



HAL
open science

A note on existence and asymptotic behavior of Lagrangian equilibria for first-order optimal-exit mean field games

Guilherme Mazanti

► **To cite this version:**

Guilherme Mazanti. A note on existence and asymptotic behavior of Lagrangian equilibria for first-order optimal-exit mean field games. 2024. hal-04726422

HAL Id: hal-04726422

<https://hal.science/hal-04726422v1>

Preprint submitted on 8 Oct 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A note on existence and asymptotic behavior of Lagrangian equilibria for first-order optimal-exit mean field games

Guilherme Mazanti*

Abstract

In this paper, we consider a first-order mean field game model motivated by crowd motion in which agents evolve in a (not necessarily compact) metric space and wish to reach a given target set. Each agent aims to minimize the sum of their travel time and an exit cost which depends on their exit position on the target set. Agents interact through their dynamics, the maximal speed of an agent being assumed to be a function of their position and the distribution of other agents. This interaction may model, in particular, congestion phenomena. Under suitable assumptions on the model, we prove existence of Lagrangian equilibria, analyze the asymptotic behavior for large time of the distribution of agents, and study the dependence of equilibria and asymptotic limits on the initial distribution of the agents.

2020 Mathematics Subject Classification. 49N80, 91A16, 93C15.

Keywords. Mean field games, Lagrangian equilibria, congestion games, existence of equilibria, asymptotic behavior.

1 Introduction

Since their introduction around 2006 by the simultaneous works of Peter E. Caines, Minyi Huang, and Roland P. Malhamé [69–71] and of Jean-Michel Lasry and Pierre-Louis Lions [76–78], following previous works in the economics literature on games with infinitely many agents [9, 73], mean field games (MFGs) have attracted the interest of a large number of researchers, due both to the interesting and challenging theoretical questions raised by their analysis and their wide spectrum of applications, ranging from engineering and economics to the modeling of crowd motion and epidemics [10, 69, 74, 78]. Works on this topic have considered questions such as existence and uniqueness of their equilibria, approximation of games with many players by MFGs, numerical methods for approximating equilibria of MFGs, or the characterization of equilibria through the master equation, among others [1, 33, 39, 62, 79, 86]. We refer to [2, 40, 41] for more details on recent topics on mean field games.

While the majority of the works on mean field games consider that the agents of the game evolve in a given time interval $[0, T]$, many applications involve agents that may leave the game before its terminal time T , and also games in which a terminal time T is not prescribed and agents evolve in $[0, +\infty)$, leaving the game at some point. This is the case, for instance, of [23, 42], which consider mean field games with applications to bank run, i.e., situations in which clients of a bank, believing that the bank is about to fail, withdraw all their money, and try to choose the time to withdraw the money in an

*Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France.

optimal way. These models belong to a more general class of mean field game problems known as mean field games of optimal stopping, in which the main choice of an agent is when to stop the game [17, 20, 61, 85]. Other works, such as [63], consider economic models for the production of exhaustible resources, in which firms, who wish to maximize their profit, produce goods based on exhaustible resources, and they leave the game when they deplete their capacities. These games can be seen as mean field games with an absorbing boundary, i.e., in which agents who reach a certain part of the boundary of the domain immediately leave the game [24]. An important kind of optimal control problem with an absorbing boundary is that of conditional exit control problems, studied in details in [43], in which the running cost of an agent at time t is conditioned to the fact that the agent is still in the domain at time t . In the sequel, we refer to MFGs in which agents choose the time at which they leave the game as *optimal-exit* MFGs.

This paper considers a first-order optimal-exit MFG in which a continuum of rational agents evolve in a given metric space X , the aim of each agent being to reach a given target set $\Gamma \subset X$ while minimizing the sum of the time they take to reach Γ and a cost depending on the arrival position at Γ . We also assume that, at each time, the velocity of an agent is bounded by a function depending on their position and the current distribution of other agents, that is,

$$|\dot{\gamma}|(t) \leq K(m_t, \gamma(t)), \quad (1.1)$$

where γ is the trajectory of the agent, $|\dot{\gamma}|$ is its metric derivative, m_t is a probability measure on X describing the distribution of agents at time t , and K is a function taking positive values. This MFG model is inspired by the study of crowd motion: agents of the game may represent pedestrians moving in the spatial domain X who wish to reach an exit of X , the set of possible exits of X being Γ . In this context, the bound on the velocity of an agent from (1.1) can be interpreted as a model for congestion, which, roughly speaking, should model the fact that an agent is physically blocked by other agents in regions of large density, and thus they cannot move faster than a certain maximal speed, which depends on their position and the density of other agents around their position.

Motivated by understanding and, if possible, controlling and optimizing the flow of large groups of people, several works have addressed the mathematical modeling of crowd motion [67, 68, 82]. Among the diversity of crowd motion models available in the literature, those of interest when dealing with MFGs are the *macroscopic* models, in which the crowd at a given time t is represented by a probability measure m_t on the space of possible positions X , which evolves according to some conservation law, typically a continuity equation of the form $\partial_t m + \operatorname{div}(mV) = 0$, where V is the velocity field followed by the agents. While most macroscopic crowd motion models consider a given velocity field V constructed from modeling assumptions, the mean field game approach consists instead in considering that each agent will choose their trajectory by solving some optimal control problem, the velocity field V being a consequence of the optimal choices of the agents. In other words, MFG models for crowd motion usually try to capture strategic choices of the crowd based on the rational anticipation by an agent of the behavior of others.

Up to the author's knowledge, the first work to be fully dedicated to a mean field game model for crowd motion is [74], which proposes an MFG model for a two-population crowd with trajectories perturbed by additive Brownian motion and considers both their stationary distributions and their evolution on a prescribed time interval. Since then, many other works have studied MFG models for (or related to) crowd motion with a diversity of modeling perspectives and assumptions, such as [11, 15, 22, 36, 49, 53, 55, 83, 89–91]. A key feature for an MFG model for crowd motion is that it should take into account the difficulty of pedestrians of passing through congested regions, and several works in the

literature do that by a suitable penalization term in the cost function, yielding the so-called *MFGs of congestion* [3, 56, 60]. These games typically consider costs which include a product of the form $m_t^\alpha |\dot{\gamma}(t)|^\beta$ (for some exponents $\alpha, \beta > 0$), where m_t is evaluated at or in a neighborhood of $\gamma(t)$, meaning that high velocities are costly, and that they are even more costly in the presence of high concentrations. This is not the point of view considered in this paper, which models congestion as a constraint on the velocity of agents through (1.1).

In order to properly model congestion, the function K in (1.1) should compute $K(\mu, x)$ for some distribution of agents μ and some position x by evaluating μ at or around x and giving as a result some nonincreasing function of this evaluation, meaning that the maximal speed of an agent is a nonincreasing function of some “average density” around x . A natural choice, for instance, would be $K(\mu, x) = \max(0, 1 - \mu(x))$ if μ is absolutely continuous with respect to the Lebesgue measure, with a continuous density μ , and $+\infty$ otherwise, which is the one adopted in Hughes’ model for crowd motion [72]. However, due to the lack of regularity of this choice of K — and of any other choice of K depending on a local evaluation of μ at a given position x —, there are no general mathematical results on the existence of solutions for neither Hughes’ nor MFG models, apart from some results for Hughes’ model in dimension 1 [5, 7, 52]. The interested reader can find an overview of mathematical results for the Hughes’ model in the recent survey [4] and further links between Hughes’ model and MFGs in [59]. We also highlight that the choice of K in Hughes’ model is an indirect way to model a density constraint, since it imposes that agents can only move with a positive speed in the domain $\{x \in X \mid \mu(x) < 1\}$. We refer to [36, 45, 50, 80, 84] for mean field games and multi-agent optimal control problems with density constraints.

To avoid issues concerning the regularity of K , our model is motivated by the case where K is nonlocal. A typical kind of K one can keep in mind is

$$K(\mu, x) = \kappa \left(\int_X \chi(x, y) \eta(y) d\mu(y) \right), \quad (1.2)$$

where $\chi: X \times X \rightarrow [0, +\infty)$ is a kernel, $\eta: X \rightarrow [0, +\infty)$ may serve as a weight on X or as a cut-off function to discount some part of X , and the nonincreasing function $\kappa: [0, +\infty) \rightarrow [0, +\infty)$ provides the maximal speed in terms of the average density computed by the integral. Even though the results presented in this paper do not assume a particular form for K , most of our assumptions are introduced having in mind that they should be verified for (1.2) under suitable regularity assumptions on κ , χ , and η .

The MFG model considered here is an extension of those treated in [55, 83, 89–91] and is also related to [53], which considers a second-order version of the model. We refer to those references, and in particular to the introduction of [53], for further discussion on the model and more details on its relations to other mean field games considered in the literature. A detailed comparison between our model and our results with those from [55, 83, 89–91] is provided in Section 3.3.

In this work, as in [55, 83, 89–91], we adopt a Lagrangian framework for describing equilibria of the mean field game we consider: instead of representing the movement of the agents as a time-dependent probability measure m_t on X , we make use instead of a probability measure Q on the set of all possible trajectories of the agents in X . The Lagrangian formulation is a classical approach in optimal transport problems (see, e.g., [6, 92]), which has been used for instance in [21] to study incompressible flows, in [38] for Wardrop equilibria in traffic flow, or in [16] for branched transport problems. The use of the Lagrangian approach in mean field games dates back at least to [32, 36], and since then it has been used in several works, such as [15, 25–27, 55, 57, 83, 89–91]. Some

of these references call equilibria in a Lagrangian framework as *Lagrangian equilibria* but, for simplicity, we will only use the term *equilibria* in the sequel of this paper.

In addition to proving existence of equilibria, we will also consider in this paper the asymptotic behavior at an equilibrium of the probability measure m_t describing the distribution of agents at time t as $t \rightarrow +\infty$. The study of the asymptotic behavior of mean field games and optimal control problems is a classical problem that was addressed in several works in the literature. In the context of optimal control, a standard question, addressed for instance in [12, 14, 28, 51, 65, 75, 88, 95], is to understand the large-time behavior of the value function of an optimal control problem, and studying more precisely whether the ergodic limit of such a value function converges to the value function of a suitable stationary problem, described as the solution of a stationary Hamilton–Jacobi equation. As for mean field games, a frequent question, addressed for instance in [13, 31, 34, 35, 37, 81], is that of the average behavior of equilibria of games on a finite time interval $[0, T]$ as the time horizon T tends to $+\infty$, most works being interested in whether such a limit can be characterized by a system of stationary PDEs, which can itself be interpreted as a stationary mean field game in infinite horizon. Some works also study the so-called *turnpike property* [46, 58, 86, 93, 94], which consists on the fact that, for optimal control problems or mean field games in a large time interval $[0, T]$, the behavior of the solution in an interval of the form $[\varepsilon, T - \varepsilon]$ for some suitable $\varepsilon > 0$ can be approached by the solution of a stationary problem. The asymptotic analysis carried out in this paper is more closely related to the classical asymptotic analysis of dynamical systems [66, 87], as it can be seen as the computation of the ω -limit set of optimal trajectories, the relation with the asymptotic analysis of optimal control problems and mean field games coming from the fact that the dynamics of the distribution of agents $t \mapsto m_t$ is obtained through the solution of optimal control problems.

The paper is organized as follows. Section 2 sets the notation and provides definitions important for the sequel of the paper, such as those of Wasserstein distance and metric derivative. The mean field game model considered in this paper is described in details in Section 3, which also considers an associated optimal control problem. We also provide, in Section 3, the main assumptions used in the paper, and compare our setting and our results to those of [55, 83, 89–91]. Preliminary properties of the mean field game and the optimal control problem considered in the paper are provided in Section 4. Finally, our main results are presented in Section 5: existence of equilibria of the game is shown in Section 5.1, the asymptotic behavior of the distribution of agents m_t as $t \rightarrow +\infty$ is the subject of Section 5.2, and the dependence of equilibria and asymptotic limits on the initial distribution of agents is studied in Section 5.3.

2 Notation and definitions

In this paper, we denote by \mathbb{N} , \mathbb{N}^* , \mathbb{R}_+ , and \mathbb{R}_+^* the sets of nonnegative integers, positive integers, nonnegative real numbers, and positive real numbers, respectively. Given a subset A of a topological space X , its boundary and closure are denoted by ∂A and \bar{A} , respectively. The open and closed balls in a metric space X centered at $x \in X$ and with radius $r \geq 0$ are denoted by $B_X(x, r)$, $\bar{B}_X(x, r)$, respectively. Denoting by \mathbf{d} the metric of X , given $x \in X$ and $A \subset X$, we write $\mathbf{d}(x, A)$ for the distance from the point x to the set A , defined by $\mathbf{d}(x, A) = \inf_{y \in A} \mathbf{d}(x, y)$. We use $|x|$ to denote the Euclidean norm of a vector $x \in \mathbb{R}^d$.

Given two metric spaces X and Y , we use $\mathcal{C}(X, Y)$ (respectively, $\text{Lip}(X, Y)$) to denote the set of continuous (respectively, Lipschitz continuous) functions from X to Y . Given

$c \geq 0$, we define $\text{Lip}_c(X, Y)$ as the subset of $\text{Lip}(X, Y)$ made of those functions whose Lipschitz constant is upper bounded by c . For ease of notation, we write simply \mathcal{C}_X (respectively, $\text{Lip}(X)$, $\text{Lip}_c(X)$) for $\mathcal{C}(\mathbb{R}_+, X)$ (respectively, $\text{Lip}(\mathbb{R}_+, X)$, $\text{Lip}_c(\mathbb{R}_+, X)$). We always assume that \mathcal{C}_X , $\text{Lip}(X)$, and $\text{Lip}_c(X)$ are endowed with the topology of uniform convergence on compact sets and we recall that, when (X, \mathbf{d}) is a complete and separable metric space, \mathcal{C}_X is a Polish space (see, e.g., [19, Corollary 3, page X.9; Corollary, page X.20; and Corollary, page X.25]). Whenever needed, we endow \mathcal{C}_X with the complete distance

$$\mathbf{d}_{\mathcal{C}_X}(\gamma_1, \gamma_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{t \in [0, n]} \mathbf{d}(\gamma_1(t), \gamma_2(t)).$$

Recall that, if X is compact, then, thanks to Arzelà–Ascoli theorem [19, Corollary 3, page X.19], $\text{Lip}_c(X)$ is compact. For $t \in \mathbb{R}_+$, we denote by $e_t: \mathcal{C}_X \rightarrow X$ the evaluation map given by $e_t(\gamma) = \gamma(t)$.

We will make use in this paper of set-valued maps, and we denote $f: A \rightrightarrows B$ to say that f is a set-valued map defined on A and taking as values subsets of B , i.e., $f(a) \subset B$ for every $a \in A$.

Given a Polish space X , the set of all Borel probability measures defined on X is denoted by $\mathcal{P}(X)$. The support of a measure $\mu \in \mathcal{P}(X)$ is denoted by $\text{supp}(\mu)$. Given two measurable spaces X and Y , a measurable map $f: X \rightarrow Y$, and a measure μ in X , we use $f_{\#}\mu$ to denote the pushforward of μ through f , i.e., the measure on Y defined by $f_{\#}\mu(A) = \mu(f^{-1}(A))$ for every measurable set $A \subset Y$.

We will always consider in the sequel that, for a Polish space X , the set $\mathcal{P}(X)$ is endowed with the topology of weak convergence of measures. Recall that, by the Portmanteau theorem (see, e.g., [18, Chapter 1, Theorem 2.1]), a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{P}(X)$ converges weakly to some $\mu \in \mathcal{P}(X)$ if and only if one of the following equivalent conditions is satisfied:

- $\lim_{n \rightarrow +\infty} \int_X f(x) d\mu_n(x) = \int_X f(x) d\mu(x)$ for every continuous and bounded function $f: X \rightarrow \mathbb{R}$;
- $\limsup_{n \rightarrow +\infty} \mu_n(F) \leq \mu(F)$ for every closed set $F \subset X$;
- $\liminf_{n \rightarrow +\infty} \mu_n(G) \geq \mu(G)$ for every open set $G \subset X$.

For a Polish space X endowed with a complete metric \mathbf{d} , we define, for $p \in [1, +\infty)$,

$$\mathcal{P}_p(X) = \left\{ \mu \in \mathcal{P}(X) \mid \int_X \mathbf{d}(x, \bar{x})^p d\mu(x) < +\infty \text{ for some } \bar{x} \in X \right\}.$$

Clearly, if $\int_X \mathbf{d}(x, \bar{x})^p d\mu(x) < +\infty$ for some $\bar{x} \in X$, then the same is true also for every $\bar{x} \in X$. We endow $\mathcal{P}_p(X)$ with the usual Wasserstein distance \mathbf{W}_p , defined by

$$\mathbf{W}_p(\mu, \nu) = \inf \left\{ \int_{X \times X} \mathbf{d}(x, y)^p d\lambda(x, y) \mid \lambda \in \Pi(\mu, \nu) \right\}^{1/p}, \quad (2.1)$$

where $\Pi(\mu, \nu) = \{ \lambda \in \mathcal{P}(X \times X) \mid \pi_{1\#}\lambda = \mu, \pi_{2\#}\lambda = \nu \}$ and $\pi_1, \pi_2: X \times X \rightarrow X$ denote the canonical projections onto the first and second factors of the product $X \times X$, respectively.

Let X be a metric space with metric \mathbf{d} , $a, b \in \mathbb{R}$ with $a < b$, and $\gamma: (a, b) \rightarrow X$. The *metric derivative* of γ at a point $t \in (a, b)$ is defined by

$$|\dot{\gamma}|(t) = \lim_{s \rightarrow t} \frac{\mathbf{d}(\gamma(s), \gamma(t))}{|s - t|}$$

whenever this limit exists. Recall that, if γ is absolutely continuous, then $|\dot{\gamma}|(t)$ exists for almost every $t \in (a, b)$ (see, e.g., [6, Theorem 1.1.2]).

3 The model

3.1 Description of the model

Let (X, \mathbf{d}) be a complete and separable metric space, $\Gamma \subset X$ be nonempty and closed, $K: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$, $g: \Gamma \rightarrow \mathbb{R}_+$, and $m_0 \in \mathcal{P}(X)$. We consider in this paper the following mean field game, denoted by $\text{MFG}(X, \Gamma, K, g, m_0)$. Agents evolve in X , their distribution at time $t \in \mathbb{R}_+$ being given by a probability measure $m_t \in \mathcal{P}(X)$, with $m_0 \in \mathcal{P}(X)$ being given. The goal of each agent is to minimize the sum of the time at which they reach the exit Γ with the exit cost g computed at the position at which they first reach Γ , and we assume that the speed of an agent at a position x in time t is bounded by $K(m_t, x)$.

Before providing a more mathematically precise definition of this mean field game and its equilibria, we first introduce an associated optimal control problem where agents evolving in X want to reach Γ while minimizing the sum of their arrival time and a function of their arrival position, their speed being bounded by some time- and state-dependent function $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$. Here k will not depend on the density of the agents, and we consider instead that the dependence of k with respect to time is known. This optimal control problem is denoted in the sequel by $\text{OCP}(X, \Gamma, k, g)$.

Definition 3.1 ($\text{OCP}(X, \Gamma, k, g)$). Let (X, \mathbf{d}) be a complete and separable metric space, $\Gamma \subset X$ be nonempty and closed, and $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ and $g: \Gamma \rightarrow \mathbb{R}_+$ be continuous.

(a) A curve $\gamma \in \text{Lip}(X)$ is said to be *k-admissible* for $\text{OCP}(X, \Gamma, k, g)$ if its metric derivative $|\dot{\gamma}|$ satisfies $|\dot{\gamma}|(t) \leq k(t, \gamma(t))$ for almost every $t \in \mathbb{R}_+$. The set of all *k-admissible* curves is denoted by $\text{Adm}(k)$.

(b) Let $t_0 \in \mathbb{R}_+$. The *first exit time* after t_0 of a curve $\gamma \in \mathcal{C}_X$ is the number $\tau(t_0, \gamma) \in [t_0, +\infty]$ defined by

$$\tau(t_0, \gamma) = \inf\{t \geq 0 \mid \gamma(t + t_0) \in \Gamma\}, \quad (3.1)$$

with the convention $\inf \emptyset = +\infty$.

