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Automatica 173 (2025) 112077

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# New negative definiteness conditions for quadratic functions with illustration in LPV sampled-data control☆

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ARTICLE INFO

*Article history:*
Received 12 October 2023
Received in revised form 19 July 2024
Accepted 1 November 2024
Available online xxxx

*Keywords:*
Quadratic functions
Negative definiteness
Monotonic convergent conditions
LPV systems
Time-varying delay systems

ABSTRACT

This paper addresses negative definiteness conditions of quadratic functions, common in control problems with time-varying delay systems. Existing geometric conditions, relaxed by partitioning techniques, may lack monotonic convergence, making their optimality questionable. Alternative conditions based on the generalized S-procedure are known to be necessary and sufficient when coefficients are not dependent on uncertain parameters; otherwise, they are sufficient only. We propose new approaches to mitigate these issues, unexplored in previous studies, demonstrated in stability analysis of linear parameter varying sampled-data control systems. First, we rewrite quadratic polynomial inequalities as homogeneous ones, deriving relaxed conditions using Young's inequality or Polya's theorem, offering recursive monotonic convergent relaxations. Then, based on Bézier curve equivalence and the de Casteljau algorithm, we provide further relaxed recursive monotonic convergent conditions. Two numerical examples illustrate the effectiveness and improvements over previous related studies.


## 1. Introduction and problem statement

Consider quadratic polynomial inequalities given by:

$$\mathcal{P}(\tau) = \tau^2\Phi_2 + \tau\Phi_1 + \Phi_0 < 0,\ \forall\tau \in [\underline{\tau}, \bar{\tau}], \tag{1}$$

where $\Phi_i \in \mathbb{R}^{n\times n}$ $(i = 0, 1, 2)$ and $\tau$ is a real bounded scalar quantity. Such constraints play an important role in control theory, especially for stability analysis of dynamic systems with time-varying delays (see, e.g., Chen, Park, Xu, and Zhang (2022), Zhang, Han, and Ge (2022)). However, since these matrix inequalities are not convex in $\tau$, ongoing research focuses on providing conditions based on linear matrix inequality (LMI) to satisfy (1) with minimal conservatism. Widely used approaches involve geometrical considerations to establish sufficient conditions, summarized in the following lemmas, which include the pioneer and most recent related results from the literature; see, e.g., Chen, Park, and Xu (2019), Kim (2011, 2016), Liu, Liu, Li, and Sidorov (2023), Yang, He, Kang, and Pan (2019), Yang and Zhang (2014), Zhang, Long, He, Yao, Jiang, and Wu (2020) and the surveys in Chen, Park, Xu, and Zhang (2022), He, Liang, Yang, and Wei (2023), Zhang et al. (2022) for a complete overview.

**Lemma 1.** *Let $\Delta\tau = \bar{\tau} - \underline{\tau}$. The quadratic inequality (1) holds if one of the following set of conditions (i.e., **(i)** (Kim, 2011; Yang & Zhang, 2014), **(ii)** (Kim, 2016; Yang et al., 2019), **(iii)** (Zhang et al., 2020) or **(iv)** (Liu et al., 2023) is satisfied.*

*(i)* $\mathcal{P}(\underline{\tau}) < 0$, $\mathcal{P}(\bar{\tau}) < 0$ *and* $\Phi_2 \geq 0$.
*(ii)* $\mathcal{P}(\underline{\tau}) < 0$, $\mathcal{P}(\bar{\tau}) < 0$ *and* $\mathcal{P}(\underline{\tau}) - \Delta\tau^2\Phi_2 < 0$.
*(iii)* $\exists\mu \in [0, 1]$ *such that* $\mathcal{P}(\underline{\tau}) < 0$, $\mathcal{P}(\bar{\tau}) < 0$, $\mathcal{P}(\underline{\tau}) - \mu^2\Delta\tau^2\Phi_2 < 0$ *and* $\mathcal{P}(\bar{\tau}) - (1-\mu)^2\Delta\tau^2\Phi_2 < 0$.
*(iv)* $\exists\mu \in [0, 1]$ *such that* $\mathcal{P}(\underline{\tau}) < 0$, $\mathcal{P}(\bar{\tau}) < 0$, $\mathcal{P}(\underline{\tau}) + \mathcal{P}(\underline{\tau} + \mu\Delta\tau) - \mu^2\Delta\tau^2\Phi_2 < 0$ *and* $\mathcal{P}(\bar{\tau}) + \mathcal{P}(\underline{\tau} + \mu\Delta\tau) - (1-\mu)^2\Delta\tau^2\Phi_2 < 0$.

In Lemma 1, conditions **(i)** require convexity (in $\tau$) of the scalar polynomial $\xi^\top\mathcal{P}(\tau)\xi$, $\forall\xi \in \mathbb{R}^n \setminus \{0\}$, often overly conservative. Therefore, relaxed conditions **(ii)** were introduced to accommodate convex or concave polynomials. To further relax these, conditions **(iii)** and **(iv)** incorporate a parameter $\mu$ that must be tuned within $[0, 1]$ or treated as a decision variable, leading to bilinear matrix inequality (BMI) constraints, making the application for computationally complex control problems harder. It is important to note that these geometrical conditions

☆ This work was partially supported by Centro Federal de Educação Tecnológica de Minas Gerais (CEFET-MG), and Brazilian Agencies FAPEMIG (APQ-02100-22), CNPq, Brazil (311891/2022-5 and 408411/2023-6), and Project Stic-Amsud/CAPES NetConHybSDP, code 22-STIC-09. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Bin Zhou under the direction of Editor Florian Dorfler.

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https://doi.org/10.1016/j.automatica.2024.112077

are only sufficient. Hence, improvements have been recently proposed considering partitioning approaches (Chen et al., 2019; He et al., 2023; Liu et al., 2023), summarized in the following lemma, which consists in splitting the interval of the definition of $\tau$ into $N \in \mathbb{N}^*$ equally spaced partitions, i.e., $[\underline{\tau}, \bar{\tau}] = \cup_{i=1}^{N} [\underline{\tau}_i, \bar{\tau}_i]$ with:

$$\underline{\tau}_1 = \underline{\tau},\ \bar{\tau}_i = \underline{\tau}_{i+1} = \underline{\tau} + i\frac{\Delta\tau}{N} \text{ and } \Delta\tau = \bar{\tau} - \underline{\tau} \tag{2}$$

In the sequel, $\mathbb{I}_N$ denotes the set of positive integers from 0 to $N$ and $\mathbb{I}_N^* = \mathbb{I}_N \setminus \{0\}$.

**Lemma 2.** *For a given $N \in \mathbb{N}^*$, the quadratic inequality (1) holds if one of the following set of conditions (i.e., **(i)** (Chen et al., 2019), **(ii)** (He et al., 2023) or **(iii)** (Liu et al., 2023)) is satisfied, $\forall i \in \mathbb{I}_N^*$.*

***(i)*** *$\mathcal{P}(\underline{\tau}) < 0$, $\mathcal{P}(\bar{\tau}_i) < 0$, and $\mathcal{P}(\bar{\tau}_{i-1}) - \frac{\Delta\tau^2}{N^2}\Phi_2 < 0$.*

***(ii)*** *$\mathcal{P}(\underline{\tau}) < 0$, $\mathcal{P}(\bar{\tau}) < 0$ and $\mathcal{P}\left(\bar{\tau}_i - \frac{\Delta\tau}{2N}\right) - \frac{\Delta\tau^2}{4N^2}\Phi_2 < 0$.*

***(iii)*** *$\mathcal{P}(\underline{\tau}) < 0$, $\mathcal{P}(\bar{\tau}) < 0$, and $\mathcal{P}(\bar{\tau}_{i-1}) + \mathcal{P}(\bar{\tau}_i) - \frac{\Delta\tau^2}{N^2}\Phi_2 < 0$.*

This partitioning approaches offer recursive relaxations and might approximate necessary and sufficient conditions for large $N$, when $\Phi_0$, $\Phi_1$, and $\Phi_2$ are constant matrices. However, they suffer from the following fact (illustrated by a numerical example given in Section 4.1).

