



**HAL**  
open science

## Teaching Divisibility and Binomials with Coq

Sylvie Boldo, François Clément, David Hamelin, Micaela Mayero, Pierre Rousselin

### ► To cite this version:

Sylvie Boldo, François Clément, David Hamelin, Micaela Mayero, Pierre Rousselin. Teaching Divisibility and Binomials with Coq. 13th International Workshop on Theorem proving components for Educational software - ThEdu 2024, Julien Narboux; Walther Neuper; Pedro Quaresma, Jul 2024, Nancy, France. pp.124 - 139, <10.4204/EPTCS.419.8>. <hal-04725586v3>

**HAL Id: hal-04725586**

**<https://hal.science/hal-04725586v3>**

Submitted on 14 May 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

# Teaching Divisibility and Binomials with Coq\*

Sylvie Boldo

Université Paris-Saclay, Inria,  
CNRS, ENS Paris-Saclay, LMF  
91190 Gif-sur-Yvette, France.  
sylvie.boldo@inria.fr

François Clément

Inria  
48 rue Barrault, 75013 Paris.  
CERMICS, École des Ponts  
77455 Marne-la-Vallée, France.  
francois.clement@inria.fr

David Hamelin

Université Paris-Saclay, Inria,  
CNRS, ENS Paris-Saclay, LMF  
91190 Gif-sur-Yvette, France.  
david.hamelin@inria.fr

Micaela Mayero

Université Sorbonne Paris Nord, CNRS,  
Laboratoire d'Informatique de Paris Nord, LIPN,  
F-93430 Villetaneuse, France.  
Université Paris-Saclay, Inria,  
CNRS, ENS Paris-Saclay, LMF  
91190 Gif-sur-Yvette, France.  
mayero@lipn.univ-paris13.fr

Pierre Rousselin

Université Sorbonne Paris Nord, CNRS,  
Laboratoire Analyse, Géométrie et Applications, LAGA,  
F-93430 Villetaneuse, France.  
Inria  
48 rue Barrault, 75013 Paris.  
CERMICS, École des Ponts  
77455 Marne-la-Vallée, France.  
rousselin@univ-paris13.fr

The goal of this contribution is to provide worksheets in Coq for students to learn about divisibility and binomials. These basic topics are a good case study as they are widely taught in the early academic years (or before in France). We present here our technical and pedagogical choices, the numerous exercises we developed and a small experiment we conducted on two students. As expected, it required additional Coq material such as other lemmas and dedicated tactics. The worksheets are freely available and flexible in several ways.

## 1 Introduction

This work was done in the context of the LiberAbaci project<sup>1</sup> aimed at improving the accessibility of the Coq proof system for mathematics students in the early academic years.

A difficulty in France is that there are many different curricula for the first academic years, so it was difficult to choose which topic to cover. A solution was to focus on just before higher education. At 17–18 years old, French students are usually at the last year of high school (*lycée*), this year being called *terminale* is ending by an exam called *baccalauréat*. With the high school reform from 2020 in France, students may choose mathematics as major (*spécialité*), possibly with additional mathematics during the last year. We have focused on these national curricula, more precisely on integers: divisibility and binomials. It is not the first time Coq tackles mathematics from *baccalauréat*: in 2013, the real analysis exercise was done in Coq at the same time as students [11] as a test of the Coquelicot library [4].

As for the exercises, they follow the spirit from Software Foundations [13] that was also used at Sorbonne Paris-Nord University [10] of a Coq file with holes. We provide exercises with comments both in French and English (as the language should not be the main difficulty for the students). Depending on the difficulty of the exercise, we may provide hints or explanations.

---

\*This work was supported by the Inria Challenge LiberAbaci.

This work was also partly supported by the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme – Grant Agreement n°810367.

<sup>1</sup><https://liberabaci.gitlabpages.inria.fr/>

We provide several levels of leeway to the teacher. First, the teacher may choose among the exercise (and their order) from the worksheets (files named `WS_*.v`). Second, some lemmas of the core library can be seen as exercises, such as the committee-chair identity about binomials in Section 3.2. Third, the level of automation is also up to the teacher. We provide all the exercise proofs without automation such as `lia` but it may be allowed for more advanced proofs in order to shorten them. Last but not least, the teacher chooses the environment to be used by the students, both the IDE (possibly `jsCoq`) and the required libraries (with a limited number of lemmas, with or without automation and so on).

Our goal is that formal proofs be as near as possible to mathematical proofs. `Coq` is here to prevent the student from making errors, forgetting a sub-case, and so on, as much as the student needs, and without getting on the teacher's nerves. We want the statements and proofs to be as near as possible to the mathematical reasonings and pedagogy. Even if we want to increase the abstraction level of students, we chose not to rely on too much abstraction in these exercises. A first reason is that it is probably too early in the academic years and a second reason is that error messages would be impossible to understand. This is why we rely on plain Peano naturals, and not on an abstract commutative monoid (where we could have defined sums for instance). We therefore only depend upon the `Coq` standard library, and not on `math-comp` or `Coquelicot` (for sums for instance) that abstract too much for our purpose. See also Section 3.1.

Divisibility and binomials are also commonly studied in the first academic year by students in mathematics and the contributions of this article range from *terminale* to the early academic years. Section 2 is about divisibility and Section 3 about binomials. Section 4 describes a small experiment on two students in June 2024 that has led to a several improvements of the worksheets.

All the `Coq` material (and a useful script explained later) is freely available at the following address:  
<https://liberabaci.gitlabpages.inria.fr/contenus/ThEdu24/ThEdu24.tar.gz>  
 that compiles with `Coq` versions 8.16 to 8.20. The worksheets are also available (with their proof for the demo) to be tested in a browser thanks to `jsCoq` [8] at:

<https://liberabaci.gitlabpages.inria.fr/contenus/ThEdu24/div.html>  
<https://liberabaci.gitlabpages.inria.fr/contenus/ThEdu24/binom.html>.

## 2 Divisibility

The divisibility relation on integers and basic arithmetic in general are often taught in high school and at the beginning of mathematically oriented higher education. Without neglecting historical (arithmetic was, along with geometry, one of the main mathematical sciences in the Greek antiquity) and practical (e.g. in cryptography) aspects, there are also important pedagogical reasons to teach arithmetic. Indeed, even a very basic course in arithmetic will expose students to many proof schemes, potent and nontrivial algorithms, and predicates of very different forms (primality, divisibility) which require the students to design careful rigorous proofs (sadly, maybe for the first time). In particular, the divisibility relation has many pedagogical benefits : working with an existentially quantified predicate, showing the power of symbolic calculus to prove unexpected facts and using one of many different tools (explicit witness, remainder computation, modular arithmetics, prime factor decomposition, and so on) to solve an exercise. It also serves as an example of a non-total order relation, with algebraic compatibility results.

