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Uniqueness for Quasilinear Elliptic Problems in a Two-Component Domain with L^1 data

Rheadel G. Fulgencio* and Olivier Guibé†

Abstract

In the present paper, we prove the uniqueness of the renormalized solution of the class of quasilinear elliptic problems with L^1 data given by

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases}$$

The open sets Ω_1 and Ω_2 , with Γ as the interface between them, are the two components of the domain Ω . The data f is in $L^1(\Omega)$. In addition to uniform ellipticity, we also prescribe the assumption that the matrix field B is locally Lipschitz continuous with respect to the second variable.

1 Introduction

In the present paper, we study the uniqueness of the renormalized solution of the following class of quasilinear elliptic problems:

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases} \quad (1.1)$$

The domain Ω can be written as the disjoint union of Ω_1 , Ω_2 and Γ , where Ω_1 and Ω_2 are the two open components of Ω , and Γ is the interface between them. The matrix field B is a Carathéodory function that is uniformly elliptic (see assumption (A3)). The function h is in $L^\infty(\Gamma)$ while f belongs to $L^1(\Omega)$.

When f belongs to $L^2(\Omega)$ and the domain is composed of only one component, that is, $-\operatorname{div}(B(x, u)\nabla u) = f$ in Ω with Dirichlet boundary conditions, the uniqueness of the solution was obtained in [1] and [6] under a global Lipschitz-kind condition on B with respect to the second variable. Some generalizations to nonlinear elliptic problems were addressed in [5], [7] and [8].

As far as the two-component domain is concerned, additional difficulties arise due to the jump at the interface. When $f \in L^2(\Omega)$, the uniqueness of the variational solution was proved in [2] by adapting the method of Chipot introduced in [7] (see also [12] for (1.1) with a singular term).

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Elliptic problems in the usual one component domain and L^1 data, that is, $-\operatorname{div}(B(x, u)\nabla u) = f$ in Ω with Dirichlet boundary conditions, are not in the standard variational setting. Furthermore, in the sense of distribution, we do not have uniqueness of the solution (see [17]). Thus, we need a convenient framework to prove the uniqueness of the solution.

Uniqueness results were proved by using the notion of entropy solutions (see [16]) or by using the (equivalent) notion of renormalized solutions (see [3], [10], and [13]).

Since we consider L^1 data, we choose the appropriate framework of renormalized solutions (see [9, 15]). The existence of a renormalized solution (which is motivated by homogenization, see [11]) has been obtained in [14] (see Definition 2.2).

The main novelty of the present paper is the uniqueness of the renormalized solution under a fairly used assumption on the matrix field $B(x, s)$ in s (similar to [10], see assumption (A4)). With respect to the already mentioned references, let us point out that mixing technical test functions developed in [3] for L^1 problem and the jump give additional difficulties.

In particular, we cannot expect to control the sign of the contribution of the interface terms. To overcome this, we first prove in Lemma 3.3 that if u and v are two renormalized solutions of (1.1), then $u_1 - v_1$ and $u_2 - v_2$ have the same sign on the interface Γ . This sign property is crucial to prove the uniqueness result, Theorem 4.2, which we accomplish by adapting the method of [10].

The present paper is organized as follows. The next section is devoted to the assumptions and the definitions that are necessary to achieve our aim. Here, we present the definition of a renormalized solution of (1.1) (see Definition 2.2). Section 3 is devoted to prove some properties of the renormalized solution to (1.1), in particular, the sign property (see Lemma 3.3) mentioned above. Our uniqueness result (see Theorem 4.2) is proved in Section 4.

2 Assumptions and Definitions

We now present the assumptions and some definitions for our problem. The domain Ω is a connected bounded open set in \mathbb{R}^N with its boundary $\partial\Omega$. We can write $\overline{\Omega}$ as the disjoint union $\overline{\Omega} = \Omega_1 \cup \Omega_2 \cup \Gamma$, where Ω_2 is an open set such that $\overline{\Omega_2} \subset \Omega$ with a Lipschitz continuous boundary Γ and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$ (see figure below).

We denote by $u_i = u|_{\Omega_i}$ the restriction of u in Ω_i , where u is any measurable function defined on $\Omega \setminus \Gamma$.

We prescribe the following assumptions on f , h and B :

(A1) The function f is in $L^1(\Omega)$.

(A2) The function h belongs to $L^\infty(\Gamma)$ and for some $h_0 > 0$,

$$h(y) \geq h_0 \quad \text{for a.e. } y \in \Gamma. \quad (2.1)$$

(A3) The matrix field B is a Carathéodory function, that is,

- (a) the map $r \mapsto B(x, r)$ is continuous for a.e. $x \in \Omega$;
- (b) the map $x \mapsto B(x, r)$ is measurable for every $r \in \mathbb{R}$,

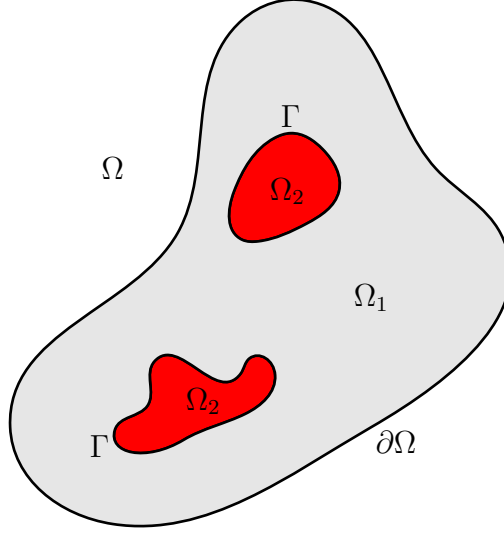


Figure 1: The two-component domain Ω

and it satisfies the following properties:

- (a) $B(x, r)\xi \cdot \xi \geq \alpha|\xi|^2$, for some $\alpha > 0$ a.e. $x \in \Omega$, $\forall r \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$;
- (b) for any $k > 0$, $B(x, r) \in L^\infty(\Omega \times (-k, k))^{N \times N}$;

(A4) $B(x, r)$ is Locally Lipschitz with respect to r , that is, for any compact subset K of \mathbb{R} , there exists $M_K > 0$ such that

$$|B(x, r) - B(x, s)| \leq M_K |r - s|, \quad \forall r, s \in K, \text{ a.e. } x \in \Omega. \quad (2.2)$$

Due to the jump of a solution on the interface Γ , the usual Sobolev spaces are not suitable to work with for our problem. Hence, we need to define a special normed space V .

Let V_1 be the normed space defined as

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

The space V is defined as

$$V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\},$$

equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2. \quad (2.3)$$

As presented in [14], since the data f is in $L^1(\Omega)$, we do not expect a solution u of (1.1) to be in any L^p -space. Moreover, it is also not expected to have the regularity required to have a gradient and trace in the usual sense. The following proposition was proved in [14] to give a definition for the gradient and trace of any measurable function. This proposition made use of the truncation function $T_k : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$T_k(t) = \begin{cases} -k, & \text{if } t \leq -k, \\ t, & \text{if } -k \leq t \leq k, \\ k, & \text{if } t \geq k. \end{cases} \quad (2.4)$$

Proposition 2.1 ([14]). *Let $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$ be a measurable function such that $T_k(u) \in V$ for every $k > 0$.*

1. *For $i = 1, 2$, there exists a unique measurable function $G_i : \Omega_i \rightarrow \mathbb{R}^N$ such that for all $k > 0$,*

$$\nabla T_k(u_i) = G_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i, \quad (2.5)$$

where $\chi_{\{|u_i| < k\}}$ denotes the characteristic function of

$$\{x \in \Omega_i : |u_i(x)| < k\}.$$

We define G_i as the gradient of u_i and write $G_i = \nabla u_i$.

2. *If*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty, \quad (2.6)$$

then there exists a unique measurable function $w_i : \Gamma \rightarrow \mathbb{R}$, for $i = 1, 2$, such that for all $k > 0$

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma, \quad (2.7)$$

where $\gamma_i : H^1(\Omega_i) \rightarrow L^2(\Gamma)$ is the trace operator. We define the function w_i as the trace of u_i on Γ and set

$$\gamma_i(u_i) = w_i.$$

With this proposition, we can now present the definition of a renormalized solution of (1.1) given in [14].

