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Strong asymptotic freeness of Haar unitaries in quasi-exponential dimensional representations

Michael Magee and Mikael de la Salle

Abstract

We prove almost sure strong asymptotic freeness of i.i.d. random unitaries with the following law: sample a Haar unitary matrix of dimension n and then send this unitary into an irreducible representation of $U(n)$. The strong convergence holds as long as the irreducible representation arises from a pair of partitions of total size at most $n^{\frac{1}{24}-\varepsilon}$ and is uniform in this regime.

Previously this was known for partitions of total size up to $\asymp \log n / \log \log n$ by a result of Bordenave and Collins.

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1 Introduction

Let $U(n)$ denote the group of complex $n \times n$ unitary matrices. For each $k, \ell \in \mathbb{N}$ there is a unitary representation

$$\pi_{k,\ell}^0 : U(n) \rightarrow U\left((\mathbb{C}^n)^{\otimes k} \otimes ((\mathbb{C}^n)^\vee)^{\otimes \ell}\right).$$

This representation has a non-zero invariant vector for $U(n)$ if and only if $k = \ell$, and in that case the space of invariant vectors is understood¹. Let $\pi_{k,\ell}$ be the restriction of $\pi_{k,\ell}^0$ to the orthocomplement to the invariant vectors.

Fix $r \in [n]$. The main theorem of this paper is the following.

¹The space of invariant vectors is the image of $\mathbb{C}[S_k]$ in $\text{End}((\mathbb{C}^n)^{\otimes k}) \cong (\mathbb{C}^n)^{\otimes k} \otimes ((\mathbb{C}^n)^\vee)^{\otimes k}$. If $n \geq k$ this is isomorphic to $\mathbb{C}[S_k]$ and we will always be in this regime in this paper. If $n < k$ the dimension of the space is still understood as the number of permutations in S_k with longest increasing subsequence $\leq n$ [BR01, §8].

Theorem 1.1. *Let $U_1^{(n)}, \dots, U_r^{(n)}$ denote i.i.d. Haar distributed elements of $U(n)$. For any $A < \frac{1}{24}$ the following holds almost surely. For any non-commutative $*$ -polynomial² p*

$$\sup_{k+\ell \leq n^A} \left\| \left\| \pi_{k,\ell} \left(p \left(U_1^{(n)}, \dots, U_r^{(n)} \right) \right) \right\| - \|p(x_1, \dots, x_r)\| \right\| = o(1)$$

where x_1, \dots, x_r are generators of a free group \mathbf{F}_r and the norm on the right is the one in $C_{\text{red}}^*(\mathbf{F}_r)$.

The analog of Theorem 1.1 with $U(n)$ replaced by S_n will appear in forthcoming work of E. Cassidy.

Theorem 1.1 gives a clean statement about strong convergence of i.i.d. Haar elements of $SU(n)$ in representations of quasi-exponential dimension in the following form:

Corollary 1.2. *Let $U_1^{(n)}, \dots, U_r^{(n)}$ denote i.i.d. Haar distributed elements of $SU(n)$. For any $A < \frac{1}{24}$ the following holds almost surely. For any non-commutative $*$ -polynomial p*

$$\sup_{\substack{\pi \in \widehat{SU(n)} \setminus \text{triv} \\ \dim(\pi) \leq \exp(n^A)}} \left\| \left\| \pi \left(p \left(U_1^{(n)}, \dots, U_r^{(n)} \right) \right) \right\| - \|p(x_1, \dots, x_r)\| \right\| = o(1).$$

Our proof of Theorem 1.1 relies on two things:

- a recent breakthrough of Chen, Garza-Vargas, Tropp, and van Handel [CGVTvH24] who found a new (and remarkable) approach to strong convergence based on ‘differentiation with respect to n^{-1} ’. We add to this method in various ways in the sequel. One of the conceptual differences is a new criterion that we find for temperedness of a unitary representation of a free group (or more generally a group with the rapid decay property), and therefore for strong convergence towards the regular representation, see §5. This allows us to bypass the use of Pisier’s linearization argument, which was essential in many proofs of strong convergence so far (a notable exception is the work of Parraud and Collins-Guionnet-Parraud [CGP22, Par23a, Par23b, Par22]), and replace it by considerations on random walks on free groups, see §6.

²That is, a polynomial in r non-commuting indeterminates X_1, \dots, X_r and their formal adjoints X_1^*, \dots, X_r^* .

- The method above is given as input rapid decay estimates for expected values of stable characters of word maps on $U(n)$. The key feature of these estimates is that, up to a point, they actually improve with the complexity of the representations that we consider. This feature is intimately related to the concept of *stable commutator length* in free groups as uncovered in [MP19].

Theorem 1.1 was proved when $k = 1, \ell = 0$ by Collins and Male in [CM14]. This resolved a problem left open in Haagerup and Thorbjørnsen's breakthrough work [HT05]. Bordenave and Collins [BC22] prove Theorem 1.1 in the regime

$$k + \ell \leq c \frac{\log(n)}{\log \log(n)}$$

for some positive c .

Theorem 1.1 can be written in the following two alternate ways.

- Let $V_i^{(n)} \stackrel{\text{def}}{=} \bigoplus_{k+\ell \leq n^A} \pi_{k,\ell}(U_i^{(n)})$. The random matrices $V_i^{(n)}$ are (almost surely) *strongly asymptotically free*.
- The random unitary representations of \mathbf{F}_r described by $x_i \mapsto V_i^{(n)}$ almost surely *strongly converge* to the regular representation of \mathbf{F}_r .

The following simple case of Theorem 1.1 is both new and important.

Corollary 1.3. *For any $A < \frac{1}{24}$, almost surely, for all $k, \ell \in \mathbb{N} \cup \{0\}$ such that $k + \ell \leq n^A$*

$$\left\| \sum_{i=1}^r \pi_{k,\ell}(U_i^{(n)}) + \pi_{k,\ell}(U_i^{(n)})^{-1} \right\| = 2\sqrt{2r-1} + o_{n \rightarrow \infty}(1).$$

We highlight this corollary in connection with the following well known question of Gamburd, Jakobson, and Sarnak [GJS99, pg. 57] — do generic in (Haar) measure $(u_1, \dots, u_r) \in \text{SU}(2)^r$ have some $\epsilon > 0$ such that for all $k > 0$

$$\left\| \sum_{i=1}^r \pi_k(u_i) + \pi_k(u_i)^{-1} \right\| \leq 2r - \epsilon?$$

It was proved by Bourgain and Gamburd in [BG08] for u_i with algebraic entries that generate a dense subgroup of $\text{SU}(2)$. This was extended to

$SU(n)$ for general $n \geq 3$ by Bourgain and Gamburd [BG12] — in this case the spectral gap is uniform over all representations $\pi_{k,\ell}$ with $k + \ell > 0$. The analogous result for all compact simple Lie groups was obtained by Benoist and de Saxce [BdS16].

Of course, Corollary 1.3 does not advance these questions for any fixed n as it deals only with infinite sequences of random matrices of dimension $n \rightarrow \infty$ but it does make significant progress on a relaxation of the problem. The motif of Theorem 1.1 is that we have only a small amount of randomness of our matrices compared to their dimensions.

The method of proof also allows us to obtain estimates for large matrix coefficients. For example, we have the following result, which relies on the ideas from [CGVTvH24] and the results of [Par23a].

Theorem 1.4. *Let $k_n \leq \exp(n^{\frac{1}{2}}(\log n)^{-4})$. For every q and every sequence P_n of non-commutative $*$ -polynomial of degree $\leq q$ and with coefficients in $M_{k_n}(\mathbb{C})$, almost surely*

$$\lim_n \frac{\|P_n(U_1^{(n)}, \dots, U_r^{(n)})\|}{\|P_n(x_1, \dots, x_r)\|} = 1.$$

Bordenave and Collins [BC23, Theorem 1.1] proved the same theorem for $q = 1$ and $k_n \leq \exp(n^{\frac{1}{32r+160}})$ instead of $k_n = \exp(n^{\frac{1}{2}+o(1)})$; this result gives yet another proof of the result from [BC24], which has important consequences in the theory of von Neumann algebras [Hay22, HJE24]. It is an intriguing question whether the exponent $\frac{1}{2}$ is optimal or not. As observed by Pisier [Pis14], the optimal exponent is necessarily at most 2.

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2 Preliminaries

2.1 Representation theory of $U(n)$

For partitions $\lambda = (\lambda_1, \dots, \lambda_p) \vdash k, \mu = (\mu_1, \dots, \mu_q) \vdash \ell$, for $n \geq p + q$ let $s_{\lambda, \mu} : U(n) \rightarrow \mathbb{C}$ denote the character of the representation

$$(\pi_{\lambda, \mu}, V^{\lambda, \mu})$$

with dominant weight

$$(\lambda_1, \lambda_2, \dots, \lambda_p, \underbrace{0, 0, \dots, 0}_{n - (p + q)}, -\mu_q, -\mu_{q-1}, \dots, -\mu_1).$$

Because this character interpolates for all $n \geq p + q$, we refer to it as a *stable* character. In the paper we write $|\lambda| = k$ for the size of λ .

2.2 Representation theory of $SU(n)$

The irreducible unitary representations of $SU(n)$ are closely related to those of $U(n)$: for every partitions λ, μ as before with $n \geq p + q$, the restriction of $\pi_{\lambda, \mu}$ to $SU(n)$ is an irreducible representation; all irreducible representations of $SU(n)$ appear in this way; and the restrictions of $\pi_{\lambda, \mu}$ to $SU(n)$ and $\pi_{\lambda', \mu'}$ to $SU(n)$ coincide if and only if the dominant weights differ by a multiple of $(1, \dots, 1)$. In other words, the irreducible unitary representations of $SU(n)$ are indexed by the sets of integral dominant weights of the root system A_{n-1} , which can be parametrized by the set of n -tuples of integers $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \mathbb{Z}^n$ with $\Lambda_1 \geq \dots \geq \Lambda_n$, modulo the subgroup $\mathbb{Z}(1, \dots, 1)$.

Write $\|\Lambda\|_1$ for the norm on $\mathbb{R}^n / \mathbb{R}(1, \dots, 1)$

$$\|\Lambda\|_1 = \inf_{t \in \mathbb{R}} \sum_i |\Lambda_i - t|.$$

It is the natural norm in the dual of the subspace of $(\mathbb{R}^n, \|\cdot\|_\infty)$ for which the coordinates sum to 0. In what follows, whenever Λ belongs to the

quotient $\mathbb{R}^n/(\mathbb{R}(1, \dots, 1))$, we will chose a representative $(\Lambda_1, \dots, \Lambda_n) \in \mathbb{R}^n$ such that $\sum_i |\Lambda_i| = \|\Lambda\|_1$. If Λ belongs to the image of \mathbb{Z}^n , we can and will assume that $(\Lambda_1, \dots, \Lambda_n)$ has integer coordinates.

By Weyl's dimension formula, if π is an irreducible representation of $\mathrm{SU}(n)$ with highest weight Λ ,

$$\dim(\pi) = \prod_{1 \leq i < j \leq n} \frac{j - i + \Lambda_i - \Lambda_j}{j - i}. \quad (2.1)$$

Lower bounds for $\dim(\pi)$ in terms of Λ have been obtained in [GLM12], but we will need a bound that is more precise³ in the regime $\|\Lambda\|_1 = o(n)$.

Proposition 2.1. *There is a constant $c > 0$ such that for every integer $n \geq 2$ and every irreducible representation π of $\mathrm{SU}(n)$,*

$$\exp(c \min(\|\Lambda\|_1, n)) \leq \dim(\pi) \leq \exp(\|\Lambda\|_1 \log n),$$

where Λ is the highest weight of π .

Proof. The upper bound is because π appears as a sub-representation of the representation of $\mathrm{SU}(n)$ on $(\mathbb{C}^n)^k \otimes (\overline{\mathbb{C}^n})^{\otimes \ell}$, where k is the sum of the positive Λ_i 's and $-\ell$ is the sum of the negative Λ_i 's. And $(\mathbb{C}^n)^k \otimes (\overline{\mathbb{C}^n})^{\otimes \ell}$ has dimension $(k + \ell)n = \exp(\|\Lambda\|_1 \log n)$.

For the lower bound, by Weyl's dimension formula (2.1) we have to prove

$$c \min(\|\Lambda\|_1, n) \leq \sum_{i < j} \log \left(1 + \frac{\Lambda_i - \Lambda_j}{j - i} \right). \quad (2.2)$$

We will prove this inequality for every $\Lambda \in \mathbb{R}^n$ with $\Lambda_1 \geq \dots \geq \Lambda_n$. In that case, since the right-hand side increases if we replace Λ by $t\Lambda$ for $t > 1$, it is enough to consider the case when $\|\Lambda\|_1 \leq n$. Finally, replacing Λ by $(-\Lambda_n, \dots, -\Lambda_1)$, we can assume that the sum of the positive entries of Λ is at least $\|\Lambda\|_1/2$.

Let $n_1 = \lfloor \frac{n}{2} \rfloor$ and $n_2 = \lfloor \frac{3n}{4} \rfloor$. Expressing that the ℓ_1 norm of $(\Lambda_1, \dots, \Lambda_n)$ is at most the ℓ_1 norm of $(\Lambda_1 - t, \dots, \Lambda_n - t)$ for every $t \neq 0$, we see that Λ has at most $\frac{n}{2}$ positive entries, and at most $\frac{n}{2}$ negative entries. In particular, we have

$$\sum_{i=1}^{n_1} \Lambda_i \geq \frac{1}{2} \|\Lambda\|_1, \quad (2.3)$$

³For example if $\Lambda = (k, 0, \dots, 0)$, our bound gives $\dim(\pi) \geq \exp(ck)$, whereas [GLM12] gives $\dim(\pi) \geq (k+1)^{\log(n-1)}$.

and $\Lambda_j \leq 0$ for every $n_2 < j \leq n$. If $i \leq n_1$ and $j > n_2$, we have

$$\frac{\Lambda_i - \Lambda_j}{j - i} \geq \frac{\Lambda_i}{n}.$$

We deduce the obvious bound

$$\sum_{i < j} \log \left(1 + \frac{\Lambda_i - \Lambda_j}{j - i} \right) \geq \sum_{i \leq n_1, n_2 < j} \log \left(1 + \frac{\Lambda_i}{n} \right).$$

By the bound $\log(1 + t) \geq t \log(2)$ for every $0 \leq t \leq 1$ and the inequality $0 \leq \Lambda_i \leq \|\Lambda\|_1 = n$, this is at most

$$\log(2) \sum_{1 \leq i \leq n_1} \sum_{n_2 < j \leq n} \frac{\Lambda_i}{n} \geq \frac{\log(2)}{4} \sum_{i=1}^{n_1} \Lambda_i.$$

By (2.3), we obtain (2.2) with $c = \frac{\log(2)}{8}$. \square

Corollary 2.2. *For every $0 < A \leq 1$ and every $n \geq N$, for every irreducible representation π of $\mathrm{SU}(n)$:*

$$\dim(\pi) < \exp(cn^A)$$

implies

$$\pi \subset \bigoplus_{k+\ell \leq n^A} \pi_{k,\ell},$$

which in turn implies

$$\dim(\pi) \leq \exp(n^A \log n).$$

3 Matrix integral results

Let \mathbf{F}_r denote the free group on a fixed generating set $X \stackrel{\text{def}}{=} \{x_1, \dots, x_r\}$. For $w \in \mathbf{F}_r$ let

$$w : \mathrm{U}(n)^r \rightarrow \mathrm{U}(n)$$

denote the induced word map defined by substituting in an r -tuple of unitaries for the elements of X appearing in a reduced expression of w . For example if $w = x_1 x_2 x_1^{-1} x_2^{-1}$ then $w(u_1, u_2, \dots, u_r) = u_1 u_2 u_1^{-1} u_2^{-1}$. Let $|w|$ denote the word length of w w.r.t. X .