**Fact 1.** *With the above considered partitioning approaches, there is no guarantee that, if a solution exists for a given $N$, then there is a solution for $N+1$.*

Therefore, these partitioning-based conditions will be said to be non-monotonic convergent (non-MC) as $N$ expands, which hinders the search for a large enough $N$ to closely approximate the optimal solution.

In addition to extensive research efforts to relax geometrical conditions, recent studies established milestone results through the application of the generalized S-procedure (Chen, Park, & Xu, 2022; de Oliveira & Souza, 2020; Kim, 2021; Park & Park, 2020; Rouamel, Oliveira, Bourahala, Guelton, & Motchon, 2023), summarized in the next lemma.

**Lemma 3** (*Kim, 2021; Park & Park, 2020; Rouamel et al., 2023*)**.** *The quadratic inequality (1) holds if there exists $0 \leq M + M^\top \in \mathbb{R}^{n\times n}$ such that:*

$$\begin{bmatrix} \mathcal{P}(\underline{\tau}) & \frac{1}{2}\Phi_1 + \underline{\tau}\Phi_2 + \Delta\tau M \\ \star & \Phi_2 - M - M^\top \end{bmatrix} < 0.$$

Lemma 3 has the advantage of being necessary and sufficient when $\Phi_0$, $\Phi_1$, and $\Phi_2$ are constant matrices, without the need for partitioning approaches. However, it requires the introduction of additional decision variables. As claimed by many authors of recent studies (see, e.g. Chen and Li (2021), Chen, Park, Xu, and Zhang (2022), He et al. (2023), Zeng, Lin, He, Teo, and Wang (2020), Zhang et al. (2022)), this can be highly demanding and impose significant computational burden, particularly for large matrix-valued inequalities (1) with numerous decision variables, as commonly involved in the time-delay systems framework (e.g, when applying Jensen, Wirtinger or Bessel–Legendre inequalities (Kim, 2021; Liu, Seuret, & Xia, 2017; Lopes, Guelton, Arcese, & Leite, 2021; Rouamel et al., 2023; Seuret & Gouaisbaut, 2015)).

**Fact 2.** *If (1) involves uncertain parameter-dependent matrices $\Phi_0(\rho)$, $\Phi_1(\rho)$, and $\Phi_2(\rho)$, where $\rho \in \mathbb{R}^p$ with bounded entries $\rho_i \in [\underline{\rho}_i, \bar{\rho}_i]$, $\forall i \in \mathbb{I}_p^*$, then the conditions in Lemmas 2 and 3 are no longer necessary, but only sufficient.*

Furthermore, to reduce the number of decision variables (i.e., to lower computational complexity) compared to Lemma 3 (Kim, 2021; Park & Park, 2020; Rouamel et al., 2023), matrix injection approaches were used in Xie, Chen, Jin, Zhang, and He (2023, Lemma 4). Nonetheless, these conditions (not reproduced here for space reasons) also suffer from Fact 2.

According to Facts 1 and 2, it appears that there is still room for improvement. Therefore, the problem statement of this paper can be summarized as follows.

**Problem statement** *Provide relaxed negative definiteness conditions for (1), which are (when possible) monotonic convergent (MC) as $N$ (or any other recursive parameter) expands to an optimal solution (i.e., ensure that if a solution exists for $N$, then it also exists for $N+1$).*

MC conditions might help to closely approximate optimal solutions when there is no significant improvement for a sufficiently large $N$. This paper addresses this problem by introducing two previously unexplored novel approaches. The first considers rewriting $\mathcal{P}(\tau)$ in (1) as a homogeneous polynomial matrix inequality (see e.g., Oliveira and Peres (2005)), then applying recursive relaxations from the application of Young's inequality or Polya's Theorem (Hardy, Littlewood, & Pólya, 1952). The second alternative consists in rewriting $\mathcal{P}(\tau)$ in (1) as a Bézier curve (Farin, 2002), then applying recursive relaxations from the application of de Casteljau's algorithm (Farin, 2002). To the best of the authors' knowledge, investigating the negative definiteness of quadratic functions (1) from homogeneous polynomials or Bézier curve approaches has never been done in the literature. Bézier curves, independently introduced by Paul de Casteljau and Pierre Bézier, respectively, in 1959 and 1960, have numerous applications in image synthesis and font rendering (Farin, 2002). Except in a very recent work dealing with Takagi–Sugeno model-based control (Bainier, Marx, & Ponsart, 2024), their use within linear matrix inequality (LMI) relaxation has not been explored so far.

The remainder of this paper is structured as follows. Section 2 presents the proposed relaxed non-MC and MC conditions for quadratic polynomials (1), derived in homogeneous polynomials or Bézier curve frameworks. Section 3 highlights the case where (1) involves uncertain parameter-dependent matrices (see Fact 2), by considering the stability analysis of closed-loop sampled-data linear parameter-varying (LPV) systems, via a looped Lyapunov–Krasovskii functional (LKF) inspired by Seuret (2012), which also demonstrates the application of this proposal in time-varying delay systems. Section 4 offers two numerical examples to illustrate the efficacy of our proposal compared to previous prominent conditions in the literature. The first example considers simple scalar polynomials, while the second showcases the efficiency of our proposal on an LPV sampled-data control benchmark. Finally, the conclusion provides perspectives on this work.

## 2. New negative definiteness conditions

This section presents new non-MC and MC conditions for (1) from the perspective of homogeneous polynomial and the Bézier curve approaches and discusses their computational complexity w.r.t. previous results.

### 2.1. Homogeneous polynomial approach

For any given $N \in \mathbb{N}^*$, consider equally spaced partitions of $[\underline{\tau}, \bar{\tau}]$ defined in (2). $\forall \tau \in [\underline{\tau}_i, \bar{\tau}_i]$, let $\alpha_{1i} = \frac{(\tau - \underline{\tau}_i)N}{\Delta\tau} \in [0,1]$ and $\alpha_{2i} = \frac{(\bar{\tau}_i - \tau)N}{\Delta\tau} \in [0,1]$, where $\alpha_{1i} + \alpha_{2i} = 1$. Hence, $\tau = \alpha_{1i}\bar{\tau}_i + \alpha_{2i}\underline{\tau}_i$ and (1) can be rewritten as:

$$(\alpha_{1i}\bar{\tau}_i + \alpha_{2i}\underline{\tau}_i)^2\Phi_2 + (\alpha_{1i}\bar{\tau}_i + \alpha_{2i}\underline{\tau}_i)\Phi_1 + \Phi_0 < 0,\ \forall i \in \mathbb{I}_N^*. \tag{3}$$

By multiplying the second and the third terms in (3) respectively by $\alpha_{1i}+\alpha_{2i}$ and $(\alpha_{1i}+\alpha_{2i})^2$, it is equivalent to the second-order homogeneous polynomial inequality:

$$\alpha_{1i}^2\mathcal{P}(\bar{\tau}_i)+\alpha_{1i}\alpha_{2i}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\alpha_{2i}^2\mathcal{P}(\underline{\tau}_i)<0,\ \forall i\in\mathbb{I}_N^*, \tag{4}$$

with $\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)=2\underline{\tau}_i\bar{\tau}_i\Phi_2+(\underline{\tau}_i+\bar{\tau}_i)\Phi_1+2\Phi_0$.

