density is high  $\sim 0.5$ .

the universe “started” in an improbable low-entropy state; as the existence of an arrow of time just follows from the existence of a minimal region, which itself is a direct consequence of the dynamical law.

## 4 Conclusion

This paper provides a first rigorous proof that an arrow of time typically emerges in some time-reversible dynamics, without the need for a past hypothesis. The reversible dynamics in question is cast in the setting of reversible causal graph dynamics ; the methods used pertain to the field of theoretical Computer Science. The proof works by showing that graph size increases, and that entropy increases with size, where entropy is defined as for perfect gases. It provides a local explanation for the origin of the arrow of time, by tracing it back to local, reversible expansion mechanisms acting over configurations of finite but unbounded size.

This explanation resolves two of the three main criticism to the standard of the standard Boltzmann argument: there is no recurrence time, and no need to assume atypical initial conditions. One criticism still holds as there are successive configurations of minimal entropy from which two arrows of time flow in opposite directions. This idea is present in some cosmological models and popularized under the name or ‘Janus point’.

We also show, numerically, in the longer version of this paper, that the explanation is compatible with fully time-symmetric dynamics, as well as periods of exponential growth.

In Thermodynamics, entropy increases globally, but locally in may well decrease and stabilize close to zero, reaching the so-called ‘thermal death’. At which stage it becomes impossible to witness an arrow of time locally. The longer version of

this paper includes a theorem about the fatality of thermal death, and discusses delaying it.

We wonder whether the same argument can be made base on other notions of entropy, such as: metric entropy, topological entropy, or Von Neumann entropy (in the quantum regime).

### Acknowledgements

We wish to thank Marios Christodoulou, Pierre Guillon, Benjamin Hellouin, Carlo Rovelli and Francesca Vidotto for many discussions and insights. This project/publication was made possible through the support of the ID# 62312 grant from the John Templeton Foundation, as part of the ‘The Quantum Information Structure of Spacetime’ Project (QISS). The opinions expressed in this project/publication are those of the author(s) and do not necessarily reflect the views of the John Templeton Foundation.

### References

1. David Z Albert. Time and chance, 2001.
2. Pablo Arrighi. An overview of quantum cellular automata. *Natural Computing*, 18:885, 2019. arXiv preprint arXiv:1904.12956.
3. Pablo Arrighi, Marios Christodoulou, and Amélia Durbec. On quantum superpositions of graphs, no-signalling and covariance. *CoRR*, abs/2010.13579, 2020.
4. Pablo Arrighi, Amélia Durbec, and Matt Wilson. Quantum networks theory. *CoRR*, abs/2110.10587, 2021.
5. Pablo Arrighi, Amélia Durbec, and Aurélien Emmanuel. Reversibility vs local creation/destruction. In Michael Kirkedal Thomsen and Mathias Soeken, editors, *Reversible Computation - 11th International Conference, RC 2019, Lausanne, Switzerland, June 24-25, 2019, Proceedings*, volume 11497 of *Lecture Notes in Computer Science*, pages 51–66. Springer, Springer, 2019.
6. Pablo Arrighi, Nicolas Durbec, and Aurélien Emmanuel. Reversibility vs local creation/destruction. In *Proceedings of RC 2019, LLNCS*, volume 11497, pages 51–66. Springer, 2019.
7. Pablo Arrighi, Simon Martiel, and Vincent Nesme. Cellular automata over generalized cayley graphs. *Mathematical Structures in Computer Science*, 28(3):340–383, 2018.
8. Pablo Arrighi, Simon Martiel, and Simon Perdrix. Reversible causal graph dynamics: invertibility, block representation, vertex-preservation. *Natural Computing*, 19(1):157–178, 2020. Pre-print arXiv:1502.04368.
9. Julian Barbour. *The Janus point: A new theory of time*. Random House, 2020.
10. Julian Barbour, Tim Koslowski, and Flavio Mercati. Identification of a gravitational arrow of time. *Phys. Rev. Lett.*, 113:181101, Oct 2014.
11. Julian Barbour, Tim Koslowski, and Flavio Mercati. Janus points and arrows of time. *arXiv preprint arXiv:1604.03956*, 2016.
12. Karin Baur, Jeffrey M Rabin, and David A Meyer. Periodicity and growth in a lattice gas with dynamical geometry. *Physical Review E*, 73(2):026129, 2006.
13. Ludwig Boltzmann. *Vorlesungen Über Gastheorie*. Barth, Leipzig, 1. auflage edition, 1896.