(c) The *final cost function* is the function $G: \mathbb{R}_+ \times \mathcal{C}_X \rightarrow [0, +\infty]$ defined, for $(t_0, \gamma) \in \mathbb{R}_+ \times \mathcal{C}_X$, by

$$G(t_0, \gamma) = \begin{cases} g(\gamma(t_0 + \tau(t_0, \gamma))) & \text{if } \tau(t_0, \gamma) < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

(d) Let $t_0 \in \mathbb{R}_+$ and $x_0 \in X$. A curve $\gamma \in \text{Lip}(X)$ is said to be an *optimal curve* or *optimal trajectory* for (k, g, t_0, x_0) if $\gamma \in \text{Adm}(k)$, $\gamma(t) = x_0$ for every $t \in [0, t_0]$, $\tau(t_0, \gamma) < +\infty$, $\gamma(t) = \gamma(t_0 + \tau(t_0, \gamma)) \in \Gamma$ for every $t \in [t_0 + \tau(t_0, \gamma), +\infty)$, and

$$\tau(t_0, \gamma) + G(t_0, \gamma) = \min_{\substack{\beta \in \text{Adm}(k) \\ \beta(t_0) = x_0}} \tau(t_0, \beta) + G(t_0, \beta). \quad (3.2)$$

The set of all optimal curves for (k, g, t_0, x_0) is denoted by $\text{Opt}(k, g, t_0, x_0)$.

(e) The *value function* of the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ is the function $\varphi: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined for $(t_0, x_0) \in \mathbb{R}_+ \times X$ by

$$\varphi(t_0, x_0) = \inf_{\substack{\gamma \in \text{Adm}(k) \\ \gamma(t_0) = x_0}} \tau(t_0, \gamma) + G(t_0, \gamma). \quad (3.3)$$

Remark 3.2. When $\tau(t_0, \gamma) < +\infty$, the infimum in (3.1) is a minimum, since γ is continuous and Γ is closed. In addition, it follows immediately from (3.1) that, for every $h \geq 0$, we have

$$t_0 + \tau(t_0, \gamma) \leq t_0 + h + \tau(t_0 + h, \gamma),$$

with equality if $\gamma(t) \notin \Gamma$ for every $t \in [t_0, t_0 + h]$.

Remark 3.3. If X is the closure of an open subset of \mathbb{R}^d , the constraint $|\dot{\gamma}|(t) \leq k(t, \gamma(t))$, $t \geq 0$, imposed on a k -admissible curve can be interpreted as the fact that γ is a solution to a control system, justifying thus the optimal control terminology used in Definition 3.1. Indeed, a curve γ is k -admissible if and only if there exists a measurable function $u: \mathbb{R}_+ \rightarrow \overline{B}_{\mathbb{R}^d}(0, 1)$ such that

$$\dot{\gamma}(t) = k(t, \gamma(t))u(t). \quad (3.4)$$

Remark 3.4. The fact that we only consider trajectories $\gamma \in \mathcal{C}_X$ implies that $\gamma(t) \in X$ for every $t \in \mathbb{R}_+$, and this condition can often be interpreted as a *state constraint* of the optimal control problem $\text{OCP}(X, \Gamma, k, g)$. Indeed, in many applications, X is a subset of a larger space Y (for instance, $X = \overline{\Omega}$ and $Y = \mathbb{R}^d$ for some nonempty bounded open set $\Omega \subset \mathbb{R}^d$), and the condition $\gamma(t) \in X$ can be interpreted as the constraint of preventing γ to leave X and go into $Y \setminus X$.

Given a mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and denoting by $m_t \in \mathcal{P}(X)$ the distribution of agents at time $t \geq 0$, we assume that each agent of the game solves the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ with $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ given by $k(t, x) = K(m_t, x)$. An equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$ is a situation in which the solutions of these optimal control problems by all agents will induce an evolution of the distribution of agents that coincides with the one given by m_t , $t \geq 0$. More precisely, we will use in this paper the following definition.

Definition 3.5 (Equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$). Let (X, \mathbf{d}) be a complete and separable metric space, $\Gamma \subset X$ be nonempty and closed, and $K: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$ and $g: \Gamma \rightarrow \mathbb{R}_+$ be continuous functions. Let $m_0 \in \mathcal{P}(X)$, $Q \in \mathcal{P}(\mathcal{C}_X)$, and define $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ for $(t, x) \in \mathbb{R}_+ \times X$ by $k(t, x) = K(e_{t\#}Q, x)$.

(a) The measure $Q \in \mathcal{P}(\mathcal{C}_X)$ is said to be a *weak Lagrangian equilibrium* (or simply *weak equilibrium*) of $\text{MFG}(X, \Gamma, K, g, m_0)$ if $e_{0\#}Q = m_0$ and Q -almost every $\gamma \in \mathcal{C}_X$ is an optimal curve for $(k, g, 0, \gamma(0))$.

(b) The measure $Q \in \mathcal{P}(\mathcal{C}_X)$ is said to be a *strong Lagrangian equilibrium* (or simply *strong equilibrium*) of $\text{MFG}(X, \Gamma, K, g, m_0)$ if $e_{0\#}Q = m_0$ and every $\gamma \in \text{supp}(Q)$ is an optimal curve for $(k, g, 0, \gamma(0))$.

To simplify the notation, given $K: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$ and $m: \mathbb{R}_+ \rightarrow \mathcal{P}(X)$, we define $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ by $k(t, x) = K(m_t, x)$ for $(t, x) \in \mathbb{R}_+ \times X$ and say that $\gamma \in \text{Lip}(X)$ is m -admissible for $\text{MFG}(X, \Gamma, K, g, m_0)$ if it is k -admissible for $\text{OCP}(X, \Gamma, k, g)$, denoting $\text{Adm}(k)$ simply by $\text{Adm}(m)$. Given $(t_0, x_0) \in \mathbb{R}_+ \times X$, we say that $\gamma \in \text{Lip}(X)$ is an optimal trajectory for (m, g, t_0, x_0) if it is an optimal trajectory for (k, g, t_0, x_0) for the optimal control problem $\text{OCP}(X, \Gamma, k, g)$, and denote the set of optimal trajectories for (m, g, t_0, x_0) by $\text{Opt}(m, g, t_0, x_0)$. Given $Q \in \mathcal{P}(\mathcal{C}_X)$, we consider the time-dependent measure $m^Q: \mathbb{R}_+ \rightarrow \mathcal{P}(X)$ given by $m_t^Q = e_{t\#}Q$, and denote $\text{Adm}(m^Q)$ and $\text{Opt}(m^Q, g, t_0, x_0)$ simply by $\text{Adm}(Q)$ and $\text{Opt}(Q, g, t_0, x_0)$, respectively.

Remark 3.6. Since $Q(\text{supp}(Q)) = 1$, any strong equilibrium is also a weak equilibrium. The converse turns out to be true for the model we consider in this paper, as we will prove later in Corollary 4.16, a result that generalizes [55, Remark 4.6] and [91, Proposition 3.7] to the present model.

Remark 3.7. The cost $\tau_0(t_0, \gamma) + G(t_0, \gamma)$ only takes into account the values of γ on the interval $[t_0, t_0 + \tau_0(t_0, \gamma)]$ and, in particular, if $\gamma \in \text{Adm}(k)$ is a minimizer of $\tau_0(t_0, \cdot) + G(t_0, \cdot)$ with the constraint $\gamma(t_0) = x_0$, then any other trajectory $\tilde{\gamma} \in \text{Adm}(k)$ coinciding with γ in $[t_0, t_0 + \tau_0(t_0, \gamma)]$ is also a minimizer of the same cost with this constraint. In order to avoid ambiguity on the behavior of minimizers before t_0 or after $t_0 + \tau_0(t_0, \gamma)$, Definition 3.1(d) defines an *optimal trajectory* to this minimization problem as being necessarily constant in the intervals $[0, t_0]$ and $[t_0 + \tau_0(t_0, \gamma), +\infty)$.

From the point of view of the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, the above choice leads to a concentration of agents on the target set Γ : interpreting the game as a crowd motion model, this would mean that agents that leave the domain X through Γ remain stopped at their arrival position at Γ , which creates congestion in Γ . In order to consider that agents of the game “disappear” when they reach Γ , one may consider, for modeling purposes, that, for K given by (1.2), the function η is a cut-off function, equal to 1 everywhere on X except on a neighborhood of Γ and vanishing at Γ . Notice, however, that such an assumption on K is not necessary for the results proved in this paper.

3.2 Main assumptions and their consequences

Let us now present the main assumptions needed in the sequel. The following assumptions are common to $\text{MFG}(X, \Gamma, K, g, m_0)$ and $\text{OCP}(X, \Gamma, k, g)$.

- (H1) (X, \mathbf{d}) is a metric space and there exists $\mathbf{0} \in X$ such that, for every $R > 0$, the closed ball $\overline{B}_X(\mathbf{0}, R)$ is compact.
- (H2) $\Gamma \subset X$ is nonempty and closed.
- (H3) The function $g: \Gamma \rightarrow \mathbb{R}_+$ is Lipschitz continuous, with Lipschitz constant L_g .
- (H4) There exists $D > 0$ such that, for every $x, y \in X$, there exist $T \in [0, D \mathbf{d}(x, y)]$ and $\gamma \in \text{Lip}([0, T], X)$ such that $\gamma(0) = x$, $\gamma(T) = y$, and $|\dot{\gamma}|(t) = 1$ for almost every $t \in [0, T]$.

The following assumptions are specific to $\text{OCP}(X, \Gamma, k, g)$.

- (H5) The function $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ is continuous and there exist $K_{\min}, K_{\max} \in \mathbb{R}_+^*$ such that, for all $(t, x) \in \mathbb{R}_+ \times X$, one has $k(t, x) \in [K_{\min}, K_{\max}]$.
- (H6) For every $R > 0$, there exists $L_R > 0$ such that, for every $t \in \mathbb{R}_+$ and $x_1, x_2 \in \overline{B}_X(\mathbf{0}, R)$, we have
$$|k(t, x_1) - k(t, x_2)| \leq L_R \mathbf{d}(x_1, x_2).$$
- (H7) Hypotheses (H3) and (H5) are satisfied with $L_g K_{\max} < 1$.

Finally, we state the following assumptions, specific to $\text{MFG}(X, \Gamma, K, g, m_0)$.

- (H8) The function $K: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$ is continuous and there exist $K_{\min}, K_{\max} \in \mathbb{R}_+^*$ such that, for all $(\mu, x) \in \mathcal{P}(X) \times X$, one has $K(\mu, x) \in [K_{\min}, K_{\max}]$.

(H9) For every $R > 0$, there exists $L_R > 0$ such that, for every $\mu \in \mathcal{P}(X)$ and $x_1, x_2 \in \overline{B}_X(\mathbf{0}, R)$, we have

$$|K(\mu, x_1) - K(\mu, x_2)| \leq L_R \mathbf{d}(x_1, x_2).$$

(H10) Hypotheses (H3) and (H8) are satisfied with $L_g K_{\max} < 1$.

Hypotheses (H1)–(H4) are the standard assumptions used in most of the results of this paper. Hypothesis (H1) is inspired by the case where X is a closed subset of a finite-dimensional vector space, and it implies, in particular, that X is σ -compact, complete, and separable. Clearly, if (H1) is satisfied for some element $\mathbf{0} \in X$, then it is also satisfied with $\mathbf{0}$ replaced by any element $x \in X$. In the sequel, we shall always consider that $\mathbf{0}$ is a fixed element of X .

Hypothesis (H4) provides a relation between the distance $\mathbf{d}(x, y)$ and the length of curves from x to y in X , stating that the former is, up to a constant, an upper bound on the latter. This assumption is satisfied when the geodesic metric induced by \mathbf{d} is equivalent to \mathbf{d} itself. In particular, it is satisfied if X is a length space. Moreover, (H4) implies that X is path-connected.

One of the important consequences of (H5), which we will use frequently in the sequel, is that it implies that $\text{Adm}(k) \subset \text{Lip}_{K_{\max}}(X)$.

Hypotheses (H6) and (H9) can be reformulated by saying that, on every compact subset of X , the functions k and K are Lipschitz continuous with respect to their second variable, uniformly with respect to the first one.

Notice that, under suitable assumptions on κ , χ , and η , the function K defined in (1.2) satisfies (H8) and (H9). More precisely, we have the following result.

Proposition 3.8. *Let (X, \mathbf{d}) be a complete and separable metric space, $\kappa \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$, $\chi \in \mathcal{C}(X \times X, \mathbb{R}_+)$ be bounded, $\eta \in \mathcal{C}(X, \mathbb{R}_+)$ be bounded, and define $K: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$ by (1.2). Assume that, for every $R > 0$, there exists $L_{\chi, R} > 0$ such that, for every $x_1, x_2 \in \overline{B}_X(\mathbf{0}, R)$ and $y \in X$, we have*

$$|\chi(x_1, y) - \chi(x_2, y)| \leq L_{\chi, R} \mathbf{d}(x_1, x_2).$$

Then K satisfies (H8) and (H9).

A result very similar to Proposition 3.8 was shown in [83, Proposition 3.1] in the case where $X = \overline{\Omega}$ for some nonempty bounded open set $\Omega \subset \mathbb{R}^d$, and with additional assumptions on χ that also allow one to prove that K is Lipschitz continuous when considering the Wasserstein distance \mathbf{W}_1 in $\mathcal{P}(\overline{\Omega})$. We provide here an adaptation of that proof to our current setting.

Proof of Proposition 3.8. Let $E: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$ be defined by

$$E(\mu, x) = \int_X \chi(x, y) \eta(y) \, \mathrm{d}\mu(y).$$

Let $R > 0$. For every $\mu \in \mathcal{P}(X)$ and $x_1, x_2 \in \overline{B}_X(\mathbf{0}, R)$, we have

$$|E(\mu, x_1) - E(\mu, x_2)| \leq L_{\chi} \mathbf{d}(x_1, x_2) \int_X \eta(y) \, \mathrm{d}\mu(y).$$

Since η is bounded, the integral in the above expression is bounded independently of μ , and thus E is Lipschitz continuous with respect to its second variable in $\overline{B}_X(\mathbf{0}, R)$, uniformly with respect to the first one. Since $K = \kappa \circ E$, we deduce that K satisfies (H9).

Since χ and η are bounded, E is also bounded by some constant $M > 0$. Since $K = \kappa \circ E$ and κ is continuous and takes values in \mathbb{R}_+^* , we deduce that K takes values in $[K_{\min}, K_{\max}]$, where $K_{\min} = \min_{s \in [0, M]} \kappa(s) > 0$ and $K_{\max} = \max_{s \in [0, M]} \kappa(s) > 0$.

If $(\mu_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{P}(X)$ converging to some $\mu \in \mathcal{P}(X)$ and $x \in X$ is given, then, since $y \mapsto \chi(x, y)\eta(y)$ is continuous and bounded, we deduce that $E(\mu_n, x) \rightarrow E(\mu, x)$ as $n \rightarrow +\infty$. Together with the fact that $(\mu, x) \mapsto E(\mu, x)$ is Lipschitz continuous in $x \in \overline{B}_X(\mathbf{0}, R)$ uniformly in μ for every $R > 0$, we conclude that E is continuous on $\mathcal{P}(X) \times X$, and thus, since κ is continuous, we deduce that K satisfies (H8). \square

Hypotheses (H7) and (H10) can be seen as restriction on the Lipschitz constant of g in terms of the upper bound K_{\max} on k or K . This is a standard assumption in optimal control problems with free final time and boundary costs (see, e.g., [29, (8.6) and Remark 8.1.5] and [54, 55]), its importance being the following property, whose proof is straightforward.

Lemma 3.9. *Assume that (H1)–(H3) are satisfied and let $\gamma \in \mathcal{C}_X$ be Lipschitz continuous, with Lipschitz constant L_γ satisfying $L_g L_\gamma < 1$. If $t_1, t_2 \in \mathbb{R}_+$ are such that $t_1 < t_2$ and $\gamma(t_1), \gamma(t_2) \in \Gamma$, then*

$$t_1 + g(\gamma(t_1)) < t_2 + g(\gamma(t_2)).$$

As a consequence of Lemma 3.9, we also obtain the following result.

Lemma 3.10. *Assume that (H1)–(H3) are satisfied and let $C > 0$ be such that $L_g C < 1$. Then the function $(t, \gamma) \mapsto \tau(t, \gamma) + G(t, \gamma)$ is lower semicontinuous on $\mathbb{R}_+ \times \text{Lip}_C(X)$.*

Proof. Let $(t_n, \gamma_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+ \times \text{Lip}_C(X)$ converging, in the topology of $\mathbb{R}_+ \times \mathcal{C}_X$, to some $(t, \gamma) \in \mathbb{R}_+ \times \mathcal{C}_X$. Since $\text{Lip}_C(X)$ is closed, we have $\gamma \in \text{Lip}_C(X)$. We want to prove that

$$\tau(t, \gamma) + G(t, \gamma) \leq \liminf_{n \rightarrow +\infty} \tau(t_n, \gamma_n) + G(t_n, \gamma_n).$$

There is nothing to prove if the right-hand side of the above inequality is equal to $+\infty$, so we assume, with no loss of generality, that it is finite. We extract a subsequence $(t_{n_k}, \gamma_{n_k})_{k \in \mathbb{N}}$ of $(t_n, \gamma_n)_{n \in \mathbb{N}}$ such that $\tau(t_{n_k}, \gamma_{n_k}) + G(t_{n_k}, \gamma_{n_k}) < +\infty$ for every $k \in \mathbb{N}$ and

$$\lim_{k \rightarrow +\infty} \tau(t_{n_k}, \gamma_{n_k}) + G(t_{n_k}, \gamma_{n_k}) = \liminf_{n \rightarrow +\infty} \tau(t_n, \gamma_n) + G(t_n, \gamma_n).$$

For simplicity, we denote $(t_{n_k}, \gamma_{n_k})_{k \in \mathbb{N}}$ simply by $(t_n, \gamma_n)_{n \in \mathbb{N}}$ in the sequel. Also, without loss of generality, we assume that $(\tau(t_n, \gamma_n))_{n \in \mathbb{N}}$ converges to some $\tau_* \in \mathbb{R}_+$. Then $G(t_n, \gamma_n) = g(\gamma_n(t_n + \tau(t_n, \gamma_n))) \rightarrow g(\gamma(t + \tau_*))$ as $n \rightarrow +\infty$.

Since $\gamma_n(t_n + \tau(t_n, \gamma_n)) \in \Gamma$ for every $n \in \mathbb{N}$ and Γ is closed, we obtain that $\gamma(t + \tau_*) \in \Gamma$, and thus $\tau(t, \gamma) \leq \tau_*$. Hence $t + \tau(t, \gamma) \leq t + \tau_*$ and, by Lemma 3.9, we deduce that $t + \tau(t, \gamma) + G(t, \gamma) \leq t + \tau_* + g(\gamma(t + \tau_*))$, which is the desired inequality. \square

3.3 Comparison to other works

The model considered here can be seen as a generalization of the mean field game models considered in [55, 83, 89–91], and we now provide a more detailed comparison between them.

First of all, the goal of [55, 83, 89–91] is broader than that of the present paper: in addition to showing existence of equilibria, these papers also aim at proving that equilibria

necessarily satisfy a system of partial differential equations, the MFG system, showing also, when possible, the converse statement (namely, that solutions of the MFG system are equilibria). These additional properties require more assumptions than just (H1)–(H10); in particular, in order to properly write the partial differential equations in the MFG system, one usually works on the closure of an open subset of \mathbb{R}^d instead of a metric space (X, \mathbf{d}) satisfying (H1). On the other hand, concerning existence of equilibria, all of the references [55, 83, 89–91] make use of more restrictive assumptions than (H1)–(H10), which allows for simpler proofs with respect to the ones presented here.

Among these works, [83, 89–91] use only the notion of weak equilibrium, while [55] uses the notion of strong equilibrium. The fact that both notions coincide for the models treated in these references under suitable assumptions was already observed in [55, Remark 4.6] and [91, Proposition 3.7], a result that we also present later in Corollary 4.16 for the more general model we consider here.

As regards the characterization of the asymptotic behavior of equilibria, among [55, 83, 89–91], only [90] considers this question. The main reason for that is that [55, 83, 89, 91] always consider that X is compact and, in this case, one can easily prove that, at equilibrium, the distribution m_t converges to a limit distribution m_∞ in finite time, i.e., there exists $t_0 \geq 0$ such that $m_t = m_\infty$ for every $t \geq 0$ (see Theorem 5.4(c) below). The asymptotic behavior of equilibria is studied in [90] only in the case $X = \mathbb{R}^d$, and our main result on this question, Theorem 5.4, is an extension of [90, Theorem 5.8] to our more general setting. We also provide further consequences of Theorem 5.4 in Corollary 5.6, which provides explicit convergence rates for mean field games in \mathbb{R}^d , a novelty with respect to [90].

Finally, the analysis of the dependence of equilibria and their asymptotic limits on the initial distribution of agents, which is the topic of Section 5.3, is a novelty with respect to [55, 83, 89–91].

Now that we have compared the present paper with [55, 83, 89–91] in what concerns the kind of results that are shown, we proceed to compare the models treated in those references and their assumptions with the setting of this paper.

