Finite and infinite degree Thurston maps with a small postsingular set

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Abstract. We develop the theory of Thurston maps that are defined everywhere on the topological sphere S^2 with a possible exception of a single essential singularity. We establish an analog of the celebrated W. Thurston's characterization theorem for a broad class of such Thurston maps having four postsingular values. To achieve this, we analyze the corresponding pullback maps defined on the one-complex dimensional Teichmüller space. This analysis also allows us to derive various properties of Hurwitz classes of the corresponding Thurston maps.

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1. Introduction

1.1. **Thurston theory for finite and infinite degree maps.** In the one-dimensional rational dynamics, the crucial role is played by the family of *postcritically finite* (or *pcf* in short) rational maps, i.e., maps with all critical points being periodic or pre-periodic. In this context, one of the most influential ideas has been to abstract from the rigid underlying complex structure and consider the more general setup of postcritically finite *branched self-coverings* of the topological 2-sphere S^2. Nowadays, orientation-preserving pcf branched covering maps $f\colon S^2 \to S^2$ of topological degree $\deg(f) \geqslant 2$ are called *Thurston maps* (*of finite degree*), in honor of William Thurston, who introduced them to deepen the understanding of the dynamics of postcritically finite rational maps on $\widehat{\mathbb{C}}$.

These ideas can be extended to the transcendental setting to explore the dynamics of *postsingularly finite* (*psf* in short) meromorphic maps. A meromorphic map is called *postsingularly finite* if it has finitely many *singular values*, and each of them eventually becomes periodic or lands on the essential singularity under the iteration. We can generalize the notion of a Thurston map to include postsingularly finite *topologically holomorphic* non-injective maps $f\colon X \to S^2$, where X is a punctured topological sphere, f does not extend continuously to the entire S^2 and meets a technical condition of being a *parabolic type* map; see Sections 2.2 and 2.3. Note that in this case the map f must be *transcendental*, meaning that it has infinite topological degree. For simplicity, we will use the notation $f\colon S^2 \dashrightarrow S^2$ to indicate that the Thurston map f, whether finite or infinite degree, might not be defined at a single point of S^2.

For a Thurston map $f\colon S^2 \dashrightarrow S^2$, the *postsingular set* P_f is defined as the union of all orbits of the singular values of the map f. It is important to note that some of these orbits might terminate after several iterations, if a singular value reaches the point where the map f is not defined. The elements of the postsingular set P_f are called the *postsingular values* of the Thurston map f. If the map f is defined on the entire sphere S^2, we simply refer to its *postcritical set* and *postcritical values*, as the set of singular values of f coincides with the set of its *critical values*. Two Thurston maps are called *combinatorially* (or *Thurston*) *equivalent* if they are conjugate up to isotopy relative to their postsingular sets; see Definition 2.7.

A fundamental question in this context is whether a given Thurston map f can be *realized* by a psf meromorphic map with the same combinatorics, that is, if f is combinatorially equivalent to a psf meromorphic map. If the Thurston map f is not realized, then we say that f is *obstructed*. William Thurston answered this question for Thurston maps of finite degree in his celebrated *characterization of rational maps*: if a finite degree Thurston map $f\colon S^2 \to S^2$ has a *hyperbolic orbifold* (this is always true, except for some well-understood special maps), then f is realized by a pcf rational map if and only if f has no *Thurston obstruction* [DH93]. Such an obstruction is given by a finite collection of disjoint simple closed curves in $S^2 - P_f$ with certain invariance properties under the map f. In many instances, it suffices to restrict to simpler types of Thurston obstructions provided by *Levy cycles* or even *Levy fixed curves*; see Definition 2.10, and [Hub16, Theorem 10.3.8], [HP22, Corollary 1.5], or [Par23, Theorems 7.6 and 8.6] for examples of such cases.

The same characterization question can be also asked in the transcendental setting. The first breakthrough in this area was obtained in [HSS09], where it was shown that an *exponential* Thurston map is realized if and only if it has no Levy cycle. In this context, the exponential Thurston map is defined as a Thurston map with two singular values, both of which are *omitted*, and one of which is the only essential singularity of the considered

Thurston map. Furthermore, the results of [She22] and [MPR24] suggest that a Thurston-like criterion for realizability may hold in a greater generality. However, the characterization question in the transcendental setting remains largely open, as many of the techniques used in Thurston theory for finite degree maps do not extend to this context.

Thurston theory lays out the relationship between the topological properties of a map, its dynamics, and its geometry in terms of the existence of a holomorphic realization. Furthermore, it is strongly connected with the combinatorial and algebraic aspects of the dynamics of pcf rational and psf meromorphic maps. The results mentioned above have substantial applications for both rational and transcendental dynamics. For instance, Thurston's characterization result has allowed to classify various families of postcritically finite rational maps or finite degree Thurston maps in terms of combinatorial models [Poi93, Poi10, BLMW22], [DMRS19, LMS22], [Hlu19, HP22]. Building on the result of [HSS09], similar classifications were obtained in [LSV08] in terms of *kneading sequences* and in [PRS21] in terms of *homotopic Hubbard treed* for the family of postsingularly finite exponential maps. Moreover, the concept of a homotopic Hubbard tree was extended to general postsingularly finite entire maps in [Pfr19] (see also [PPS21]). It is plausible that a Thurston-like criterion is the final missing ingredient for the complete classification of the family of all psf entire maps.

1.2. **Pullback maps.** The key method in determining whether a given finite or infinite degree Thurston map $f\colon S^2 \dashrightarrow S^2$ with the postsingular set $P_f = A$ is realized by a psf meromorphic map is the analysis of the dynamics of a holomorphic operator σ_f, known as the *pullback map*, defined on a complex manifold called the *Teichmüller space* $\mathcal{T}_A$; see Sections 2.4 and 2.5. Crucially, the Thurston map f is realized if and only if the pullback map σ_f has a fixed point in $\mathcal{T}_A$; see 2.17. Moreover, the dynamics of the pullback map encodes many other properties of the corresponding Thurston map; see, for instance, [BEKP09], [KPS16], and [BDP24].

Instead of working directly in the Teichmüller space $\mathcal{T}_A$, it is often more convenient to work in a simpler complex manifold $\mathcal{M}_A$, known as a *moduli space*. This space, roughly speaking, encodes all possible complex structures biholomorhic to a punctured Riemann sphere that can be put on the punctured topological sphere $S^2 - A$ (see Section 2.4 for the precise definition). There is a natural projection map $\pi\colon \mathcal{T}_A \to \mathcal{M}_A$ that is a holomorphic universal covering. However, the map σ_f rarely descends to a map on the moduli space $\mathcal{M}_A$. Nevertheless, for a finite degree Thurston map $f\colon S^2 \to S^2$, there exists a complex manifold $\mathcal{W}_f$, known as the *Hurwitz space* of the Thurston map f, along with the holomorphic *X-map* $X_f\colon \mathcal{W}_f \to \mathcal{M}_A$, a holomorphic covering map $Y_f\colon \mathcal{W}_f \to \mathcal{M}_A$, called the *$Y$-map*, and a holomorphic covering $\omega_f\colon \mathcal{T}_A \to \mathcal{W}_f$, such that the following diagram commutes [Koc13, Section 2]:

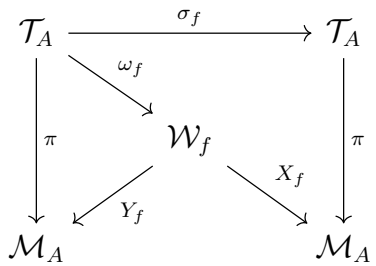

FIGURE 1. Fundamental diagram.

In other words, the pullback map σ_f is semi-conjugate to a self-correspondence $X_f \circ Y_f^{-1}$ of the moduli space $\mathcal{M}_A$. If $f\colon S^2 \to S^2$ with $P_f = A$ is a finite degree Thurston map, then the Y-map $Y_f\colon \mathcal{W}_f \to \mathcal{M}_A$ has a finite topological degree; see [Koc13, Theorem 2.6]. This observation plays a crucial role in the proof of Thurston's characterization of rational maps. In fact, it allows to conclude that for a Thurston map f with a hyperbolic orbifold, the σ_f-orbit $(\sigma_f^{\circ n}(\tau))$ of $\tau \in \mathcal{T}_A$ converges (indicating that f is realized), if the projection $(\pi(\sigma_f^{\circ n}(\tau)))$ of this orbit visits some compact set of the moduli space $\mathcal{M}_A$ infinitely many times; see [Hub16, Section 10.9 and Lemma 10.11.9] and [Sel12, Proof of Theorem 2.3, p. 20].

The objects introduced above, along with commutative diagram (1) and the fact that $Y_f\colon \mathcal{W}_f \to \mathcal{M}_A$ has a finite degree, have broad applications beyond the proof of Thurston's characterization of rational maps; see, for example, [BN06, BEKP09, Sel12, Sel13, Koc13, Lod13, KPS16, FKK$^+$17, Smi24a, Smi24b, BDP24]. These tools, for instance, allow to simultaneously study the entire *Hurwitz* (*equivalence*) *class* $\mathcal{H}_f$ of the finite degree Thurston map f. Here, two Thurston maps $f_1\colon S^2 \dashrightarrow S^2$ and $f_2\colon S^2 \dashrightarrow S^2$ with $P_{f_1} = P_{f_2}$ are said to be *Hurwitz equivalent* if there exist orientation-preserving homeomorphisms $\phi_1, \phi_2\colon S^2 \to S^2$ such that $\phi_1|P_{f_1} = \phi_2|P_{f_1}$ and $\phi_1 \circ f_1 = f_2 \circ \phi_2$; see [BN06, Lod13, KPS16] for examples of results on Hurwitz classes. For instance, one can pose a question whether a given Thurston map is *totally unobstructed*, i.e., $\mathcal{H}_f$ consists of only realized Thurston maps.

Commutative diagram (1) is particularly powerful in two specific cases. The first is when the Thurston map $f\colon S^2 \to S^2$ has the postcritical set $P_f = A$ consisting of exactly four points. In this situation, the spaces $\mathcal{T}_A$, $\mathcal{M}_A$, and $\mathcal{W}_f$ are simply Riemann surfaces. In fact, the Teichmüller space is biholomorphic to the unit disk $\mathbb{D}$ and the moduli space $\mathcal{M}_A$ is biholomorphic to the three punctured Riemann sphere $\Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$. This allows the use of powerful machinery of one-dimensional holomorphic dynamics to study pullback maps. For example, this approach was utilized in [Smi24a] to derive an alternative proof of Thurston's characterization of rational maps in the case of four postcritical values, as well as in [Smi24a, Smi24b] to investigate the *global curve attractor conjecture*, which was ultimately resolved in [BDP24] for all pcf rational maps with four postcritical values. Secondly, when the X-map X_f is injective, the "inverse" of σ_f descends to the so-called *g-map* $g_f := Y_f \circ X_f^{-1}$, which is defined on the subset $X_f(\mathcal{W}_f)$ of the moduli space $\mathcal{M}_A$; see [Koc13, Section 5] for examples of finite degree Thurston maps that satisfy this condition.

The theory of moduli maps, as outlined above, is developed for finite degree Thurston maps but has not yet been established in the context of transcendental Thurston maps. In this paper, we consider a certain family of Thurston maps, that includes maps of both finite and infinite degree, with four postsingular values and show that the corresponding pullback maps admit an analogue of commutative diagram (1), where the X-map is always injective, but the Y-map has infinite degree if the initial Thurston map $f\colon S^2 \dashrightarrow S^2$ is transcendental. Using tools of one-dimensional holomorphic dynamics and hyperbolic geometry, we establish a Thurston-like realizability criterion for this family of maps. In particular, we demonstrate that the obstacle of the Y-map having infinite degree can be finessed. Afterward, we illustrate how the developed machinery can be used to investigate the properties of the corresponding Hurwitz classes.

1.3. **Main results.** In this paper, we study the family of Thurston maps $f\colon S^2 \dashrightarrow S^2$ satisfying the following conditions:

(A) the map f has at most three singular values;

(B) the postsingular set P_f consists of exactly four points;
(C) there exists a set $B \subset P_f$ such that $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap P_f| = 3$.

Here, $\overline{f^{-1}(B)}$ is the closure of the set $f^{-1}(B)$ in the topology of S^2. In particular, it coincides with $f^{-1}(B)$ if the Thurston map f has a finite degree; otherwise, it also includes the point where f is not defined.

Clearly, conditions (B) and (C) are independent from the function-theoretical properties of the map f. More specifically, if the map f has at most three singular values, then these conditions can be verified by analyzing the dynamics of the map f on the finite set P_f. For instance, in the case of *entire* Thurston maps (those that can be restricted to self-maps of $\mathbb{R}^2$; see Section 2.2) with three singular and four postsingular values, three out of seven possible *postsingular portraits* satisfy condition (C); see Example 4.13. More examples of families of Thurston maps that meet these conditions can be found in Section 4.3.

Although conditions (A)–(C) are quite restrictive, there are still uncountably many pairwise combinatorially inequivalent both realized and obstructed Thurston maps that meet them; see Remark 4.11. Notably, these conditions are preserved under Hurwitz equivalence of Thurston maps. Finally, it is worth mentioning that all of our further results work in a slightly more general setting of *marked* Thurston maps satisfying analogous properties to (A)–(C); see Sections 2.3 and 4.

1.3.1. *Characterization problem.* We establish an analog of Thurston's characterization result for the family of Thurston maps that satisfy conditions (A)–(C). In fact, we show that it is sufficient to consider one of the simplest types of Thurston obstructions – Levy fixed curves – to determine whether such a Thurston map is realized. For a Thurston map $f\colon S^2 \dashrightarrow S^2$, a Levy fixed curve is a simple closed curve $\gamma \subset S^2 - P_f$ such that γ is *essential*, i.e., it cannot be shrinked to a point by a homotopy in $S^2 - P_f$, and there exists another simple closed curve $\widetilde{\gamma} \subset f^{-1}(\gamma)$ such that γ and $\widetilde{\gamma}$ are homotopic in $S^2 - P_f$ and $f|\widetilde{\gamma}\colon \widetilde{\gamma} \to \gamma$ is a homeomorphism. If the map f is injective on one of the connected components of $S^2 - \widetilde{\gamma}$, then we say that the Levy fixed curve γ is *weakly degenerate*.

Main Theorem A. *Let $f\colon S^2 \dashrightarrow S^2$ be a Thurston map of finite or infinite degree that satisfies conditions (A)–(C). Then f is realized if and only if it has no weakly degenerate Levy fixed curve. Moreover, if f is obstructed, then it has a unique Levy fixed curve up to homotopy in $S^2 - P_f$; otherwise, it is realized by a psf meromorphic map that is unique up to Möbuis conjugation.*

To prove Main Theorem A, we start by showing that the corresponding pullback map σ_f defined on the Teichmüller space $\mathcal{T}_A \sim \mathbb{D}$, where $A = P_f$, admits an analog of commutative diagram (1), where the X-map is injective, the Y-map is a covering, potentially of infinite degree, and the analog of the Hurwitz space $\mathcal{W}_f$ is a finitely or countably punctured Riemann sphere (see Proposition 4.1 and Remark 4.2). Further analysis reveals a crucial observation similar to that in the proof of Thurston's characterization result: if the sequence $(\pi(\sigma_f^{\circ n}(\tau)))$ with $\tau \in \mathcal{T}_A$ visits a certain compact set of the moduli space $\mathcal{M}_A$ infinitely many times, then the σ_f-orbit $(\sigma_f^{\circ n}(\tau))$ of τ converges to the unique fixed point of σ_f (see Claim 1 of the proof of Theorem 3.10).

Moreover, we establish a more refined result: if the pullback map σ_f does not have a fixed point (indicating that the Thurston map f is obstructed), then the sequence $(\pi(\sigma_f^{\circ n}(\tau)))$ converges to the same "cusp" $x \in \partial\mathcal{M}_A \sim \partial\Sigma = \{0, 1, \infty\}$ of the moduli space $\mathcal{M}_A$, regardless

of the choice of $\tau \in \mathcal{T}_A$. Furthermore, the map $g_f = Y_f \circ X_f^{-1}$ can be holomorphically extended to a neighborhood of this cusp and x becomes a repelling fixed point of g_f. It is worth noting that these results hold not only for pullback maps, but also in a broader class of holomorphic self-maps of the unit disk; see Theorem 3.10. Finally, this analysis provides a sufficient control over the dynamics of σ_f to derive the existence of a Levy fixed curve for the obstructed Thurston map $f\colon S^2 \dashrightarrow S^2$.

Main Theorem A is the first result addressing the characterization problem in the transcendental setting that is established with minimal reliance on the function-theoretical properties of the considered Thurston maps. This differs from [HSS09], which focuses on exponential Thurston maps, and [She22], which is primarily devoted to *structurally finite* Thurston maps. Note that the geometric and analytic properties of entire or meromorphic maps, even those with few singular values, can be highly varied and subtle; see [Bis15a, Bis15b, Bis17].

Main Theorem A also offers an alternative proof for the characterization of postsingularly finite exponential maps [HSS09, Theorem 2.4] for the case of four postsingular values; see Example 4.12. While the proof in [HSS09] relies on the intricate machinery of *integrable quadratic differentials* and their *thick-thin decompositions*, our approach uses more explicit techniques that shed the light on the geometry of the pullback dynamics on Teichmüller and moduli spaces. At the same time, Main Theorem A provides a novel proof of Thurston's characterization of rational maps within a broad class of examples, and this proof does not rely on the fact that Y-map has a finite degree.

1.3.2. *Hurwitz classes.* Techniques explained in Section 1.3.1 also allow us to derive several properties of Hurwitz classes:

Main Theorem B. *Let $f\colon S^2 \dashrightarrow S^2$ be a Thurston map of finite or infinite degree that satisfies conditions (A)–(C). Then*

(1) f is totally unobstructed if and only if there are no two points $a, b \in P_f$ such that $\deg(f, a) = \deg(f, b) = 1$ and $f(\{a, b\})$ equals $\{a, b\}$ or $P_f - \{a, b\}$;
(2) if f is not totally unobstructed, then its Hurwitz class $\mathcal{H}_f$ contains infinitely many pairwise combinatorially inequivalent obstructed Thurston maps;
(3) if f has infinite degree, then its Hurwitz class $\mathcal{H}_f$ contains infinitely many pairwise combinatorially inequivalent realized Thurston maps.