Considering the well-known Young's inequality, the next theorem provides relaxed conditions for (4), and so (1).

**Theorem 1.** *For a given $N\in\mathbb{N}^*$, the quadratic inequality (1) holds if $\mathcal{P}(\bar{\tau}_i)<0$, $\forall i\in\mathbb{I}_N$, $2\mathcal{P}(\bar{\tau}_i)+\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)<0$ and $2\mathcal{P}(\underline{\tau}_i)+\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)<0$, $\forall i\in\mathbb{I}_N^*$.*

**Proof.** When $\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)<0$, the inequalities $\mathcal{P}(\bar{\tau}_i)<0$, $\forall i\in\mathbb{I}_N$, provide the negativeness of (4) (and so (1)). In the other case, when $\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)\geq 0$, from the Young's inequality we have $0\leq\alpha_{1i}\alpha_{2i}\leq\frac{1}{2}(\alpha_{1i}^2+\alpha_{2i}^2)$. Hence, (4) holds if $\alpha_{1i}^2(\mathcal{P}(\bar{\tau}_i)+\frac{1}{2}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i))+\alpha_{2i}^2(\mathcal{P}(\underline{\tau}_i)+\frac{1}{2}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i))<0$, i.e., if the conditions in Theorem 1 hold. □

Theorem 1 also suffers from Fact 1 and thus provides non-MC conditions for (1) as $N$ expands (see the numerical example given in Section 4.1). Furthermore, applying Young's inequality makes it only sufficient. The following Theorem provides an alternative based on the application of Polya's Theorem (Hardy et al., 1952).

**Theorem 2.** *For a given $N\in\mathbb{N}^*$ and for a sufficiently large $d\in\mathbb{N}$, the quadratic inequality (1) holds if, $\forall i\in\mathbb{I}_N^*$, the monomial coefficients of*

$$(\alpha_{1i}^2\mathcal{P}(\bar{\tau}_i)+\alpha_{1i}\alpha_{2i}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\alpha_{2i}^2\mathcal{P}(\underline{\tau}_i))(\alpha_{1i}+\alpha_{2i})^d \tag{5}$$

*are all strictly negative.*

**Proof.** Straightforward by applying Polya's Theorem (Hardy et al., 1952, Theorem 56) on (4), since $(\alpha_{1i}+\alpha_{2i})^d=1$, $\forall d\in\mathbb{N}$. □

An important milestone of Polya's Theorem is, conversely, if all the monomial coefficients of (5) are negative for a large enough $d$, then it is necessary and sufficient to prove that (4) is negative (Oliveira & Peres, 2005). Hence, the conditions in Theorem 2 are MC and tend to an optimal solution (when it holds) as $d$ expands (with fixed $N$).

**Remark 1.** To implement the conditions of Theorem 2 into typical convex optimization tools (e.g., YALMIP (Lofberg, 2004) in MATLAB), one can expand (5) on the canonical basis of a homogeneous polynomial of order $d+2$, like in Oliveira and Peres (2005). This process is revisited for clarity in Appendix, where it is shown that, if the inequalities (A.4), (A.5), and (A.6) hold, then (5) is negative-definite (so (1) is verified).

### 2.2. Bézier curve approach

Let $\alpha=\frac{\tau-\underline{\tau}}{\Delta\tau}\in[0,1]$, inequality (1) can be rewritten as:

$$\mathrm{P}(\alpha)=\mathrm{B}_2^2(\alpha)\mathrm{P}_2^0+\mathrm{B}_1^2(\alpha)\mathrm{P}_1^0+\mathrm{B}_0^2(\alpha)\mathrm{P}_0^0<0, \tag{6}$$

where $\mathrm{B}_2^2(\alpha)=\alpha^2$, $\mathrm{B}_1^2(\alpha)=2\alpha(1-\alpha)$, and $\mathrm{B}_0^2(\alpha)=(1-\alpha)^2$ are second-order Bernstein polynomials in $\alpha$, and $\mathrm{P}_2^0=\mathcal{P}(\bar{\tau})$, $\mathrm{P}_1^0=\frac{1}{2}\mathcal{T}(\underline{\tau},\bar{\tau})$ and $\mathrm{P}_0^0=\mathcal{P}(\underline{\tau})$ provide the control points of a quadratic Bézier curve (see Farin (2002) for more details) by congruence with any $\xi\in\mathbb{R}^n\setminus\{0\}$.

**Remark 2.** The quadratic Bezier representation (6) of (1) is equivalent to its homogeneous polynomial representation (4) with $N=1$. The only difference remains on the weight of the monomials, i.e., $\mathrm{B}_1^2(\alpha)=2\alpha(1-\alpha)$ in (6) and $\alpha_1\alpha_2$ in (4). This slight difference provides an important feature for Bernstein polynomials, which belong to the unitary simplex while the monomials of homogeneous polynomials (4) do not (i.e., $\forall\alpha_1\in[0,1]$ and $\alpha_2=1-\alpha_1$, $\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2\leq 1$ but not always equal to 1).

Bezier curves always remain within the convex hull formed by their control points, leading to straightforward conditions for the negative definiteness of (1) if all control points in (6) are negative (i.e., $\mathrm{P}_2^0<0$, $\mathrm{P}_1^0<0$, and $\mathrm{P}_0^0<0$), which are equivalent to the conditions of Theorem 2 with $N=1$ and $d=0$. Furthermore, a single Bezier curve can be split into two connected curves at a common control point using de Casteljau's algorithm and recursively divided into $2^{\mathcal{N}}$ curves, with tighter control points approaching the original curve (Farin, 2002). Leveraging these properties, the next theorem provides new relaxed conditions for (1).

**Theorem 3.** *For a given $\mathcal{N}\in\mathbb{N}^*$, the quadratic inequality (1) holds if, $\forall k\in\mathbb{I}_{2^{(\mathcal{N}-1)}-1}$:*

$$(e_q^\top\otimes I)\mathbb{P}_k^{\mathcal{N}}<0,\ \forall q\in\mathbb{I}_5^*, \tag{7}$$

*where $\otimes$ denotes the Kronecker product, $\mathbb{P}_k^{\mathcal{N}}=(\tilde{D}\prod_{j=0}^{n_b}D_{k_j}\otimes I)\mathbb{P}_0$, where $(k_{n_b}\ldots k_j\ldots k_0)_2$ is the base-2 decomposition of the integers $k$ (so we have $k_j\in\mathbb{I}_1$), $e_q$ ($\forall q\in\mathbb{I}_5^*$) denotes the standard Euclidean basis vectors of $\mathbb{R}^5$, $\mathbb{P}_0^\top=\begin{bmatrix}\mathrm{P}_0^{0\top} & \mathrm{P}_1^{0\top} & \mathrm{P}_2^{0\top}\end{bmatrix}$, and:*

$$D_0=\begin{bmatrix}1&0&0\\ \frac{1}{2}&\frac{1}{2}&0\\ \frac{1}{4}&\frac{1}{2}&\frac{1}{4}\end{bmatrix},\ D_1=\begin{bmatrix}\frac{1}{4}&\frac{1}{2}&\frac{1}{4}\\ 0&\frac{1}{2}&\frac{1}{2}\\ 0&0&1\end{bmatrix},\ \tilde{D}=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{4}&0&0\\ 0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0\\ 0&0&\frac{1}{4}&\frac{1}{2}&1\end{bmatrix}^\top.$$