Definition 2.2. *Let $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$ be a measurable function. Then u is a renormalized solution of (1.1) if*

$$T_k(u) \in V, \quad \forall k > 0; \quad (2.8a)$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \forall k > 0; \quad (2.8b)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u| < n\}} B(x, u) \nabla u \cdot \nabla u \, dx = 0; \quad (2.9a)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma = 0; \quad (2.9b)$$

and for any $S_1, S_2 \in C^1(\mathbb{R})$ (or equivalently for any $S_1, S_2 \in W^{1,\infty}(\mathbb{R})$) with compact support, u satisfies

$$\begin{aligned} & \int_{\Omega_1} S_1(u_1) B(x, u_1) \nabla u_1 \cdot \nabla \psi_1 \, dx + \int_{\Omega_1} S_1'(u_1) B(x, u_1) \nabla u_1 \cdot \nabla u_1 \psi_1 \, dx \\ & + \int_{\Omega_2} S_2(u_2) B(x, u_2) \nabla u_2 \cdot \nabla \psi_2 \, dx + \int_{\Omega_2} S_2'(u_2) B(x, u_2) \nabla u_2 \cdot \nabla u_2 \psi_2 \, dx \\ & + \int_{\Gamma} h(x) (u_1 - u_2) (\psi_1 S_1(u_1) - \psi_2 S_2(u_2)) \, d\sigma \\ & = \int_{\Omega_1} f \psi_1 S_1(u_1) \, dx + \int_{\Omega_2} f \psi_2 S_2(u_2) \, dx, \end{aligned} \quad (2.10)$$

for all $\psi \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$.

Remark 2.3. *In the previous definition, conditions (2.8a) and (2.9a) are standard in the definition of renormalized solution. However, due to the presence of the boundary integral on Γ , (2.8b) and (2.9b) have to be added. In particular, since $\gamma_i(u_i) \in L^1(\Gamma)$ is not an assumption in Definition 2.2, we need (2.8b) to give sense to the integral on Γ in (2.10) (see [14, Remark 2]).*

We can avoid introducing this extra regularity condition on Γ by using the Boccardo-Gallouët estimates presented in [4] (see also Proposition 3.2). However, these estimates are heavily dependent on the Sobolev constants. With the final aim of doing the homogenization process (see [11]) or of considering more general nonlinear equations we chose not to impose it in the definition.

As presented in [14], assumptions (A1)-(A3) are enough to show the existence of a renormalized solution of (1.1) in the sense of this previous definition. However, to have uniqueness of the solution, an additional assumption on matrix B must be added (see (A4)), as will be seen in the next section.

3 Preliminary Results

In this section, we prove some properties on renormalized solutions of (1.1) (see Lemma 3.1 and Proposition 3.2), which are standard in the L^1 framework. Moreover, we prove Lemma 3.3, which states that if u and v are two renormalized solutions of (1.1) for the same data f , then we have the sign condition on the interface Γ , that is $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$ on Γ . This result is crucial for the proof of our uniqueness result (see Theorem 4.2).

Lemma 3.1. *Let u be a renormalized solution of (1.1). If φ is a bounded and increasing function belonging in $C^1(\mathbb{R})$ such that $\varphi(0) = 0$, then*

$$\varphi'(u_i)B(x, u_i)\nabla u_i \cdot \nabla u_i \in L^1(\Omega_i), \quad i = 1, 2, \quad (3.1)$$

$$(u_1 - u_2)(\varphi(u_1) - \varphi(u_2)) \in L^1(\Gamma). \quad (3.2)$$

Proof. Let φ be a bounded increasing function that belongs in $C^1(\mathbb{R})$ such that $\varphi(0) = 0$. Let $n > 0$. Define the function $\theta_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\theta_n(s) = \begin{cases} 0, & \text{if } s \leq -2n \\ \frac{s}{n} + 2, & \text{if } -2n \leq s \leq -n \\ 1, & \text{if } -n \leq s \leq n \\ -\frac{s}{n} + 2, & \text{if } n \leq s \leq 2n \\ 0, & \text{if } s \geq 2n. \end{cases} \quad (3.3)$$

We can clearly see from the definition of θ_n that it is a continuous Lipschitz function verifying

$$|\theta_n(r)| \leq 1 \quad \text{and} \quad |\theta'_n(r)| \leq \frac{1}{n}, \quad \text{a.e. in } \mathbb{R}. \quad (3.4)$$

Substituting $S_1 = S_2 = \theta_n$ and $\psi = \varphi(T_{2n}(u))$ in (2.10) of Definition 2.2, we

have

$$\begin{aligned}
& \int_{\Omega_1} \theta_n(u_1) \varphi'(u_1) B(x, u_1) \nabla u_1 \nabla u_1 dx + \int_{\Omega_2} \theta_n(u_2) \varphi'(u_2) B(x, u_2) \nabla u_2 \nabla u_2 dx \\
& + \int_{\Omega_1} \theta'_n(u_1) \varphi(u_1) B(x, u_1) \nabla u_1 \nabla u_1 dx + \int_{\Omega_2} \theta'_n(u_2) \varphi(u_2) B(x, u_2) \nabla u_2 \nabla u_2 dx \\
& + \int_{\Gamma} h(x) (u_1 - u_2) (\theta_n(u_1) \varphi(u_1) - \theta_n(u_2) \varphi(u_2)) d\sigma = \int_{\Omega} f \theta_n(u) \varphi(u) dx.
\end{aligned} \tag{3.5}$$

We now study the terms in (3.5) to pass to the limit as n goes to infinity. Regarding the third and fourth terms we have, for $i = 1, 2$,

$$\begin{aligned}
& \left| \int_{\Omega_i} \theta'_n(u_i) \varphi(u_i) B(x, u_i) \nabla u_i \nabla u_i dx \right| \\
& \leq \frac{\|\varphi\|_{L^\infty(\mathbb{R})}}{n} \int_{\{n < |u_i| < 2n\}} B(x, u_i) \nabla u_i \nabla u_i dx,
\end{aligned}$$

so that the decay of the energy of the truncates (2.9a) implies that

$$\lim_{n \rightarrow +\infty} \left| \int_{\Omega_i} \theta'_n(u_i) \varphi(u_i) B(x, u_i) \nabla u_i \nabla u_i dx \right| = 0, \quad \text{for } i = 1, 2. \tag{3.6}$$

As far as the fifth term of (3.5) is concerned, we have in view of the definition of θ_n ,

$$\begin{aligned}
& h(x) (u_1 - u_2) (\theta_n(u_1) \varphi(u_1) - \theta_n(u_2) \varphi(u_2)) \\
& = h(x) (u_1 - u_2) \theta_n(u_1) (\theta_{2n}(u_1) - \theta_{2n}(u_2)) \varphi(u_1) \\
& \quad + h(x) (u_1 - u_2) \theta_n(u_1) \theta_{2n}(u_2) (\varphi(u_1) - \varphi(u_2)) \\
& \quad - h(x) (u_1 - u_2) \theta_{2n}(u_2) (\theta_n(u_1) - \theta_n(u_2)) \varphi(u_2).
\end{aligned} \tag{3.7}$$

Since the functions θ_n and θ_{2n} are Lipschitz continuous and recalling that φ is bounded, we deduce that

$$\begin{aligned}
& \left| h(x) (u_1 - u_2) \theta_n(u_1) (\theta_{2n}(u_1) - \theta_{2n}(u_2)) \varphi(u_1) \right| \\
& \leq \frac{\|h\|_{L^\infty(\Gamma)}}{2n} (u_1 - u_2) (T_{4n}(u_1) - T_{4n}(u_2)) \|\varphi\|_{L^\infty(\mathbb{R})}, \\
& \left| h(x) (u_1 - u_2) \theta_{2n}(u_2) (\theta_n(u_1) - \theta_n(u_2)) \varphi(u_2) \right| \\
& \leq \frac{\|h\|_{L^\infty(\Gamma)}}{2n} (u_1 - u_2) (T_{2n}(u_1) - T_{2n}(u_2)) \|\varphi\|_{L^\infty(\mathbb{R})},
\end{aligned}$$

so that condition (2.9b) leads to

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \left| h(x) (u_1 - u_2) \theta_n(u_1) (\theta_{2n}(u_1) - \theta_{2n}(u_2)) \varphi(u_1) \right| dx = 0, \tag{3.8}$$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \left| h(x) (u_1 - u_2) \theta_{2n}(u_2) (\theta_n(u_1) - \theta_n(u_2)) \varphi(u_2) \right| dx = 0. \tag{3.9}$$

At last, the integral on the right-hand side of (3.5), by (3.4), is bounded by

$$\left| \int_{\Omega} f \theta_n(u) \varphi(u) dx \right| \leq \|f\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\mathbb{R})}. \tag{3.10}$$

Combining (3.5)-(3.10), we obtain

$$\begin{aligned} & \int_{\Omega_1} \theta_n(u_1) \varphi'(u_1) B(x, u_1) \nabla u_1 \nabla u_1 dx + \int_{\Omega_2} \theta_n(u_2) \varphi'(u_2) B(x, u_2) \nabla u_2 \nabla u_2 dx \\ & + \int_{\Gamma} h(x)(u_1 - u_2) \theta_n(u_1) \theta_{2n}(u_2) (\varphi(u_1) - \varphi(u_2)) d\sigma \\ & \leq \omega(n) + \|f\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

where $w(n) \rightarrow 0$ as n goes to infinity.

