The defocusing Calogero–Moser derivative nonlinear Schrödinger equation with a nonvanishing condition at infinity

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Abstract

We consider the defocusing Calogero–Moser derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - 2\Pi D\left(|u|^2\right) u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}$$

posed on $E := \left\{u \in L^\infty(\mathbb{R}) : u' \in L^2(\mathbb{R}), u'' \in L^2(\mathbb{R}), |u|^2 - 1 \in L^2(\mathbb{R})\right\}$. We prove the global well-posedness of this equation in E. Moreover, we give an explicit formula for the chiral solution to this equation.

Keywords Calogero–Moser derivative nonlinear Schrödinger equation, Global well-posedness, Explicit formula.

Contents

1 Introduction

We consider the following defocusing Calogero–Moser derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - 2\Pi D\left(|u|^2\right) u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{1.1}$$

In [9], Gérard and Lenzmann considered the focusing Calogero–Moser derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + 2\Pi D\left(|u|^2\right) u = 0 \tag{1.2}$$

on the line. They derived that for any solution $u \in H_+^s(\mathbb{R})$ with s sufficiently large, the equation (1.2) enjoys a Lax pair structure and infinite conservation laws. Based on the conservation laws, they showed the global well-posedness of (1.2) in $H_+^s(\mathbb{R})$ for any $s > \frac{1}{2}$ with small mass initial data ($\|u_0\|_{L^2}^2 \leq 2\pi$). Killip, Laurens and Vişan combined the explicit formulae of (1.1) and (1.2) with the tools and techniques of commuting flows to deduce the global well-posedness of (1.1) and (1.2) (with small mass initial data) in $H_+^s(\mathbb{R})$ ($s \geq 0$) [13]. Hogan and Kowalski showed that for arbitrarily small $\varepsilon > 0$ there exists initial data $u_0 \in H_+^\infty$ of mass $2\pi + \varepsilon$ to the equation (1.2) such that the corresponding maximal lifespan solution $u : (T_-, T_+) \times \mathbb{R} \to \mathbb{C}$ satisfies $\lim_{t\to T_\pm} \|u(t)\|_{H^s} = \infty$ for all $s > 0$ [10]. K. Kim, T. Kim and Kwon constructed smooth, chiral, and finite energy finite-time blow-up solutions to (1.2) with mass arbitrarily close to 2π [14]. Badreddine characterized the zero dispersion limit solution to (1.1) and (1.2) with initial data $u_0 \in L_+^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by explicit formulae, and she identified the zero dispersion limit in terms of the branches of the multivalued solution of the inviscid Burgers–Hopf equation [3]. Badreddine also showed the global well-posedness of (1.1) and (1.2) (with small mass initial data) on the circle in $H^s(\mathbb{T})$ for any $s \geq 0$ [1], and she characterized the traveling wave solutions to (1.1) and (1.2) on the circle [2].

In this paper, we study the Cauchy problem of (1.1) defined on the space

$$E := \left\{u \in L^\infty(\mathbb{R}) : u' \in L^2(\mathbb{R}), u'' \in L^2(\mathbb{R}), |u|^2 - 1 \in L^2(\mathbb{R})\right\}. \tag{1.3}$$

It is easy to endow the space E with a structure of complete metric space by introducing the following distance function,

$$d_E(u,\tilde{u}) = \|u-\tilde{u}\|_{L^\infty} + \|u'-\tilde{u}'\|_{L^2} + \|u''-\tilde{u}''\|_{L^2} + \left\||u|^2 - |\tilde{u}|^2\right\|_{L^2}. \tag{1.4}$$

We also introduce the following Zhidkov space (see also [19, 20])

$$X^2(\mathbb{R}) := \left\{u \in L^\infty(\mathbb{R}) : u' \in L^2(\mathbb{R}), u'' \in L^2(\mathbb{R})\right\}, \tag{1.5}$$

equipped with the norm

$$\|u\|_{X^2} = \|u\|_{L^\infty} + \|u'\|_{L^2} + \|u''\|_{L^2}.$$

It is obvious that $E = X^2(\mathbb{R}) \cap \{|u|^2 - 1 \in L^2(\mathbb{R})\}$. However, E is not a vector space.

Furthermore, we introduce the following Hardy–Zhidkov space

$$X^2_+(\mathbb{R}) := \left\{u \in L^\infty_+(\mathbb{R}) : u' \in L^2_+(\mathbb{R}), u'' \in L^2_+(\mathbb{R})\right\} \subset X^2(\mathbb{R}), \tag{1.6}$$

equipped with the same norm as $\|\cdot\|_{X^2}$. In this paper, we also study the explicit formula of the solution to (1.1) in

$$E_+ := X^2_+(\mathbb{R}) \cap \{|u|^2 - 1 \in L^2(\mathbb{R})\}, \tag{1.7}$$

where $E_+ \subset E$.

In [15], Matsuno presented the multiperiodic solutions to (1.1) and then derived the multisoliton solutions. Also, Matsuno established the linear stability of the solitary wave solutions [16] and studied the linear spectral problems of (1.1) [17].

Zhidkov introduced the space

$$X^1(\mathbb{R}) := \left\{u \in L^\infty(\mathbb{R}) : u' \in L^2(\mathbb{R})\right\}$$

in [19] and obtained the global well-posedness results [19, 20] of the Gross–Pitaevskii equation

$$i\partial_t u + \Delta u = \left(|u|^2 - 1\right) u \tag{1.8}$$

in the space

$$\left\{u \in X^1(\mathbb{R}) : |u|^2 - 1 \in L^2(\mathbb{R})\right\}.$$

Using the strategy of Kato [11], Béthuel and Saut derived the global well-posedness of (1.8) in $1 + H^1\left(\mathbb{R}^d\right)$ for $d = 2, 3$. Also, Gérard used the Strichartz estimates to obtain the global well-posedness of (1.8) in the space

$$\left\{u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d), |u|^2 - 1 \in L^2(\mathbb{R}^d)\right\}$$

for $d = 2, 3$ [7].

1.1 Main results

In this paper, we first study the Cauchy problem of (1.1) defined on E and we obtain the following global well-posedness result.

Theorem 1.1. *Given $u_0 \in E$, there exists a unique solution $u \in C(\mathbb{R}; E)$ of (1.1) with $u(0) = u_0$. Moreover, for every $T > 0$, the flow map $u_0 \in E \mapsto u \in C([-T, T]; E)$ is continuous.*

In [8], Gérard established an explicit formula for the $H^2_{\text{real}}(\mathbb{R})$ solution to the Benjamin–Ono equation on the line. According to his method, Killip, Laurens and Vişan also obtained an explicit formula for the $H^s_+(\mathbb{R})(s \geq 0)$ solution to (1.1) [13]. In this paper, we establish an explicit formula for the solution to (1.1) in E_+.

Theorem 1.2. *Given $u_0 \in E_+$, let $u \in C(\mathbb{R}; E_+)$ be the corresponding solution of (1.1). Then we give the following explicit formula: For every $t \in \mathbb{R}$ and for every $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $u(t, z)$ identifies to*

$$
\begin{aligned}
&\mathrm{e}^{it\partial_x^2}u_0(z) - \frac{t}{i\pi}I_+\left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1}\left(T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x - z}\right)\right)\right] \\
&= \mathrm{e}^{it\partial_x^2}u_0(z) \\
&- 2t\left[\left(Id + 2t\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}(G - zId)^{-1}\right)^{-1}\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x - z}\right)\right](z).
\end{aligned}
\tag{1.9}
$$

Here $u(t, z)$ and $\mathrm{e}^{it\partial_x^2}u_0(z)$ are defined by the Poisson's formulation (2.6), and the operators G, L_{u_0}, T_{u_0} are defined in the subsection 2.3.