In [83], the authors consider the mean field game $\text{MFG}(X, \Gamma, K, 0, m_0)$, i.e., there is no cost at the exit position of an agent, and each agent minimizes only their time to reach Γ . Instead of (H1), the authors assume that X is compact, which simplifies many technical details in the proofs when compared to the presentation of this paper. Further technical simplifications are also possible in [83] thanks to the fact that (H6) and (H9) are replaced by the stronger assumptions of Lipschitz continuity of k and K , respectively, when $\mathcal{P}(X)$ is endowed with the Wasserstein metric \mathbf{W}_1 .

The article [55] considers the mean field game $\text{MFG}(\overline{\Omega}, \partial\Omega, K, g, m_0)$, where $\Omega \subset \mathbb{R}^d$ is a nonempty bounded open set. Apart from this fact, the other assumptions required for the existence of equilibria in [55] are essentially the same as (H1)–(H10). The fact that $X = \overline{\Omega}$ and $\Gamma = \partial\Omega$ implies that one does not need to consider the state constraint $\gamma(t) \in X$ in the analysis of the optimal control problem $\text{OCP}(\overline{\Omega}, \partial\Omega, k, g)$, which allows for some simplifications in the proofs in [55] with respect to the ones presented here.

One of the main novelties of [90] with respect to [55, 83] is to consider a mean field game in the noncompact space \mathbb{R}^d . More precisely, [90] addresses the mean field game $\text{MFG}(\mathbb{R}^d, \Gamma, K, 0, m_0)$, where $\Gamma \subset \mathbb{R}^d$ is a nonempty closed set, and agents only minimize the time to reach Γ , with no arrival cost. Instead of (H9), [90] works with a more restrictive assumption, requiring global Lipschitz behavior in the space variable x . Hence, with respect to [90], we replace here \mathbb{R}^d by a more general metric space (X, \mathbf{d}) satisfying (H1) and we work under only local Lipschitz behavior of K with respect to its space variable.

Note also that [90] considers multipopulation mean field games, but the generalization of the techniques of this work to the multipopulation setting are straightforward, and we restrict our attention here to the single-population case in order to avoid cumbersome notation.

Finally, [89,91] consider both the same model, [91] being an extended version of the conference paper [89]. They both consider $\text{MFG}(\overline{\Omega}, \Gamma, K, 0, m_0)$, where $\Omega \subset \mathbb{R}^d$ is a nonempty bounded open set and $\Gamma \subset \overline{\Omega}$ is nonempty and closed. This model is actually a particular case of that of [83], but [89,91] work under more general assumptions on k and K , which are just the particular case of (H6) and (H9) when X is compact. The main difference of the present work with respect to those references is thus the fact that we consider here games in a more general metric space (X, \mathbf{d}) and with a cost g on the exit position.

4 Preliminary results

Before turning to the study of existence of equilibria of $\text{MFG}(X, \Gamma, k, g, m_0)$ and their asymptotic behavior, we collect some important preliminary properties of the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and the mean field game $\text{MFG}(X, \Gamma, k, g, m_0)$ that are useful for the proofs of our main results in Section 5.

4.1 Properties of the optimal control problem

Note that $\text{OCP}(X, \Gamma, k, g)$ is an optimal control problem with free final time, which is a classic subject in the optimal control literature (see, e.g., [29, 48]), but the assumptions (H1)–(H7) on $\text{OCP}(X, \Gamma, k, g)$ are more general than the ones usually considered in the literature, allowing for more general spaces X and fewer regularity assumptions on Γ and k . For this reason, we provide here a detailed presentation of the properties of $\text{OCP}(X, \Gamma, k, g)$ in order to highlight which hypotheses are required for each result.

The first result we provide is the following, on the closedness of $\text{Adm}(k)$.

Proposition 4.1. *Consider the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and assume that (H1) and (H5) are satisfied. Then the set $\text{Adm}(k)$ is closed.*

Proof. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Adm}(k)$ converging in \mathcal{C}_X to some $\gamma \in \mathcal{C}_X$. Note that, since $\text{Adm}(k) \subset \text{Lip}_{K_{\max}}(X)$ and the latter set is closed in \mathcal{C}_X , we deduce that $\gamma \in \text{Lip}_{K_{\max}}(X)$.

By [6, Theorem 1.1.2] and using the fact that $\gamma_n \in \text{Adm}(k)$, we obtain that, for every $n \in \mathbb{N}$ and $s, t \in \mathbb{R}_+$ with $s \leq t$, we have

$$\mathbf{d}(\gamma_n(s), \gamma_n(t)) \leq \int_s^t k(r, \gamma_n(r)) \, dr. \quad (4.1)$$

For every $T_0 > 0$, since $\gamma_n \rightarrow \gamma$ in \mathcal{C}_X , there exists $R > 0$ such that $\gamma_n(t) \in \overline{B}_X(\mathbf{0}, R)$ for every $n \in \mathbb{N}$ and $t \in [0, T_0]$. Since k is continuous on $[0, T_0] \times \overline{B}_X(\mathbf{0}, R)$, it is uniformly continuous on this set. Hence, $k(r, \gamma_n(r)) \rightarrow k(r, \gamma(r))$ as $n \rightarrow +\infty$, uniformly on $r \in [0, T_0]$. Thus, taking the limit as $n \rightarrow +\infty$ in (4.1), we deduce that

$$\mathbf{d}(\gamma(s), \gamma(t)) \leq \int_s^t k(r, \gamma(r)) \, dr$$

for every $s, t \in [0, T_0]$ with $s \leq t$. Since $T_0 > 0$ is arbitrary, the previous inequality holds true for every $s, t \in \mathbb{R}_+$ with $s \leq t$. The conclusion now follows from [6, Theorem 1.1.2]. \square

Our next result concerns boundedness of the value function and of optimal trajectories on bounded subsets of initial conditions x_0 in X , which adapts [90, Proposition 4.4] to the case of metric space X and an exit cost g .

Proposition 4.2. *Consider the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and assume that (H1)–(H5) and (H7) are satisfied. Then there exist two nondecreasing maps with linear growth $\psi, T: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ depending only on $\mathbf{0}, \Gamma, g, D, K_{\min}$, and K_{\max} such that, for every $R > 0$ and $(t_0, x_0) \in \mathbb{R}_+ \times \overline{B}_X(\mathbf{0}, R)$, we have $\varphi(t_0, x_0) \leq T(R)$ and, for every $\gamma \in \text{Opt}(k, g, t_0, x_0)$, we have $\gamma(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $t \geq 0$.*

Proof. Fix $y_0 \in \Gamma$ and let $G_0 = g(y_0)$. Let $R > 0$ and $(t_0, x_0) \in \mathbb{R}_+ \times \overline{B}_X(\mathbf{0}, R)$. Let $T_0 \in [0, D \mathbf{d}(x_0, y_0)]$ and $\tilde{\gamma} \in \text{Lip}([0, T_0], X)$ be such that $\tilde{\gamma}(0) = x_0$, $\tilde{\gamma}(T_0) = y_0$, and $|\dot{\tilde{\gamma}}|(t) = 1$ for almost every $t \in [0, T_0]$. Set $t_1 = t_0 + \frac{T_0}{K_{\min}}$ and let $\gamma: \mathbb{R}_+ \rightarrow X$ be defined by $\gamma(t) = x_0$ for $t \in [0, t_0]$, $\gamma(t) = \tilde{\gamma}(K_{\min}(t - t_0))$ for $t \in [t_0, t_1]$, and $\gamma(t) = y_0$ for $t \geq t_1$. Then $\gamma \in \text{Adm}(k)$, $\gamma(t_0) = x_0$, and $\tau(t_0, \gamma) \leq \frac{T_0}{K_{\min}}$. By Lemma 3.9, we have

$$t_0 + \tau(t_0, \gamma) + G(t_0, \gamma) \leq t_1 + g(y_0).$$

Hence,

$$\varphi(t_0, x_0) \leq \tau(t_0, \gamma) + G(t_0, \gamma) \leq G_0 + \frac{D \mathbf{d}(x_0, y_0)}{K_{\min}} \leq G_0 + \frac{D \mathbf{d}(\mathbf{0}, y_0)}{K_{\min}} + \frac{DR}{K_{\min}},$$

and thus the first part of the conclusion holds true with $T(R) = G_0 + \frac{D \mathbf{d}(\mathbf{0}, y_0)}{K_{\min}} + \frac{DR}{K_{\min}}$.

As for the second part of the statement, take $R > 0$, $(t_0, x_0) \in \mathbb{R}_+ \times \overline{B}_X(\mathbf{0}, R)$, and $\gamma \in \text{Opt}(k, g, t_0, x_0)$. It suffices to prove that $\gamma(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $t \in [t_0, t_0 + \tau(t_0, \gamma)]$, since γ is constant on $[0, t_0]$ and on $[t_0 + \tau(t_0, \gamma), +\infty)$. For any such t , using the facts that $\gamma \in \text{Adm}(k) \subset \text{Lip}_{K_{\max}}(X)$ and $\tau(t_0, \gamma) \leq \varphi(t_0, x_0) \leq T(R)$, we have

$$\mathbf{d}(\mathbf{0}, \gamma(t)) \leq \mathbf{d}(\gamma(t), \gamma(t_0)) + \mathbf{d}(\mathbf{0}, x_0) \leq K_{\max}(t - t_0) + R \leq K_{\max}T(R) + R, \quad (4.2)$$

and the conclusion holds true with $\psi(R) = K_{\max}T(R) + R$. \square

We can now turn to the question of existence of optimal trajectories for $\text{OCP}(X, \Gamma, k, g)$. The proof of our next result, omitted here, can be easily obtained by following the classical strategy in optimal control of considering the limit of a minimizing sequence for the right-hand side of (3.3), exploring also Proposition 4.1 and the bound on the value function from Proposition 4.2 in order to bound exit times and costs of the trajectories in the minimizing sequence.

Proposition 4.3. *Consider the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and assume that (H1)–(H5) and (H7) are satisfied. Then, for every $(t_0, x_0) \in \mathbb{R}_+ \times X$, there exists $\gamma \in \text{Opt}(k, g, t_0, x_0)$.*

Another important classical property of optimal trajectories is that the restriction of an optimal trajectory is still optimal.

Proposition 4.4. *Consider the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and assume that (H1)–(H3), (H5), and (H7) are satisfied. Let $(t_0, x_0) \in \mathbb{R}_+ \times X$ and $\gamma_0 \in \text{Opt}(k, g, t_0, x_0)$. Then, for every $t_1 \in [t_0, +\infty)$, denoting $x_1 = \gamma_0(t_1)$, the function $\gamma_1: \mathbb{R}_+ \rightarrow X$ defined by $\gamma_1(t) = x_1$ for $t \leq t_1$ and $\gamma_1(t) = \gamma_0(t)$ for $t \geq t_1$ satisfies $\gamma_1 \in \text{Opt}(k, g, t_1, x_1)$.*

Proof. If $t_1 \geq \tau(t_0, \gamma_0)$, then $x_1 \in \Gamma$ and γ_1 is the constant trajectory remaining at x_1 at all times. In particular, $\tau(t_1, \gamma_1) = 0$, $G(t_1, \gamma_1) = g(x_1)$, and the optimality of γ_1 follows from the fact that, if $\tilde{\gamma}_1 \in \text{Adm}(k)$ is such that $\tilde{\gamma}_1(t_1) = x_1$ and $\tau(t_1, \tilde{\gamma}_1) < +\infty$, then by Lemma 3.9, we have $g(x_1) \leq \tau(t_1, \tilde{\gamma}_1) + G(t_1, \tilde{\gamma}_1)$.

Let us now consider the case $t_1 < \tau(t_0, \gamma_0)$, in which case, by definition, we have $\gamma_0(t) \notin \Gamma$ for $t \in [t_0, t_1]$, and thus $\tau(t_1, \gamma_0) = \tau(t_1, \gamma_1)$, $t_1 + \tau(t_1, \gamma_1) = t_0 + \tau(t_0, \gamma_0)$, and $G(t_1, \gamma_1) = G(t_0, \gamma_0)$. Let $\tilde{\gamma}_1 \in \text{Adm}(k)$ be such that $\tilde{\gamma}_1(t_1) = x_1$. Define $\tilde{\gamma}_0: \mathbb{R}_+ \rightarrow X$ by $\tilde{\gamma}_0(t) = \gamma_0(t)$ for $t \in [0, t_1]$ and $\tilde{\gamma}_0(t) = \tilde{\gamma}_1(t)$ for $t \geq t_1$. Then $\tilde{\gamma}_0 \in \text{Adm}(k)$ and $\tilde{\gamma}_0(t_0) = x_0$, showing that $\tau(t_0, \gamma_0) + G(t_0, \gamma_0) \leq \tau(t_0, \tilde{\gamma}_0) + G(t_0, \tilde{\gamma}_0)$. By definition of $\tilde{\gamma}_0$, we have $\tau(t_1, \tilde{\gamma}_0) = \tau(t_1, \tilde{\gamma}_1)$ and $\tilde{\gamma}_0(t) \notin \Gamma$ for every $t \in [0, t_1]$, and thus $t_1 + \tau(t_1, \tilde{\gamma}_1) = t_0 + \tau(t_0, \tilde{\gamma}_0)$ and $G(t_1, \tilde{\gamma}_1) = G(t_0, \tilde{\gamma}_0)$. We thus deduce that

$$\begin{aligned} \tau(t_1, \gamma_1) + G(t_1, \gamma_1) &= t_0 - t_1 + \tau(t_0, \gamma_0) + G(t_0, \gamma_0) \\ &\leq t_0 - t_1 + \tau(t_0, \tilde{\gamma}_0) + G(t_0, \tilde{\gamma}_0) \\ &= \tau(t_1, \tilde{\gamma}_1) + G(t_1, \tilde{\gamma}_1), \end{aligned}$$

proving that $\gamma_1 \in \text{Opt}(k, g, t_1, x_1)$, as required. \square

Our next result provides the dynamic programming principle for OCP(X, Γ, k, g).

Proposition 4.5. *Consider the optimal control problem OCP(X, Γ, k, g) and assume that (H1)–(H5) and (H7) are satisfied. Then, for every $(t_0, x_0) \in \mathbb{R}_+ \times X$ and $\gamma \in \text{Adm}(k)$ with $\gamma(t_0) = x_0$, we have*

$$\varphi(t_0 + h, \gamma(t_0 + h)) + h \geq \varphi(t_0, x_0), \quad \text{for every } h \geq 0, \quad (4.3)$$

with equality for every $h \in [0, \tau(t_0, \gamma)]$ if $\gamma \in \text{Opt}(k, g, t_0, x_0)$. Moreover, if γ is constant on $[0, t_0]$ and on $[t_0 + \tau(t_0, \gamma), +\infty)$ and if equality holds in (4.3) for every $h \in [0, \tau(t_0, \gamma)]$ with $h < +\infty$, then $\tau(t_0, \gamma) < +\infty$ and $\gamma \in \text{Opt}(k, g, t_0, x_0)$.

Proof. Let $(t_0, x_0) \in \mathbb{R}_+ \times X$ and $\gamma \in \text{Adm}(k)$ with $\gamma(t_0) = x_0$. Take $h \geq 0$. Let $\hat{\gamma} \in \text{Opt}(k, g, t_0 + h, \gamma(t_0 + h))$ and define $\tilde{\gamma}: \mathbb{R}_+ \rightarrow X$ by $\tilde{\gamma}(t) = \gamma(t)$ for $t \in [0, t_0 + h]$ and $\tilde{\gamma}(t) = \hat{\gamma}(t)$ for $t \geq t_0 + h$. By definition of $\tilde{\gamma}$ and φ , we have

$$\begin{aligned} \varphi(t_0, x_0) &\leq \tau(t_0, \tilde{\gamma}) + G(t_0, \tilde{\gamma}), \\ \varphi(t_0 + h, \gamma(t_0 + h)) &= \tau(t_0 + h, \tilde{\gamma}) + G(t_0 + h, \tilde{\gamma}). \end{aligned} \quad (4.4)$$

In addition, by Remark 3.2, we have

$$t_0 + \tau(t_0, \tilde{\gamma}) \leq t_0 + h + \tau(t_0 + h, \tilde{\gamma}), \quad (4.5)$$

and thus, by Lemma 3.9,

$$t_0 + \tau(t_0, \tilde{\gamma}) + G(t_0, \tilde{\gamma}) \leq t_0 + h + \tau(t_0 + h, \tilde{\gamma}) + G(t_0 + h, \tilde{\gamma}). \quad (4.6)$$

Combining the above inequalities, we obtain (4.3).

In addition, if $\gamma \in \text{Opt}(k, g, t_0, x_0)$, we have equality in (4.4) and, if $h \in [0, \tau(t_0, \gamma)]$, then we have equality in (4.5) thanks to Remark 3.2, implying that we have equality in (4.6) since we apply Lemma 3.9 with equal times, and thus we also have equality in (4.3).

Assume now that γ is constant on $[0, t_0]$ and on $[t_0 + \tau(t_0, \gamma), +\infty)$ and that equality holds in (4.3) for every $h \in [0, \tau(t_0, \gamma)]$ with $h < +\infty$. In particular, since φ takes nonnegative and finite values, for every such h , we have $h = \varphi(t_0, x_0) - \varphi(t_0 + h, \gamma(t_0 + h))$.

$h)) \leq \varphi(t_0, x_0)$, proving that $[0, \tau(t_0, \gamma)] \subset [0, \varphi(t_0, x_0)]$ and thus $\tau(t_0, \gamma) < +\infty$. For $h = \tau(t_0, \gamma)$, we have $\gamma(t_0 + h) \in \Gamma$, thus $\varphi(t_0 + h, \gamma(t_0 + h)) = g(\gamma(t_0 + h))$ and hence (4.3) reads

$$\varphi(t_0, x_0) = \tau(t_0, \gamma) + g(\gamma(t_0 + h)) = \tau(t_0, \gamma) + G(t_0, \gamma),$$

yielding the conclusion. \square

Our next result deals with Lipschitz continuity of the value function φ . This kind of result is classical for optimal control problems with free final time (see, e.g., [29, Proposition 8.2.5] for a proof in the autonomous case), and a complete proof for the nonautonomous optimal control problem $\text{OCP}(X, \Gamma, k, g)$ was given in [83, Propositions 4.2 and 4.3] in the case where X is compact. However, that reference uses the stronger assumption that $k \in \text{Lip}(\mathbb{R}_+ \times X, \mathbb{R}_+)$. When the optimal control problem does not have state constraints, this assumption can be relaxed to (H6) by first showing Lipschitz continuity of φ with respect to x , which can be done by adapting the classical proof of [29, Proposition 8.2.5], and then using the dynamic programming principle to deduce Lipschitz continuity also with respect to t . This strategy was described in [55, Proposition 3.8] and carried out in details in [90, Lemma 4.6 and Proposition 4.7] when $X = \mathbb{R}^d$, however those proofs rely on the absence of state constraints and cannot be easily generalized to optimal control problems with state constraints. For that reason, we present here another proof, used in [91, Lemma 3.2 and Proposition 3.3] in the case where X is the closure of a nonempty bounded open subset of \mathbb{R}^d . This proof is inspired by that of [83, Propositions 4.2 and 4.3] but uses a technique introduced in the proof of [55, Proposition 3.9] in order to replace the assumption $k \in \text{Lip}(\mathbb{R}_+ \times X, \mathbb{R}_+)$ by the weaker assumption (H6). As a first step, we prove Lipschitz continuity of φ in space for fixed time.

Lemma 4.6. *Consider the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and assume that (H1)–(H7) are satisfied. Then, for every $R > 0$, there exists $C > 0$ depending only on $\mathbf{0}$, Γ , g , D , K_{\min} , K_{\max} , and R such that, for every $t_0 \in \mathbb{R}_+$ and $x_0, x_1 \in \overline{B}_X(\mathbf{0}, R)$, we have*

$$|\varphi(t_0, x_0) - \varphi(t_0, x_1)| \leq C \mathbf{d}(x_0, x_1). \quad (4.7)$$

Proof. It suffices to show that, for every $R > 0$, there exists $C > 0$ such that, for every $t_0 \in \mathbb{R}_+$ and $x_0, x_1 \in \overline{B}_X(\mathbf{0}, R)$, we have

$$\varphi(t_0, x_1) - \varphi(t_0, x_0) \leq C \mathbf{d}(x_0, x_1), \quad (4.8)$$

since, in this case, (4.7) can be deduced by exchanging the role of x_0 and x_1 .

Let $R > 0$, $t_0 \in \mathbb{R}_+$, and $x_0, x_1 \in \overline{B}_X(\mathbf{0}, R)$. Let $\gamma_0 \in \text{Opt}(k, g, t_0, x_0)$ and $t_0^* = t_0 + \tau(t_0, \gamma)$ be the time at which γ_0 arrives at the target set Γ , and set $x_0^* = \gamma_0(t_0^*) \in \Gamma$. Since $\gamma_0 \in \text{Opt}(k, g, t_0, x_0)$, we have $\varphi(t_0, x_0) = t_0^* - t_0 + g(x_0^*)$.