Since u_1 (resp. u_2) is finite almost everywhere in Ω_1 (resp. Ω_2), the definition of θ_n and Fatou's Lemma allow one to deduce that

$$\begin{aligned} & \int_{\Omega_1} B(x, u_1) \varphi'(u_1) \nabla u_1 \nabla u_1 dx + \int_{\Omega_2} B(x, u_2) \varphi'(u_2) \nabla u_2 \nabla u_2 dx \\ & + \int_{\Gamma} h(x)(u_1 - u_2) (\varphi(u_1) - \varphi(u_2)) d\sigma \leq \|f\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\mathbb{R})}. \quad (3.11) \end{aligned}$$

This gives (3.1) and (3.2). \square

As mentioned in Remark 2.3, we do not impose in Definition 2.2 that $\gamma_i(u_i)$ ($i = 1, 2$) belongs to $L^1(\Gamma)$. However, having no regularity on $\gamma_i(u_i)$, for $i = 1, 2$, seems to be an obstacle to prove Theorem 4.2. By adapting the estimates of Boccardo-Gallouët (see [4]) to our two-component domain, we are able to prove in Proposition 3.2 that $\gamma_i(u_i)$ belongs to $L^1(\Gamma)$, for $i = 1, 2$.

Proposition 3.2. *For $i = 1, 2$, let γ_i be the trace function defined on $H^1(\Omega_i)$. If u is a renormalized solution of (1.1), then $\gamma_i(u_i) \in L^1(\Gamma)$, $i = 1, 2$.*

Proof. By taking $\varphi = T_k$ in Lemma 3.1, and by observing the precise estimate (3.11) at the end of the proof of Lemma 3.1, we have

$$\|T_k(u)\|_V^2 \leq k \|f\|_{L^1(\Omega)}, \quad \forall k > 0.$$

Since on V , the norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(\Omega_1) \times H^1(\Omega_2)$, the Boccardo-Gallouët estimates hold true, so that $u_i \in W^{1,p}(\Omega_i)$, $i = 1, 2$, for any $p < \frac{N}{N-1}$. We are then able to conclude that $\gamma_i(u_i) \in L^1(\Gamma)$, $i = 1, 2$. \square

In proving the uniqueness result, one of the main difficulties we encountered is managing the integral on the interface Γ with test functions which are nonlinear with respect to the unknown. The very first step to overcome this difficulty is the following lemma which establishes a sign property of the difference of any two renormalized solutions of (1.1) on the interface. We will denote by sgn the usual sign function ($\text{sgn}(r) = r/|r|$ if $r \neq 0$ and $\text{sgn}(0) = 0$).

Lemma 3.3. *Suppose (A1)-(A4) hold. If u and v are two renormalized solutions of (1.1), then $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$ a.e. on Γ .*

Proof. Let u and v be renormalized solutions of (1.1). Writing (2.10) of Definition 2.2 for $S_1 = S_2 = \theta_n$ and $\psi = \frac{1}{k} T_k(u - v)$, where $0 < k < 1$, for u and v , and subtracting the resulting equations, we have

$$I_1^{k,n} + I_2^{k,n} + J_1^{k,n} + J_2^{k,n} + L^{k,n} = M^{k,n}, \quad (3.12)$$

where for $i = 1, 2$,

$$\begin{aligned}
I_i^{k,n} &= \frac{1}{k} \int_{\Omega_i} (\theta_n(u_i)B(x, u_i)\nabla u_i - \theta_n(v_i)B(x, v_i)\nabla v_i)\nabla T_k(u_i - v_i) dx, \\
J_i^{k,n} &= \frac{1}{k} \int_{\Omega_i} (\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i - \theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i)T_k(u_i - v_i) dx, \\
L^{k,n} &= \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)(\theta_n(u_1)T_k(u_1 - v_1) - \theta_n(u_2)T_k(u_2 - v_2)) d\sigma \\
&\quad - \frac{1}{k} \int_{\Gamma} h(x)(v_1 - v_2)(\theta_n(v_1)T_k(u_1 - v_1) - \theta_n(v_2)T_k(u_2 - v_2)) d\sigma, \\
M^{k,n} &= \frac{1}{k} \int_{\Omega} f(\theta_n(u) - \theta_n(v))T_k(u - v) dx.
\end{aligned}$$

We study the behavior of each term first as $k \rightarrow 0$ and then as $n \rightarrow \infty$.

We can write $I_i^{k,n}$, $i = 1, 2$, as

$$I_i^{k,n} = I_{i,1}^{k,n} + I_{i,2}^{k,n} + I_{i,3}^{k,n},$$

where

$$\begin{aligned}
I_{i,1}^{k,n} &= \frac{1}{k} \int_{\Omega_i} \theta_n(u_i)B(x, u_i)\nabla T_k(u_i - v_i)\nabla T_k(u_i - v_i) dx, \quad i = 1, 2, \\
I_{i,2}^{k,n} &= \frac{1}{k} \int_{\Omega_i} \theta_n(u_i)(B(x, u_i) - B(x, v_i))\nabla v_i\nabla T_k(u_i - v_i) dx, \quad i = 1, 2, \\
I_{i,3}^{k,n} &= \frac{1}{k} \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i))B(x, v_i)\nabla v_i\nabla T_k(u_i - v_i) dx, \quad i = 1, 2.
\end{aligned}$$

Clearly, $I_{i,1}^{k,n} \geq 0$, $i = 1, 2$. For $I_{i,2}^{k,n}$, $i = 1, 2$, we use (2.2) and (3.4) to obtain

$$\begin{aligned}
|I_{i,2}^{k,n}| &= \left| \frac{1}{k} \int_{\Omega_i} \theta_n(u_i)(B(x, u_i) - B(x, v_i))\nabla v_i\nabla T_k(u_i - v_i) dx \right| \\
&\leq \frac{1}{k} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| \leq 2n\} \\ \cap \{|v_i| \leq 2n+1\}}} |\theta_n(u_i)| |B(x, u_i) - B(x, v_i)| |\nabla v_i| |\nabla T_k(u_i - v_i)| dx \\
&\leq \frac{1}{k} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| \leq 2n\} \\ \cap \{|v_i| \leq 2n+1\}}} C |u_i - v_i| |\nabla v_i| |\nabla T_k(u_i - v_i)| dx \\
&\leq C \int_{\{0 < |u_i - v_i| < k\}} |\nabla T_{2n+1}(v_i)\nabla T_k(u_i - v_i)| dx.
\end{aligned}$$

From (2.8a) of Definition 2.2, we know that for any $0 < k < 1$, $i = 1, 2$,

$$|\nabla T_{2n+1}(v_i)\nabla T_k(u_i - v_i)\chi_{\{0 < |u_i - v_i| < k\}}| \leq |\nabla T_{2n+1}(v_i)\nabla T_1(u_i - v_i)| \in L^1(\Omega_i).$$

In addition, we have,

$$\nabla T_k(u_i - v_i)\chi_{\{0 < |u_i - v_i| < k\}} \rightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2.$$

By Lebesgue Dominated Convergence Theorem, we conclude that

$$C \int_{\{0 < |u_i - v_i| < k\}} |\nabla T_{2n+1}(v_i)\nabla T_k(u_i - v_i)| dx \rightarrow 0 \quad \text{as } k \rightarrow 0, \quad i = 1, 2,$$

which gives

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} I_{i,2}^{k,n} = 0, \quad i = 1, 2. \quad (3.13)$$

For $I_{i,3}^{k,n}$, noting that θ_n is Lipschitz continuous with (3.4), we have for $i = 1, 2$,

$$\begin{aligned} |I_{i,3}^{k,n}| &= \left| \frac{1}{k} \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i)) B(x, v_i) \nabla v_i \nabla T_k(u_i - v_i) dx \right| \\ &\leq \frac{1}{k} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |\theta_n(u_i) - \theta_n(v_i)| |B(x, u_i) \nabla v_i \nabla(u_i - v_i)| dx \\ &\leq \frac{1}{n} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |B(x, u_i) \nabla v_i \nabla(u_i - v_i)| dx \\ &\leq \frac{1}{n} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |B(x, v_i) \nabla u_i \nabla v_i| dx \\ &\quad + \frac{1}{n} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, v_i) \nabla v_i \nabla v_i dx \\ &\leq \frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} |B(x, v_i) \nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| dx \\ &\quad + \frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} B(x, v_i) \nabla T_{2n+1}(v_i) \nabla T_{2n+1}(v_i) dx. \end{aligned}$$

Again, from (2.8a) of Definition 2.2, we deduce that for $i = 1, 2$,

$$|\nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| \chi_{\{0 < |u_i - v_i| < k\}} \leq |\nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| \in L^1(\Omega_i)$$

and

$$|\nabla T_{2n+1}(v_i)|^2 \chi_{\{0 < |u_i - v_i| < k\}} \leq |\nabla T_{2n+1}(v_i)|^2 \in L^1(\Omega_i).$$