**Proof.** Applying de Casteljau's algorithm to the quadratic Bézier representation (6), we can divide the Bézier segments into half points without loss of generality for numerical convenience (Farin, 2002). For the first iteration ($\mathcal{N}=1$), it yields the definition of new control points $\mathrm{P}_0^1$, $\mathrm{P}_1^1$ and $\mathrm{P}_2^1$ for the subdivided Bézier curve on the left, and $\mathrm{P}_2^1$, $\mathrm{P}_3^1$ and $\mathrm{P}_4^1$ for the right one, such that $\begin{bmatrix}\mathrm{P}_0^{1\top} & \mathrm{P}_1^{1\top} & \mathrm{P}_2^{1\top}\end{bmatrix}=((D_0\otimes I)\mathbb{P}_0)^\top$ and $\begin{bmatrix}\mathrm{P}_2^{1\top} & \mathrm{P}_3^{1\top} & \mathrm{P}_4^{1\top}\end{bmatrix}=((D_1\otimes I)\mathbb{P}_0)^\top$, where $D_0$ and $D_1$ are de Casteljau's matrices for quadratic Bézier curves (Farin, 2002). Since the subdivided left and right Bézier representations share the same control point $\mathrm{P}_2^1$, they can be rearranged as $\mathbb{P}_0^{1\top}=\begin{bmatrix}\mathrm{P}_0^{1\top} & \mathrm{P}_1^{1\top} & \mathrm{P}_2^{1\top} & \mathrm{P}_3^{1\top} & \mathrm{P}_4^{1\top}\end{bmatrix}=((\tilde{D}\otimes I)\mathbb{P}_0)^\top$.

Then, for the second iteration ($\mathcal{N}=2$), we apply the same procedure on each previously obtained left and right subdivision to get $\mathbb{P}_0^2=(\tilde{D}D_0\otimes I)\mathbb{P}_0$ and $\mathbb{P}_1^2=(\tilde{D}D_1\otimes I)\mathbb{P}_0$.

Again, for the third iteration ($\mathcal{N}=3$), we obtain $\mathbb{P}_0^3=(\tilde{D}D_0D_0\otimes I)\mathbb{P}_0$, $\mathbb{P}_1^3=(\tilde{D}D_0D_1\otimes I)\mathbb{P}_0$, $\mathbb{P}_2^3=(\tilde{D}D_1D_0\otimes I)\mathbb{P}_0$ and $\mathbb{P}_3^3=(\tilde{D}D_1D_1\otimes I)\mathbb{P}_0$.

Finally, proceeding by recurrence to the $\mathcal{N}$-th iteration, we get the control points stored element-wise in $\mathbb{P}_k^{\mathcal{N}}$ (for all $k\in\mathbb{I}_{2^{(\mathcal{N}-1)}-1}$), which negativeness are guaranteed if the inequalities in (7) are satisfied. □

**Remark 3.** In Theorem 3, $\mathcal{N}$ represents the number of iterations in de Casteljau's algorithm, resulting in $2^{\mathcal{N}}$ equally spaced partitions of (6) across the parameter range $\alpha\in[0,1]$. This contrasts with the $N$ partitioning approach over the parameter range $\tau\in[\underline{\tau},\bar{\tau}]$ considered in Lemma 2 and Theorems 1, 2. Moreover, since the obtained de Casteljau's subdivisions are equivalent, all together, to the original Bézier representation (6), the conditions of Theorem 3 are MC as $\mathcal{N}$ expands.

**Remark 4.** Extensions of Theorems 2 and 3 to deal with higher-order polynomial functions than (1) are straightforward by considering their higher-order homogeneous polynomial or Bézier

**Table 1**
Number of matrix inequality rows, number of additional decision variables ($a_{dv}$) and types of negative definiteness conditions in Lemmas 1–3 and Theorems 1–3.

| Methods: | Rows | $a_{dv}$ | Type |
|---|---|---|---|
| Lemma 1, (i) (Kim, 2011; Yang & Zhang, 2014), (ii) (Kim, 2016; Yang et al., 2019) | $3n$ | – | S |
| Lemma 1, (iii) (Zhang et al., 2020), (iv) (Liu et al., 2023) | $4n$ | 1 | S |
| Lemma 2, (i) (Chen et al., 2019) | $(2N+1)n$ | – | non-MC |
| Lemma 2, (ii) (He et al., 2023), (iii) (Liu et al., 2023) | $(N+2)n$ | – | non-MC |
| Lemma 3 (Kim, 2021; Park & Park, 2020; Rouamel et al., 2023) | $3n$ | $n^2$ | S |
| Theorem 1 | $(3N+1)n$ | – | non-MC |
| Theorem 2 | $((2+d)N+1)n$ | – | MC |
| Theorem 3 | $5(2^{\mathcal{N}-1})n$ | – | MC |

curve equivalences. However, such extensions are left out of this brief for space reasons.

### 2.3. Computational complexity

The number of rows of the resulting matrix inequalities relative to $n$, $N$, $d$, and $\mathcal{N}$ are listed in Table 1 for the conditions outlined in Lemmas 1, 2, and 3, Theorems 1, 2, and 3. Lemma 1 presents simpler sufficient (S) conditions. In contrast, complexity increases with other results, required for non-MC or MC conditions. Lemma 3 conditions have a smaller number of rows than non-MC or MC conditions but introduce $n^2$ additional decision variables ($a_{dv}$), which could increase the computational burden, particularly for control problems with numerous decision variables. This computational complexity of Lemma 3 can be reduced by considering the matrix injection approach as proposed in Xie et al. (2023). In such a case, the reduced number of additional variables will depend on the rank of $\Phi_2$. Also, due to Fact 2, Lemma 3, as well as the conditions in Xie et al. (2023) can be more conservative than non-MC or MC approaches for some control problems, as illustrated in Section 4.2 with the stability analysis of LPV sampled-data control systems. Theorems 2 and 3 may also be seen as computationally extensive. However, drawing such a conclusion would be a shortcut since we would also have to consider the effective $N$, $d$, or $\mathcal{N}$ to reach an optimal solution with the benefit of the MC characteristics, as demonstrated via numerical examples in Section 4. Furthermore, the overall complexity cannot be evaluated without considering the specificity of each control problem treated with the help of negativeness conditions for (1), as it also affects the total number of rows and decision variables. In this context, the specific control problem considered in the next section will show that Theorems 2 and 3 can provide less conservative results at the price of increasing computational complexity. Finally, note that the overall complexity factors can also differ depending on the solver used to cope with the convex optimization problems. For more information on this issue, the reader can refer to the detailed discussion given in Lee and Joo (2015, Remark 1), where it is highlighted that the choice of the most convenient solver can be made according to the balance between the number of decision variables and LMI rows.

## 3. Application to LPV sampled-data control

Consider the class of LPV systems given by:

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t), \tag{8}$$

where $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^m$ are the state and control vectors, $A(\rho(t)) = \sum_{i=1}^{\nu} \rho_i(t) A_i \in \mathbb{R}^{n_x \times n_x}$ and $B(\rho(t)) = \sum_{i=1}^{\nu} \rho_i(t) B_i \in \mathbb{R}^{n_x \times m}$ are polytopic matrices, parameterized by $\rho(t) = \begin{bmatrix} \rho_1(t) & \dots & \rho_\nu(t) \end{bmatrix} \in \mathbb{R}^\nu$, which belongs to the unitary simplex ($\sum_{i=1}^{\nu} \rho_i(t) = 1$, $\rho_i(t) \geq 0$).