Let $D > 0$ be the constant from (H4), T and ψ be the functions from Proposition 4.2, and $L_{\psi(R)}$ be the constant obtained from (H6) applied to $\psi(R)$, which we denote simply by L in the sequel. Note that, by Proposition 4.2, we have $\gamma_0(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $t \geq 0$.

Applying (H4) to x_0, x_1 and renormalizing the speed of the curve whose existence is asserted in that hypothesis, we obtain the existence of $t_1 \geq t_0$ and a Lipschitz continuous curve $\sigma: [t_0, t_1] \rightarrow X$ such that $\sigma(t_0) = x_1$, $\sigma(t_1) = x_0$, $|\dot{\sigma}|(t) = K_{\min}$ for almost every $t \in [t_0, t_1]$, and $t_1 - t_0 \leq \frac{D \mathbf{d}(x_0, x_1)}{K_{\min}}$.

Let us now define $\phi: [t_1, +\infty) \rightarrow [t_0, +\infty)$ as a solution of the problem

$$\begin{cases} \dot{\phi}(t) = \frac{k(t, \gamma_0(\phi(t)))}{k(\phi(t), \gamma_0(\phi(t)))} & \text{for } t \geq t_1, \\ \phi(t_1) = t_0. \end{cases} \quad (4.9)$$

Note that, since $(t, s) \mapsto \frac{k(t, \gamma_0(s))}{k(s, \gamma_0(s))}$ is continuous (but not necessarily Lipschitz continuous in its second argument), a solution ϕ to the above problem exists and is of class \mathcal{C}^1 (but it may not be unique). Moreover, $\dot{\phi}(t) \in \left[\frac{K_{\min}}{K_{\max}}, \frac{K_{\max}}{K_{\min}}\right]$ for every $t \in [t_1, +\infty)$, which implies that $\phi: [t_1, +\infty) \rightarrow [t_0, +\infty)$ is increasing and surjective, and hence invertible, and both ϕ and ϕ^{-1} are Lipschitz continuous, with Lipschitz constant $\frac{K_{\max}}{K_{\min}}$. We define $\sigma_1: [t_1, +\infty) \rightarrow X$ by $\sigma_1(t) = \gamma_0(\phi(t))$, which, by construction, satisfies $\sigma_1(t_1) = x_0$, $\sigma_1 \in \text{Lip}(X)$, and $|\dot{\sigma}_1|(t) = |\dot{\gamma}_0|(\phi(t))\dot{\phi}(t) \leq k(t, \sigma_1(t))$ for a.e. $t \in [t_1, +\infty)$. We define $t_1^* = \phi^{-1}(t_0^*)$ and remark that $\sigma_1(t_1^*) = \gamma_0(t_0^*) = x_0^* \in \Gamma$.

Finally, we define $\gamma_1 \in \text{Lip}(X)$ by $\gamma_1(t) = x_1$ for $t \in [0, t_0]$, $\gamma_1(t) = \sigma(t)$ for $t \in [t_0, t_1]$, and $\gamma_1(t) = \sigma_1(t)$ for $t \in [t_1, +\infty)$. By construction, we have $\gamma_1 \in \text{Adm}(k)$ and $\tau(t_0, \gamma_1) \leq t_1^* - t_0$.

Note that, for every $t \in [t_1, +\infty)$, we have $k(\phi(t), \gamma_0(\phi(t)))\dot{\phi}(t) = k(t, \gamma_0(\phi(t)))$ and thus, by integrating this identity and performing a change of variables, we deduce that, for every $t \geq t_1$,

$$\int_{t_0}^{\phi(t)} k(s, \gamma_0(s)) ds = \int_{t_1}^t k(s, \gamma_0(\phi(s))) ds. \quad (4.10)$$

Let $H: [t_0, +\infty) \rightarrow [0, +\infty)$ be defined for $\theta \geq t_0$ by

$$H(\theta) = \int_{t_0}^{\theta} k(s, \gamma_0(s)) ds.$$

Then H is differentiable, with $\dot{H}(\theta) = k(\theta, \gamma_0(\theta)) \in [K_{\min}, K_{\max}]$ for every $\theta \geq t_0$. In particular, H is K_{\max} -Lipschitz continuous, invertible, and its inverse is $\frac{1}{K_{\min}}$ -Lipschitz continuous. Moreover, using (4.10), we have, for every $t \geq t_1$, that

$$H(\phi(t)) = \int_{t_1}^t k(s, \gamma_0(\phi(s))) ds, \quad H(t) = \int_{t_0}^t k(s, \gamma_0(s)) ds.$$

Hence, for every $t \geq t_1$, we have

$$\begin{aligned} |\phi(t) - t| &= \left| H^{-1} \left(\int_{t_1}^t k(s, \gamma_0(\phi(s))) ds \right) - H^{-1} \left(\int_{t_0}^t k(s, \gamma_0(s)) ds \right) \right| \\ &\leq \frac{1}{K_{\min}} \left| \int_{t_1}^t k(s, \gamma_0(\phi(s))) ds - \int_{t_0}^t k(s, \gamma_0(s)) ds \right| \\ &\leq \frac{1}{K_{\min}} \int_{t_1}^t |k(s, \gamma_0(\phi(s))) - k(s, \gamma_0(s))| ds + \frac{1}{K_{\min}} \int_{t_0}^{t_1} k(s, \gamma_0(s)) ds \\ &\leq \frac{LK_{\max}}{K_{\min}} \int_{t_1}^t |\phi(s) - s| ds + \frac{K_{\max}}{K_{\min}}(t_1 - t_0). \end{aligned} \quad (4.11)$$

Thus, by Grönwall's inequality, we have, for every $t \geq t_1$,

$$|\phi(t) - t| \leq (t_1 - t_0) \frac{K_{\max}}{K_{\min}} e^{\frac{LK_{\max}}{K_{\min}}(t-t_1)},$$

which yields, for $t = t_1^*$, that

$$|t_1^* - t_0^*| \leq (t_1 - t_0) \frac{K_{\max}}{K_{\min}} e^{\frac{LK_{\max}}{K_{\min}}(t_1^* - t_1)}.$$

Note that $0 \leq t_1^* - t_1 = \phi^{-1}(t_0^*) - \phi^{-1}(t_0) \leq \frac{K_{\max}}{K_{\min}}(t_0^* - t_0) \leq \frac{K_{\max}}{K_{\min}}\varphi(t_0, x_0) \leq \frac{T(R)K_{\max}}{K_{\min}}$. Recalling that $0 \leq t_1 - t_0 \leq \frac{D\mathbf{d}(x_1, x_0)}{K_{\min}}$, we deduce that

$$t_1^* - t_0^* \leq C\mathbf{d}(x_0, x_1),$$

with $C = \frac{DK_{\max}}{K_{\min}^2} \exp\left(\frac{T(R)LK_{\max}^2}{K_{\min}^2}\right)$. This implies (4.8) since, using Lemma 3.9, we have $\varphi(t_0, x_1) \leq \tau(t_0, \gamma_1) + G(t_0, \gamma_1) \leq t_1^* - t_0 + g(x_0^*)$ and $\varphi(t_0, x_0) = t_0^* - t_0 + g(x_0^*)$. \square

Now, exactly as in [90, Proposition 4.7] and [91, Proposition 3.3], we can deduce Lipschitz continuity of φ by using Lemma 4.6 and the dynamic programming principle from Proposition 4.5. The proof of our next result, omitted here, can be obtained through a straightforward adaptation of [91, Proposition 3.3] to our more general setting.

Proposition 4.7. *Consider the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and assume that (H1)–(H7) are satisfied. Then, for every $R > 0$, there exists $M > 0$ depending only on $\mathbf{0}$, Γ , g , D , K_{\min} , K_{\max} , and R such that, for every $(t_0, x_0), (t_1, x_1) \in \mathbb{R}_+ \times \overline{B}_X(\mathbf{0}, R)$, we have*

$$|\varphi(t_0, x_0) - \varphi(t_1, x_1)| \leq M(|t_0 - t_1| + \mathbf{d}(x_0, x_1)).$$

We conclude these preliminary results on $\text{OCP}(X, \Gamma, k, g)$ with the following property of the value function φ . This property was first established in [83, Proposition 4.4] for compact X and with $g = 0$, but using the stronger assumption that $k \in \text{Lip}(\mathbb{R}_+ \times X, \mathbb{R}_+)$. We present here instead a proof based on the sharper result [55, Proposition 3.9], which uses only an assumption on the Lipschitz behavior of k similar to the weaker assumption (H6) but was shown in that reference only in the case where X is the closure of a nonempty bounded open subset of \mathbb{R}^d and without state constraints.

Proposition 4.8. *Consider the optimal control problem $\text{OCP}(X, \Gamma, k, g)$ and assume that (H1)–(H7) are satisfied. Then, for every $R > 0$, there exists $c > 0$ such that, for every $x \in \overline{B}_X(\mathbf{0}, R)$ and $t_0, t_1 \in \mathbb{R}_+$ with $t_0 \neq t_1$, we have*

$$\frac{\varphi(t_0, x) - \varphi(t_1, x)}{t_0 - t_1} \geq c - 1. \quad (4.12)$$

Proof. Let $R > 0$, T and ψ be the functions from Proposition 4.2, and $L_{\psi(R)}$ be the constant from (H6) applied with R replaced by $\psi(R)$. We denote $L_{\psi(R)}$ simply by L in the sequel for simplicity. Let $x \in \overline{B}_X(\mathbf{0}, R)$ and $t_0, t_1 \in \mathbb{R}_+$ with $t_0 \neq t_1$ and suppose, without loss of generality, that $t_1 < t_0$.

Let $\gamma_0 \in \text{Opt}(k, g, t_0, x)$ and $x_0 = \gamma_0(t_0 + \tau(t_0, \gamma_0)) \in \Gamma$ and recall that, by Proposition 4.2, we have $\gamma_0(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $t \in \mathbb{R}_+$. Similarly to the proof of Lemma 4.6, let $\phi: [t_1, +\infty) \rightarrow [t_0, +\infty)$ be a function satisfying (4.9). Note that, with no loss of generality, we may assume that $\phi(t) \geq t$ for every $t \in [t_1, +\infty)$. Indeed, $\phi(t_1) = t_0 > t_1$ and, if there exists $t > t_1$ such that $\phi(t) < t$, then, setting $t_2 = \inf\{t \in [t_1, +\infty) \mid \phi(t) = t\}$, we define $\tilde{\phi}: [t_1, +\infty) \rightarrow [t_0, +\infty)$ by $\tilde{\phi}(t) = \phi(t)$ for $t \in [t_1, t_2]$ and $\tilde{\phi}(t) = t$ for $t \geq t_2$. It is immediate to verify that $\tilde{\phi}$ is a solution of (4.9) with $\tilde{\phi}(t) \geq t$ for every $t \in [t_1, +\infty)$. We thus assume in the sequel that $\phi(t) \geq t$ for every $t \in [t_1, +\infty)$.

Define $\gamma_1: \mathbb{R}_+ \rightarrow X$ by $\gamma_1(t) = x$ for $t \in [0, t_1]$ and $\gamma_1(t) = \gamma_0(\phi(t))$ for all $t \geq t_1$. By construction, we have $\gamma_1 \in \text{Adm}(k)$, $t_0 + \tau(t_0, \gamma_0) = \phi(t_1 + \tau(t_1, \gamma_1))$, and $t_0 + \varphi(t_0, x) = t_0 + \tau(t_0, \gamma_0) + g(x_0) = \phi(t_1 + \tau(t_1, \gamma_1)) + g(x_0)$. For $i \in \{1, 2\}$, we set $t_i^* = t_i + \tau(t_i, \gamma_i)$, so that $\phi(t_1^*) = t_0^*$. In particular, $t_0^* \geq t_1^*$.

Similarly to the proof of Lemma 4.6, we have the identity

$$\int_{\phi(t)}^{t_0^*} k(s, \gamma_0(s)) ds = \int_t^{t_1^*} k(s, \gamma_0(\phi(s))) ds \quad \text{for every } t \in [t_1, t_1^*].$$

Defining this time $H: [0, t_0^*] \rightarrow \mathbb{R}_+$ for $\theta \in [0, t_0^*]$ by

$$H(\theta) = \int_{\theta}^{t_0^*} k(s, \gamma_0(s)) ds,$$

we have

$$\begin{aligned} H(\phi(t)) &= \int_t^{t_1^*} k(s, \gamma_0(\phi(s))) ds, & \text{for } t \in [t_1, t_1^*], \\ H(t) &= \int_t^{t_0^*} k(s, \gamma_0(s)) ds, & \text{for } t \in [0, t_0^*], \end{aligned}$$

and thus, proceeding as in the estimate (4.11) from the proof of Lemma 4.6, we deduce that, for every $t \in [t_1, t_1^*]$,

$$0 \leq \phi(t) - t \leq \frac{LK_{\max}}{K_{\min}} \int_t^{t_1^*} (\phi(s) - s) ds + \frac{K_{\max}}{K_{\min}} (t_0^* - t_1^*).$$

Defining $\lambda(r) = \phi(t_1^* - r) - t_1^* + r$ for $r \in [0, \tau(t_1, \gamma_1)]$, rewriting the above inequality in terms of λ , and using Grönwall's inequality, we deduce that, for every $t \in [t_1, t_1^*]$, we have

$$0 \leq \phi(t) - t \leq \frac{K_{\max}}{K_{\min}} (t_0^* - t_1^*) e^{\frac{LK_{\max}}{K_{\min}} (t_1^* - t)},$$

and thus

$$t_0 - t_1 \leq \frac{K_{\max}}{K_{\min}} (t_0^* - t_1^*) e^{\frac{LK_{\max}}{K_{\min}} \tau(t_1, \gamma_1)}. \quad (4.13)$$

Using Proposition 4.2 and the fact that ϕ^{-1} is $\frac{K_{\max}}{K_{\min}}$ -Lipschitz continuous, we get the estimate

$$\tau(t_1, \gamma_1) = \phi^{-1}(t_0^*) - t_1 = \phi^{-1}(t_0^*) - \phi^{-1}(t_0) \leq \frac{K_{\max}}{K_{\min}} \tau(t_0, \gamma_0) \leq \frac{K_{\max}}{K_{\min}} \varphi(t_0, x) \leq \frac{K_{\max}}{K_{\min}} T(R),$$

and thus, setting $c = \frac{K_{\min}}{K_{\max}} \exp\left(-\frac{T(R)LK_{\max}^2}{K_{\min}^2}\right) > 0$, we deduce from (4.13) that $c(t_0 - t_1) \leq t_0^* - t_1^*$, that is,

$$(c - 1)(t_0 - t_1) \leq \tau(t_0, \gamma_0) - \tau(t_1, \gamma_1),$$

and the conclusion follows by noticing that $\varphi(t_0, x) = \tau(t_0, \gamma_0) + g(x_0)$ and $\varphi(t_1, x) \leq \tau(t_1, \gamma_1) + g(x_0)$. \square

4.2 Properties of the mean field game

Using the properties of the optimal control problem $\text{OCP}(X, \Gamma, k, g)$, we now establish, in this section, some properties of the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$. Since X is not assumed to be compact, we need here results that are finer than those from [55, 83, 89, 91], and, for this reason, we follow the approach from [90]. However, some adaptations are needed, since [90] only deals with the case of the Euclidean space $X = \mathbb{R}^d$, without considering state constraints, and it also only considers the minimal-time problem, while we also have the exit cost g here.

Recall that, according to Definition 3.5, (weak or strong) equilibria of $\text{MFG}(X, \Gamma, K, g, m_0)$ are described in terms of a measure $Q \in \mathcal{P}(\mathcal{C}_X)$. Given such a measure Q , we

shall consider the optimal control problem $\text{OCP}(X, \Gamma, k_Q, g)$, with k_Q given by $k_Q(t, x) = K(e_{t\#}Q, x)$ for $(t, x) \in \mathbb{R}_+ \times X$. We will denote the value function of $\text{OCP}(X, \Gamma, k_Q, g)$ by φ_Q , and we omit Q from the notation of both k_Q and φ_Q when it is clear from the context.

Given $Q \in \mathcal{C}_X$, we denote in the sequel by $\mathbf{Opt}(Q)$ the set

$$\mathbf{Opt}(Q) = \bigcup_{x \in X} \text{Opt}(Q, g, 0, x), \quad (4.14)$$

i.e., $\mathbf{Opt}(Q)$ is the set of optimal trajectories for $\text{OCP}(X, \Gamma, k_Q, g)$ starting at time 0. Note that Q is a weak equilibrium if and only if $Q(\mathbf{Opt}(Q)) = 1$, and it is a strong equilibrium if and only if $\text{supp}(Q) \subset \mathbf{Opt}(Q)$.

We start with the following preliminary result, whose proof is straightforward.

Lemma 4.9.

- (a) For every $Q \in \mathcal{P}(\mathcal{C}_X)$, the function $\mathbb{R}_+ \ni t \mapsto e_{t\#}Q \in \mathcal{P}(X)$ is continuous.
- (b) For every $t \in \mathbb{R}_+$, the function $\mathcal{P}(\mathcal{C}_X) \ni Q \mapsto e_{t\#}Q \in \mathcal{P}(X)$ is continuous.

As a consequence, we immediately obtain the following result.

Corollary 4.10. *Let $K: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$ be a function satisfying (H8) and (H9), $Q \in \mathcal{P}(\mathcal{C}_X)$, and $k_Q: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ be defined for $(t, x) \in \mathbb{R}_+ \times X$ by $k_Q(t, x) = K(e_{t\#}Q, x)$. Then k_Q satisfies (H5) and (H6) with the same constants K_{\min} , K_{\max} , and L_R as K .*

The next result, which is a consequence of Proposition 4.2, states an important a priori property of equilibria, which will allow us in the sequel to restrict our search for equilibria to a compact space. This result was already stated, without proof, in [90, Lemma 5.3], and we provide its proof below for completeness.

Proposition 4.11. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4), (H8), and (H10) are satisfied. Then there exists a nondecreasing function with linear growth $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, depending only on $\mathbf{0}$, Γ , g , D , K_{\min} , and K_{\max} , such that, for every weak equilibrium $Q \in \mathcal{P}(\mathcal{C}_X)$ of $\text{MFG}(X, \Gamma, K, g, m_0)$ and $R > 0$, we have*

$$Q\left(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R)))\right) \geq m_0(\overline{B}_X(\mathbf{0}, R)).$$

In particular, we have $e_{t\#}Q(\overline{B}_X(\mathbf{0}, \psi(R))) \geq m_0(\overline{B}_X(\mathbf{0}, R))$ for every $t \in \mathbb{R}_+$.

Proof. Let $Q \in \mathcal{P}(\mathcal{C}_X)$ be a weak equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$, $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ be defined for $(t, x) \in \mathbb{R}_+ \times X$ by $k(t, x) = K(e_{t\#}Q, x)$, and ψ be the function obtained by applying Proposition 4.2 to $\text{OCP}(X, \Gamma, k, g)$. Recalling Corollary 4.10, we observe that ψ only depends on $\mathbf{0}$, Γ , g , D , K_{\min} , and K_{\max} , and, in particular, it is independent of Q . By Proposition 4.2, we have

$$\{\gamma \in \mathcal{C}_X \mid \gamma(0) \in \overline{B}_X(\mathbf{0}, R)\} \cap \mathbf{Opt}(Q) \subset \text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R))) \cap \mathbf{Opt}(Q),$$

where $\mathbf{Opt}(Q)$ is the set defined in (4.14). Recall that, Q being a weak equilibrium, we have $Q(\mathbf{Opt}(Q)) = 1$, and thus the above inclusion implies that

$$\begin{aligned} Q\left(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R)))\right) &= Q\left(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R))) \cap \mathbf{Opt}(Q)\right) \\ &\geq Q(\{\gamma \in \mathcal{C}_X \mid \gamma(0) \in \overline{B}_X(\mathbf{0}, R)\} \cap \mathbf{Opt}(Q)) \\ &= Q(\{\gamma \in \mathcal{C}_X \mid \gamma(0) \in \overline{B}_X(\mathbf{0}, R)\}) = m_0(\overline{B}_X(\mathbf{0}, R)), \end{aligned}$$

since $e_{0\#}Q = m_0$.

Finally, the last part of the conclusion follows as an immediate consequence of the inclusion $\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R))) \subset e_t^{-1}(\overline{B}_X(\mathbf{0}, \psi(R)))$. \square

It follows from Proposition 4.11 that it suffices to look for equilibria of the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ in the set

$$\begin{aligned} \mathfrak{Q} = \left\{ Q \in \mathcal{P}(\mathcal{C}_X) \mid e_{0\#}Q = m_0 \text{ and} \right. \\ \left. \forall R > 0, Q\left(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R)))\right) \geq m_0(\overline{B}_X(\mathbf{0}, R)) \right\}, \quad (4.15) \end{aligned}$$

where ψ is the function from the statement of Proposition 4.11. The next result provides elementary properties of \mathfrak{Q} . Its proof, omitted here, is identical to that of [90, Lemma 5.4], since the fact that [90] considers $X = \mathbb{R}^d$ and $g = 0$ plays no particular role in the proof.