Furthermore,

$$\nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i) \chi_{\{0 < |u_i - v_i| < k\}} \longrightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2,$$

and

$$|\nabla T_{2n+1}(v_i)|^2 \chi_{\{0 < |u_i - v_i| < k\}} \longrightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2.$$

Using the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} \frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} |B(x, v_i) \nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| dx \\ + \frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} B(x, v_i) |\nabla T_{2n+1}(v_i)|^2 dx \longrightarrow 0 \quad \text{as } k \rightarrow 0. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} I_{i,3}^{k,n} = 0, \quad i = 1, 2. \quad (3.14)$$

For $J_i^{k,n}$, $i = 1, 2$, since we have (3.4) and

$$|T_k(r)| \leq k, \quad \forall r \in \mathbb{R}, \quad \forall k > 0, \quad (3.15)$$

we obtain for $i = 1, 2$,

$$\begin{aligned}
|J_i^{k,n}| &= \left| \frac{1}{k} \int_{\Omega_i} (\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i - \theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i)T_k(u_i - v_i) dx \right| \\
&\leq \int_{\Omega_i} |\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i| dx + \int_{\Omega_i} |\theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i| dx \\
&\leq \frac{1}{n} \int_{\{|u_i| < 2n\}} B(x, u_i)\nabla u_i\nabla u_i dx + \frac{1}{n} \int_{\{|v_i| < 2n\}} B(x, v_i)\nabla v_i\nabla v_i dx.
\end{aligned}$$

By (2.9a) of Definition 2.2, it follows that

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} J_i^{k,n} = 0, \quad i = 1, 2. \quad (3.16)$$

For the integral on the boundary, we use Proposition 3.2 to pass to the limit. Note that by (3.4) and (3.15), we have

$$\left| \frac{1}{k} h(x)(u_1 - u_2)\theta_n(u_1)T_k(u_1 - v_1) \right| \leq \|h\|_{L^\infty(\Gamma)} |u_1 - u_2| \in L^1(\Gamma).$$

Furthermore,

$$\frac{1}{k} h(x)(u_1 - u_2)\theta_n(u_1)T_k(u_1 - v_1) \xrightarrow[k \rightarrow 0]{} h(x)(u_1 - u_2)\theta_n(u_1) \operatorname{sgn}(u_1 - v_1)$$

a.e. on Γ , and

$$h(x)(u_1 - u_2)\theta_n(u_1) \operatorname{sgn}(u_1 - v_1) \xrightarrow[n \rightarrow \infty]{} h(x)(u_1 - u_2) \operatorname{sgn}(u_1 - v_1),$$

a.e. on Γ . The Lebesgue Dominated Convergence Theorem implies

$$\begin{aligned}
\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)\theta_n(u_1)T_k(u_1 - v_1) d\sigma \\
= \int_{\Gamma} h(x)(u_1 - u_2) \operatorname{sgn}(u_1 - v_1) d\sigma,
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)\theta_n(u_2)T_k(u_2 - v_2) d\sigma \\
= \int_{\Gamma} h(x)(u_1 - u_2) \operatorname{sgn}(u_2 - v_2) d\sigma, \\
\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)\theta_n(v_i)T_k(u_2 - v_2) d\sigma \\
= \int_{\Gamma} h(x)(u_1 - u_2) \operatorname{sgn}(u_2 - v_2) d\sigma, \quad i = 1, 2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} L^{k,n} \\
= \int_{\Gamma} h(x)[(u_1 - v_1) - (u_2 - v_2)](\operatorname{sgn}(u_1 - v_1) - \operatorname{sgn}(u_2 - v_2)) d\sigma. \quad (3.17)
\end{aligned}$$

For the integral on the right-hand side of (3.12),

$$\begin{aligned} |M^{k,n}| &= \left| \frac{1}{k} \int_{\Omega} f(\theta_n(u) - \theta_n(v)) T_k(u - v) dx \right| \\ &\leq \int_{\Omega} |f| |\theta_n(u) - \theta_n(v)| dx. \end{aligned}$$

Note that $|f| |\theta_n(u) - \theta_n(v)| \leq 2|f| \in L^1(\Omega)$ with

$$\theta_n(u) - \theta_n(v) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad \text{a.e. in } \Omega.$$

Thus, by the Lebesgue Dominated Convergence Theorem,

$$\int_{\Omega} |f| |\theta_n(u) - \theta_n(v)| dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which gives

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} M^{k,n} = 0. \quad (3.18)$$

From (3.12) and the fact that $I_{i,1}^{k,n} \geq 0$, we obtain

$$I_{i,2}^{k,n} + I_{i,3}^{k,n} + X^{k,n} + L^{k,n} \leq M^{k,n}.$$

Taking the limit of both sides of the last inequality first as $k \rightarrow 0$ then as $n \rightarrow \infty$, we get

$$\int_{\Gamma} h(x)((u_1 - v_1) - (u_2 - v_2))(\text{sgn}(u_1 - v_1) - \text{sgn}(u_2 - v_2)) d\sigma = 0.$$

That is, $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$ on Γ . □

4 Main Result

This section is devoted to our main result, Theorem 4.2, namely the uniqueness of the renormalized solution under assumptions (A1)-(A4). The proof of this uniqueness result makes use of the results of the previous section and the method developed in [3, 10]. The following proposition, proved in [10], states that assuming a very local Lipschitz control of $B(x, s)$ with respect to s , we have the existence of a function φ which controls the Lipschitz continuous character of the matrix field B through very technical conditions.

Proposition 4.1 ([10]). *Suppose that (2.2) holds. Then there exists a function $\varphi \in C^1(\mathbb{R})$ that satisfies the following properties:*

$$\varphi(0) = 0 \quad \text{and} \quad \varphi' \geq 1. \quad (4.1)$$

In addition, there are constants $\delta > 1/2$, $0 < k_0 < 1$, and $L > 0$ such that

$$\frac{\varphi'}{(1 + |\varphi|)^{2\delta}} \in L^\infty(\mathbb{R}), \quad (4.2)$$

and for any $r, s \in \mathbb{R}$ satisfying $|\varphi(r) - \varphi(s)| \leq k$, for $0 < k < k_0$,

$$\left| \frac{B(x, r)}{\varphi'(r)} - \frac{B(x, s)}{\varphi'(s)} \right| \leq \frac{1}{\varphi'(s)} \frac{Lk}{(1 + |\varphi(r)| + |\varphi(s)|)^\delta} \quad (4.3)$$

and

$$\frac{1}{L} \leq \frac{\varphi'(s)}{\varphi'(r)} \leq L. \quad (4.4)$$

We now state and prove the main theorem.

Theorem 4.2. *If assumptions (A1)–(A4) hold, then the renormalized solution of (1.1) is unique.*

Proof. In view of Theorem 1 in [14], assumptions (A1)–(A3) are sufficient to give the existence of at least one solution to (1.1). Let u and v be two renormalized solutions of (1.1).

Since (2.2) holds, by Proposition 4.1, we can find a function $\varphi \in C^1(\mathbb{R})$, such that for some constants $\delta > 1/2$, $0 < k_0 < 1$, and $L > 0$, φ satisfies (4.1)–(4.4).

The proof is then decomposed into two steps. Step 1 is devoted to show the very technical result (4.5). Roughly speaking, (4.5) is an extension of the method developed by Artola in [1] (see also [5]), and allows one to consider very general dependency of $B(x, s)$ with respect to s and L^1 data. Limit (4.5) was also derived in [3] (see also [10]) for elliptic equations with Dirichlet boundary condition. Since we have to deal with the boundary term, we give here a complete proof of (4.5). In Step 2, we are able to conclude that $u = v$ a.e. in Ω .

Step 1. In this step we prove that

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{\Omega_i} \left(\frac{1}{\varphi'(u_i)} + \frac{1}{\varphi'(v_i)} \right) |\nabla T_k(\varphi(u_i) - \varphi(v_i))|^2 dx = 0, \quad i = 1, 2. \quad (4.5)$$

Writing (2.10) of Definition 2.2 for u and v , with $S_1 = S_2 = \theta_n$ and $\psi = W_k := T_k(\varphi(T_{3n}(u)) - \varphi(T_{3n}(v)))$, where $n \in \mathbb{N}$, $n \geq 1$ and $0 < k < 1$, and subtracting the resulting equations, we have

$$A_1^{k,n} + A_2^{k,n} + B_1^{k,n} + B_2^{k,n} + C^{k,n} = D^{k,n}, \quad (4.6)$$

where

$$\begin{aligned} A_i^{k,n} &= \int_{\Omega_i} (\theta_n(u_i)B(x, u_i)\nabla u_i - \theta_n(v_i)B(x, v_i)\nabla v_i)\nabla W_k dx, \quad i = 1, 2, \\ B_i^{k,n} &= \int_{\Omega_i} (\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i - \theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i)W_k dx, \quad i = 1, 2, \\ C^{k,n} &= \int_{\Gamma} h(x)(u_1 - u_2)(W_{k,1}\theta_n(u_1) - W_{k,2}\theta_n(u_2)) d\sigma \\ &\quad - \int_{\Gamma} h(x)(v_1 - v_2)(W_{k,1}\theta_n(v_1) - W_{k,2}\theta_n(v_2)) d\sigma, \\ D^{k,n} &= \int_{\Omega} fW_k(\theta_n(u) - \theta_n(v)) dx. \end{aligned}$$

We will first look at the limit of each term as $n \rightarrow \infty$.