14. Ludwig Boltzmann. *Lectures on gas theory*. Courier Corporation, 2012.
15. Latham Boyle, Kieran Finn, and Neil Turok. Cpt-symmetric universe. *Physical review letters*, 121(25):251301, 2018.
16. Sean M Carroll and Jennifer Chen. Spontaneous inflation and the origin of the arrow of time. *arXiv preprint hep-th/0410270*, 2004.
17. Sean M Carroll and Jennifer Chen. Does inflation provide natural initial conditions for the universe? *International Journal of Modern Physics D*, 14(12):2335–2339, 2005.
18. Shagnik Das. A brief note on estimates of binomial coefficients. <https://www.semanticscholar.org/paper/A-brief-note-on-estimates-of-binomial-coefficients-Das/e0e703e1bbc914e563afb72480d7f915df79b834>, 2015.
19. Anahí Gajardo, Jarkko Kari, and Andrés Moreira. On time-symmetry in cellular automata. *Journal of Computer and System Sciences*, 78(4):1115–1126, 2012.
20. Sheldon Goldstein, Roderich Tumulka, and Nino Zanghì. Is the hypothesis about a low entropy initial state of the universe necessary for explaining the arrow of time? *Phys. Rev. D*, 94:023520, Jul 2016.
21. J. Hardy, Y. Pomeau, and O. de Pazzis. Time Evolution of a Two-Dimensional Classical Lattice System. *Physical Review Letters*, 31(5):276–279, July 1973.
22. J. B. Hartle and S. W. Hawking. Wave function of the universe. *Phys. Rev. D*, 28:2960–2975, Dec 1983.
23. Brosl Hasslacher and David A. Meyer. Modelling dynamical geometry with lattice gas automata. Expanded version of a talk presented at the Seventh International Conference on the Discrete Simulation of Fluids held at the University of Oxford, June 1998.
24. Tim A Koslowski, Flavio Mercati, and David Sloan. Through the big bang: Continuing einstein’s equations beyond a cosmological singularity. *Physics Letters B*, 778:339–343, 2018.
25. AD Sakharov. Cosmological models of the universe with reversal of time’s arrow. *JETP*, 52:349–351, 1980.
26. C. E. Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27(3):379–423, July 1948.
27. Alexander Vilenkin. Arrows of time and the beginning of the universe. *Physical Review D*, 88(4):043516, 2013.
28. Robert M. Wald. The arrow of time and the initial conditions of the universe. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 37(3):394–398, 2006. The arrows of time, 2006.
29. Christopher Gregory Weaver. On the carroll–chen model. *Journal for General Philosophy of Science*, 48(1):97–124, 2017.
30. H Dieter Zeh. Comment on the” janus point” explanation of the arrow of time. *arXiv preprint arXiv:1601.02790*, 2016.

## A Named graphs

Say as in Fig. 2b that some quantum evolution splits a vertex  $u$  into two. We need to name the two infants in a way that avoids name conflicts with the vertices of the rest of the graph. But if the evolution is locally-causal, we are unable to just ‘pick a fresh name out of the blue’, because we do not know which names are available. Thus, we have to construct new names locally. A natural choice is to use the names  $u.l$  and  $u.r$  (for left and right respectively). Similarly, say

that some other evolution merges two vertices  $u, v$  into one. A natural choice is to call the resultant vertex  $u \vee v$ , where the symbol  $\vee$  is intended to represent a merger of names.

This is, in fact, what the inverse evolution will do to vertices  $u.l$  and  $u.r$  that were just split: merge them back into a single vertex  $u.l \vee u.r$ . But, then, in order to get back where we came from, we need that the equality  $u.l \vee u.r = u$  holds. Moreover, if the evolution is time-reversible, then this inverse evolution does exist, therefore we are compelled to accept that vertex names obey this algebraic rule.

