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Statistical indicators for the optimal prediction of failure times of stochastic reliability systems: A rational expectations-based approach

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ABSTRACT

We introduce a method to estimate the failure time of a class of weighted *k*-out-of-*n* systems using the idea of rational expectations, which to the best of our knowledge is a new approach, not found elsewhere in the existing literature. This paper explores the predictive power of several statistical indicators (variance, skewness, kurtosis, Gini coefficient, entropy) and shows how they perform differently as the system approaches global failure. The proposed method is shown to outperform a benchmark prediction model obtained without rational expectations, and our results offer a panoramic view of the predictive power of the statistical indicators under different assumptions about the initial weight distributions.

1. Introduction

The aim of this paper is to estimate the optimal failure time of a class of stochastic reliability systems – with a special focus on the weighted *k*-out-of-*n* systems, defined in [45] – by exploiting the potential of certain statistical measures using a rational expectations-based approach. Indeed, we assume that the number and importance (relative weights) of the components are fundamental to determining the failure of every single system so that the reliability of the overall system depends on the lifetimes of its components. We assign to each component a *weight*, to identify the relevance of its failure to the failure of the overall system.

Reliability theory and prediction of system failure times is of particular interest, both for the methodology and for practical applications. A nonparametric Bayesian approach is exploited in [43], where the authors studied the failure of systems with heterogeneous components using prior knowledge of the failure distributions of the components. Within the Bayesian framework, [9] and [44] evaluate the failure of reliability systems using mean and variance and applying a new method based on the estimation of the probability of failure. A first-passage problem is addressed by [23] through numerical experiments, to increase the reliability of a complex dynamical system using a stochastic averaging method. Li et al. [24] proposed a model to evaluate the reliability of multi-state deteriorating systems. In [14], many strategies are implemented to model the vulnerability of interdependent systems using numerical simulations, with the specific aim of demonstrating how cascading failures can be significantly decreased. In the field of

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fuzzy multi-state systems, [11] and [33] assessed the reliability of systems with uncertain state probabilities. Reliability problems for systems with multi-state components are addressed using fuzzy probability theory in [26]. Other interesting papers dealing with the reliability of particular types of systems are [20], [28], [29], and [48].

Among the existing approaches for predicting failure times of stochastic systems, we propose to analyze the failure patterns among the components of the given reliability systems. This analysis is done using a rational expectations-based approach – in other words, we provide an estimation conditioned on past statistical information collected during the evolution of the simulated systems.

A word of clarification is in order on what we mean by "rational expectations-based" in this context. Indeed, "rational expectations" is a term used in economics, where we may quote the original statement in [30], namely that it is *often necessary to make sensible predictions about the way expectations would change when either the amount of available information or the structure of the system is changed*. Starting from the general equilibrium, the insights of rational expectations can be deduced analytically. From the equation of supply and demand and the assumptions of rational expectations, we obtain $p_t^\star = \mathbb{E}[p_t|\mathbb{I}_{t-1}]$, where $p_t^\star$ is the price expectation, $\mathbb{E}$ is the expected value operator, p_t is the price at time t, and $\mathbb{I}_{t-1}$ is all the information available at time $t-1$ [30]. Muth's original idea finds applications in macroeconomics and financial markets theory (see the classical contributions by [7], [27], and [36]). Rational expectations are also exploited in complex economic and financial systems, described in [15] and [12]. The first paper studied bounded rational expectations assuming that the set of available information is not perfect, while the second investigated systems with heterogeneous interacting components by applying rational heuristics and demonstrating high prediction accuracy. In the context of agent-based models with heterogeneous households, rational expectations are addressed, for example, in [10], [21], [32], and [38]. The interested reader is also referred to the rich critical survey in [5].

In this paper we take the basic idea of rational expectations and apply it in the context of reliability theory. The main motivation behind the present study is to exploit the insights coming from rational expectations. The idea is that every new piece of information about a system can be used to give optimal predictions about the state of the system. Rational expectations are particularly useful in finance; in this case, the idea is that new information about a company's performance should immediately be reflected in a new stock price. The word "rational" in "rational expectations" refers to the assumption that, even though experts will in general attribute different price changes when they see the same new information, the fact that they are "rational" implies that there will be no consistent bias; thus, when the average (the expectation) is taken, the right new price will be obtained. Here, we exploit the same idea to give optimal predictions about failure of a system: each time a node fails, it fails with a specific configuration of the weights of the nodes. This is new information, and it can be used in different ways (via statistical indicators) to give a new optimal prediction of the failure time.