Let

$$\mathbb{E}_n[s_{\lambda,\mu}(w)] \stackrel{\text{def}}{=} \int_{U(n)^r} s_{\lambda,\mu}(w(u_1, u_2, \dots, u_r)) du_1 \cdots du_r. \quad (3.1)$$

For $L \in \mathbb{N}$ let

$$g_L(x) \stackrel{\text{def}}{=} \prod_{c=1}^L (1 - c^2 x^2)^{\lfloor \frac{L}{c} \rfloor}. \quad (3.2)$$

The input into our analysis is the following result about expected values of stable characters of word maps, building on the works [MP19, Mag22, Mag21].

Theorem 3.1. *Let $w \in \mathbf{F}_r$ be not the identity with $|w| \leq q$.*

1. *There is a polynomial $P_{\lambda,\mu,w} \in \mathbb{Q}[x]$ such that for $n \geq (k + \ell)|w|$,*

$$\mathbb{E}_n[s_{\lambda,\mu}(w)] = \frac{P_{\lambda,\mu,w}\left(\frac{1}{n}\right)}{g_{(k+\ell)q}\left(\frac{1}{n}\right)},$$

with

$$\deg(P_{\lambda,\mu,w}) \leq 3(k + \ell)q(1 + \log((k + \ell)q)).$$

2. *If w is not a proper power in \mathbf{F}_r , then $\mathbb{E}_n[s_{\lambda,\mu}(w)] = O(n^{-\frac{1}{3}(k+\ell)})$.*
3. *If w is not a proper power in \mathbf{F}_r and $\mu = \emptyset$, then $\mathbb{E}_n[s_{\lambda,\emptyset}(w)] = O(n^{-(k+\ell)})$.*

It is probable that the conclusion of Part 2 also holds with $O(n^{-(k+\ell)})$, but we are unsure how to prove this at the moment, and it just changes constants in our theorems.

Part 3 of Theorem 3.1 follows from combining a result of [MP19] with one of Duncan and Howie [DH91] as explained in §11.2.

Theorem 3.1 will be proved in the later part of the paper (§9-11).

In §4-7 we prove Theorem 1.1 from Theorem 3.1.

4 Markov brothers inequality

We use the following fundamental inequality by Andrey and Vladimir Markov, see [CGVTvH24, Section 4.1] for references. If P is a degree $\leq D$ polynomial

in one variable, then for every integer k ,

$$\sup_{[-1,1]} |P^{(k)}| \leq \frac{D^2(D^2-1)\dots(D^2-(k-1)^2)}{(2k-1)!!} \sup_{[-1,1]} |P| \quad (4.1)$$

where $(2k-1)!! = 1 \cdot 3 \dots (2k-1)$. By an affine change of variable, for a general interval (4.2) becomes:

$$\sup_{[a,b]} |P^{(k)}| \leq \frac{2^k}{(b-a)^k} \frac{D^2(D^2-1)\dots(D^2-(k-1)^2)}{(2k-1)!!} \sup_{[a,b]} |P|. \quad (4.2)$$

We deduce the following, which is essentially [CGVTvH24, Lemma 4.2].

Lemma 4.1. *For every polynomial P of degree $\leq D$ and every integer $N \geq D^2$*

$$\sup_{[0, \frac{1}{N}]} |P| \leq \frac{1}{1 - \frac{D^2}{N+1}} \sup_{n \geq N} \left| P \left(\frac{1}{n} \right) \right|.$$

Proof. Every element of the interval $[0, \frac{1}{N}]$ is at distance at most $\frac{1}{2N(N+1)}$ from $\{\frac{1}{n} \mid n \geq N\}$. Therefore, by the fundamental inequality of calculus, we have

$$\sup_{[0, \frac{1}{N}]} |P| \leq \sup_{n \geq N} \left| P \left(\frac{1}{n} \right) \right| + \frac{1}{2N(N+1)} \sup_{[0, \frac{1}{N}]} |P'|.$$

By the Markov brothers inequality (4.2) with $k = 1$, this is less than

$$\sup_{n \geq N} \left| P \left(\frac{1}{n} \right) \right| + \frac{1}{2N(N+1)} \cdot 2ND^2 \sup_{[0, \frac{1}{N}]} |P|.$$

The lemma follows. \square

As a consequence,

Lemma 4.2. *For every polynomial P of degree $\leq D$ and every integer $k \leq D$,*

$$\sup_{[0, \frac{1}{2D^2}]} |P^{(k)}| \leq \frac{2^{2k+1} D^{4k}}{(2k-1)!!} \sup_{n \geq D^2} \left| P \left(\frac{1}{n} \right) \right|.$$

Proof. By (4.2) with $a = 0$ and $b = \frac{1}{2D^2}$,

$$\sup_{[0, \frac{1}{2D^2}]} |P^k| \leq (4D^2)^k \frac{D^2(D^2-1)\dots(D^2-(k-1)^2)}{(2k-1)!!} \sup_{[0, \frac{1}{2D^2}]} |P|.$$

The lemma follows from the bound $D^2(D^2-1)\dots(D^2-(k-1)^2) \leq D^{2k}$ and Lemma 4.1. \square

5 A criterion for strong convergence

In this section, Γ will be a finitely generated group with a fixed finite generating set and corresponding word-length. We denote by $\mathbb{C}_{\leq q}[\Gamma]$ the subspace of the elements of the group algebra of Γ supported in the ball of radius q .

Definition 5.1. A function $u: \Gamma \rightarrow \mathbb{C}$ is tempered if

$$\limsup_{n \rightarrow \infty} |u((x^*x)^n)|^{\frac{1}{2n}} \leq \|\lambda(x)\|$$

for every $x \in \mathbb{C}[\Gamma]$. Here λ is the left-regular representation of Γ on $\ell_2(\Gamma)$ and the norm is the operator norm.

A function $u: \Gamma \rightarrow \mathbb{C}$ is completely tempered if

$$\limsup_{n \rightarrow \infty} |\text{Tr} \otimes u((x^*x)^n)|^{\frac{1}{n}} \leq \|(\text{id} \otimes \lambda)(x)\|$$

for every integer $k \geq 1$ and every $x \in M_k(\mathbb{C}) \otimes \mathbb{C}[\Gamma]$, where Tr is the trace on $M_k(\mathbb{C})$.

For example, if $u(\gamma) = \langle \pi(\gamma)\xi, \xi \rangle$ is a matrix coefficient of a unitary representation of Γ , then u is tempered if and only if it is completely tempered, if and only if the cyclic representation generated by ξ is weakly-contained in the left-regular representation. So Definition 5.1 can be seen as an adaptation to functions that are not related to representations of the classical notion of tempered representation.

This section is devoted to the proof of the following criterion for strong convergence in probability of a sequence of random representations.

Proposition 5.2. *Let π_n be a sequence of random unitary representations of finite nonrandom dimension, $u_n: \Gamma \rightarrow \mathbb{C}$ be functions and $\varepsilon_n > 0$. Assume that*

- u_n is tempered and there is a polynomial P_n such that for every q and every $x \in \mathbb{C}_{\leq q}[\Gamma]$, $|u_n(x)| \leq P_n(q)\|x\|_{C^*(\Gamma)}$,
- $|\mathbb{E}\text{Tr}(\pi_n(x)) - u_n(x)| \leq \varepsilon_n \exp(\frac{q}{\log(2+q)^2})\|x\|_{C^*(\Gamma)}$ for every q and every $x \in \mathbb{C}_{\leq q}[\Gamma]$,
- $\lim_n \varepsilon_n = 0$.

Then for every $y \in \mathbb{C}[\Gamma]$, and every $\delta > 0$,

$$\lim_n \mathbb{P}(\|\pi_n(y)\| \geq \|\lambda(y)\| + \delta) = 0.$$

If $C_\lambda^*(\Gamma)$ has a unique trace, then the conclusion becomes that

$$\lim_n \mathbb{P}(|\|\pi_n(y)\| - \|\lambda(y)\|| \geq \delta) = 0.$$

Remark 5.3. The choice of $w(q) = \exp\left(\frac{q}{\log(2+q)^2}\right)$ is an arbitrary choice that is convenient for our applications, the crucial property that is used is that $\sum_q \frac{1}{1+q^2} \log w(q) < \infty$, reminiscent of the Beurling-Malliavin Theorem. It could be replaced by $\exp(q^2 u_q)$ for an arbitrary decreasing and summable sequence $(u_q)_{q \geq 0}$.

Remark 5.4. Proposition 5.2 is a variant with high derivatives of the criterion for strong convergence that appeared implicitly, for polynomial w , in [HT05] with $u_n(\gamma) = 1_{\gamma=1}$, and [Sch05] for general u_n .

Remark 5.5. The condition on the existence of P_n is probably not needed, but will be trivially satisfied in the applications. It allows to use standard results on distributions rather than ad-hoc proofs.

We have the following variant for matrix coefficients.

Proposition 5.6. *Let $\pi_n, d_n, u_n, \varepsilon_n$ be as in Proposition 5.2. Assume moreover that u_n is completely tempered. Let k_n be a sequence of integers such that $\lim_n \varepsilon_n k_n = 0$.*

Then for every integer q , $\delta > 0$ and every sequence $y_n \in \mathbb{C}_{\leq q}[\Gamma] \otimes M_{k_n}(\mathbb{C})$ with $\|y_n\|_{C^(\Gamma) \otimes M_{k_n}} \leq 1$,*

$$\lim_n \mathbb{P}(\|\pi_n(y_n)\| \geq \|\lambda(y_n)\| + \delta) = 0.$$

If $C_\lambda^(\Gamma)$ has a unique trace and is exact, then the conclusion becomes that*

$$\lim_n \mathbb{P}(|\|\pi_n(y_n)\| - \|\lambda(y_n)\|| \geq \delta) = 0.$$

We first collect a few ingredients that we need for the proof.

5.1 Bump functions with small Fourier coefficient

A nonzero compactly supported continuous function cannot have a Fourier transform that decays too fast at infinity, as it has to satisfy $\int \frac{\log|\hat{f}(t)|}{1+t^2} dt > -\infty$. In other words, the condition $\int_0^\infty \frac{\varphi(t)}{1+t^2} dt < \infty$ on a function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a necessary condition for the existence of a nonzero compactly supported continuous function f such that $|\hat{f}(t)| \leq \exp(-\varphi(|t|))$ for every t . The Beurling-Malliavin theorem asserts that this is often also a sufficient condition, see for example [MNH06]. The following elementary fact is an illustration of the same phenomenon. The small nuance compared to the Beurling-Malliavin theorem is that we need that f be nonnegative. The same proof works for $\varphi(t) = \frac{t}{(\log(2+|t|))^{1+\varepsilon}}$ replaced by any continuous function $[0, \infty) \rightarrow [0, \infty)$ such that $\frac{\varphi(t)}{t^2}$ is non-increasing and integrable on $[1, \infty)$.

Lemma 5.7. *For every $\varepsilon > 0$, there is a real number $M > 0$ and an even function C^∞ function $f: \mathbb{R} \rightarrow \mathbb{R}$ that is strictly positive on $(-1, 1)$, zero outside of $(-1, 1)$, and such that*

$$|\hat{f}(t)| \leq \exp\left(-M \frac{|t|}{(\log(2+|t|))^{1+\varepsilon}}\right).$$

Proof. The proof is a standard construction of a smooth bump function as infinite convolution product of indicator functions [Kat04, Lemma V.2.7]. Define $a_j = \frac{c}{j(\log 2+j)^{1+\varepsilon}}$, where $c > 0$ is chosen such that $\sum_{j \geq 1} a_j = 1$. Let $(X_j)_{j \geq 1}$ be a sequence of independent random variables, all uniform in $[-1, 1]$. Let μ be the law of $\sum_j a_j X_j$. It is a measure with full support in $[-\sum_j a_j, \sum_j a_j] = [-1, 1]$. Then

$$\hat{\mu}(t) = \mathbb{E} \exp(-it \sum a_j X_j) = \prod_{j \geq 1} \frac{\sin(ta_j)}{ta_j}.$$

The inequality

$$|\hat{\mu}(t)| \leq \prod_{j \leq \frac{t}{(\log 2+t)^{1+\varepsilon}}} \frac{1}{|t|a_j} \leq \exp\left(-M \frac{|t|}{(\log 2+t)^{1+\varepsilon}}\right)$$

is a computation. It implies that μ is absolutely continuous with respect to the Lebesgue measure and that its Radon-Nikodym derivative, which is our f , is C^∞ . f does not vanish on the interval $(-1, 1)$ because f is a log-concave function as a limit of convolutions of log-concave functions. \square

Remark 5.8. In this lemma, we have tried to optimize the rate of decay of the Fourier transform of f , because this is what allows to take A the largest in Theorem 1.1. But this is reflected by the fact that $f(t)$ is extremely small when $t < 1$ is close to 1, something like $\exp(-\exp(O(\frac{1}{1-t})))$, which gives rather bad quantitative estimates for the speed of the convergence. For smaller values of $K \ll n^A$, it would better to use functions with slower decay of \hat{f} .

We can periodize the above function:

Lemma 5.9. *For every $0 < \alpha < \pi$, there is a nonnegative real-valued even function $\varphi_\alpha \in \mathbb{C}^\infty(\mathbb{R}/2\pi\mathbb{Z})$ that is strictly positive on $(-\alpha, \alpha) + 2\pi\mathbb{Z}$, zero outside of $(-\alpha, \alpha) + 2\pi\mathbb{Z}$ with support equal to $[\alpha, \alpha]$ and such that*

$$|\hat{\varphi}_\alpha(k)| \leq \exp\left(-M \frac{|k|\alpha}{\log(2 + |k|\alpha)^{1+\varepsilon}}\right)$$

for every $k \in \mathbb{Z}$.

Proof. Set $\varphi_\alpha(t + 2\pi\mathbb{Z}) = f(t/\alpha)$ for every $t \in [-\pi, \pi]$. □

5.2 Sobolev algebras in the full group C^* algebra

Let $(w(q))_{q \in \mathbb{N}}$ be a non-decreasing sequence of positive real numbers satisfying $w(q_1 + q_2) \leq w(q_1)w(q_2)$. Set

$$\mathcal{S}_w(\Gamma) = \left\{ x \in C^*(\Gamma) \mid x = \sum_q x_q, x_q \in \mathbb{C}_{\leq q}[\Gamma], \sum_q w(q) \|x_q\|_{C^*(\Gamma)} < \infty \right\}.$$

It is a Banach algebra for the norm

$$\|x\|_w := \inf \left\{ \sum_q w(q) \|x_q\|_{C^*(\Gamma)} \mid x = \sum_q x_q, x_q \in \mathbb{C}_{\leq q}[\Gamma] \right\}.$$

For $w(q) = (1 + q)^i$ we denote the corresponding space $\mathcal{S}_i(\Gamma)$.