Reciprocally, say that some evolution merges two vertices  $u, v$  into one and calls them  $u \vee v$ . Now say that some other evolution splits them back, calling them  $(u \vee v).l$  and  $(u \vee v).r$ . This is, in fact, what the inverse evolution will do to the vertex  $u \vee v$ , split it back into  $(u \vee v).l$  and  $(u \vee v).r$ . But then, in order to get back where we came from, we need the equalities  $(u \vee v).l = u$  and  $(u \vee v).r = v$ .

**Definition 7 (Names).** *Let  $\mathbb{K}$  be a countable set. The name algebra  $\mathcal{N}[\mathbb{K}]$  has terms given by the grammar*

$$u, v ::= c \mid u.t \mid u \vee v \quad \text{with } c \in \mathbb{K}, t \in \{l, r\}^*$$

*and is endowed with the following equality theory over terms (with  $\varepsilon$  the empty word):*

$$(u \vee v).l = u \quad (u \vee v).r = v \quad u.\varepsilon = u \quad u.l \vee u.r = u$$

*We define  $\mathcal{V} := \mathcal{N}[\mathbb{K}]$ .*

The fact that this algebra is well-defined was proven in [6]. Now that we have a set of possible names for our vertices, we can readily define ‘port graphs’ (aka ‘generalized Cayley graphs’ [7]. Condition (1) will just ensure that names do not intersect, e.g. forbidding that there be a name  $u \vee v$  and another  $v \vee w$ , so as to avoid name collisions should they split.

**Definition 8 (Named graphs).** *Let  $\Sigma$  be the set of internal states and  $\pi$  be the set of ports. A graph  $G$  is given by a finite set of vertices  $V_G \subseteq \mathcal{V}$  such that for all  $v, v' \in V_G$  and for all  $t, t' \in \{l, r\}^*$ ,*

$$v.t = v'.t' \text{ implies } v = v' \text{ and } t = t' \tag{6}$$

*together with*

- $\sigma_G : V_G \rightarrow \Sigma$  its internal states
- $E_G$  a set of non-intersecting two element subsets of  $V_G : \pi$ , its edges.

*In other words an edge  $e$  is of the form  $\{x : a, y : b\}$  and  $\forall e, e' \in E_G, e \cap e' \neq \emptyset \Rightarrow e = e'$ .*

## B Proof of Th. 1

*Proof.* First, it is argued that there is a time step  $m$  after which no more vertex mergers will occur. We denote  $n_f(X)$  the number of merger patterns in  $X$ , and  $d_f(X)$  the minimum distance between a merger pattern and a particle moving towards it (this includes a particle present in a merger pattern, and itself if there are no other particles). We will show that the pair  $(n_f(\sqrt{\tau}I)^n(X), d_f(\sqrt{\tau}I)^n(X))$  decreases strictly in lexicographic order.

As we have seen in Lem. 1, a merger pattern cannot be created, and is destroyed on collision. We only need to prove that  $d_{fu}((\sqrt{\tau}I)^n X)$  decreases strictly when  $p(\sqrt{\tau}I)^n X$  remains constant. Let  $u, v$  be two vertices of a merger pattern and  $p$  a particle such that  $u, v$  and  $p$  realise the distance  $d_f(X)$ . Two cases can occur, either  $p$  is itself part of a merger pattern, in which case there are no particles between the two merger patterns, or  $p$  is free moving, in which case there are only particles going in the opposite direction between  $p$  and  $u, v$ . In the first case, the two perform a fusion and there are no particles between the two merger patterns (so there is no division); the distance between the two patterns therefore decreases by 1. In the second case, the particle  $p$  will move towards the merger pattern, decreasing  $d_{fu}((\sqrt{\tau}I)^n X)$ .