We assume that the states are available at periodic sampling instants $t_k$, such that $t \in \cup_{k\in\mathbb{N}}[t_k, t_{k+1}) = [0, +\infty)$, with $t_{k+1} - t_k = \eta$. Moreover, the control signal at $t = t_k$ is kept constant by a zero-order holder until the next sample. We consider the sampled-data state feedback control law (Fridman, 2010; Seuret, 2012):

$$u(t) = Kx(t_k) = Kx(t - \tau(t)), \tag{9}$$

where $K \in \mathbb{R}^{m \times n_x}$ is a robust control gain matrix and, $\forall t \in [t_k, t_{k+1})$, $\tau(t) = t - t_k \in [0, \eta)$ with $\dot{\tau}(t) = 1$. Yields the closed-loop dynamics:

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))Kx(t - \tau(t)). \tag{10}$$

In this application, we assume that the robust control gain $K$ is known and $A(\rho(t)) + B(\rho(t))K$ is Hurwitz stable (Seuret, 2012). To provide the closed-loop stability conditions of the sampled data system, summarized in the next theorem, we follow a path similar to that of Seuret (2012), but with a slightly modified looped LKF given by:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \tag{11}$$

with $V_1(t) = x^\top(t)Px(t)$, $V_2(t) = (\eta - \tau(t))\Delta x^\top(t)S\Delta x(t)$, $V_3(t) = 2(\eta - \tau(t))\Delta x^\top(t)Qx(t - \tau(t))$, $V_4(t) = (\eta - \tau(t))\tau(t)\zeta^\top(t)U\zeta(t)$, $V_5(t) = (\eta - \tau(t))\int_{t-\tau(t)}^{t} \dot{x}^\top(s)R\dot{x}(s)ds$, where $\Delta x(t) = x(t) - x(t - \tau(t))$, $\zeta(t) = \text{col}\{x(t), x(t - \tau(t))\}$.

**Theorem 4.** *Let us denote $\mathcal{H}(M) = M + M^\top$, for any square matrix $M$. For given $\eta \geq 0$ and $K \in \mathbb{R}^{m \times n_x}$, the closed-loop LPV sampled-data system (10) is asymptotically stable, if there exist the matrices $0 < R = R^\top \in \mathbb{R}^{n_x \times n_x}$, $0 < P = P^\top \in \mathbb{R}^{n_x \times n_x}$, $S = S^\top \in \mathbb{R}^{n_x \times n_x}$, $U = U^\top \in \mathbb{R}^{2n_x \times 2n_x}$, $Q \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{3n_x \times n_x}$, and $X \in \mathbb{R}^{3n_x \times n_x}$ satisfying:*

$$\tau^2(t)\Phi_2 + \tau(t)\Phi_1 + \Phi_0(\rho(t)) < 0, \ \forall \tau(t) \in [0, \eta] \tag{12}$$

*with $\Phi_2 = -\mathcal{H}(E_1^\top U E_2)$, $\Phi_1 = \mathcal{H}((\varepsilon_2 - \varepsilon_1)^\top S\varepsilon_3 - \varepsilon_3^\top Q\varepsilon_2 + \eta E_1^\top U E_2) - 2E_1^\top U E_1 - \varepsilon_3^\top R\varepsilon_3 - LR^{-1}L^\top$, $\Phi_0 = \mathcal{H}(\varepsilon_3^\top(P\varepsilon_1 + \eta Q\varepsilon_2) + (\varepsilon_1 - \varepsilon_2)^\top(Q\varepsilon_2 + \eta S\varepsilon_3 - L^\top)) - (\varepsilon_1 - \varepsilon_2)^\top S(\varepsilon_1 - \varepsilon_2) + \eta(E_1^\top U E_1 + \varepsilon_3^\top R\varepsilon_3) + \mathcal{H}(X\mathcal{G}(\rho(t)))$, $\mathcal{G}(\rho(t)) = \begin{bmatrix} A(\rho(t)) & B(\rho(t))K & -I \end{bmatrix}$, $\varepsilon_1 = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$, $\varepsilon_2 = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, $\varepsilon_3 = \begin{bmatrix} 0 & 0 & I \end{bmatrix}$, $E_1^\top = \begin{bmatrix} \varepsilon_1^\top & \varepsilon_2^\top \end{bmatrix}$ and $E_2^\top = \begin{bmatrix} \varepsilon_3^\top & 0 \end{bmatrix}$.*

*Since $\tau(t) \in [0, \eta)$, and if $A(\rho(t)) + B(\rho(t))K$ is Hurwitz stable, the maximum allowed upper bound $\bar{\eta}$ can be obtained by linear search, i.e., by increasing $\eta \to \bar{\eta}$ until the quadratic inequality (12) remains feasible.*

**Proof.** Assume that $P = P^\top > 0$, then $V(t)$ is continuous and positive at every $t_k$ since $V_1(t_k) > 0$ and $V_i(t_k) = V_i(t_{k+1}) = 0$, $\forall i \in \{2, 3, 4, 5\}$. Therefore, the closed-loop system (10) is asymptotically stable if, $\forall t \in [t_k, t_{k+1})$:

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \dot{V}_5(t) < 0. \tag{13}$$

Let $\xi(t) = \text{col}\{x(t), x(t - \tau(t)), \dot{x}(t)\}$. The derivatives of each $V_i(t)$ are given by $\dot{V}_1(t) = 2\xi^\top(t)\varepsilon_1^\top P\varepsilon_3\xi(t)$, $\dot{V}_2(t) = -\xi^\top(t)(\varepsilon_1 - \varepsilon_2)^\top S(\varepsilon_1 - \varepsilon_2)\xi(t) + 2(\eta - \tau(t))\xi^\top(t)(\varepsilon_1 - \varepsilon_2)^\top S\varepsilon_3\xi(t)$, $\dot{V}_3(t) = -2\xi^\top(t)(\varepsilon_1 - \varepsilon_2)^\top Q\varepsilon_2\xi(t) + 2(\eta - \tau(t))\xi^\top(t)\varepsilon_3^\top Q\varepsilon_2\xi(t)$, $\dot{V}_4(t) = (\eta - 2\tau(t))\xi^\top(t)E_1^\top U E_1\xi(t) + 2(\eta - \tau(t))\tau(t)\xi^\top(t)E_1^\top U E_2\xi(t)$, and $\dot{V}_5(t) = (\eta - \tau(t))\xi^\top(t)\varepsilon_3^\top R\varepsilon_3\xi(t) - \int_{t-\tau(t)}^{t} \dot{x}^\top(s)R\dot{x}(s)ds$.

Assuming $R = R^\top > 0$, we apply Jensen's inequality (see Theorem 4.1 in Briat (2011)), so that $-\int_{t-\tau(t)}^{t}\dot{x}^\top(s)R\dot{x}(s)ds \leq -2\xi^\top(t)L(\varepsilon_1-\varepsilon_2)\xi(t)+\tau(t)\xi^\top(t)LR^{-1}L^\top\xi(t)$. Then, by rewriting (10) as $\mathcal{G}(\rho(t))\xi(t)=0$, we apply Finsler's Lemma (Skelton, Iwasaki, & Grigoriadis, 1998), leading to (12) with the introduction of the slack decision matrix $X \in \mathbb{R}^{3n\times n}$. □

**Remark 5.** Condition (12) is a quadratic constraint (1) with a time-varying parameter-dependent $\Phi_0(\rho(t))$. To address the conditions outlined in Theorem 4, we can apply any conditions of Lemmas 1, 2, Theorems 1, 2 or 3, for all vertices of $(A(\rho(t)), B(\rho(t))) \in \text{co}\{(A_i, B_i)_{i\in\mathbb{I}_{\nu*}}\}$ (with extra Schur complements to handle the term $LR^{-1}L^\top$). See also Remark 1 and Appendix for more details on how to apply Theorem 2 to solve (12).