2 Preliminaries

2.1 The space E and the Sobolev spaces

Firstly, we give the following estimate.

Lemma 2.1. *For any $u \in E$, we have*

$$
\|u\|_{L^\infty} \leq 3 + C(\||u|^2 - 1\|_{L^2}^2 + \||u|^2 - 1\|_{L^2}^{2-2s}\|u'\|_{L^2}^{2s})^{1/2} \tag{2.1}
$$

for any $\frac{1}{2} < s \leq 1$.

Proof. Let $\chi \in C_0^\infty(\mathbb{C})$ such that $0 \leqslant \chi \leqslant 1, \chi(z) = 1$ for $|z| \leqslant 2$, and $\chi(z) = 0$ for $|z| \geqslant 3$. Let us decompose

$$
u = u_1 + u_2, \quad u_1 = \chi(u)u, \quad u_2 = (1 - \chi(u))u.
$$

We have

$$\|u_1\|_{L^\infty} \leqslant 3.$$

Since $|u| \geq 2$ on the support of u_2, we have

$$\|u_2\|_{L^2} \leqslant \big\||u|^2 - 1\big\|_{L^2}.$$

On the other hand,

$$u_2' = (1 - \chi(u) - u\partial\chi(u))u' - u\bar{\partial}\chi(u)\bar{u}'.$$

This implies easily, for some $A = A(\chi)$,

$$\big\|u_2'\big\|_{L^2} \leqslant A\|u'\|_{L^2}.$$

Hence

$$\begin{aligned}\|u\|_{L^\infty} &\leq \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} \\ &\leq \|u_1\|_{L^\infty} + \|u_2\|_{H^s} \\ &\leq 3 + C(\||u|^2 - 1\|_{L^2}^2 + \||u|^2 - 1\|_{L^2}^{2-2s}\|u'\|_{L^2}^{2s})^{1/2}\end{aligned}$$

for any $\frac{1}{2} < s \leq 1$. □

Then we introduce the following lemma which will be useful in the sequel.

Lemma 2.2. *We have $E + H^2(\mathbb{R}) \subset E$, and for every $v \in E, w \in H^2(\mathbb{R})$,*

$$\big\||v+w|^2 - 1\big\|_{L^2} \leq \big\||v|^2 - 1\big\|_{L^2} + 2\|v\|_{L^\infty}\|w\|_{L^2} + \|w\|_{L^4}^2. \tag{2.2}$$

Moreover, for every $v \in E, \tilde{v} \in E$ and $w \in H^2, \tilde{w} \in H^2$, we have

$$\begin{aligned}d_E(v+w, \tilde{v}+\tilde{w}) \leq\ & C(1 + \|w\|_{L^2} + \|\tilde{w}\|_{L^2})d_E(v, \tilde{v}) \\ & + C(1 + \|v\|_{L^\infty} + \|\tilde{v}\|_{L^\infty} + \|w\|_{H^1} + \|\tilde{w}\|_{H^1})\|w - \tilde{w}\|_{H^2}.\end{aligned} \tag{2.3}$$

Proof. Given $v \in E, w \in H^2$, we expand

$$|v+w|^2 - 1 = |v|^2 - 1 + 2\,\mathrm{Re}(\bar{v}w) + |w|^2,$$

we get the corresponding estimates (2.2) since we have $v \in L^\infty(\mathbb{R})$ and $w \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$. Then we can easily deduce that $E + H^2(\mathbb{R}) \subset E$.

Also, the estimate of $\big\||v+w|^2 - |\tilde{v}+\tilde{w}|^2\big\|_{L^2}$ can be established along the same line as (2.2), so we can easily deduce (2.3). □

2.2 Action of the linear Schrödinger group on the space E

We start with the following lemma.

Lemma 2.3. *For any $f \in E$, we have $\mathrm{e}^{it\partial_x^2} f - f \in H^2(\mathbb{R})$. Moreover, we have*

$$\|\mathrm{e}^{it\partial_x^2} f - f\|_{L^2} \leqslant C|t|^{1/2}\|f'\|_{L^2} \tag{2.4}$$

and

$$\|\mathrm{e}^{it\partial_x^2} f - f\|_{H^2} \to 0 \text{ as } t \to 0. \tag{2.5}$$

Proof. Let $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ near the origin, we can write

$$\mathrm{e}^{-\mathrm{i}t|\xi|^2} - 1 = g(t,\xi)\xi$$

with

$$g(t,\xi) = -\mathrm{i}t\chi\left(t\xi^2\right)\xi\int_0^1 \mathrm{e}^{-\mathrm{i}st\xi^2}\,\mathrm{d}s + \frac{1-\chi\left(t\xi^2\right)}{\xi}\left(\mathrm{e}^{-\mathrm{i}t\xi^2} - 1\right) = \mathcal{O}\left(|t|^{1/2}\right).$$

Consequently,

$$\mathrm{e}^{it\partial_x^2} f - f = -ig(t,D)f' \in L^2(\mathbb{R})$$

and this yields (2.4).

Also, we can easily observe that

$$\left\|\partial_x^2(\mathrm{e}^{it\partial_x^2} f - f)\right\|_{L^2} \leqslant C\left\|f''\right\|_{L^2} \quad \text{and} \quad \left\|\partial_x^2(\mathrm{e}^{it\partial_x^2} f - f)\right\|_{L^2} \underset{t\to 0}{\longrightarrow} 0,$$

combined with (2.4), we infer (2.5). □

Combining the previous lemmas, it is now possible to show that the space E is kept invariant by the Schrödinger group.

Lemma 2.4. *For every $t \in \mathbb{R}$, $\mathrm{e}^{it\partial_x^2}(E) \subset E$ and for every $u_0 \in E$, the map*

$$t \in \mathbb{R} \mapsto \mathrm{e}^{it\partial_x^2} u_0 \in E$$

is continuous. Moreover, for every R, for every $T > 0$, there exists $C > 0$ such that, for every $u_0, \tilde{u}_0 \in E$ such that $\|u_0\|_{X^2} \leqslant R$ and $\|\tilde{u}_0\|_{X^2} \leqslant R$,

$$\sup_{|t|\leqslant T} d_E\left(\mathrm{e}^{it\partial_x^2} u_0, \mathrm{e}^{it\partial_x^2}\tilde{u}_0\right) \leqslant C d_E\left(u_0, \tilde{u}_0\right).$$

Proof. We write

$$e^{it\partial_x^2}u_0 = u_0 + \left(e^{it\partial_x^2}u_0 - u_0\right).$$

By using Lemma 2.2 and Lemma 2.3, we can easily see that $e^{it\partial_x^2}(E) \subset E$, while the continuity property

$$e^{it\partial_x^2}u_0 \underset{t\to 0}{\longrightarrow} u_0 \quad \text{in } E.$$

is a consequence of (2.3) and (2.5). The last argument in Lemma 2.4 also comes from (2.3) and (2.5). □

Remark 2.5. *The argument in Lemma 2.4 is also valid for $u_0 \in X^2(\mathbb{R})$.*