We have the following elementary fact:

Lemma 5.10. *A function $u: \Gamma \rightarrow \mathbb{C}$ extends by linearity to a linear map $\mathcal{S}_w(\Gamma) \rightarrow \mathbb{C}$ of norm $\leq N$ if and only if for every q and every $x \in \mathbb{C}_{\leq q}[\Gamma]$,*

$$|u(x)| := \left| \sum_\gamma x(\gamma)u(\gamma) \right| \leq Nw(q) \|x\|_{C^*(\Gamma)}.$$

The next lemma is useful and suggests the terminology of Sobolev algebras for the algebras $\mathcal{S}_i(\Gamma)$.

Lemma 5.11. *Let $y = y^* \in \mathbb{C}_{\leq q}[\Gamma]$ and $f : [-\|y\|_{C^*(\Gamma)}, \|y\|_{C^*(\Gamma)}] \rightarrow \mathbb{R}$ be a continuous function. Let $w : \mathbb{N} \rightarrow \mathbb{R}$ as above. Assume that the function $\varphi(\theta) = f(\|y\|_{C^*(\Gamma)} \cos \theta)$ satisfies $\sum_{n \in \mathbb{Z}} w(q|n|)|\hat{\varphi}(n)| < \infty$. Then $f(y) \in \mathcal{S}_w(\Gamma)$ and*

$$\|f(y)\|_w \leq \sum_{n \in \mathbb{Z}} w(q|n|)|\hat{\varphi}(n)|.$$

In particular, if f is C^{i+1} , then $f(y) \in \mathcal{S}_i(\Gamma)$ with norm $\leq C_i(1 + \|y\|_{C^(\Gamma)})^{i+1}(1+q)^i\|f\|_{C^{i+1}}$.*

Proof. In the proof, we write $\|y\|$ for $\|y\|_{C^*(\Gamma)}$. The assumption implies that the Fourier coefficients of φ are absolutely summable, which gives rise to a norm-converging expansion

$$f(y) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n) T_n(y/\|y\|)$$

where T_n is the n -th Chebyshev polynomial $T_n(\cos \theta) = \cos(n\theta)$. So $T_n(y/\|y\|)$ belongs to $\mathbb{C}_{\leq q|n|}[\Gamma]$, has norm ≤ 1 in $C^*(\Gamma)$ and

$$\sum_n w(q|n|)|\hat{\varphi}(n)| \|T_n(y/\|y\|)\|_{C^*(\Gamma)} \leq \sum_n w(q|n|)|\hat{\varphi}(n)|.$$

This proves the first part of the lemma. For the second part, if f is C^{i+1} , then so is φ , and by the Cauchy-Schwarz inequality

$$\sum_n (1+q|n|)^i |\hat{\varphi}(n)| \leq \left(\sum_n (1+q|n|)^{-2} \right)^{\frac{1}{2}} \left(\sum_n (1+q|n|)^{2i+2} |\hat{\varphi}(n)|^2 \right)^{\frac{1}{2}},$$

which is finite. □

Proposition 5.12. *Let $u : \Gamma \rightarrow \mathbb{C}$. Assume that u extends to a continuous linear map $u : \mathcal{S}_i(\Gamma) \rightarrow \mathbb{C}$. Then u is tempered if and only if there is n such that for every selfadjoint $x \in \mathcal{S}_i(\Gamma) \cap \ker \lambda$, $u(x^n) = 0$. In that case, this holds for all $n \geq i+2$.*

Furthermore, u is completely tempered if and only if there is n such that for every k and every selfadjoint $x \in M_k(\mathbb{C}) \otimes (\mathcal{S}_i(\Gamma) \cap \ker \lambda)$, $(\text{Tr} \otimes u)(x^n) = 0$. In that case, this holds for all $n \geq i+2$.

Proof. We prove the equivalence for temperedness, the proof for complete temperedness is identical.

We start with a preliminary observation. Lemma 5.11 tells us that for every self-adjoint $y \in \mathbb{C}_{\leq q}(\Gamma)$, the function $f \in C^\infty(\mathbb{R}) \mapsto u(f(y))$ is a compactly supported distribution of order $\leq i + 1$, and the largest symmetric interval containing its support is $[-\limsup |u(y^k)|^{\frac{1}{k}}, \limsup |u(y^k)|^{\frac{1}{k}}]$, see [CGVTvH24, Lemma 4.9].

Given this observation, the if direction is direct. Let $\varepsilon > 0$, and pick f be a C^∞ function equal to 0 on the interval $[-\|\lambda(y)\|^2, \|\lambda(y)\|^2]$ and to 1 outside of the ε -neighbourhood of it. For k large enough (so that $t \mapsto |t|^{k/n}$ is C^{i+2}), $x = (y^*y)^{k/n} f(y^*y)$ belongs to $\mathcal{S}_i(\Gamma) \cap \ker \lambda$. As a consequence, $u(x^n) = 0$ and therefore

$$u((y^*y)^k) = u((y^*y)^k(1 - f(y^*y)^n)).$$

But the C^{i+1} norm of $t \mapsto t^k(1 - f(t)^n)$ is $\leq C(i, \varepsilon, y)(\|\lambda(y)\|^2 + \varepsilon)^k$, which implies that

$$\limsup_k |u((y^*y)^k)|^{1/2k} \leq \sqrt{\|\lambda(y)\|^2 + \varepsilon}.$$

This proves that u is tempered.

For the converse, consider a self-adjoint $x \in \mathcal{S}_i(\Gamma) \cap \ker \lambda$. Decompose $x = \sum_q x_q$ as in the definition of $\mathcal{S}_i(\Gamma)$. We can moreover assume that each x_q is self-adjoint. Let $y_q = \sum_{s \leq q} x_s \in \mathbb{C}_{\leq q}[\Gamma]$. Then we have

$$\|\lambda(y_q)\| = \|\lambda(y_q - x)\| \leq \sum_{q > n} \|x_s\| = o((1 + q)^{-1}),$$

because $\sum_q (1 + q)^i \|x_q\| < \infty$. Let φ be a C^∞ function equal to 1 on $[-1, 1]$ and 0 outside of $[-2, 2]$, and let $\varphi_q(t) = \varphi(\frac{t}{\|\lambda(y_q)\|})$. We know from the preliminary observation that $\Lambda_q(f) = u(f(y_q))$ is a distribution with support inside $[-\|\lambda(y_q)\|, \|\lambda(y_q)\|]$ and such that $|\Lambda_q(f)| \leq C(1 + q)^i \|f\|_{C^{i+1}(\mathbb{R})}$ for some C independent from q . Therefore, for every n we have

$$u(y_q^n) = u(y_q^n \varphi_q(y_q)) = (1 + q)^i O_{q \rightarrow \infty}(\|\lambda(y_q)\|^{n-i-1}).$$

If $n \geq i - 2$ this goes to 0. Therefore, we have

$$u(x^n) = u(\lim_q y_q^n) = \lim_q u(y_q^n) = 0.$$

The proposition is proved. □

5.3 Unique trace and exactness

We use the following easy consequence of the operator-space reformulation of exactness, in the line of [BC23, Theorem 7.3].

Lemma 5.13. *Let A be an exact C^* -algebra. For every finite family a_1, \dots, a_n in A and every $\varepsilon > 0$, there is an integer d such that for every k and every $b_1, \dots, b_n \in M_k(\mathbb{C})$, there are norm ≤ 1 matrices $u, v \in M_{d,k}(\mathbb{C})$ such that*

$$(1 - \varepsilon) \left\| \sum_i a_i \otimes b_i \right\| \leq \left\| \sum_i a_i \otimes u b_i v^* \right\|.$$

Proof. If a_i belong to $M_N(\mathbb{C})$, the lemma is easy with $d = N$, because the norm of $X = \sum a_i \otimes b_i$ is the supremum of $\langle X\xi, \eta \rangle$ over norm 1 elements of $\mathbb{C}^N \otimes \mathbb{C}^k$, and every element of $\mathbb{C}^N \otimes \mathbb{C}^k$ belongs to $\mathbb{C}^N \otimes E$ for a subspace $E \subset \mathbb{C}^k$ of dimension $\leq N$. The general case follows by Kirchberg's characterization of exact C^* -algebras, which implies that for every $\varepsilon > 0$, there is N such that the operator space spanned by a_1, \dots, a_n is at completely bounded distance $\leq 1 + \varepsilon$ from a subspace of $M_N(\mathbb{C})$ [Pis03, Corollary 17.5]. \square

Lemma 5.14. *Let Γ be a finitely generated group. Assume that $C_\lambda^*(\Gamma)$ has a unique trace. Then for every integer q and $\varepsilon > 0$, there exist $y_1, \dots, y_n \in \mathbb{C}[\Gamma]$ and $\delta > 0$ such that, for every finite dimensional unitary representation π of Γ , if*

$$\max_i \frac{\|\pi(y_i)\|}{\|\lambda(y_i)\|} \leq 1 + \delta$$

then

$$\inf_{y \in \mathbb{C}_{\leq q}[\Gamma]} \frac{\|\pi(y)\|}{\|\lambda(y)\|} \geq 1 - \varepsilon. \quad (5.1)$$

If moreover $C_\lambda^*(\Gamma)$ is exact, the conclusion (5.1) can be strengthened to

$$\inf_{\substack{d \geq 1 \\ y \in M_d(\mathbb{C}) \otimes \mathbb{C}_{\leq q}[\Gamma]}} \frac{\|(\text{id} \otimes \pi)(y)\|}{\|(\text{id} \otimes \lambda)(y)\|} \geq 1 - \varepsilon.$$

Proof. We use the following fact: $C_\lambda^*(\Gamma)$ has a unique trace if and only if for every $\gamma \in \Gamma \setminus \{1\}$ and every ε , there is $\delta > 0$ and a finite family $y_1, \dots, y_k \in \mathbb{C}[\Gamma]$ such that, for every unitary representation $\pi: \Gamma \rightarrow A$ to a C^* -algebra A with a tracial state σ , $\max_i \frac{\|\pi(y_i)\|}{\|\lambda(y_i)\|} \leq 1 + \delta$ implies $|\sigma(\pi(\gamma))| \leq \varepsilon$. The only if direction is direct: if σ is an arbitrary tracial state on $C_\lambda^*(\Gamma)$, by applying the

criterion to $A = C_\lambda^*(\Gamma)$ and π the left-regular representation, we obtain that $|\sigma(\lambda(\gamma))| \leq \varepsilon$ for every $\gamma \neq 1$ and $\varepsilon > 0$, that is σ is the standard trace. The converse is proved by standard ultraproduct arguments (see [LM23, Lemma 6.1] for the details).

The first conclusion of the lemma follows quite easily from this characterization of the unique trace property. Indeed, if $y \in \mathbb{C}[\Gamma]$, there is an integer p such that $\|\lambda(y)\| \leq (1 + \varepsilon)\tau((y^*y)^p)^{\frac{1}{2p}}$, and by compactness the same p can be taken for every $y \in \mathbb{C}_{\leq q}[\Gamma]$. Now applying the criterion for every γ in the ball of radius $2qp$, we obtain that there is a finite family $y_1, \dots, y_k \in \mathbb{C}[\Gamma]$ such that, for π, A, σ as above, $\max_i \frac{\|\pi(y_i)\|}{\|\lambda(y_i)\|} \leq 1 + \delta$ implies that for every $y \in \mathbb{C}_{\leq q}[\Gamma]$, $\tau((y^*y)^p)^{\frac{1}{2p}} \leq (1 + \varepsilon)\sigma(\pi((y^*y)^p))^{\frac{1}{2p}}$, and in particular $\|\lambda(y)\| \leq (1 + \varepsilon)^2\|\pi(y)\|$. This is (5.1), up to a change of ε .

Observe that exactly the same argument also shows that, without any further assumption on $C_\lambda^*(\Gamma)$, for any fixed d , (5.1) can be replaced by

$$\inf_{y \in M_d(\mathbb{C}) \otimes \mathbb{C}_{\leq q}[\Gamma]} \frac{\|(\text{id} \otimes \pi)(y)\|}{\|(\text{id} \otimes \lambda)(y)\|} \geq 1 - \varepsilon.$$

But of course, y_i and δ a priori depend on d . If $C_\lambda^*(\Gamma)$ is exact, Lemma 5.13 allows to remove this dependence and to conclude the proof of the lemma. \square

5.4 Proof of Proposition 5.2 and Proposition 5.6

Proof of Proposition 5.2. Let i_n be the degree of P_n and $w(q) = \exp(\frac{q}{\log(2+q)^2})$. Lemma 5.10 implies that u_n extends by continuity to $\mathcal{S}_i(\Gamma)$, and that on its subspace $\mathcal{S}_w(\Gamma)$,

$$|\mathbb{E}\text{Tr}(\pi_n(x)) - u_n(x)| \leq \varepsilon_n \|x\|_w. \quad (5.2)$$

Without loss of generality, we can take $y \neq 0$ and normalize it so that $\|y\|_{C^*(\Gamma)} = 1$.

By Lemma 5.11, the map $f \mapsto u_n(f(y^*y/2))$, is continuous for the $C^{i_n+1}([-1/2, 1/2])$ topology, so it extends to a distribution. The assumption that u_n is tempered in turn implies that its support is contained in $[-\|\lambda(y)\|^2/2, \|\lambda(y)\|^2/2]$ (see [CGVTvH24, Lemma 4.9]). In particular, using $\|\lambda(y)\| > 0$, we have that $u_n(f(y^*y)) = 0$ for every f that vanishes on $[-\|\lambda(y)\|^2/2, \|\lambda(y)\|^2/2]$ [Hör03, Theorem 2.3.3].

Let $\alpha = \arccos(\|\lambda(y^*y)\|/2) \in [\frac{\pi}{3}, \frac{\pi}{2}]$. Let φ_α be the function given by Lemma 5.9, but for $\varepsilon = \frac{1}{2}$. It is even, so it is of the form $\varphi_\alpha(\theta) = f(\cos(\theta))$ for

a continuous function $f: [-1, 1] \rightarrow \mathbb{R}$ that is zero on $[-\|\lambda(y)\|^2/2, \|\lambda(y)\|^2/2]$, strictly positive outside of this interval, and C^∞ on $(-1, 1)$. We take $x = f(y^*y)$. By the preceding discussion, $u_n(x) = 0$ for every n . Moreover, by Lemma 5.11, x belongs to $\mathcal{S}_w(\Gamma)$, because

$$\sum_{N \geq 1} \exp\left(\frac{qN}{\log(1+qN)^2} - M \frac{\alpha N}{(\log(1+\alpha N))^{1+\frac{1}{2}}}\right) < \infty.$$

So the inequality (5.2) together with the obvious inequality $\|\pi_n(x)\| \leq \text{Tr} \pi_n(x)$ gives

$$\mathbb{E} \|\pi_n(x)\| \leq \varepsilon_n \|x\|_w,$$

which goes to zero by the last assumption.

Now, since f is strictly positive on $[-1, 1] \setminus [-\|\lambda(y)\|^2/2, \|\lambda(y)\|^2/2]$, there is $c > 0$ such that $f(t/2) > c$ if $|t| \geq (\|\lambda(y)\| + \delta)^2$. In particular, we have

$$\mathbb{P}(\|\pi_n(y)\| > \|\lambda(y)\| + \delta) \leq \mathbb{P}(\|\pi_n(x)\| > c) \leq \frac{1}{c} \varepsilon_n \|x\|_w,$$

which goes to 0. This proves the first part of the proposition.

The second half follows from the first half of Lemma 5.14. \square

Proof of Proposition 5.6. The first part is identical to that of Proposition 5.2, considering the distribution $f \mapsto \text{Tr} \otimes u_n(f(y_n^*y_n))$.