**Remark 6.** The looped-LKF (11) is standard, lacking additional terms such as multiple integrals as often seen in previous studies to reduce conservatism (e.g., Lee, Lee, Park, and Kwon (2018), Lee and Park (2017)). The proof of Theorem 4 is also simple, employing Jensen's inequality (see Theorem 4.1 in Briat (2011)), while other studies typically use tighter bounding techniques such as Wirtinger (Seuret & Gouaisbaut, 2013), Bessel–Legendre inequalities (Gao, Liu, He, Wu, & Navaratne, 2020; Lee & Park, 2017), or combinations, as relaxation techniques. In fact, in most previous related studies that provide new negative definiteness conditions for (1), comparisons are made with other studies employing different relaxation techniques (see, e.g., Chen et al. (2019), Chen, Park, Xu, and Zhang (2022), He et al. (2023), Kim (2011, 2016), Liu et al. (2023), Yang et al. (2019), Yang and Zhang (2014), Zhang et al. (2022, 2020)). Hence, in these previous studies, it is difficult to distinguish conservative improvements due only to the proposed negative definiteness conditions for (1). In this paper, with the same framework of the stability analysis of LPV sampled data systems, our objective is to fairly compare the different approaches presented above to address (1). The numerical results will be presented in the next section.

## 4. Illustrative benchmark examples

In this section, the example of a quadratic scalar inequality (1) is first considered, then another one dealing with the stability analysis of an LPV sampled-data control plant. MATLAB is used for computation and LMI conditions are solved using YALMIP (Lofberg, 2004) and MOSEK (Andersen & Andersen, 2000). All numerical tests are performed on an MSI GS60 2PL laptop equipped with a 2.5 GHz Intel Core i7-4710HQ processor and 12 GB of RAM, running Windows 10.

### 4.1. Numerical example of a scalar quadratic inequality

Let us consider the scalar quadratic inequality:

$$\mathcal{P}(\tau) = \tau^2 10a + \tau 10 + b - a < 0, \quad \tau \in [0,1], \tag{14}$$

where $a$ and $b$ are real parameters used to evaluate and compare the feasibility fields of the different negative definiteness conditions. Because (14) is a scalar, we can assert that, for $\tau \in [0,1]$, $\mathcal{P}(\tau) < 0$ if $(a,b) \in \mathcal{S}$, with $\mathcal{S}$ defined by:

$$\mathcal{S} = \left\{ (a,b)\in\mathbb{R}^2 \;\middle|\; \begin{array}{l} \mathcal{P}(0) = b-a<0, \\ \mathcal{P}(1) = 9a+b+10<0, \\ b-a-\frac{5}{2a}<0, \text{ if } -\frac{1}{2a}\in[0,1]. \end{array} \right\} \tag{15}$$

This characterization of the domain $\mathcal{S}$ where (14) holds is used to compare the conservatism, in terms of feasibility fields, of Lemmas 1–3 and Theorems 1–3.

**Table 2**
Feasibility of the different negative definiteness conditions (w.r.t. $N$, $d$ or $\mathcal{N}$) with (14), $a = -0.66$ and $b = -4.46$.

| Methods/$N$ (or $\mathcal{N}$) : | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Lemma 2, (i) (Chen et al., 2019) | ∅ | ∅ | ∅ | ∅ | ∅ | ∅ |
| Lemma 2, (ii) (He et al., 2023) | ∅ | ∅ | ∅ | Feas | ∅ | ∅ |
| Lemma 2, (iii) (Liu et al., 2023) | ∅ | ∅ | ∅ | Feas | ∅ | ∅ |
| Theorem 1 | ∅ | Feas | ∅ | Feas | ∅ | Feas |
| Theorem 2, $d = 0$ | ∅ | ∅ | ∅ | Feas | ∅ | ∅ |
| Theorem 2, $d = 1$ | ∅ | ∅ | ∅ | Feas | ∅ | ∅ |
| Theorem 2, $d = 2$ | ∅ | ∅ | ∅ | Feas | ∅ | ∅ |
| Theorem 2, $d = 3$ | ∅ | ∅ | ∅ | Feas | Feas | Feas |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Theorem 2, $d = 9$ | ∅ | ∅ | ∅ | Feas | Feas | Feas |
| Theorem 2, $d = 10$ | ∅ | ∅ | Feas | Feas | Feas | Feas |
| Theorem 3 | ∅ | Feas | Feas | Feas | Feas | Feas |

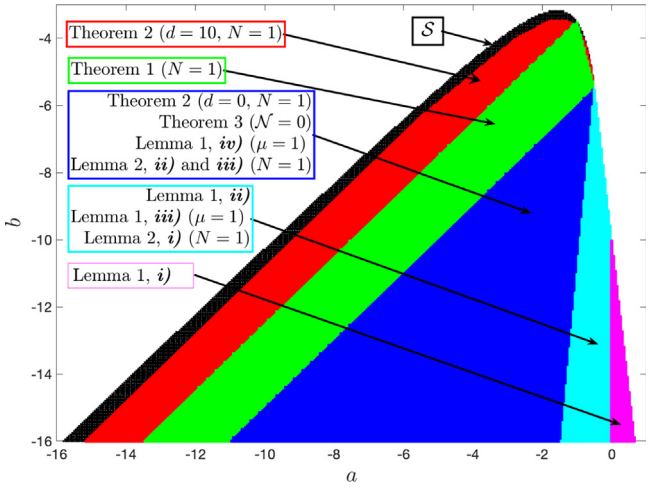


**Fig. 1.** Comparison of the feasibility fields obtained from the different approaches with no partitions ($N = 1$ or $\mathcal{N} = 0$).

First, consider the special case where $a = -0.66$ and $b = -4.46$. Table 2 shows feasible solutions for some $N$ under partitioning conditions in Lemma 2, Theorem 1, and Theorem 2 (with fixed $d$), while $N+1$ does not yield solutions (∅). This highlights the non-MC behavior of these conditions, and so confirms Fact 1. The MC behaviors of Theorem 2 (w.r.t. $d$ for fixed $N$) and Theorem 3 (w.r.t. $\mathcal{N}$) will be demonstrated in the sequel.

Then, for $a \in [-16, 1]$, $b \in [-16, -3]$, and without partition ($N = 1$), a comparison of the feasibility fields obtained from the different negative definiteness conditions is shown in Fig. 2. Lemma 1 provides the most conservative results, while Theorem 2 (with $d = 0$ and $N = 1$) matches Theorem 3 (with $\mathcal{N} = 0$) and the best geometric conditions in Lemmas 1 and 2. Furthermore, Theorem 1 encompasses the latter results, while Theorem 2 (with $d = 10$ and $N = 1$) provides a wider feasibility field. Hence, the present proposals consistently outperform or equalize the results obtained from Lemmas 1 and 2. Note that Lemma 3 allows recovery of $\mathcal{S}$ with this simple example. However, this may not always be true, due to Fact 2 (this will be emphasized in Section 4.2).