Section 2.1 presents the state of the art and some technical choices, notably on the use of either  $\mathbb{N}$  or  $\mathbb{Z}$ . A point on primality is given in Section 2.2. For binomials, we did not notice any tactics missing as is the case here. For the simplicity of proofs, we defined several dedicated tactics and constructs in Section 2.3, before describing the worksheet in Section 2.4.

## 2.1 State of the Art and Technical Choices

We chose definitions and basic lemmas from the Coq standard library<sup>4</sup> to back our exercises. It has many benefits, together with some drawbacks (which will be discussed shortly): it is widely used, does not expose the user to many abstractions and is also used in the series “Software Foundations” by Pierce *et al.* [13].

Another choice to make is whether we work with natural numbers or integers. Natural numbers are easier to work with in some respects, for instance there is no need to consider units in prime number decompositions and there is only one possible choice for Euclidean division. However, they become impracticable whenever subtraction might be involved, for instance in Bezout’s lemma. We chose to stick with integers, a last argument being that the standard library offers much more in terms of arithmetics for integer than for natural numbers.

The standard library defines (in `Numbers.BinNums`) the type `Z` of integers in two steps. First, positive (nonzero) natural numbers are defined as non-empty strings of bits, starting with 1. Then, an integer number in `Z` is 0 or a positive number with a sign. This representation of integers has many qualities, the first being that equality needs not be redefined (there is no need to use a quotient as is the case for instance with the usual mathematical construction of  $\mathbb{Z}$  as equivalence classes of couples of natural numbers). The other one, maybe not relevant for an introductory mathematics course, is that this makes operations quite fast.

The bulk of the theory of integers in the standard library is actually implementation independent. (Most) lemmas and conventions which are shared with natural numbers are proved in the `Numbers.NatInt` sub-theory following an axiomatic module-based approach. Lemmas specific to integers are proved (still in an axiomatic way) in the `Numbers.Integer.Abstract` sub-theory. Finally, these theories are included in the concrete module `ZArith.BinInt.Z`. This axiomatic module-based approach makes it possible to factor many lemmas which are common between natural numbers and integers and also transfer for free all the theory, say of natural numbers, to any concrete implementation (say Peano natural numbers or binary natural numbers).

This module-based approach has some benefits (naming consistency, code factoring) as well as downsides: many concrete lemmas have useless hypotheses (it is frustrating, for instance, to have to prove  $0 \leq n$  when  $n \in \mathbb{N}$ ). Another source of disappointment is the power operation, which is specified to be of type  $\tau \rightarrow \tau \rightarrow \tau$ , with  $\tau$  being e.g. Peano’s natural numbers (which is fine, then) or integers (which is certainly not mathematically natural and adds some noise in the proofs). In addition, some lemmas are weaker than expected, even with this factorization effort in mind. In practice, we have to provide a thin layer on top of the standard library to remedy these defects. In addition to this general theory of integers, we use the `ZArith.Znumtheory` module which contains, for instance Gauss’ and Bezout’s lemmas.

Finally, let us discuss Euclidean division conventions on  $\mathbb{Z}$ . This is mostly hidden under the rug in usual mathematics courses. Indeed the mathematics *spécialité* national curriculum<sup>2</sup> only mentions Euclidean division of an integer by a positive natural number. In programming languages there are mostly two flavors of Euclidean divisions on integers : “floor division”, where the quotient of  $a$  by  $b$  is the greatest integer smaller or equal to the rational  $a/b$  and “trunc division” where it is the value of the ration  $a/b$  truncated towards 0. This is discussed at length in the standard library file `Numbers.Integer.Abstract.ZDivFloor` which also links to the useful reference [5]. We chose the floor division provided by the standard library. It does not have any mathematical advantage over the others, but its notations are less surprising and it is the sole division considered in `Znumtheory`. In Coq, there is no primitive notion of partial function, so one should decide the result of the Euclidean division

<sup>2</sup><https://eduscol.education.fr/document/24574/download>.

by 0. The Coq development team chose wisely  $\forall a : \mathbb{Z}, a / 0 = 0$  and  $a \bmod 0 = 0$ , so that the formula  $\forall a b : \mathbb{Z}, a = b * (a / b) + a \bmod b$  holds even when  $b = 0$ . This actually simplifies greatly many proofs.

## 2.2 Computation and prime numbers

One of the distinguishing traits of Coq is its orientation towards computation. Most of the operations are actually defined by (terminating) programs which can compute if given explicit arguments (and usually perform as many reduction steps as they can). For instance the `mod` operation on  $\mathbb{Z}$  actually computes the remainder of an Euclidean division, using an efficient algorithm. Then `mod` is proved to satisfy its specification and all the subsequent theory follows.

In contrast to computation, `Prop`-valued predicates such as the divisibility relation in  $\mathbb{Z}$  express a property which is (in general) not obtained by computation. In the standard library, it is expressed in a very natural way with an existential quantifier:

**Definition** `divide n m =  $\exists p, m = n * p$ .`

The usual notation for divisibility allows for more concise statements:

**Infix** `"|" := divide (at level 0).`

Some lemmas serve as a bridge between the computational and the non-computational world:

**Check** `Z.mod_div :  $\forall a b : \mathbb{Z}, a \bmod b = 0 \leftrightarrow (b | a)$ .`

This means that, during the course of an exercise, one can prove explicit divisibility relations simply by computation:

**Lemma** `example_div : (31 | 62744).`

**Proof.** `apply Z.mod_div; reflexivity. Qed.`

Here, the `reflexivity` tactic checks that  $62744 \bmod 31$  and  $0$  are *convertible*, in short, in our case, that the result of the computation of the left operand is indeed  $0$ . This kind of translation between properties and computations is the basis of the “Small Scale Reflection” methodology used in the math-comp library [9]: mathematical propositions having two sides, one computational and one logical, with reflection lemmas (akin to `Z.mod_div` here, but more elaborate) to toggle between those.

While doing some exercises, we noticed that the standard library lacks a function to decide the primality of an integer. The `prime` predicate in `Znumtheory` is itself quite surprising<sup>3</sup>:

**Definition** `prime (p : Z) : Prop :=  $1 < p \wedge \forall n, 1 \leq n < p \rightarrow \text{Z.gcd } n \text{ } p = 1$ .`

In practice, it proved to be as inconvenient as it is pedagogically debatable. Notice the choice to consider negative numbers non-prime which contradicts more advanced ideal based definitions of primality, but is common in introductory mathematical courses on arithmetic. We provide some alternative formulations, as well as a Boolean-valued function `primeb`, which computes:

**Lemma** `prime_no_strict_divisor (p : Z) :`

`prime p  $\leftrightarrow 1 < p \wedge \forall a, (a | p) \rightarrow a = -1 \vee a = 1 \vee a = p \vee a = -p$ .`

**Lemma** `prime_no_strict_divisor_below_sqrt (p : Z) :`

`prime p  $\leftrightarrow 1 < p \wedge \forall a, 1 < a \leq \text{Z.sqrt } p \rightarrow \neg (a | p)$ .`

**Lemma** `primeb_correct (n : Z) : primeb n = true  $\leftrightarrow$  prime n.`

<sup>3</sup>We have modified it slightly for simplification purposes.