The second part follows, using the full statement of Lemma 5.14. \square

6 Random walks on free groups

The content of this section will play a key rôle in the proof of Theorem 1.1 (see Lemma 7.4).

6.1 Proper powers

Let μ be a symmetric probability measure on \mathbf{F}_r , whose support is finite, contains the identity element and generates \mathbf{F}_r . To save space, we will call such a measure **reasonable**. Let $(g_n)_{n \geq 0}$ be the corresponding random walk on \mathbf{F}_r , that is $g_n = s_1 s_2 \dots s_n$ for iid s_i distributed as μ . Let $\rho = \rho(\mu)$ be the spectral radius, that is the norm of $\lambda(\mu)$ on $\ell_2(\mathbf{F}_r)$. We have the easy bound

$$\forall g \in \mathbf{F}_r, \mathbb{P}(g_n = g) = \langle \lambda(\mu)^n \delta_e, \delta_g \rangle \leq \rho^n. \quad (6.1)$$

We say that an element of \mathbf{F}_r is a proper power if it is of the form h^d for $h \in \mathbf{F}_r$ and $d \geq 2$.

Proposition 6.1. *There is a constant $C = C(\mu)$ such that*

$$\mathbb{P}(g_n \text{ is a proper power}) \leq Cn^5\rho^n.$$

The proof uses that the Cayley graph of \mathbf{F}_r is a tree, in the following form:

Lemma 6.2. *There is a constant C and an integer a such that for every $h \in \mathbf{F}_r$ and every h' on the segment from the identity to h ,*

$$\mathbb{P}(g_n = h) \leq C \sum_{k=0}^{n+a} \mathbb{P}(g_{n+a} = h \text{ and } g_k = h').$$

Proof. When μ is supported on the generators, this is clear with $C = 1$ and $a = 0$: any nearest-neighbor path from e to h has to pass through h' . In the general case, the idea is that such a path has to pass not too far from h' , and forcing it to pass exactly through h' does not cost too much, neither in time nor in probability. Let us make this idea precise. By our assumptions on μ , there is an integer r such that the support of μ is contained in the ball of radius r , and another integer a_0 such that $c := \inf_{g \in B(e,r)} \mathbb{P}(g_{a_0} = g) > 0$.

Let T be the first hitting time of the ball of radius r around h' , so that if $g_n = h$ then $T \leq n$. The lemma will follow from the following observation: if we add $2a_0$ steps to the random walk after time T , with probability at least c^2 we will be at h' at time $T + a_0$ and back to g_T at time $T + 2a_0$, and then we can run the random walk as before. More formally, define another realization of the random walk as follows: let g'_i be an independent copy of the random walk, and define

$$\tilde{g}_n = \begin{cases} g_n & \text{if } n \leq T \\ g_T g'_{n-T} & \text{if } T \leq n \leq T + 2a_0 \\ g_T g'_{2a_0} g_T^{-1} g_{n-2a_0} & \text{if } T + 2a_0 \leq n. \end{cases}$$

By the Markov property (T is a stopping time) $(\tilde{g}_n)_{n \geq 0}$ is distributed as the random walk with step distribution μ .

Conditionally to T , the event $A = \{g'_{2a_0} = e \text{ and } g'_{a_0} = g_T^{-1} h'\}$ happens with probability $\geq c^2$, and when it happens we have $\tilde{g}_{n+2a_0} = g_n$ for every $n \geq T$ and $\tilde{g}_{T+a_0} = h'$. Therefore, we can bound the probability of the event

$$B = \{\tilde{g}_{n+2a_0} = h \text{ and } h' \in \{\tilde{g}_k, 0 \leq k \leq n + 2a_0\}\}$$

as follows

$$\mathbb{P}(B) \geq \mathbb{P}(g_n = h \text{ and } A) \geq c^2 \mathbb{P}(g_n = h).$$

By the union bound we obtain

$$c^2 \mathbb{P}(g_n = h) \leq \sum_{k=0}^{n+2a_0} \mathbb{P}(\tilde{g}_{n+2a_0} = h \text{ and } \tilde{g}_k = h'),$$

which is the content of the lemma with $C = c^{-2}$ and $a = 2a_0$. \square

Proof of Proposition 6.1. Denote by p_n the probability that g_n is a cyclically reduced proper power, that is $g_n = h^d$ for a cyclically reduced h and $d \geq 2$. We shall prove the following two inequalities, from which the proposition follows immediately: there are constants C_1, C_2 such that

$$\mathbb{P}(g_n \text{ is a proper power}) \leq C_2 n^2 \max_{k \leq n+2a} p_{n+2a-k} \rho^k. \quad (6.2)$$

$$p_n \leq C_1 n^3 \rho^n, \quad (6.3)$$

We start with the proof of (6.3). Let h be cyclically reduced and $d \geq 2$. The assumption that h is cyclically reduced means that h belongs to the segment between 1 and h^d , and that h^{d-1} belongs to the segment between h and h^d (tautologically if $d = 2$). By two applications of Lemma 6.2, we have

$$\begin{aligned} \mathbb{P}(g_n = h^d) &\leq C^2 \sum_{k_1+k_2+k_3=n+2a} \mathbb{P}(g_{k_1} = h, g_{k_1+k_2} = h^{d-1}, g_{k_1+k_2+k_3} = h^d) \\ &= C^2 \sum_{k_1+k_2+k_3=n+2a} \mathbb{P}(g_{k_1} = h) \mathbb{P}(g_{k_2} = h^{d-2}) \mathbb{P}(g_{k_3} = h). \end{aligned}$$

The inequality (6.1) allows us to bound the middle term by ρ^{k_2} , and by symmetry we have

$$\mathbb{P}(g_{k_1} = h) \mathbb{P}(g_{k_3} = h) = \mathbb{P}(g_{k_1} = h) \mathbb{P}(g_{k_3} = h^{-1}) = \mathbb{P}(g_{k_1} = h \text{ and } g_{k_1+k_2} = e).$$

Summing over all cyclically reduced words, we get

$$\sum_{h \text{ cyclically reduced}} \mathbb{P}(g_n = h^d) \leq C^2 \sum_{k_1+k_2+k_3=n+2a} \rho^{k_2} \mathbb{P}(g_{k_1+k_3} = e).$$

By (6.1), this is less than $C^2(n+2a+1)^2 \rho^{n+2a}$. Summing over all $d \leq n$ we obtain (6.3).

We now move to (6.2). Let $h \in \mathbf{F}_r$, not necessarily cyclically reduced. Write $h = w\tilde{h}w^{-1}$ its reduced expression with \tilde{h} cyclically reduced. Then $h^d = w\tilde{h}^dw^{-1}$ is the reduced expression of h^d . In particular w and $w\tilde{h}^d$ are on the segment from the identity to h^d , and by two applications of Lemma 6.2 we obtain

$$\mathbb{P}(g_n = h^d) \leq C^2 \sum_{k_1+k_2+k_3 \leq n+2a} \mathbb{P}(g_{k_1} = w)\mathbb{P}(g_{k_2} = \tilde{h}^d)\mathbb{P}(g_{k_3} = w^{-1}).$$

If we first sum over all \tilde{h} such that $w\tilde{h}w^{-1}$ is reduced, and then over w , we obtain

$$\mathbb{P}(g_n \text{ is a proper power}) \leq C^2 \sum_{k_1+k_2+k_3 \leq n+2a} p_{k_2} \sum_w \mathbb{P}(g_{k_1} = w)\mathbb{P}(g_{k_3} = w^{-1}).$$

The formula (6.2) follows because $\sum_w \mathbb{P}(g_{k_1} = w)\mathbb{P}(g_{k_3} = w^{-1}) = \mathbb{P}(g_{k_1+k_3} = e) \leq \rho^{k_1+k_3}$, by (6.1). \square

6.2 Random walks and tempered functions

The following is a useful criterion for a function on a group with the rapid decay property to be tempered. Recall a group is said to have the rapid decay property if it admits a finite generating set and a polynomial P such that, for every $a \in \mathbb{C}_{\leq q}[\Gamma]$,

$$\|\lambda(a)\| \leq P(R) \left(\sum_{\gamma} |a(\gamma)|^2 \right)^{\frac{1}{2}}. \quad (6.4)$$

Haagerup's inequality [Haa79, Lemma 1.5] asserts that free groups with their standard generating sets have the rapid decay property, with $P(R) = 3(1 + R^2)$.

Proposition 6.3. *Let Γ be a finitely generated group with the rapid decay property, and $u: \Gamma \rightarrow \mathbb{C}$ a function. Assume that for every reasonable probability measure μ on Γ , if $(\gamma_n)_{n \geq 0}$ is the associated random walk on Γ ,*

$$\limsup_n \left(\mathbb{E}|u(\gamma_n)| \right)^{\frac{1}{n}} \leq \rho(\mu). \quad (6.5)$$

Then u is tempered, and even completely tempered.

Proof. Before we start, let us observe that the assumption (6.5) in fact holds for every symmetric finitely supported probability measure μ , even if it not reasonable in the sense that its support is not generating or does not contain the identity. Indeed, let ν be a fixed reasonable probability measure. Then for every $\varepsilon > 0$, $\mu_\varepsilon = \frac{1}{1+\varepsilon}(\mu + \varepsilon\nu)$ is a reasonable probability measure, so that (6.5) holds for μ_ε . If \mathbb{E}_ε and \mathbb{E} denote the law of the random walk with transition μ_ε and μ respectively, the inequality $\mu \leq (1 + \varepsilon)\mu_\varepsilon$ gives

$$\mathbb{E}|u(\gamma_n)| \leq (1 + \varepsilon)^n \mathbb{E}_\varepsilon|u(\gamma_n)|,$$

so we obtain

$$\limsup_n (\mathbb{E}|u(\gamma_n)|)^{\frac{1}{n}} \leq (1 + \varepsilon)\rho(\mu_\varepsilon) \leq \rho(\mu) + \varepsilon\rho(\nu).$$

This proves that (6.5) holds for μ by taking $\varepsilon \rightarrow 0$.

We now move to the proof of the proposition. We claim that for every $k, q \geq 1$ and every $x = x^* \in M_k(\mathbb{C}) \otimes \mathbb{C}_{\leq q}[\Gamma]$,

$$\limsup_{n \rightarrow \infty} |\mathrm{Tr} \otimes u(x^n)|^{\frac{1}{n}} \leq P(q)\sqrt{k}\|(\mathrm{id} \otimes \lambda)(x)\|. \quad (6.6)$$

Here P is the polynomial satisfying (6.4), given by the assumption that Γ has the rapid decay property.

By a standard *tensor-power* trick, (6.6) implies that u is completely tempered. Indeed, given an arbitrary $x \in M_k(\mathbb{C}) \otimes \mathbb{C}_{\leq q}[\Gamma]$, applying (6.6) to $y = (x^*x)^m$ (which belongs to $M_k(\mathbb{C}) \otimes \mathbb{C}_{\leq 2mq}[\Gamma]$ and satisfies $\|(\mathrm{id} \otimes \lambda)(y)\| = \|(\mathrm{id} \otimes \lambda)(x)\|^{2m}$) and raising to the power $\frac{1}{2m}$, it implies

$$\limsup_{n \rightarrow \infty} |\mathrm{Tr} \otimes u((x^*x)^n)|^{\frac{1}{2n}} \leq (P(2mq)\sqrt{k})^{\frac{1}{2m}} \|(\mathrm{id} \otimes \lambda)(x)\|,$$

and in the $m \rightarrow \infty$ limit

$$\limsup_{n \rightarrow \infty} |\mathrm{Tr} \otimes u((x^*x)^n)|^{\frac{1}{2n}} \leq \|(\mathrm{id} \otimes \lambda)(x)\|.$$

So it remains to prove (6.6). We can normalize x so that $\sum_\gamma \|x(\gamma)\| = 1$, where $\|\cdot\|$ is the operator norm on $M_k(\mathbb{C})$. Let $\mu(s) = \|x(s)\|$; it is a symmetric finitely supported probability measure on Γ . By the triangle inequality and the inequality $|\mathrm{Tr}(x(s_1) \dots x(s_n))| \leq k\|x(s_1)\| \dots \|x(s_n)\|$, we have

$$|\mathrm{Tr} \otimes u(x^n)| \leq k \sum_{s_1, \dots, s_n \in \Gamma} \mu(s_1) \dots \mu(s_n) |u(s_1 \dots s_n)| = k\mathbb{E}|u(\gamma_n)|.$$

By the assumption in the extended form above, we obtain

$$\limsup_{n \rightarrow \infty} |\mathrm{Tr} \otimes u(x^n)|^{\frac{1}{n}} \leq \rho(\mu).$$

By the rapid decay assumption (6.4), this is less than

$$P(q) \left(\sum_s \mu(s)^2 \right)^{\frac{1}{2}}.$$

By the basic fact $\mu(s)^2 = \|x(s)\|^2 \leq \mathrm{Tr}(x(s)^*x(s))$, we have

$$\sum_s \mu(s)^2 \leq (\mathrm{Tr} \otimes \lambda)(x^*x) \leq k \|(\mathrm{id} \otimes \lambda)(x)\|^2,$$

which proves the claimed inequality (6.6). \square

7 Proof of Theorem 1.1 from Theorem 3.1

Let $K \geq 1$ be an integer. Most of the objects will depend on K but our notation will not reflect this. The only exception is the letter C , which will always denote a constant that does not depend on anything, but that can change from one line to the next.

The aim of this section is to explain why Theorem 3.1 implies the strong convergence result from Theorem 1.1.

Let

$$\sigma_n \stackrel{\mathrm{def}}{=} \bigoplus_{|\lambda|+|\mu|=K} \pi_{\lambda,\mu}.$$

It is a sub-representation of $\bigoplus_{k=0}^{\ell} \pi_{k,K-k}^0$, therefore

$$\dim(\sigma_n) \leq (K+1)n^K. \tag{7.1}$$

Recall the definition of g_L from §3. Theorem 3.1 immediately implies the following result about σ_n . Throughout this section and as in §3 \mathbb{E}_n or simply \mathbb{E} will denote the integral with respect to the Haar measure on $U(n)^r$.

Theorem 7.1. *For every $w \in \mathbf{F}_r$, there is a rational function $\varphi_w \in \mathbb{Q}(x)$ such that*

1. For every integer $n \geq K \max(|w|, 1)$,

$$\varphi_w \left(\frac{1}{n} \right) = \frac{1}{n^K} \mathbb{E}_n [\text{Tr}(\sigma_n(u_1, \dots, u_r))].$$

2. If $w \neq e$ has length $\leq q$, then $g_{Kq}\varphi_w$ is a polynomial of degree $\leq D_q = 3Kq \log(1 + Kq)$.

3. If w is not a proper power, then $\varphi_w^{(i)}(0) = 0$ for all $i < K + \frac{K}{3}$.

We collect in the next lemma some of the basic properties of g_L that we need.