The main tool for proving Main Theorem B is the relationship between fixed points of the map $g_f = Y_f \circ X_f^{-1}$ and the elements of the Hurwitz class $\mathcal{H}_f$. Let $f\colon S^2 \dashrightarrow S^2$ be a Thurston map with $P_f = A$ satisfying assumptions (A)–(C). We show that if the corresponding map g_f can be holomorphically extended to a neighborhood of $x \in \partial\mathcal{M}_A \sim \partial\Sigma = \{0, 1, \infty\}$ and x becomes a repelling fixed point of g_f, then the Hurwitz class $\mathcal{H}_f$ must contain an obstructed Thurston map. Moreover, f is totally unobstructed if and only if none of the "cusps" of the moduli space $\mathcal{M}_A$ exhibit this behavior; see Proposition 4.6. Using additional properties of the map g_f, we establish a simple criterion, as in item (1) of Main Theorem B, for determining whether a given Thurston map with properties (A)–(C) is totally unobstructed. Furthermore, with the understanding of possible obstructions provided by Main Theorem A, we can construct infinitely many pairwise combinatorially inequivalent obstructed Thurston maps within the Hurwitz class $\mathcal{H}_f$ starting from just one of them.

To prove item (3) of Main Theorem B, we show that a fixed point $x \in \mathcal{M}_A$ of the map g_f corresponds to a realized Thurston map within the Hurwitz class $\mathcal{H}_f$; see Proposition 4.5. Furthermore, only finitely many fixed points of g_f can correspond to the same Thurston

map up to combinatorial equivalence (see the proof of Theorem 4.8). The desired result then follows because the map g_f has infinitely many fixed points when f is transcendental; see Proposition 4.1 and Lemma A.2.

Similar connections between the fixed points of the map g_f and Thurston maps within the Hurwitz class $\mathcal{H}_f$ are already established in the context of finite degree Thurston maps (cf. [Koc13, Propositions 4.3 and 4.4] and [KPS16, Theorem 9.1]). However, their extensions to the setting of transcendental Thurston maps are novel contributions. Additionally, as an application of Main Theorem B, we can obtain the following result regarding the structure of parameter spaces of finite-type meromorphic maps (see Definition 4.9):

Corollary 1.1. *Let $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic map such that $|S_g| \leqslant 3$. Then its parameter space* $\mathrm{Par}(g)$ *contains infinitely many pairwise (topologically or conformally) non-conjugate psf maps with four postsingular values.*

1.4. **Organization of the paper.** Our paper is organized as follows. In Section 2, we review some general background. In Section 2.1, we fix the notation and state some basic definitions. We discuss topologically holomorphic maps in Section 2.2. The necessary background on Thurston maps is covered in Section 2.3. Section 2.4 introduces the Teichmüller and moduli spaces of a marked topological sphere. Finally, in Section 2.5, we define pullback maps, discuss their basic properties and their relations with the associated Thurston maps.

In Section 3, we present several results concerning the hyperbolic geometry and dynamics of holomorphic self-maps of the unit disk. Section 3.1 provides tools for identifying obstructions for Thurston maps with four postsingular values. We establish some estimates for hyperbolic contraction of inclusion maps between two hyperbolic Riemann surfaces in Section 3.2. In Section 3.3, we investigate dynamics of holomorphic self-maps of the unit disk satisfying certain additional assumptions.

Further, in Section 4, we develop the Thurston theory for a family of Thurston maps satisfying condition (A)–(C). In particular, in Section 4.1, we address the characterization problem for this class of Thurston maps and prove Main Theorem A. We study properties of Hurwitz classes and prove Main Theorem B and Corollary 1.1 in Section 4.2. Finally, in Section 4.3, we provide and analyze various examples.

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2. Background

2.1. **Notation and basic concepts.** The sets of positive integers, non-zero integers, integers, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}^*$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. We use the notation $\mathbb{I} := [0,1]$ for the closed unit interval on the real line, $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ for the open unit disk in the complex plane, $\mathbb{D}^* := \mathbb{D} - \{0\}$ for the punctured unit disk, $\mathbb{C}^* := \mathbb{C} - \{0\}$ for the punctured complex plane, $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ for the upper half-plane, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, and Σ for the three-punctured Riemann

sphere $\widehat{\mathbb{C}} - \{0, 1, \infty\}$. The open and closed disks of radius $r > 0$ centered at 0 are denoted by $\mathbb{D}_r$ and $\overline{\mathbb{D}}_r$, respectively. Finally, $\arg(z) \in [0, 2\pi)$ and $|z|$ denote the argument and the absolute value, respectively, of the complex number z.

We denote the oriented 2-dimensional sphere by S^2. In this paper, we treat it as a purely topological object. In particular, our convention is to write $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ or $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ to indicate that g is holomorphic, and $f\colon S^2 \to S^2$ if f is only continuous. The same rule applies to the notation $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ and $f\colon S^2 \dashrightarrow S^2$ (see Section 2.2 for the details).

The cardinality of a set A is denoted by $|A|$ and the identity map on A by id_A. If $f\colon U \to V$ is a map and $W \subset U$, then $f|W$ stands for the restriction of f to W. If U is a topological space and $W \subset U$, then $\overline{W}$ denotes the closure and ∂W the boundary of W in U.

A subset D of $\widehat{\mathbb{C}}$ is called an *open Jordan region* if there exists an injective continuous map $\varphi\colon \overline{\mathbb{D}} \to \widehat{\mathbb{C}}$ such that $D = \varphi(\mathbb{D})$. In this case, $\partial D = \varphi(\partial\mathbb{D})$ is a simple closed curve in $\widehat{\mathbb{C}}$. A domain $U \subset \widehat{\mathbb{C}}$ is called an *annulus* if $\widehat{\mathbb{C}} - U$ has two connected components. The *modulus* of an annulus U is denoted by $\mathrm{mod}(U)$ (see [Hub06, Proposition 3.2.1] for the definition).

Let U and V be topological spaces. A continuous map $H\colon U \times \mathbb{I} \to V$ is called a *homotopy* from U to V. When $U = V$, we simply say that H is a homotopy *in* U. Given a homotopy $H\colon U \times \mathbb{I} \to V$, for each $t \in \mathbb{I}$, we associate the *time-t map* $H_t := H(t, \cdot)\colon U \to V$. Sometimes it is convenient to think of the homotopy H as a continuous family of its time maps $(H_t)_{t\in\mathbb{I}}$. The homotopy H is called an (*ambient*) *isotopy* if the map $H_t\colon U \to V$ is a homeomorphism for each $t \in \mathbb{I}$. Suppose A is a subset of U. An isotopy $H\colon U \times \mathbb{I} \to V$ is said to be an isotopy *relative to* A (abbreviated "H is an isotopy rel. A") if $H_t(p) = H_0(p)$ for all $p \in A$ and $t \in \mathbb{I}$.

Given $M, N \subset U$, we say that M *is homotopic* (*in* U) to N if there exists a homotopy $H\colon U \times \mathbb{I} \to U$ with $H_0 = \mathrm{id}_U$ and $H_1(M) = N$. Two homeomorphisms $\varphi_0, \varphi_1\colon U \to V$ are called *isotopic* (*rel.* $A \subset U$) if there exists an isotopy $H\colon U \times \mathbb{I} \to V$ (rel. A) with $H_0 = \varphi_0$ and $H_1 = \varphi_1$.

We assume that every topological surface X is oriented. We denote by $\mathrm{Homeo}^+(X)$ and $\mathrm{Homeo}^+(X, A)$ the group of all orientation-preserving self-homeomorphisms of X and the group of all orientation-preserving self-homeomorphisms of X fixing A pointwise, respectively. We use the notation $\mathrm{Homeo}_0^+(X, A)$ for the subgroup of $\mathrm{Homeo}^+(X, A)$ consisting of all homeomorphisms isotopic rel. A to id_X.

2.2. **Topologically holomorphic maps.** In this section, we briefly recall the definition of a topologically holomorphic map and some of its basic properties (for more detailed discussion see [MPR24, Section 2.3]; see also [LP20]).

Definition 2.1. Let X and Y be two connected topological surfaces. A map $f\colon X \to Y$ is called *topologically holomorphic* if it satisfies one of the following four equivalent conditions:

(1) for every $p \in X$ there exist $d \in \mathbb{N}$, a neighborhood U of x, and two orientation-preserving homeomorphisms $\psi\colon U \to \mathbb{D}$ and $\varphi\colon f(U) \to \mathbb{D}$ such that $\psi(p) = \varphi(f(p)) = 0$ and $(\varphi \circ f \circ \psi^{-1})(z) = z^d$ for all $z \in \mathbb{D}$;
(2) f is continuous, open, *discrete* (i.e., $f^{-1}(q)$ is discrete in X for very $q \in Y$), and for every $p \in X$ such that f is locally injective at p, there exists a neighborhood U of p for which $f|U\colon U \to f(U)$ is an orientation-preserving homeomorphism;
(3) there exist Riemann surfaces S_X and S_Y and orientation-preserving homeomorphisms $\varphi\colon Y \to S_Y$ and $\psi\colon X \to S_X$ such that $\varphi \circ f \circ \psi^{-1}\colon S_X \to S_Y$ is a holomorphic map;

(4) for every orientation-preserving homeomorphism $\varphi\colon Y \to S_Y$, where S_Y is a Riemann surface, there exist a Riemann surface S_X and an orientation-preserving homeomorphism $\psi\colon X \to S_X$ such that $\varphi \circ f \circ \psi^{-1}\colon S_X \to S_Y$ is a holomorphic map.

Note that in condition (4) of Definition 2.1, the homeomorphism ψ is defined uniquely up to post-composition with a conformal automorphism of S_X for fixed φ and S_X.

It is straightforward to define the concepts of *regular*, *singular*, *critical*, and *asymptotic values*, as well as *regular* and *critical points* and their *local degrees* (denoted by $\deg(f,\cdot)$) for the topologically holomorphic map f (see [MPR24, Definition 2.7]). We denote by $S_f \subset Y$ the *singular set* of f, i.e., the union of all singular values of the topologically holomorphic map $f\colon X \to Y$. We say that the map f is *of finite type* or belongs to the *Speiser class* $\mathcal{S}$ if the set S_f is finite.

In this paper, we study topologically holomorphic maps $f\colon X \to S^2$, where X is either the sphere S^2 or the punctured sphere $S^2 - \{e\}$. In the latter case, we assume that f cannot be extended as a topologically holomorphic map to the entire sphere S^2. For the sake of simplicity, we are going to use the notation $f\colon S^2 \dashrightarrow S^2$ in order to indicate that f might not be defined at a single point $e \in S^2$. Similar to the holomorphic case, the point e is referred as the *essential singularity* of the map f. Likewise, for a holomorphic map g defined everywhere on $\widehat{\mathbb{C}}$ with the possible exception of a single essential singularity, we use the notation $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$, and we say that $g\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic.

It is possible to derive the following isotopy lifting property for topologically holomorphic maps as above in the case when they are of finite type (cf. [ERG15, Propostion 2.3]).

Proposition 2.2. *Let $f\colon X \to S^2$ and $\widetilde{f}\colon \widetilde{X} \to S^2$ be topologically holomorphic maps of finite type, where X and $\widetilde{X}$ are either topological spheres or punctured topological spheres. Suppose that $\varphi_0 \circ f = \widetilde{f} \circ \psi_0$ for some $\varphi_0, \psi_0 \in \mathrm{Homeo}^+(S^2)$. Let $A \subset S^2$ be a finite set containing S_f and $\varphi_1 \in \mathrm{Homeo}^+(S^2)$ is isotopic rel. A to φ_0. Then $\varphi_1 \circ f = \widetilde{f} \circ \psi_1$ for some $\psi_1 \in \mathrm{Homeo}^+(S^2)$ isotopic rel. $f^{-1}(A) \cup (S^2 - X) \supset \overline{f^{-1}(A)}$ to ψ_0.*

Proof. Let $(\varphi_t)_{t\in\mathbb{I}}$ be the corresponding isotopy. From the definition of a singular value, it follows that the restrictions $\varphi_t \circ f|Y\colon Y \to Z$ are covering maps for each $t \in \mathbb{I}$, where $Y := X - f^{-1}(A)$ and $Z := S^2 - A$. Therefore, [ABF21, Lemma 2.7] implies the existence of an isotopy $(\phi_t)_{t\in\mathbb{I}}$ in Y such that $\varphi_t \circ f = \varphi_0 \circ f \circ \phi_t$. Each homeomorphism $\phi_t\colon Y \to Y$ extends to a self-homeomorphism of the entire sphere S^2 since all but at most one point of the set $S^2 - Y$ are isolated. Moreover, it is straightforward to check that $\phi_t|f^{-1}(A) \cup (S^2 - X) = \phi_0|f^{-1}(A) \cup (S^2 - X)$ for each $t \in \mathbb{I}$. In other words, the homotopy $(\phi_t)_{t\in\mathbb{I}}$ can be viewed as an isotopy in S^2 rel. $f^{-1}(A) \cup (S^2 - X)$.

At the same time, $\varphi_0 \circ f = \varphi_0 \circ f \circ \phi_0$ and, therefore, we have the following:

$$\varphi_1 \circ f = \varphi_0 \circ f \circ \phi_1 = (\varphi_0 \circ f \circ \phi_0) \circ \phi_0^{-1} \circ \phi_1 = \varphi_0 \circ f \circ \phi_0^{-1} \circ \phi_1 = \widetilde{f} \circ (\psi_0 \circ \phi_0^{-1} \circ \phi_1).$$

Thus, we can set $\psi_1 := \psi_0 \circ \phi_0^{-1} \circ \phi_1$, and $(\psi_0 \circ \phi_0^{-1} \circ \phi_t)_{t\in\mathbb{I}}$ provides the required isotopy rel. $f^{-1}(A) \cup (S^2 - X)$. Clearly, if $p \in S^2$ is an accumulation point of the set $f^{-1}(A)$, then $p \notin X$, which implies $\overline{f^{-1}(A)} \subset f^{-1}(A) \cup (S^2 - X)$. Finally, ψ_1 is orientation-preserving since f and $\widetilde{f}$ are local orientation-preserving homeomorphisms outside the sets of their critical points. □

Corollary 2.3. *Let $f\colon X \to S^2$ be a topologically holomorphic map of finite type, where $X = S^2$ or $X = S^2 - \{e\}$, $e \in S^2$, and $A \subset S^2$ be a finite set containing S_f. Suppose that*

γ_0 is a simple closed curve in $S^2 - A$, and let $\widetilde{\gamma}_0 \subset f^{-1}(\gamma)$ be a simple closed curve with $\deg(f|\widetilde{\gamma}_0 \colon \widetilde{\gamma}_0 \to \gamma_0) = d$. If γ_1 is a simple closed curve that is homotopic in $S^2 - A$ to γ_0, then there exists a simple closed curve $\widetilde{\gamma}_1 \subset f^{-1}(\gamma_1)$ such that $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are homotopic in $X - f^{-1}(A) \subset S^2 - \overline{f^{-1}(A)}$ and $\deg(f|\widetilde{\gamma}_1 \colon \widetilde{\gamma}_1 \to \gamma_1) = d$.

Proof. According to [Bus10, Theorem A.3] (see also [FM12, Sections 1.2.5 and 1.2.6]), there exists an isotopy $(\varphi_t)_{t\in\mathbb{I}}$ rel. A in S^2 such that $\varphi_0 = \mathrm{id}_X$ and $\varphi_1(\gamma_0) = \gamma_1$. Since $\varphi_0 = \mathrm{id}_{S^2}$ is orientation-preserving, then φ_t is also orientation-preserving for each $t \in \mathbb{I}$. By Proposition 2.2, there exists a homeomorphism $\psi_1 \in \mathrm{Homeo}^+(S^2, f^{-1}(A) \cup (S^2 - X))$ such that $\varphi_1 \circ f = f \circ \psi_1$. Thus, we can take $\widetilde{\gamma}_1 := \psi_1(\widetilde{\gamma}_0)$. Finally, $X - f^{-1}(A) \subset S^2 - \overline{f^{-1}(A)}$, since any accumulation point $p \in S^2$ of the set $f^{-1}(A)$ cannot be in X. □

Due to the Uniformization Theorem and item (4) of Definition 2.1, in the case when $X = S^2$, a topologically holomorphic map $f \colon X \to S^2$ can be written as $f = \varphi \circ g \circ \psi^{-1}$, where $g \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a non-constant rational map and $\varphi, \psi \colon S^2 \to \widehat{\mathbb{C}}$ are orientation-preserving homeomorphisms. In fact, in this case $f \colon S^2 \to S^2$ is simply a *branched self-covering* of S^2, which is always of finite type and has finite topological degree.

Similarly, in the case when $X = S^2 - \{e\}$, we can write f as $\varphi \circ g \circ \psi^{-1}$ such that $g \colon R \to \widehat{\mathbb{C}}$ is a non-constant meromorphic map, $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ and $\psi \colon S^2 - \{e\} \to R$ are orientation-preserving homeomorphisms, where $R = \mathbb{C}$ or $R = \mathbb{D}$. Suppose that the map f is of finite type. Then the image of ψ above does not depend on the choice of the homeomorphism φ (see [Ere04, pp. 3-4]; it essentially follows from Proposition 2.2 and some well-known facts from the theory of *quasiconformal mappings*). Thus, finite-type topologically holomorphic maps for which the image of ψ is $\mathbb{C}$ are referred to as *parabolic type* maps, while those for which the image of ψ is $\mathbb{D}$ are called *hyperbolic type* maps.

Further, we assume that every topologically holomorphic map $f \colon S^2 \dashrightarrow S^2$ we consider either has no essential singularities or is a finite-type topologically holomorphic map of parabolic type. Definition 2.1 and Great Picard's Theorem imply that in any neighborhood of an essential singularity e of such a map f, every value is attained infinitely often with at most two exceptions. In particular, f can have at most two *omitted values*, i.e., points p in S^2 such that the preimage $f^{-1}(p)$ is empty. Furthermore, each omitted value is an asymptotic value of f. Additionally, observe that if $A \subset S^2$ is a finite set with $|A| \geqslant 3$ and $S_f \subset A$, then the restriction

$$f|S^2 - \overline{f^{-1}(A)} \colon S^2 - \overline{f^{-1}(A)} \to S^2 - A$$

is a covering map. Note that the closure $\overline{f^{-1}(A)}$ equals $f^{-1}(A)$ if $f \colon S^2 \to S^2$ has no essential singularity. Otherwise, $\overline{f^{-1}(A)}$ consists of $f^{-1}(A)$ and the essential singularity $e \in S^2$ of f due to Great Picard's Theorem and the assumption $|A| \geqslant 3$.