Our `primeb` function simply computes the remainders of the division of an integer  $p \geq 2$  by  $2, 3, \dots, \sqrt{p}$ , where  $\sqrt{p}$  is the integer (floor) square root of  $p$ . This way, it becomes possible in exercises to prove well-known primality results by computation:

**Lemma** `prime_101` : `prime 101`.

**Proof.** `apply primeb_correct; reflexivity. Qed.`

Of course, our algorithm is quite naive and pales in comparison to state-of-the-art methods such as what can be found, for instance, in the `coqprime` library [15]. It is simple, probably what the students expect at this point, and can compute reasonably fast up to integers of around 8 decimal digits, and does not require to import a large development.

### 2.3 Tactics

After the ability to compute whether an integer is prime, another gap in the existing tactics is the ability to easily split into various subcases. More precisely, a recurring theme in divisibility exercises is to study the various possible results of a modulo. For example, an integer has only seven possibilities of values for its modulo 7. This is used in the following exercise:

$$\forall n > 0, (\exists k > 0, \exists q > 0, n = k^2 \wedge n = q^3) \Rightarrow (n \equiv 0[7] \vee n \equiv 1[7]).$$

It can be proven by considering the modulo 7 remainders of  $k^2$  and  $q^3$ . The mathematical proof is then the following table and the fact that only 0 and 1 appear in both lines.

$k, q$	0	1	2	3	4	5	6
$k^2 \bmod 7$	0	1	4	2	2	4	1
$q^3 \bmod 7$	0	1	1	6	1	6	6

The Coq formalization of the property is straightforward, but the proof is tedious as we have to split many cases without any help from the tool:

**Lemma** `exercise11` (`n k q` : `Z`) (`h0n` : `0 ≤ n`) (`h0k` : `0 ≤ k`) (`h0q` : `0 ≤ q`)

(`hk` : `n = k * k`) (`hq` : `n = q * q * q`) : `n mod 7 = 0 ∨ n mod 7 = 1`.

**Proof.** `assert (n mod 7 = 0 ∨ n mod 7 = 1 ∨ n mod 7 = 2 ∨ n mod 7 = 4) as h1.`

– `rewrite !hk, Z.mul_mod.`

`destruct (Z.eq_dec (k mod 7) 0) as [hk0|hk0].`

  + `rewrite hk0. tauto. (* solves the goal for k mod 7 = 0 *)`

  + `destruct (Z.eq_dec (k mod 7) 1) as [hk1|hk1].`

    \* `rewrite hk0. tauto. (* solves the goal for k mod 7 = 1 *)`

`(*... same for 2,3,4,5 and 6: we have k mod 7 ≠ 0, k mod 7 ≠ 1, ... k mod 7 ≠ 6... *)`

    \* `pose proof (mod_pos_bound k 7); nia. (*... but k mod 7 < 7, so the goal is absurd *)`

– `assert ((n mod 7) = 0 ∨ (n mod 7) = 1 ∨ (n mod 7) = 6) as h2. (* ... same as h1 *)`

**lia. Qed.**

To solve this issue, we have added the `study` tactic, that fits the table construction by making the adequate splitting. It considerably shortens the proof:

`assert (n mod 7 = 0 ∨ n mod 7 = 1 ∨ n mod 7 = 2 ∨ n mod 7 = 4) as h1.`

– `rewrite !hk, Z.mul_mod.`

`study (k mod 7) between 0 and 6 as h2.`

  8: `pose proof (Z.mod_pos_bound k 7) as bounds; lia.`

  all: `rewrite ← h2; tauto.`

– `assert (n mod 7 = 0 ∨ n mod 7 = 1 ∨ n mod 7 = 6) as h2.`

```

+ rewrite !hq, (Z.mul_mod (q * q)), (Z.mul_mod q q).
  study (q mod 7) between 0 and 6 as h2.
  8: pose proof (Z.mod_pos_bound q 7) as bounds; lia.
  all: rewrite ← h2; tauto.
+ lia.

```

Generating a large number of Coq goals may seem hard to handle for students, but it should motivate them to learn selectors such as `all:` or `1,2:` in order to avoid repetition in the proofs.

```

study x between 0 and 8 as h3.
10: apply Z.divide_pos_le in h2; lia.
all: subst x.
1: now apply Z.divide_0_1 in h2.
1, 2, 4, 8: now repeat constructor.

```

Listing 1: An example from exercise15

**Finding all solutions** This is not a tactic but a definition that aims to stick to mathematical exercise statements. They will often ask to determine the possible numbers satisfying a given property; for instance “*Find all natural numbers such that  $n+8$  is a multiple of  $n$* ”. Such statements imply that there exists a finite set satisfying the property: it is the student’s job to construct it. To express those problems, we added a new definition `findall`, used as follows:

**Lemma** `exercise15` : `findall` ( $\lambda n \Rightarrow 0 \leq n \wedge n \mid n + 8$ ).

where `findall {E} (P : E → Prop) := { l : list E & Forall P l ∧  $\forall (x : E), P x \rightarrow \text{In } x \text{ l}$  }`. In other words, the student must provide a list of all the elements satisfying the given property. We use a Sigma-type, which lives in `Type` rather than `exists` which lives in `Prop`: this forces the student to actually construct the object, rather than using a proof by contradiction.

## 2.4 Worksheet

We were surprised by the variety and number of exercises we found about divisibility. This has led to a large variety in the formalization and pedagogy of the formal counterpart. Some exercises were straightforward to formalize; for instance,  $(6 \mid n) \leftrightarrow (3 \mid n) \wedge (2 \mid n)$  is stated exactly like we would on paper, and the formal proof uses the same arguments. Some of the exercises become straightforward thanks to additional lemmas and the use of the tactics explained in Section 2.3, such as “find all the  $n$  so that  $n \mid n + 8$ ”.

We found out exercises become pedagogical failures when formalized. This is the case of computational exercises such as  $17 \mid 35^{228} + 84^{501}$ : Coq computes the modulo without the students having to think about a way to do it themselves. We think one such exercise is interesting for outlining computation, but they miss how to cleverly compute  $a^b \bmod c$ .