In order to preserve the readability of the notations, we will denote  $(u_n)$  the sequence  $(|V((\sqrt{\tau}I)^n(X))|)$ . Thanks to the previous point, we know that there is a time step  $m \in \mathbb{N}$  from which each collision of particles will cause the creation of an additional vertex, so the sequence  $u_n$  is necessarily increasing for all  $n > m$ . Since  $X$  contains at least one particle of each type, we have that for all  $n \geq m$ , the evolution of  $(\sqrt{\tau}I)^n X$  during  $u_n$  time steps causes at least one collision. Similarly, we know that at most  $c = 2n_a n_b$  collisions occur in the same time frame where  $n_a$  (resp.  $n_b$ ) is the number of particles on the port  $a$  (resp.  $b$ ). This allows us to obtain the following inequalities:

$$u_n + 1 \leq u_{n+u_n} \leq u_n + c \quad (7)$$

Let  $(v_k)_{k \in \mathbb{N}}$  be the sub-sequence such that  $v_0 = u_m$  and for all  $k \in \mathbb{N}$ ,  $v_{k+1} = u_{\text{ind}(k)+v_k}$ , where  $\text{ind}(k)$  is the function such that  $\text{ind}(0) = m$  and for all  $k \in \mathbb{N}$ ,  $\text{ind}(k) = \sum_{i=0}^{k-1} v_i$ . By recurrence, we prove that  $u_{\text{ind}(k)} = v_k$  :

$$u_{\text{ind}(k+1)} = u_{\sum_{i=0}^k v_i} = u_{v_k + \sum_{i=0}^{k-1} v_i} = u_{\text{ind}(k)+v_k} = v_{k+1} \quad (8)$$

This allows us to apply the inequality (2) on  $v_{k+1} = u_{\text{ind}(k)+v_k}$ . Combining (2) and (3), we obtain the linear growth of  $(v_k)_{k \in \mathbb{N}}$ :

$$v_k + 1 \leq v_{k+1} = u_{\text{ind}(k)+v_k} \leq v_k + c \quad (9)$$

Let us now focus on the growth of the index of  $(v_k)_{k \in \mathbb{N}}$ . By applying the inequalities of (4) to the definition of  $\text{ind}(k)$  we obtain :

$$\sum_{i=0}^k (v_0 + i) \leq \sum_{i=0}^k (v_i) \leq \sum_{i=0}^k (v_0 + ci) \quad (10)$$

$$ku_m + \frac{k(k-1)}{2} = \sum_{i=0}^k (v_0 + i) \leq \text{ind}(k) \leq \sum_{i=0}^k (v_0 + ci) = ku_m + c\frac{k(k-1)}{2} \quad (11)$$

To conclude, let us return to the main sequence  $(u_n)_{n \in \mathbb{N}}$ . For a sufficiently large  $n$ , there exists  $k \geq 4u_m + c$  such that :

$$\text{ind}(k) \leq n \leq \text{ind}(k+1)$$

This gives us, considering that  $(u_n)_{n \in \mathbb{N}}$  is increasing, the previous equation (6) and that  $k \geq 4u_m + c$  the following inequalities:

$$\text{ind}(k) \leq n \leq \text{ind}(k+1) \quad (12)$$

$$\implies ku_m + \frac{k(k-1)}{2} \leq n \leq (k+1)u_m + c\frac{k(k+1)}{2} \quad (13)$$

$$\implies k(2u_m + k - 1) \leq 2n \leq k(4u_m + c + ck) \quad (14)$$

$$\implies 2k^2 \leq 2n \leq (c+1)k^2 \quad (15)$$

$$\implies k \leq \sqrt{n} \leq \sqrt{\frac{c+1}{2}}k \quad (16)$$

$$\implies \sqrt{\frac{2}{c+1}}\sqrt{n} \leq k \leq \sqrt{n} \quad (17)$$

Since the sequence  $u_n$  is increasing, and using the inequalities (12) and (4), we can conclude with the following inequalities:

$$u_m + \sqrt{\frac{2}{c+1}}\sqrt{n} \leq u_m + k \leq v_k \leq u_n \leq v_{k+1} \leq u_m + c(k+1) \leq u_m + c + c\sqrt{n}$$

Thus,  $u_n$  is of the order of  $(\sqrt{n})$ .