We say that a topologically holomorphic map $f \colon S^2 \dashrightarrow S^2$ is *transcendental* if it has an essential singularity. Given our previous assumptions on the map f, this is equivalent to saying that f has infinite topological degree. The map $f \colon S^2 \dashrightarrow S^2$ is called *entire* if either f has finite topological degree and there exists a point $p \in S^2$ such that $f^{-1}(p) = \{p\}$ (in which case f is called a *topological polynomial*), or f has infinite topological degree and $f^{-1}(e) = \varnothing$, where e is the essential singularity of f. We can view entire topologically holomorphic maps as topologically holomorphic self-maps of $\mathbb{R}^2$.

2.3. **Thurston maps.** Let $f\colon S^2 \dashrightarrow S^2$ be a topologically holomorphic map. Then the *postsingular set* P_f of the map f is defined as

$$P_f := \{q \in S^2 : q = f^{\circ n}(p) \text{ for some } n \geqslant 0 \text{ and } p \in S_f\}.$$

In other words, the postsingular set P_f is the union of all forward orbits of the singular values of f. It is worth noting that some of these orbits might terminate after several iterations if a singular value reaches the essential singularity of the map f.

We say that $f\colon S^2 \dashrightarrow S^2$ is *postsingularly finite* (*psf* in short) if the set P_f is finite, i.e., f has finitely many singular values and each of them eventually becomes periodic or lands on the essential singularity of f under the iteration. Postsingularly finite topologically holomorphic maps of finite degree are also called *postcritically finite* (*pcf* in short), and their *postsingular values* are called *postcritical*, as their singular values are always critical.

Now we are ready to state one of the key definitions of this section.

Definition 2.4. A non-injective topologically holomorphic map $f\colon S^2 \dashrightarrow S^2$ is called a *Thurston map* if it is postsingularly finite and either f has no essential singularity or f is a parabolic type map.

Given a finite set $A \subset S^2$ such that $P_f \subset A$ and every $a \in A$ is either the essential singularity of f or $f(a) \in A$, we call the pair (f, A) a *marked Thurston map* and A its *marked set*.

We often consider marked Thurston maps in the same way as usual Thurston maps and use the notation $f\colon (S^2, A) \circlearrowleft$ while still assuming that f might not be defined on the entire sphere S^2. If no specific marked set is mentioned, we assume it to be P_f. Note that when the marked set A contains the essential singularity of the map f, the set A is not forward invariant with respect to f in the usual sense. However, if $|A| \geqslant 3$, then $A \subset \overline{f^{-1}(A)}$.

The dynamics of a Thurston map on its marked set or some other finite subsets of S^2 can also be represented graphically, in a way that turns out to be useful in study. Suppose that $f\colon S^2 \dashrightarrow S^2$ is a Thurston map and $A \subset S^2$ is a finite set such that every $a \in A$ is either the essential singularity of f or $f(a) \in A$. Then the *dynamical portrait* of the map f on the set A is a directed abstract graph with the vertex set A, where for each vertex $v \in A$ that is not the essential singularity of f, there is a unique directed edge from v to $f(v)$, and if $v \in A$ is the essential singularity of f, there are no outgoing edges from v. If the set A coincides with the postsingular set P_f, the dynamical portrait of f on the set A is called the *postsingular portrait* of the Thurston map f.

Definition 2.5. Two Thurston maps $f_1\colon (S^2, A) \circlearrowleft$ and $f_2\colon (S^2, A) \circlearrowleft$ are called *isotopic* (*rel.* A) if there exists $\phi \in \mathrm{Homeo}_0^+(S^2, A)$ such that $f_1 = f_2 \circ \phi$.

Remark 2.6. Let $f_1\colon (S^2, A) \circlearrowleft$ and $f_2\colon (S^2, A) \circlearrowleft$ be two Thurston maps satisfying the relation $\phi_1 \circ f_1 = f_2 \circ \phi_2$ for some $\phi_1, \phi_2 \in \mathrm{Homeo}_0^+(S^2, A)$. Then it follows from Proposition 2.2 that f_1 and f_2 are isotopic rel. A.

The notion of isotopy for Thurston maps depends on their common marked set. Consequently, we sometimes refer to isotopy *relative* A (or rel. A for short) to specify which marked set is being considered. This applies to other notions introduced below that also depend on the choice of the marked set.

We say that two (marked) Thurston maps are *combinatorially equivalent* if they are "topologically conjugate up to isotopy":

Definition 2.7. Two Thurston maps $f_1\colon (S^2, A_1) \circlearrowleft$ and $f_2\colon (S^2, A_2) \circlearrowleft$ are called combinatorially (or *Thurston*) equivalent if there exist two Thurston maps $\widetilde{f}_1\colon (S^2, A_1) \circlearrowleft$ and $\widetilde{f}_2\colon (S^2, A_2) \circlearrowleft$ such that:

- f_i and $\widetilde{f}_i$ are isotopic rel. A_i for each $i = 1, 2$, and
- $\widetilde{f}_1$ and $\widetilde{f}_2$ are conjugate via a homeomorphsim $\phi \in \mathrm{Homeo}^+(S^2)$, i.e., $\phi \circ \widetilde{f}_1 = \widetilde{f}_2 \circ \phi$, such that $\phi(A_1) = A_2$.

Remark 2.8. Definition 2.7 can be reformulated in a more classical way. Thurston maps $f_1\colon (S^2, A_1) \circlearrowleft$ and $f_2\colon (S^2, A_2) \circlearrowleft$ are combinatorially equivalent if and only if there exist two homeomorphisms $\phi_1, \phi_2 \in \mathrm{Homeo}^+(S^2)$ such that $\phi_1(A_1) = \phi_2(A_1) = A_2$, ϕ_1 and ϕ_2 are isotopic rel. A, and $\phi_1 \circ f_1 = f_2 \circ \phi_2$.

Remark 2.9. If $A_1 = P_{f_1}$ and $A_2 = P_{f_2}$, then the condition that $\phi(A_1) = A_2$ in Definition 2.7 and the condition $\phi_1(A_1) = \phi_2(A_1) = A_2$ in Remark 2.8 can be removed since they are automatically satisfied if all other conditions hold.

A Thurston map $f\colon (S^2, A) \circlearrowleft$ is said to be *realized* if it is combinatorially equivalent to a postsingularly finite holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$. If $f\colon (S^2, A) \circlearrowleft$ is not realized, we say that it is *obstructed*.

Let $A \subset S^2$ be a finite set. We say that a simple closed curve $\gamma \subset S^2 - A$ is *essential* in $S^2 - A$ if each connected component of $S^2 - \gamma$ contains at least two points of the set A. In other words, γ is essential in $S^2 - A$ if it cannot be shrinked to a point via a homotopy in $S^2 - A$.

Definition 2.10. Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. We say that a simple closed curve γ *forms a Levy cycle* for $f\colon (S^2, A) \circlearrowleft$ if γ is essential in $S^2 - A$ and there exists another simple closed curve $\widetilde{\gamma} \subset f^{-n}(\gamma)$ for some $n \geqslant 1$ such that γ and $\widetilde{\gamma}$ are homotopic in $S^2 - A$ and $\deg(f^{\circ n}|\widetilde{\gamma}\colon \widetilde{\gamma} \to \gamma) = 1$.

If $n = 1$ in Definition 2.10, then γ is called a *Levy fixed* (or *Levy invariant*) curve. Levy fixed curve γ is called *weakly degenerate* if f is injective on one of the connected components of $S^2 - \widetilde{\gamma}$. If additionally the image of this connected component U under f contains the same points of the set A as U, i.e., $U \cap A = f(U) \cap A$, we say that γ is a *degenerate* Levy fixed curve for the Thurston map $f\colon (S^2, A) \circlearrowleft$.

The following observation is widely known in the context of finite degree Thurston maps [Hub16, Exercise 10.3.6], and its proof extends to the case of transcendental Thurston maps as well (see Section 3.1 for the proof).

Proposition 2.11. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. If there exists a simple closed curve $\gamma \subset S^2$ forming a Levy cycle for $f\colon (S^2, A) \circlearrowleft$, then f is obstructed rel. A.*

We require one more notion of equivalence between Thurston maps.

Definition 2.12. We say that two Thurston maps $f_1\colon (S^2, A) \circlearrowleft$ and $f_2\colon (S^2, A) \circlearrowleft$ are *pure Hurwitz equivalent* (or simply *Hurwitz equivalent*) if there exist two homeomorphisms $\phi_1, \phi_2 \in \mathrm{Homeo}^+(S^2, A)$ such that $\phi_1 \circ f_1 = f_2 \circ \phi_2$.

If $f\colon (S^2, A) \circlearrowleft$ is a Thurston map, then the *Hurwitz class* $\mathcal{H}_{f,A}$ of f is the union of all Thurston maps with the marked set A that are Hurwitz equivalent to f. If A coincides with the postsingular set P_f of f, we simply use the notation $\mathcal{H}_f$. We say that a Thurston map $f\colon (S^2, A) \circlearrowleft$ is *totally unobstructed* if every Thurston map in $\mathcal{H}_{f,A}$ is unobstructed.

Remark 2.13. According to [FM12, Proposition 2.3], if two orientation-preserving homeomorphisms $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ and $\psi\colon S^2 \to \widehat{\mathbb{C}}$ agree on the set $A \subset S^2$ with $|A| \leqslant 3$, they are isotopic rel. A. This observation can be used to show that any Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized when the marked set A contains three or fewer points. Similarly, in thise case, one can show that $\mathcal{H}_{f,A}$ consists of a single Thurston map up to a combinatorial equivalence rel. A. However, when $|A| = 4$, the question of realizability already becomes significantly more challenging.

2.4. **Teichmüller and moduli spaces.** Let $A \subset S^2$ be a finite set containing at least three points. Then the *Teichmüller space of the sphere S^2 with the marked set A* is defined as

$$\mathcal{T}_A := \{\varphi\colon S^2 \to \widehat{\mathbb{C}} \text{ is an orientation-preserving homeomorphism}\}/\sim$$

where $\varphi_1 \sim \varphi_2$ if there exists a Möbius transformation M such that φ_1 is isotopic rel. A to $M \circ \varphi_2$.

Similarly, we define the *moduli space of the sphere S^2 with the marked set A*:

$$\mathcal{M}_A := \{\eta\colon A \to \widehat{\mathbb{C}} \text{ is injective}\}/\sim,$$

where $\eta_1 \sim \eta_2$ if there exists a Möbius transformation M such that $\eta_1 = M \circ \eta_2$.

Further, $[\cdot]$ denotes an equivalence class corresponding to a point of either the Teichmüller space $\mathcal{T}_A$ or the moduli space $\mathcal{M}_A$. Note that there is an obvious map $\pi\colon \mathcal{T}_A \to \mathcal{M}_A$ defined as $\pi([\varphi]) = [\varphi|A]$. According to [FM12, Proposition 2.3], when $|A| = 3$, both the Teichmüller space $\mathcal{T}_A$ and the moduli space $\mathcal{M}_A$ are just single points. Therefore, for the rest of this section, we assume that $|A| \geqslant 4$.

It is known that the Teichmüller space $\mathcal{T}_A$ admits a complete metric d_T, known as the *Teichmüller metric* [Hub06, Proposition 6.4.4]. Moreover, with respect to the topology induced by this metric, $\mathcal{T}_A$ is a contractible space [Hub06, Corollary 6.7.2]. At the same time, both $\mathcal{T}_A$ and $\mathcal{M}_A$ admit structures of $(|A|-3)$-complex manifolds (see [Hub06, Theorem 6.5.1]) so that the map $\pi\colon \mathcal{T}_A \to \mathcal{M}_A$ becomes a holomorphic universal covering map [Hub16, Section 10.9].

Moreover, the complex structure of $\mathcal{M}_A$ is quite explicit in the general case. Let $A = \{a_1, a_2, \dots, a_k, a_{k+1}, a_{k+2}, a_{k+3}\}$, $k \geqslant 1$, where the indexing of the points of A is chosen arbitrarily. Define the map $h\colon \mathcal{M}_A \to \mathbb{C}^k - \mathcal{L}_k$ by

$$h([\varphi]) = (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_k)),$$

where the representative $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi(a_{k+1}) = 0, \varphi(a_{k+2}) = 1$, and $\varphi(a_{k+3}) = \infty$, and where $\mathcal{L}_k$ is the subset of $\mathbb{C}^k$ defined by

$$\mathcal{L}_k := \{(z_1, z_2, \dots, z_k) \in \mathbb{C}^k : z_i = z_j \text{ for some } i \neq j, \text{ or } z_i = 0, \text{ or } z_i = 1\}.$$

It is known that the map h provides a biholomorphism between $\mathcal{M}_A$ and $\mathbb{C}^k - \mathcal{L}_k$ (see [Hub16, Section 10.9]).

Our focus in this paper is on the case when $|A| = 4$. In this situation, the Teichmüller space $\mathcal{T}_A$ is biholomorphic to $\mathbb{D}$, with the metric d_T coinciding with the usual hyperbolic metric on $\mathbb{D}$; see [Hub06, Corollary 6.10.3 and Theorem 6.10.6]. Furthermore, the moduli space $\mathcal{M}_A$ is biholomorphic to the three-punctured Riemann sphere Σ.

2.5. **Pullback maps.** In this section, we illustrate how the notions introduced in Section 2.4 can be applied for studying the properties of Thurston maps. Most importantly, using Definition 2.1 and Proposition 2.2, we can introduce the following crucial concept (see [MPR24, Proposition 2.21] for the proof; note that this is where the parabolic type condition in the Definition 2.4 plays a crucial role).

Proposition 2.14. *Suppose that $f\colon (S^2, A) \circlearrowleft$ is a Thurston map, or $f \in \mathrm{Homeo}^+(S^2)$ and $f(A) = A$, where $3 \leqslant |A| < \infty$. Let $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ be an orientation-preserving homeomorphism. Then there exists an orientation-preserving homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $g_\varphi := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. In other words, the following diagram commutes*

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\widehat{\mathbb{C}}, \psi(A)) \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g_\varphi} \\ (S^2, A) & \xrightarrow{\varphi} & (\widehat{\mathbb{C}}, \varphi(A)) \end{array}$$

The homeomorphism ψ is unique up to post-composition with a Möbius transformation. Different choices of φ that represent the same point in $\mathcal{T}_A$ yield maps ψ that represent the same point in $\mathcal{T}_A$.

In other words, we have a well-defined map $\sigma_f\colon \mathcal{T}_A \to \mathcal{T}_A$ such that $\sigma_f([\varphi]) = [\psi]$, called the pullback map (or the σ-map) associated with the Thurston map $f\colon (S^2, A) \circlearrowleft$. As φ ranges across all maps representing a single point in $\mathcal{T}_A$, the map g_φ is uniquely defined up to pre- and post-composition with Möbius transformations.

Remark 2.15. Let $\phi\colon S^2 \to S^2$ be an orientation-preserving homeomorphism with $\phi(A) = A$ and $|A| \geqslant 3$. It is straightforward to verify that if $\tau = [\varphi] \in \mathcal{T}_A$, then $\sigma_\phi([\varphi]) = [\varphi \circ \phi]$. Moreover, if $f\colon (S^2, A) \circlearrowleft$ is a Thurston map, it is easy to see that $\sigma_{\phi \circ f} = \sigma_f \circ \sigma_\phi$ and $\sigma_{f \circ \phi} = \sigma_\phi \circ \sigma_f$.

Proposition 2.16. *Suppose that we are in the setting of Proposition 2.14. If there exists a subset $B \subset A$, such that $S_f \subset B$ and $|B| = 3$, then g_φ, up to pre-composition with a Möbius transformation, depends only on $\varphi|B$. Furthermore, if there exists a subset $C \subset A$ such that $|C| = 3$ and $C \subset \overline{f^{-1}(B)}$, then g_φ is uniquely determined by $\varphi|B$ and $\psi|C$, and if $A \subset \overline{f^{-1}(B)}$, then σ_f is a constant map.*

Proof. Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2\colon S^2 \to \widehat{\mathbb{C}}$ are orientation-preserving homeomorphisms such that $\varphi_1|B = \varphi_2|B$, and the maps $g_{\varphi_1} = \varphi_1 \circ f \circ \psi_1^{-1}$ and $g_{\varphi_2} = \varphi_2 \circ f \circ \psi_2^{-1}$ are holomorphic possibly outside of single points in $\widehat{\mathbb{C}}$. One can easily see that we have the following:

$$g_{\varphi_1} = (\varphi_1 \circ \varphi_2^{-1}) \circ g_{\varphi_2} \circ (\psi_2 \circ \psi_1^{-1}),$$

where the homeomorphism $\varphi := \varphi_1 \circ \varphi_2^{-1}$ fixes each point of the set $\varphi_1(B) = \varphi_2(B)$. Since $|B| = 3$, [FM12, Proposition 2.3] implies that φ is isotopic rel. $\varphi_1(B)$ to $\mathrm{id}_{\widehat{\mathbb{C}}}$. According to Proposition 2.2, this isotopy can be lifted, leading to the relation $g_{\varphi_1} = g_{\varphi_2} \circ \psi$, where $\psi \in \mathrm{Homeo}^+(S^2)$. It is easy to see that the homeomorphism ψ is a Möbius transformation.

Now, suppose that there exists a subset $C \subset A$ such that $|C| = 3$ and $C \subset \overline{f^{-1}(B)}$, and $\psi_1|C = \psi_2|C$. Note that the homeomorphism ψ is isotopic rel. $\overline{g_{\varphi_1}^{-1}(\varphi_1(B))} = \psi_1(\overline{f^{-1}(B)})$ to $\psi_2 \circ \psi_1^{-1}$ due to Proposition 2.2. Consequently, $\psi|\psi_1(C) = (\psi_2 \circ \psi_1^{-1})|\psi_1(C) = \mathrm{id}_{\psi_1(C)}$. Since

ψ is a Möbius transformation fixing three distinct points in $\widehat{\mathbb{C}}$, it must be the identity $\mathrm{id}_{\widehat{\mathbb{C}}}$. Thus, the maps g_{φ_1} and g_{φ_2} coincide.

If $A \subset \overline{f^{-1}(B)}$, then $\psi_2 \circ \psi_1^{-1}$ is isotopic rel. $\psi_1(A)$ to the Möbius transformation $\psi = \mathrm{id}_{\widehat{\mathbb{C}}}$. Thus, $\sigma_f([\varphi_1]) = [\psi_1] = [\psi_2] = \sigma_f([\varphi_2])$ in the Teichmüller space $\mathcal{T}_A$, and the rest follows. □

The following observation provides the most crucial property of pullback maps (see [MPR24, Proposition 2.24] for the proof).

Proposition 2.17. *A Thurston map $f\colon (S^2, A) \circlearrowleft$ with $|A| \geqslant 3$ is realized if and only if the pullback map σ_f has a fixed point in the Teichmüller space $\mathcal{T}_A$.*

To illustrate the principle formulated in Proposition 2.17, we present the following remark.