2.3 The Lax pair

We denote by $L^p_+(\mathbb{R})(1 \le p \le \infty)$ the Hardy space corresponding to $L^p(\mathbb{R})$ functions having a Fourier transform supported in the domain $\xi \ge 0$. Equivalently, the space $L^p_+(\mathbb{R})$ comprised of $f \in L^p(\mathbb{R})$ whose Poisson integral

$$f(z) = \int \frac{\operatorname{Im} z}{\pi|x-z|^2} f(x)dx \tag{2.6}$$

is holomorphic in the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. With the Poisson's formulation (2.6), the Holder's inequality implies

$$|f(z)| \lesssim (\operatorname{Im} z)^{-\frac{1}{p}} \|f\|_{L^p}.$$

The Riesz-Szegő projector Π is the orthogonal projector from $L^2(\mathbb{R})$ onto $L^2_+(\mathbb{R})$. It is given by

$$\forall f \in L^2(\mathbb{R}), \quad \forall z \in \mathbb{C}_+, \quad \Pi f(z) = \frac{1}{2\pi}\int_0^\infty e^{iz\xi}\hat{f}(\xi)d\xi = \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{f(y)}{y-z}dy. \tag{2.7}$$

The Toeplitz operator T_b on $L^2(\mathbb{R})$ associated to a function $b \in L^\infty(\mathbb{R})$ is defined by

$$T_b f := \Pi(bf), \quad f \in L^2(\mathbb{R}).$$

For $u \in E$, the operator L_u is defined by

$$\forall f \in \operatorname{Dom}(L_u) = H^1(\mathbb{R}), \quad L_u f := Df + uT_{\bar{u}}f \text{ with } D := \frac{1}{i}\frac{d}{dx}.$$

We also consider, for $u \in E$, the bounded operator defined by

$$B_u = -uT_{\partial_x \bar{u}} + \partial_x u T_{\bar{u}} + i\left(uT_{\bar{u}}\right)^2.$$

Also, we recall the definition of G in [8],

$$\forall f \in \mathrm{Dom}\,(G) := \left\{ f \in L^2_+(\mathbb{R}) : \hat{f} \in H^1(0,\infty) \right\}, \quad \widehat{Gf}(\xi) := i\frac{d}{d\xi}[\hat{f}(\xi)]\mathbf{1}_{\xi>0}.$$

Here G is the adjoint of the operator of multiplication by x on $L^2_+(\mathbb{R})$, and $(\mathrm{Dom}\,(G), -iG)$ is maximally dissipative. Notice that $-iG$ is the infinitesimal generator of the adjoint semi–group of contractions $(S(\eta)^*)_{\eta\geq 0}$ with

$$\forall \eta \geq 0, \quad S(\eta)^* = \mathrm{e}^{-i\eta G}.$$

We also notice that

$$\forall f \in \mathrm{Dom}\,(G), \left|\hat{f}\left(0^+\right)\right|^2 = -4\pi\,\mathrm{Im}\,\langle Gf \mid f\rangle \leq 4\pi \|Gf\|_{L^2}\|f\|_{L^2}.$$

Therefore, we can define

$$\forall f \in \mathrm{Dom}\,(G), \quad I_+(f) := \hat{f}\left(0^+\right).$$

In fact, as observed in [12, Lemma 3.4], the resolvent of G is given by

$$\forall z \in \mathbb{C}_+, \quad \forall f \in L^2_+(\mathbb{R}), \quad (G - z\,\mathrm{Id})^{-1} f(x) = \frac{f(x) - f(z)}{x - z}, \tag{2.8}$$

and we have

$$\forall z \in \mathbb{C}_+, \quad \forall f \in L^2_+(\mathbb{R}), \quad f(z) = \frac{1}{2i\pi} I_+\left((G - z\,\mathrm{Id})^{-1} f\right). \tag{2.9}$$

3 Well-posedness and conservation laws

3.1 Local well-posedness

In this subsection, we consider the local well-posedness of (1.1) with initial data in E, where Kato's classical iterative scheme for quasilinear evolution equations can be utilized.

We first write (1.1) as

$$\partial_t u = i\partial_x^2 u - 4u\Pi(\Re(\bar{u}\partial_x u)). \tag{3.1}$$

In view of (3.1), we consider the following iteration scheme

$$\partial_t u^{k+1} = i\partial_x^2 u^{k+1} - 4u^k\Pi(\Re(\bar{u}^k\partial_x u^{k+1})) \tag{3.2}$$

with initial datum $u^{k+1}(0,x) = u_0(x) \in E$. A standard energy method yields the following result.

Lemma 3.1. *Let $w \in C([-T,T],E))$ with some $T>0$ and $w_0 \in E$. Then there exists a unique solution $w \in C([-T,T];E)$ such that*

$$\partial_t w = i\partial_x^2 w - 4u\Pi(\Re(\bar{u}\partial_x w)), \quad w(0,x) = w_0(x).$$

Furthermore, if we write

$$w = \mathrm{e}^{it\partial_x^2} w_0 + v,$$

then we have

$$\sup_{|t|\le T} \|v(t)\|_{H^2} \le C\mathrm{e}^{C\int_{-T}^{T}\|u\|_{X^2}^2 dt}\left(\|w_0'\|_{H^1} + \int_{-T}^{T}\|u\|_{X_+^2}^2\|w_0'\|_{H^1}dt\right), \tag{3.3}$$

Proof. By Duhamel's formula, we have

$$v(t) = \int_0^t \mathrm{e}^{i(t-s)\partial_x^2}(-4u\Pi(\Re(\bar{u}\partial_x w)))ds,$$

so we can deduce that

$$\|v(t)\|_{L^2} \le C\int_0^t \|u\|_{L^\infty}^2(\|\partial_x v\|_{L^2} + \|w_0'\|_{L^2})ds \tag{3.4}$$

Also, a standard energy method yields (see [6, Lemma 5.4])

$$\|\partial_x w(t)\|_{H^1} \le \|w_0'\|_{H^1} + C\int_0^t \|u\|_{X^2}^2\|\partial_x w\|_{H^1}\,ds,$$

which implies

$$\|v(t)\|_{H^2} \le C\|w_0'\|_{H^1} + C\int_0^t \left(\|u\|_{X^2}^2\|v\|_{H^2} + \|u\|_{X^2}^2\left\|w_0'\right\|_{H^1}\right)ds, \tag{3.5}$$

Combining (3.4) and (3.5), by Grönwall's inequality, we can deduce that

$$\sup_{|t|\le T} \|v(t)\|_{H^2} \le C\mathrm{e}^{C\int_{-T}^{T}\|u\|_{X^2}^2 dt}\left(\|w_0'\|_{H^1} + \int_{-T}^{T}\|u\|_{X^2}^2\|w_0'\|_{H^1}dt\right),$$

which yields (3.3). □

Now we can introduce the following local well-posedness result.

Proposition 3.2. *For any $R>0$, there is some $T=T(R)>0$ such that, for every $u_0 \in E$ with $\|u_0\|_{X^2}^2 \le R$, there exists a unique solution $u \in C([-T,T];E)$ of (1.1) with $u(0)=u_0$. Moreover, the flow map $u_0 \in E \mapsto u \in C([-T,T];E)$ is continuous.*

Proof. We consider the iteration scheme

$$\partial_t u^{k+1} = i\partial_x^2 u^{k+1} - 4u^k \Pi(\Re(\bar{u}^k \partial_x u^{k+1})) \tag{3.6}$$

with initial datum $u^{k+1}(0,x) = u_0(x) \in E$. Lemma 3.1 allows us to construct by induction a sequence u^k with $u^0(t,x) = \mathrm{e}^{\mathrm{i}t\partial_x^2} u_0(x)$.