Here, using rational expectations, we can predict the failure times of a particular set of systems – the so-called *in-vivo* systems – on the basis of knowledge about the past experience of synthetically created stochastic systems – the so-called *catalog* systems. In this way, we obtain a Bayesian estimate of the failure time of the given systems. The approach is based on an extensive simulation procedure, where the *in-vivo* systems are compared with the *catalog*. This is similar to forecasting the failure times of stochastic systems by merging a mathematical model with scenario analyses (see, e.g., [6], [24], [46] and [47]). We assume that the dependence of the components is due to the structure of the given reliability systems, while the failure times of the system's components are independent random variables. The independence of the components' failure times is one of our key assumptions and refers to reliability systems in which the status of the generic component – i.e., whether the component has failed or not – does not influence the status of the other components. There exist contexts where there are strong interactions among the components of the system, so that the failure of one portion of it may generate cascade failures of other components. This important case is of particular interest in the literature on reliability systems because it allows the estimation of the failure time of the system to be more complex (see, e.g., [8] and references therein). This would go well beyond the scope of the present paper.

Here, we introduce a new methodology that takes its cue from the approach introduced in [2] and [40], based on a method for the prediction of earthquakes proposed in [18] and [39] and referred to in the literature as reverse tracing of precursors (RTP). We estimate the (average) failure times of the *in-vivo* systems using information collected in the *catalog* systems. The information concerns all the component weights at each time before the system failure – with weights evolving as the individual components fail. Indeed, as explained in detail below, we introduce a *failed component allocation rule*, which states that the weight of the failed components is redistributed across the weights of the still active components. We store the component weights and the final failure time in the information set. It is worth noticing that, unlike Andersen and Sornette [40], we insert an interaction into our systems so that the failure of a component affects the rest of the still-active system.

To proceed, we aggregate the component weights through several statistical indicators. In so doing, we gain in computational tractability of the Bayesian estimation procedure, because the aggregation of the weights allows us to reduce the number of *catalog* systems required to constitute a reliable information set. Moreover, we are also able to establish the failure-time forecasting power of the individual statistical indicators, according to existing literature. The selected statistical indicators are: variance, skewness, kurtosis, Gini coefficient, and Shannon entropy. They are basically the most commonly used indicators, but one can use other indicators to provide insights into a system's reliability. In this respect, this paper is a first step toward exploring this aspect of the problem, and the search for and analysis of other statistical indicators to predict the failure of reliability system is already on our research agenda. We pay attention to the dependence of our results on the initial distribution of the weights. To this end, we test five paradigmatic cases: the uniform distribution and four types of Beta distribution, assigning values to the parameters to obtain cases of symmetry and asymmetry. In so doing, we interpret the findings on the basis of the reference literature (see e.g., [4], [13], [25], [37] and [42]).

This paper is thus based on a fusion of the rational expectations and reliability theory frameworks. Our recent paper [3] is quite similar in this respect. Indeed, it describes a rational expectations-based perspective for estimating the failure time of weighted k-out-of-n reliability systems. Moreover, we also proceed here by creating a synthetic information set – the *catalog* – as in [3]. However, we

extend and complement [3] in some important ways. First, we avoid the explicit introduction of time in the forecasting exercises; by doing this, we are able to identify the predictive power of the statistical indicators used to build the information set. Here, we note that time plays a relevant role in our study, in that it governs the consecutive failures of the system components (see the conclusions for details). Second, we provide complete information on how the levels of prediction errors vary according to the sizes of the statistical indicators. Third, we can relate the functioning of the system (active or failed) and its failure probability to the value of the statistic – hence, leading to a sort of ranking of the failure risk on the basis of information available about the component weights. This is very important for forecasting; some of the themes covered here are barely discussed in the reference literature, in particular for certain statistical indicators.

The rest of the paper is organized as follows. Section 2 outlines the reliability model we use. Some details relating to this model can be found in [3]. Section 3 describes the simulation procedure. Section 4 contains the results of the study and discussion. Section 5 offers some concluding remarks. The supplementary material contains the code used to run the simulations of the proposed methodology.

2. The reliability model

We consider the weighted k-out-of-n type system $\mathbf{S}$ as a unified entity with n individual interconnected components denoted by $C_1, \dots, C_n$. The system fails when some of its components fail. The *state* of $\mathbf{S}$ and of $C_1, \dots, C_n$ is a binary variable (active/inactive). It changes over time. At the starting point (time $t = 0$), the system and the components are naturally assumed to work (to be active). We label the active components by A and the inactive ones by NA. We define a *configuration* as a vector $\gamma \in \{A, NA\}^n$ describing the states of the components of the system. The configuration at time $t = 0$ is called the *initial configuration*. If we maintain the same labels used for the components for active and inactive systems (A and NA, respectively), then the arguments above can be formalized by introducing the so-called *structure function* $\phi : \{A, NA\}^n \to \{A, NA\}$. By definition, the structure function provides complete information on the configurations leading to failure of the system. Indeed, configurations at a given time can be sorted into two groups: in the first, we collect the configurations leading to system failure, while in the second, we consider the configurations corresponding to a still active system. As mentioned above, we assume that the initial configuration belongs to the second group.