Proposition 4.12. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, assume that (H1)–(H4), (H8), and (H10) are satisfied, and let \mathfrak{Q} be the set defined in (4.15). Then \mathfrak{Q} is nonempty, convex, and compact.*

We now prove an additional continuity property of the value function, stating that it is continuous when it is regarded as a function not only of (t, x) but also of the measure Q . This property was stated in [83, Proposition 5.2] and [90, Lemma 5.2] in the case $g = 0$, but none of these proofs can be immediately applied to our setting: [83, Proposition 5.2] makes the additional assumption of Lipschitz behavior of K with respect to its first variable, deducing Lipschitz behavior of φ with respect to Q , which is not necessarily the case in our context, while [90, Lemma 5.2] makes use of the Euclidean structure of \mathbb{R}^d and the absence of state constraints in some steps of the proof, exploiting also deeply the fact that $g = 0$. The proof we present here makes use of a different strategy, its ideas being inspired by a combination of those of [83, Proposition 5.2] and [55, Proposition 3.9], and it is reminiscent of the strategy of proof used in [55, Lemma 4.5].

Proposition 4.13. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4) and (H8)–(H10) are satisfied. Then $(t, x, Q) \mapsto \varphi_Q(t, x)$ is continuous on $\mathbb{R}_+ \times X \times \mathcal{P}(\mathcal{C}_X)$.*

Proof. Let $(t_n, x_n, Q_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+ \times X \times \mathcal{P}(\mathcal{C}_X)$ converging to some (t_*, x_*, Q_*) . For $n \in \mathbb{N}$ and $(t, x) \in \mathbb{R}_+ \times X$, define $k_n(t, x) = K(e_{t\#}Q_n, x)$ and $k_*(t, x) = K(e_{t\#}Q_*, x)$. Note that, by Lemma 4.9, we have that $k_n(t, x) \rightarrow k_*(t, x)$ for every $(t, x) \in \mathbb{R}_+ \times X$. For simplicity of notation, we write φ_n and φ_* for φ_{Q_n} and φ_{Q_*} , respectively.

Since $x_n \rightarrow x_*$ as $n \rightarrow +\infty$, there exists $R > 0$ such that $x_n \in \overline{B}_X(\mathbf{0}, R)$ for every $n \in \mathbb{N}$ and $x_* \in \overline{B}_X(\mathbf{0}, R)$. Let T and ψ be the functions obtained when one applies Proposition 4.2 to the functions k_n , $n \in \mathbb{N}$, and k_* ; note that, by Proposition 4.2, these functions do not depend on n , since k_n , $n \in \mathbb{N}$, and k_* all satisfy (H5) and (H6) with the same constants K_{\min} , K_{\max} , and L_R , thanks to Corollary 4.10. Similarly, Proposition 4.7 applied to $\text{OCP}(X, \Gamma, k_n, g)$, $n \in \mathbb{N}$, and to $\text{OCP}(X, \Gamma, k_*, g)$ yields a constant $M > 0$ independent of n such that φ_n , $n \in \mathbb{N}$, and φ_* are all M -Lipschitz continuous in $\mathbb{R}_+ \times \overline{B}_X(\mathbf{0}, R)$. In particular, we have

$$|\varphi_n(t_n, x_n) - \varphi_*(t_*, x_*)| \leq M(|t_n - t_*| + \mathbf{d}(x_n, x_*)) + |\varphi_n(t_*, x_*) - \varphi_*(t_*, x_*)|.$$

Thus, to conclude the proof, it suffices to show that $\varphi_n(t_*, x_*) \rightarrow \varphi_*(t_*, x_*)$ as $n \rightarrow +\infty$. We split this proof into two parts, showing first that $\varphi_*(t_*, x_*) \leq \liminf_{n \rightarrow +\infty} \varphi_n(t_*, x_*)$ and then that $\limsup_{n \rightarrow +\infty} \varphi_n(t_*, x_*) \leq \varphi_*(t_*, x_*)$.

Let us first show that $\varphi_*(t_*, x_*) \leq \liminf_{n \rightarrow +\infty} \varphi_n(t_*, x_*)$. Let $\underline{\varphi} = \liminf_{n \rightarrow +\infty} \varphi_n(t_*, x_*)$ and consider a subsequence of $(\varphi_n(t_*, x_*))_{n \in \mathbb{N}}$ converging to $\underline{\varphi}$, which we still denote by $(\varphi_n(t_*, x_*))_{n \in \mathbb{N}}$ for simplicity. For $n \in \mathbb{N}$, let $\gamma_n \in \text{Opt}(Q_n, g, t_*, x_*)$ and recall that, by Proposition 4.2, we have $\gamma_n(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Since in addition γ_n is K_{\max} -Lipschitz continuous for every $n \in \mathbb{N}$, by Arzelà–Ascoli theorem, up to extracting a subsequence, there exists $\gamma_* \in \text{Lip}_{K_{\max}}(X)$ such that $\gamma_n \rightarrow \gamma_*$ as $n \rightarrow +\infty$ in the topology of \mathcal{C}_X and $\gamma_*(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $t \in \mathbb{R}_+$.

Using the facts that $k_n(t, x) \rightarrow k_*(t, x)$ for every $(t, x) \in \mathbb{R}_+ \times X$ and that $k_n, n \in \mathbb{N}$, and k_* are Lipschitz continuous in their second argument in $\overline{B}_X(\mathbf{0}, \psi(R))$, uniformly with respect to the first argument in \mathbb{R}_+ and to $n \in \mathbb{N}$, we deduce, proceeding as in the proof of Proposition 4.1, that $\gamma_* \in \text{Adm}(k_*)$. In addition, since $\gamma_n(t_*) = x_*$ for every $n \in \mathbb{N}$, we have $\gamma_*(t_*) = x_*$. Thus, $\varphi_*(t_*, x_*) \leq \tau(t_*, \gamma_*) + G(t_*, \gamma_*)$ and, by Lemma 3.10, we have $\tau(t_*, \gamma_*) + G(t_*, \gamma_*) \leq \liminf_{n \rightarrow +\infty} \tau(t_*, \gamma_n) + G(t_*, \gamma_n) = \underline{\varphi}$, yielding the conclusion.

Let us now prove that $\limsup_{n \rightarrow +\infty} \varphi_n(t_*, x_*) \leq \varphi_*(t_*, x_*)$. Let $\gamma_* \in \text{Opt}(Q_*, g, t_*, x_*)$ and recall that, by Proposition 4.2, we have $\gamma_*(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $t \in \mathbb{R}_+$. In the sequel, we let L denote the constant $L_{\psi(R)}$ from (H9). For $n \in \mathbb{N}$, let $\phi_n: [t_*, +\infty) \rightarrow [t_*, +\infty)$ be a function of class \mathcal{C}^1 satisfying

$$\begin{cases} \dot{\phi}_n(t) = \frac{k_n(t, \gamma_*(\phi_n(t)))}{k_*(\phi_n(t), \gamma_*(\phi_n(t)))} & \text{for } t \geq t_*, \\ \phi_n(t_*) = t_*. \end{cases}$$

Let $\gamma_n: \mathbb{R}_+ \rightarrow X$ be defined by $\gamma_n(t) = x_*$ for $t \in [0, t_*]$ and $\gamma_n(t) = \gamma_*(\phi_n(t))$ for $t \in [t_*, +\infty)$. By construction, we have $\gamma_n \in \text{Adm}(Q_n)$ and $t_* + \tau(t_*, \gamma_*) = \phi_n(t_* + \tau(t_*, \gamma_n))$.

We now prove that $\phi_n(t) \rightarrow t$ as $n \rightarrow +\infty$, uniformly for t on a compact interval of $[t_*, +\infty)$. To do so, we proceed as in the proof of Lemma 4.6: for every $t \in [t_*, +\infty)$, we have

$$\int_{t_*}^{\phi_n(t)} k_*(s, \gamma_*(s)) ds = \int_{t_*}^t k_n(s, \gamma_*(\phi_n(s))) ds.$$

Defining $H: [t_*, +\infty) \rightarrow [t_*, +\infty)$ for $\theta \in [t_*, +\infty)$ by

$$H(\theta) = \int_{t_*}^{\theta} k_*(s, \gamma_*(s)) ds,$$

we then have the estimate, for every $t \in [t_*, +\infty)$,

$$\begin{aligned} |\phi_n(t) - t| &= \left| H^{-1} \left(\int_{t_*}^t k_n(s, \gamma_*(\phi_n(s))) ds \right) - H^{-1} \left(\int_{t_*}^t k_*(s, \gamma_*(s)) ds \right) \right| \\ &\leq \frac{1}{K_{\min}} \int_{t_*}^t |k_n(s, \gamma_*(\phi_n(s))) - k_*(s, \gamma_*(s))| ds \\ &\leq \frac{1}{K_{\min}} \int_{t_*}^t |k_n(s, \gamma_*(\phi_n(s))) - k_n(s, \gamma_*(s))| ds \\ &\quad + \frac{1}{K_{\min}} \int_{t_*}^t |k_n(s, \gamma_*(s)) - k_*(s, \gamma_*(s))| ds \\ &\leq \frac{LK_{\max}}{K_{\min}} \int_{t_*}^t |\phi_n(s) - s| ds + \frac{1}{K_{\min}} \int_{t_*}^t |k_n(s, \gamma_*(s)) - k_*(s, \gamma_*(s))| ds, \end{aligned}$$

which yields, by Grönwall's inequality, that, for every $t \in [t_*, +\infty)$,

$$|\phi_n(t) - t| \leq \frac{1}{K_{\min}} e^{\frac{LK_{\max}}{K_{\min}}(t-t_*)} \int_{t_*}^t |k_n(s, \gamma_*(s)) - k_*(s, \gamma_*(s))| ds.$$

Since $k_n(s, \gamma_*(s)) \rightarrow k_*(s, \gamma_*(s))$ for every $s \in [t_*, t]$ and $k_n, n \in \mathbb{N}$, and k_* are positive functions upper bounded by K_{\max} , it follows from Lebesgue's dominated convergence theorem that the integral in the right-hand side of the above inequality converges to 0 as $n \rightarrow +\infty$, and such a convergence can be made uniformly in t for t on a compact interval of $[t_*, +\infty)$. Hence, $\phi_n(t) \rightarrow t$ as $n \rightarrow +\infty$, uniformly in t for t on a compact interval of $[t_*, +\infty)$, as required. Such a convergence also implies that $\phi_n^{-1}(t) \rightarrow t$ as $n \rightarrow +\infty$ uniformly in t for t on a compact interval of $[t_*, +\infty)$, since $|\phi_n^{-1}(t) - t| \leq \frac{K_{\max}}{K_{\min}}|t - \phi_n(t)|$, using the fact that ϕ_n^{-1} is $\frac{K_{\max}}{K_{\min}}$ -Lipschitz continuous.

Now, recalling that $t_* + \tau(t_*, \gamma_*) = \phi_n(t_* + \tau(t_*, \gamma_n))$, we have $\tau(t_*, \gamma_n) = \phi_n^{-1}(t_* + \tau(t_*, \gamma_*)) - t_* \rightarrow \tau(t_*, \gamma_*)$ as $n \rightarrow +\infty$. Thus, $G(t_*, \gamma_n) = g(\gamma_n(t_* + \tau(t_*, \gamma_n))) \rightarrow G(t_*, \gamma_*)$ as $n \rightarrow +\infty$. As a conclusion,

$$\limsup_{n \rightarrow +\infty} \varphi_n(t_*, x_*) \leq \lim_{n \rightarrow +\infty} [\tau(t_*, \gamma_n) + G(t_*, \gamma_n)] = \tau(t_*, \gamma_*) + G(t_*, \gamma_*) = \varphi_*(t_*, x_*). \quad \square$$

One of the consequences of Proposition 4.13 is the following property of the graph of the set-valued map **Opt** defined in (4.14).

Proposition 4.14. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, assume that (H1)–(H4) and (H8)–(H10) are satisfied, and let $\mathbf{Opt}: \mathcal{P}(\mathcal{C}_X) \rightrightarrows \mathcal{C}_X$ be the set-valued map defined for $Q \in \mathcal{P}(\mathcal{C}_X)$ by (4.14). Then **Opt** has closed graph.*

Proof. Let $(Q_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{P}(\mathcal{C}_X)$ with $Q_n \rightarrow Q$ as $n \rightarrow +\infty$ for some $Q \in \mathcal{P}(\mathcal{C}_X)$ and $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{C}_X with $\gamma_n \in \mathbf{Opt}(Q_n)$ for every $n \in \mathbb{N}$ and $\gamma_n \rightarrow \gamma$ as $n \rightarrow +\infty$ for some $\gamma \in \mathcal{C}_X$. For every $n \in \mathbb{N}$, since $\gamma_n \in \mathbf{Opt}(Q_n) \subset \text{Adm}(Q_n)$, we have in particular that $\gamma_n \in \text{Lip}_{K_{\max}}(X)$, and thus $\gamma \in \text{Lip}_{K_{\max}}(X)$. Proceeding as in the proof of Propositions 4.1 and 4.13, we obtain that $\gamma \in \text{Adm}(Q)$. Using Lemma 3.10 and Proposition 4.13, we have that

$$\tau(0, \gamma) + G(0, \gamma) \leq \liminf_{n \rightarrow +\infty} \tau(0, \gamma_n) + G(0, \gamma_n) = \liminf_{n \rightarrow +\infty} \varphi_{Q_n}(0, \gamma_n(0)) = \varphi_Q(0, \gamma(0)),$$

and thus, up to modifying γ on the interval $[\tau(0, \gamma), +\infty)$ in order for it to be constant there, we deduce that $\gamma \in \text{Opt}(Q, g, 0, \gamma(0))$, thus $\gamma \in \mathbf{Opt}(Q)$, as required. \square

As a consequence of Proposition 4.14, we deduce that the set-valued map $(Q, x) \mapsto \text{Opt}(Q, g, 0, x)$ also has closed graph. This property was shown in [55, Lemma 4.5] for the mean field game model considered in that reference, using a direct proof relying neither on the closedness of the graph of **Opt** nor on the continuity of $(Q, t, x) \mapsto \varphi_Q(t, x)$. Thanks to our previous results, we provide here a much simpler proof.

Corollary 4.15. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4) and (H8)–(H10) are satisfied. Then the set-valued map $\text{Opt}(\cdot, g, 0, \cdot): \mathcal{P}(\mathcal{C}_X) \times X \rightrightarrows \mathcal{C}_X$ has closed graph.*

Proof. Note that, if $(Q_n, x_n, \gamma_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{P}(\mathcal{C}_X) \times X \times \mathcal{C}_X$ converging to some element $(Q, x, \gamma) \in \mathcal{P}(\mathcal{C}_X) \times X \times \mathcal{C}_X$ and with $\gamma_n \in \text{Opt}(Q_n, g, 0, x_n)$ for every $n \in \mathbb{N}$, then $\gamma_n \in \mathbf{Opt}(Q_n)$ for every $n \in \mathbb{N}$ and, by Proposition 4.14, we have $\gamma \in \mathbf{Opt}(Q)$. Since $\gamma(0) = \lim_{n \rightarrow +\infty} \gamma_n(0) = \lim_{n \rightarrow +\infty} x_n = x$, we deduce that $\gamma \in \text{Opt}(Q, g, 0, x)$, as required. \square

Another important consequence of Proposition 4.14 is that the notions of weak and strong equilibria coincide, a fact already stated in [55, Remark 4.6] and [91, Proposition 3.7] for the models considered in those references. We deduce it here as a trivial consequence of Proposition 4.14.

Corollary 4.16. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4) and (H8)–(H10) are satisfied. Let $Q \in \mathcal{P}(\mathcal{C}_X)$. Then Q is a weak equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$ if and only if it is a strong equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$.*

Proof. Recall that, as stated in Remark 3.6, any strong equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$ is also a weak equilibrium. To prove the converse implication, let $Q \in \mathcal{P}(\mathcal{C}_X)$ be a weak equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$. Take $\gamma \in \text{supp}(Q)$. Then, by definition of support and using the fact that $Q(\mathbf{Opt}(Q)) = 1$, there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in $\mathbf{Opt}(Q)$ such that $\gamma_n \rightarrow \gamma$ as $n \rightarrow +\infty$. It follows from Proposition 4.14 that $\mathbf{Opt}(Q)$ is closed, hence $\gamma \in \mathbf{Opt}(Q)$. Thus $\text{supp}(Q) \subset \mathbf{Opt}(Q)$, concluding the proof. \square

Finally, another consequence that we can get from Proposition 4.14 is that, given $Q \in \mathcal{P}(\mathcal{C}_X)$, one can find a Borel-measurable function that, with each $x \in X$, associates an optimal trajectory $\gamma \in \text{Opt}(Q, g, 0, x)$. A similar result was stated in [83, Proposition 5.3], but with a weaker conclusion, asserting measurability only with respect to a σ -algebra larger than that of Borel in X . We provide here more detailed arguments that allow one to obtain Borel measurability, based on the ideas given in the proof of [55, Lemma 4.7] and adapting them to the case of a space X not necessarily compact.

Corollary 4.17. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4) and (H8)–(H10) are satisfied. For every $Q \in \mathcal{P}(\mathcal{C}_X)$, there exists a Borel-measurable function $\Phi: X \rightarrow \mathbf{Opt}(Q)$ such that $\Phi(x) \in \text{Opt}(Q, g, 0, x)$ for every $x \in X$.*

Proof. Take $Q \in \mathcal{P}(\mathcal{C}_X)$ and let $k: \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ be defined by $k(t, x) = K(e_{t\#}Q, x)$. Let $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the function obtained by applying Proposition 4.2 to $\text{OCP}(X, \Gamma, k, g)$.

Let $\Phi: X \rightrightarrows \mathbf{Opt}(Q)$ be the set-valued function defined for $x \in X$ by $\Phi(x) = \text{Opt}(Q, g, 0, x)$ and note that, as an immediate consequence of Corollary 4.15, Φ has closed graph. In addition, by Proposition 4.3, $\Phi(x) \neq \emptyset$ for every $x \in X$.

We claim that Φ is upper semicontinuous. Indeed, if it were not the case, there would exist $x \in X$, an open set U containing $\Phi(x)$ in \mathcal{C}_X , a sequence $(x_n)_{n \in \mathbb{N}}$ in X converging to x , and a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in \mathcal{C}_X such that $\gamma_n \in \Phi(x_n) = \text{Opt}(Q, g, 0, x_n)$ for every $n \in \mathbb{N}$ and $\gamma_n \notin U$. Note that, in particular, $\gamma_n \in \text{Lip}_{K_{\max}}(X)$.

Let $R > 0$ be such that $x_n \in \overline{B}_X(\mathbf{0}, R)$ for every $n \in \mathbb{N}$ and $x \in \overline{B}_X(\mathbf{0}, R)$. Then, by Proposition 4.2, we have $\gamma_n(t) \in \overline{B}_X(\mathbf{0}, \psi(R))$ for every $n \in \mathbb{N}$ and $t \geq 0$, i.e., $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence in $\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R)))$. Since the latter set is compact, up to extracting a subsequence, there exists $\gamma \in \text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R)))$ such that $\gamma_n \rightarrow \gamma$ as $n \rightarrow +\infty$. Hence, the sequence $(Q, x_n, \gamma_n)_{n \in \mathbb{N}}$ converges to (Q, x, γ) and satisfies $\gamma_n \in \text{Opt}(Q, g, 0, x_n)$ for every $n \in \mathbb{N}$, showing, by Corollary 4.15, that $\gamma \in \text{Opt}(Q, g, 0, x) \subset U$. Since $\gamma_n \rightarrow \gamma$ as $n \rightarrow +\infty$ and U is open, we deduce that $\gamma_n \in U$ for n large enough, which contradicts the fact that $\gamma_n \notin U$ for every $n \in \mathbb{N}$. This contradiction establishes the fact that Φ is upper semicontinuous.

Since Φ is upper semicontinuous, by [8, Proposition 1.4.4], the set $\Phi^{-1}(A) = \{x \in X \mid \Phi(x) \cap A \neq \emptyset\}$ is closed for every closed subset A of \mathcal{C}_X . Hence, by [44, Proposition III.11], the set $\Phi^{-1}(B)$ is open for every open subset B of \mathcal{C}_X . Thus, Φ is measurable, and we obtain the desired function Φ as a Borel-measurable selection of Φ , which exists thanks to [8, Theorem 8.1.3]. \square

5 Main results

Thanks to the preliminary results from Section 4, we are now in position to state and prove the main results of this paper. In all the discussions in this section, we assume that

(H1)–(H4) and (H8)–(H10) are satisfied and, in particular, by Corollary 4.16, the notions of strong and weak equilibria for $\text{MFG}(X, \Gamma, K, g, m_0)$ coincide. For that reason, we refer in the sequel to a strong or weak equilibrium simply as *equilibrium*.