We write

$$u^k = \mathrm{e}^{\mathrm{i}t\partial_x^2} u_0 + v^k.$$

From Lemma 3.1, we have

$$\sup_{|t|\leqslant T} \|v^{k+1}(t)\|_{H^2} \leq C\mathrm{e}^{C\int_{-T}^{T} \|u^k\|_{X^2}^2 dt} \left(\left\|u_0'\right\|_{H^1} + \int_{-T}^{T} \|u^k\|_{X^2}^2 \|u_0'\|_{H^1} dt \right). \tag{3.7}$$

Given $u_0 \in E$ such that $\|u_0\|_{X^2} \leqslant R$, by Lemma 2.3, for $T = T(R)$ with respect to R,

$$\left\| \mathrm{e}^{it\partial_x^2} u_0 \right\|_{X^2} \leqslant 2R.$$

Assume that $\sup_{|t|\leqslant T} \|u^k\|_{X^2} \leq R_1$, and the size of R_1 will be determined later. By (3.7), we have

$$\sup_{|t|\leqslant T} \|v^{k+1}\|_{H^2} \leq C_1 e^{2C_1 T R_1^2} (R + 2TR_1^2 R) := R_2.$$

By the triangle inequality, we know that

$$\sup_{|t|\leqslant T} \left\|u^{k+1}\right\|_{X^2} \leq 2R + C_2 R_2.$$

Let $R_1 = (3 + C_1 C_2)R$, then we can choose $T = T(R)$ such that

$$2R + C_2 R_2 \leq R_1.$$

Then by an elementary induction argument, we find that $\sup_{|t|\leqslant T} \|u^k\|_{X^2} \leq R_1$ for all k and $\sup_{|t|\leqslant T} \|v^k\|_{H^2} \leq R_2$ for all k.

Next, we show that we have a contraction property of the sequence v_k in $L^2(\mathbb{R})$. Observe that

$$\begin{aligned} &\partial_t \left(u^{k+1} - u^k\right) = i\partial_x^2 \left(u^{k+1} - u^k\right) - 4u^k \Pi \left(\Re(\bar{u}^k \partial_x (u^{k+1} - u^k))\right) \\ &+4u^{k-1} \Pi \left(\Re(\bar{u}^{k-1} \partial_x u^k)\right) - 4u^k \Pi \left(\Re(\bar{u}^k \partial_x u^k)\right) \end{aligned}.$$

A standard energy method yields

$$\sup_{|t|\leq T}\left\|u^{k+1}(t)-u^{k}(t)\right\|_{L^2}\leq KT\sup_{|t|\leq T}\left\|u^{k}(t)-u^{k-1}(t)\right\|_{L^2}.$$

So we take T small enough to ensure that $KT<1$, and we then we can deduce that $\sup_{|t|\leq T(R)}\left\|u^{k+1}(t)-u^{k}(t)\right\|_{L^2}=\sup_{|t|\leq T(R)}\left\|v^{k+1}(t)-v^{k}(t)\right\|_{L^2}$ is geometrically convergent.

Finally, the sequence $v_k(t)$ is uniformly weakly convergent in $C\left([-T,T];H^2(\mathbb{R})\right)$ and strongly convergent in $C\left([-T,T];L^2(\mathbb{R})\right)$. We note the limit of $v_k(t)$ as $v(t)$. By adapting an argument due to Bona-Smith [5](or Tao's frequency envelope method [18]), we can deduce that the limit $v(t)$ actually belongs to $C\left([-T,T];H^2(\mathbb{R})\right)$. Thus $u(t)=\mathrm{e}^{it\partial_x^2}u_0+v(t)\in C([-T,T];E)$ solves (1.1).

Uniqueness and continuity of the flow map follow along the same lines as the contraction property in $L^2(\mathbb{R})$. The proof of Proposition 3.2 is complete. □

Remark 3.3. *In fact, by adapting the same argument used in the proof of Proposition 3.2, we can also deduce the local well-posedness of (1.1) in* $\{u\in E:u^{(3)}\in L^2(\mathbb{R})\}$.

3.2 Conservation laws

For $u\in E$, we recall the operators L_u and B_u acting on $L^2(\mathbb{R})$,

$$L_u=D+uT_{\bar{u}}\quad\text{ and }\quad B_u=-uT_{\partial_x\bar{u}}+\partial_x uT_{\bar{u}}+i\left(uT_{\bar{u}}\right)^2.$$

We infer the following Lax equation for (1.1).

Lemma 3.4. *(Lax Equation) If $u\in C\left([0,T];E\right)$ solves (1.1), then it holds*

$$\frac{d}{dt}L_u=\left[B_u,L_u\right]. \tag{3.8}$$

Proof. We refer to [9, Lemma 2.3] for the proof. □

As a consequence of the Lax equation, we obtain a hierarchy of conservation laws of (1.1), and this implies the a-priori bound on the solution.

Lemma 3.5. *Let $u_0\in E$ and $u\in C\left(I,E\right)$ be the correspoding solution of (1.1) with $u(0,x)=u_0(x)$, where $I\subset\mathbb{R}$ denotes the maximal time interval of existence of the solution. Then the quantities*

$$I_1(u):=\left\|Du+u\Pi\left(|u|^2-1\right)\right\|_{L^2}$$

and

$$I_2(u) := \left\| D^2 u + u T_{\bar{u}} D\,(u) + D(u\Pi((|u|^2-1))) + Du + u\Pi(|u|^2\Pi(|u|^2-1)) + u\Pi(|u|^2-1) \right\|_{L^2}$$

are conserved.

Also, we have the following a-priori bound on the solution u,

$$\sup_{t\in I} \|u\|_{X^2} \leq C(u_0). \tag{3.9}$$

Proof. We observe that (1.1) can be written as

$$\partial_t u = -iD^2 u - 2iu\Pi D(|u|^2) \tag{3.10}$$

Let

$$\chi_\varepsilon(x) := \frac{1}{1-i\varepsilon x}, \tag{3.11}$$

then we have

$$\partial_t(u\chi_\varepsilon) = \chi_\varepsilon \partial_t u,$$

$$-iD^2(u\chi_\varepsilon) = i\partial_x^2(u\chi_\varepsilon) = i\chi_\varepsilon \partial_x^2 u + 2i\partial_x u \partial_x \chi_\varepsilon + iu\partial_x^2 \chi_\varepsilon$$

and

$$-2iu\Pi D(|u|^2\chi_\varepsilon) = -2iu\Pi(D(|u|^2)\chi_\varepsilon) - 2iu\Pi(|u|^2 D(\chi_\varepsilon)).$$

So we have

$$\begin{aligned} r_\varepsilon &:= \partial_t(u\chi_\varepsilon) - B_u(u\chi_\varepsilon) + i(L_u)^2(u\chi_\varepsilon) \\ &= \partial_t(u\chi_\varepsilon) + iD^2(u\chi_\varepsilon) + 2itu\Pi D(|u|^2\chi_\varepsilon) \\ &= -2i\partial_x u \partial_x \chi_\varepsilon - iu\partial_x^2\chi_\varepsilon + 2iu\Pi(|u|^2 D(\chi_\varepsilon)) + 2iu\Pi(D(|u|^2)\chi_\varepsilon) - 2i\chi_\varepsilon u\Pi(D(|u|^2)). \end{aligned}$$