At time t, the generic configuration is denoted by $\gamma(t)$. We define the *system lifetime* by

$$\mathcal{T} := \inf\{t \geq 0 | \phi(\gamma(t)) = 0\}. \tag{1}$$

As in standard reliability theory, we assume that the components are weighted, and system failure depends on the aggregate weight of the active components. In this respect, the weights express the fact that the different components of the system are not equally relevant. Furthermore, the relevance of the components is assumed to vary over time according to the changes in status of the full set of components. Here are the details.

For each $j = 1, \dots, n$ and $t \geq 0$, the *relative importance of the component* C_j within the full system at time t is measured through $\alpha_j(t)$, where $\alpha_j : [0, +\infty) \to [0, 1]$ and $\sum_{j=1}^{n} \alpha_j(t) = 1$, for each t. The term $\alpha_j(t)$ represents the weight of component C_j at time t. For each $t \geq 0$, we collect the weights in a time-varying vector $\mathbf{a}(t) = (\alpha_j(t))_j$, where

$$\mathbf{a} : [0, +\infty) \to [0, 1]^n \qquad \text{such that} \qquad t \mapsto \mathbf{a}(t). \tag{2}$$

If a component is inactive at time t, then its relevance for the system is null. Moreover, each active component of the system has positive relative relevance (weight).

$$\alpha_j(t) = 0 \Leftrightarrow C_j \text{ has status } NA \text{ at time } t. \tag{3}$$

Condition (3) allows us to interpret $\mathbf{a}(t)$ as a further specification of the configuration $\gamma(t)$, obtained with the introduction of the weights. For this reason and without loss of generality, we will also call the vector $\mathbf{a}(t)$ the *configuration* at time t. Once a component fails, its weight is reallocated to the surviving components in proportion to their weights.

A rational expectations-based approach is used to compute the expectation of the random time in which the system fails. Specifically, we will compute the expected value of $\mathcal{T}$ conditioned on the specific values of the weights $\mathbf{a}$. We denote by RE the rational expectation of the time $\mathcal{T}$ given all the possible configurations of the system. Specifically, for $t \geq 0$, we set

$$RE = \left\{ \mathbb{E}\left[\mathcal{T} \mid \tilde{\mathbf{a}}\right] \; : \; \tilde{\mathbf{a}} \in [0, 1]^n \right\}, \tag{4}$$

where $\mathbb{E}$ is the expected value operator. Formula (4) provides the expected value of the lifetime of $\mathbf{S}$ for any configuration of the system.

3. Simulation procedure for validating the rational expectations-based estimation

This section aims to validate the rational expectations-based estimation of the failure time of the system through an extensive set of numerical experiments. To this end, we first provide a detailed description of the numerical procedure and the steps in the algorithm, then describe the parameter set.

In each experiment, we assume that the system fails the first time the number of failed components is greater than the number of active components. In doing so, the weights of the individual components are not relevant here. Those weights are used only to express the failure of their associated component. This is done as follows. We assume a discrete-time procedure, with $t \in \mathbb{N}$, and at

Table 1
Parameter set.

Parameter	Value
Number of components	10
$K\_simulations$	10000 (*Catalog* systems)
$X\_simulations$	10000 (*In-vivo* systems)

a given time t, we select one of the active components randomly according to a uniform distribution – say, C_j. Then, we extract a random number θ from a uniform distribution in $[0,1]$ and say that C_j fails at time t if and only if $\theta > \alpha_j(t)$.

To avoid the computational complexity associated with the analysis of (4) for all possible values of the configurations, we synthesize them using statistical indicators to describe the central tendency, variability, shape, and concentration of the information content of the component weight distribution. Specifically, we use the following: (i) the variance, to evaluate the variability of the configurations and therefore the dispersion of the weights; (ii) the skewness, to provide a measure of the symmetry of the weight distributions; (iii) the kurtosis, to see whether the weights are heavy-tailed or light-tailed with respect to a normal distribution; (iv) the Gini coefficient, to evaluate the inequalities between the weights; (v) the Shannon entropy, to estimate the degree of complexity of information and the shape of the weight distribution.