Many exercises rely on the decimal representation of number, which is out of the scope of the standard library. We therefore developed definitions and lemmas to state results such as the following ones: 91 divides any number written  $abcabc$ ; 2 divides  $n$  if and only if 2 divides the last digit of  $n$ ; 3 divides  $n$  if and only if 3 divides the sum of the digits of  $n$ . We chose to define our own inductive type `Digits` rather than using a list of digits, so that the student will not have to deal with empty digits list (for the last digit for instance).

We explored the option of only allowing a canonical representation of decimal numbers by forbidding

trailing zeroes. This proved to be a bad idea in practice, so we went back to a simpler representation. Another consideration was choosing whether our list of digits was little or big endian. Both approaches have strengths and weaknesses; it is easier to convert a little endian digit list into a number :

```
Fixpoint Z_of_Digits_alt (l : Digits): Z :=
match l with
| dsingleton d => (Z_of_Digit d)
| dcons x xs => (Z_of_Digit x) + (Z_of_Digits xs)*10
end.
```

Listing 2: Little endian

```
Fixpoint Z_of_Digits (l : Digits): Z :=
match l with
| dsingleton d => (Z_of_Digit d)
| dcons x xs =>
(Z_of_Digit x)*10^(digits_len xs) +
(Z_of_Digits xs)
end.
```

Listing 3: Big endian

The big endian function is both more computationally expensive and conceptually more complex than the little endian one. Despite that, we choose the big endian representation, because it is more natural to read numbers left-to-right, and the proofs are not much more difficult.

We also initially had two coercions  $\text{Digit} \rightarrow \text{Digits}$  and  $\text{Digits} \rightarrow \mathbb{Z}$  : this made writing the properties easier, however we removed it after testing the worksheets on students as they found it too confusing (in Section 4).

### 3 Binomials

After a mathematical introduction, Section 3.1 presents a (large) state of the art about binomials. Section 3.2 describes numerous additional needed lemmas, especially on sums and factorials, before going into the details of the worksheet in Section 3.3.

Given natural numbers  $n$  and  $k$  satisfying  $k \leq n$ , the *binomial coefficient*  $\binom{n}{k}$  is the coefficient of the monomial  $a^k b^{n-k}$  in the expansion of the binomial  $(a + b)$  to the power  $n$ , i.e. we have

$$\forall a, b \in \mathbb{R}, \quad (a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

The notation  $\binom{n}{k}$  usually reads “ $n$  choose  $k$ ”. Indeed, the binomial coefficient also represents the number of subsets of  $k$  elements of a set of  $n$  elements. Note that this second definition extends naturally to the *irregular* case  $n < k$  with the value 0, thus making the binomial coefficient a total function over  $\mathbb{N}^2$ . Note also that this combinatorial aspect is left aside here since the formalization of subsets is itself a nontrivial issue when teaching purposes are at stake.

Binomial coefficients appear in several fields of mathematics such as combinatorics, probability distributions, and series expansion. They can be generalized to real and complex numbers, by using the gamma function instead of the factorial. Their teaching is of great importance; it usually starts in high school, for example for computing probabilities.

There are mainly two ways to define the binomial coefficients. Either the multiplicative way using factorials or products,

$$\forall n, k \in \mathbb{N}, \quad \binom{n}{k} \stackrel{\text{def.}}{=} \frac{n!}{k!(n-k)!} \quad \text{when } k \leq n, \quad \binom{n}{k} \stackrel{\text{def.}}{=} 0 \quad \text{when } n < k, \quad (1)$$

or its variant where the left clause (when  $k \leq n$ ) is replaced with  $\binom{n}{k} \stackrel{\text{def.}}{=} \prod_{i=0}^{k-1} \frac{n-i}{k-i}$ . Or the additive way

using sums through Pascal’s rule,

$$\forall n, k \in \mathbb{N}, \quad \binom{n+1}{k+1} \stackrel{\text{def.}}{=} \binom{n}{k} + \binom{n}{k+1}, \quad (2)$$

together with the initializations  $\binom{n}{0} \stackrel{\text{def.}}{=} 1$  for  $n \in \mathbb{N}$ , and  $\binom{0}{k} \stackrel{\text{def.}}{=} 0$  for  $k > 0$ . Of course, these definitions are equivalent, and choosing one makes the other a lemma. While the second one naturally defines a natural number, the first one a priori lives in the field of rational numbers. Establishing that the division is actually exact needs a demonstration. Note that the expression using factorials is actually valid on  $\mathbb{N}^2$ , except when  $(n, k) = (0, 1)$ , provided the use of the Euclidean division and of both conventions  $0! \stackrel{\text{def.}}{=} 1$  and  $n - k \stackrel{\text{def.}}{=} 0$  when  $n < k$ .

Many properties of binomial coefficients involve finite summations of natural numbers, which are not always easy to deal with in a proof assistant. Note also that exercises given to students often deal with general purpose properties that may be reused in other proofs. Then, teachers would be free to give students a truncated library file and ask them to prove the missing results.

### 3.1 State of the Art

As explained above, binomial coefficient are inseparable from factorials and finite summations of natural numbers. Several implementations of these are available through the Coq community.

To start with, the offer provided by the Coq standard library<sup>4</sup> is not satisfactory, for three main reasons. First, the corpus of lemmas is extremely poor: three lemmas about factorials, five about binomial coefficients, and none about finite summations of natural numbers. Second, Pascal’s triangle exhibits in (2) an intrinsic integer nature of binomials. Thus, it is not reasonable for teaching purposes to search for assets in the Reals section of the Coq standard library. Indeed, the binomial coefficient is defined as `C` in `Reals.Binomial`, and the finite sum of endofunctions on natural numbers as `sum_nat_f` in `Reals.Rfunctions`. Moreover, binomial `C` is defined in  $\mathbb{R}$  through the factorial formula (1) and no projection on natural numbers is provided. And third, some lemmas could have more friendly names: `lt_0_fact` could be renamed `fact_gt_0`, as it is a good policy to take the main notion as prefix, and `pascal_step{1,2,3}` should have more explicit names related to their semantics.

As a consequence, several public developments in Coq also provide binomial coefficients, as they actually need them. For instance, we may cite `RSA`<sup>5</sup> for a proof of correctness of the RSA algorithm by J. C. Almeida and L. Théry in 1999, `Bertrand`<sup>6</sup> for a proof of Bertrand’s postulate by L. Théry in 2002. Both formalize the summation  $f(a) + f(a+1) + \dots + f(a+n)$ , and provide the possibility to extract first and last terms, additivity, distributivity and extensionality results, an induction principle, and an index shift. `Bertrand` adds splitting, monotony and subtractivity results, and some properties about double summation. Both define the binomials using Pascal’s triangle (2) and provide basic properties such as Pascal’s rule, a property using factorial, and the binomial formula (8). `Bertrand` adds positiveness, symmetry and monotony results, as well as an upper bound when  $n$  is odd and a lower bound when it is even.