We can write $A_i^{k,n}$ as

$$\begin{aligned} A_i^{k,n} &= \int_{\Omega_i} \theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \nabla W_k \, dx \\ &\quad + \int_{\Omega_i} \theta_n(u_i) \left(\frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right) \varphi'(v_i) \nabla v_i \nabla W_k \, dx \\ &\quad + \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i)) B(x, v_i) \nabla v_i \nabla W_k \, dx, \quad i = 1, 2. \end{aligned}$$

Using the symmetry with respect to v_i , we obtain

$$A_i^{k,n} = A_{i,1}^{k,n} + A_{i,2}^{k,n} + A_{i,3}^{k,n}, \quad i = 1, 2,$$

where

$$\begin{aligned} A_{i,1}^{k,n} &= \frac{1}{2} \int_{\Omega_i} \left(\theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} + \theta_n(v_i) \frac{B(x, v_i)}{\varphi'(v_i)} \right) (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \nabla W_k \, dx, \\ A_{i,2}^{k,n} &= \frac{1}{2} \int_{\Omega_i} \left(\frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right) \\ &\quad \times (\theta_n(u_i) \varphi'(v_i) \nabla v_i + \theta_n(v_i) \varphi'(u_i) \nabla u_i) \nabla W_k \, dx, \\ A_{i,3}^{k,n} &= \frac{1}{2} \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i)) (B(x, u_i) \nabla u_i + B(x, v_i) \nabla v_i) \nabla W_k \, dx. \end{aligned}$$

For $i = 1, 2$, let us define $U_i^k = \{x \in \Omega_i : 0 < |\varphi(u_i) - \varphi(v_i)| < k\}$. For any $k > 0$ small enough, since $\text{supp } \theta_n = [-2n, 2n]$, we have a.e. in U_i^k ,

$$\begin{aligned} \theta_n(u_i) \nabla T_k(\varphi(u_i) - \varphi(v_i)) &= \theta_n(u_i) \nabla T_k(\varphi(T_{3n}(u_i)) - \varphi(T_{3n}(v_i))) \\ &= \theta_n(u_i) \nabla W_k, \quad i = 1, 2. \end{aligned} \quad (4.7)$$

As a consequence of (4.7), for any $k > 0$ small enough, we get for $i = 1, 2$,

$$\begin{aligned} A_{i,1}^{k,n} &= \frac{1}{2} \int_{\Omega_i} \theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \nabla W_k \, dx \\ &\quad + \frac{1}{2} \int_{\Omega_i} \theta_n(v_i) \frac{B(x, v_i)}{\varphi'(v_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \nabla W_k \, dx \\ &= \frac{1}{2} \int_{U_i^k} \theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \, dx \\ &\quad + \frac{1}{2} \int_{U_i^k} \theta_n(v_i) \frac{B(x, v_i)}{\varphi'(v_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \, dx. \end{aligned}$$

Using the coercivity of B we obtain that

$$\frac{\alpha}{2} \int_{U_i^k} \left(\frac{\theta_n(u_i)}{\varphi'(u_i)} + \frac{\theta_n(v_i)}{\varphi'(v_i)} \right) |\nabla(\varphi(u_i) - \varphi(v_i))|^2 \, dx \leq A_{i,1}^{k,n}, \quad i = 1, 2. \quad (4.8)$$

As far as $A_{i,2}^{k,n}$ are concerned, by (4.3), we have for $i = 1, 2$, and any $k > 0$ small

enough

$$\begin{aligned}
|A_{i,2}^{k,n}| &\leq \frac{1}{2} \int_{U_i^k} \left| \frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right| |\theta_n(u_i) \nabla \varphi(v_i) + \theta_n(v_i) \nabla \varphi(u_i)| \\
&\quad \times |\nabla(\varphi(u_i) - \varphi(v_i))| dx \\
&\leq \frac{1}{2} \int_{U_i^k} \frac{1}{\varphi'(u_i)} \frac{Lk}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \\
&\quad \times |\theta_n(u_i) \nabla \varphi(v_i) + \theta_n(v_i) \nabla \varphi(u_i)| |\nabla(\varphi(u_i) - \varphi(v_i))| dx \\
&\leq \frac{1}{2} \int_{U_i^k} \frac{1}{\varphi'(u_i)} \frac{Lk}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \\
&\quad \times \theta_n(u_i) |\nabla \varphi(v_i)| |\nabla(\varphi(u_i) - \varphi(v_i))| dx \\
&\quad + \frac{1}{2} \int_{U_i^k} \frac{1}{\varphi'(u_i)} \frac{Lk}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \\
&\quad \times \theta_n(v_i) |\nabla \varphi(u_i)| |\nabla(\varphi(u_i) - \varphi(v_i))| dx.
\end{aligned}$$

For $\varepsilon > 0$ (which will be chosen later), Young's inequality leads to, for $i = 1, 2$,

$$\begin{aligned}
|A_{i,2}^{k,n}| &\leq \int_{U_i^k} \frac{\theta_n(u_i) \varphi'(v_i)}{2} \left[\frac{1}{\varepsilon} \left(\frac{Lk}{\varphi'(u_i)} \frac{|\nabla \varphi(v_i)|}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \right)^2 \right. \\
&\quad \left. + \varepsilon \left(\frac{1}{\varphi'(v_i)} |\nabla \varphi(u_i) - \nabla \varphi(v_i)| \right)^2 \right] dx \\
&\quad + \int_{U_i^k} \frac{\theta_n(v_i) \varphi'(u_i)}{2} \left[\frac{1}{\varepsilon} \left(\frac{Lk}{\varphi'(u_i)} \frac{|\nabla \varphi(u_i)|}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \right)^2 \right. \\
&\quad \left. + \varepsilon \left(\frac{1}{\varphi'(u_i)} |\nabla \varphi(u_i) - \nabla \varphi(v_i)| \right)^2 \right] dx \\
&\leq C_1 k^2 \int_{U_i^k} \frac{(\theta_n(u_i) \varphi'(v_i) |\nabla \varphi(v_i)|^2 + \theta_n(v_i) \varphi'(u_i) |\nabla \varphi(u_i)|^2)}{(\varphi'(u_i))^2 (1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
&\quad + C_2 \varepsilon \int_{U_i^k} \left(\frac{\theta_n(u_i)}{\varphi'(u_i)} + \frac{\theta_n(v_i)}{\varphi'(v_i)} \right) |\nabla \varphi(u_i) - \nabla \varphi(v_i)|^2 dx.
\end{aligned}$$

In view of assumption (4.4), we deduce that, for $i = 1, 2$,

$$\begin{aligned}
&C_1 k^2 \int_{U_i^k} \frac{[\theta_n(u_i) \varphi'(v_i) |\nabla \varphi(v_i)|^2 + \theta_n(v_i) \varphi'(u_i) |\nabla \varphi(u_i)|^2]}{(\varphi'(u_i))^2 (1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
&\leq C_1 k^2 \int_{U_i^k} \frac{(L^2 \theta_n(u_i) \varphi'(v_i) |\nabla v_i|^2 + \theta_n(v_i) \varphi'(u_i) |\nabla u_i|^2)}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
&\leq C_3 k^2 \int_{U_i^k} (\theta_n(u_i) + \theta_n(v_i)) \frac{\varphi'(u_i) |\nabla u_i|^2 + \varphi'(v_i) |\nabla v_i|^2}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
|A_{i,2}^{k,n}| &\leq C_3 k^2 \int_{U_i^k} (\theta_n(u_i) + \theta_n(v_i)) \frac{\varphi'(u_i) |\nabla u_i|^2 + \varphi'(v_i) |\nabla v_i|^2}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
&\quad + C_2 \varepsilon \int_{U_i^k} \left(\frac{\theta_n(u_i)}{\varphi'(u_i)} + \frac{\theta_n(v_i)}{\varphi'(v_i)} \right) |\nabla \varphi(u_i) - \nabla \varphi(v_i)|^2 dx, \quad i = 1, 2,
\end{aligned} \tag{4.9}$$

where C_2 and C_3 are positive constants independent of k and n (with C_2 also independent of ε).