Remark 2.18. Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| \geqslant 3$, and suppose that there is a subset $B \subset A$ such that $S_f \subset B$, $|B| = 3$, and $A \subset \overline{f^{-1}(B)}$. Then f is realized rel. A because σ_f is a constant map according to Proposition 2.16. Additionally, by applying Remark 2.15 and Proposition 2.16, it is easy to show that the Hurwitz class $\mathcal{H}_{f,A}$ consists of a single Thurston map, up to combinatorial equivalence rel. A.

However, Thurston maps that satisfy these conditions are somewhat artificial. For such a marked Thurston map $f\colon (S^2, A) \circlearrowleft$, it must hold $P_f \subset B$ and for every $a \in A - B$, either a is the essential singularity of f or $f(a) \in B$. For instance, this scenario is impossible for unmarked Thurston maps with at least four postsingular values.

Proposition 2.19. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| \geqslant 3$. Then the pullback map σ_f is holomorphic.*

Proof. This result is rather well-known in the context of finite degree Thurston maps (see [BCT14, Section 1.3] and [Hub16, Sections 10.6 and 10.7]), and it can be extended analogously to the transcendental setting (see, for instance, [Ast, Lemma 3.3]). □

Remark 2.20. Proposition 2.19 and [Hub06, Corollary 6.10.7] imply that the map σ_f is 1-*Lipschitz*, meaning $d_T(\sigma_f(\tau_1), \sigma_f(\tau_2)) \leqslant d_T(\tau_1, \tau_2)$ for every $\tau_1, \tau_2 \in \mathcal{T}_A$. In fact, in many cases, such as when the Thurston map $f\colon (S^2, A) \circlearrowleft$ is transcendental, it can be shown (see [HSS09, Section 3.2], [Pfr19, Chapter 5.1], or [Ast, Sections 2.3 and 3.1]) that σ_f is actually *distance-decreasing*, i.e., $d_T(\sigma_f(\tau_1), \sigma_f(\tau_2)) < d_T(\tau_1, \tau_2)$ for every distinct $\tau_1, \tau_2 \in \mathcal{T}_A$. This property of pullback maps can be used to obtain certain rigidity results for transcendental postsingularly finite meromorphic maps (cf. [Hub16, Corollary 10.7.8] and [MPR24, Proposition 2.26]).

However, we do not require these results and the observation of Proposition 2.19 for our further arguments since we mostly work with families of Thurston maps satisfying additional assumptions. For these families, we will directly observe all the properties mentioned above.

3. Hyperbolic tools

Let U be a hyperbolic Riemann surface, and let d_U denote the distance function of the hyperbolic metric on U. For any rectifiable curve α in U, we denote the length of α with respect to the hyperbolic metric by $\ell_U(\alpha)$. When we refer to γ as a geodesic in U, we always mean that γ is a geodesic with respect to the hyperbolic metric. Also, let $B_U(z, r)$ be the hyperbolic ball in U with center $z \in U$ and radius r. If U is a subset of $\mathbb{C}$, then the hyperbolic metric on U, as a conformal metric, can be written as $\rho_U(z)|dz|$, where $\rho_U\colon U \to [0, +\infty)$ is the *density* of the hyperbolic metric on U.

For a holomorphic map $g\colon U \to V$ between two hyperbolic Riemann surfaces, we denote by $\|\mathrm{D}g(z)\|_U^V$ the norm of the derivative of g with respect to the hyperbolic metrics on the domain U and the range V. More precisely, this norm is given by $\|\mathrm{D}g(z)\|_U^V = \|\mathrm{D}_z g(v)\|_V / \|v\|_U$, where $v \in T_z U$ is any non-zero vector, and $\|\cdot\|_U$ represents the length of a tangent vector to U with respect to the hyperbolic metric. If $U = V$, we simply use the notation $\|\mathrm{D}g(z)\|_U$.

Schwarz-Pick's lemma [Hub06, Proposition 3.3.4] implies that for a holomorphic map $g\colon U \to V$ between two hyperbolic Riemann surfaces, we have $\|\mathrm{D}g(z)\|_U^V \leqslant 1$ for every $z \in U$. Furthermore, if g is a covering map, this inequality becomes an equality; otherwise, g is *locally uniformly contracting*, i.e., for every compact set $K \subset U$, there exists a constant $\lambda_K < 1$ such that $\|\mathrm{D}g(z)\|_U^V \leqslant \lambda_K$ for all $z \in K$. Suppose that α is a C^1-curve in U and $\|\mathrm{D}g(z)\|_U^V \leqslant \lambda$ for all $z \in \alpha$. Then it is straightforward to check that $\ell_V(g(\alpha)) \leqslant \lambda \ell_U(\alpha)$. In particular, Schwarz-Pick's lemma implies that the map $g\colon U \to V$ is always 1-Lipschitz and, if g is not a covering map, then g is *locally uniformly distance-decreasing*.

We will be particularly interested in the case when the map g mentioned above is simply the inclusion map $I\colon U \hookrightarrow V$. We are going to denote $\|\mathrm{D}I(z)\|_U^V$ by $c_U^V(z)$. Clearly, if $U \subset V \subset \mathbb{C}$, then $c_U^V(z) = \rho_V(z)/\rho_U(z)$.

3.1. **Levy cycles.** Let δ be an essential simple closed curve in the punctured Riemann sphere $X := \widehat{\mathbb{C}} - P$, where $3 \leqslant |P| < \infty$. According to [Hub06, Proposition 3.3.8], there exists a unique closed geodesic in X that is homotopic (in X) to δ. Note that this geodesic should be necessarily simple [Hub06, Proposition 3.3.9].

In order to illustrate the utility of the hyperbolic tools, we prove that Levy cycles are obstructions for Thurston maps of both finite and infinite degree.

Proof of Proposition 2.11. Let γ be a simple closed curve forming a Levy cycle for the Thurston map $f\colon (S^2, A) \circlearrowleft$. We assume that $n = 1$ in Definition 2.10, i.e., γ is a Levy fixed curve. The general case can be handled in a similar manner.

Suppose that $f\colon (S^2, A) \circlearrowleft$ is combinatorially equivalent to a postsingularly finite holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$. Based on Definition 2.7 and Remark 2.8, it is easy to see that $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$ also has a Levy fixed curve $\delta \subset \widehat{\mathbb{C}} - P$. Note that $|P| \geqslant 4$ since, otherwise, δ would not be essential in $\widehat{\mathbb{C}} - P$. According to the previous discussion and Corollary 2.3, we can assume that δ is a simple closed geodesic in $\widehat{\mathbb{C}} - P$.

Let $\widetilde{\delta}$ be a connected component of $g^{-1}(\delta)$ such that δ and $\widetilde{\delta}$ are homotopic in $\widehat{\mathbb{C}} - P$, and $\deg(g|\widetilde{\delta}\colon \widetilde{\delta} \to \delta) = 1$. By Schwarz-Pick's lemma, it follows that

$$\ell_{\widehat{\mathbb{C}}-P}(\delta) = \ell_{\widehat{\mathbb{C}}-\overline{g^{-1}(P)}}(\widetilde{\delta}) > \ell_{\widehat{\mathbb{C}}-P}(\widetilde{\delta}),$$

where the last inequality is strict since $\overline{g^{-1}(P)} - P$ is non-empty because of Great Picard's theorem and the Riemann-Hurwitz formula [Hub06, Appendix A.3]. However, δ is the unique geodesic in its homotopy class in $\widehat{\mathbb{C}} - P$. Thus, $\ell_{\widehat{\mathbb{C}}-P}(\delta) \leqslant \ell_{\widehat{\mathbb{C}}-P}(\widetilde{\delta})$, and it leads to a contradiction. □

Let γ be an essential simple closed curve in $S^2 - A$, and $\tau = [\varphi]$ be a point in the Teichmüller space $\mathcal{T}_A$. We define $l_\gamma(\tau)$ as the length of the unique hyperbolic geodesic in $\widehat{\mathbb{C}} - \varphi(A)$ that is homotopic in $\widehat{\mathbb{C}} - \varphi(A)$ to $\varphi(\alpha)$. Additionally, we introduce $w_\gamma(\tau) := \log l_\gamma(\tau)$. It is known that $w_\gamma\colon \mathcal{T}_A \to \mathbb{R}$ is a 1-Lipschitz function [Hub06, Theorem 7.6.4].

Let X be a hyperbolic Riemann surface, and suppose $\alpha \subset X$ is a simple closed geodesic with $\ell_X(\alpha) < \ell^*$, where $\ell^* := \log(3 + 2\sqrt{2})$. In this case, we say that α is *short*. As stated

in [Hub06, Proposition 3.3.8 and Corollary 3.8.7], two short simple closed geodesics on a hyperbolic Riemann surface X are either disjoint and non-homotopic in X, or they coincide. Therefore, [Hub06, Proposition 3.3.8] implies that a punctured Riemann sphere $\widehat{\mathbb{C}}-P$, where $3 \leqslant |P| < \infty$, can have at most $|P|-3$ distinct short simple closed geodesics.

The following result allows us to identify a Levy fixed curve for a Thurston map based on the behavior of the corresponding pullback map.

Proposition 3.1. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| = 4$, and $\tau = [\varphi]$ and $\sigma_f(\tau) = [\psi]$ be points in the Teichmüller space $\mathcal{T}_A$, where the representatives $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that the map $g := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. Suppose that there exists an annulus $U \subset \widehat{\mathbb{C}}$ such that:*

- *each connected component of $\widehat{\mathbb{C}} - U$ contains two points of $\psi(A)$;*
- *$\operatorname{mod}(U) > 5\pi e^{d_0}/\ell^*$, where $d_0 = d_T(\tau, \sigma_f(\tau))$;*
- *g is defined and injective on U.*

Then $f\colon (S^2, A) \circlearrowleft$ has a Levy fixed curve. Moreover, if g is defined and injective outside a single connected component of $\widehat{\mathbb{C}} - U$, then $f\colon (S^2, A) \circlearrowleft$ has a weakly degenerate Levy fixed curve.

Proof. Since g is defined and injective on U, the annulus U contains at most 4 points of the set $\overline{g^{-1}(\varphi(A))}$. Therefore, we can find a parallel subannulus V of U such that $\operatorname{mod}(V) \geqslant \operatorname{mod}(U)/5 > \pi e^{d_0}\ell^*$, and V does not contain any points of $\overline{g^{-1}(\varphi(A))}$. Denote $X := \widehat{\mathbb{C}} - \psi(A)$, $Y := \widehat{\mathbb{C}} - \varphi(A)$, and $Z := \widehat{\mathbb{C}} - \overline{g^{-1}(\varphi(A))}$. In particular, $g|Z\colon Z \to X$ is a holomorphic covering map.

Let α be a unique hyperbolic geodesic of V. It is known that α is a simple closed curve that forms a core curve of the annulus V, and its length in V is given by $\ell_V(\alpha) = \pi/\operatorname{mod}(V)$; see [Hub06, Proposition 3.3.7]. Let β denote the curve $g(\alpha)$. Since α is a simple closed curve and $g|V$ is injective, then β is also a simple closed curve. At the same time, β must be essential in Y since, otherwise, α would not be essential in X (see, for instance, [For91, Theorems 5.10 and 5.11]). Define $\widetilde{\alpha} := \psi^{-1}(\alpha)$ and $\widetilde{\beta} := \varphi^{-1}(\beta)$, both of which are essential simple closed curves in $S^2 - A$. It is straightforward to verify that $f(\widetilde{\alpha}) = \widetilde{\beta}$ and $\deg(f|\widetilde{\alpha}\colon \widetilde{\alpha} \to \widetilde{\beta}) = 1$.

According to Schwarz-Pick's lemma and the choice of $\alpha \subset Z$, we have the following inequality:

$$\ell_X(\alpha) \leqslant \ell_Z(\alpha) \leqslant \ell_V(\alpha) = \pi/\operatorname{mod}(V) < e^{-d_0}\ell^*.$$

Therefore, since the function $w_{\widetilde{\alpha}}\colon \mathcal{T}_A \to \mathbb{R}$ is 1-Lipschitz, it follows that $l_{\widetilde{\alpha}}(\tau) < \ell^*$. Hence, there exists a simple closed geodesic δ in Y, homotopic in Y to $\varphi(\widetilde{\alpha})$, such that $\ell_Y(\delta) < \ell^*$. At the same time, $\ell_Y(\beta) = \ell_Z(\alpha) < e^{-d_0}\ell^*$. Therefore, since both β and δ are essential in Y, they are homotopic in Y to short simple closed geodesics. However, as discussed previously, the four-punctured Riemann sphere Y can have only one short simple closed geodesic. This implies that β and δ are homotopic in Y, which in turn means that the curves $\widetilde{\alpha}$ and $\widetilde{\beta}$ are homotopic in $S^2 - A$. Hence, $\widetilde{\beta}$ provides a Levy fixed curve for the Thurston map $f\colon (S^2, A) \circlearrowleft$. It is also straightforward to verify that if g is defined and injective outside a single connected component of $\widehat{\mathbb{C}} - U$, then this Levy fixed curve is weakly degenerate. □

The following result guarantees the uniqueness of a Levy fixed curve for a Thurston map with four marked points, given a specific technical condition.

Proposition 3.2. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with $|A| = 4$ that has a Levy fixed curve. Suppose that there exists a point τ in the Teichmüller space $\mathcal{T}_A$ such that the sequence $(\pi(\sigma_f^{\circ n}(\tau)))$ eventually leaves every compact subset of the moduli space $\mathcal{M}_A$. Then $f\colon (S^2, A) \circlearrowleft$ has a unique Levy fixed curve up to homotopy in $S^2 - A$.*

Proof. Let γ be a Levy fixed curve for the Thurston map $f\colon (S^2, A) \circlearrowleft$. Choose a point $\mu \in \mathcal{T}_A$ such that $l_\gamma(\mu) < \ell^*$. Define $\tau_n := \sigma_f^{\circ n}(\tau) = [\varphi_n]$ and $\mu_n := \sigma_f^{\circ n}(\mu) = [\psi_n]$ for all $n \geqslant 0$. According to Proposition 2.14, we can assume that $g_n := \psi_n \circ f \circ \psi_{n+1}^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic for every $n \geqslant 0$. We also define $X_n := \widehat{\mathbb{C}} - \varphi_n(A)$, $Y_n := \widehat{\mathbb{C}} - \psi_n(A)$, and $Z_n := \widehat{\mathbb{C}} - \overline{g_n^{-1}(\psi_n(A))} = \widehat{\mathbb{C}} - \overline{\psi_{n+1}(f^{-1}(A))}$ for all $n \geqslant 0$. In particular, $g_n|Z_n\colon Z_n \to Y_n$ is a holomorphic covering map.

Claim. We have $l_\gamma(\mu_n) < \ell^*$ for all $n \geqslant 0$.

Proof. Clearly, $l_\gamma(\mu_0) < \ell^*$ by the choice of μ. We will prove that $l_\gamma(\mu_1) < \ell^*$, and the rest easily follows by induction on n. There exists a short simple closed geodesic β in Y_0 such that β and $\psi_0(\gamma)$ are homotopic in Y_0, and $\ell_{Y_0}(\beta) = l_\gamma(\mu_0)$. Define $\widetilde{\beta} := \psi_0^{-1}(\beta)$. Since $\widetilde{\beta}$ and γ are homotopic in $S^2 - A$, then by Corollary 2.3, there exists a simple closed curve $\widetilde{\alpha} \subset f^{-1}(\widetilde{\alpha})$ that is homotopic to γ in $S^2 - A$, since γ is a Levy fixed curve for $f\colon (S^2, A) \circlearrowleft$, and $\deg(f|\widetilde{\alpha}\colon \widetilde{\alpha} \to \widetilde{\beta}) = 1$. Now, define $\alpha := \psi_1(\widetilde{\alpha})$, which is homotopic in Y_1 to $\psi_1(\gamma)$. Clearly, $g(\alpha) = \beta$, $\deg(g|\alpha\colon \alpha \to \beta) = 1$, and $\alpha \subset Z_0$. Therefore, by Schwarz-Pick's lemma, we have

$$\ell_{Y_1}(\alpha) \leqslant \ell_{Z_0}(\alpha) = \ell_{Y_0}(\beta) < \ell^*.$$

Finally, $l_\gamma(\mu_1) = l_{\widetilde{\alpha}}(\mu_1) < \ell^*$. ■

Since $(\pi(\tau_n))$ eventually leaves every compact subset of $\mathcal{M}_A$, Mumford's compactness theorem [Hub06, Theorem 7.3.3] states that the length of the shortest simple closed geodesic δ_n in X_n tends to zero. Then $w_{\varphi_n^{-1}(\delta_n)}(\mu_n) < \log \ell^*$ for any sufficiently large n, given that $w_{\varphi_n^{-1}(\delta)}\colon \mathcal{T}_A \to \mathbb{R}$ is 1-Lipschitz and $d_T(\mu_n, \tau_n) \leqslant d_T(\tau, \mu)$ by Proposition 2.19 and Schwarz-Pick's lemma. Thus, $\psi_n(\gamma)$ and $\psi_n(\varphi_n^{-1}(\delta_n))$ are homotopic in Y_n to short simple closed geodesics. Since $Y_n = \widehat{\mathbb{C}} - \psi_n(A)$ is a four-punctured Riemann sphere, it follows that γ and $\varphi_n^{-1}(\delta_n)$ are homotopic in $S^2 - A$ for any n large enough.

This argument applies to any Levy fixed curve of $f\colon (S^2, A) \circlearrowleft$. Therefore, it has to be unique up to homotopy in $S^2 - A$. □

Remark 3.3. If an obstructed Thurston map $f\colon (S^2, A) \circlearrowleft$ with $|A| \geqslant 3$ has a finite degree and *hyperbolic orbifold*, then it is known (see [Hub16, Section 10.9 and Lemma 10.11.9] or [Sel12, Proof of Theorem 2.3, p. 20]) that every point τ in the Teichmüller space $\mathcal{T}_A$ satisfies the condition of Proposition 3.2.

Remark 3.4. Suppose that we are in the setting of Proposition 3.2. It is clear that it suffices to require that the sequence $(\pi(\sigma_f^{\circ n}(\tau)))$ admits a subsequence eventually leaving every compact subset of the moduli space $\mathcal{M}_A$.

3.2. **Estimating contraction.** The following proposition provides an estimate for the contraction of an inclusion between hyperbolic Riemann surfaces. Although this result is well-known [McM94, Theorem 2.25], we include the proof here for the sake of completeness.

Proposition 3.5. *Suppose that U and V are two hyperbolic Riemann surfaces such that $U \subsetneq V$. Let $z \in U$ and $s := d_V(z, V - U)$. Then*

$$c_U^V(z) \leqslant \frac{2r|\log r|}{1 - r^2},$$

where $r := \tanh(s/2)$.