Lemma 7.2. *Let $L \geq 1$ be an integer. The polynomial $g_L(t) = \prod_{c=1}^L (1 - c^2 t^2)^{\lfloor \frac{L}{c} \rfloor}$ has the following properties, for every $0 \leq t \leq \frac{1}{2L^2}$ and every integer $i \geq 0$:*

$$\frac{1}{2} \leq g_L(t) \leq 1, \quad (7.2)$$

$$|g_L^{(i)}(t)| \leq (3iL^{\frac{3}{2}})^i, \quad (7.3)$$

$$\left| \left(\frac{1}{g_L} \right)^{(i)}(t) \right| \leq 2 \cdot i! \cdot (C\sqrt{i}L^{\frac{3}{2}})^i. \quad (7.4)$$

Proof. Fix $0 \leq t \leq \frac{1}{2L^2}$. The inequality $g_L(t) \leq 1$ is clear and only uses $|t| \leq \frac{1}{L}$. For the converse, use that $(1 - u) \geq e^{-2u}$ for $0 < u < \frac{1}{4}$ to bound

$$g_L(t) \geq \exp\left(-2 \sum_{c=1}^L \frac{L}{c} c^2 t^2\right) = \exp(-L^2(L+1)t^2) \geq \frac{1}{2}.$$

We now prove (7.3). Let $N = \sum_{c=1}^L \lfloor \frac{L}{c} \rfloor$ and write $g_L(t) = \prod_{k=1}^N (1 - c_k^2 t^2)$. Each of the factors $h_k(t) = (1 - c_k^2 t^2)$ is a degree 2 polynomial, so when we differentiate i times g_L using the Leibniz rule, we obtain a big sum of terms, in which some factors are derived twice (so equal to $h_k''(t) = -2c_k^2$), some are derived once (so equal to $h_k'(t) = -2c_k^2 t$) and all the other are not derived. Gathering the terms according to how many factors are derived twice, we can write

$$\frac{g_L^{(i)}}{g_L} = \sum_{s=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i}{2s} (2s-1)!! \sum_{k_1, \dots, k_s, \ell_1, \dots, \ell_{i-2s} \text{ all distinct}} \prod_{\alpha=1}^s \frac{h_{k_\alpha}''}{h_{k_\alpha}} \cdot \prod_{\beta=1}^{i-2s} \frac{h_{\ell_\beta}'}{h_{\ell_\beta}}.$$

The term $\binom{i}{2s}(2s-1)!!$ appears as the number of ways to partition a set of size i (the steps of derivation) into s sets of size 2 (the steps when a factor h_k is derived twice) and $i-2s$ sets of size 1 (the steps when a factor h_k is derived once). Forgetting the condition that the k_α and ℓ_β are distinct, and bounding $(1-c_k^2 t^2) \leq 1$ and $(2s-1)!! \leq i^{\frac{i}{2}}$, we can bound the preceding, in absolute value, as follows:

$$|g_L^{(i)}(t)| \leq i^{\frac{i}{2}} \sum_{s=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i}{2s} t^{i-2s} \left(\sum_{k=1}^N 2c_k^2 \right)^{i-s} \leq i^{\frac{i}{2}} \sum_{j=0}^i \binom{i}{j} t^{i-j} \left(\sum_{k=1}^N 2c_k^2 \right)^{i-\frac{j}{2}}.$$

Using

$$\sum_{k=1}^N 2c_k^2 = \sum_{c=1}^L 2 \lfloor \frac{L}{c} \rfloor c^2 \leq L^2(L+1) \leq 2L^3,$$

we obtain

$$|g_L^{(i)}(t)| \leq i^{\frac{i}{2}} \sum_{j=0}^i \binom{i}{j} \left(\frac{1}{2L^2} \right)^{i-j} (2L^3)^{i-\frac{j}{2}} = i^{\frac{i}{2}} \left(L + \sqrt{2}L^{\frac{3}{2}} \right)^i \leq \left(3\sqrt{i}L^{\frac{3}{2}} \right)^i.$$

This proves (7.3).

The last inequality (7.4) is a formal consequence of the first two. Indeed, by induction we see that, for any function g we can write the i^{th} derivative of $\frac{1}{g}$ as a product of $(2i-1)!!$ terms, each of which is of the form

$$\frac{\pm g^{(\alpha_1)} \dots g^{(\alpha_i)}}{g^{i+1}}$$

where $(\alpha_1, \dots, \alpha_i)$ are nonnegative integers that sum to i . For $g = g_L$, by (7.2) and (7.3), each of these terms is bounded above that $\left(3\sqrt{i}L^{\frac{3}{2}} \right)^i 2^{i+1}$. \square

We extend the map $w \mapsto \varphi_w$ by linearity, setting $\varphi_x = \sum_{w \in \mathbf{F}_r} x(w) \varphi_w$ for $x \in \mathbb{C}[\mathbf{F}_r]$.

Lemma 7.3. *If $x \in \mathbb{C}_{\leq q}[\mathbf{F}_r]$, then for any i ,*

$$\sup_{t \in [0, \frac{1}{2D_q^2}]} \frac{|\varphi_x^{(i)}(t)|}{i!} \leq p(i, q) \|x\|_{C^*(\mathbf{F}_r)},$$

where

$$p(i, q) = \begin{cases} (2K + 2) \left(\frac{CD_q^4}{i^2} \right)^i & \text{if } i \leq D_q, \\ (2K + 2) (CD_q^2)^{D_q} \left(C\sqrt{i}D_q^{\frac{3}{2}} \right)^{i-D_q} & \text{if } i > D_q. \end{cases}$$

Proof. Let $P = g_{Kq}\varphi_x$. We know that P is a polynomial of degree $\leq D_q$, so to bound its derivatives using the Markov brothers inequality, we need to derive an a priori bound on the values of P . For $n \geq Kq$, we have $g_{Kq}(\frac{1}{n}) \in [0, 1]$ and therefore

$$\left| P \left(\frac{1}{n} \right) \right| \leq \left| \varphi_x \left(\frac{1}{n} \right) \right| = n^{-K} \left| \mathbb{E}[\text{Tr}(\sigma_n(x(u_1, \dots, u_r)))] \right|.$$

For every $(u_1, \dots, u_r) \in \mathcal{U}(n)^r$, the map $w \mapsto \sigma_n(w(u_1, \dots, u_r))$ is a unitary representation of \mathbf{F}_r (hence of $C^*(\mathbf{F}_r)$), so $\|\sigma_n(x(u_1, \dots, u_r))\| \leq \|x\|_{C^*(\mathbf{F}_r)}$ almost surely. Bounding the trace of a matrix by its norm times its size, we obtain $|\mathbb{E}[\text{Tr}(\sigma_n(x(u_1, \dots, u_r)))]| \leq \dim(\sigma_n)\|x\|_{C^*(\mathbf{F}_r)}$, and we conclude by the dimension bound (7.1) that

$$\left| P \left(\frac{1}{n} \right) \right| \leq (K + 1)\|x\|_{C^*(\mathbf{F}_r)}.$$

We deduce by Lemma 4.2 that for any integer j ,

$$\sup_{t \in [0, \frac{1}{2D_q^2}]} \frac{|P^{(j)}(t)|}{j!} \leq (2K + 2) \left(\frac{CD_q^4}{j^2} \right)^j \|x\|_{C^*(\mathbf{F}_r)}. \quad (7.5)$$

By the Leibniz rule, remembering that P is a polynomial of degree $\leq D_q$,

$$\varphi_x^{(i)} = \sum_{j=0}^{\min(i, D_q)} \binom{i}{j} P^{(j)} \left(\frac{1}{g_{Kq}} \right)^{(i-j)}.$$

We see from Lemma 7.2 and (7.5) that the leading term is for $j = \min(i, D_q)$. The lemma follows directly if $i \leq D_q$. If $i > D_q$, we obtain the lemma by bounding $Kq \leq D_q$ and $i - D_q \leq i$, so that

$$\sup_{t \in [0, \frac{1}{2D_q^2}]} \frac{1}{(i - D_q)!} \left| \left(\frac{1}{g_{Kq}} \right)^{(i-D_q)} (t) \right| \leq (CD_q^2)^i \left(\frac{i}{D_q} \right)^{\frac{i-D_q}{2}}. \quad \square$$

We deduce two facts from this lemma. In the first, the temperedness of v_i is where most of the results of this paper are used. It is worth noting here that, in the particular case of $K = 1$, a much stronger result is known: all the v_i 's are tempered, as proven by Parraud [Par23a]. We will see below that they are even completely tempered, see Proposition 7.9.

Lemma 7.4. *For every integer $i \geq 0$, let $v_i: w \in \mathbf{F}_r \mapsto \frac{\varphi_w^{(K+i)}(0)}{(K+i)!}$. There is a polynomial P of degree $4K + Ki + 1$ such that $|v_i(x)| \leq P_n(q)\|x\|_{C^*(\mathbf{F}_r)}$ for every q and every $x \in \mathbb{C}_{\leq q}[\mathbf{F}_r]$. Moreover, v_i is tempered if $i < \frac{K}{3}$.*

Proof. The existence of P_n is a consequence of Lemma 7.3 and the bound $D_q \leq CKq \log(1 + K) \log(1 + q)$, which implies $\sup_{q \geq 1} \frac{p(i,q)}{(1+q)^{4i+1}} < \infty$.

Let us explain why v_i is tempered if $i < \frac{K}{3}$. The proof combines several ingredients: the first is Haagerup's inequality, which tells us that we can apply the criterion for temperedness given in Proposition 6.3. The second is the polynomial growth of v_i : as a particular case of what we have just shown, there is a constant C_i such that for every q and every $w \in \mathbf{F}_r$ in the ball of radius q , $|v_i(w)| \leq C_i(1 + q)^j$, where $j = 4i + 4K + 1$. The third ingredient is the small support of v_i given by item 3 in Theorem 3.1. The last is the random walk results from Section 6.

Let us put all these ingredients together. Let μ be a reasonable probability measure on \mathbf{F}_r , and $(\gamma_n)_n$ be the associated random walk on \mathbf{F}_r . If μ is supported in the ball of radius q , we know that γ_n belongs to the ball of radius qn , so that

$$\mathbb{E}|v_i(\gamma_n)| \leq C_i(1 + qn)^j \mathbb{P}(v_i(\gamma_n) \neq 0) \leq C_i C(\mu)(1 + qn)^j n^5 \rho(\mu)^n.$$

We deduce

$$\limsup_n (\mathbb{E}|v_i(\gamma_n)|)^{\frac{1}{n}} \leq \rho(\mu),$$

so v_i is tempered by Proposition 6.3. \square

The second consequence is an analogue in our context to the master inequality in [CGVTvH24, Theorem 7.1]. In the particular case $K = 1$, a similar result was obtained in [Par23a].

Lemma 7.5. *Let $w(q) = \exp(\frac{q}{\log(2+q)^2})$. For every integers $r, q, n \geq 1$ and every $x \in \mathbb{C}_{\leq q}[\Gamma]$*

$$\begin{aligned} & \left| \mathbb{E} \text{Tr}(\sigma_n(x(u_1, \dots, u_r)) - \tau(x) \text{Id}) - \sum_{i=0}^{r-1} \frac{v_i(x)}{n^i} \right| \\ & \leq \frac{(C(K+r)^2 \log(1+K+r)^{12} K^4 \log(1+K)^4)^{K+r}}{n^r} w(q) \|x\|_{C^*(\mathbf{F}_r)}. \end{aligned} \quad (7.6)$$

Proof. If $n \geq Kq$, by (1) in Theorem 7.1, we know that the left-hand side of (7.6) is equal to

$$n^K \left| \varphi_x\left(\frac{1}{n}\right) - \sum_{i=0}^{K+r-1} \frac{\varphi_x^{(i)}(0)}{i! n^i} \right|.$$

By Taylor's inequality and Lemma 7.3, if we moreover assume that $n \geq 2D_q^2$, this is less than

$$\frac{1}{n^r} p(K+r, q) \|x\|_{C^*(\mathbf{F}_r)}. \quad (7.7)$$

If $n \leq 2D_q^2$, we claim that this bound is still valid. Indeed, we can bound the left-hand side of (7.6) by

$$2 \dim(\sigma_n) \|x\|_{C^*(\mathbf{F}_r)} + \sum_{i=0}^{r-1} \frac{|v_i(x)|}{n^i}.$$

By Lemma 7.3 and (7.1), this is less than

$$n^K (p(0, q) + \sum_{i=0}^{r-1} \frac{1}{n^{K+i}} p(K+i, q)) \|x\|_{C^*(\mathbf{F}_r)},$$

which is indeed bounded above by (7.7), at least if the constant C appearing there is large enough : for example $C \geq 2e^2$ guarantees that $\frac{1}{n^{j+1}} p(j+1, q) \geq 2 \frac{1}{n^j} p(j, q)$ for every j .

The statement of the lemma follows because

$$\sup_q p(i, q) \exp\left(-\frac{q}{\log(2+q)^2}\right) \leq (CK^4 i^2 \log(1+K)^4 \log(1+i)^{12})^i. \quad \square$$

We can now deduce the main result of this section. Now we make K vary, so we write $\sigma_{K,n}$ the representation that was denoted σ_n so far, and $v_{K,i}$ the function denoted v_i so far.

Lemma 7.6. *Let $A < \frac{1}{24}$. Then for every $x \in \mathbb{C}[\mathbf{F}_r]$,*

$$\limsup_n \sup_{K \leq n^A} \mathbb{E} \left| \|\sigma_{n,K}(x(u_1, \dots, u_r))\| - \|x\|_{C_\lambda^*(\mathbf{F}_r)} \right| = 0.$$

Proof. Let $1 \leq K_n \leq n^A$ for every n , and let π_n be the random representation of \mathbf{F}_r : $\pi_n(w) = \sigma_{K,n}(w(u_1, \dots, u_r))$. Apply Lemma 7.5 for the smallest integer $r \geq \frac{K}{3}$: for every q and every $x \in \mathbb{C}_{\leq q}[\Gamma]$,

$$\left| \mathbb{E} \text{Tr}(\pi_n(x)) - u_n(x) \right| \leq \varepsilon_n w(q) \|x\|_{C^*(\mathbf{F}_r)}.$$

where

$$\varepsilon_n = \frac{(CK_n^6 (\log K_n)^{12+8\varepsilon})^{K_n+r}}{n^r} \leq (Cn^{24A-1} (\log n)^{12+8\varepsilon})^{\frac{1}{3}} \rightarrow 0$$

and

$$u_n(x) = \dim(\sigma_{n,k})\tau(x) + \sum_{i=0}^{r-1} \frac{v_{K_n,i}(x)}{n^i}.$$

By Lemma 7.4, the function u_n is tempered as a finite sum of tempered function, and it satisfies

$$|u_n(x)| \leq C(n)(1+q)^{4r+4K_n-3}.$$

Moreover, $C_\lambda^*(\Gamma)$ has a unique trace (it is even simple [Pow75]). So all the hypotheses of Proposition 5.2 are satisfied; its conclusion proves the lemma. \square

7.1 From convergence in probability to almost sure convergence

Lemma 7.6 is not exactly our main result, because we have convergence in expected value. However, by the concentration of measure phenomenon in the groups $U(n)$ (see [Mec19, Theorem 5.17]), we can improve our results to almost sure convergence.