We now turn to the term $A_{i,3}^{k,n}$. By (4.1) and (4.4), we have

$$|u_i - v_i| \leq \frac{L}{\varphi'(u_i)} |\varphi(u_i) - \varphi(v_i)| \quad \text{a.e. in } U_i^k, \quad i = 1, 2, \quad (4.10)$$

and since θ_n is a Lipschitz continuous function verifying $|\theta'_n(r)| = \frac{1}{n} \chi_{\{n < |r| < 2n\}}$ a.e. in \mathbb{R} , we obtain

$$|\theta_n(u_i) - \theta_n(v_i)| \leq \frac{1}{n} |u_i - v_i| \leq \frac{Lk}{n\varphi'(u_i)} \quad \text{a.e. in } U_i^k, \quad i = 1, 2. \quad (4.11)$$

Observe that this inequality still holds if the roles of u_i and v_i are interchanged. Therefore using (4.11), we obtain, for $i = 1, 2$,

$$\begin{aligned} |A_{i,3}^{k,n}| &\leq \frac{1}{2} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |\theta_n(u_i) - \theta_n(v_i)| \left| B(x, u_i) \varphi'(u_i) \nabla u_i \nabla u_i \right. \\ &\quad \left. + B(x, u_i) \varphi'(u_i) \nabla u_i \nabla v_i - B(x, v_i) \varphi'(v_i) \nabla u_i \nabla v_i \right. \\ &\quad \left. - B(x, v_i) \varphi'(v_i) \nabla v_i \nabla v_i \right| dx \\ &\leq \frac{Lk}{2n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) \nabla u_i \nabla u_i + B(x, v_i) \nabla v_i \nabla v_i) dx \\ &\quad + \frac{Lk}{2n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) |\nabla u_i \nabla v_i| + B(x, v_i) |\nabla u_i \nabla v_i|) dx \end{aligned}$$

Applying Young's Inequality on the second term of the previous inequality, we get, for $i = 1, 2$,

$$\begin{aligned} &\frac{Lk}{2n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) |\nabla u_i \nabla v_i| + B(x, v_i) |\nabla u_i \nabla v_i|) dx \\ &\leq \frac{Lk}{4n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} \left[B(x, u_i) (\nabla u_i \nabla u_i + \nabla v_i \nabla v_i) \right. \\ &\quad \left. + [B(x, v_i) (\nabla u_i \nabla u_i + \nabla v_i \nabla v_i)] \right] dx. \end{aligned}$$

It follows that, for $i = 1, 2$,

$$\begin{aligned} |A_{i,3}^{k,n}| &\leq \frac{C_4 k}{n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) \nabla u_i \nabla u_i + B(x, v_i) \nabla v_i \nabla v_i) dx \\ &\quad + \frac{C_5 k}{n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) \nabla v_i \nabla v_i + B(x, v_i) \nabla u_i \nabla u_i) dx. \end{aligned} \quad (4.12)$$

By (2.9a) of Definition 2.2, the first term of the right-hand side of (4.12) goes to zero as n goes to infinity. It is worth noting that the second term of the right-hand side of (4.12) contains non symmetric terms in u_i and v_i , so that without any bound on B , the behavior of this term is not a direct consequence of the decay of the truncate energy (2.9a). Using (2.9a) and condition (4.3), we claim that the second term also goes to zero as n goes to infinity.

Indeed, writing for $i = 1, 2$,

$$\begin{aligned} & \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, u_i) \nabla v_i \nabla v_i \, dx \\ &= \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} \left(\frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right) \varphi'(u_i) \nabla v_i \nabla v_i \, dx \\ & \quad + \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} \frac{B(x, v_i)}{\varphi'(v_i)} \varphi'(u_i) \nabla v_i \nabla v_i \, dx, \end{aligned}$$

and using (4.3) and (4.4), we have, for $i = 1, 2$,

$$\begin{aligned} & \frac{C_5 k}{n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, u_i) \nabla v_i \nabla v_i \, dx \\ & \leq \frac{C_5 L k^2}{n} \int_{\{|v_i| < 2n+1\}} \frac{|\nabla v_i|^2}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \, dx \\ & \quad + \frac{C_5 L k}{n} \int_{\{|v_i| < 2n+1\}} B(x, v_i) \nabla v_i \nabla v_i \, dx. \end{aligned}$$

It follows that by (2.9a) of Definition 2.2, we have

$$\lim_{n \rightarrow \infty} \frac{C}{n} \int_{\substack{U_i^k \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, u_i) \nabla v_i \nabla v_i \, dx = 0, \quad i = 1, 2.$$

By similar computations, it can be shown that

$$\lim_{n \rightarrow \infty} \frac{C}{n} \int_{\substack{U_i^k \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, v_i) \nabla u_i \nabla u_i \, dx = 0, \quad i = 1, 2.$$

Consequently,

$$\lim_{n \rightarrow \infty} A_{i,3}^{k,n} = 0, \quad i = 1, 2. \quad (4.13)$$

Regarding the term $B_i^{k,n}$, $i = 1, 2$, we have

$$\begin{aligned} |B_i^{k,n}| &= \left| \int_{\Omega_i} (\theta'_n(u_i) B(x, u_i) \nabla u_i \nabla u_i - \theta'_n(v_i) B(x, v_i) \nabla v_i \nabla v_i) W_k \, dx \right| \\ &\leq \int_{\Omega_i} |\theta'_n(u_i)| B(x, u_i) \nabla u_i \nabla u_i |W_k| \, dx + \int_{\Omega_i} |\theta'_n(v_i)| B(x, v_i) \nabla v_i \nabla v_i |W_k| \, dx \\ &\leq \frac{k}{n} \int_{\{|u_i| < 2n\}} B(x, u_i) \nabla u_i \nabla u_i \, dx + \frac{k}{n} \int_{\{|v_i| < 2n\}} B(x, v_i) \nabla v_i \nabla v_i \, dx. \end{aligned}$$

These last two integrals go to zero as n goes to infinity by (2.9a) of Definition 2.2. Thus,

$$\lim_{n \rightarrow \infty} B_i^{k,n} = 0, \quad i = 1, 2. \quad (4.14)$$

To pass to the limit of $C^{k,n}$ as n goes to ∞ , we use Proposition 3.2. Note that

$$|h(x)(u_1 - u_2)W_{k,1}\theta_n(u_1)| \leq \|h\|_{L^\infty(\Gamma)}|u_1 - u_2|k \in L^1(\Gamma),$$

and

$$h(x)(u_1 - u_2)W_{k,1}\theta_n(u_1) \xrightarrow[n \rightarrow \infty]{} h(x)(u_1 - u_2)T_k(\varphi(u_1) - \varphi(v_1)) \quad \text{a.e. on } \Gamma.$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\Gamma} h(x)(u_1 - u_2)W_{k,1}\theta_n(u_1) d\sigma = \int_{\Gamma} h(x)(u_1 - u_2)T_k(\varphi(u_1) - \varphi(v_1)) d\sigma.$$

Using similar arguments, we obtain that

$$\lim_{n \rightarrow \infty} \int_{\Gamma} h(x)(u_1 - u_2)W_{k,2}\theta_n(u_2) d\sigma = \int_{\Gamma} h(x)(u_1 - u_2)T_k(\varphi(u_2) - \varphi(v_2)) d\sigma$$

and for $i = 1, 2$,

$$\lim_{n \rightarrow \infty} \int_{\Gamma} h(x)(v_1 - v_2)W_{k,i}\theta_n(v_i) d\sigma = \int_{\Gamma} h(x)(v_1 - v_2)T_k(\varphi(v_i) - \varphi(v_i)) d\sigma.$$

Therefore, we conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} C^{k,n} &= \int_{\Gamma} h(x)[(u_1 - u_2) - (v_1 - v_2)] \\ &\quad \times [T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))] d\sigma. \end{aligned} \quad (4.15)$$

Finally, concerning $D^{k,n}$, since

$$|fW_k(\theta_n(u) - \theta_n(v))| \leq 2k|f| \in L^1(\Omega),$$

while

$$\theta_n(u) - \theta_n(v) \longrightarrow 0 \quad \text{a.e. in } \Omega \text{ as } n \longrightarrow \infty,$$

the Lebesgue Dominated Convergence Theorem leads to

$$\lim_{n \rightarrow \infty} D^{k,n} = 0. \quad (4.16)$$

Combining (4.8), (4.9), (4.13), (4.14), and (4.16), and choosing ε small enough, we obtain

$$\begin{aligned} &\int_{U_1^k} \left(\frac{\theta_n(u_1)}{\varphi'(u_1)} + \frac{\theta_n(v_1)}{\varphi'(v_1)} \right) |\nabla\varphi(u_1) - \nabla\varphi(v_1)|^2 dx \\ &\quad + \int_{U_2^k} \left(\frac{\theta_n(u_2)}{\varphi'(u_2)} + \frac{\theta_n(v_2)}{\varphi'(v_2)} \right) |\nabla\varphi(u_2) - \nabla\varphi(v_2)|^2 dx + C^{k,n} \\ &\leq Ck^2 \int_{U_1^k} (\theta_n(u_1) + \theta_n(v_1)) \frac{\varphi'(u_1)|\nabla u_1|^2 + \varphi'(v_1)|\nabla v_1|^2}{(1 + |\varphi(u_1)| + |\varphi(v_1)|)^{2\delta}} dx \\ &\quad + Ck^2 \int_{U_2^k} (\theta_n(u_2) + \theta_n(v_2)) \frac{\varphi'(u_2)|\nabla u_2|^2 + \varphi'(v_2)|\nabla v_2|^2}{(1 + |\varphi(u_2)| + |\varphi(v_2)|)^{2\delta}} dx + \rho(n), \end{aligned} \quad (4.17)$$

where C is a positive constant independent of k and n , and where $\rho(n)$ goes to zero as n goes to ∞ .