5.1 Existence of equilibria

As in [55, 83, 89–91], our strategy to prove existence of equilibria consists in recasting the notion of equilibrium in terms of fixed points of a set-valued map and then applying to the latter a suitable fixed-point theorem which, in our case, will be Kakutani fixed-point theorem (see, e.g., [64, § 7, Theorem 8.6]).

Let \mathbf{Opt} and \mathfrak{Q} be defined as in (4.14) and (4.15), respectively. We define the set-valued map $F: \mathfrak{Q} \rightrightarrows \mathfrak{Q}$ by setting, for $Q \in \mathfrak{Q}$,

$$F(Q) = \left\{ \tilde{Q} \in \mathfrak{Q} \mid \tilde{Q}(\mathbf{Opt}(Q)) = 1 \right\}. \quad (5.1)$$

Clearly, Q is an equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$ if and only if $Q \in F(Q)$. To apply Kakutani fixed-point theorem to F , we first prove the following properties of F . These properties were shown in [83, Lemmas 5.3 and 5.5], [55, Lemma 4.7], and [90, Lemma 5.6] for the mean field game models considered in those papers, and we proceed here by following the line of the proof of [90, Lemma 5.6], adapting arguments to our more general setting when needed.

Lemma 5.1. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, assume that (H1)–(H4) and (H8)–(H10) are satisfied, and let $F: \mathfrak{Q} \rightrightarrows \mathfrak{Q}$ be defined as in (5.1). Then F is upper semicontinuous and, for every $Q \in \mathfrak{Q}$, $F(Q)$ is nonempty, compact, and convex.*

Proof. Let $Q \in \mathfrak{Q}$. Clearly, by (5.1), the set $F(Q)$ is convex. To prove that it is nonempty, let $\Phi: X \rightarrow \mathbf{Opt}(Q)$ be the map from Corollary 4.17 and set $\tilde{Q} = \Phi_{\#} m_0$. By construction, $\tilde{Q} \in \mathcal{P}(\mathcal{C}_X)$, $e_{0\#} \tilde{Q} = m_0$, and $\tilde{Q}(\mathbf{Opt}(Q)) = 1$. Using the latter fact and proceeding as in the proof of Proposition 4.11, we deduce that $\tilde{Q}(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R)))) \geq m_0(\overline{B}_X(\mathbf{0}, R))$ for every $R > 0$, and thus $\tilde{Q} \in \mathfrak{Q}$. Hence $\tilde{Q} \in F(Q)$, so that $F(Q)$ is nonempty.

Let us now show that $F(Q)$ is compact. Since $F(Q) \subset \mathfrak{Q}$ and \mathfrak{Q} is compact by Proposition 4.12, it suffices to prove that $F(Q)$ is closed. Let $(\tilde{Q}_n)_{n \in \mathbb{N}}$ be a sequence in $F(Q)$ converging as $n \rightarrow +\infty$ to some $\tilde{Q} \in \mathcal{P}(\mathcal{C}_X)$. Since $\tilde{Q}_n \in F(Q) \subset \mathfrak{Q}$ and \mathfrak{Q} is closed, we deduce that $\tilde{Q} \in \mathfrak{Q}$. Note also that, as a consequence of Proposition 4.14, $\mathbf{Opt}(Q)$ is closed. Hence, by the Portmanteau theorem, we have

$$\tilde{Q}(\mathbf{Opt}(Q)) \geq \limsup_{n \rightarrow +\infty} \tilde{Q}_n(\mathbf{Opt}(Q)) = 1,$$

which proves that $\tilde{Q}(\mathbf{Opt}(Q)) = 1$, and thus $\tilde{Q} \in F(Q)$. Hence $F(Q)$ is closed.

Let us finally show that F is upper semicontinuous. Since \mathfrak{Q} is compact, it suffices to show, by [8, Proposition 1.4.8], that the graph of F is closed. Let $(Q_n, \tilde{Q}_n)_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{Q} \times \mathfrak{Q}$ with $(Q_n, \tilde{Q}_n) \rightarrow (Q, \tilde{Q})$ for some $(Q, \tilde{Q}) \in \mathfrak{Q} \times \mathfrak{Q}$ and such that $\tilde{Q}_n \in F(Q_n)$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, since $\tilde{Q}_n \in F(Q_n)$, we have $\tilde{Q}_n(\mathbf{Opt}(Q_n)) = 1$ and, since $\tilde{Q}_n \in \mathfrak{Q}$, we also have that $\tilde{Q}_n(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R)))) \geq m_0(\overline{B}_X(\mathbf{0}, R))$ for every $R > 0$, where ψ is the function from Proposition 4.11.

For every $\varepsilon \in (0, 1)$, let $V_\varepsilon = \{\gamma \in \mathcal{C}_X \mid \mathbf{d}_{\mathcal{C}_X}(\gamma, \mathbf{Opt}(Q)) \leq \varepsilon\}$. Let $R_0 > 0$ be such that $m_0(\overline{B}_X(\mathbf{0}, R_0)) \geq 1 - \varepsilon$. We claim that there exists $N_\varepsilon \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$

with $n \geq N_\varepsilon$, we have

$$\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0))) \cap \mathbf{Opt}(Q_n) \subset V_\varepsilon. \quad (5.2)$$

Indeed, if it were not the case, then there would exist a subsequence of $(Q_n)_{n \in \mathbb{N}}$, still denoted by $(Q_n)_{n \in \mathbb{N}}$ for simplicity, and a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in $\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0)))$ such that, for every $n \in \mathbb{N}$, we have $\gamma_n \in \mathbf{Opt}(Q_n)$ and $\gamma_n \notin V_\varepsilon$. Since $\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0)))$ is compact, up to extracting a subsequence, there exists $\gamma \in \text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0)))$ such that $\gamma_n \rightarrow \gamma$ as $n \rightarrow +\infty$. By Proposition 4.14, we also deduce that $\gamma \in \mathbf{Opt}(Q)$. Since $\gamma_n \notin V_\varepsilon$ for every $n \in \mathbb{N}$, we have $\mathbf{d}_{\mathcal{C}_X}(\gamma_n, \mathbf{Opt}(Q)) > \varepsilon$ for every $n \in \mathbb{N}$, thus $\mathbf{d}_{\mathcal{C}_X}(\gamma, \mathbf{Opt}(Q)) \geq \varepsilon$, which contradicts the fact that $\gamma \in \mathbf{Opt}(Q)$. This contradiction establishes that there exists $N_\varepsilon \in \mathbb{N}$ such that (5.2) holds true for every $n \in \mathbb{N}$ with $n \geq N_\varepsilon$.

Since $\tilde{Q}_n \in \mathfrak{Q}$ and $\tilde{Q}_n(\mathbf{Opt}(Q_n)) = 1$, we obtain from (5.2) that, for every $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, we have

$$\tilde{Q}_n(V_\varepsilon) \geq \tilde{Q}_n(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0))) \cap \mathbf{Opt}(Q_n)) \geq m_0(\overline{B}_X(\mathbf{0}, R_0)) \geq 1 - \varepsilon.$$

Since $\tilde{Q}_n \rightarrow \tilde{Q}$ and V_ε is closed, we have from the Portmanteau theorem that $\tilde{Q}(V_\varepsilon) \geq \limsup_{n \rightarrow +\infty} \tilde{Q}_n(V_\varepsilon) \geq 1 - \varepsilon$. On the other hand, since $\mathbf{Opt}(Q)$ is closed and $(V_\varepsilon)_{\varepsilon \in (0,1)}$ is a nondecreasing family of sets with $\bigcap_{\varepsilon \in (0,1)} V_\varepsilon = \mathbf{Opt}(Q)$, we conclude that $\tilde{Q}(\mathbf{Opt}(Q)) = \lim_{\varepsilon \rightarrow 0} \tilde{Q}(V_\varepsilon) = 1$. Hence $\tilde{Q} \in F(Q)$, concluding the proof that the graph of F is closed. \square

Using Lemma 5.1, we are finally in position to state and prove our main result on the existence of equilibria for $\text{MFG}(X, \Gamma, K, g, m_0)$. Recall that, by Corollary 4.16, weak and strong equilibria of $\text{MFG}(X, \Gamma, K, g, m_0)$ coincide under our assumptions.

Theorem 5.2. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4) and (H8)–(H10) are satisfied. Then there exists an equilibrium $Q \in \mathcal{P}(\mathcal{C}_X)$ of $\text{MFG}(X, \Gamma, K, g, m_0)$.*

Proof. Let $F: \mathfrak{Q} \rightrightarrows \mathfrak{Q}$ be the set-valued map defined by (5.1). By Proposition 4.12, Lemma 5.1, and the Kakutani fixed-point theorem (see, e.g., [64, § 7, Theorem 8.6]), F admits a fixed point, which, as discussed before, is an equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$. \square

Remark 5.3. Given $m_0 \in \mathcal{P}(X)$, one may have several equilibria of $\text{MFG}(X, \Gamma, K, g, m_0)$, as one may see from the following example taken from [83, Remark 7.1]. Let $X = [0, 1]$, $\Gamma = \{0, 1\}$, and K and g be constant functions equal, respectively, to 1 and 0. Assume that m_0 is the Dirac delta measure on the point $\frac{1}{2}$. Let $\gamma_\ell, \gamma_r \in \mathcal{C}_X$ be given for $t \in \mathbb{R}_+$ by $\gamma_\ell(t) = \max\left(\frac{1}{2} - t, 0\right)$ and $\gamma_r(t) = \min\left(\frac{1}{2} + t, 1\right)$. Then any $Q \in \mathcal{P}(\mathcal{C}_X)$ concentrated on γ_ℓ, γ_r (i.e., satisfying $Q(\{\gamma_\ell, \gamma_r\}) = 1$) is an equilibrium for $\text{MFG}(X, \Gamma, K, g, m_0)$. This example can be generalized for $X = \overline{\Omega}$ and $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^d$ is a nonempty bounded open set, by taking K and g as before and considering initial distributions $m_0 \in \mathcal{P}(\overline{\Omega})$ concentrated on the set where the distance function $\mathbf{d}(\cdot, \partial\Omega)$ is not differentiable.

For other models of mean field games, uniqueness of equilibria is typically obtained under some monotonicity assumptions on functions appearing in the cost of each player (see, e.g., [30, Proposition 2.9 and Theorem 3.6], [76, Theorem 4.1], and [77, Theorem 3.1]). Typically, these monotonicity assumptions mean that players tend to avoid congested regions, and they are important for uniqueness since games in which players tend to aggregate may present several equilibria (see, e.g., [47]). In our setting, it is not clear whether suitable congestion-avoidance assumptions should be sufficient for obtaining uniqueness of equilibria.

5.2 Asymptotic behavior

Given an equilibrium $Q \in \mathcal{P}(\mathcal{C}_X)$ of a mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, we now wish to understand the asymptotic behavior of the distribution of agents at time t , $m_t = e_{t\#}Q$, as $t \rightarrow +\infty$. It is natural to expect that m_t converges, as $t \rightarrow +\infty$, to a measure m_∞ concentrated in Γ and, in addition to prove this, the goal of this section is to provide also the convergence rate of m_t to m_∞ in the Wasserstein distance. This question was addressed in [90, Section 5.2] for the mean field game $\text{MFG}(\mathbb{R}^d, \Gamma, K, 0, m_0)$ (in a multipopulation setting), and it turns out that it is not difficult to generalize the arguments from [90] to a set X satisfying (H1) and to the presence of an exit cost g satisfying (H3). For sake of completeness, we provide below the details of this generalization.

In order to characterize the limit of m_t as $t \rightarrow +\infty$, we will need some additional notation, which we now state. Let $\mathcal{C}_{\text{lim}}(X) = \{\gamma \in \mathcal{C}_X \mid \lim_{t \rightarrow +\infty} \gamma(t) \text{ exists}\}$, which is a Borel subset of \mathcal{C}_X , and define $e_\infty: \mathcal{C}_{\text{lim}}(X) \rightarrow X$ by $e_\infty(\gamma) = \lim_{t \rightarrow +\infty} \gamma(t)$, which is a Borel-measurable function. Note that $\mathcal{C}_{\text{lim}}(X)$ is nonempty, since it contains, for instance, all constant trajectories (and also all continuous trajectories that become constant after some time). In addition, by definition of optimal trajectories, we have $\mathbf{Opt}(Q) \subset \mathcal{C}_{\text{lim}}(X)$ for every $Q \in \mathcal{P}(\mathcal{C}_X)$, and thus $e_{\infty\#}Q \in \mathcal{P}(X)$ is well-defined for every equilibrium Q of a mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$.

Our main result on the asymptotic behavior of equilibria is the following.

Theorem 5.4. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4) and (H8)–(H10) are satisfied. Let $Q \in \mathcal{P}(\mathcal{C}_X)$ be an equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$ and, for $t \in [0, +\infty]$, define $m_t = e_{t\#}Q$. Let ψ be the function obtained by applying Proposition 4.2 to $\text{OCP}(X, \Gamma, k, g)$, where $k(t, x) = K(m_t, x)$ for $(t, x) \in \mathbb{R}_+ \times X$.*

(a) *We have $m_t \rightarrow m_\infty$ as $t \rightarrow +\infty$.*

(b) *Let $p \in [1, +\infty)$ and assume that $m_0 \in \mathcal{P}_p(X)$. Then, for every $t \in [0, +\infty]$, we have $m_t \in \mathcal{P}_p(X)$. Moreover, there exist constants $\alpha > 0$ and $t_0 \geq 0$ only depending on $\mathbf{0}, \Gamma, g, D, K_{\min}$, and K_{\max} such that, for all $t \geq t_0$, we have*

$$\mathbf{W}_p(m_t, m_\infty)^p \leq 2^p \int_{X \setminus \bar{B}_X(\mathbf{0}, \alpha(t-t_0))} \psi(\mathbf{d}(\mathbf{0}, x))^p dm_0(x). \quad (5.3)$$

(c) *Assume that m_0 is compactly supported. Then, for every $t \in [0, +\infty]$, m_t is compactly supported and there exists $t_* \geq 0$ such that, for every $t \geq t_*$, we have*

$$m_t = m_\infty.$$

Proof. Let T be the function obtained by applying Proposition 4.2 to $\text{OCP}(X, \Gamma, k, g)$, and let $\alpha > 0$ and $t_0 \geq 0$ be such that $T(R) \leq \frac{R}{\alpha} + t_0$ for every $R > 0$. Note that, thanks to Corollary 4.10, α and t_0 can be chosen to depend only on $\mathbf{0}, \Gamma, g, D, K_{\min}$, and K_{\max} .

To show (a), let $f: X \rightarrow \mathbb{R}$ be continuous and bounded. We then have, using the continuity and boundedness of f and Lebesgue's dominated convergence theorem, that

$$\begin{aligned} \int_X f(x) dm_t(x) &= \int_{\mathcal{C}_{\text{lim}}(X)} f(\gamma(t)) dQ(\gamma) \\ &\xrightarrow{t \rightarrow +\infty} \int_{\mathcal{C}_{\text{lim}}(X)} f\left(\lim_{t \rightarrow +\infty} \gamma(t)\right) dQ(\gamma) = \int_X f(x) dm_\infty(x), \end{aligned}$$

yielding the required convergence.

Let us now prove (b). By Proposition 4.2, for every $\gamma \in \mathbf{Opt}(Q)$, we have that $\gamma(t) \in \overline{B}_X(\mathbf{0}, \psi(\mathbf{d}(\mathbf{0}, \gamma(0))))$ for every $t \in \mathbb{R}_+$, and the same is still true for $t = +\infty$ by taking the limit $t \rightarrow +\infty$. Hence, for $t \in [0, +\infty]$, we have

$$\begin{aligned} \int_X \mathbf{d}(\mathbf{0}, x)^p dm_t(x) &= \int_{\mathbf{Opt}(Q)} \mathbf{d}(\mathbf{0}, \gamma(t))^p dQ(\gamma) \\ &\leq \int_{\mathbf{Opt}(Q)} \psi(\mathbf{d}(\mathbf{0}, \gamma(0)))^p dQ(\gamma) = \int_X \psi(\mathbf{d}(\mathbf{0}, x))^p dm_0(x), \end{aligned}$$

where $\gamma(\infty)$ is defined for $\gamma \in \mathbf{Opt}(Q)$ as $\lim_{t \rightarrow +\infty} \gamma(t)$. Since ψ has linear growth, it follows that $m_t \in \mathcal{P}_p(X)$ for every $t \in [0, +\infty]$.

Let $t \in [t_0, +\infty)$. Note that, using the notation introduced in Section 2, we have $(e_t, e_\infty)_{\#} Q \in \Pi(m_t, m_\infty)$ and thus, by (2.1), we have

$$\mathbf{W}_p(m_t, m_\infty)^p \leq \int_{X \times X} \mathbf{d}(x, y)^p d(e_t, e_\infty)_{\#} Q(x, y) = \int_{\mathbf{Opt}(Q)} \mathbf{d}(e_t(\gamma), e_\infty(\gamma))^p dQ(\gamma).$$

If $\gamma \in \mathbf{Opt}(Q)$ is such that $\mathbf{d}(\mathbf{0}, \gamma(0)) \leq \alpha(t - t_0)$, then, since $T(\mathbf{d}(\mathbf{0}, \gamma(0))) \leq t$, we have, as a consequence of Proposition 4.2, that $\gamma(t) \in \Gamma$ and γ is constant on $[t, +\infty)$, yielding that $e_t(\gamma) = e_\infty(\gamma)$. Thus

$$\mathbf{W}_p(m_t, m_\infty)^p \leq \int_{\mathbf{Opt}(Q) \cap \{\gamma \in \mathcal{C}_X \mid \mathbf{d}(\mathbf{0}, \gamma(0)) > \alpha(t - t_0)\}} \mathbf{d}(e_t(\gamma), e_\infty(\gamma))^p dQ(\gamma).$$

Recalling that $e_t(\gamma) = \gamma(t) \in \overline{B}_X(\mathbf{0}, \psi(\mathbf{d}(\mathbf{0}, \gamma(0))))$ for every $t \in [0, +\infty]$ and $\gamma \in \mathbf{Opt}(Q)$, one has $\mathbf{d}(\mathbf{0}, e_t(\gamma)) \leq \psi(\mathbf{d}(\mathbf{0}, \gamma(0)))$ and thus

$$\begin{aligned} \mathbf{W}_p(m_t, m_\infty)^p &\leq \int_{\mathbf{Opt}(Q) \cap \{\gamma \in \mathcal{C}_X \mid \mathbf{d}(\mathbf{0}, \gamma(0)) > \alpha(t - t_0)\}} 2^p \psi(\mathbf{d}(\mathbf{0}, \gamma(0)))^p dQ(\gamma) \\ &= 2^p \int_{X \setminus \overline{B}_X(\mathbf{0}, \alpha(t - t_0))} \psi(\mathbf{d}(\mathbf{0}, x))^p dm_0(x), \end{aligned}$$

as required.

Finally, to prove (c), let $R_0 > 0$ be such that the support of m_0 is included in $\overline{B}_X(\mathbf{0}, R_0)$ and notice that, as a consequence of Proposition 4.2, the support of m_t is included in the compact set $\overline{B}_X(\mathbf{0}, \psi(R_0))$ for every $t \in [0, +\infty]$. Letting $t_* = T(R_0)$, we deduce that, for every $t \geq t_*$ and $\gamma \in \mathbf{Opt}(Q)$ with $\gamma(0) \in \overline{B}_X(\mathbf{0}, R_0)$, we have $e_t(\gamma) = e_\infty(\gamma)$, which concludes the proof since Q is supported in $\mathbf{Opt}(Q) \cap \{\gamma \in \mathcal{C}_X \mid \gamma(0) \in \overline{B}_X(\mathbf{0}, R_0)\}$. \square

Remark 5.5. The limit $m_\infty \in \mathcal{P}(X)$ from Theorem 5.4(a) is characterized as $m_\infty = e_{\infty \#} Q$, and hence it is uniquely determined by Q . However, as equilibria Q of a given mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ are not necessarily unique, m_∞ is also not necessarily uniquely determined by m_0 . Indeed, for the example of mean field game considered in Remark 5.3, m_∞ can be any measure of the form $\alpha \delta_0 + (1 - \alpha) \delta_1$ for $\alpha \in [0, 1]$, where δ_a denotes the Dirac delta measure at a .

We conclude this section by illustrating how (5.3) can be used to obtain explicit convergence rates of m_t to m_∞ in the Wasserstein distance \mathbf{W}_p in the case $X = \mathbb{R}^d$.