Proposition 7.7. *For every $x \in \mathbb{C}_{\leq q}[\mathbf{F}_r]$, there is a constant $C(x) = \sum_w |x(w)||w|$ such that*

$$\begin{aligned} & \mathbb{P} \left(\|\sigma_{n,K}(x(U_1^{(n)}, \dots, U_r^{(n)}))\| \geq \mathbb{E} \|\sigma_{n,K}(x(U_1^{(n)}, \dots, U_r^{(n)}))\| + \varepsilon \right) \\ & \leq \exp \left(- \frac{(n-2)\varepsilon^2}{24C(x)^2 K^2} \right). \end{aligned}$$

Proof. Equip $U(n)^r$ with the L_2 -sum of Hilbert-Schmidt distances. The function $(U_1^{(n)}, \dots, U_r^{(n)}) \mapsto \|\sigma_{n,K}(x(U_1^{(n)}, \dots, U_r^{(n)}))\|$ is $KC(x)$ -Lipschitz, so the bound is [Mec19, Theorem 5.17] (for the original see [Voi91, Proof of Thm. 3.9]). \square

Proof of Theorem 1.1. Theorem 1.1 now follows by combining Proposition 7.7 and Lemma 7.6. \square

7.2 Variants and operator coefficients

If we only consider the representation $\pi_{k,\ell}$ from the introduction with $\ell = 0$, we have the stronger conclusion 3 in Theorem 3.1 instead of 2. Taking this improvement into account, with the notation of Theorem 1.1, the conclusion becomes that if $A < \frac{1}{12}$,

$$\sup_{k \leq n^A} \left| \left\| \pi_{k,0} \left(p \left(U_1^{(n)}, \dots, U_r^{(n)} \right) \right) \right\| - \left\| p(x_1, \dots, x_r) \right\| \right| = o(1).$$

This improvement illustrates that the more of the higher derivatives $v_{K,i}$ from Lemma 7.4 are shown to be tempered, the stronger the conclusion. For example, if we knew that $v_{K,i}$ is tempered for every i and every K (which Parraud has proved in the case $K = 1$ [Par23a]), or at least for every $i \leq K/o(1)$, then the condition on A in Theorem 1.1 would be $A < \frac{1}{6}$.

Theorem 7.8. *Let $A < \frac{1}{6}$ and for every n , let $K_n \leq n^A$. Assume that $v_{K_n,i}$ is tempered for every n and every i . Let $k_n = \exp(n^{\frac{1}{2}-2A}(\log n)^{-4})$.*

For every q and every sequence $y_n \in M_{k_n} \otimes \mathbb{C}_{\leq q}(\mathbf{F}_r)$, almost surely

$$\lim_n \frac{\|(\text{id} \otimes \sigma_{K_n,n})(y_n)\|}{\|(\text{id} \otimes \lambda)(y_n)\|} = 1.$$

If the assumption was that $v_{K_n,i}$ is completely tempered, the proof of the theorem would be a straightforward adaptation of the proof of Theorem 1.1. So our main task is to prove the following:

Proposition 7.9. *For every K, I , if $v_{K,0}, \dots, v_{K,I}$ are all tempered, then they are all completely tempered.*

For the proof, we need the following result. Here if \mathcal{A} is an algebra, \mathcal{A}^n denotes its subalgebra equal to the linear span of $\{x_1 \dots x_n \mid x_i \in \mathcal{A}\}$.

Lemma 7.10. *Let \mathcal{A} be a $*$ -algebra and $u: \mathcal{A} \rightarrow \mathbb{C}$ a positive linear map: $u(x^*x) \geq 0$ for every $x \in \mathcal{A}$. If there is i such that $u(x^i) = 0$ for every self-adjoint x , then $u = 0$ on \mathcal{A}^3 .*

Proof. The form $\langle a, b \rangle = u(a^*b)$ is a scalar product. By the Cauchy-Schwarz inequality, if $k \geq 2$, if $a^* = x^*xx^* \dots$ ($k+1$) terms and $b = \dots x$ ($k-1$) terms,

$$|u((x^*x)^k)|^2 = |\langle a, b \rangle|^2 \leq u((x^*x)^{k+1})u((x^*x)^{k-1}),$$

so by induction we have $u((x^*x)^2) = 0$, and by the Cauchy-Schwarz inequality again, $u(x^*xz) = 0$ for every $z \in \mathcal{A}$. By polarization we deduce $u(xyz) = 0$ for every x, y, z . \square

Proof of Proposition 7.9. We use the notation and results from Section 5.2. In particular, Lemma 7.4 tells us that $v_{K,i}$ extends by continuity to a linear map $\mathcal{S}_{4K+4i+1}(\mathbf{F}_r) \rightarrow \mathbb{C}$. Let $\mathcal{A}_i = \mathcal{S}_{4K+4i+1}(\mathbf{F}_r) \cap \ker \lambda$. We prove by induction that $v_{K,i}$ vanishes on $(\mathcal{A}_i)^{3^{i+1}}$ for every $i \leq I$. This readily implies that $\text{Tr} \otimes v_{K,i}$ vanishes on $M_k \otimes (\mathcal{A}_i)^{3^{i+1}}$ for all k . In particular, $v_{K,i}$ is completely tempered by Proposition 5.12.

So let us assume that $v_{K,j}$ vanishes on $(\mathcal{A}_j)^{3^{j+1}}$ for every $j < i$ (we assume nothing if $i = 0$), and let us show that it is true for $j = i$.

Using the induction hypothesis, it follows from Lemma 7.5 that on the $*$ -algebra $(\mathcal{A}_i)^{3^i}$ (which is contained in $\ker \tau$ and $(\mathcal{A}_j)^{3^{j+1}}$ for every $j < i$),

$$v_{K,i}(x) = \lim_n n^i \mathbb{E} \text{Tr} \sigma_{K,n}(x).$$

In particular, $v_{K,i}$ is positive on $(\mathcal{A}_i)^{3^i}$ as a limit of positive maps. Moreover, the fact that $v_{K,i}$ is tempered implies that $v_{K,i}(x^n) = 0$ for all $n \geq 4K + 4i + 3$, see Proposition 5.12. By Lemma 7.10, we obtain that $v_{K,i}$ vanishes on $(\mathcal{A}_i)^{3^{i+1}}$. This concludes the proof of the proposition. \square

Proof of Theorem 7.8. Thanks to Proposition 7.9, the temperedness assumption is automatically upgraded to complete temperedness. We apply Lemma 7.5 with $r = n^{\frac{1}{2}-2A}(\log n)^{-\frac{7}{2}}$, which is chosen so that

$$k_n \frac{(C(K+r)^2 \log(1+K+r))^{12} K^4 \log(1+K)^4)^{K+r}}{n^r} \rightarrow 0.$$

So by Proposition 5.6 we deduce the convergence in probability. The almost sure convergence is obtained by the concentration of measure phenomenon from Proposition 7.7. \square

For $K_n = 1$ (that is, $A = 0$), Parraud proved that $v_{1,i}$ are tempered for every i . As a consequence, we obtain the unconditional result from Theorem 1.4.

8 Proof of Corollary 1.2

Corollary 1.2 follows easily from the proof of Theorem 1.1 and the following lemma:

Lemma 8.1. *For every partitions $\lambda \vdash k, \mu \vdash \ell$, every $w \in \mathbf{F}_r$ and every $n > (k + \ell)|w|$, we have*

$$\mathbb{E}_n[s_{\lambda,\mu}(w)] = \int_{\mathrm{SU}(n)^r} s_{\lambda,\mu}(w(v_1, v_2, \dots, v_r)) dv_1 \cdots dv_r.$$

Proof. Let $z_1, \dots, z_r, z'_1, \dots, z'_r, v_1, \dots, v_r$ be independent random variables where z_i are uniform in the center of $U(n)$ (the complex numbers of modulus one), z'_i are uniform in the center of $SU(n)$ (the n -th roots of unity) and $v_i \in \mathrm{SU}(n)$ are Haar distributed, so that $(z_1 v_1, \dots, z_r v_r)$ are independent Haar-distributed variables in $U(n)$, and $(z'_1 v_1, \dots, z'_r v_r)$ are independent Haar-distributed variables in $SU(n)$.

By Schur's lemma, for every z in the center of $U(n)$, $\pi_{\lambda,\mu}(z)$ is a multiple of the identity. On the other hand, we know that $(\lambda_1, \dots, \lambda_p, \dots, -\mu_q, \dots, -\mu_1)$ is the maximal weight of $\pi_{\lambda,\mu}$, so the corresponding character of the maximal torus appears in the restriction of $\pi_{\lambda,\mu}$. This means that the scalar $\pi_{\lambda,\mu}(z)$ is $z^{\lambda_1 + \dots + \lambda_p - \mu_q - \dots - \mu_1} = z^{k-\ell}$ for every z in the center of $U(n)$. Putting these two facts together, we see that

$$\mathbb{E}_n[s_{\lambda,\mu}(w)] = \mathbb{E}[s_{\lambda,\mu}(w(z_1 v_1, \dots, z_r v_r))]$$

decomposes by independence as the product

$$\mathbb{E}_n[s_{\lambda,\mu}(w)] = \mathbb{E}[s_{\lambda,\mu}(w(v_1, \dots, v_r))] \cdot \mathbb{E}[w(z_1, \dots, z_r)^{k-\ell}],$$

and similarly

$$\mathbb{E}[s_{\lambda,\mu}(w(z'_1 v_1, \dots, z'_r v_r))] = \mathbb{E}[s_{\lambda,\mu}(w(v_1, \dots, v_r))] \cdot \mathbb{E}[w(z'_1, \dots, z'_r)^{k-\ell}].$$

But $\mathbb{E}w(z_1, \dots, z_r)^{k-\ell}$ is equal to 1 if $w^{k-\ell}$ vanishes in the abelianization \mathbb{Z}^r of \mathbf{F}_r , and is equal to 0 otherwise. Similarly, $\mathbb{E}w(z'_1, \dots, z'_r)^{k-\ell}$ is equal to 1

if $w^{k-\ell}$ vanishes in $(\mathbb{Z}/n\mathbb{Z})^r$ and 0 otherwise. These two conditions coincide if $n > |k - \ell| \cdot |w|$, and in particular if $n > (k + \ell)|w|$. \square

Proof of Corollary 1.2. Using Lemma 8.1, we see that Lemma 7.5 also holds if \mathbb{E}_n denotes the integration over $\mathrm{SU}(n)^r$: the validity of (7.7) when $x \in \mathbb{C}_{\leq q}[\mathbf{F}_r]$ with $n \geq 2D_q^2$ is by Lemma 8.1, the extension for arbitrary x follows with the same proof. Therefore, Lemma 7.6 also holds for $\mathrm{SU}(n)$. Moreover, the concentration of measure from Proposition 7.7 also holds for $\mathrm{SU}(n)$ (same reference as there), so Theorem 1.1 holds also for independent variables in $\mathrm{SU}(n)$. Corollary 1.2 follows by Corollary 2.2. \square

9 Transverse maps

We now fix the free group we work with \mathbf{F}_r and its generators $\{x_1, \dots, x_r\}$. Here we introduce a framework, following [MP19], that we use to prove Theorem 3.1.

In the sequel, a marked rose refers to a finite CW -complex structure R_r whose underlying topological space is the wedge of r circles, with the base-point being the wedge point, denoted o , and with a marking $\pi_1(R_r, o) \cong \mathbf{F}_r$ that identifies the generators $\{x_1, \dots, x_r\}$ with oriented circles of the rose.

The following definition will not be used in the paper but gives important background motivation for the definitions to follow.

Definition 9.1 (Motivational). A transverse map is a manifold M , a based marked rose R_r and a continuous function

$$f : M \rightarrow R_r$$

transverse to all the vertices of R_r .

In the rest of the paper we work with *classes of transverse maps*. Instead of getting into what is an isotopy of transverse maps, etc, we make the following equivalent combinatorial definition.

Definition 9.2. An (*isotopy*) class of (filling) transverse map from a surface with boundary to a marked rose R_r is the finitary combinatorial data:

- a marked base-point and orientation for each boundary component

- a ribbon graph structure on the surface, where vertices are discs and edges correspond to interfaces between discs on their boundaries; such an interface is called an *arc*
- information about which arc is in the preimage of each vertex of R_r other than o , and which sides of the vertex correspond to which sides of the arc
- up to isomorphism of the above data under decoration respecting homeomorphisms.

We refer to these simply as *classes of transverse maps*.

Definition 9.3 (Strict transverse maps). A class of transverse map f as in Definition 9.2 is *strict* if for every two vertices p_1, p_2 of R_r that are consecutive on some circle (and not o), do not have parallel preimages in the following sense. Suppose the orientation of their circle in R_r points from p_1 to p_2 . The preimages of p_1 and p_2 are *parallel* if for every arc α in the preimage of p_1 , there is a parallel⁴ arc in the preimage of p_2 on the side of α ‘towards’ p_2 .

Given a class $[f]$ of transverse map on surface with boundary there is an obvious way to get a boundary class of transverse map on a union of circles, denoted $\partial[f]$.

Given $\kappa : [r] \rightarrow \mathbb{N} \cup \{0\}$ let R_r^κ be the (isomorphism class of) rose with $\kappa(i) + 1$ vertices in the interior of the circle corresponding to x_i . Let $|\kappa| = \sum_{i \in [r]} \kappa_i$.

Given every non-identity $w \in F_r$, let $\mathfrak{w} : S^1 \rightarrow R_r$ denote a fixed immersion⁵ from a based oriented circle to the based rose such that if γ is the generator of $\pi_1(S^1, \text{basepoint})$ corresponding to the chosen orientation,

$$\mathfrak{w}_*(\gamma) = w \in \pi_1(R_r, o).$$

⁴Cobounding a rectangle.

⁵Away from the base point of S^1 , where there could be backtracking if w is not cyclically reduced.

10 Proof of Theorem 3.1 Part 1

Suppose $|w| \leq q$, $\lambda \vdash k, \mu \vdash \ell$ as before. By a result of Koike [Koi89, eq. (0.3)] it is possible to write for $g \in U(n)$

$$s_{\lambda, \mu}(w) = \sum_{|\lambda'| \leq k, |\mu'| \leq \ell} \alpha_{\lambda', \mu'}^{\lambda, \mu} s_{\lambda'}(w) s_{\mu'}(w^{-1})$$

where the coefficients $\alpha_{\lambda', \mu'}^{\lambda, \mu}$ are given in (*ibid.*) as sums of products of Littlewood–Richardson coefficients and importantly, do not depend on n . Then by base change between Schur polynomials as above and power sum symmetric polynomials, we obtain

$$s_{\lambda, \mu}(w) = \sum_{|\lambda'| \leq k, |\mu'| \leq \ell} \beta_{\lambda', \mu'}^{\lambda, \mu} p_{\lambda'}(w) p_{\mu'}(w^{-1}) \quad (10.1)$$

where again the coefficients do not depend on n . This tells us the poles of $\mathbb{E}[s_{\lambda, \mu}(w)]$ are at most the union of every possible pole of

$$\mathbb{E}_n[p_{\lambda'}(w) p_{\mu'}(w^{-1})]$$

with $k' = |\lambda'| \leq k, \ell' = |\mu'| \leq \ell$.

For the purpose of finding the poles, we can here use the Weingarten calculus in the most naive possible way (later, we need to do something more sophisticated). To this end, [MP19, Theorem 2.8] yields

$$\mathbb{E}_n[p_{\lambda'}(w) p_{\mu'}(w^{-1})] = \sum_{\substack{[f: \Sigma \rightarrow R_r^{(2)}] \\ \partial f \cong \mathbf{w} \circ \varphi_{\lambda', \mu'}}} \left(\prod_{1 \leq i \leq r} \text{Wg}_{(k'+\ell')L_i(w)}(\pi_{f,i}) \right) n^{N(f)} \quad (10.2)$$

where $R_r^{(2)} = R_r^{(\kappa)}$ with $\kappa = (2, 2, \dots, 2)$, $\pi_{f,i} \in S_{(k'+\ell')L_i(w)}$ are permutations determined from the combinatorial structure of f , and $N(f) \in \mathbb{N}$ is similar — neither depend on n . It is a finite sum.