Let $\tau : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\tau(r) = \int_0^r \frac{\varphi'(t)}{(1 + |\varphi(t)|)^{2\delta}} dt.$$

Clearly, τ is an increasing $C^1(\mathbb{R})$ -function and since $2\delta > 1$, τ is bounded. Then, by Lemma 3.1 and (4.1) of Proposition 4.1, we deduce that

$$\frac{\varphi'(u_i)|\nabla u_i|^2}{(1 + |\varphi(u_i)|)^{2\delta}} \quad \text{and} \quad \frac{\varphi'(v_i)|\nabla v_i|^2}{(1 + |\varphi(v_i)|)^{2\delta}} \in L^1(\Omega_i). \quad (4.18)$$

Hence, we can pass to the limit on the right-hand side of (4.17). Furthermore,

$$\theta_n(u_i) \rightarrow 1 \quad \text{as } n \rightarrow \infty, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2.$$

By Fatou's Lemma and (4.15), we have

$$\begin{aligned} & \int_{U_1^k} \left(\frac{1}{\varphi'(u_1)} + \frac{1}{\varphi'(v_1)} \right) |\nabla \varphi(u_1) - \nabla \varphi(v_1)|^2 dx \\ & \quad + \int_{U_2^k} \left(\frac{1}{\varphi'(u_2)} + \frac{1}{\varphi'(v_2)} \right) |\nabla \varphi(u_2) - \nabla \varphi(v_2)|^2 dx + C^k \\ & \leq Ck^2 \int_{U_1^k} \frac{\varphi'(u_1)|\nabla u_1|^2 + \varphi'(v_1)|\nabla v_1|^2}{(1 + |\varphi(u_1)| + |\varphi(v_1)|)^{2\delta}} dx \\ & \quad + Ck^2 \int_{U_2^k} \frac{\varphi'(u_2)|\nabla u_2|^2 + \varphi'(v_2)|\nabla v_2|^2}{(1 + |\varphi(u_2)| + |\varphi(v_2)|)^{2\delta}} dx, \end{aligned} \quad (4.19)$$

where

$$C^k = \int_{\Gamma} h(x)[(u_1 - v_1) - (u_2 - v_2)][T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))] d\sigma.$$

Dividing both sides of (4.19) by k^2 and noting that $\chi_{U_i^k} \rightarrow 0$ a.e. in Ω_i as $k \rightarrow 0$, (4.18) and the Lebesgue Dominated Convergence Theorem allow one to conclude that

$$\begin{aligned} & \limsup_{k \rightarrow 0} \left(\frac{1}{k^2} \int_{U_1^k} \left(\frac{1}{\varphi'(u_1)} + \frac{1}{\varphi'(v_1)} \right) |\nabla \varphi(u_1) - \nabla \varphi(v_1)|^2 dx \right. \\ & \quad \left. + \frac{1}{k^2} \int_{U_2^k} \left(\frac{1}{\varphi'(u_2)} + \frac{1}{\varphi'(v_2)} \right) |\nabla \varphi(u_2) - \nabla \varphi(v_2)|^2 dx + \frac{1}{k^2} C^k \right) \leq 0. \end{aligned} \quad (4.20)$$

As a consequence, proving (4.5) is equivalent to showing that

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} C^k \geq 0. \quad (4.21)$$

We now study the behavior of C^k/k^2 as k goes to zero. To shorten the notation, we will denote by g_k the function given by

$$g_k = h[(u_1 - v_1) - (u_2 - v_2)] \times [T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))].$$

The main difficulty in managing this term is its non-linearity. Indeed, even if $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$ a.e. on Γ , there is no reason to have $g_k \geq 0$ nor to give a bound of g_k/k^2 . In order to study the behavior of C^k/k^2 , we decompose the integral on Γ into the integral on different subsets. Since from Lemma 3.3, $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$ a.e. on Γ , in view of the symmetry of g_k with respect to u_i and v_i ($i = 1, 2$), proving (4.21) is equivalent to prove

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} \int_{\{u_1 - v_1 > 0\}} g_k d\sigma \geq 0.$$

We now split the set $\{x \in \Gamma; u_1(x) - v_1(x) > 0\}$ (up to a zero measure subset) into 4 subsets,

$$\{u_1 - v_1 > 0\} = P_1 \cup P_2 \cup P_3 \cup P_4,$$

where

$$\begin{aligned} P_1 &:= \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}, \\ P_2 &:= \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\ P_3 &:= \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\ P_4 &:= \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}. \end{aligned}$$

Since we have

$$T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2)) = 0 \text{ a.e. on } P_1,$$

we obtain that

$$\frac{1}{k^2} \int_{P_1} g_k d\sigma = 0. \quad (4.22)$$

As far as $\int_{P_2} g_k d\sigma$ is concerned, recalling that $\varphi \in C^1(\mathbb{R})$ with $\varphi'(t) \geq 1$ for any $t \in \mathbb{R}$ gives

$$0 < u_1 - v_1 \leq \varphi(u_1) - \varphi(v_1) < k \text{ a.e. on } P_2$$

and

$$0 < u_2 - v_2 \leq \varphi(u_2) - \varphi(v_2) < k \text{ a.e. on } P_2.$$

As a consequence, we deduce that

$$\begin{aligned} \frac{1}{k^2} |g_k| &= \frac{1}{k^2} |h| |(u_1 - v_1) - (u_2 - v_2)| |(\varphi(u_1) - \varphi(v_1)) - (\varphi(u_2) - \varphi(v_2))| \\ &\leq \|h\|_{L^\infty(\Gamma)} \text{ a.e. on } P_2. \end{aligned}$$

Since

$$\chi_{P_2} \rightarrow 0 \text{ as } k \rightarrow 0 \text{ a.e. on } \Gamma,$$

the Lebesgue Dominated Convergence Theorem leads to

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{P_2} g_k d\sigma = 0. \quad (4.23)$$

We now study $\int_{P_3} g_k d\sigma$ by splitting P_3 into $P_3 \cap \{u_1 - v_1 \geq k\}$ and $P_3 \cap \{u_1 - v_1 < k\}$. With already used arguments, we have

$$0 < u_2 - v_2 < k \text{ a.e. on } P_3.$$

It follows that

$$[(u_1 - v_1) - (u_2 - v_2)] \geq 0 \text{ a.e. on } P_3 \cap \{u_1 - v_1 \geq k\},$$

so that

$$g_k \geq 0 \text{ a.e. on } P_3 \cap \{u_1 - v_1 \geq k\}.$$

On the other hand, we have

$$\begin{aligned} \frac{1}{k^2} |g_k| &= \frac{1}{k^2} |h| |(u_1 - v_1) - (u_2 - v_2)| (k - (\varphi(u_2) - \varphi(v_2))) \\ &\leq \|h\|_{L^\infty(\Gamma)} \text{ a.e. on } P_3 \cap \{0 < u_1 - v_1 < k\}. \end{aligned}$$

Since

$$\chi_{P_3} \rightarrow 0 \text{ as } k \rightarrow 0, \text{ a.e. on } \Gamma,$$

the Lebesgue Dominated Convergence Theorem leads to

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{P_3 \cap \{0 < u_1 - v_1 < k\}} g_k d\sigma = 0.$$

Noting that we can write

$$\int_{P_3} g_k d\sigma = \int_{P_3 \cap \{u_1 - v_1 \geq k\}} g_k d\sigma + \int_{P_3 \cap \{0 < u_1 - v_1 < k\}} g_k d\sigma,$$

we deduce that

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} \int_{P_3} g_k d\sigma \geq 0. \quad (4.24)$$

At last, by writing

$$\int_{P_4} g_k d\sigma = \int_{P_4 \cap \{u_2 - v_2 \geq k\}} g_k d\sigma + \int_{P_4 \cap \{0 < u_2 - v_2 < k\}} g_k d\sigma,$$

and by proving with similar arguments that

$$\int_{P_4 \cap \{u_2 - v_2 \geq k\}} g_k d\sigma \geq 0$$

and

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{P_4 \cap \{0 < u_2 - v_2 < k\}} g_k d\sigma = 0,$$

yield

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} \int_{P_4} g_k d\sigma \geq 0. \quad (4.25)$$

The results (4.22)–(4.25) give

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} C^k \geq 0. \quad (4.26)$$

Therefore, (4.5) holds true.