Proof. Suppose that $z_0 \in V - U$ is a point such that $s = d_V(z, z_0)$. Let $p\colon \mathbb{D} \to V$ be a holomorphic universal covering with $p(0) = z_0$. Let us chose $w \in p^{-1}(z)$ such that $d_{\mathbb{D}}(0, w) = s$. Finally, we denote by $\widetilde{U} \subset \mathbb{D}$ the connected component of w in $p^{-1}(U)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{D}^* & \longleftarrow & \widetilde{U} & \xrightarrow{p|\widetilde{U}} & U \\ & \searrow & \downarrow & & \downarrow \\ & & \mathbb{D} & \xrightarrow{p} & V \end{array}$$

Applying Schwarz-Pick's lemma and recalling that $p\colon \mathbb{D} \to V$ and $p|\widetilde{U}\colon \widetilde{U} \to U$ are holomorphic covering maps, we obtain the following:

$$c_U^V(z) = c_{\widetilde{U}}^{\mathbb{D}}(z) = \frac{\rho_{\mathbb{D}}(w)}{\rho_{\widetilde{U}}(w)} \leqslant \frac{\rho_{\mathbb{D}}(w)}{\rho_{\mathbb{D}^*}(w)}.$$

The rest easily follows since $\rho_{\mathbb{D}}(w) = 2/(1 - |w|^2)$, $\rho_{\mathbb{D}^*}(w) = 1/|w \log |w||$ ([Hub06, Example 3.3.2]), and $s = d_{\mathbb{D}}(0, w) = 2\tanh^{-1}(|w|)$ due to [Hub06, Exercise 2.1.8]. □

Remark 3.6. Suppose that we are in the setting of Proposition 3.5. It shows that there exists an upper bound $\lambda(s) < 1$ for $c_U^V(z)$ depending only on $s = d_V(z, V - U)$ and not on U, V, or z. Moreover, $\lambda(s) \sim |s \log(s)|$ as $s \to 0$. In particular, $\lambda(s) \to 0$ as $s \to 0$.

Proposition 3.5 and Remark 3.6 allow us to estimate the contraction of certain inclusions between countably and finitely punctured Riemann spheres (see [Rem09, Lemma 2.1] for the result of a similar nature).

Proposition 3.7. *Suppose that (z_n) is a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} z_n = \infty$ and $\lim_{n\to\infty} \log|z_{n+1}|/\log|z_n| = 1$. Let P be a finite set consisting of at least two elements of (z_n). Then $c_U^V(z)$ converges to 0 as $|z|$ tends to ∞ for $z \in U$, where $U := \mathbb{C} - \{z_n : n \in \mathbb{N}\}$ and $V := \mathbb{C} - P$.*

Proof. Without loss of generality, we can assume that $P \subset \mathbb{D}$ and the sequence of absolute values $(|z_n|)$ is non-decreasing. Let $Q := \{z_n : n \in \mathbb{N}\}$, and let $p\colon \mathbb{H} \to \mathbb{C} - \overline{\mathbb{D}}$ be the holomorphic universal covering defined as $p(z) = \exp(-iz)$ for $z \in \mathbb{H}$.

According to Proposition 3.5 and Remark 3.6, it suffices to demonstrate that for any $r > 0$, the set $\bigcup_{x \in Q-P} B_V(x, r)$ covers some punctured neighborhood of ∞ in $\widehat{\mathbb{C}}$. Alternatively, it is sufficient to show that for any $r > 0$, there exists $t > 0$ so that

$$\bigcup_{x \in p^{-1}(Q - \overline{\mathbb{D}})} B_{\mathbb{H}}(x, r) \supset \mathbb{H}_t := \{z \in \mathbb{C} : \operatorname{Im}(z) > t\}. \tag{1}$$

Indeed, $p(\mathbb{H}_t) = \mathbb{C} - \overline{\mathbb{D}}_{e^t}$ and $p(B_{\mathbb{H}}(x, r)) \subset B_V(p(x), r)$ due to Schwarz-Pick's lemma.

Note that for sufficiently large n, we have that $|z_n| > 1$ and

$$\pi^{-1}(z_n) = \{2\pi k - \arg(z_n) + i\log(|z_n|) : k \in \mathbb{Z}\} =: \{z_{n,k} : k \in \mathbb{Z}\}. \tag{2}$$

Observation (2) implies that

$$d_{n,k} := \operatorname{diam}_{\mathbb{H}}\left(\{z : 2\pi k \leqslant \operatorname{Re}(z) \leqslant 2\pi(k+1), \operatorname{Im}(z_{n,k}) \leqslant \operatorname{Im}(z) \leqslant \operatorname{Im}(z_{n+1,k})\}\right) \leqslant \tag{3}$$
$$\leqslant \log\left(\frac{\log|z_{n+1}|}{\log|z_n|}\right) + \frac{2\pi}{\log|z_n|}.$$

Estimate (3) is derived from the following well-known facts about the hyperbolic distance between two points $z, w \in \mathbb{H}$:

- if $\operatorname{Re}(z) = \operatorname{Re}(w)$, then $d_{\mathbb{H}}(z,w) = |\log(\operatorname{Im}(z)/\operatorname{Im}(w))|$, and
- if $\operatorname{Im}(z) = \operatorname{Im}(w)$, then $d_{\mathbb{H}}(z,w) \leqslant |\operatorname{Re}(z) - \operatorname{Re}(w)|/\operatorname{Im}(z)$. In fact, the upper estimate is the hyperbolic length of the horizontal segment connecting z and w.

Finally, estimate (3) implies that $d_{n,k}$ tends to zero independently from k when $n \to \infty$. Thus, inclusion (1) holds for certain $t > 0$, and the desired result follows. □

Remark 3.8. Suppose that we are in the setting of Proposition 3.7. Assume that we know that $\log|z_{n+1}|/\log|z_n|$ is uniformly bounded from above but does not necessarily converge to 1 as $n \to \infty$. Following the proof of Proposition 3.7, one can show that there exists $\lambda < 1$ so that $c_U^V(z) < \lambda$ for any $z \in W \cap U$, where $W \subset \widehat{\mathbb{C}}$ is a neighborhood of ∞.

As an application of Proposition 3.7, we can obtain the following result.

Proposition 3.9. *Suppose that $g\colon U \to V$ is a holomorphic covering map, where $U \subset V$ is a domain of $\widehat{\mathbb{C}}$ and $V = \widehat{\mathbb{C}} - P$ with $3 \leqslant |P| < \infty$. Let $x \in P$ be an accumulation point of the set U. Then exactly one of the following two possibilities is satisfied:*

(1) x is an isolated removable singularity of $g\colon U \to V$, or
(2) $c_U^V(z)$ converges to 0 as $|z - x|$ tends to 0 for $z \in U$.

Proof. Without loss of generality, we can assume that $x = \infty$. Suppose that condition (1) is not satisfied. We will prove that there exists a constant $a > 1$ such that for every sufficiently large $r > 0$, the annulus $A(r/a, ra) := \{z \in \mathbb{C} : r/a < |z| < ra\}$ contains a point of $\mathbb{C} - U$ (see [BR20, Lemma 3.2] for a similar result in the setting of entire maps). Suppose the contrary, i.e., that there exist sequences (a_n) and (r_n) such that $a_n > 1$, $a_n \to \infty$, $r_n \to \infty$, and $A_n := A(r_n/a_n, r_n a_n) \subset U$ for every $n \in \mathbb{N}$.

In particular, g is defined and meromorphic on A_n, allowing us to consider the sequence of meromorphic functions $g_n\colon A(1/a_n, a_n) \to \widehat{\mathbb{C}}$, $n \in \mathbb{N}$, where $g_n(z) = g(r_n z)$ for every $z \in A(1/a_n, a_n)$. By Montel's theorem, the family (g_n) is normal since each g_n omits the values in the set P. Thus, there exists a subsequence of (g_n) that converges locally uniformly on the punctured plane $\mathbb{C}^*$. Without loss of generality, we assume that this subsequence coincides with the original sequence (g_n). Let $h\colon \mathbb{C}^* \to \widehat{\mathbb{C}}$ be the limiting function. If h is not constant, then by Hurwitz's theorem [Gam01, p. 231], h also omits the values in the set P. This leads to a contradiction because either h would have an isolated essential singularity at 0 or ∞, contradicting Great Picard's theorem, or both 0 and ∞ would be removable singularities or poles of h, and thus, h is a rational map, which cannot omit any values.

Therefore, h is a constant map. Denote by $w \in \widehat{\mathbb{C}}$ its unique value. Choose a simply connected domain $W \ni w$ such that $\overline{W} \cap P = \{w\} \cap P$. From the previous discussion, $g(A(2r_n/a_n, a_n r_n/2)) \subset W$ for all n large enough. Note that every connected component of $g^{-1}(W) \subset U$ is either a simply connected domain or a simply connected domain with a single puncture (see, for instance, [For91, Theorems 5.10 and 5.11]). Hence, for every sufficiently

large n, either all but at most one of the points in $\mathbb{D}_{a_n r_n/2}$ belong to $g^{-1}(W)$, or the same is true for the set $\widehat{\mathbb{C}} - \overline{\mathbb{D}}_{2r_n/a_n}$. The first case cannot happen for infinitely many n, so there must be some $n \in \mathbb{N}$ for which the second case occurs. It immediately leads to a contradiction, as we initially assumed that x is not an isolated removable singularity of the map $g\colon U \to V$.

Now, we can take z_n to be a point of $\widehat{\mathbb{C}} - U$ in $A(a^{2n}, a^{2n+2})$ for every sufficiently large n. It is easy to see that the sequence (z_n) satisfies the conditions of Proposition 3.7. Given that

$$c_U^V(z) = \frac{\rho_V(z)}{\rho_U(z)} \leqslant \frac{\rho_V(z)}{\rho_{\mathbb{C}-Q}(z)},$$

where $Q := P \cup \{z_n : n \in \mathbb{N}\}$, item (2) follows from Proposition 3.7. □

3.3. **Iteration on the unit disk.** If $h\colon \mathbb{D} \to \mathbb{D}$ is a non-injective holomorphic map, the Denjoy-Wolff theorem [Aba23, Theorem 3.2.1] states that any point $z \in \mathbb{D}$ converges under iteration of h to a point $z_0 \in \overline{\mathbb{D}}$ that is independent of the initial choice of z. In this section, we explore holomorphic maps on the unit disk that satisfy stronger assumptions, allowing us to achieve more precise control on the behavior of their orbits. Many examples of such maps will appear in Section 4. Specifically, many pullback maps of (marked) Thurston maps with a marked set A, where $|A| = 4$, satisfy these conditions.

We say that $z \in \widehat{\mathbb{C}}$ is a *regular point* of a holomorphic map $g\colon U \to V$, where U and V are domains of $\widehat{\mathbb{C}}$, if either $z \in U$ and $\deg(g, z) = 1$, or z is an isolated removable singularity of g and, after extending g holomorphically to a neighborhood of z, $\deg(g, z) = 1$. A point $z \in \widehat{\mathbb{C}}$ is a *fixed point* of the map $g\colon U \to V$ if either $z \in U$ and $g(z) = z$, or z is an isolated removable singularity of g and, after extending g holomorphically to a neighborhood of z, we have $g(z) = z$. The concepts of repelling or attracting fixed points can be generalized in a similar way.

Theorem 3.10. *Let $h\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map, and $\pi\colon \mathbb{D} \to V$ and $g\colon U \to V$ are holomorphic covering maps, where $U \subset V$ is a domain of $\widehat{\mathbb{C}}$ and $V = \widehat{\mathbb{C}} - P$ with $3 \leqslant |P| < \infty$. Suppose that $\pi(h(\mathbb{D})) \subset U$ and $\pi = g \circ \pi \circ h$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{h} & h(\mathbb{D}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ V & \xleftarrow[g]{} & U \end{array}$$

If the map $g\colon U \to V$ is non-injective, then exactly one of the following two possibilities is satisfied:

(1) for every $z \in \mathbb{D}$, the h-orbit of z converges to the unique fixed point of h, or
(2) the sequence $(\pi(h^{\circ n}(z)))$ converges to the same repelling fixed point $x \in P$ of the map g, regardless of the choice of $z \in \mathbb{D}$.

Proof. Given that the maps $\pi\colon \mathbb{D} \to V$ and $g \circ \pi|\widetilde{U}\colon \widetilde{U} \to V$ are covering maps, where $\widetilde{U} := \pi^{-1}(U)$, it follows [Hat02, Section 1.3, Exercise 16] that $h\colon \mathbb{D} \to \widetilde{U}$ is also a covering map and, in particular, $\widetilde{U}$ is connected. Therefore, according to Schwarz-Pick's lemma, we have $\|\mathrm{D}h(z)\|_{\mathbb{D}} = c_{\widetilde{U}}^{\mathbb{D}}(h(z)) = c_U^V(\pi(h(z)))$ for every $z \in \mathbb{D}$. At the same time, Great Picard's theorem and the Riemann-Hurwitz formula [Hub06, Appendix A.3] imply that $V - U$ contains at least one point. Hence, the inclusion $I\colon U \hookrightarrow V$ is locally uniformly contracting with respect to the hyperbolic metrics on U and V. As a result, h is locally

uniformly contracting with respect to the hyperbolic metric on $\mathbb{D}$. In particular, h has at most one fixed point, and if there exists a point $z \in \mathbb{D}$ such that its orbit $(h^{\circ n}(z))$ converges in $\mathbb{D}$, then every orbit of h converges to the unique fixed point of h.

Let us pick an arbitrary point $z_0 \in \mathbb{D}$. Define $z_n = h^{\circ n}(z_0)$ and $x_n = \pi(h^{\circ n}(z_0))$ for $n \geqslant 0$. Connect the points z_0 and z_1 by the hyperbolic geodesic $\delta_0 \subset \mathbb{D}$. We denote by $\delta_n \subset \mathbb{D}$ the curve $h^{\circ n}(\delta)$ that connects z_n and z_{n+1}. By Schwarz-Pick's lemma, the sequence $(\ell_{\mathbb{D}}(\delta_n))$ is non-increasing. In particular, if $d_0 = d_{\mathbb{D}}(z_0, z_1)$, then $d_{\mathbb{D}}(z_n, z_{n+1}) \leqslant \ell_{\mathbb{D}}(\delta_n) \leqslant d_0$ and $d_V(x_n, x_{n+1}) \leqslant \ell_V(\pi(\delta_n)) \leqslant d_0$ for every $n \geqslant 0$.

Further, we structure the proof as a series of claims.

Claim 1. If there exists a compact set $K \subset V$ such that $x_n \in K$ for infinitely many n, then the sequence (z_n) converges in $\mathbb{D}$.

Proof. Since $\ell_V(\pi(\delta_n)) \leqslant d_0$, we can enlarge K so that $\pi(\delta_n) \subset K$ for infinitely many n. First, we demonstrate that there exists $\lambda < 1$ so that $c_U^V(z) \leqslant \lambda$ for all $z \in K \cap U$. Indeed, as we mentioned earlier, there exists a point $w \in U - V$. Therefore, for any $z \in K$, the distance $d_V(z, V - U) \leqslant d_V(z, w)$ is uniformly bounded from above. Hence, Proposition 3.5 and Remark 3.6 imply that such λ exists. Finally, from the previous discussions, if $\pi(\delta_{n+1}) \subset K$, then $\ell_{\mathbb{D}}(\delta_{n+1}) \leqslant \lambda \ell_{\mathbb{D}}(\delta_n)$. In particular, this shows that $(\ell_{\mathbb{D}}(\delta_n))$ converges to 0 as $n \to \infty$.

There exists a subsequence of (x_n) that converges to a point $x \in K$. By enlarging K, we can assume that $B_V(x, 2r) \subset K$ for a certain $r > 0$. Now, we choose m such that $\ell_{\mathbb{D}}(\delta_m) < r(1-\lambda)/2$ and $d_V(x_m, x) < r$. Notice that m is chosen so that $\ell_{\mathbb{D}}(\delta_m) \sum_{i=0}^{\infty} \lambda^i < r/2$. Using this fact and applying induction (see [Sel12, Proof of Theorem 2.3, p. 20] for a similar argument), it can be shown that $\pi(\delta_n) \subset B_V(x, r)$ and $\ell_V(\pi(\delta_{n+1})) \leqslant \lambda \ell_V(\pi(\delta_n))$ for all $n \geqslant m$. In other words, the distance between z_n and z_{n+1} decreases exponentially, and the convergence follows from the completeness of the hyperbolic metric on $\mathbb{D}$. ■

Claim 2. If the sequence (x_n) converges to $x \in P$ along some subsequence, then the entire sequence (x_n) converges to x.

Proof. For any $x \in P$, we choose a neighborhood $V_x \subset \widehat{\mathbb{C}}$ so that if $x' \neq x''$, then $d_V(y_1, y_2) > d_0$ for any $y_1 \in V_{x'} - P$ and $y_2 \in V_{x''} - P$. Now, suppose that the sequence (x_n) has subsequences that converge to different limits $y' \in \widehat{\mathbb{C}}$ and $y'' \in \widehat{\mathbb{C}}$, respectively. If either of y' or y'' does not belong to P, then by Claim 1, the sequence (z_n) must converge to a limit in $\mathbb{D}$, which leads to a contradiction. If we instead assume that $y', y'' \in P$, then it follows that $x_n \in V - \bigcup_{x \in P} V_x$ for infinitely many n. Once again, Claim 1 implies that (z_n) must converge to a limit in $\mathbb{D}$, and it also leads to a contradiction. Thus, the entire sequence (x_n) converges to $x \in P$ as $n \to \infty$. ■

Claim 3. If the sequence (x_n) converges to $x \in P$, then x is a fixed point of the map $g \colon U \to V$.

Proof. Suppose $x \in P$ is not an isolated removable singularity of g. Observe that for sufficiently large n, $\pi(\delta_n)$ lies within any given neighborhood of x. Therefore, based on Proposition 3.9 and the fact that $\|\mathrm{D}h(z)\|_{\mathbb{D}} = c_U^V(\pi(h(z)))$ for all $z \in \mathbb{D}$, we have $\ell_{\mathbb{D}}(\delta_{n+1}) \leqslant \lambda \ell_{\mathbb{D}}(\delta_n)$ for n large enough and some $\lambda < 1$. This means that the distance between z_n and z_{n+1} decreases exponentially, so the sequence (z_n) must have a limit in $\mathbb{D}$, leading to a contradiction. Therefore, $x \in P$ is an isolated removable singularity of g. Finally, since $x_n = g(x_{n+1})$ for all $n \geqslant 0$, x must be a fixed point of the map g. ■

Claim 4. If the sequence (x_n) converges to a fixed point $x \in P$ of the map g, then x is a repelling fixed point of g.