Each term

$$\text{Wg}_{(K+\ell)L_i(w)}(\pi_{f,i}) \in \mathbb{Q}(n)$$

is given by the formula for the *Weingarten function* [CS06, eq. (9)]

$$\text{Wg}_L(\pi) = \frac{1}{(L!)^2} \sum_{\lambda \vdash L} \frac{\chi_\lambda(1)^2}{s_\lambda(1)} \chi_\lambda(\pi); \quad (10.3)$$

here we are viewing $s_\lambda(1)$ as the formal element of $\mathbb{Q}(n)$

$$s_\lambda(1) \stackrel{\text{def}}{=} \frac{\prod_{\square \in \lambda} (n + c(\square))}{\prod_{\square \in \lambda} h_\lambda(\square)},$$

$c(\square)$ is the content of the box,

$$c(\square) = j(\square) - i(\square)$$

and $h_\lambda(\square)$ is the hook-length of the box. The only terms in the formula for the Weingarten function that depend on n are therefore the denominators $\prod_{\square \in \lambda} (n + c(\square))$

If $c \geq 0$ then the factor $(n + c(\square))$ appears at most $d(c, L)$ times in the denominator of $\text{Wg}_L(\pi)$ where $d(c, L)$ is the largest natural number d such that

$$d(d + c) \leq L.$$

Clearly $d(c, L) \leq \frac{L}{c}$ for $c > 0$. If $c < 0$ then a similar argument says $(n + c)$ appears at most $d(n, -c)$ times.

This means that $\prod_{\square \in \lambda} (n + c(\square))$ can be written as $n^L Q_\lambda(\frac{1}{n})$ where Q_λ is a polynomial that divides $g_L(\frac{1}{n})$ where (as in (3.2))

$$g_L(x) \stackrel{\text{def}}{=} \prod_{c=1}^L (1 - c^2 x^2)^{\lfloor \frac{L}{c} \rfloor}.$$

Therefore, $\frac{g_L(\frac{1}{n})}{\prod_{\square \in \lambda} (n + c(\square))} = \frac{1}{n^L} \frac{g_L}{Q_\lambda}(\frac{1}{n})$ is a polynomial in $\frac{1}{n}$ of degree $\leq L + \deg(g_L)$. By 10.3 the same is true for $g_L(\frac{1}{n}) \text{Wg}_L(\pi)$. Since

$$\prod_{i=1}^r g_{(k'+\ell')L_i}$$

divides $g_{(k+\ell)q}$, we see from (10.1) that for $n \geq (k + \ell)q$, the rational function agreeing with $g_{(k+\ell)q}(\frac{1}{n}) \mathbb{E}[s_{\lambda, \mu}(w)]$ in this range can be written as a sum of a polynomial in $\frac{1}{n}$ of degree $\leq (k + \ell)q + \deg(g_{(k+\ell)q})$ and possibly a polynomial in n (because of the terms $n^{N(f)}$). However, the polynomial in n is necessarily constant because it is known (see Theorem 1.7 in [MP19, Thm. 1.7]) that $g_{(k+\ell)q}(\frac{1}{n}) \mathbb{E}[s_{\lambda, \mu}(w)]$ remains bounded as $n \rightarrow \infty$.

To conclude, observe that we can bound the degree of g_L by

$$2 \sum_{c=1}^L \lfloor \frac{L}{c} \rfloor \leq 2 \sum_{c=1}^L \frac{L}{c} \leq 2L(1 + \log L),$$

so that $(k + \ell)q + \deg(g_{(k+\ell)q}) \leq 3L(1 + \log L)$.

11 Proof of Theorem 3.1 Part 2 and 3

In the paper [Mag21] the integrals

$$\int_{U(n)^4} \text{Tr}(\tilde{w}(u_1, u_2, u_3, u_4)) s_{\lambda, \mu}([u_1, u_2][u_3, u_4]) du_1 du_2 du_3 du_4$$

w.r.t Haar measure are calculated in a very specific way that exhibits cancellations that cannot be easily seen⁶ using the formula (10.1). The initial part of this calculation is very general and does not depend on \tilde{w} or the word $w = [x_1, x_2][x_3, x_4]$. So the method generalizes as-is to the case when the $\text{Tr}(\tilde{w})$ term is not present and $[x_1, x_2][x_3, x_4]$ is replaced by any reduced word w . That is, to integrals of the form

$$\mathbb{E}_n[s_{\lambda, \mu}(w)] \stackrel{\text{def}}{=} \int_{U(n)^r} s_{\lambda, \mu}(w(u_1, \dots, u_r)) du_1 \cdots du_r$$

where w is any element of \mathbf{F}_r , viewed as a reduced word in the generators.

The analog of [Mag21, Cor. 4.7] here is the following bound. Let R_r^0 be the based CW -complex structure on R_r with one interior vertex per circle (and a vertex for the wedge point o). Recall the immersion $\mathfrak{w} : (S^1, \text{basepoint}) \rightarrow (R_r^0, o)$ from §9. Let

$$\varphi_{\lambda, \mu} : \prod_{i=1}^{\ell(\lambda)} S^1 \sqcup \prod_{i=1}^{\ell(\mu)} S^1 \rightarrow S^1$$

denote the map whose components are

- multiplication by λ_i on the i^{th} circle, if $i \leq \ell(\lambda)$,
- and multiplication by $-\mu_i$ (so orientation reversing) on the $(i - \ell(\lambda))^{\text{th}}$ circle for $i > \ell(\lambda)$.

⁶In fact, we do not know how to.

This map is a convenient way of book-keeping winding numbers.

Proposition 11.1. *We have*

$$\mathbb{E}_n[s_{\lambda,\mu}(w)] = O_{k,\ell,w}(n^{\max_{w,k,\ell} \chi(\Sigma)})$$

where $\max_{w,k,\ell}$ is the maximum over classes of transverse maps $[f]$ from a surface with boundary to R_r^0 such that

Σ is an oriented surface with compatibly oriented boundary

For some $\lambda' \vdash k$ and $\mu' \vdash \ell$, $\partial[f]$ factors as

$$\partial[f] = [\mathfrak{w} \circ \varphi_{\lambda',\mu'}] \tag{11.1}$$

where $[\mathfrak{w} \circ \varphi_{\lambda',\mu'}]$ is the (unique) class of transverse map to R_r^0 obviously associated to the immersion $\mathfrak{w} \circ \varphi_{\lambda',\mu'}$. Components of $\partial\Sigma$ mapped in an orientation respecting (resp. non-orientation respecting) way by $\varphi_{\lambda',\mu'}$ are called positive (resp. negative).

Forbidden Matchings In the factorization of ∂f above, no such arc has endpoints in a positive and negative component of $\partial\Sigma$ that are mapped to the same point under $\varphi_{\lambda',\mu'}$.

This proposition follows formally from [Mag21, §§3.1 - §§4.3], but we give most details in the Appendix, §12.. The **Forbidden Matchings** property is completely crucial and the main point of the method introduced in (*ibid.*). The following topological result is a new input to that method.

Proposition 11.2. *If w is not a proper power in \mathbf{F}_r and cyclically reduced, then any class of transverse map $[f]$ on underlying surface Σ satisfying the conditions of Proposition 11.1 has*

$$\chi(\Sigma) \leq -\frac{k+\ell}{3}.$$

Corollary 11.3 (Theorem 3.1 Part 2). *If w is not a proper power in \mathbf{F}_r then*

$$\mathbb{E}_n[s_{\lambda,\mu}(w)] = O_{k,\ell,w}\left(n^{-\frac{1}{3}(k+\ell)}\right).$$

N.B. One can assume without loss of generality in Corollary 11.3 that w is cyclically reduced, as conjugating w does not change $\mathbb{E}_n[s_{\lambda,\mu}(w)]$.

11.1 Proof of Proposition 11.2

Suppose w is cyclically reduced and not a proper power or the identity, with word length $|w|$. Suppose that some class of transverse map $[f : \Sigma \rightarrow R_r^0]$ satisfies **P1-P4** of Proposition 11.1.

Let $L(e)$ denote the number of edges in the chain defining e .

Let V^*, E^* denote the topological vertices and edges respectively. We have

$$2E^* = \sum_{v \in V^*} d(v) \geq 3V^*$$

and hence

$$-\chi(\Sigma) = E^* - V^* \geq \frac{E^*}{3}. \quad (11.2)$$

We now aim to show E^* is large under the previous assumptions.

Each class of strict transverse map f is a (isomorphism class of) ribbon graph \mathcal{R} whose edges correspond to the arcs of f . Each edge of the ribbon graph inherits a direction and labeling by $\{x_1, \dots, x_r\}$ from the manner in which f crosses the (sole) interior vertex of the i^{th} circle in R_r^0 along this edge.

Say that a vertex v of \mathcal{R} is a *topological vertex* if it has valence $d(v) \geq 3$. A *topological edge* e is a maximal chain of edges incident at vertices of valence 2. (As w is cyclically reduced, there are no vertices of valence 1).

Every topological edge e is bordered by two segments of $\partial\mathcal{R}$ that each spell a subword of w^{k_i} or $w^{-\ell_i}$ for some i , reading along the fixed orientation of the boundary.

Suppose that for some topological edge e , $L(e) > |w|$. For a reduced word v in the generators of \mathbf{F}_r , write \bar{v} for its mirror (reversing order and replacing generators by inverses). Then both boundary segments spell words

$$u_1 w^a u_2, \quad v_1 w^b v_2$$

with $a, b \in \mathbb{Z} \setminus \{0\}$, $|u_1|, |u_2|, |v_1|, |v_2| < |w|$.

Case 1. If a, b have same sign. In this case one arrives at

$$wu = v\bar{w}$$

with $|u|, |v| < w$. (see Figure 11.1). This is impossible since then v is a prefix of w and \bar{v} is too, meaning $v = \bar{v}$ since they have the same length, so v is empty. Similarly u is empty and $w = \bar{w}$ contradicting w being the identity.

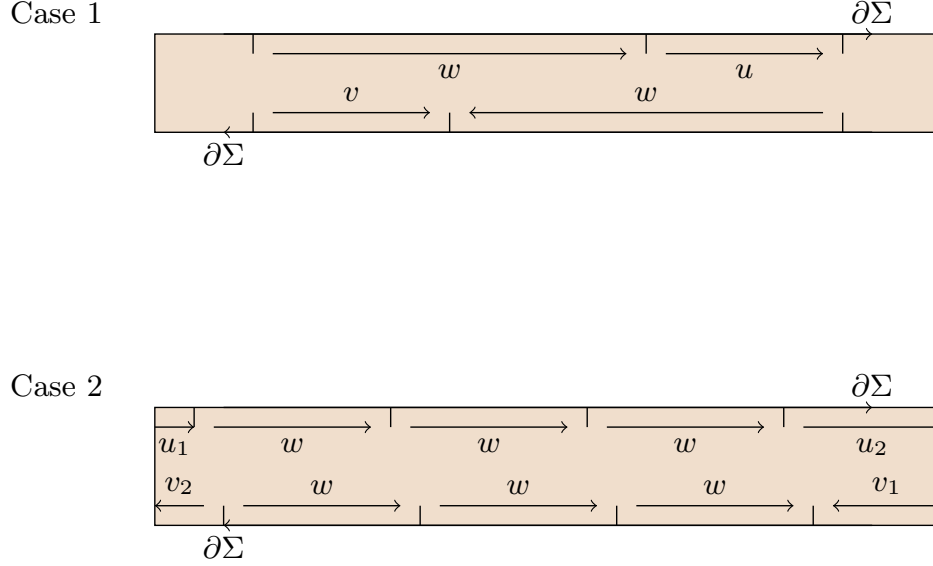


Figure 11.1: Auxiliary to proof of Proposition 11.2.

Case 2. If a, b have opposite signs. So without loss of generality assume $a > 0$ and $b = -B$ with $B > 0$, then we have

$$u_1 w^a u_2 = \bar{v}_2 w^B \bar{v}_1.$$

Condition **P4** implies that $|u_1| \neq |\bar{v}_2|$. Again without loss of generality assume $|u_1| < |\bar{v}_2|$. Write $\bar{v}_2 = u_1 w_1$.

If $|u_2| > |\bar{v}_1|$ then write $u_2 = w_2 \bar{v}_1$ to obtain

$$w^a w_2 = w_1 w^B \tag{11.3}$$

as reduced words. This immediately implies $w_2 = w_1$, as both are prefixes of w of the same length. Since w is not a proper power, e.g. [Raz14, Lemma 2.2] implies that w_1 is a power of w , which must be empty since its length is less than w .

If $|u_2| < |\bar{v}_1|$ then write $w_2 u_2 = \bar{v}_1$ to arrive at

$$w^a = w_1 w^B w_2.$$

This implies w_1 and w_2 are both prefixes and suffixes of w .

Write $w = w_1 w_3$ and obtain

$$w_3 w^{a-1} = w^B w_2.$$

Since $B > 0$, $a > 1$ here and we argue as we did from (11.3) to get a contradiction.

The upshot of this argument is that each topological edge e has

$$L(e) \leq |w|.$$

Since the total lengths of topological edges is $(k+\ell)|w|$ this gives $|E^*| \geq k+\ell$. Hence from (11.2)

$$-\chi(\Sigma) \geq \frac{k+\ell}{3}.$$

11.2 Proof of Theorem 3.1 Part 3

If $\mu = \emptyset$, i.e. $s_{\lambda,\mu} = s_\lambda$ is a polynomial stable character, then instead we can use the following two results.

For $w \in \mathbf{F}_r$ the commutator length of w , denoted by $\text{cl}(w) \in \mathbb{N} \cup \{\infty\}$ is defined by

$$\text{cl}(w) = \inf \{ g : w = [u_1, v_1] \cdots [u_g, v_g], u_i, v_i \in \mathbf{F}_r \}$$

interpreted as ∞ if it is not possible to write w as a product of commutators, i.e. $w \notin [\mathbf{F}_r, \mathbf{F}_r]$. The *stable commutator length* of w , denoted $\text{scl}(w)$, is defined as

$$\text{scl}(w) \stackrel{\text{def}}{=} \lim_{m \rightarrow \infty} \frac{\text{cl}(w^m)}{m}.$$

On \mathbf{F}_r , scl takes values in \mathbb{Q} by a result of Calegari [Cal09]. Duncan and Howie proved in [DH91] that

$$\text{scl}(w) \geq \frac{1}{2} \tag{11.4}$$

for all $w \in \mathbf{F}_r$. This bound can be combined with the following result of the first named author and Puder following immediately from [MP19, Cor. 1.11].

Theorem 11.4. *We have*

$$\mathbb{E}_n[s_\lambda(w)] = O\left(\frac{1}{n^{2k\text{scl}(w)}}\right). \tag{11.5}$$

In fact, [MP19, Cor. 1.11] says that for every $w \in [\mathbf{F}_r, \mathbf{F}_r]$, there is some λ such that the bound (11.5) is saturated, but we do not use this here.

Combining (11.4) with (11.5) gives

$$\mathbb{E}_n[s_\lambda(w)] = O(n^{-k})$$

as required.

12 Appendix: Extension of the method of Random Unitary Representations of Surface Groups II

In this section we explain the extension of the methodology of [Mag21] to bound the integrals $\mathbb{E}_n[s_{\lambda,\mu}(w)]$ defined in (3.1). We import some results from [Mag21].