The last test compares the feasibility fields of Theorems 1–3 (for various $N$, $d$, and $\mathcal{N}$) with the most effective previous partitioning approaches. The results are shown in Fig. 1. In Fig. 1(a), as $N$ increases, Lemma 2 **(ii)**, ***iii)***, and Theorem 2 (with $d = 0$), exhibit identical results. Notably, expanding $N$ does not cover all areas consistently, as $N$+1 misses certain solutions found at lower $N$ values, again confirming Fact 1. In Fig. 1(b), Theorem 1 also demonstrates non-MC behavior, but the coverage area for a given

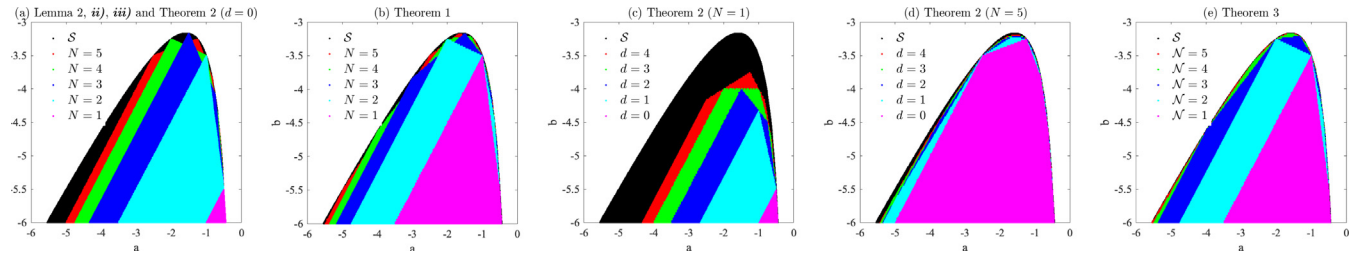


**Fig. 2.** Comparison of the feasibility fields satisfying (14), obtained from Lemmas 2 and Theorems 1–3, for various $N$, $d$, or $\mathcal{N}$.

$N$ consistently exceeds that of Fig. 1(a), indicating a reduction in conservatism. In contrast, Theorem 2 provides MC conditions by increasing $d$, as shown in Figs. 1(c) and (d) for fixed $N=1$ and 5, respectively, where the areas for $d+1$ always include those for $d$. Then, Theorem 2 with $N=5$ and $d=4$ closely approximates $\mathcal{S}$. Fig. 1(e) shows results from Theorem 3, increasing de Casteljau's iterations from $\mathcal{N}=1$ to 5 (with tighter approximations achieved for large $\mathcal{N}$), outperforming all the other tested results in terms of coverage area. Furthermore, the inclusion of the results for $\mathcal{N}+1$ in $\mathcal{N}$ confirms the MC nature of Theorem 3.

This first example demonstrates the effectiveness and advantages of the proposed conditions, highlighting Fact 1 alongside Lemma 2, Theorem 1, and Theorem 2 (w.r.t. $N$ for fixed $d$). Additionally, it reveals the MC behavior of Theorem 2 (w.r.t. $d$ for fixed $N$) and Theorem 3, addressing the drawback of non-MC conditions.

### 4.2. Benchmark of a sampled-data control system

In this example, we investigate the stability, for the largest allowable sampling interval $\bar{\eta}$, of a Hurwitz stable LPV sampled-data closed-loop dynamics (10) with the vertices:

$$A_i=\begin{bmatrix}\gamma_i & 0\\ 0 & \theta_i\end{bmatrix},\ B_iK=\begin{bmatrix}-1 & 0\\ -1 & \lambda_i\end{bmatrix}$$

where $\gamma_i=-1.8$ for $i\in\{1,5,2,6\}$, $\gamma_i=-2.2$ for $i\in\{3,7,4,8\}$, $\theta_i=-0.8$ for $i\in\{1,5,3,7\}$, $\theta_i=-1$ for $i\in\{2,6,4,8\}$, $\lambda_i=-0.8$ for $i\in\{1,2,3,4\}$ and $\lambda_i=-1.2$ for $i\in\{5,6,7,8\}$.

According to Remark 5, the quadratic conditions of Theorem 4 are addressed using Lemmas 1–3, Xie et al. (2023, Lemma 4), and Theorems 1–3. Table 3 shows the maximum allowable sampling period $\bar{\eta}$ achieved with various values of $N$, $d$, or $\mathcal{N}$. For each method presented in Table 3, the maximum sampling period was determined through a linear search on $\bar{\eta}$. The highest value obtained is $\bar{\eta}=1.950$. Among geometric conditions in Lemmas 1–3, only those in Lemma 2, **(ii)** and **(iii)**, reach this maximum with $N=29$ partitions (indicating conservatism in the other conditions). Moreover, despite the fact that Lemma 3 is reputed to be necessary and sufficient when $\Phi_0$, $\Phi_1$, and $\Phi_2$ are parameter independent, it is only sufficient for the uncertain parameter-dependent case studied in this example (as well as Xie et al. (2023, Lemma 4)). This confirms Fact 2 since the achieved values $\bar{\eta}=1.926$ with Lemma 3, and $\bar{\eta}=1.872$ with Xie et al. (2023, Lemma 4), are less than the largest value $\bar{\eta}=1.950$ obtained with Lemma 2, **(ii)** or **(iii)** or Theorems 1–3. In particular, Theorems 1–3 reach the largest value with fewer (or equal) partitions (or de Casteljau's iterations) than Lemma 2, **(ii)** and **(iii)**. The last column of Table 3 shows the average solver time, $t_{sol}$, which is the time required to obtain a solution using the respective numerical method. It is computed from 20 consecutive executions of the method listed in the first column of Table 3, with the corresponding $\bar{\eta}$. It is fair to say that the conditions in Theorems 1–3 are more computationally demanding, as revealed by comparing the solver time $t_{sol}$ required for each result that reaches the maximum $\bar{\eta}=1.950$ (with minimal $N$, polynomial degree $d$, or de Casteljau iterations $\mathcal{N}$). However, it is important to recall that the conditions of Lemmas 1 and 2 are non-MC, so reaching the maximum does not guarantee optimality. On the contrary, Theorems 2 and 3 being MC, ensure that $\bar{\eta}=1.950$ is the optimal solution for the quadratic constraints in Theorem 4. Furthermore, Theorems 2 and 3 offer more flexibility in adjusting $d$, $N$, or $\mathcal{N}$, to balance conservatism improvements and computational complexity. Hence, we can claim that the increase in computational complexity is the price to pay for relaxed MC conditions (unexplored in the previous literature) to provide guaranteed optimal solutions, which is an important feature in various control problems involving quadratic constraints.

## 5. Conclusion

This paper introduces new approaches to tackle the negative definiteness of quadratic functions, crucial in stability analysis and control design for time-varying delayed systems. These are based on rewriting the quadratic functions in (1) as homogeneous polynomials or Bézier curves. First, relaxed non-MC conditions are suggested using Young's inequality, succeeded by MC conditions via Polya's relaxations or de Casteljau's algorithm. Unlike prior non-MC partitioning methods, the proposed relaxed MC conditions, based on Polya's Theorem or de Casteljau's algorithm, ensure optimal solutions for sufficiently large $d$ (Polya's degree) or $\mathcal{N}$ (number of de Casteljau's iterations). This important feature for control problems is illustrated first through a scalar quadratic function example and then applied to an LPV sampled-data control benchmark. Both examples demonstrate the advantages, despite increased computational complexity, over the most effective negative definiteness conditions from previous literature. Future work aims to generalize these conditions for higher-order polynomial constraints and to explore more complex control problems, e.g., involving multiple polynomial-dependent parameters.