Proof. Without loss of generality, we assume $x = 0$. Let us choose a continuous parametrization $\delta_0 \colon \mathbb{I} \to \mathbb{D}$ for the arc δ_0. We then define a continuous curve $\delta \colon [0, +\infty) \to \mathbb{D}$ by setting $\delta(t + n) = h^{\circ n}(\delta_0(t))$ for any non-negative integer n and $t \in \mathbb{I}$. It is straightforward to see that $\pi(\delta(t))$ converges to 0 as t tends to ∞, since $\pi(\delta(n)) = x_n \to x = 0$ as $n \to \infty$ and $\ell_V(\pi(\delta([n, n + 1]))) \leqslant d_0$ for all $n \geqslant 0$. Furthermore, note that $\pi(\delta(t)) = g(\pi(\delta(t + 1))$ for $t \geqslant 0$. Applying the Snail Lemma [Mil06, Lemma 13.2, Corollrary 13.3] to the curve $\pi \circ \delta$ and the map g in a neighborhood of 0, we obtain that either $|g'(0)| > 1$ or $g'(0) = 1$.

Suppose $g'(0) = 1$. Further, we assume that g is extended holomorphically to a neighborhood 0. If we choose an arbitrary simply connected domain $D \subset \widehat{\mathbb{C}}$ such that $D \cap P = \{0\}$, then $g^{-1}(D)$ has a connected component D' containing 0. Moreover, $D' \subset U \cup \{0\}$ is simply connected and $g|D' \colon D' \to D$ is a biholomorphism that fixes 0. Similarly, we can define the local inverse branches $\varphi_n \colon D \to U \cup \{0\}$ of $g^{\circ n}$ in a neighborhood of 0. In other words, $\varphi_n := (g^{\circ n}|D_n)^{-1}$, where D_n is the connected component of 0 in $g^{-n}(D)$. In particular, φ_n is a biholomorphism, $\varphi_n(0) = 0$, and $(\varphi_n)'(0) = 1/(g^{\circ n})'(0) = 1$ for every $n \in \mathbb{N}$.

Due to the previous discussions, $U \cup \{0\}$ does not contain at least three points of $\widehat{\mathbb{C}}$. Hence, Montel's theorem implies that the family (φ_n) is normal. Therefore, up to a subsequence, it converges locally uniformly on D to some holomorphic map φ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$. Since $\varphi'(0) = 1$, φ is injective in a neighborhood of 0. Thus, the iterates $(g^{\circ n})$ converge uniformly in a neighborhood of 0 to φ^{-1}, up to a subsequence. However, if

$$g(z) = z + a_k z^k + \dots,$$

where $a_k \neq 0$, $k \geqslant 2$, and three dots represent higher order terms, then

$$g^{\circ n}(z) = z + n a_k z^k + \dots,$$

so $(g^{\circ n})^{(k)}(0)$ diverges as $n \to \infty$. This leads to a contradiction ruling out the possibility $g'(0) = 1$. Thus, $x = 0$ must be a repelling fixed point of the map g. ■

Claims 1-4 imply that if the sequence $(h^{\circ n}(z_0))$ diverges in $\mathbb{D}$, then the sequence $(\pi(h^{\circ n}(z_0)))$ converges to a repelling fixed point $x \in P$ of the map g. Furthermore, in this case $(\pi(h^{\circ n}(z)))$ converges to x for every $z \in \mathbb{D}$. Indeed, this follows easily since $d_V(\pi(h^{\circ n}(z_0)), \pi(h^{\circ n}(z)))$ is bounded from above by $d_{\mathbb{D}}(z_0, z)$ due to Schwarz-Pick's lemma. □

4. Thurston theory

In this section, we focus on the study of (marked) Thurston maps $f \colon (S^2, A) \circlearrowleft$ that satisfy the following two conditions:

(I) the marked set A contains exactly four points, and
(II) there exists a set $B \subset A$ such that $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap A| = 3$.

It is evident that when the marked set A coincides with the postsingular set P_f, conditions (I) and (II) are equivalent to conditions (A)–(C) from Section 1.3. Also, it is worth noting that the case when $|\overline{f^{-1}(B)} \cap A| = 4$ or, equivalently, $A \subset \overline{f^{-1}(B)}$, is rather trivial due to Remark 2.18.

Using the tools developed Section 3 and the theory of iteration of meromorphic functions, we analyze the corresponding pullbacks map defined on the one-complex dimensional Teichmüller space. It allows us to derived several properties of the corresponding Thurston maps and their Hurwitz classes. In particular, in Section 4.1 we prove Main Theorem A

(see Theorem 4.4), and in Section 4.2 we prove Main Theorem B (see Theorem 4.8) and Corollary 1.1.

Let $A = \{a_1, a_2, a_3, a_4\}$, $B = \{a_{i_1}, a_{i_2}, a_{i_3}\}$, and $C := \overline{f^{-1}(B)} \cap A = \{a_{j_1}, a_{j_2}, a_{j_3}\}$, where $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$. Additionally, assume that i and j are the indices so that $a_i \in A - B$ and $a_j \in A - C$. Under this conventions, we have that $f(a_j) = a_i$ and $\deg(f, a_j) = 1$.

It is important to note that there may be multiple choices for the set B. However, when $|S_f| = 3$, the set B is uniquely determined by the properties described in condition (II). Also, another choice that we made, which will be relevant in our further arguments, is the indexing for the set A.

Now we are ready to introduce the following objects:

- the map $\pi_B \colon \mathcal{T}_A \to \Sigma$, where $\Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$, is defined by $\pi_B([\varphi]) = \varphi(a_i)$, where the representative $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi(a_{i_1}) = 0$, $\varphi(a_{i_2}) = 1$, and $\varphi(a_{i_3}) = \infty$;
- the map $\pi_C \colon \mathcal{T}_A \to \Sigma$ is defined by $\pi_C([\varphi]) = \varphi(a_j)$, where the representative $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi(a_{j_1}) = 0$, $\varphi(a_{j_2}) = 1$, and $\varphi(a_{j_3}) = \infty$;
- the map $\omega_f \colon \mathcal{T}_A \to \Sigma$ is defined by $\omega_f = \pi_C \circ \sigma_f$;
- the map $F_f \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is a unique (which follows from Proposition 2.16) holomorphic map such that $F_f = \varphi \circ f \circ \psi^{-1}$, where $\varphi, \psi \colon S^2 \to \widehat{\mathbb{C}}$ are orientation-preserving homeomorphisms satisfying $\varphi(a_{i_1}) = \psi(a_{j_1}) = 0$, $\varphi(a_{i_2}) = \psi(a_{j_2}) = 1$, and $\varphi(a_{i_3}) = \psi(a_{j_3}) = \infty$;
- the map $M_{i,j} \colon \Sigma \to \Sigma$ is defined as $h_i \circ h_j^{-1}$, where $h_i \colon \mathcal{M}_A \to \Sigma$ is defined by $h_i([\eta]) = \eta(a_i)$, with $\eta \colon A \to \widehat{\mathbb{C}}$ chosen so that $\eta(a_{i_1}) = 0$, $\eta(a_{i_2}) = 1$, $\eta(a_{i_3}) = \infty$, and the map $h_j \colon \mathcal{M}_A \to \Sigma$ is defined analogously.
- the set $W_f := \widehat{\mathbb{C}} - \overline{F_f^{-1}(\{0, 1, \infty\})}$ is a domain of $\widehat{\mathbb{C}}$;
- the map $G_f := F_f \circ M_{i,j}^{-1} \colon M_{i,j}(W_f) \to \Sigma$.

Of course, the maps F_f and G_f, as well as the domain W_f, also depend on the choice of the subset B, and many other objects defined above depend on the indexing of A. However, for the simplicity, we are going to exclude these dependencies from the notation. Throughout this section, we will maintain the notation and conventions established above. Specifically, if $\widetilde{f} \colon (S^2, A) \circlearrowleft$ is any Thurston map that is Hurwitz equivalent to $f \colon (S^2, A) \circlearrowleft$, then we use the same set B and the same indexing of the set A when we work with the map $\widetilde{f}$ as we do it with f.

Proposition 4.1. *The objects introduced above satisfy the following properties:*

(1) diagram (2) commutes.
(2) $S_{F_f} \subset \{0, 1, \infty\}$ and $F_f \colon W_f \to \Sigma$ is a covering map;
(3) $\pi_B \colon \mathcal{T}_A \to \Sigma$, $\pi_C \colon \mathcal{T}_A \to \Sigma$, and $\omega_f \colon \mathcal{T}_A \to W_f$ are holomorphic covering maps;
(4) $M_{i,j}$ extends to a Möbius transformation such that $\{0, 1, \infty\}$ is an $M_{i,j}$-invariant subset;
(5) $\sigma_f(\mathcal{T}_A) \subsetneq \mathcal{T}_A$ is open and dense in $\mathcal{T}_A$ and $\sigma_f \colon \mathcal{T}_A \to \sigma_f(\mathcal{T}_A)$ is a holomorphic covering map;
(6) the maps F_f and G_f, as well as the domain W_f, depend only on the Hurwitz equivalence class of the Thurston map f.

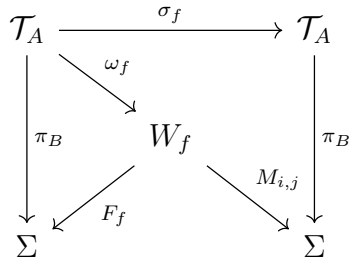

FIGURE 2. Fundamental diagram for Thurston maps satisfying conditions (I) and (II).

Proof. Let $\tau = [\varphi] \in \mathcal{T}_A$ and $\sigma_f(\tau) = [\psi] \in \mathcal{T}_A$, where the representatives $\varphi, \psi \colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that $\varphi(a_{i_1}) = \psi(a_{j_1}) = 0$, $\varphi(a_{i_2}) = \psi(a_{j_2}) = 1$, $\varphi(a_{i_3}) = \psi(a_{j_3}) = \infty$, and the map $\varphi \circ f \circ \psi^{-1} \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. In particular, $\varphi \circ f \circ \psi^{-1}$ coincides with the map F_f. Since, $\omega_f([\varphi]) = \pi_C([\psi]) = \psi(a_j)$, we have the following

$$(F_f \circ \omega_f)([\varphi]) = F_f(\psi(a_j)) = \varphi(f(a_j)) = \varphi(a_i) = \pi_B(\varphi).$$

Thus, $\pi_B = F_f \circ \omega_f$. At the same time, it is straightforward to verify that $\pi_B = M_{i,j} \circ \pi_C$. Finally, $M_{i,j} \circ \omega_f = M_{i,j} \circ \pi_C \circ \sigma_f = \pi_B \circ \sigma_f$, and item (1) follows.

Item (2) directly follows from the definition of a singular set. Maps π_B and π_C are holomorphic coverings, as discussed in Section 2.4. The map $\omega_f \colon \mathcal{T}_A \to W_f$ is also a holomorphic covering since both $\pi_B \colon \mathcal{T}_A \to \Sigma$ and $F_f \colon W_f \to \Sigma$ are holomorphic covering maps, and $\pi_B = F_f \circ \omega_f$ (see [Hat02, Section 1.3, Excercise 16]). Hence, item (3) follows.

The discussion of Section 2.4 shows that the maps $h_i, h_j \colon \mathcal{M}_A \to \Sigma$ are holomorphic. Consequently, $M_{i,j}$ is a conformal automorphism of Σ that extends to a Möbius transformation of $\widehat{\mathbb{C}}$ permuting 0, 1, and ∞. Alternative way to prove item (4) is through direct computation. For example, when $i = j$, then $M_{i,j} = \mathrm{id}_{\widehat{\mathbb{C}}}$; if $i = 1$ and $j = 2$, then $M_{1,2}(z) = z/(z-1)$, and so on.

Since $M_{i,j} \circ \omega_f = \pi_B \circ \sigma_f$, where ω_f and π_B are holomorphic covering maps and $M_{i,j}$ is a Möbius transformation, it follows that $\sigma_f \colon \mathcal{T}_A \to \sigma_f(\mathcal{T}_A)$ is also a holomorphic covering map, where $\sigma_f(\mathcal{T}_A) = \pi_B^{-1}(M_{i,j}(W_f))$. Note that by Great Picard's Theorem and the Riemann-Hurwitz formula [Hub06, Appendix A.3], the set $\widehat{\mathbb{C}} - W_f$ contains at least one point different from the points $\{0, 1, \infty\}$. Therefore, $\sigma_f(\mathcal{T}_A)$ is different from $\mathcal{T}_A$. Since W_f is open and dense in $\widehat{\mathbb{C}}$, it follows that $\sigma_f(\mathcal{T}_A)$ is open and dense in $\mathcal{T}_A$, establishing item (5).

Finally, let $\widetilde{f}$ be a Thurston map Hurwitz equivalent rel. A to f. Suppose that $\phi_1 \circ \widetilde{f} = f \circ \phi_2$, where $\phi_1, \phi_2 \in \mathrm{Homeo}_0^+(S^2, A)$. Then $F_f = \widetilde{\varphi} \circ \widetilde{f} \circ \widetilde{\psi}^{-1}$, where $\widetilde{\varphi} = \varphi \circ \phi_1$ and $\widetilde{\psi} = \psi \circ \phi_2$. In particular, $\widetilde{\varphi}(a_{i_1}) = \widetilde{\psi}(a_{j_1}) = 0$, $\widetilde{\varphi}(a_{i_2}) = \widetilde{\psi}(a_{j_2}) = 1$, and $\widetilde{\varphi}(a_{i_3}) = \widetilde{\psi}(a_{j_3}) = \infty$. Therefore, F_f and $F_{\widetilde{f}}$ coincide, as well as W_f and $W_{\widetilde{f}}$, and item (6) follows. □

Remark 4.2. Proposition 4.1 shows that for Thurston maps satisfying conditions (I) and (II), the corresponding pullback maps admit a commutative diagram analogous to diagram (1) from Section 1.2. In this context, the Möbius transformation $M_{i,j}$ serves the role of the X-map, the map F_f takes the place of the Y-map, and the domain W_f is an analog of the Hurwitz space. In particular, σ_f has the "g-map" $G_f = F_f \circ M_{i,j}^{-1} \colon M_{i,j}(W_f) \to \Sigma$.

In contrast to the finite degree case, when $f \colon (S^2, A) \circlearrowleft$ is transcendental, the map $F_f \colon W_f \to \Sigma$ is a covering of infinite degree and W_f is a countably punctured Riemann sphere.

Remark 4.3. Suppose that we are in the setting of Proposition 4.1. We have observed that $\sigma_f(\mathcal{T}_A) = \pi_B^{-1}(M_{i,j}(W_f))$ and the complement of the set $M_{i,j}(W_f)$ in Σ contains at least one point. Given that $\pi_B\colon \mathcal{T}_A \to \Sigma$ is a covering map of infinite degree, the pullback map σ_f has infinitely many omitted values, i.e., the points of $\mathcal{T}_A - \sigma_f(\mathcal{T}_A)$, and they are not compactly contained in $\mathcal{T}_A$. Moreover, the set of omitted values of σ_f is not discrete in the Teichmüller space $\mathcal{T}_A$ if the essential singularity of the map f lies within the set $S^2 - A$.

4.1. **Characterization problem.** In this section, we present and prove a slightly stronger version of Main Theorem A utilizing the tools developed in Sections 3.1 and 3.3 along with the properties of pullback maps established in Proposition 4.1.

We recall that the Teichmüller space $\mathcal{T}_A$, where $|A| = 4$, is biholomorphic to the unit disk $\mathbb{D}$, and the metric d_T defined in Section 2.4 coincides with the hyperbolic metric $d_{\mathbb{D}}$. If $f\colon (S^2, A) \circlearrowleft$ is a Thurston map satisfying assumptions (I) and (II), then the the corresponding pullback map $\sigma_f\colon \mathcal{T}_A \to \mathcal{T}_A$ is holomorphic. It can be established in two ways: either through the general approach outlined in Proposition 2.19 or by more elementary methods as in item (5) of Proposition 4.1. It is worth mentioning that item (5) of Proposition 4.1 and Schwarz-Pick's lemma imply that σ_f is distance-decreasing on the Teichmüller space $\mathcal{T}_A$ (cf. Remark 2.20).

Theorem 4.4. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree that satisfies properties* (I) *and* (II)*. Then f is realized rel. A if and only if it has no weakly degenerate Levy fixed curve. Moreover,*

(1) if f is realized rel. A by postsingularly finite holomorphic maps $g_1\colon (\widehat{\mathbb{C}}, P_1) \circlearrowleft$ and $g_2\colon (\widehat{\mathbb{C}}, P_2) \circlearrowleft$, then g_1 and g_2 are conjugate by a Möbius transformation M, i.e., $M \circ g_1 = g_2 \circ M$, such that $M(P_1) = P_2$;
(2) if f is obstructed rel. A, then it has a unique Levy fixed curve up to homotopy in $S^2 - A$.

Proof. Suppose that the Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized. According to Proposition 2.11, $f\colon (S^2, A) \circlearrowleft$ cannot have a Levy fixed curve. From Proposition 2.17, it follows that σ_f has a fixed point in the Teichmüller space $\mathcal{T}_A$. As it was discussed previously, σ_f is distance-decreasing on $\mathcal{T}_A$, which implies that it has a unique fixed point. Now, it is straightforward to verify using Proposition 2.14 that item (1) holds (cf. [Hub16, Corollary 10.7.8] and [MPR24, Proposition 2.26]).

Now, suppose that the Thurston map $f\colon (S^2, A) \circlearrowleft$ is obstructed. Choose an arbitrary point $\tau_0 \in \mathcal{T}_A$ and define $\tau_n := \sigma_f^{\circ n}(\tau_0)$, $x_n := \pi_B(\tau_n)$, and $y_n := \pi_C(\tau_{n+1})$ for every $n \geqslant 0$. Let $\tau_n = [\varphi_n] \in \mathcal{T}_A$, where the representative $\varphi_n\colon S^2 \to \widehat{\mathbb{C}}$ is chosen so that $\varphi_n(a_{i_1}) = 0$, $\varphi_n(a_{i_2}) = 1$, and $\varphi_n(a_{i_3}) = \infty$. Denote by $\psi_n\colon S^2 \to \widehat{\mathbb{C}}$ the unique (due to Proposition 2.14) orientation-preserving homeomorphism so that $g_n := \varphi_n \circ f \circ \psi_n^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic and $\psi_n(a_{j_1}) = 0$, $\psi_n(a_{j_2}) = 1$, and $\psi_n(a_{j_3}) = \infty$. According to the definition of the map F_f, it must coincide with the map g_n for every $n \geqslant 0$. Moreover, it follows that $\tau_{n+1} = [\psi_n]$. From this, we observe that $\varphi_n(a_i) = x_n$, $\psi_n(a_j) = y_n$, and $F_f(y_n) = x_n$ for every $n \geqslant 0$.