We can easily observe that

$$\|r_\varepsilon\|_{H^1} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Then we have

$$\begin{aligned} \frac{d}{dt}\langle u\chi_\varepsilon, u\chi_\varepsilon\rangle &= 2\Re\langle \partial_t u\chi_\varepsilon, u\chi_\varepsilon\rangle \\ &= 2\Re\left\langle (B_u - iL_u^2)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + 2\Re\langle r_\varepsilon, u\chi_\varepsilon\rangle \\ &= 2\Re\langle r_\varepsilon, u\chi_\varepsilon\rangle. \end{aligned}$$

Also, by the Lax pair equation (3.8) and (3.10), we have

$$\begin{aligned}
&\frac{d}{dt}\left\langle L_u(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&= \left\langle L_u(u\chi_\varepsilon), \partial_t(u\chi_\varepsilon)\right\rangle + \left\langle L_u(\partial_t(u\chi_\varepsilon)), u\chi_\varepsilon\right\rangle + \left\langle \partial_t(L_u)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&= \left\langle L_u(u\chi_\varepsilon), (B_u - iL_u^2)(u\chi_\varepsilon)\right\rangle + \left\langle L_u(B_u - iL_u^2)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&+ \left\langle (B_uL_u - L_uB_u)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + 2\Re\left\langle L_u\left(u\chi_\varepsilon\right), r_\varepsilon\right\rangle \\
&= 2\Re\left\langle L_u\left(u\chi_\varepsilon\right), r_\varepsilon\right\rangle \\
&= 2\Re\left\langle D\left(u\chi_\varepsilon\right) + u\Pi(|u|^2\chi_\varepsilon), r_\varepsilon\right\rangle \\
&= 2\Re\left\langle D\left(u\chi_\varepsilon\right) + u\Pi\left((|u|^2-1)\chi_\varepsilon\right), r_\varepsilon\right\rangle + 2\Re\left\langle u\chi_\varepsilon, r_\varepsilon\right\rangle,
\end{aligned}$$

where

$$2\Re\left\langle D\left(u\chi_\varepsilon\right) + u\Pi\left((|u|^2-1)\chi_\varepsilon\right), r_\varepsilon\right\rangle \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Similarly, we have

$$\begin{aligned}
&\frac{d}{dt}\left\langle L_u^2(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&= \left\langle L_u^2(u\chi_\varepsilon), \partial_t(u\chi_\varepsilon)\right\rangle + \left\langle L_u^2(\partial_t(u\chi_\varepsilon)), u\chi_\varepsilon\right\rangle + \left\langle \partial_t(L_u^2)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&= \left\langle L_u^2(u\chi_\varepsilon), (B_u - iL_u^2)(u\chi_\varepsilon)\right\rangle + \left\langle L_u^2(B_u - iL_u^2)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&+ \left\langle (B_uL_u^2 - L_u^2B_u)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + 2\Re\left\langle L_u^2\left(u\chi_\varepsilon\right), r_\varepsilon\right\rangle \\
&= 2\Re\left\langle L_u\left(u\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle \\
&= 2\Re\left\langle D\left(u\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle + 2\Re\left\langle u\Pi\left(|u|^2\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle \\
&= 2\Re\left\langle D\left(u\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle + 2\Re\left\langle u\Pi\left((|u|^2-1)\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle + 2\Re\left\langle L_u(u\chi_\varepsilon), r_\varepsilon\right\rangle,
\end{aligned}$$

where

$$2\Re\left\langle D\left(u\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle + 2\Re\left\langle u\Pi\left((|u|^2-1)\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Thus we infer that

$$\frac{d}{dt}\left(\left\langle L_u^2(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle - 2\left\langle L_u(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + \left\langle u\chi_\varepsilon, u\chi_\varepsilon\right\rangle\right) \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.12}$$

In fact,

$$\begin{aligned}
&\left\langle L_u^2(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle - 2\left\langle L_u(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + \left\langle u\chi_\varepsilon, u\chi_\varepsilon\right\rangle \\
&= \left\langle L_u(u\chi_\varepsilon), L_u(u\chi_\varepsilon)\right\rangle - 2\left\langle L_u(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + \left\langle u\chi_\varepsilon, u\chi_\varepsilon\right\rangle \\
&= \left\langle (D+uT_{\bar u})(u\chi_\varepsilon), (D+uT_{\bar u})(u\chi_\varepsilon)\right\rangle - 2\left\langle (D+uT_{\bar u})(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + \left\langle u\chi_\varepsilon, u\chi_\varepsilon\right\rangle \\
&= \left\langle D(u\chi_\varepsilon) + u\Pi\left((|u|^2-1)\chi_\varepsilon\right) + u\chi_\varepsilon, D(u\chi_\varepsilon) + u\Pi\left((|u|^2-1)\chi_\varepsilon\right) + u\chi_\varepsilon\right\rangle \\
&- 2\left\langle D(u\chi_\varepsilon) + u\Pi\left((|u|^2-1)\chi_\varepsilon\right) + u\chi_\varepsilon, u\chi_\varepsilon\right\rangle + \left\langle u\chi_\varepsilon, u\chi_\varepsilon\right\rangle \\
&= \|D(u\chi_\varepsilon) + u\Pi\left((|u|^2-1)\chi_\varepsilon\right)\|_{L^2}^2.
\end{aligned}$$

Then by (3.12), we can deduce that

$$\|D(u(t)\chi_\varepsilon)+u(t)\Pi\left((|u(t)|^2-1)\chi_\varepsilon\right)\|_{L^2}^2-\|D(u_0\chi_\varepsilon)+u_0\Pi\left((|u_0|^2-1)\chi_\varepsilon\right)\|_{L^2}^2\underset{\varepsilon\to 0}{\longrightarrow}0,$$

and thus

$$I_1(u):=\|Du(t)+u(t)\Pi\left(|u(t)|^2-1\right)\|_{L^2}=\|Du_0+u_0\Pi\left(|u_0|^2-1\right)\|_{L^2}=I_1(u_0),$$

which yields the first conservation law in Lemma 3.5.

Following the modified gauge transform

$$v:=u(x)\mathrm{e}^{\frac{i}{2}\int_0^x(|u(y)|^2-1)dy},$$

we know that

$$I_1(u)=\|Du+u\Pi\left(|u|^2-1\right)\|_{L^2}=\|\partial_x v-\frac{1}{2}vH\left(|v|^2-1\right)\|_{L^2}.$$