## Appendix. Equivalent inequality conditions to the statement of Theorem 2

Note that $(\alpha_{1i}+\alpha_{2i})^d=\sum_{k=0}^{d}\mathcal{C}_d^k\alpha_{1i}^{d-k}\alpha_{2i}^k$. Hence, similarly to the way borrow in Oliveira and Peres (2005), (5) can be rewritten as:

$$\begin{aligned}&\left(\alpha_{1i}^2\mathcal{P}(\bar{\tau}_i)+\alpha_{1i}\alpha_{2i}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\alpha_{2i}^2\mathcal{P}(\underline{\tau}_i)\right)\sum_{k=0}^{d}\mathcal{C}_d^k\alpha_{1i}^{d-k}\alpha_{2i}^k\\ &=\sum_{k=0}^{d}\mathcal{C}_d^k\left(\alpha_{1i}^{d-k+2}\alpha_{2i}^k\mathcal{P}(\bar{\tau}_i)+\alpha_{1i}^{d-k+1}\alpha_{2i}^{k+1}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\alpha_{1i}^{d-k}\alpha_{2i}^{k+2}\mathcal{P}(\underline{\tau}_i)\right).\end{aligned}\tag{A.1}$$

Let, for $j\in\mathbb{Z}$ and $|j|\le d\in\mathbb{N}$:

$$\mathscr{C}_d^j=\begin{cases}\mathcal{C}_d^j=\frac{d!}{j!(d-j)!}, & \text{if } j\ge 0,\\ 0, & \text{otherwise.}\end{cases}\tag{A.2}$$

**Table 3**
Maximum allowable sampling period ($\bar{\eta}$) and solver time ($t_{sol}$), obtained from Lemmas 2–3, Xie et al. (2023, Lemma 4), and Theorems 1–3 (with respect to $d$, $N$ or $\mathcal{N}$).

| Methods | $\bar{\eta}$ \| $N$ or $\mathcal{N}$ \| $t_{sol}(s)$ | |
|---|---|---|
| Lemma 1, (ii) (Kim, 2016; Yang et al., 2019) | 1.851 \| - \| 0.061 | – |
| Lemma 1, (iii) (Zhang et al., 2020), $\mu = 0$ | 1.865 \| - \| 0.079 | – |
| Lemma 1, (iii) (Zhang et al., 2020), $\mu = 1$ | 1.851 \| - \| 0.048 | – |
| Lemma 1, (iii) (Zhang et al., 2020), $\mu = .15$ | 1.881 \| - \| 0.067 | – |
| Lemma 1, (iv) (Liu et al., 2023), $\mu = 0$ | 1.867 \| - \| 0.059 | – |
| Lemma 1, (iv) (Liu et al., 2023), $\mu = 1$ | 1.867 \| - \| 0.056 | – |
| Lemma 1, (iv) (Liu et al., 2023), $\mu = 0.2$ | 1.888 \| - \| 0.061 | – |
| Lemma 2, (i) (Chen et al., 2019) | 1.851 \| 1 \| 0.043 | 1.921 \| 9 \| 0.385 |
| Lemma 2, (ii) (He et al., 2023) | 1.867 \| 1 \| 0.045 | **1.950** \| 29 \| 0.595 |
| Lemma 2 (iii) (Liu et al., 2023) | 1.867 \| 1 \| 0.047 | **1.950** \| 29 \| 0.646 |
| Lemma 3 (Kim, 2021; Park & Park, 2020; Rouamel et al., 2023) | 1.926 \| - \| 0.038 | – |
| (Xie et al., 2023, Lemma 4) | 1.872 \| - \| 0.024 | – |
| Theorem 1 | 1.885 \| 1 \| 0.057 | **1.950** \| 15 \| 1.281 |
| Theorem 2, $d = 0$ | 1.867 \| 1 \| 0.042 | **1.950** \| 29 \| 1.756 |
| Theorem 2, $d = 1$ | 1.870 \| 1 \| 0.061 | **1.950** \| 18 \| 1.534 |
| Theorem 2, $d = 5$ | 1.906 \| 1 \| 0.145 | **1.950** \| 11 \| 2.539 |
| Theorem 2, $d = 10$ | 1.925 \| 1 \| 0.286 | **1.950** \| 6 \| 2.591 |
| Theorem 3 | 1.885 \| 1 \| 0.086 | **1.950** \| 5 \| 2.098 |

We have $\mathscr{C}_d^k = \mathscr{C}_d^{d-k}$, $\mathscr{C}_d^0 = 1$. Thus, by symmetry, (A.1) can be rewritten as:

$$\sum_{k=0}^{\beta_d} \alpha_{1i}^{d+2-k}\alpha_{2i}^{k}(\mathscr{C}_d^k\mathcal{P}(\bar{\tau}_i) + \mathscr{C}_d^{k-1}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\mathscr{C}_d^{k-2}\mathcal{P}(\underline{\tau}_i))+$$
$$\mathbf{1}_{\mathbb{N}}\left(\tfrac{d}{2}\right)\alpha_{1i}^{\left\lfloor\frac{d}{2}\right\rfloor+1}\alpha_{2i}^{\left\lfloor\frac{d}{2}\right\rfloor+1}(\mathscr{C}_d^{\left\lfloor\frac{d}{2}\right\rfloor-1}\mathcal{P}(\bar{\tau}_i)+\mathscr{C}_d^{\left\lfloor\frac{d}{2}\right\rfloor}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\mathscr{C}_d^{\left\lfloor\frac{d}{2}\right\rfloor-1}\mathcal{P}(\underline{\tau}_i)) \tag{A.3}$$
$$+\sum_{k=0}^{\beta_d}\alpha_{1i}^{k}\alpha_{2i}^{d+2-k}(\mathscr{C}_d^{k-2}\mathcal{P}(\bar{\tau}_i) + \mathscr{C}_d^{k-1}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i) + \mathscr{C}_d^{k}\mathcal{P}(\underline{\tau}_i)).$$

where $\mathbf{1}_{\mathbb{N}}(\cdot)$ is the indicator function of the subset $\mathbb{N}$ of $\mathbb{R}$, i.e. $\mathbf{1}_{\mathbb{N}}\left(\frac{d}{2}\right) = 1$ if $d$ is even or 0, and 0 otherwise. $\lfloor\gamma\rfloor$ and $\lceil\gamma\rceil$ are the floor and ceiling of a real number $\gamma$, respectively.

Finally, if all the monomial coefficients in (A.3) are negative, then the inequality (1) holds, i.e., if, $\forall(i,k)\in\mathbb{I}_N\times\mathbb{I}_{\beta_d}$:

$$\mathscr{C}_d^k\,\mathcal{P}(\bar{\tau}_i)+\mathscr{C}_d^{k-1}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\mathscr{C}_d^{k-2}\mathcal{P}(\underline{\tau}_i)<0, \tag{A.4}$$
$$\mathscr{C}_d^{k-2}\,\mathcal{P}(\bar{\tau}_i)+\mathscr{C}_d^{k-1}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i)+\mathscr{C}_d^{k}\mathcal{P}(\underline{\tau}_i) < 0, \tag{A.5}$$

and, if $\mathbf{1}_{\mathbb{N}}\left(\frac{d}{2}\right) = 1$ ($d$ is even or 0):

$$\mathscr{C}_d^{\left\lfloor\frac{d}{2}\right\rfloor-1}\mathcal{P}(\bar{\tau}_i) + \mathscr{C}_d^{\left\lfloor\frac{d}{2}\right\rfloor}\mathcal{T}(\underline{\tau}_i,\bar{\tau}_i) + \mathscr{C}_d^{\left\lfloor\frac{d}{2}\right\rfloor-1}\mathcal{P}(\underline{\tau}_i)<0 \tag{A.6}$$

are satisfied with $\beta_d = \left\lfloor\frac{d-1}{2}\right\rfloor + 1$. Therefore, (A.4), (A.5) and (A.6) provide equivalent inequality constraints to the statement of Theorem 2 with the Polya's expansion of $\mathcal{P}(\tau)$ given in (5).

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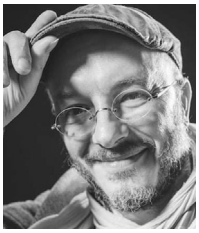

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