According to item (1) of Proposition 4.1, we have $\pi_B(\sigma_f(\mathcal{T}_A)) \subset M_{i,j}(W_f)$ and $\pi_B = G_f \circ \pi_B \circ \sigma_f$. Since G_f is not injective, items (2)-(5) of Proposition 4.1 allow us to apply Theorem 3.10 to the pullback map σ_f. This shows that the sequence (x_n) converges to a repelling fixed point $x \in \{0, 1, \infty\}$ of the map G_f. Given that $x_{n+1} = M_{i,j}(y_n)$, the sequence (y_n) converges to a regular point $y \in \{0, 1, \infty\}$ of the map F_f due to item (4) of Proposition 4.1.

We assume that the map F_f is extended holomorphically to a neighborhood of y. Then there exists a disk $D \subset \widehat{\mathbb{C}}$ such that $D \cap \{0,1,\infty\} = \{y\}$ and F_f is injective on D. Consider another disk D' such that $\overline{D'} \subset D$ and the annulus $D - \overline{D'}$ has modulus greater than $5\pi e^{d_0}/\ell^*$, where $d_0 = d_T(\tau_0, \tau_1)$ and $\ell^* = \log(3 + 2\sqrt{2})$. Observe that $y_n \in D'$ for all n large enough and, in particular, each connected component of $\widehat{\mathbb{C}} - (D - \overline{D'})$ contains two points of the set $\psi_n(A)$.

Finally, by Schwarz-Pick's lemma, we have $d_T(\tau_n, \tau_{n+1}) \leqslant d_T(\tau_0, \tau_1)$ for every $n \geqslant 0$. The existence of a weakly degenerate Levy fixed curve for the Thurston map $f\colon (S^2, A) \circlearrowleft$ then follows from Proposition 3.1, applied to $\tau_n = [\varphi_n]$ and $\tau_{n+1} = [\psi_n]$, where n is taken sufficiently large, and the annulus $D - \overline{D'}$ as above. The uniqueness part follows from Proposition 3.2 since the sequence $(\pi(\sigma_f^{\circ n}(\tau_0)))$ clearly leaves every compact set of the moduli space $\mathcal{M}_A$. □

4.2. **Hurwitz classes.** In Section 4.1, we demonstrated how Proposition 4.1, especially commutative diagram (2), can be helpful for studying Thurston maps that satisfy assumptions (I) and (II). In this section, we further develop this idea by showing the significance of the dynamical properties of the map G_f in understanding the Hurwitz class $\mathcal{H}_{f,A}$ of a Thurston map $f\colon (S^2, A) \circlearrowleft$ that satisfies properties (I) and (II). In particular, we prove Main Theorem B (see Theorem 4.8) and Corollary 1.1. However, before proceeding with their proofs, we present two propositions that relate the fixed points of the map G_f to the Thurston maps in the Hurwitz class $\mathcal{H}_{f,A}$, which are either obstructed or realized depending on the properties of the corresponding fixed point.

Proposition 4.5. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map that satisfies conditions* (I) *and* (II)*. Suppose that $x \in \Sigma$ is a fixed point of the map G_f. Then there exists a homeomorphism $\phi \in \mathrm{Homeo}^+(S^2, A)$ such that the Thurston map $\widetilde{f} := \phi \circ f\colon (S^2, A) \circlearrowleft$ is realized by a holomorphic map $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$, where $P = \{0, 1, \infty, x\}$.*

Proof. Proposition 4.1 suggests that there exist points τ_0 and τ_1 in the Teichmüller space $\mathcal{T}_A$ so that $\sigma_f(\tau_0) = \tau_1$ and $\pi_B(\tau_0) = \pi_B(\tau_1) = x$. Indeed, choose an arbitrary point τ_1 of $\pi_B^{-1}(x)$. Since $x \in M_{i,j}(W_f)$, $\sigma_f^{-1}(\tau_1)$ is non-empty, and moreover, $\pi_B(\sigma_f^{-1}(\tau_1)) = \{x\}$. Thus, we can take τ_0 to be any point of $\sigma_f^{-1}(\tau_1)$.

Let $\tau_0 = [\varphi]$ and $\tau_1 = [\psi]$, where the representatives $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ are chosen such that $g := \varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic, $\varphi(a_{i_1}) = \psi(a_{i_1}) = 0$, $\varphi(a_{i_2}) = \psi(a_{i_2}) = 1$, $\varphi(a_{i_3}) = \psi(a_{i_3}) = \infty$, and $\varphi(a_i) = \psi(a_i) = x$. It is straightforward to verify that $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$ is a postsingularly finite holomorphic map, where $P = \varphi(A) = \psi(A) = \{0, 1, \infty, x\}$.

Now, define $\phi := \psi^{-1} \circ \varphi \in \mathrm{Homeo}^+(S^2, A)$ and $\widetilde{f} := \phi \circ f\colon (S^2, A) \circlearrowleft$. It is easy to see that $\psi \circ \widetilde{f} \circ \psi^{-1} = g$. Therefore, the Thurston map $\widetilde{f}\colon (S^2, A) \circlearrowleft$ is combinatorially equivalent to $g\colon (\widehat{\mathbb{C}}, P) \circlearrowleft$. □

Proposition 4.6. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map that satisfies conditions* (I) *and* (II)*. If G_f has a repelling fixed point $x \in \{0, 1, \infty\}$, then there exists a homeomorphism $\phi \in \mathrm{Homeo}^+(S^2, A)$ such that the Thurston map $\widetilde{f} := \phi \circ f\colon (S^2, A) \circlearrowleft$ is obstructed. Moreover, f is totally unobstructed rel. A if and only if none of the points* 0, 1, *or ∞ is a repelling fixed point of the map G_f.*

Proof. Suppose that $x \in \{0, 1, \infty\}$ is a repelling fixed point of the map G_f. Assume that G_f is extended holomorphically to a neighborhood x. Let $U \subset M_{i,j}(W_f) \cup \{x\}$ be a neighborhood

of x where G_f is injective and $\overline{U} \subset G_f(U)$. Define the local inverse branch $g\colon G_f(U) \to U$ of G_f at x, i.e., $g := (G_f|U)^{-1}$. Note that every orbit of g converges to x, since g is uniformly distance-decreasing with respect to the hyperbolic metric on U according to Schwarz-Pick's lemma, Proposition 3.5, and Remark 3.6.

Claim. The distance $d_\Sigma(y, g(y))$ converges to 0 as $y \in G_f(U)$ tends to x.

Proof. Without loss of generality, assume that $x = 0$ and $y, g(y) \in \mathbb{D}$. Since $x = 0$ is a repelling fixed point of the map G_f, by choosing y sufficiently small, we can ensure that $\lambda|y| \leqslant |g(y)| \leqslant |y|$ for some λ, where $0 < \lambda < 1$. Let $p\colon \mathbb{H} \to \mathbb{D}$ be the holomorphic universal covering defined as $p(z) = \exp(iz)$ for $z \in \mathbb{H}$. Define $y_1 := \arg(y) - i\log(|y|) \in p^{-1}(y)$ and $y_2 := \arg(g(y)) - i\log(|g(y)|) \in p^{-1}(g(y))$. Similarly to the proof of Proposition 3.7, we have

$$d_{\mathbb{H}}(y_1, y_2) \leqslant \log\left(\frac{\log|g(y)|}{\log|y|}\right) + \frac{2\pi}{|\log|y||} \leqslant \log\left(\frac{\log|y| + \log\lambda}{\log|y|}\right) + \frac{2\pi}{|\log|y||}.$$

This shows that the distance $d_{\mathbb{H}}(y_1, y_2)$ converges to 0 as y approaches x. According to Schwarz-Pick's lemma, the same holds for $d_{\mathbb{D}}(y, g(y))$ and $d_\Sigma(y, g(y))$. ∎

Therefore, by making U even smaller, we can assume that $d_\Sigma(y, g(y)) < d_\Sigma(y, y')$, where $y \in G_f(U)$ is any point other than x and $y' \in G_f^{-1}(y)$ with $y \neq g(y)$.

Now, we choose $x_1 \in U - \{x\}$ and let $x_0 = G_f(x_1)$. Similarly to the proof of Proposition 4.5, there exists two points τ_1 and τ_2 of the Teichmüller space $\mathcal{T}_A$ so that $\sigma_f(\tau_0) = \tau_1$, $\pi_B(\tau_0) = x_0$, and $\pi_B(\tau_1) = x_1$. Since $\pi_B\colon \mathcal{T}_A \to \Sigma$ is a holomorphic covering map, then there also exists a point $\tau \in \mathcal{T}_A$ so that $d_T(\tau, \tau_1) = d_\Sigma(x_0, x_1)$ and $\pi_B(\tau) = x_0$.

Let $\tau = [\varphi]$ and $\tau_0 = [\psi]$, where the representatives $\varphi, \psi\colon S^2 \to \widehat{\mathbb{C}}$ are chosen so that $\varphi|A = \psi|A$. Define the homeomorphism $\phi := \varphi^{-1} \circ \psi \in \mathrm{Homeo}^+(S^2, A)$. According to Remark 2.15, $\sigma_\phi(\tau) = \tau_0$ and $\sigma_{\widetilde{f}}(\tau) = \sigma_f(\sigma_\phi(\tau)) = \sigma_f(\tau_0) = \tau_1$, where $\widetilde{f} := \phi \circ f\colon (S^2, A) \circlearrowleft$ is a Thurston map.

Schwarz-Pick's lemma, along with items (3) and (5) of Proposition 4.1, implies that

$$d_\Sigma(\pi_B(\sigma_{\widetilde{f}}^{\circ n}(\tau)), \pi_B(\sigma_{\widetilde{f}}^{\circ(n+1)}(\tau))) \leqslant d_T(\tau, \sigma_{\widetilde{f}}(\tau)) = d_T(\tau, \tau_1) = d_\Sigma(x_0, x_1).$$

At the same time, it follows from items (1) and (6) of Proposition 4.1 that

$$\pi_B(\sigma_{\widetilde{f}}^{\circ(n+1)}(\tau)) \in G_f^{-1}(\pi_B(\sigma_{\widetilde{f}}^{\circ n}(\tau))).$$

Since $\pi_B(\tau) = x_0 \in G_f(U)$, and based on the previous assumptions, we have $\pi_B(\sigma_{\widetilde{f}}^{\circ n}(\tau)) = g^{\circ n}(x_0)$. Thus, $\pi_B(\sigma_{\widetilde{f}}^{\circ n}(\tau))$ converges to x as $n \to \infty$. Given that $\sigma_{\widetilde{f}}$ is 1-Lipschitz, it cannot have a fixed point. Hence, by Proposition 2.17, the Thurston map $\widetilde{f}\colon (S^2, A) \circlearrowleft$ must be obstructed.

Suppose none of points 0, 1, or ∞ is a repelling fixed point of the map G_f. Let $\widehat{f}$ be any Thurston map Hurwitz equivalent rel. A to f. Then, according to item (6) of Proposition 4.1, we have $W_f = W_{\widehat{f}}$ and $G_f = G_{\widehat{f}}$. Taking into account Proposition 4.1 and applying Theorem 3.10, we see that $\sigma_{\widehat{f}}$ must have a fixed point. Thus, it follows from Proposition 2.17 that $\widehat{f}$ is realized rel. A. □

Remark 4.7. It is clear that G_f extends to a postsingularly finite holomorphic map having at most one essential singularity and a postsingular set contained within $\{0, 1, \infty\}$. Therefore, by Lemma A.1, every fixed point of G_f is either superattracting or repelling. Furthermore, the only possible superattracting fixed points are 0, 1, and ∞.

Now we are ready to state and prove a slightly stronger version of Main Theorem B.

Theorem 4.8. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map of finite or infinite degree that satisfies conditions* (I) *and* (II)*. Then*

(1) f is totally unobstructed rel. A if and only if there are no two points $a, b \in A$ such that $\deg(f, a) = \deg(f, b) = 1$ and $f(\{a, b\})$ equals $\{a, b\}$ or $A - \{a, b\}$;
(2) if f is not totally unobstructed rel. A, then its Hurwitz class $\mathcal{H}_{f,A}$ contains infinitely many pairwise combinatorially inequivalent obstructed Thurston maps;
(3) if f has infinite degree, then its Hurwitz class $\mathcal{H}_{f,A}$ contains infinitely many pairwise combinatorially inequivalent realized Thurston maps.

Proof. Without loss of generality, we assume that A is indexed so that $B = \{a_1, a_2, a_3\}$, and therefore, $i = 4$ (see the beginning of Section 4). We then analyze four different cases based on the value of $j, 1 \leqslant j \leqslant 4$, to find out when one of the points 0, 1, or ∞ is a repelling fixed point of the map G_f (this analysis will be needed to apply Proposition 4.6). We also recall that $\deg(f, a_j) = 1$ and $f(a_j) = a_i$.

- For the case $j = 1$, we have $M_{4,1}(z) = 1/z$. Therefore,
$$G_f(0) = F_f(\infty) = (\varphi \circ f \circ \psi^{-1})(\infty) = \varphi(f(a_4)).$$
This means that 0 is a fixed point of G_f is and only if $f(a_4) = \varphi^{-1}(0) = a_1$. Furthermore, according to Remark 4.7, 0 is a repelling fixed point of G_f if and only if $f(a_4) = a_1$ and a_4 is a regular point of f. Similarly, 1 is a repelling fixed point of G_f if and only if $f(a_3) = a_2$ and $\deg(f, a_3) = 1$. Lastly, ∞ is a repelling fixed point of the map G_f if and only if $f(a_2) = a_3$ and $\deg(f, a_2) = 1$.
- For the case $j = 2$, we have $M_{4,2}(z) = (z - 1)/z$. Similarly to the previous case, one of the points 0, 1, and ∞ is a repelling fixed point of G_f if and only if $f(a_3) = a_1$ and $\deg(f, a_3) = 1$, or $f(a_4) = a_2$ and $\deg(f, a_4) = 1$, or $f(a_1) = a_3$ and $\deg(f, a_1) = 1$.
- For the case $j = 3$, we have $M_{4,3}(z) = 1 - z$. Here, one of the points 0, 1, and ∞ is a repelling fixed point of G_f if and only if $f(a_2) = a_1$ and $\deg(f, a_2) = 1$, or $f(a_1) = a_2$ and $\deg(f, a_1) = 1$, or $f(a_4) = a_3$ and $\deg(f, a_4) = 1$.
- For the case $j = 4$, we have $M_{4,4} = \mathrm{id}_{\widehat{\mathbb{C}}}$. In this case, one of the points 0, 1, and ∞ is a repelling fixed point of G_f if and only if $f(a_k) = a_k$ and $\deg(f, a_k) = 1$ for some $k = 1, 2, 3$.

Summarizing the calculations above and applying Proposition 4.6, we obtain item (1).

To establish item (2), we largely follow the approach used in the proof of [KPS16, Theorem 9.2(V)]. Suppose that f is not totally unobstructed rel. A, i.e., there exists an obstructed Thurston map in $\mathcal{H}_{f,A}$. Without loss of generality, we assume that this map is $f\colon (S^2, A) \circlearrowleft$. Theorem 4.4 shows that there exists a Levy fixed curve γ for $f\colon (S^2, A) \circlearrowleft$. Define $f_n := T_\gamma^{\circ n} \circ f\colon (S^2, A) \circlearrowleft$, where $n \in \mathbb{Z}$ and $T_\gamma \in \mathrm{Homeo}^+(S^2, A)$ is the *Dehn twist* about a curve γ. Clearly, each Thurston map f_n has a Levy fixed curve γ, and therefore, is obstructed rel. A by Proposition 2.11. We will show that these Thurston maps are pairwise combinatorially inequivalent rel. A.

Suppose the contrary. Then there exist two homeomorphisms $\phi_1, \phi_2 \in \mathrm{Homeo}^+(S^2)$ such that $\phi_1(A) = \phi_2(A) = A$, ϕ_1 is isotopic rel. A to ϕ_2, and $f_n = \phi_1 \circ f_m \circ \phi_2^{-1}$ for some $m \neq n$. Therefore, $f_n\colon (S^2, A) \circlearrowleft$ has $\phi_1(\gamma)$ as a Levy fixed curve. However, Theorem 4.4 states that this Levy fixed curve is unique up to homotopy in $S^2 - A$. This implies that γ and $\phi_1(\gamma)$ are homotopic in $S^2 - A$, and thus, ϕ_1 and ϕ_2 are isotopic rel. A to $T_\gamma^{\circ k}$ for some $k \in \mathbb{Z}$.

One can easily see that f commutes with the Dehn twist T_γ up to isotopy rel. A, meaning $T_\gamma \circ f$ is isotopic rel. A to $f \circ T_\gamma$. Indeed, up to isotopy rel. A, we can assume that f is the identity on a certain annulus in $S^2 - A$ with a core curve γ. Considering the previous discussion, we conclude that f is isotopic rel. A to $f \circ T_\gamma^{\circ(m-n)}$. The following claim proves that it is not possible, and item (2) follows.

Claim. Suppose that f is isotopic rel. A to $f \circ \phi$, where $\phi \in \mathrm{Homeo}^+(S^2, A)$. Then ϕ is isotopic rel. A to id_{S^2}.

Proof. According to Definition 2.5, we can assume without loss of generality that $f = f \circ \phi$. There exist orientation-preserving homeomorphisms $\varphi, \psi \colon \widehat{\mathbb{C}} \to S^2$ such that the map $g := \varphi \circ f \circ \psi^{-1} \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic. One can easily check that $g = g \circ h$, where $h := \psi \circ \phi \circ \psi^{-1} \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Since h must be a Möbius transformation and h fixes the points of the set $\psi(A)$, it follows that $h = \mathrm{id}_{\widehat{\mathbb{C}}}$. Thus, ϕ is also the identity map, proving the claim. ■

Lemma A.2 implies that the map G_f has infinitely many fixed points when the map f is transcendental. According to Proposition 4.5, every such fixed point, apart from 0, 1, and ∞, corresponds to a realized Thurston map in $\mathcal{H}_{f,A}$. However, some of these maps might be combinatorially equivalent rel. A. Nevertheless, we will show that only finitely many of them can be pairwise combinatorially equivalent rel. A.