From [9, Appendix C] we infer

$$\begin{aligned}
&\Re\left\langle\partial_x v,\mathrm{H}\left(|v|^2-1\right)v\right\rangle\\
=&\left\langle\Re\left[\bar{v}\partial_x v\right],\mathrm{H}\left(|v|^2-1\right)\right\rangle\\
=&\frac{1}{2}\left\langle\partial_x\left(|v|^2-1\right),\mathrm{H}\left(|v|^2-1\right)\right\rangle\\
=&-\frac{1}{2}\left\langle|v|^2-1,H\partial_x\left(|v|^2-1\right)\right\rangle\\
=&-\frac{1}{2}\left\langle|v|^2-1,|D|\left(|v|^2-1\right)\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle\mathrm{H}\left(|v|^2-1\right)v,\mathrm{H}\left(|v|^2-1\right)v\right\rangle\\
=&\left\langle|v|^2,\left(\mathrm{H}\left(|v|^2-1\right)\right)^2\right\rangle\\
=&\left\langle|v|^2-1,\left(\mathrm{H}\left(|v|^2-1\right)\right)^2\right\rangle+\left\langle1,\left(\mathrm{H}\left(|v|^2-1\right)\right)^2\right\rangle\\
=&\left\langle|v|^2-1,\left(\mathrm{H}\left(|v|^2-1\right)\right)^2\right\rangle+\left\langle1,\left(\mathrm{H}\left(|v|^2-1\right)\right)^2\right\rangle\\
=&\frac{1}{3}\int_{\mathbb{R}}(|v|^2-1)^3dx+\||v|^2-1\|_{L^2}^2,
\end{aligned}$$

so we have

$$\begin{aligned}
&\|\partial_x v - \frac{1}{2} v H\left(|v|^2-1\right)\|_{L^2}^2 \\
&= \|\partial_x v\|_{L^2}^2 + \frac{1}{2}\||v|^2-1\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{1}{12}\int_{\mathbb{R}}(|v|^2-1)^3 dx + \frac{1}{4}\||v|^2-1\|_{L^2}^2 \\
&\geq \frac{1}{12}\int_{|v|\geq 1}(|v|^2-1)^3 dx + \frac{1}{12}\int_{|v|<1}(|v|^2-1)^3 dx + \frac{1}{4}\||v|^2-1\|_{L^2}^2 \\
&\geq \frac{1}{4}\int_{|v|\geq 1}(|v|^2-1)^2 dx + \frac{1}{6}\int_{|v|<1}(|v|^2-1)^2 dx \\
&\geq \frac{1}{6}\||v|^2-1\|_{L^2}^2,
\end{aligned}$$

Thus

$$I_1(u_0)^2 = I_1(u)^2 = \|\partial_x v - \frac{1}{2} v H\left(|v|^2-1\right)\|_{L^2}^2 \geq \frac{1}{6}\||v|^2-1\|_{L^2}^2 = \frac{1}{6}\||u|^2-1\|_{L^2}^2,$$

which infers that $\sup_{t\in I}\left\||u(t)|^2-1\right\|_{L^2} \leq C\left(u_0\right)$.

According to Lemma 2.1, for every $\frac{1}{2} < s < 1$, we have

$$\begin{aligned}
&\|Du + u\Pi\left(|u|^2-1\right)\|_{L^2} \\
&\geq \|\partial_x u\|_{L^2}^2 - 2\|\partial_x u\|\|u\|_{L^\infty}\|\Pi\left(|u|^2-1\right)\|_{L^2} \\
&\geq \|\partial_x u\|_{L^2}^2 - 6\|\partial_x u\|_{L^2}\|\Pi\left(|u|^2-1\right)\|_{L^2} \\
&- C\|\partial_x u\|_{L^2}\left(\left\||u|^2-1\right\|_{L^2}^2 + \left\||u|^2-1\right\|_{L^2}^{2-2s}\|\partial_x u\|_{L^2}^{2s}\right)^{1/2}\|\Pi\left(|u|^2-1\right)\|_{L^2}.
\end{aligned}$$

From the a-priori bound on $\left\||u|^2-1\right\|_{L^2}$, we readily infer that $\sup_{t\in I}\|\partial_x u(t)\|_{L^2} \leq C\left(u_0\right)$. Again by Lemma 2.1, we can deduce that $\sup_{t\in I}\|u(t)\|_{L^\infty} \leq C\left(u_0\right)$.

Now we assume moreover that $u_0^{(3)} \in L^2(\mathbb{R})$. By Remark 3.3, we know that $\partial_x^3 u(t) \in L^2(\mathbb{R})$, and then we can easily observe that

$$\|r_\varepsilon\|_{H^2} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

in this case.

Now we have

$$
\begin{aligned}
&\frac{d}{dt}\left\langle L_u^4(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&= \left\langle L_u^4(u\chi_\varepsilon), (\partial_t u\chi_\varepsilon)\right\rangle + \left\langle L_u^4(\partial_t(u\chi_\varepsilon)), u\chi_\varepsilon\right\rangle + \left\langle \partial_t(L_u^4)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&= \left\langle L_u^4(u\chi_\varepsilon), (B_u - iL_u^2)(u\chi_\varepsilon)\right\rangle + \left\langle L_u^4(B_u - iL_u^2)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle \\
&+ \left\langle (B_u L_u^4 - L_u^4 B_u)(u\chi_\varepsilon), u\chi_\varepsilon\right\rangle + 2\Re\left\langle L_u^4\left(u\chi_\varepsilon\right), r_\varepsilon\right\rangle \\
&= 2\Re\left\langle L_u^2\left(u\chi_\varepsilon\right), L_u^2(r_\varepsilon)\right\rangle \\
&= 2\Re\left\langle D^2\left(u\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle + 2\Re\left\langle uT_{\bar{u}}D\left(u\chi_\varepsilon\right), L_u^2(r_\varepsilon)\right\rangle + 2\Re\left\langle D\left(uT_{\bar{u}}\left(u\chi_\varepsilon\right)\right), L_u^2(r_\varepsilon)\right\rangle \\
&+ 2\Re\left\langle u\Pi\left(|u|^2\Pi\left(|u|^2-1\right)\chi_\varepsilon\right)\right), L_u^2(r_\varepsilon)\right\rangle + 2\Re\left\langle u\Pi((|u|^2-1)\chi_\varepsilon), L_u^2(r_\varepsilon)\right\rangle \\
&+ 2\Re\left\langle L_u\left(u\chi_\varepsilon\right), L_u\left(r_\varepsilon\right)\right\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
&2\Re\left\langle D^2\left(u\chi_\varepsilon\right), L_u(r_\varepsilon)\right\rangle + 2\Re\left\langle uT_{\bar{u}}D\left(\chi_\varepsilon\right), L_u^2(r_\varepsilon)\right\rangle + 2\Re\left\langle D\left(uT_{\bar{u}}\left(u\chi_\varepsilon\right)\right), L_u^2(r_\varepsilon)\right\rangle \\
&+ 2\Re\left\langle u\Pi\left(|u|^2\Pi\left(|u|^2-1\right)\chi_\varepsilon\right)\right), L_u^2(r_\varepsilon)\right\rangle + 2\Re\left\langle u\Pi((|u|^2-1)\chi_\varepsilon), L_u^2(r_\varepsilon)\right\rangle \underset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}
$$

Then we infer that

$$
\frac{d}{dt}\left(\left\langle L_u^4\left(u\chi_\varepsilon\right), u\chi_\varepsilon\right\rangle - 2\left\langle L_u^2\left(u\chi_\varepsilon\right), u\chi_\varepsilon\right\rangle + \left\langle u\chi_\varepsilon, u\chi_\varepsilon\right\rangle\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.
$$

We also observe that

$$
\begin{aligned}
&\left\langle L_u^4\left(u\chi_\varepsilon\right), u\chi_\varepsilon\right\rangle - 2\left\langle L_u^2\left(u\chi_\varepsilon\right), u\chi_\varepsilon\right\rangle + \left\langle u\chi_\varepsilon, u\chi_\varepsilon\right\rangle \\
&= \|D^2(u\chi_\varepsilon) + uT_{\bar{u}}D\left(u\chi_\varepsilon\right) + D(u\Pi((|u|^2-1)\chi_\varepsilon)) + D(u\chi_\varepsilon) + u\Pi(|u|^2\Pi((|u|^2-1)\chi_\varepsilon)) \\
&+ u\Pi((|u|^2-1)\chi_\varepsilon)\|_{L^2}^2 \\
&:= I_{2,\varepsilon}(u)^2.
\end{aligned}
$$

Then we can deduce that

$$
I_{2,\varepsilon}(u(t))^2 - I_{2,\varepsilon}(u_0)^2 \underset{\varepsilon\to 0}{\longrightarrow} 0,
$$

which implies that

$$
I_2(u(t)) = I_2(u_0). \tag{3.13}
$$

By the density and the continuity of the flow map, we can deduce (3.13) for $u_0 \in E$, which yields the second conservation law in Lemma 3.5.