We may also cite the `FSets`<sup>7</sup> library for finite sets over ordered types [7], in which both definitions

<sup>4</sup><https://coq.inria.fr/doc/V8.19.1/stdlib/>.

<sup>5</sup><https://github.com/coq-contribs/rsa/>, see file `Binomial.v`.

<sup>6</sup><https://github.com/coq-community/bertrand/>, see files `Summation.v` and `Binomial.v`.

<sup>7</sup><https://github.com/coq-contribs/fsets/>, see file `PowerSet.v`.

of the binomials are provided (using Pascal’s triangle (2) and using factorials with the Euclidean division (1)), as well as the proof that they agree.

The Coquelicot<sup>8</sup> library [4] defines finite sums in the abstract setting of any abelian monoid. They are formalized as the summation  $f(a) + f(a + 1) + \dots + f(b)$ , which is zero, the neutral element of the monoid, when  $b < a$ . The support includes the extraction of first and last term, an index shift, splitting, additivity and extensionality results.

On the other end, the math-comp<sup>9</sup> library [12] provides comprehensive support for the three features, respectively accessible through the notations  $n!$  for `factorial n`,  $\sum_{(m \leq i < n)} f$  for an instantiation of `bigop` for the addition of natural numbers, and  $\binom{n}{k}$  for `binomial n k`.

To sum up, none seem suitable for our teaching purposes about finite sums of natural numbers, factorials and binomials. The Coq standard library is not competitive on these topics. Contributions such as RSA, Bertrand, or FSets would come with too many out-of-scope lemmas. The formalizations in Coquelicot [4] and math-comp [12] are way too complex. These provide desirable automation for professionals, but makes it challenging to be used with first-years students. We do not want to expose them to a hierarchy of abstract algebraic structures, advanced syntax, or too general definitions.

### 3.2 Provided Library about Sums, Factorials, and Binomials

We have developed library-like modules `Nat_Compl`, `Nat_Sum`, `Nat_Factorial` and `Nat_Binomial`, and worksheet-like modules `WS_Helper` and `WS_Binomial`. The library modules contain definitions and the associated main properties. The module `Nat_Compl` is for complementary properties of basic operations on natural numbers. The module `WS_Binomial` collects proposals for exercises and their possible solution, including statements from the library part and specific results that only have a teaching interest. The module `WS_Helper` aims at listing relevant results about natural numbers for solving the exercises, and at allowing their use without qualification with the `Nat.` prefix. It is the only module required in `WS_Binomial`.

The library modules are only based on `PeanoNat`, `Compare_dec` and `Wf_nat` from the Coq standard library. We provide definitions that are structurally identical to their usual mathematical counterparts, i.e. with increasingly indexed sums and products. There is no addition of superficial syntax, such as implicit arguments. There is no use of automation through tactics `auto`, `lia`, `now` or `easy`. Thus, all proofs are completely explicit, and use plain Coq style. It is straightforward to incorporate automation in an existing proof, but the other way around is quite cumbersome, and is provided.

We have also taken a particular care in naming the provided lemmas. In module `Nat_Compl`, the names of lemmas are prefixed by `nat_`, as if they come from module `Arith.PeanoNat.Nat` from the Coq standard library. In the other library modules, we use the following naming rules: the main notion of the lemma is always used as prefix (here `sum_n_` and `sum_range_` in `Nat_Sum`, `fact_` in `Nat_Factorial` and `binom_` in `Nat_Binomial`), results with reverse implication have suffix `_rev`, equivalence results have suffix `_equiv`, contrapositive version has suffix `_contra`, alternate version has suffix `_alt`, significant ingredients of the conclusion are used to fill the names, and finally, we use common names when they exists.

The module `Nat_Sum` is dedicated to iterated summations on natural numbers. Given a function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , among various alternatives, we choose a definition that takes the first index  $a$  and the number of

<sup>8</sup><https://gitlab.inria.fr/coquelicot/coquelicot/>, see file `Hierarchy.v`.

<sup>9</sup><https://github.com/math-comp/math-comp/>, see files `ssrnat.v`, `bigop.v` and `binomial.v`.

summands  $n$ , and that is structurally equal to  $f(a) + f(a+1) + \dots + f(a+n-1)$ . For convenience, we also provide the sum over a range (with both bounds included).

**Fixpoint** `sum_n (a n : nat) (f : nat → nat) : nat :=  
 match n with 0 ⇒ 0 | S n' ⇒ sum_n a n' f + f (a + n') end.`

**Definition** `sum_range a b f := sum_n a (S b - a) f.`

Thus, we have `sum_range 0 n f = sum_n 0 (S n)`. We provide extensionality results, extraction of the first and last terms, concatenation, splitting and extraction of any term, weak and strict monotony results, the zero-sum property, additivity and subtractivity, left and right distributivity of the multiplication, left and right arbitrary index shifts, and a double summation result for the multiplication of polynomials.

The module `Nat_Factorial` is dedicated to the support for factorials. Given  $n \in \mathbb{N}$ , we define `fact n` to structurally represent  $1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n$ . The Coq implementation is as follows.

**Fixpoint** `fact (n : nat) : nat := match n with 0 ⇒ 1 | S n' ⇒ fact n' * S n' end.`

We provide positiveness, non-nullity and division results, weak and strict monotony results, injectivity results, and some divisibility results.

The module `Nat_Binomial` is dedicated to the support for binomial coefficients. The definition follows the recurrence relation in Pascal's triangle (2). The Coq implementation is as follows.

**Fixpoint** `binom (n p : nat) : nat :=  
 match n, p with _, 0 ⇒ 1 | 0, _ ⇒ 0 | S n', S p' ⇒ binom n' p' + binom n' p end.`

Among other basic results, we provide Pascal's rule, nullity and positiveness results, weak and strict monotony results, the committee-chair identity (3) and its iterated version (4), equality with the factorial formula (1), the cancellation identity (5), rising sums (6), the hockey stick identity (7), the binomial formula (8), Vandermonde's identity (9), and the formulas for the sum of binomials (10) and of their square (11).