Background

In this section, fix $\lambda \vdash k$ and $\mu \vdash \ell$, and assume $n \geq k + \ell$. Let $D_{\lambda,\mu}(n) = s_{\lambda,\mu}(1)$. Write χ_λ for the character of S_k associated to the irreducible representation W^λ that corresponds to λ . Let $d_\lambda \stackrel{\text{def}}{=} \chi_\lambda(\text{id}) = \dim W^\lambda$. Given $\lambda \vdash k$, the element

$$\mathfrak{p}_\lambda \stackrel{\text{def}}{=} \frac{d_\lambda}{k!} \sum_{\sigma \in S_k} \chi_\lambda(\sigma) \sigma \in \mathbb{C}[S_k]$$

is the central projection in $\mathbb{C}[S_k]$ to the W^λ -isotypic component. We view $S_k \times S_\ell$ as a subgroup of $S_{k+\ell}$ in the standard way.

Let

$$\mathfrak{p}_{\lambda \otimes \mu} \stackrel{\text{def}}{=} \mathfrak{p}_\lambda \mathfrak{p}'_\mu$$

where \mathfrak{p}'_μ is the image of \mathfrak{p}_μ under the inclusion $S_\ell \leq S_k \times S_\ell \leq S_{k+\ell}$. We define Young subgroups, for $\lambda \vdash k$

$$S_\lambda \stackrel{\text{def}}{=} S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_{\ell(\lambda)}} \leq S_k.$$

Let

$$\mathcal{T}_n^{k,\ell} \stackrel{\text{def}}{=} (\mathbb{C}^n)^{\otimes k} \otimes ((\mathbb{C}^n)^\vee)^{\otimes \ell}.$$

For $I = (i_1, \dots, i_{k+\ell})$ let

$$\begin{aligned} I'(I; \pi) &\stackrel{\text{def}}{=} i_{\pi(1)}, \dots, i_{\pi(k)}, \\ J'(I; \pi) &\stackrel{\text{def}}{=} i_{\pi(k+1)}, \dots, i_{\pi(k+\ell)}. \end{aligned}$$

For $\pi \in S_{k+\ell}$ let

$$\Phi(\pi) \stackrel{\text{def}}{=} \sum_{I=(i_1, \dots, i_k), J=(j_{k+1}, \dots, j_{k+\ell})} e^{J_{I' \sqcup J; \pi}} \otimes \check{e}_I^{J'(I \sqcup J; \pi)} \in \text{End}(\mathcal{T}_n^{k, \ell})$$

and extend the map Φ linearly to $\mathbb{C}[S_{k+\ell}]$.

Recall the definition of the Weingarten function from (10.3). Let

$$\begin{aligned} z &\stackrel{\text{def}}{=} \sum_{\tau \in S_{k+\ell}} z(\tau) \tau \tag{12.1} \\ &\stackrel{\text{def}}{=} \frac{[S_k : S_\lambda][S_\ell : S_\mu]}{d_\lambda d_\mu} \mathfrak{p}_{\lambda \otimes \mu} \left(\sum_{\sigma \in S_\lambda \times S_\mu} \sigma \right) \mathfrak{p}_{\lambda \otimes \mu} \text{Wg}_{n, k+\ell} \in \mathbb{C}[S_{k+\ell}]. \end{aligned}$$

One has the following bound on the coefficients of z [Mag21, Lemma 3.3]

$$z(\tau) = O_{k, \ell}(n^{-k-\ell-\|\tau\|_{k, \ell}}). \tag{12.2}$$

Let

$$\mathfrak{q} \stackrel{\text{def}}{=} D_{\lambda, \mu}(n) \Phi(z).$$

Theorem 12.1.

1. The operator \mathfrak{q} is an orthogonal projection with $U(n)$ -invariant image that is isomorphic to $V^{\lambda, \mu}$ as a $U(n)$ -representation.
2. For any $i_1, \dots, i_{k+\ell}, j_1, \dots, j_{k+\ell}$ and any $p \in [k + \ell]$,

$$\sum_u \mathfrak{q}_{(i_1 \dots i_{p-1} u i_{p+1} \dots i_{k+\ell}), (j_1 \dots j_{p-1} u j_{p+1} \dots j_{k+\ell})} = 0.$$

Part 1 is the combination of [Mag21, Lemma 2.3, eq. (3.11), Prop. 3.2, eq. (3.12)].

Part 2 arises from the fact that \mathfrak{q} is zero on the orthocomplement to the contraction free subspace $\dot{\mathcal{T}}_n^{k, \ell}$ from [Mag21].

Combinatorial integration

We write w in reduced form:

$$w = f_1^{\epsilon_1} f_2^{\epsilon_2} \dots f_q^{\epsilon_q}, \quad \epsilon_u \in \{\pm 1\}, f_u \in \{x_i\}, \quad (12.3)$$

where if $f_u = f_{u+1}$, then $\epsilon_u = \epsilon_{u+1}$. For $f \in \{a, b, c, d\}$ let p_f denote the number of occurrences of f^{+1} in (12.3). The expression (12.3) implies that for $u \stackrel{\text{def}}{=} (u_i : i \in [r]) \in \mathbb{U}(n)^r$,

$$s_{\lambda, \mu}(w(u)) = \text{Tr}_{\mathcal{T}_n^{k, \ell}} \left(\mathbf{q} f_1^{\epsilon_1} \mathbf{q} f_2^{\epsilon_2} \mathbf{q} \dots \mathbf{q} f_{|w|}^{\epsilon_{|w|}} \right) \quad (12.4)$$

$$= \sum_{I_j \in [n]^k, J_j \in [n]^\ell} \prod_{u=1}^q \mathbf{q}_{K_i \sqcup L_i, \mathbf{q}_{I_{i+1} \sqcup J_{i+1}}} \\ (u_{f_1}^{\epsilon_1})_{I_1 \sqcup J_1, K_1 \sqcup L_1} (u_{f_2}^{\epsilon_2})_{I_2 \sqcup J_2, K_2 \sqcup L_2} \dots (u_{f_q}^{\epsilon_q})_{I_q \sqcup J_q, K_q \sqcup L_q} \quad (12.5)$$

$$= \sum_{\pi_1, \dots, \pi_1 \in \mathcal{S}_{k+\ell}} \prod_{i=1}^q z(\pi_i) \sum_{I_j \in [n]^k, J_j \in [n]^\ell} \mathbf{q}_{K_i \sqcup L_i, \mathbf{q}_{I_{i+1} \sqcup J_{i+1}}} \\ (u_{f_1}^{\epsilon_1})_{I_1 \sqcup J_1, K_1 \sqcup L_1} (u_{f_2}^{\epsilon_2})_{I_2 \sqcup J_2, K_2 \sqcup L_2} \dots (u_{f_q}^{\epsilon_q})_{I_q \sqcup J_q, K_q \sqcup L_q} \\ \mathbf{1} \{ K_i \sqcup J_{i+1} = (I_{i+1} \sqcup L_i) \circ \pi_i : i \in [q] \}. \quad (12.6)$$

In the last line above the indices run mod q . Each product of matrices here can be integrated using the Weingarten calculus.

We view all I_u etc as functions from *indices* (the domain) to $[n]$. We view all domains for distinct u as disjoint as possible.

There are however, fixed matchings between the domains of each

$$I_u \sqcup J_u \text{ and } K_u \sqcup L_u.$$

These will come into play later.

We now define sub-collections of all the indices that are treated as ‘the same type’ by the Weingarten calculus.

For each $i \in [r]$, let:

- \mathcal{S}_i be all indices of I_u such that $f_u = x_i$ and $\epsilon_u = +1$ and indices of L_u such that $f_u = x_i$ and $\epsilon_u = -1$,
- \mathcal{S}_i^* be all indices of J_u such that $f_u = x_i$ and $\epsilon_u = +1$ and indices of K_u such that $f_u = x_i$ and $\epsilon_u = -1$,

- \mathcal{T}_i be all indices of K_u such that $f_u = x_i$ and $\epsilon_u = +1$ and indices of J_u such that $f_u = x_i$ and $\epsilon_u = -1$,
- \mathcal{T}_i^* be all indices of L_u such that $f_u = x_i$ and $\epsilon_u = +1$ and indices of I_u such that $f_u = x_i$ and $\epsilon_u = -1$.

Let \mathbf{M} denote the set of data consisting of

- for each $i \in [r]$, σ_i a bijection from \mathcal{S}_i to \mathcal{S}_i^* ,
- for each $i \in [r]$, τ_i a bijection from \mathcal{T}_i to \mathcal{T}_i^* .

Doing the integral of terms in either (12.5) or (12.6) replaces

$$(u_{f_1}^{\epsilon_1})_{I_1 \sqcup J_1, K_1 \sqcup L_1} (u_{f_2}^{\epsilon_2})_{I_2 \sqcup J_2, K_2 \sqcup L_2} \cdots (u_{f_q}^{\epsilon_q})_{I_q \sqcup J_q, K_q \sqcup L_q}$$

by

$$\sum_{\mathbf{D} \in \mathbf{M}} \prod_{i \in [r]} W_{g_{k+\ell}(\sigma_i \tau_i^{-1})} \mathbf{1}\{\mathbf{D} \text{ 'respected' by } I_u, J_u, K_u, L_u\}.$$

Before going on, we make a key argument. Suppose that some index of I_2 is matched to an index of J_2 by \mathbf{D} , for example the two first indices. Then the result of integrating (12.5), the terms corresponding to \mathbf{D} contain all contain factors

$$\sum_{a=I_2(1)=J_2(1)} \mathfrak{q}_{K_1 \sqcup L_1}, \mathfrak{q}_{I_2 \sqcup J_2}$$

where all indices but the first indices of I_2 and J_2 are frozen (conditioned upon). This is zero by Theorem 12.1 part 2.

This means, going back to (12.6), if we define \mathbf{M}^* to be the subset of \mathbf{M} such that

- σ_i never matches indices I_u to those of J_u for any u ,
- σ_i never matches indices of L_u to those of K_u for any u ,
- τ_i never matches indices of K_u to those of L_u for any u ,
- τ_i never matches indices of J_u to those of I_u for any u ,

then we obtain

$$\mathbb{E}_n[s_{\lambda,\mu}(w(u))] = \sum_{\pi_1, \dots, \pi_q \in S_{k+\ell}} \sum_{\mathbf{D} \in \mathbf{M}^*} \prod_{i=1}^q z(\pi_i) \prod_{i \in [r]} W_{g_{k+\ell}(\sigma_i \tau_i^{-1})} \mathcal{N}(\pi_1, \dots, \pi_q, \mathbf{D}) \quad (12.7)$$

where $\mathcal{N}(\pi_1, \dots, \pi_q, \mathbf{D})$ is the number of choices of I, J, K, L such that

$$K_u \sqcup J_{u+1} = (I_{u+1} \sqcup L_u) \circ \pi_u : u \in [q], \quad (12.8)$$

indices matched by \mathbf{D} have the same value.

Surface construction

Now construct a surface as follows.

Begin with a vertex for every index. Add an edge (called \mathbf{D} -edge) between all matched indices (by \mathbf{D}). Add an edge (called w -edge) between indices paired by the fixed identifications

$$I_u \sqcup J_u \cong [k + \ell] \cong K_u \sqcup L_u$$

and direct this edge from the indices on the left hand side above to those on the right hand side.

Add also an edge (called π -edge) between indices matched by (12.8). We now have a trivalent graph.

We now glue two types of discs to this graph following [Mag21, §§4.1]. The boundaries of the discs are glued along two types of cycles in the graph:

Type-I Cycles that alternate between π -edges and \mathbf{D} edges. Such cycles are disjoint.

Type-II Cycles that alternate between w -edges and \mathbf{D} -edges. Again, such cycles are disjoint.

The resulting glued discs therefore meet only along the \mathbf{D} -edges and the resulting total object is a topological surface we call

$$\Sigma(\mathbf{D}, \{\pi_i\}).$$

The boundary cycles of this surface alternate between w -edges and π -edges.

For $\sigma \in S_{k+\ell}$, let $\|\sigma\|_{k,\ell}$ denote the minimum m for which

$$\sigma = \sigma_0 t_1 t_2 \cdots t_m$$

where $\sigma_0 \in S_k \times S_\ell$ and t_1, \dots, t_m are transpositions in $S_{k+\ell}$. The analog of [Mag21, Thm. 4.3] is that, after an elementary calculation using (12.2) and bounds for the Weingarten function one obtains

$$\prod_{i=1}^q z(\pi_i) \prod_{i \in [r]} \text{Wg}_{\mathfrak{S}_{k+\ell}}(\sigma_i \tau_i^{-1}) \mathcal{N}(\pi_1, \dots, \pi_q, \mathbf{D}) \ll_{w,k,\ell} n^{-\sum_i \|\pi_i\|_{k,\ell}} n^{\chi(\Sigma(\mathbf{D}, \{\pi_i\}))},$$

hence from (12.7)

$$\mathbb{E}_n[s_{\lambda,\mu}(w(u))] \ll_{w,k,\ell} \sum_{\pi_1, \dots, \pi_q \in S_{k+\ell}} \sum_{\mathbf{D} \in \mathbf{M}^*} n^{-\sum_i \|\pi_i\|_{k,\ell}} n^{\chi(\Sigma(\mathbf{D}, \{\pi_i\}))}.$$

Surfaces with large contribution

It is shown in [Mag21, §§4.2] — the same proof applies without change to the current setting — that given any

$$\pi_1, \dots, \pi_q \in S_{k+\ell}, \mathbf{D} \in \mathbf{M}^*,$$

it is possible to modify these so that

$$\pi'_1, \dots, \pi'_q \in S_k \times S_\ell, \quad \sum_i \|\pi'_i\|_{k,\ell} = 0 \quad (12.9)$$

$$\mathbf{D}' = \{\sigma'_i, \tau'_i\} \in \mathbf{M}^* \text{ with } \sigma'_i = \tau'_i \text{ for all } i \in [r] \quad (12.10)$$

and there exists an inequality between corresponding terms

$$n^{\chi(\Sigma(\mathbf{D}', \{\pi'_i\}))} \geq n^{-\sum_i \|\pi_i\|_{k,\ell}} n^{\chi(\Sigma(\mathbf{D}, \{\pi_i\}))}.$$

The condition (12.10) means all type-II cycles are now rectangles with two (non-consecutive) edges in the boundary; we now replace each rectangle with an arc connecting the two boundary segments of the rectangle.

Connection to transverse maps

We now create a class of transverse map on the surface $\Sigma(\mathbf{D}', \{\pi'_i\})$ as follows (cf. Definition 9.2).

The ribbon graph structure is the one dictated by the arcs we just prior constructed. By construction, they cut the surface into discs. For each

boundary component, place a marked point in some (it does not matter) π -edge that arose from π'_q . Recall the CW -complex rose R_r^0 with one point (call it z_i) in the interior of the circle corresponding to x_i . If an arc arose from a rectangle with edges that arose from $\sigma'_i = \tau'_i$ then we declare our function to take the constant value z_i on this arc and the transverse map will traverse the arc from the σ'_i side to the τ'_i side.

The property (12.9) implies that for every boundary component of the surface, the w -edges are all directed the same way along this boundary component and hence give an orientation to the boundary. With respect to this orientation, the isotopy class of SF transverse map satisfies (11.1).

The fact that $\mathbf{D}' \in \mathbf{M}^*$, rather than \mathbf{M} , implies that the class of transverse map has the crucial **Forbidden Matchings** property.

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