Then from the a-priori bounds on $\|u(t)\|_{L^\infty}$, $\|\partial_x u(t)\|_{L^2}$ and $\||u|^2-1\|_{L^2}$, we can infer that $\sup_{t\in I}\left\|\partial_x^2 u(t)\right\|_{L^2} \le C\left(u_0\right)$, thus we have (3.9).

The proof is complete. □

3.3 Global well-posedness

As an application of Lemma 3.5, we deduce the following global well-posedness result.

Theorem 3.6. *Given $u_0 \in E$, there exists a unique solution $u \in C(\mathbb{R}; E)$ of (1.1) with $u(0) = u_0$. Moreover, for every $T > 0$, the flow map $u_0 \in E \mapsto u \in C([-T, T]; E)$ is continuous.*

Proof. From Lemma 3.5, we know that $\|u(t)\|_{X^2}$ is uniformly bounded in t, then by the classical bootstrap argument, we infer that the solution exists globally in time, which yields the global well-posedness of (1.1) in E. □

4 Explicit formula

In [13], Killip, Laurens and Vişan established an explicit formula of the $H^s_+(\mathbb{R})(s \geq 0)$ solution to (1.1). In this section, we establish an explicit formula of the solution to (1.1) in E_+.

Theorem 4.1. *Given $u_0 \in E_+$, let $u \in C(\mathbb{R}; E_+)$ be the corresponding solution of (1.1). Then we give the following explicit formula: For every $t \in \mathbb{R}$ and for every $z \in \mathbb{C}_+$, $u(t, z)$ identifies to*

$$
\begin{aligned}
&\mathrm{e}^{it\partial_x^2}u_0(z) - \frac{t}{i\pi}I_+\left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1}\left(T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x - z}\right)\right)\right] \\
&= \mathrm{e}^{it\partial_x^2}u_0(z) \\
&- 2t\left[\left(Id + 2t\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}(G - zId)^{-1}\right)^{-1}\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x - z}\right)\right](z).
\end{aligned}
\tag{4.1}
$$

Here $u(t, z)$ and $\mathrm{e}^{it\partial_x^2}u_0(z)$ are defined by the Poisson's formulation (2.6).

Proof. We satrt with the derivation method as in [8, Section 3]. For every $f \in L^2_+(\mathbb{R})$, from (2.7) we know that

$$
\forall z \in \mathbb{C}_+, \quad f(z) = \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{iz\xi}\hat{f}(\xi)d\xi.
$$

While, in view of the Plancherel theorem, we have, in $L^2(0, +\infty)$,

$$
\hat{f}(\xi) = \lim_{\delta\to 0}\int_{\mathbb{R}}\mathrm{e}^{-ix\xi}\frac{f(x)}{1 + i\delta x}dx = \lim_{\delta\to 0}\langle S(\xi)^* f, \chi_\delta\rangle,
$$

where

$$\chi_\delta(x) := \frac{1}{1-i\delta x}.$$

Plugging the second formula into the first one, we infer

$$\begin{aligned} f(z) &= \lim_{\delta\to 0} \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{iz\xi} \left\langle S(\xi)^* f, \chi_\delta \right\rangle d\xi \\ &= \lim_{\delta\to 0} \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{iz\xi} \left\langle \mathrm{e}^{-i\xi G} f, \chi_\delta \right\rangle d\xi \\ &= \lim_{\delta\to 0} \frac{1}{2i\pi} \left\langle (G - z\mathrm{Id})^{-1} f, \chi_\delta \right\rangle. \end{aligned}$$

We then use the family $U(t)$ of unitary operators defined by the linear initial value problem in $\mathscr{L}\left(L^2_+(\mathbb{R})\right)$,

$$U'(t) = B_{u(t)}U(t), U(0) = \mathrm{Id}.$$

For every $z \in \mathbb{C}_+$, we have

$$\begin{aligned} (u\chi_\varepsilon)(t,z) &= \lim_{\delta\to 0} \frac{1}{2i\pi} \left\langle U(t)^*(G - z\mathrm{Id})^{-1}(u\chi_\varepsilon)(t), U(t)^*\chi_\delta \right\rangle \\ &= \lim_{\delta\to 0} \frac{1}{2i\pi} \left\langle (U(t)^*GU(t) - z\mathrm{Id})^{-1} U(t)^*(u\chi_\varepsilon)(t), U(t)^*\chi_\delta \right\rangle. \end{aligned}$$

It is not difficult to see that

$$[G, B_u] = 2L_u + i\left[G, (L_u)^2\right], \tag{4.2}$$

then we calculate

$$\begin{aligned} \frac{d}{dt}U(t)^*GU(t) &= U(t)^*\left[G, B_{u(t)}\right]U(t) \\ &= U(t)^*\left(2L_{u(t)} + i\left[G, L^2_{u(t)}\right]\right)U(t) \\ &= 2L_{u_0} + i\left[U(t)^*GU(t), L^2_{u_0}\right]. \end{aligned}$$

Integrating this ODE, we get

$$U(t)^*GU(t) = 2tL_{u_0} + \mathrm{e}^{-itL^2_{u_0}} G \mathrm{e}^{itL^2_{u_0}}.$$

Let us determine the other terms in the inner product. We have

$$\begin{aligned} \frac{d}{dt}U(t)^*(u\chi_\varepsilon) &= U(t)^*\left(\partial_t(u\chi_\varepsilon) - B_{u(t)}(u\chi_\varepsilon)\right) \\ &= -iU(t)^*L^2_{u(t)}(u\chi_\varepsilon) + U(t)^*r_\varepsilon \\ &= -iL^2_{u_0}U(t)^*(u\chi_\varepsilon) + U(t)^*r_\varepsilon, \end{aligned}$$

from which we infer

$$R_\varepsilon := U(t)^*(u\chi_\varepsilon) - \mathrm{e}^{-itL_{u_0}^2}(u_0\chi_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$$

in $L^2_+(\mathbb{R})$.

Moreover, we can easily observe that

$$B_u\chi_\delta - iL_u^2\chi_\delta \underset{\delta\to 0}{\longrightarrow} 0$$

in $L^2_+(\mathbb{R})$.