Corollary 5.6. *Let $d \in \mathbb{N}^*$ and assume that \mathbb{R}^d is endowed with the Euclidean distance. Consider the mean field game $\text{MFG}(\mathbb{R}^d, \Gamma, K, g, m_0)$ and assume that (H2), (H3), and (H8)–(H10) are satisfied. Let $Q \in \mathcal{P}(\mathcal{C}_X)$ be an equilibrium of $\text{MFG}(X, \Gamma, K, g, m_0)$ and, for $t \in [0, +\infty]$, define $m_t = e_{t \#} Q$. Fix $p \in [1, +\infty)$.*

(a) Assume that there exist $R_0 > 0$, $C_0 > 0$, and $\beta \in (p+d, +\infty)$ such that the restriction of m_0 to $\mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0, R_0)$ is absolutely continuous with respect to the Lebesgue measure and its density, still denoted by m_0 , satisfies $m_0(x) \leq \frac{C_0}{|x|^\beta}$ for almost every $x \in \mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0, R_0)$. Then there exist $T_* > 0$ (only depending on Γ , g , K_{\min} , K_{\max} , and R_0) and $C_* > 0$ (only depending on Γ , g , K_{\min} , K_{\max} , C_0 , d , p , and β) such that, for every $t > T_*$, we have

$$\mathbf{W}_p(m_t, m_\infty)^p \leq \frac{C_*}{t^{\beta-p-d}}.$$

(b) Assume that there exist $R_0 > 0$, $C_0 > 0$, and $\gamma_0 > 0$ such that the restriction of m_0 to $\mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0, R_0)$ is absolutely continuous with respect to the Lebesgue measure and its density, still denoted by m_0 , satisfies $m_0(x) \leq C_0 e^{-\gamma_0|x|}$ for almost every $x \in \mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0, R_0)$. Then there exist $T_* > 0$ (only depending on Γ , g , K_{\min} , K_{\max} , and R_0), $C_* > 0$ (only depending on Γ , g , K_{\min} , K_{\max} , C_0 , γ_0 , d , and p), and $\gamma_* > 0$ (only depending on Γ , g , K_{\min} , K_{\max} , and γ_0) such that, for every $t > T_*$, we have

$$\mathbf{W}_p(m_t, m_\infty)^p \leq C_* t^{p+d-1} e^{-\gamma_* t}.$$

Proof. We first note that, for $X = \mathbb{R}^d$ endowed with the Euclidean distance, (H1) is satisfied (and we take $\mathbf{0}$ as the origin 0 of \mathbb{R}^d in the sequel) and (H4) is satisfied with $D = 1$ (by taking γ to be the straight line connecting x to y with unit speed). Moreover, in both (a) and (b), we have $m_0 \in \mathcal{P}_p(\mathbb{R}^d)$. Indeed, this is immediate in (b), while, in (a), it is a consequence of the assumption that $\beta \in (p+d, +\infty)$.

Take $R_0 > 0$ and $C_0 > 0$ as in (a) or (b). Let ψ be as in Theorem 5.4 and note that, by Proposition 4.2 and Corollary 4.10, ψ satisfies $\psi(R) \leq AR + B$ for every $R > 0$, for some nonnegative constants A and B only depending on Γ , g , K_{\min} , and K_{\max} . In particular, $\psi(R) \leq A_* R$ for every $R \geq 1$, where $A_* = A + B$. Let $\alpha > 0$ and $t_0 \geq 0$ be as in Theorem 5.4(b) and set $T_0 = t_0 + \frac{1}{\alpha} \max(R_0, 1)$. Hence, for every $t > T_0$, we have $\alpha(t - t_0) > \max(R_0, 1)$, and thus, by Theorem 5.4(b), we have

$$\mathbf{W}_p(m_t, m_\infty)^p \leq (2A_*)^p \int_{\mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0, \alpha(t-t_0))} |x|^p dm_0(x). \quad (5.4)$$

In the sequel, we denote by ω_d the surface of the unit sphere in \mathbb{R}^d .

Let us prove (a). Let $T_* = \max(T_0, 2t_0)$. For every $t > T_*$, we have from (5.4) that

$$\begin{aligned} \mathbf{W}_p(m_t, m_\infty)^p &\leq (2A_*)^p C_0 \int_{\mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0, \alpha(t-t_0))} \frac{1}{|x|^{\beta-p}} dx \\ &= (2A_*)^p C_0 \omega_d \int_{\alpha(t-t_0)}^{+\infty} \frac{1}{r^{\beta-p-d+1}} dr \\ &= \frac{(2A_*)^p C_0 \omega_d}{(\beta-p-d)\alpha^{\beta-p-d}(t-t_0)^{\beta-p-d}} \leq \frac{2^{\beta-d} A_*^p C_0 \omega_d}{(\beta-p-d)\alpha^{\beta-p-d} t^{\beta-p-d}}, \end{aligned}$$

where we use in the last inequality that $t - t_0 > \frac{t}{2}$ for $t > T_*$. We then get the desired conclusion with $C_* = \frac{2^{\beta-d} A_*^p C_0 \omega_d}{(\beta-p-d)\alpha^{\beta-p-d}}$.

Let us now prove (b). Let $T_* = T_0$. For every $t > T_*$, we have from (5.4) that

$$\begin{aligned} \mathbf{W}_p(m_t, m_\infty)^p &\leq (2A_*)^p C_0 \int_{\mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0, \alpha(t-t_0))} |x|^p e^{-\gamma_0|x|} dx \\ &= (2A_*)^p C_0 \omega_d \int_{\alpha(t-t_0)}^{+\infty} r^{p+d-1} e^{-\gamma_0 r} dr. \end{aligned}$$

Performing a change of variables in the integral, we can rewrite the last term of the above equation as

$$(2A_*)^p C_0 \omega_d (\alpha(t-t_0))^{p+d-1} e^{-\gamma_0 \alpha(t-t_0)} \int_0^{+\infty} \left(1 + \frac{r}{\alpha(t-t_0)}\right)^{p+d-1} e^{-\gamma_0 r} dr$$

and, recalling that $1 < \alpha(t-t_0) \leq \alpha t$, we deduce that

$$\mathbf{W}_p(m_t, m_\infty)^p \leq (2A_*)^p C_0 \omega_d (\alpha t)^{p+d-1} e^{-\gamma_0 \alpha(t-t_0)} \int_0^{+\infty} (1+r)^{p+d-1} e^{-\gamma_0 r} dr,$$

which is the desired conclusion with $\gamma_* = \gamma_0 \alpha$ and

$$C_* = (2A_*)^p C_0 \omega_d \alpha^{p+d-1} e^{\gamma_0 \alpha t_0} \int_0^{+\infty} (1+r)^{p+d-1} e^{-\gamma_0 r} dr. \quad \square$$

5.3 Dependence on the initial distribution of agents

In Section 5.1, we have established that, under our standing assumptions (H1)–(H4) and (H8)–(H10), for any $m_0 \in \mathcal{P}(X)$, the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ admits an equilibrium $Q \in \mathcal{P}(\mathcal{C}_X)$ and, in Section 5.2, we have considered the asymptotic behavior of such equilibria, i.e., the behavior of $m_t = e_{t\#}Q$ as $t \rightarrow +\infty$, showing that these measures converge to the limit measure $m_\infty \in \mathcal{P}(X)$ characterized by $m_\infty = e_{\infty\#}Q$. Our next goal is to understand how equilibria Q and limit measures m_∞ depend on m_0 . Since both Q and m_∞ are not uniquely determined by m_0 , we will make use a set-valued framework.

More precisely, given a metric space (X, \mathbf{d}) , a nonempty closed subset Γ of X , and functions $K: \mathcal{P}(X) \times X \rightarrow \mathbb{R}_+$ and $g: \Gamma \rightarrow \mathbb{R}_+$, we introduce the set-valued map $\text{Eq}: \mathcal{P}(X) \rightrightarrows \mathcal{P}(\mathcal{C}_X)$ defined, for $m_0 \in \mathcal{P}(X)$, by

$$\text{Eq}(m_0) = \{Q \in \mathcal{P}(\mathcal{C}_X) \mid Q \text{ is an equilibrium of } \text{MFG}(X, \Gamma, K, g, m_0)\}. \quad (5.5)$$

Note that $\text{Eq}(m_0)$ can be equivalently rewritten as

$$\text{Eq}(m_0) = \{Q \in \mathcal{P}(\mathcal{C}_X) \mid e_{0\#}Q = m_0 \text{ and } Q(\mathbf{Opt}(Q)) = 1\}.$$

We also introduce the set-valued map $\text{Lim}: \mathcal{P}(X) \rightrightarrows \mathcal{P}(X)$ by setting, for $m_0 \in \mathcal{P}(X)$,

$$\text{Lim}(m_0) = \{e_{\infty\#}Q \mid Q \in \text{Eq}(m_0)\}. \quad (5.6)$$

Our main result on the set-valued map Eq is the following one.

Theorem 5.7. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, assume that (H1)–(H4) and (H8)–(H10) are satisfied, and let $\text{Eq}: \mathcal{P}(X) \rightrightarrows \mathcal{P}(\mathcal{C}_X)$ be the set-valued map defined in (5.5). Then $\text{Eq}(m_0)$ is nonempty and compact for every $m_0 \in \mathcal{P}(X)$ and Eq is upper semicontinuous. In particular, Eq has closed graph.*

Proof. The fact that $\text{Eq}(m_0)$ is nonempty for every $m_0 \in \mathcal{P}(X)$ is an immediate consequence of Theorem 5.2.

We next show that Eq has closed graph. To do that, let $(m_{0,n})_{n \in \mathbb{N}}$ be a sequence in $\mathcal{P}(X)$ converging to some $m_0 \in \mathcal{P}(X)$ and $(Q_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{P}(\mathcal{C}_X)$ converging to some $Q \in \mathcal{P}(\mathcal{C}_X)$ and with $Q_n \in \text{Eq}(m_{0,n})$ for every $n \in \mathbb{N}$. Hence $e_{0\#}Q_n = m_{0,n}$ and $Q_n(\mathbf{Opt}(Q_n)) = 1$ for every $n \in \mathbb{N}$.

Since $e_{0\#}Q_n = m_{0,n}$ for every $n \in \mathbb{N}$, it follows from Lemma 4.9 that $e_{0\#}Q = m_0$. To prove that $Q(\mathbf{Opt}(Q)) = 1$, let $\varepsilon \in (0, 1)$ and define the set V_ε by $V_\varepsilon = \{\gamma \in \mathcal{C}_X \mid$

$\mathbf{d}_{\mathcal{C}_X}(\gamma, \mathbf{Opt}(Q)) \leq \varepsilon\}$. Let $R_0 > 0$ be such that $m_0(B_X(\mathbf{0}, R_0)) \geq 1 - \frac{\varepsilon}{2}$ and note that, by the Portmanteau theorem, we have $\liminf_{n \rightarrow +\infty} m_{0,n}(B_X(\mathbf{0}, R_0)) \geq m_0(B_X(\mathbf{0}, R_0))$, hence there exists $N_\varepsilon \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, we have $m_{0,n}(B_X(\mathbf{0}, R_0)) \geq 1 - \varepsilon$.

Proceeding exactly as in the proof of Lemma 5.1, we deduce that, up to increasing N_ε , we have

$$\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0))) \cap \mathbf{Opt}(Q_n) \subset V_\varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, where ψ is the function from Proposition 4.11. Since $Q_n \in \text{Eq}(m_{0,n})$, we obtain, by combining the above inclusion with Proposition 4.11 and the fact that $Q_n(\mathbf{Opt}(Q_n)) = 1$, that, for every $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, we have

$$\begin{aligned} Q_n(V_\varepsilon) &\geq Q_n\left(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0))) \cap \mathbf{Opt}(Q_n)\right) \\ &\geq m_{0,n}(\overline{B}_X(\mathbf{0}, R_0)) \geq m_{0,n}(B_X(\mathbf{0}, R_0)) \geq 1 - \varepsilon. \end{aligned}$$

Proceeding once again as in the proof of Lemma 5.1, we deduce that $Q(V_\varepsilon) \geq 1 - \varepsilon$ and thus $Q(\mathbf{Opt}(Q)) = \lim_{\varepsilon \rightarrow 0} Q(V_\varepsilon) = 1$, completing the proof that $Q \in \text{Eq}(m_0)$. Hence, Eq has closed graph.

Since Eq has closed graph, we deduce that $\text{Eq}(m_0)$ is closed for every $m_0 \in \mathcal{P}(X)$. On the other hand, by Proposition 4.11, $\text{Eq}(m_0)$ is a subset of the set \mathfrak{Q} defined in (4.15), which is compact thanks to Proposition 4.12, implying thus that $\text{Eq}(m_0)$ is compact.

To conclude the proof, assume, to obtain a contradiction, that Eq is not upper semicontinuous. Hence there exist $m_0 \in \mathcal{P}(X)$, a neighborhood U of $\text{Eq}(m_0)$ in $\mathcal{P}(\mathcal{C}_X)$, a sequence $(m_{0,n})_{n \in \mathbb{N}}$ converging to m_0 , and a sequence $(Q_n)_{n \in \mathbb{N}}$ in $\mathcal{P}(\mathcal{C}_X)$ such that $Q_n \in \text{Eq}(m_{0,n}) \setminus U$ for every $n \in \mathbb{N}$.

We claim that the sequence $(Q_n)_{n \in \mathbb{N}}$ is tight. Indeed, let $\varepsilon \in (0, 1)$ and take $R_0 > 0$ such that $m_0(B_X(\mathbf{0}, R_0)) \geq 1 - \frac{\varepsilon}{2}$. By the Portmanteau theorem, we have

$$\liminf_{n \rightarrow +\infty} m_{0,n}(B_X(\mathbf{0}, R_0)) \geq m_0(B_X(\mathbf{0}, R_0)) \geq 1 - \frac{\varepsilon}{2},$$

hence there exists $N_\varepsilon \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, we have $m_{0,n}(B_X(\mathbf{0}, R_0)) \geq 1 - \varepsilon$. Since $Q_n \in \text{Eq}(m_{0,n})$, we deduce from Proposition 4.11 that

$$Q_n\left(\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0)))\right) \geq 1 - \varepsilon$$

for every $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, and thus the tightness of $(Q_n)_{n \in \mathbb{N}}$ follows as a consequence of the compactness of $\text{Lip}_{K_{\max}}(\overline{B}_X(\mathbf{0}, \psi(R_0)))$. Hence, by Prokhorov's theorem (see, e.g., [6, Theorem 5.1.3]), up to extracting a subsequence, which we still denote by $(Q_n)_{n \in \mathbb{N}}$ for simplicity, there exists $Q \in \mathcal{P}(\mathcal{C}_X)$ such that $Q_n \rightarrow Q$ as $n \rightarrow +\infty$. Since Eq has closed graph, we conclude that $Q \in \text{Eq}(m_0)$. This, however, contradicts the fact that $Q_n \notin U$ for every $n \in \mathbb{N}$, yielding the desired conclusion. \square

Note that the set-valued map Lim can be seen as the composition of the set-valued map Eq: $\mathcal{P}(X) \rightrightarrows \mathcal{P}(\mathcal{C}_X)$ with the operation of pushforward by e_∞ . The main difficulty in proving a result similar to Theorem 5.7 for the set-valued map Lim is that e_∞ is not defined on the whole space \mathcal{C}_X and, even in the subset where it is defined, it is not continuous. We will prove, however, that the operation of pushforward by e_∞ is continuous on the range of Eq. For that purpose, let us first introduce the set

$$\mathbf{Eq} = \bigcup_{m_0 \in \mathcal{P}(X)} \text{Eq}(m_0), \quad (5.7)$$

which is the range of Eq. Our first preliminary result, obtained as a simple consequence of Theorem 5.7, is that this set is closed.

Lemma 5.8. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$ and assume that (H1)–(H4) and (H8)–(H10) are satisfied. Then the set \mathbf{Eq} defined in (5.7) is closed.*

Proof. Let $(Q_n)_{n \in \mathbb{N}}$ be a sequence in \mathbf{Eq} converging to some Q in $\mathcal{P}(\mathcal{C}_X)$. Clearly, $Q_n \in \text{Eq}(e_{0\#}Q_n)$ and, by Lemma 4.9, we have $e_{0\#}Q_n \rightarrow e_{0\#}Q$ as $n \rightarrow +\infty$. Hence, by Theorem 5.7, we deduce that $Q \in \text{Eq}(e_{0\#}Q)$, showing that $Q \in \mathbf{Eq}$, as required. \square

We shall also need the following preliminary result.

Lemma 5.9. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, assume that (H1)–(H4) and (H8)–(H10) are satisfied, and let \mathbf{Eq} be the set defined in (5.7). Let $Q \in \mathbf{Eq}$ and denote $m_0 = e_{0\#}Q$. Then, for every $R > 0$, we have*

$$Q(\{\gamma \in \mathcal{C}_X \mid \gamma \text{ is constant on } [T(R), +\infty)\}) \geq m_0(\overline{B}_X(\mathbf{0}, R)),$$

where T is the function from Proposition 4.2.

Remark 5.10. The function T from Proposition 4.2 is associated with an optimal control problem $\text{OCP}(X, \Gamma, k, g)$ satisfying (H1)–(H5) and (H7) and it depends only on $\mathbf{0}$, Γ , g , D , K_{\min} , and K_{\max} . In the context of Lemma 5.9, instead of an optimal control problem, we consider a mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$. The function T in its statement should be understood as the function T associated with the optimal control problem $\text{OCP}(X, \Gamma, k_Q, g)$, where k_Q is defined from K and Q as in Corollary 4.10, and it follows from the latter result that T depends only on $\mathbf{0}$, Γ , g , D , K_{\min} , and K_{\max} from (H1)–(H4) and (H8)–(H10).

Proof of Lemma 5.9. Let $R > 0$ and take $\gamma \in \mathbf{Opt}(Q)$ such that $\gamma(0) \in \overline{B}_X(\mathbf{0}, R)$. It follows from Definition 3.1 and Proposition 4.2 that $\tau(0, \gamma) \leq T(R)$ and that γ is constant on $[\tau(0, \gamma), +\infty)$. Hence, for every $R > 0$, we have

$$\{\gamma \in \mathcal{C}_X \mid \gamma \in \mathbf{Opt}(Q) \text{ and } \gamma(0) \in \overline{B}_X(\mathbf{0}, R)\} \subset \{\gamma \in \mathcal{C}_X \mid \gamma \text{ is constant on } [T(R), +\infty)\},$$

showing that

$$\begin{aligned} Q(\{\gamma \in \mathcal{C}_X \mid \gamma \text{ is constant on } [T(R), +\infty)\}) \\ \geq Q(\{\gamma \in \mathcal{C}_X \mid \gamma \in \mathbf{Opt}(Q) \text{ and } \gamma(0) \in \overline{B}_X(\mathbf{0}, R)\}). \end{aligned}$$

Since $Q \in \mathbf{Eq}$, we have $Q(\mathbf{Opt}(Q)) = 1$, thus

$$\begin{aligned} Q(\{\gamma \in \mathcal{C}_X \mid \gamma \in \mathbf{Opt}(Q) \text{ and } \gamma(0) \in \overline{B}_X(\mathbf{0}, R)\}) &= Q(\{\gamma \in \mathcal{C}_X \mid \gamma(0) \in \overline{B}_X(\mathbf{0}, R)\}) \\ &= m_0(\overline{B}_X(\mathbf{0}, R)), \end{aligned}$$

yielding the conclusion. \square

We are now in position to show that the operation of pushforward by e_∞ is continuous in the range of Eq.

Lemma 5.11. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, assume that (H1)–(H4) and (H8)–(H10) are satisfied, and let \mathbf{Eq} be the set defined in (5.7). Then the map $\mathbf{Eq} \ni Q \mapsto e_{\infty\#}Q \in \mathcal{P}(X)$ is continuous.*

Proof. Let $(Q_n)_{n \in \mathbb{N}}$ be a sequence in **Eq** converging to some $Q \in \mathbf{Eq}$. Define $m_0 = e_{0\#}Q$ and, for $n \in \mathbb{N}$, set $m_{0,n} = e_{0\#}Q_n$. Note that $m_{0,n} \rightarrow m_0$ in $\mathcal{P}(X)$ as $n \rightarrow +\infty$ thanks to Lemma 4.9.