Step 2. In this step, we prove that $u_1 = v_1$ a.e. in Ω_1 and $u_2 = v_2$ a.e. in Ω_2 .

We first show that $u_1 = v_1$ a.e. in Ω_1 . To do this, we consider the function

$$\theta_n(u_1)T_k(\varphi(u_1) - \varphi(v_1)) \in L^\infty(\Omega_1) \cap V_1.$$

Since $u_1 = v_1 = 0$ on $\partial\Omega$, we can apply Poincaré inequality which leads to

$$\begin{aligned} & \int_{\Omega_1} (\theta_n(u_1))^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx \\ & \leq C \int_{\Omega_1} (\theta_n(u_1))^2 \left(\frac{\nabla T_k(\varphi(u_1) - \varphi(v_1))}{k} \right)^2 dx \\ & \quad + C \int_{\Omega_1} (\theta'_n(u_1))^2 |\nabla u_1|^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx, \end{aligned} \quad (4.27)$$

where $C > 0$ does not depend on k and n . The second integral on the right-hand side of (4.27) can be bounded by

$$\int_{\Omega_1} (\theta'_n(u_1))^2 |\nabla u_1|^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx \leq \frac{1}{n^2} \int_{\{|u_1| < 2n\}} |\nabla u_1|^2 dx.$$

The integral on the right-hand side goes to zero as n goes to ∞ by (2.9a) of Definition 2.2. This implies

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \int_{\Omega_1} (\theta'_n(u_1))^2 |\nabla u_1|^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx = 0. \quad (4.28)$$

For the first integral on the right-hand side of (4.27), we have

$$\begin{aligned} & \int_{\Omega_1} (\theta_n(u_1))^2 \left(\frac{\nabla T_k(\varphi(u_1) - \varphi(v_1))}{k} \right)^2 dx \\ & \leq \frac{\max_{s \in [-2n, 2n]} \varphi'(s)}{k^2} \int_{U_k} \frac{1}{\varphi'(u_1)} |\nabla \varphi(u_1) - \nabla \varphi(u_2)|^2 dx. \end{aligned}$$

The integral on the right-hand side of this inequality goes to 0 as k goes to 0 by (4.5). Thus,

$$\begin{aligned} & \int_{\Omega_1} \chi_{\{\varphi(u_1) \neq \varphi(v_1)\}} dx \\ & = \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \int_{\Omega_1} (\theta_n(u_1))^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx = 0, \end{aligned} \quad (4.29)$$

that is, $\varphi(u_1) = \varphi(v_1)$ a.e. in Ω_1 . Since $\varphi' \geq 1$, we have $u_1 = v_1$ a.e. in Ω_1 . As a consequence, $\gamma_1(u_1) = \gamma_1(v_1)$, that is, $u_1 = v_1$ a.e. on Γ .

From Lemma 3.3, $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$ on Γ . Since $u_1 - v_1 = 0$ a.e. on Γ , we also have $u_2 - v_2 = 0$ a.e. on Γ . Thus, $u_2 = v_2$ a.e. on Γ .

It only remains to prove that $u_2 = v_2$ in Ω_2 . Consider the function

$$\theta_n(u_2)T_k(\varphi(u_2) - \varphi(v_2)) \in L^\infty(\Omega_2) \cap H^1(\Omega_2),$$

which has a zero value a.e. on Γ since $u_2 = v_2$ a.e. on Γ . We can then apply Poincaré inequality which implies that

$$\begin{aligned} & \int_{\Omega_2} (\theta_n(u_2))^2 \left| \frac{T_k(\varphi(u_2) - \varphi(v_2))}{k} \right|^2 dx \\ & \leq C \int_{\Omega_2} (\theta_n(u_2))^2 \left(\frac{|\nabla T_k(\varphi(u_2) - \varphi(v_2))|}{k} \right)^2 dx \\ & \quad + C \int_{\Omega_2} (\theta'_n(u_2))^2 |\nabla u_2|^2 \left| \frac{T_k(\varphi(u_2) - \varphi(v_2))}{k} \right|^2 dx. \end{aligned}$$

Using the same arguments to show (4.29), we conclude that

$$\int_{\Omega_2} \chi_{\{\varphi(u_2) \neq \varphi(v_2)\}} dx = \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \int_{\Omega_2} (\theta_n(u_2))^2 \left| \frac{T_k(\varphi(u_2) - \varphi(v_2))}{k} \right|^2 dx = 0.$$

This implies $\varphi(u_2) = \varphi(v_2)$ a.e. in Ω_2 . Therefore, since $\varphi' \geq 1$, $u_2 = v_2$ a.e. in Ω_2 .

This concludes the proof of the uniqueness of the renormalized solution of (1.1). \square

References

- [1] M. ARTOLA, *Sur une classe de problèmes paraboliques quasi-linéaires.*, Boll. Unione Mat. Ital., VI. Ser., B, 5 (1986), pp. 51–70.
- [2] R. BELTRAN, *Homogenization of a quasilinear elliptic problem in a two-component domain with an imperfect interface*, Master's thesis, University of the Philippines - Diliman, 2014.
- [3] D. BLANCHARD, F. DÉSIR, AND O. GUIBÉ, *Quasi-linear degenerate elliptic problems with L^1 data*, Nonlinear Analysis: Theory, Methods and Applications, 60 (2005), pp. 557–587.
- [4] L. BOCCARDO AND T. GALLOUËT, *Nonlinear elliptic and parabolic equations involving measure data.*, J. Funct. Anal., 87 (1989), pp. 149–169.
- [5] L. BOCCARDO, T. GALLOUËT, AND F. MURAT, *Unicité de la solution de certaines équations elliptiques non linéaires. (Uniqueness of the solution of some nonlinear elliptic equations).*, C. R. Acad. Sci., Paris, Sér. I, 315 (1992), pp. 1159–1164.
- [6] J. CARRILLO AND M. CHIPOT, *On some nonlinear elliptic equations involving derivatives of the nonlinearity.*, Proc. R. Soc. Edinb., Sect. A, Math., 100 (1985), pp. 281–294.
- [7] M. CHIPOT, *Elliptic equations. An introductory course.*, Basel: Birkhäuser, 2009.
- [8] M. CHIPOT AND G. MICHAILLE, *Uniqueness of monotonicity properties for strongly nonlinear elliptic variational inequalities.*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser., 16 (1989), pp. 137–166.

- [9] G. DAL MASO, F. MURAT, L. ORSINA, AND A. PRIGNET, *Renormalized solutions of elliptic equations with general measure data*, Annali della Scuola normale superiore di Pisa, Classe di scienze, 28 (1999), pp. 741–808.
- [10] R. DI NARDO, F. FEO, AND O. GUIBÉ, *Uniqueness of renormalized solutions to nonlinear parabolic problems with lower-order terms.*, Proc. R. Soc. Edinb., Sect. A, Math., 143 (2013), pp. 1185–1208.
- [11] P. DONATO, R. FULGENCIO, AND O. GUIBÉ, *Homogenization results for quasilinear elliptic problems in a two-component domain with L^1 data.* (submitted), 2020.
- [12] P. DONATO AND F. RAIMONDI, *Uniqueness result for a class of singular elliptic problems in two-component domains.*, J. Elliptic Parabol. Equ., 5 (2019), pp. 349–358.
- [13] F. FEO AND O. GUIBÉ, *Uniqueness for elliptic problems with locally Lipschitz continuous dependence on the solution.*, J. Differ. Equations, 262 (2017), pp. 1777–1798.
- [14] R. FULGENCIO AND O. GUIBÉ, *Quasilinear elliptic problem in a two-component domain with L^1 data*, in Donato P., Luna-Laynez M. (eds) Emerging Problems in the Homogenization of Partial Differential Equations. SEMA SIMAI Springer Series, vol. 10, Springer, 2021, pp. 59–83.
- [15] F. MURAT, *Soluciones renormalizadas de EDP elípticas no lineales*, Tech. Rep. R93023, Laboratoire d’Analyse Numérique, Paris VI, 1993.
- [16] A. PORRETTA, *Uniqueness of solutions for some nonlinear Dirichlet problems.*, NoDEA, Nonlinear Differ. Equ. Appl., 11 (2004), pp. 407–430.
- [17] J. SERRIN, *Pathological solutions of elliptic differential equations.*, Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser., 18 (1964), pp. 385–387.