So we have

$$\begin{aligned}\frac{d}{dt}U(t)^*\chi_\delta &= -U(t)^*B_u\chi_\delta \\ &= -i\lim_{\delta\to 0} U(t)^*(L_u^2\chi_\delta) \\ &= -i\lim_{\delta\to 0} L_{u_0}^2 U(t)^*\chi_\delta\end{aligned}$$

By integrating in time, we infer

$$U(t)^*\chi_\delta - \mathrm{e}^{-itL_{u_0}^2}\chi_\delta \underset{\delta\to 0}{\longrightarrow} 0$$

in $L^2_+(\mathbb{R})$. Plugging these informations into the formula which gives $u\chi_\varepsilon$, we infer

$$\begin{aligned}&(u\chi_\varepsilon)(t,z) \\ &= \lim_{\delta\to 0}\frac{1}{2i\pi}\left\langle (U(t)^*GU(t) - z\mathrm{Id})^{-1}U(t)^*(u\chi_\varepsilon)(t), U(t)^*\chi_\delta\right\rangle \\ &= \lim_{\delta\to 0}\frac{1}{2i\pi}\left\langle \left(\mathrm{e}^{-itL_{u_0}^2}G\mathrm{e}^{itL_{u_0}^2} + 2tL_{u_0} - z\mathrm{Id}\right)^{-1}\left(\mathrm{e}^{-itL_{u_0}^2}(u_0\chi_\varepsilon) + R_\varepsilon\right), \mathrm{e}^{-itL_{u_0}^2}\chi_\delta\right\rangle \\ &= \lim_{\delta\to 0}\frac{1}{2i\pi}\left\langle (G + 2tL_{u_0} - z\mathrm{Id})^{-1}\left(u_0\chi_\varepsilon + \mathrm{e}^{itL_{u_0}^2}R_\varepsilon\right), \chi_\delta\right\rangle \\ &= \frac{1}{2i\pi}I_+\left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1}\left(u_0\chi_\varepsilon + \mathrm{e}^{itL_{u_0}^2}R_\varepsilon\right)\right]\end{aligned}$$

We can easily see that

$$(u\chi_\varepsilon)(t,z) \underset{\varepsilon\to 0}{\longrightarrow} u(t,z), \quad \forall z\in\mathbb{C}_+$$

and

$$\frac{1}{2i\pi}I_+\left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1}R_\varepsilon\right] \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad \forall z\in\mathbb{C}_+,$$

so we only need to give the limit of $\frac{1}{2i\pi} I_+ \left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1} (u_0 \chi_\varepsilon) \right]$ as ε tends to 0.

In fact, we have

$$\begin{aligned} G + 2tL_{u_0} - z &= G + 2tD + 2tT_{u_0}T_{\bar{u}_0} - zId \\ &= \mathrm{e}^{-it\partial_x^2} G \mathrm{e}^{it\partial_x^2} + 2tT_{u_0}T_{\bar{u}_0} - zId \\ &= \mathrm{e}^{-it\partial_x^2} \left(G + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} - zId \right) \mathrm{e}^{it\partial_x^2}. \end{aligned}$$

Combining the above formula and (2.9), we infer

$$\begin{aligned} &\frac{1}{2i\pi} I_+ \left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1} (u_0 \chi_\varepsilon) \right] \\ &= \frac{1}{2i\pi} I_+ \left[\left(G + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} - z\mathrm{Id} \right)^{-1} \mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon) \right] \\ &= \left[\left(Id + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1} \right)^{-1} \mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon) \right] (z). \end{aligned}$$

From the identity

$$(Id + A)^{-1} = Id - (Id + A)^{-1} A,$$

we know that

$$\begin{aligned} &\left(Id + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1} \right)^{-1} \\ &= Id - 2t \left(Id + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1} \right)^{-1} \mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1}. \end{aligned}$$

Then we infer

$$\begin{aligned} &\left(Id + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1} \right)^{-1} \mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon) \\ &= \mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon) \\ &- 2t \left(Id + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1} \right)^{-1} \mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1} \mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon) \\ &= \mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon) \\ &- 2t \left(Id + 2t\mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} (G - zId)^{-1} \right)^{-1} \mathrm{e}^{it\partial_x^2} T_{u_0} T_{\bar{u}_0} \mathrm{e}^{-it\partial_x^2} \left(\frac{\mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon)(x) - \mathrm{e}^{it\partial_x^2} (u_0 \chi_\varepsilon)(z)}{x - z} \right). \end{aligned}$$

The second equality above comes from (2.8).

Let ε tend to 0, we know that

$$u_0 \chi_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u_0 \text{ pointwisely}$$

and

$$(u_0\chi_\varepsilon)' \underset{\varepsilon\to 0}{\longrightarrow} u_0' \text{ in } H^1_+(\mathbb{R}).$$

By Lemma 2.3, we infer

$$\mathrm{e}^{it\partial_x^2}(u_0\chi_\varepsilon) = u_0\chi_\varepsilon + \left(\mathrm{e}^{it\partial_x^2}(u_0\chi_\varepsilon) - u_0\chi_\varepsilon\right) \underset{\varepsilon\to 0}{\longrightarrow} u_0 + \left(\mathrm{e}^{it\partial_x^2}u_0 - u_0\right) = \mathrm{e}^{it\partial_x^2}u_0 \text{ pointwisely.}$$

Then from the Poisson's formulation (2.6) and the dominated convergence theorem, we can deduce that

$$\mathrm{e}^{it\partial_x^2}(u_0\chi_\varepsilon)(z) \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{e}^{it\partial_x^2}u_0(z), \quad \forall z \in \mathbb{C}_+.$$

Again by the dominated convergence theorem, we can deduce that, for every $z \in \mathbb{C}_+$,

$$\frac{\mathrm{e}^{it\partial_x^2}(u_0\chi_\varepsilon)(x) - \mathrm{e}^{it\partial_x^2}(u_0\chi_\varepsilon)(z)}{x-z} \underset{\varepsilon\to 0}{\longrightarrow} \frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x-z} \text{ in } L^2_+(\mathbb{R}).$$

Thus we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{2i\pi} I_+ \left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1}(u_0\chi_\varepsilon)\right] \\
&= \mathrm{e}^{it\partial_x^2}u_0(z) \\
&- 2t\left[\left(Id + 2t\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}(G - zId)^{-1}\right)^{-1}\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x-z}\right)\right](z) \\
&= \mathrm{e}^{it\partial_x^2}u_0(z) - \frac{t}{i\pi} I_+ \left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1}\left(T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x-z}\right)\right)\right].
\end{aligned}$$

The proof is complete. □

Remark 4.2. *In fact, the formula (4.1) is also valid for the $H^s_+(\mathbb{R})(s \geq 1)$ solution to (1.1).*

Remark 4.3. *From Theorem 4.1, if we write $u(t,x) = \mathrm{e}^{it\partial_x^2}u_0(x) + v(t,x)$ as in the proof of Proposition 3.2, we can also establish an explicit formula for $v(t,x) \in C\left(\mathbb{R}; H^2_+(\mathbb{R})\right)$: For every $t \in \mathbb{R}$ and for every $z \in \mathbb{C}_+$, $v(t,z)$ identifies to*

$$\begin{aligned}
&-\frac{t}{i\pi} I_+ \left[(G + 2tL_{u_0} - z\mathrm{Id})^{-1}\left(T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x-z}\right)\right)\right] \\
&= -2t\left[\left(Id + 2t\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}(G - zId)^{-1}\right)^{-1}\mathrm{e}^{it\partial_x^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{-it\partial_x^2}\left(\frac{\mathrm{e}^{it\partial_x^2}u_0(x) - \mathrm{e}^{it\partial_x^2}u_0(z)}{x-z}\right)\right](z).
\end{aligned}$$

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