Consider two Thurston maps f_1 and f_2 realized rel. A by postsingularly finite holomorphic maps $g_1 \colon (\widehat{\mathbb{C}}, P_1) \circlearrowleft$ and $g_2 \colon (\widehat{\mathbb{C}}, P_2) \circlearrowleft$, respectively, where $P_1 = \{0, 1, \infty, x_1\}$ and $P_2 = \{0, 1, \infty, x_2\}$, with x_1 and x_2 being distinct fixed points of the map G_f. If f_1 and f_2 are combinatorially equivalent rel. A, then f_1 is realized rel. A by both $g_1 \colon (\widehat{\mathbb{C}}, P_1) \circlearrowleft$ and $g_2 \colon (\widehat{\mathbb{C}}, P_2) \circlearrowleft$. By item (1) of Theorem 4.4, there exists of a Möbius transformation M such that $M \circ g_1 = g_2 \circ M$ and $M(P_1) = P_2$. In particular, M is not the identity map and $\{0, 1, \infty\} \subset M(\{0, 1, \infty, x_1\})$. Since a Möbius transformation is uniquely determined by its values at three distinct points of $\widehat{\mathbb{C}}$, there can be at most 24 such Thurston maps that are pairwise combinatorially equivalent rel. A, and item (3) follows □

Let us now proceed to prove Corollary 1.1 from Section 1.3.2. First of all, we recall the definition of a *parameter space*.

Definition 4.9. Let $g \colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic map of finite type. Then the parameter space of g is defined as follows:

$$\mathrm{Par}(g) := \{\varphi \circ g \circ \psi \colon \mathbb{C} \to \widehat{\mathbb{C}} \text{ holomorphic for some } \varphi \in \mathrm{Homeo}^+(\widehat{\mathbb{C}}) \text{ and } \psi \in \mathrm{Homeo}^+(\mathbb{C})\}.$$

Proof of Corollary 1.1. Note that g should have at least two singular values and, according to Great Picard's Theorem, at most two *exceptional* values, i.e., points $w \in \widehat{\mathbb{C}}$ such that the preimage $g^{-1}(w)$ is finite. Clearly, every exceptional value is an asymptotic value of g. Therefore, by post-composing g with a Möbius transformation, we can assume that $\{0, \infty\} \subset S_g \subset \{0, 1, \infty\}$ and $g^{-1}(1)$ is infinite. By pre-composing g with an affine transformation, we can assume that $g(0) = 1$ and $g(1) = 1$. Let $x, y \neq 0, 1, \infty$ be two points in $\widehat{\mathbb{C}}$ so that $g(x) = y$.

Next, choose four distinct point a, b, c, and d in S^2, and two orientation-preserving homeomorphisms $\varphi, \psi \colon S^2 \to \widehat{\mathbb{C}}$ such that:

- $\varphi(a) = 0$, $\varphi(b) = 1$, $\varphi(c) = y$, and $\varphi(d) = \infty$, and
- $\psi(a) = 0$, $\psi(b) = x$, $\psi(c) = 1$, and $\psi(d) = \infty$.

Then $f := \varphi^{-1} \circ g \circ \psi \colon S^2 \dashrightarrow S^2$ is a topologically holomorphic map. Moreover, $\{a, d\} \subset S_f \subset \{a, b, d\}$, and $f(a) = b$, $f(b) = c$, and $f(c) = b$, while $d \in S_f$ is the essential singularity of the map f. In other words, f is a Thurston map with the postsingular set $P_f = \{a, b, c, d\}$. Moreover, it is easy to see that f satisfies conditions (I) and (II). Indeed, by setting $B = \{a, b, d\}$, we have $\overline{f^{-1}(B)} \cap P_f = \{a, c, d\}$.

Item (3) of Theorem 4.8 implies that the Hurwitz class $\mathcal{H}_f$ contains infinitely many realized Thurston maps that are pairwise combinatorially inequivalent. Clearly, each of these maps is realized by a postsingularly finite map from $\mathrm{Par}(g)$. Obviously, these maps must be pairwise (topologically or conformally) non-conjugate, leading to the desired result. □

Remark 4.10. If $g \colon \mathbb{C} \to \mathbb{C}$ is a non-constant entire map of finite type, its *entire parameter space* is defined by

$$\mathrm{Par}_E(g) := \{\varphi \circ g \circ \psi \colon \mathbb{C} \to \mathbb{C} \text{ holomorphic for some } \varphi, \psi \in \mathrm{Homeo}^+(\mathbb{C})\}.$$

Following the proof of Corollary 1.1, one can show that if g is a transcendental and $|S_g| \leqslant 3$, then $\mathrm{Par}_E(g)$ contains infinitely many postsingularly finite entire maps with four postsingular values that are pairwise non-conjugate.

Remark 4.11. Using the framework of *line complexes* (see [GO08, Section XI] or [MPR24, Section 2.7]), it can be shown that there are uncountably many distinct parameter spaces. Therefore, Corollary 1.1 implies that conditions (I) and (II) are met by uncountably many pairwise combinatorially inequivalent realized Thurston maps with four postsingular values. Furthermore, by applying item (1) of Theorem 4.4, it is easy to verify that the Thurston maps constructed in the proof Corollary 1.1 are not totally unobstructed if $z = 1$ is a regular point of the map g. This shows that the family of Thurston maps under consideration also includes uncountably many pairwise combinatorially inequivalent obstructed Thurston maps. According to Remark 4.10, the same observations hold even if we restrict to the class of entire Thurston maps.

4.3. **Examples.** In this section, we provide examples of several families of Thurston maps that satisfy conditions (I) and (II). We also demonstrate how the framework of Sections 4.1 and 4.2 applies to these rather concrete cases.

Example 4.12 (Exponential maps). Let $f \colon S^2 \dashrightarrow S^2$ be an entire Thurston map with $P_f = \{a, b, c, d\}$ and $S_f = \{a, d\}$, where d is the essential singularity of f. We recall that Thurston maps of this type are called *exponential* Thurston maps. It is easy to see that a must be an omitted value of the map f. In particular, f has one of the following two dynamical portraits on the set $\{a, b, c\}$ as illustrated in Figure 3: either the singular value a has pre-period 1 and period 2, or it has pre-period 2 and period 1.

FIGURE 3. Possible orbit of the singular value of an exponential Thurston map with four postsingular values.

In both cases, the map f satisfies properties (I) and (II). In particular, Theorem 4.4 shows that f is realized if and only if it has no Levy fixed curve, which must, in fact, be degenerate.

This, in particular, provides a new proof for the more general result [HSS09, Theorem 2.4] in the case of four postsingular values.

According to item (1) of Theorem 4.8, if the singular value a has pre-period 2 and period 1, then f it totally unobstructed. However, if a has pre-period 1 and period 2, the Thurston map f is never totally unobstructed since $f(\{b,c\}) = \{b,c\}$ and $\deg(f,b) = \deg(f,c) = 1$ because otherwise either b or c would be a singular value of f. Moreover, in this case the Hurwitz class $\mathcal{H}_f$ contains infinitely many pairwise combinatorially inequivalent obstructed Thurston maps by item (2) of Theorem 4.8. In both cases, item (3) of Theorem 4.8 states that the Hurwitz class of f contains infinitely many pairwise combinatorially inequivalent realized Thurston maps.

We further assume that $a = a_1$, $b = a_2$, $c = a_3$, and $d = a_4$, and adopt the notation introduced at the beginning of Section 4. Our goal is to derive explicit formula for the map G_f, analyze its dynamics, and observe the phenomena described in Propositions 4.5 and 4.6, as well as in the proof of Theorem 4.8.

First, we consider the case when the singular value a has pre-period 1 and period 2. Let $B = \{a,b,d\} = \{a_1,a_2,a_4\}$, and then $C = \overline{f^{-1}(B)} \cap P_f = \{a,c,d\} = \{a_1,a_3,a_4\}$. In particular, here $i = 3$ and $j = 2$.

Let us compute the map F_f. It is evident that F_f is a transcendental entire function. Moreover, $S_{F_f} = \{0,\infty\}$. By the classical theory of covering maps, $F_f(z) = \alpha\exp(\lambda z)$ for some $\alpha,\lambda \in \mathbb{C}^*$. Given that $F_f(0) = 1$ and $F_f(1) = 1$, it follows that $\alpha = 1$ and $\lambda = 2\pi i k, k \in \mathbb{Z}^*$, where k is determined by the Hurwitz equivalence class of f, as stated in item (6) of Proposition 4.1. At the same time, $W_f = \mathbb{C} - \{l/k : l \in \mathbb{Z}\}$ and $M_{3,2}(z) = 1/z$. Therefore, $G_f(z) = \exp(2\pi i k/z)$.

In particular, 0 is the essential singularity of G_f, 1 is a fixed repelling fixed point of G_f of multiplier $2\pi i k$, and $G_f(\infty) = 1$. Moreover, by Lemma A.2, the map G_f has infinitely many repelling fixed points. Thus, Propositions 4.5 and 4.6 already imply that the Hurwitz class $\mathcal{H}_f$ contains both realized and obstructed Thurston maps.

Now, let the singular value a has pre-period 2 and period 1. Let $B = \{a,c,d\} = \{a_1,a_3,a_4\}$ and then $C = \{b,c,d\} = \{a_2,a_3,a_4\}$. Here, $i = 2$ and $j = 1$. Similarly to the previous case, we find that $F_f(z) = \exp(2\pi i k z)$ with $k \in \mathbb{Z}^*$, $W_f = \mathbb{C} - \{l/k : l \in \mathbb{Z}\}$, $M_{2,1}(z) = z/(z-1)$, and $G_f(z) = \exp(2\pi i k/(z-1))$. In particular, 1 is the essential singularity of G_f, and $G_f(0) = G_f(\infty) = 1$. Thus, none of the points 0, 1, or ∞ is a fixed point of the map G_f. Therefore, Proposition 4.6 implies that f is indeed totally unobstructed.

Example 4.13 (Entire maps with three singular values). Let $f\colon S^2 \dashrightarrow S^2$ be an entire Thurston map with the postsingular set $P_f = \{a,b,c,d\}$, where $S_f = \{a,b,d\}$ and d is the essential singularity of f. If f satisfies condition (II), then it should have one of the three (up to relabeling) possible dynamical portraits on the set $\{a,b,c\}$ as illustrated in Figure 4. Additionally, there are four more dynamical portraits when condition (II) is not satisfied.

Theorem 4.4 states that a Thurston map with one of the dynamical portraits as in Figure 4 is realized if and only if it has no weakly degenerate Levy fixed curve. Furthermore, according to item (1) of Theorem 4.8, f is totally unobstructed for the first two dynamical portraits (from left to right) in Figure 4 if and only if $\deg(f,c) = 1$, and f is always totally unobstructed for the third dynamical portrait.

Let $a = a_1$, $b = a_2$, $c = a_3$, and $d = a_4$. Then we take $B = S_f = \{a,b,d\} = \{a_1,a_2,a_4\}$, and then $C = \{a,c,d\} = \{a_1,a_3,a_4\}$. In particular, $i = 3$ and $j = 2$. It can can be verified that F_f is an entire function with $S_{F_f} = \{0,1,\infty\}$. At the same time, $M_{3,2}(z) = 1/z$, and

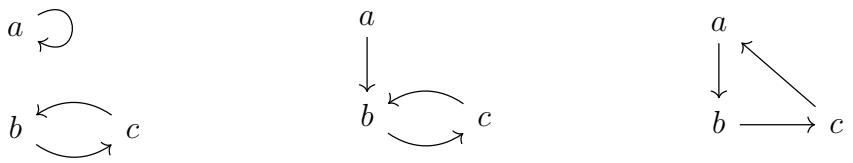

FIGURE 4. Possible dynamical portraits for an entire Thurston map with three singular and four postsingular values that satisfies condition (II).

therefore $G_f(z) = F_f(1/z)$. In particular, 0 is the essential singularity of G_f. Thus, we have the following behavior of the map G_f on the "cusps" 0 and 1 of the moduli space $\Sigma \sim \mathcal{M}_A$:

- for the first dynamical portrait, $G_f(\infty) = 0$ and 1 is a fixed point of G_f that, according to Lemma A.1, is repelling if $\deg(f, c) = 1$, or superattracting otherwise;
- for the second dynamical portrait, $G_f(\infty) = 1$ and 1 is a fixed point of G_f that is repelling if and only if $\deg(f, c) = 1$, or superattracting otherwise;
- for the third dynamical portrait, $G_f(1) = 0$ and $G_f(\infty) = 1$. In particular, neither of 0, 1, and ∞ is a fixed point of the map G_f.

Example 4.14 (Non-entire examples). Most of the observations in Example 4.13 do not depend on the condition $f^{-1}(d) = \varnothing$, i.e., that the Thurston map f is entire. Furthermore, there are more non-entire examples of Thurston maps that satisfy conditions (I) and (II). For instance, if $f\colon S^2 \dashrightarrow S^2$ is a Thurston map with $|S_f| \leqslant 3$, where the postsingular set $P_f = \{a, b, c, d\}$ does not contain an essential singularity (e.g., f could be a finite degree map), and f has one of the postsingular portraits shown in Figure 5. In particular, Theorem 4.4 provides a novel proof of celebrated Thurston's characterization theorem [DH93, Theorem 1] for a specific family of finite degree Thurston maps with four postcritical values.

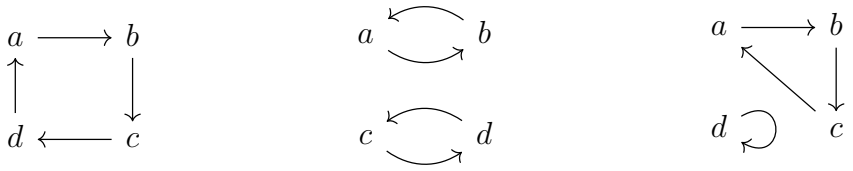

FIGURE 5. Examples of postsingular portraits of non-entire Thurston maps that satisfy conditions (I) and (II).

Of course, there are more examples, e.g., a Thurston map f with $P_f = \{a, b, c, d\}$ satisfies condition (II) if $S_f = \{a, b, c\}$, d is the essential singularity of the map f, $f(a) = d$, and b and c form a 2-cycle for the map f.

Example 4.15 (Maps with three postsingular values). Suppose that $f\colon (S^2, A) \circlearrowleft$ is a Thurston map such that $|A| = 4$ and $|P_f| \leqslant 3$. It is easy to see that if there exists a marked point $a \in A - P_f$ that is not periodic (i.e., it is either pre-periodic or lands to the essential singularity of f under the iteration), then the pullback map σ_f is constant by Proposition 2.16 and the Thurston map $f\colon (S^2, A) \circlearrowleft$ is realized, as noted in Remark 2.18.

On the other hand, if every marked point $a \in A - P_f$ is periodic, then conditions (I) and (II) are clearly satisfied. For instance, if $|P_f| = 3$, we can simply take $B = P_f$.

Appendix A. Few facts about dynamics of meromorphic maps

We require the following two results regarding the dynamics of meromorphic maps. Although these results are mostly folklore, we provide short proofs for the completeness.

Lemma A.1. *Every postsingularly finite meromorphic function has only finitely many superattracting periodic orbits, and all other periodic orbits are repelling.*

Proof. Let $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a postsingularly finite meromorphic function. It is evident that g can have only a finite number of superattracting periodic orbits, as the points of each such orbit belong to the postsingular set of g.

Now, consider a periodic point $z \in \mathbb{C}$ of g. If z is attracting (but not superattracting), then according to [Ber93, Theorem 7], the corresponding cycle of immediate attracting basins contains a singular value $w \in S_g$ that has an infinite orbit, leading to a contradiction. Therefore, if z is in the Fatou set of g, then it is the center of a cycle of Siegel disks $U_1, U_2, \dots, U_k$, and postsingular values of g are dense in ∂U_i for each $i = 1, 2, \dots, k$ [Ber93, Theorem 7]. This implies that the postsingular set of g would be infinite. If z is in the Julia set of g, then it is either a Cremer periodic point or it lies on the boundary of a cycle of parabolic basins [Mil06, Theorem 7.2]. In both cases, z is an accumulation point of the postsingular values of g (see [Ber93, Theorem 7] and [Eps93, Proposition 16]; see also [Eps93, Lemma 72]). Thus, z is either a superattracting or repelling periodic point of the function g. □

The next result that we require states that a transcendental meromorphic function of finite type has infinitely many repelling fixed points. This was established in the more general context of finite type maps in [Eps93, Proposition 14]. Furthermore, in the paper [Ben16], it was shown that the same is true for transcendental meromorphic functions of *bounded type* (i.e., having bounded singular set) under the assumption the ∞ is a *logarithmic singularity* of the considered function. In the following lemma, we show that this assumption can be removed and, in fact, the result holds for an arbitrary transcendental meromorphic function of bounded type.

Lemma A.2. *Every transcendental meromorphic function of bounded type has infinitely many repelling fixed points.*

Proof. Let $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function of bounded type and $D \subset \widehat{\mathbb{C}}$ be an open Jordan region containing ∞ such that $S_g \cap \overline{D} = \{\infty\} \cap \overline{D}$. If $f^{-1}(D)$ has a connected component that is unbounded (in $\mathbb{C}$), then the result follows directly from [Ben16]. Now, suppose that every connected component of $f^{-1}(D)$ is bounded. In this case, $f^{-1}(D)$ has infinitely many connected components, and all but finitely many of them are compactly contained in D. Let U be one such component, i.e., U is a connected component of $g^{-1}(D)$ such that $\overline{U} \subset D$. Let z be a unique pole of g in U and let $d := \deg(g, z)$. Note that U is an open Jordan region and $g|U - \{z\}\colon U - \{z\} \to D - \{\infty\}$ is covering map of degree d.

Consider a Jordan arc $\alpha \subset \overline{D}$ connecting ∞ with a point on ∂D, with the conditions that $|\alpha \cap \partial D| = 1$ and $\alpha \cap \overline{U} = \varnothing$. Then $g^{-1}(\alpha)$ subdivides U in d simply connected domains $U_1, U_2, \dots, U_d$. Furthermore, the restriction $g|U_i\colon U_i \to D - \alpha$ is a biholomorphism. Proposition 3.5 and Remark 3.6 imply that the inverse $(g|U_i)^{-1}$ is uniformly distance-decreasing with respect to the hyperbolic metric on $D - \alpha$, because $U_i = g(D - \alpha)$ is compactly contained in $D - \alpha$. Therefore, by the Banach fixed point theorem, g has a fixed point in each U_i for

$i = 1, 2, \ldots, d$. These fixed points are attracting for $(g|U_i)^{-1}$ and thus repelling for the map g. By applying the same argument to every connected component of $g^{-1}(D)$ that is compactly contained in D, we conclude that the map g has infinitely many repelling fixed points. □

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