Take $\varepsilon > 0$ and let $R_\varepsilon > 0$ be such that $m_0(B_X(\mathbf{0}, R_\varepsilon)) \geq 1 - \frac{\varepsilon}{4}$. Using the fact that $m_{0,n} \rightarrow m_0$ in $\mathcal{P}(X)$ as $n \rightarrow +\infty$ and the Portmanteau theorem, we deduce that there exists $N_\varepsilon \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq N_\varepsilon$, we have

$$m_{0,n}(B_X(\mathbf{0}, R_\varepsilon)) \geq 1 - \frac{\varepsilon}{2}.$$

Let T be the function defined in Proposition 4.2 (considered here in the sense of Remark 5.10) and set $T_\varepsilon = T(R_\varepsilon)$ and

$$C_\varepsilon = \{\gamma \in \mathcal{C}_X \mid \gamma \text{ is constant on } [T_\varepsilon, +\infty)\}.$$

Note that C_ε is closed and, from Lemma 5.9, we have

$$Q_n(C_\varepsilon) \geq 1 - \frac{\varepsilon}{2} \text{ for every } n \in \mathbb{N} \text{ with } n \geq N_\varepsilon. \quad (5.8)$$

Let $F \subset X$ be a closed set. For every $n \in \mathbb{N}$, we have

$$e_{\infty\#}Q_n(F) = Q_n(e_\infty^{-1}(F)) = Q_n(e_\infty^{-1}(F) \cap C_\varepsilon) + Q_n(e_\infty^{-1}(F) \setminus C_\varepsilon). \quad (5.9)$$

Note that, since any trajectory $\gamma \in C_\varepsilon$ is constant on $[T_\varepsilon, +\infty)$, we have $e_\infty^{-1}(F) \cap C_\varepsilon = e_{T_\varepsilon}^{-1}(F) \cap C_\varepsilon$. Since e_{T_ε} is continuous by Lemma 4.9 and F is closed, we have that $e_{T_\varepsilon}^{-1}(F)$ is closed, and thus so is $e_{T_\varepsilon}^{-1}(F) \cap C_\varepsilon$. Hence, by the Portmanteau theorem, up to increasing N_ε (in a way that also depends on F), we have that $Q_n(e_\infty^{-1}(F) \cap C_\varepsilon) \leq Q(e_\infty^{-1}(F) \cap C_\varepsilon) + \frac{\varepsilon}{2}$. Combining this with (5.8) and (5.9), we deduce that, for $n \geq N_\varepsilon$, we have

$$e_{\infty\#}Q_n(F) \leq Q(e_\infty^{-1}(F) \cap C_\varepsilon) + \varepsilon \leq Q(e_\infty^{-1}(F)) + \varepsilon = e_{\infty\#}Q(F) + \varepsilon,$$

which shows that $\limsup_{n \rightarrow +\infty} e_{\infty\#}Q_n(F) \leq e_{\infty\#}Q(F)$. Since this holds for any closed set $F \subset X$, it follows from the Portmanteau theorem that $e_{\infty\#}Q_n \rightarrow e_{\infty\#}Q$ as $n \rightarrow +\infty$, as required. \square

We can now deduce, as an immediate consequence of Theorem 5.7 and Lemma 5.11, that the counterpart of the former also holds for the set-valued Lim . More precisely, we have the following result.

Corollary 5.12. *Consider the mean field game $\text{MFG}(X, \Gamma, K, g, m_0)$, assume that (H1)–(H4) and (H8)–(H10) are satisfied, and let $\text{Lim}: \mathcal{P}(X) \rightrightarrows \mathcal{P}(X)$ be the set-valued map defined in (5.6). Then $\text{Lim}(m_0)$ is nonempty and compact for every $m_0 \in \mathcal{P}(X)$ and Lim is upper semicontinuous. In particular, Lim has closed graph.*

Remark 5.13. In addition to upper semicontinuity, there are several other notions of continuity for set-valued maps, such as lower semicontinuity, Lipschitz continuity, or also continuity with respect to the Hausdorff distance (see, e.g., [8]). In general, however, no further continuity properties should be expected for the maps Eq and Lim . Consider, for instance, a modification of the mean field game from Remark 5.3 in which the initial distribution of agents m_0 is replaced by the measure $m_{0,a} = \delta_a$, where $a \in [0, 1]$ and δ_a denotes the Dirac delta measure at a . One can then compute that, for this mean field game,

$$\text{Eq}(\delta_a) = \begin{cases} \{\delta_{\gamma_{\ell,a}}\} & \text{if } 0 \leq a < \frac{1}{2}, \\ \{\alpha\delta_{\gamma_{\ell,a}} + (1-\alpha)\delta_{\gamma_{r,a}} \mid \alpha \in [0, 1]\} & \text{if } a = \frac{1}{2}, \\ \{\delta_{r,a}\} & \text{if } \frac{1}{2} < a \leq 1, \end{cases}$$

$$\text{Lim}(\delta_a) = \begin{cases} \{\delta_0\} & \text{if } 0 \leq a < \frac{1}{2}, \\ \{\alpha\delta_0 + (1 - \alpha)\delta_1 \mid \alpha \in [0, 1]\} & \text{if } a = \frac{1}{2}, \\ \{\delta_1\} & \text{if } \frac{1}{2} < a \leq 1, \end{cases}$$

where $\gamma_{\ell,a}, \gamma_{r,a} \in \mathcal{C}_X$ are the trajectories defined for $t \in \mathbb{R}_+$ by $\gamma_{\ell,a}(t) = \max(a - t, 0)$ and $\gamma_{r,a}(t) = \min(a + t, 1)$. It is immediate to verify from the above expressions that Eq and Lim are not lower semicontinuous, Lipschitz continuous, nor continuous with respect to the Hausdorff distance at $\delta_{\frac{1}{2}}$.

Acknowledgment

The author wishes to thank the anonymous reviewers of this paper, whose remarks led to important improvements. In particular, Section 5.3 was written following a question by a reviewer on the stability of m_∞ with respect to m_0 .

References

- [1] Y. Achdou and I. Capuzzo-Dolcetta. Mean field games: numerical methods. *SIAM J. Numer. Anal.*, 48(3):1136–1162, 2010.
- [2] Y. Achdou, P. Cardaliaguet, F. Delarue, A. Porretta, and F. Santambrogio. *Mean field games*, volume 2281 of *Lecture Notes in Mathematics*. Springer, Cham; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence, 2020. Notes from the CIME School held in Cetraro, June 2019, Edited by Cardaliaguet and Porretta, Fondazione CIME/CIME Foundation Subseries.
- [3] Y. Achdou and A. Porretta. Mean field games with congestion. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(2):443–480, 2018.
- [4] D. Amadori, B. Andreianov, M. Di Francesco, S. Fagioli, T. Girard, P. Goatin, P. A. Markowich, J.-F. Pietschmann, M. D. Rosini, G. Russo, G. Stivaletta, and M.-T. Wolfram. The mathematical theory of Hughes’ model: A survey of results. In N. Bellomo and L. Gibelli, editors, *Crowd Dynamics, Volume 4: Analytics and Human Factors in Crowd Modeling*, pages 9–53. Springer International Publishing, Cham, 2023.
- [5] D. Amadori, P. Goatin, and M. D. Rosini. Existence results for Hughes’ model for pedestrian flows. *J. Math. Anal. Appl.*, 420(1):387–406, 2014.
- [6] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
- [7] B. Andreianov, M. D. Rosini, and G. Stivaletta. On existence, stability and many-particle approximation of solutions of 1D Hughes’ model with linear costs. *J. Differential Equations*, 369:253–298, 2023.
- [8] J.-P. Aubin and H. Frankowska. *Set-valued analysis*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition [MR1048347].
- [9] R. J. Aumann. Markets with a continuum of traders. *Econometrica*, 32:39–50, 1964.

- [10] A. Aurell, R. Carmona, G. Dayanikli, and M. Laurière. Optimal incentives to mitigate epidemics: a Stackelberg mean field game approach. *SIAM J. Control Optim.*, 60(2):S294–S322, 2022.
- [11] F. Bagagiolo, S. Faggian, R. Maggistro, and R. Pesenti. Optimal control of the mean field equilibrium for a pedestrian tourists’ flow model. *Netw. Spat. Econ.*, 22:243–266, 2022.
- [12] M. Bardi. Asymptotic properties of non-coercive hamiltonians with drift. *Discrete Contin. Dyn. Syst.*, In press.
- [13] M. Bardi and H. Kouhikouh. Long-time behavior of deterministic mean field games with nonmonotone interactions. *SIAM J. Math. Anal.*, 56(4):5079–5098, 2024.
- [14] G. Barles and P. E. Souganidis. On the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM J. Math. Anal.*, 31(4):925–939, 2000.
- [15] J.-D. Benamou, G. Carlier, and F. Santambrogio. Variational mean field games. In *Active particles. Vol. 1. Advances in theory, models, and applications*, Model. Simul. Sci. Eng. Technol., pages 141–171. Birkhäuser/Springer, Cham, 2017.
- [16] M. Bernot, V. Caselles, and J.-M. Morel. *Optimal transportation networks*, volume 1955 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Models and theory.
- [17] C. Bertucci. Optimal stopping in mean field games, an obstacle problem approach. *J. Math. Pures Appl. (9)*, 120:165–194, 2018.
- [18] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [19] N. Bourbaki. *Topologie Générale. Chapitres 5 à 10*. Éléments de Mathématique. Springer, 2007. Reprint of the 1974 original.
- [20] G. Bouveret, R. Dumitrescu, and P. Tankov. Mean-field games of optimal stopping: a relaxed solution approach. *SIAM J. Control Optim.*, 58(4):1795–1821, 2020.
- [21] Y. Brenier. The least action principle and the related concept of generalized flows for incompressible perfect fluids. *J. Amer. Math. Soc.*, 2(2):225–255, 1989.
- [22] M. Burger, M. Di Francesco, P. A. Markowich, and M.-T. Wolfram. On a mean field game optimal control approach modeling fast exit scenarios in human crowds. In *52nd IEEE Conference on Decision and Control*. IEEE, dec 2013.
- [23] M. Burzoni and L. Campi. Mean field games with absorption and common noise with a model of bank run. *Stochastic Process. Appl.*, 164:206–241, 2023.
- [24] L. Campi and M. Fischer. N -player games and mean-field games with absorption. *Ann. Appl. Probab.*, 28(4):2188–2242, 2018.
- [25] P. Cannarsa and R. Capuani. Existence and uniqueness for mean field games with state constraints. In *PDE models for multi-agent phenomena*, volume 28 of *Springer INdAM Ser.*, pages 49–71. Springer, Cham, 2018.

- [26] P. Cannarsa, R. Capuani, and P. Cardaliaguet. $C^{1,1}$ -smoothness of constrained solutions in the calculus of variations with application to mean field games. *Math. Eng.*, 1(1):174–203, 2019.
- [27] P. Cannarsa, R. Capuani, and P. Cardaliaguet. Mean field games with state constraints: from mild to pointwise solutions of the PDE system. *Calc. Var. Partial Differential Equations*, 60(3):Paper No. 108, 33, 2021.
- [28] P. Cannarsa and C. Mendico. Asymptotic analysis for Hamilton-Jacobi-Bellman equations on Euclidean space. *J. Differential Equations*, 332:83–122, 2022.
- [29] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2004.
- [30] P. Cardaliaguet. Notes on mean field games. Available at <https://www.ceremade.dauphine.fr/~cardaliaguet/MFG20130420.pdf>.
- [31] P. Cardaliaguet. Long time average of first order mean field games and weak KAM theory. *Dyn. Games Appl.*, 3(4):473–488, 2013.
- [32] P. Cardaliaguet. Weak solutions for first order mean field games with local coupling. In *Analysis and geometry in control theory and its applications*, volume 11 of *Springer INdAM Ser.*, pages 111–158. Springer, Cham, 2015.
- [33] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. *The master equation and the convergence problem in mean field games*, volume 201 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2019.
- [34] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta. Long time average of mean field games with a nonlocal coupling. *SIAM J. Control Optim.*, 51(5):3558–3591, 2013.
- [35] P. Cardaliaguet and C. Mendico. Ergodic behavior of control and mean field games problems depending on acceleration. *Nonlinear Anal.*, 203:Paper No. 112185, 40, 2021.
- [36] P. Cardaliaguet, A. R. Mészáros, and F. Santambrogio. First order mean field games with density constraints: pressure equals price. *SIAM J. Control Optim.*, 54(5):2672–2709, 2016.
- [37] P. Cardaliaguet and A. Porretta. Long time behavior of the master equation in mean field game theory. *Anal. PDE*, 12(6):1397–1453, 2019.
- [38] G. Carlier, C. Jimenez, and F. Santambrogio. Optimal transportation with traffic congestion and Wardrop equilibria. *SIAM J. Control Optim.*, 47(3):1330–1350, 2008.
- [39] E. Carlini and F. J. Silva. A fully discrete semi-Lagrangian scheme for a first order mean field game problem. *SIAM J. Numer. Anal.*, 52(1):45–67, 2014.
- [40] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I*, volume 83 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean field FBSDEs, control, and games.
- [41] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. II*, volume 84 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean field games with common noise and master equations.

- [42] R. Carmona, F. Delarue, and D. Lacker. Mean field games of timing and models for bank runs. *Appl. Math. Optim.*, 76(1):217–260, 2017.
- [43] R. Carmona, M. Lauriere, and P.-L. Lions. Non-standard stochastic control with nonlinear Feynman-Kac costs. Preprint arXiv:2312.00908.
- [44] C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions*. Lecture Notes in Mathematics, Vol. 580. Springer-Verlag, Berlin-New York, 1977.
- [45] A. Cesaroni and M. Cirant. One-dimensional multi-agent optimal control with aggregation and distance constraints: qualitative properties and mean-field limit. *Nonlinearity*, 34(3):1408–1447, 2021.
- [46] M. Cirant and A. Porretta. Long time behavior and turnpike solutions in mildly non-monotone mean field games. *ESAIM Control Optim. Calc. Var.*, 27:Paper No. 86, 40, 2021.
- [47] M. Cirant and D. Tonon. Time-dependent focusing mean-field games: the sub-critical case. *J. Dynam. Differential Equations*, 31(1):49–79, 2019.
- [48] F. H. Clarke. *Optimization and nonsmooth analysis*, volume 5 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1990.
- [49] E. Cristiani, A. de Santo, and M. Menci. A generalized mean-field game model for the dynamics of pedestrians with limited predictive abilities. *Commun. Math. Sci.*, 21(1):65–82, 2023.
- [50] S. Daudin. Optimal control of the Fokker-Planck equation under state constraints in the Wasserstein space. *J. Math. Pures Appl. (9)*, 175:37–75, 2023.
- [51] A. Davini, A. Fathi, R. Iturriaga, and M. Zavidovique. Convergence of the solutions of the discounted Hamilton-Jacobi equation: convergence of the discounted solutions. *Invent. Math.*, 206(1):29–55, 2016.
- [52] M. Di Francesco, P. A. Markowich, J.-F. Pietschmann, and M.-T. Wolfram. On the Hughes’ model for pedestrian flow: the one-dimensional case. *J. Differential Equations*, 250(3):1334–1362, 2011.
- [53] R. Ducasse, G. Mazanti, and F. Santambrogio. Second order local minimal-time mean field games. *NoDEA Nonlinear Differential Equations Appl.*, 29:Paper No. 37, 32 pp., 2022.
- [54] S. Dweik. Optimal transportation with boundary costs and summability estimates on the transport density. *J. Convex Anal.*, 25(1):135–160, 2018.
- [55] S. Dweik and G. Mazanti. Sharp semi-concavity in a non-autonomous control problem and L^p estimates in an optimal-exit MFG. *NoDEA Nonlinear Differential Equations Appl.*, 27(2):Paper No. 11, 59 pp., 2020.
- [56] D. Evangelista, R. Ferreira, D. A. Gomes, L. Nurbekyan, and V. K. Voskanyan. First-order, stationary mean-field games with congestion. *Nonlinear Anal.*, 173:37–74, 2018.

- [57] M. Fischer and F. J. Silva. On the asymptotic nature of first order mean field games. *Appl. Math. Optim.*, 84(2):2327–2357, 2021.
- [58] B. Geshkovski and E. Zuazua. Turnpike in optimal control of PDEs, ResNets, and beyond. *Acta Numer.*, 31:135–263, 2022.
- [59] M. Ghattassi and N. Masmoudi. Non-separable mean field games for pedestrian flow: Generalized Hughes model. Preprint arXiv:2310.04702.
- [60] D. A. Gomes and H. Mitake. Existence for stationary mean-field games with congestion and quadratic Hamiltonians. *NoDEA Nonlinear Differential Equations Appl.*, 22(6):1897–1910, 2015.
- [61] D. A. Gomes and S. Patrizi. Obstacle mean-field game problem. *Interfaces Free Bound.*, 17(1):55–68, 2015.
- [62] D. A. Gomes and J. Saúde. Numerical methods for finite-state mean-field games satisfying a monotonicity condition. *Appl. Math. Optim.*, 83(1):51–82, 2021.
- [63] P. J. Graber and C. Mouzouni. On mean field games models for exhaustible commodities trade. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 11, 38, 2020.
- [64] A. Granas and J. Dugundji. *Fixed point theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.
- [65] L. Grüne. On the relation between discounted and average optimal value functions. *J. Differential Equations*, 148(1):65–99, 1998.
- [66] P. Hartman. *Ordinary differential equations*. John Wiley & Sons, Inc., New York-London-Sydney, 1964.
- [67] D. Helbing and P. Molnár. Social force model for pedestrian dynamics. *Physical Review E*, 51(5):4282–4286, 1995.
- [68] L. F. Henderson. The statistics of crowd fluids. *Nature*, 229(5284):381–383, 1971.
- [69] M. Huang, P. E. Caines, and R. P. Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, volume 1, pages 98–103. IEEE, 2003.
- [70] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ -Nash equilibria. *IEEE Trans. Automat. Control*, 52(9):1560–1571, 2007.
- [71] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.
- [72] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transport. Res. B-Meth*, 36(6):507–535, 2002.
- [73] B. Jovanovic and R. W. Rosenthal. Anonymous sequential games. *J. Math. Econom.*, 17(1):77–87, 1988.

- [74] A. Lachapelle and M.-T. Wolfram. On a mean field game approach modeling congestion and aversion in pedestrian crowds. *Transport. Res. B-Meth*, 45(10):1572–1589, 2011.
- [75] J.-M. Lasry. Contrôle stationnaire asymptotique. In A. Bensoussan and J.-L. Lions, editors, *Control theory, numerical methods and computer systems modelling*, volume Vol. 107 of *Lecture Notes in Econom. and Math. Systems*, pages 296–313. Springer, Berlin-New York, 1975.
- [76] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.
- [77] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.
- [78] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.
- [79] M. Laurière and L. Tangpi. Convergence of large population games to mean field games with interaction through the controls. *SIAM J. Math. Anal.*, 54(3):3535–3574, 2022.
- [80] H. Lavenant and F. Santambrogio. New estimates on the regularity of the pressure in density-constrained mean field games. *J. Lond. Math. Soc. (2)*, 100(2):644–667, 2019.
- [81] M. Masoero. On the long time convergence of potential MFG. *NoDEA Nonlinear Differential Equations Appl.*, 26(2):Paper No. 15, 45, 2019.
- [82] B. Maury and S. Faure. *Crowds in equations. An introduction to the microscopic modeling of crowds*. Advanced Textbooks in Mathematics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019.
- [83] G. Mazanti and F. Santambrogio. Minimal-time mean field games. *Math. Models Methods Appl. Sci.*, 29(8):1413–1464, 2019.
- [84] A. R. Mészáros and F. J. Silva. A variational approach to second order mean field games with density constraints: the stationary case. *J. Math. Pures Appl. (9)*, 104(6):1135–1159, 2015.
- [85] M. Nutz. A mean field game of optimal stopping. *SIAM J. Control Optim.*, 56(2):1206–1221, 2018.
- [86] A. Porretta. On the turnpike property for mean field games. *Minimax Theory Appl.*, 3(2):285–312, 2018.
- [87] C. Robinson. *Dynamical systems: Stability, symbolic dynamics, and chaos*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1999.
- [88] J.-M. Roquejoffre. Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations. *J. Math. Pures Appl. (9)*, 80(1):85–104, 2001.
- [89] S. Sadeghi Arjmand and G. Mazanti. On the characterization of equilibria of non-smooth minimal-time mean field games with state constraints. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 5300–5305, 2021.

- [90] S. Sadeghi Arjmand and G. Mazanti. Multipopulation minimal-time mean field games. *SIAM J. Control Optim.*, 60(4):1942–1969, 2022.
- [91] S. Sadeghi Arjmand and G. Mazanti. Nonsmooth mean field games with state constraints. *ESAIM Control Optim. Calc. Var.*, 28:Paper No. 74, 42 pp., 2022.
- [92] F. Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.
- [93] J. Sun and J. Yong. Turnpike properties for mean-field linear-quadratic optimal control problems. *SIAM J. Control Optim.*, 62(1):752–775, 2024.
- [94] E. Trélat and C. Zhang. Integral and measure-turnpike properties for infinite-dimensional optimal control systems. *Math. Control Signals Systems*, 30(1):Art. 3, 34, 2018.
- [95] B. Ziliotto. Convergence of the solutions of the discounted Hamilton-Jacobi equation: a counterexample. *J. Math. Pures Appl. (9)*, 128:330–338, 2019.