$$\forall n, k \in \mathbb{N}, \quad (k+1) \binom{n}{k+1} = n \binom{n-1}{k}, \quad (3)$$

$$\forall n, k, i \in \mathbb{N}, \quad (k \neq 0 \vee i \leq n) \implies (k+i)!(n-i)! \binom{n}{k+i} = n!k! \binom{n-i}{k}, \quad (4)$$

$$\forall n, k, i \in \mathbb{N}, \quad \binom{n}{k+i} \binom{k+i}{i} = \binom{n}{i} \binom{n-i}{k}, \quad (5)$$

$$\forall m, n \in \mathbb{N}, \quad \sum_{i=0}^n \binom{m+i}{m} = \binom{m+n+1}{m+1}, \quad (6)$$

$$\forall n, k \in \mathbb{N}, \quad \sum_{i=k}^n \binom{i}{k} = \binom{n+1}{k+1}, \quad (7)$$

$$\forall a, b, n \in \mathbb{N}, \quad (a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}, \quad (8)$$

$$\forall m, n, p \in \mathbb{N}, \quad \binom{m+n}{p} = \sum_{i=0}^p \binom{m}{i} \binom{n}{p-i}, \quad (9)$$

$$\forall n \in \mathbb{N}, \quad \sum_{k=0}^n \binom{n}{k} = 2^n, \quad (10)$$

$$\forall n \in \mathbb{N}, \quad \sum_{k=0}^n \binom{n}{k}^2 = \binom{2*n}{n}. \quad (11)$$

The mathematical proofs of these results do not deserve to be given here. However, basic hints are written as comments in `Nat_Binomial`, and more detailed guidelines are provided in the worksheet `WS_Binomial`, as presented in the next section.

### 3.3 Worksheet

Note that most general properties provided in `Nat_Binomial` are commonly considered as good exercises for the training of students. As such, we provide a few hints and/or comments (both in French and in English) for several lemmas on binomials, that can be transmitted to students with statements stripped of their proofs. The proofs in `Nat_Binomial` are fully detailed, with no use of automation. Of course, teachers are free to let students use different levels of automation, in particular if they are sufficiently advanced in the usage of Coq.

The module `WS_Helper` provides a sufficient corpus of results from the standard library both to reproduce the proofs of the provided library (see Section 3.2) as well as to solve the proposed exercises. Thus, it is the only module required in the worksheet `WS_Binomial`. Teachers may tune the helper module to the exercises they extract from the library modules. They can also choose to hide these lemmas when students use the `Search` command by modifying the `Add Search Blacklist` command. We can notice that the corpus mentioned above actually behaves as a whitelist (see the second example `binom_sum_sqr` below).

The worksheet `WS_Binomial` includes several exercises (lemmas) with their possible solution (proofs). They are equipped with comments to help students. Comments are provided in several languages (here in English and in French), so as not to introduce a potential additional difficulty. Comments may be provided with several levels of mathematical details. A script automatically generates the student version with the selected options. It replaces the tagged solution with the command `Admitted` so that the file is still compiling. It is also capable of outputting HTML using `jsCoq`, turning the comments into rich text using the `coqdoc [6]` format.

We detail now three examples from `WS_Binomial`. The first one, `binom_ex01a` and its variant `binom_ex01a_89`, illustrate on a binomial manipulation exercise the possibility to give students more or less precise indications, tagged here `fr1/en1` for a precise indication, and `fr2/en2` for more advanced students (not yet handled by the script).

```
(*fr L'exercice suivant provient de *) (*en The following exercise is taken from *)
(* Hyperbole Terminale - Spécialité Mathématiques, Nathan, 2020 (in French). *)

(*fr Faire la preuve du lemme suivant et remplacer "Admitted" par "Qed". *)
(*en Fill the proof of next lemma and replace "Admitted" by "Qed". *)
Lemma binom_ex01a_89 :                               Lemma binom_ex01a :
  ∃ n, 5 ≤ n ∧ binom n 5 = 17 * binom n 4.           ∃ n, 5 ≤ n ∧ binom n 5 = 17 * binom n 4.
Proof.                                             Proof.
(*fr1 Indication : essayer n = 89. *)                (*fr2 Indication : trouver une valeur pour n *)
(*en1 Hint: try n = 89. *)                          (* en utilisant un papier et un crayon. *)
                                                    (*en2 Hint: find a value for n *)
                                                    (* by using pen and paper. *)

  (* Begin solution for the teacher. *)
  exists 89. split.
+ apply sub_0_1e. rewrite 5!sub_succ. apply sub_0_1.
+ apply (mul_cancel_l _ 5). discriminate.
  rewrite binom_mul_S_r.
```

```

rewrite 4!sub_succ, sub_0_r, mul_assoc.
reflexivity.
Qed.
(* End solution for the teacher. *)

```

For this kind of basic exercise, it could be more convenient to deal with the factorial expression (1) with the division of real numbers (instead of the Euclidean division). But this would add the difficulty to deal with the injection from  $\mathbb{N}$  to  $\mathbb{R}$ .

The second example, `binom_sum_sqr` (from the library), illustrates the possibility to give the teacher several proofs, here with automation (on the right) or without (on the left). We have chosen a mild level of automation, where only trivial notions with no consequence on the mathematical path are proved with automatic tactics (here `lia` and `now`). We can also imagine a version where the Coq statement is written by the student from the mathematical one (11).

```

(*fr Faire la preuve du lemme suivant et remplacer ‘‘Admitted’’ par ‘‘Qed’’.*
(*en Fill the proof of next lemma and replace ‘‘Admitted’’ by ‘‘Qed’’.*
Lemma binom_sum_sqr n : sum_range 0 n ( $\lambda$  k  $\Rightarrow$  (binom n k) ^ 2) = binom (2 * n) n.
Proof.
(*fr Indications :
- Transformer (2*n) en une somme.
- Utiliser l’identité de Vandermonde.
- Réduire l’égalité de somme à l’égalité des fonctions.
- Utiliser la symétrie des coefficients binomiaux.
- Terminer avec les conditions d’application des lemmes utilisés. *)
(*en Hints:
- Transform (2*n) into a sum.
- Use Vandermonde’s identity.
- Reduce equality on sums to equality on functions.
- Use symmetry of binomial coefficients.
- End with application conditions of used lemmas. *)

(* Begin solution for the teacher. *)      (* Begin other solution for the teacher. *)
(* Without automation. *)                  (* With automation. *)
replace (2 * n) with (n + n)
  by apply double_twice.
rewrite binom_vandermonde.
apply sum_range_ext. intros k [_ Hk].
rewrite  $\leftarrow$  binom_sym.
2: assumption.
rewrite pow_2_r. reflexivity.
Qed.
(* End solution for the teacher. *)

replace (2 * n) with (n + n) by lia.
rewrite binom_vandermonde.
apply sum_range_ext. intros k Hk.
rewrite  $\leftarrow$  binom_sym by lia.
now rewrite pow_2_r.
Qed.
(* End other solution for the teacher. *)

```

We can note that the `WS_Helper` whitelist mentioned above can reduce here to:

```
Export Nat (double_twice, le_add_r, lt_succ_r, pow_2_r).
```

Additional hint in the worksheet could be the output of a Coq command, for instance:

```

(* Check binom_vandermonde:
 $\forall$  a b n, binom (a + b) n = sum_range 0 n ( $\lambda$  k  $\Rightarrow$  binom a k * binom b (n - k)) *)

```

The third example, `binom_hockey_stick` (also from the library), illustrates the possibility to give teachers three levels of hints (from most to least detailed) on a more elaborate property.

```

(*fr Formule de la gouttière. *) (*en Hockey stick identity. *)
(*fr Faire la preuve du lemme suivant et remplacer "Admitted" par "Qed". *)
(*en Fill the proof of next lemma and replace "Admitted" by "Qed". *)
Lemma binom_hockey_stick n k : sum_range k n ( $\lambda$  i  $\Rightarrow$  binom i k) = binom (S n) (S k).
Proof.
(*fr1 Indications :
- Commencer par énoncer une version équivalente en remplaçant "sum_range k n" par
  "sum_n k p", où p représente le nombre de termes sommés, c'est-à-dire n-k+1.
  Donc "binom (S n) (S k)" devient "binom (k + p) (S k)" et il faut prouver
  " $\forall$  p k, sum_n k p ( $\lambda$  i  $\Rightarrow$  binom i k) = binom (k + p) (S k)".
- Prouver cette version par induction sur p, en utilisant la formule de Pascal pour
  le cas successeur.
- Puis faire apparaître "sum_n", utiliser le résultat prouvé précédemment, et
  discuter suivant que "S n < k" ou pas. *)
(*en1 Hints:
- First state an equivalent version by replacing "sum_range k n" by "sum_n k p",
  where p stands for the number of added terms, ie n-k+1.
  Thus "binom (S n) (S k)" becomes "binom (k + p) (S k)" and we have to prove
  " $\forall$  p k, sum_n k p ( $\lambda$  i  $\Rightarrow$  binom i k) = binom (k + p) (S k)".
- Prove this version by induction on p. Use Pascal's rule for the successor case.
- Then, make "sum_n" visible, use the result proved previously, and discuss
  whether "S n < k" or not. *)

(*fr2 Indications :
- Commencer par énoncer une version équivalente en remplaçant "sum_range k n" par
  "sum_n k p", où p représente le nombre de termes sommés, c'est-à-dire n-k+1,
  donc "binom (S n) (S k)" devient "binom (k + p) (S k)".
- Prouver cette version par induction sur p (penser à la formule de Pascal).
- Utiliser le résultat prouvé précédemment, et discuter suivant que "S n < k" ou pas. *)
(*en2 Hints:
- First state an equivalent version by replacing "sum_range k n" by "sum_n k p",
  where p stands for the number of added terms, ie n-k+1, thus "binom (S n) (S k)"
  becomes "binom (k + p) (S k)".
- Prove this version by induction on p (think about Pascal's rule).
- Then, use the result proved previously, and discuss whether "S n < k" or not. *)

(*fr3 Indications :
- Commencer par énoncer une version équivalente en remplaçant "sum_range k n" par
  "sum_n k p", où p représente le nombre de termes sommés.
- Prouver cette version par induction sur p, puis utiliser ce résultat intermédiaire. *)
(*en3 Hints:
- First state an equivalent version by replacing "sum_range k n" by "sum_n k p",
  where p stands for the number of added terms.
- Prove this version by induction on p, then use this intermediary result. *)

(* Begin solution for the teacher. *)
assert (H :  $\forall$  p k, sum_n k p ( $\lambda$  i  $\Rightarrow$  binom i k) = binom (k + p) (S k)).
intros p. induction p.
- intros. simpl. rewrite binom_eq_0.
  reflexivity.
  rewrite add_0_r. apply lt_succ_diag_r.
- intros. simpl. rewrite IHp.

```

```

rewrite add_succ_r. rewrite ← binom_pascal. apply add_comm.
(* *)
- unfold sum_range; rewrite H. destruct (nat_lt_le_dec (S n) k) as [H1 | H1].
+ rewrite !binom_eq_0. (* lia solves next subgoals. *)
  reflexivity.
  apply lt_trans with k. assumption. apply lt_succ_diag_r.
  rewrite (nat_sub_0_lt _ _ H1). rewrite add_0_r. apply lt_succ_diag_r.
+ f_equal. (* lia solves the next subgoal. *)
  rewrite (add_sub_assoc _ _ _ H1).
  rewrite add_comm, add_sub. reflexivity.
Qed.
(* End solution for the teacher. *)

```

Another question is the place of variables in statements. In Coq, we may write

`Lemma name :  $\forall$  n, ...` or `Lemma name n : ...`

Both scripts are equivalent, and have their pedagogical advantages and disadvantages that are not discussed here. We have made an arbitrary choice that can be modified by the teacher.

## 4 Experiment

In June 2024, some of us co-supervised, with Christine Paulin, two first-year undergraduate students during two weeks. They come from a dual degree curriculum, and had already completed a course on formal methods using Lean and a waterproof-like library as support. They expressed interest in an internship related to formal methods so we took this opportunity to let them test our worksheets.

They did not have any experience using Coq. We installed CoqIde on their machine, and let them finish an introductory worksheet [14] to get to know Coq. In less than two weeks, they managed to finish the two worksheets we had created. They were very autonomous for first-year undergraduate students.

We took into account several unsatisfactory points:

- The students relied on the `Check` command to interpret what they were seeing in the goal view. This is generally fine, except when coercions are involved. They did not understand how statements using coercions could possibly typecheck, and were generally very confused about them. We therefore removed the coercions (see Section 2.4), even though it made the statements longer. The students preferred this, and were able to complete the exercises.
- Another command the students used was `Search`. While they preferred using it rather than searching manually, they complained that it showed too much theorems that were unrelated. Our solution to this problem was to use the `Add Search Blacklist` command. This removes from the search result all theorems from the standard library. Then, we made our own module `Z`, which contains only theorems and definitions we want the students to use. This solves the issue, and `Search` now only shows relevant definitions and lemmas. This is especially the case for binomials, where the line is blurred between a (standard) library and some of our exercises.

Another interesting point is that, for the exercise `binom_committee_chair_alt`, they managed to complete the proof using another longer way we had not thought about.

## 5 Conclusion

We have shown a Coq development about divisibility and binomials. It consists in additional lemmas, dedicated tactics and 49 exercises in two worksheets. The total is more than 250 lemmas including the

49 exercises, and is 3,000 lines of Coq long, including about 1,300 lines of specification and comments. Ideas of exercises either come from textbooks [2, 1] or the internet, mostly `bibmath.net`; a few directly come from a worksheet of a teacher.

This library is free. Teachers may use part or all of it, with several levels of leeway as explained above: which exercises (from the worksheet or even the library), which automation, which environment, which level of hints. This pedagogical freedom is important to adapt the contents to (i) the level of the students and (ii) the goal of the course. We focus on the early academic years or even a year before, but it means a span of four years leading to more or less mature students. The goal may be either purely mathematics, or mainly logic or a trade-off between both. For example, a teacher may want to prove more advanced lemmas with as much automation as possible while another may prevent automation to focus on deduction rules and logical quantifiers.

A difficult point is how this work relates with the Coq standard library. It may be inconvenient to require the standard library with its bunch of lemmas that are probably too much for a beginning student (in a `Search` for instance). But it is inconvenient too to not require it and have a parallel development, to be maintained too. Another difficult point is the level of notation and abstraction allowed. It is tempting to abstract mathematical objects and make the proofs once, but it creates strange hypotheses as seen in Section 2.1. If a student tries to unfold all definitions, both abstraction and notation may make the goal impossible to read and understand.

The scope of this work is only integer properties. But we think we have spanned over many types of exercises, as done in mathematics. We have studied the ones for which we were not satisfied by the proof in order to develop either additional lemmas or tactics to make them reachable. The contribution is therefore both a Coq library and several Coq worksheets, done with a focus on flexibility and a pedagogical insight.

**Perspectives** A first perspective is of course for this material to be used more intensively on real students. It may help sharpen some of the exercises, by their hints, stating or expected difficulty.

A second perspective is about the script used to generate the version for students. By now, it is designed to keep only one language and it would be interesting to keep at least two of them. For instance, when selecting the local language, it would be interesting to also keep English, e.g. for the foreign students. Moreover, as presented with the worksheet about binomials, it will be interesting to handle the possibility to provide several levels of indications, e.g. with the tags `fr1/en1`, `fr2/en2` and so on.

Another perspective is more advanced mathematics and we plan to develop a worksheet on the Lebesgue integral. This is at the other end of the early academic years as possible (usually done at the end of the bachelor, three years after *terminale*). It deals with advanced concepts such as real numbers, functions, convergence. The mathematics usually rely on implicit axioms such as extensionality or classical logic. We plan to use a Coq library developed for purely research purpose [3] and see how it can be adapted to obtain pedagogical contents.

## Acknowledgments

We are very thankful to Gilles Delabrouille for letting us use his paper worksheet exercises for divisibility.

We are also very thankful to Julie Delafosse and Noé Briand for testing our worksheets during their internship. Their insights and efficiency have impressed us. Finally, we are very thankful to Christine Paulin for co-supervising the two students of the experiment.

## References

- [1] (2020): *Hyperbole Terminale - Option Mathématiques Expertes*. Nathan.
- [2] (2020): *Hyperbole Terminale - Spécialité Mathématiques*. Nathan.
- [3] Sylvie Boldo, François Clément, Florian Faissole, Vincent Martin & Micaela Mayero (2022): *A Coq Formalization of Lebesgue Integration of Nonnegative Functions*. *Journal of Automated Reasoning* 66, pp. 175–213, doi:10.1007/s10817-021-09612-0. Available at <https://inria.hal.science/hal-03471095/>.
- [4] Sylvie Boldo, Catherine Lelay & Guillaume Melquiond (2015): *Coquelicot: A User-Friendly Library of Real Analysis for Coq*. *Mathematics in Computer Science* 9(1), pp. 41–62, doi:10.1007/s11786-014-0181-1. Available at <https://inria.hal.science/hal-00860648/>.
- [5] Raymond T. Boute (1992): *The Euclidean definition of the functions div and mod*. *ACM Transactions on Programming Languages and Systems (TOPLAS)* 14(2), pp. 127–144, doi:10.1145/128861.128862.
- [6] Jean-Christophe Filliâtre: *coqdoc: a documentation tool for Coq*. Available at <https://usr.lmf.cnrs.fr/~jcf/coqdoc/manual.html>.
- [7] Jean-Christophe Filliâtre & Pierre Letouzey (2004): *Functors for Proofs and Programs*. In David Schmidt, editor: *Proc. of the 13th European Symposium on Programming (ESOP 2004)*, LNCS 2986, Springer, Berlin - Heidelberg, pp. 370–384, doi:10.1007/978-3-540-24725-8\_26.
- [8] Emilio Jesús Gallego Arias, Benoît Pin & Pierre Jouvelot (2017): *jsCoq: Towards Hybrid Theorem Proving Interfaces*. In S. Autexier & P. Quresma, editors: *Proc. of the 12th Workshop on User Interfaces for Theorem Provers (UITP 2016)*, *Electronic Proceedings in Theoretical Computer Science* 239, Open Publishing Association, pp. 15–27, doi:10.4204/EPTCS.239.2.
- [9] Georges Gonthier, Assia Mahboubi & Enrico Tassi (2016): *A Small Scale Reflection Extension for the Coq System*. Research Report RR-6455, Inria Saclay - Île-de-France. Available at <https://inria.hal.science/inria-00258384>.
- [10] Marie Kerjean, Frédéric Le Roux, Patrick Massot, Micaela Mayero, Zoé Mesnil, Simon Modeste, Julien Narboux & Pierre Rousselin (2022): *Utilisation des assistants de preuves pour l'enseignement en LI*. *Gazette de la Société Mathématique de France* 174. Available at <https://smf.emath.fr/publications/la-gazette-de-la-societe-mathematique-de-france-174-octobre-2022>.
- [11] Catherine Lelay (2015): *Repenser la bibliothèque réelle de Coq : vers une formalisation de l'analyse classique mieux adaptée*. Thèse, Université Paris Sud - Paris XI. Available at <https://theses.hal.science/tel-01228517>.
- [12] Assia Mahboubi & Enrico Tassi (2022): *Mathematical Components*. Zenodo, doi:10.5281/zenodo.7118596. Available at <https://math-comp.github.io/mcb/>.
- [13] Benjamin C. Pierce, Arthur Azevedo de Amorim, Chris Casinghino, Marco Gaboardi, Michael Greenberg, Cătălin Hrițcu, Vilhelm Sjöberg & Brent Yorgey (2023): *Software Foundations, Electronic textbooks*. Available at <https://softwarefoundations.cis.upenn.edu>.
- [14] Pierre Rousselin: *Initiation aux preuves formelles*. Available at <https://www.math.univ-paris13.fr/~rousselin/ipf.html>.
- [15] Laurent Théry & Guillaume Hanrot (2007): *Primality Proving with Elliptic Curves*. In Klaus Schneider & Jens Brandt, editors: *Proc. of the 20th International Conference on Theorem Proving in Higher Order Logics (TPHOL 2007)*, LNCS 4732, Springer, Berlin - Heidelberg, pp. 319–333, doi:10.1007/978-3-540-74591-4\_24.