In the simulation procedure, two sets of systems are considered: one set (the *catalog* systems) to record the information and calculate RE in (4), and another (the *in-vivo* systems) to validate RE by comparing the failure times of real systems with those predicted using the rational expectations-based procedure. The prediction errors are obtained by comparing the *catalog* systems with the *in-vivo* systems.

Specifically, we simulated $K_{simulations}$ *catalog* systems from $t = 0$ until failure. In each case, we stored the information on the value of the statistical indicator registered for the components at a given time t and the related residual failure time of the system – i.e., the time period from t to the failure of the system. Then, we stored all the residual failure times of the systems having a specific value $\bar{\star}_C$ of the given statistic, where $\star$ =variance, skewness, kurtosis, Gini coefficient, or Shannon entropy. The arithmetic mean of the system failure times is $\mathbb{E}\left[\mathcal{T} \mid \star(\mathbf{a} = \bar{\star}_C\right]$ in (4). This procedure was applied to all the observed $\bar{\star}_C$'s. We implemented the same procedure to simulate $X_{simulations}$ in-vivo systems and denote the generic specific value of the given statistical indicator by $\bar{\star}_V$. Then, we matched the observed values of the statistical indicator of the *in-vivo* system – say, $\bar{\star}_V$ – with those of the catalog by minimizing the distance between $\bar{\star}_V$ and the observed values in the catalog, thus assigning each $\bar{\star}_V$ to a specific $\bar{\star}_C$ in the catalog. Such an assignment allows us to speak about the "same configuration".

For each $\bar{\star}_C$, we created a distribution of errors between the expected residual times calculated using rational expectations and the residual failure times of the *in-vivo* systems. For comparison purposes, we decided to focus our attention on fixed quantiles at level q of the error distributions, taking $q = 10\%$, $q = 50\%$, and $q = 90\%$. Accordingly, E_{RE_10}, E_{RE_50}, and E_{RE_90} are 10^{th} (Errors 1), 50^{th} (Errors 2), and 90^{th} (Errors 3) percentiles of the error distribution, respectively. These quantiles take into account all the information recorded in the catalog over time. To obtain good readability of the results, we ranked the $\bar{\star}_C$'s in ascending order and grouped them into equal class intervals. We chose a class width of 50 elements – except for the last class, which included the values excluded from the previous classes. Then, we evaluated the arithmetic mean of the values of the statistical indicators and of the corresponding errors within each class.

The parameters are reported in Table 1.

The initial distributions of the components' weights were selected to provide a detailed view of the phenomenon under investigation, according to five possible cases:

- Uniform distribution in (0,1);
- Beta distribution with support in (0,1), specified by two positive shape parameters, γ and β, that represent the exponents of the random variable and determine the shape of the distribution. Depending on the values assigned to the two parameters, we specify four distributions from which we can randomly extract our weights:
  - $\gamma = 1$ and $\beta = 3$, an asymmetric distribution on the left;
  - $\gamma = \beta = 0.5$, a symmetric distribution concentrated on the extremes;
  - $\gamma = \beta = 2$, a symmetric distribution concentrated on the central values of the distribution;
  - $\gamma = 1$ and $\beta = 0.5$, an asymmetric distribution on the right.

As mentioned above, computational complexity imposes a strict limit on the present study, particularly for systems with a large number of components. Indeed, the runtime of our algorithm grows as n increases. Table 2 shows how the computational complexity of the algorithm varies with time n, for the different initial weight distributions. The runtime is taken to be the time required to simulate 5000 *catalog* systems – those used to record the information and calculate rational expectations – and 5000 *in-vivo* systems. Note that the runtime increases more than proportionally as n grows (see Fig. 1).

All experiments were carried out on a laptop with an Intel(R) Core(TM) i7-7500U CPU @ 2.70 GHz processor and 8 GB of RAM, using MATLAB R2022b.

Table 2
Computational complexity as a function of time. The runtime is expressed in seconds, the number of components is n, and we show the runtime for four different beta distributions and the uniform distribution.

n	$\alpha = 1\ \beta = 0.5$	$\alpha = 0.5\ \beta = 0.5$	$\alpha = 1\ \beta = 3$	$\alpha = 2\ \beta = 2$	Uniform
4	90.1175	101.1130	102.0817	76.2790	49.6270
6	97.8881	110.4299	110.3591	84.1229	47.4580
8	108.2468	121.0559	119.0824	98.5971	50.0671
10	126.5494	134.5019	133.8177	115.3221	57.5471
12	138.0501	163.4102	168.9586	149.0571	79.7389
14	182.3550	169.8030	182.9890	168.5507	89.2785
16	197.9418	194.2455	193.4784	189.8436	100.1458
18	224.7270	238.3337	222.2043	217.6308	125.1377
20	262.6765	274.3270	259.1357	254.4160	144.0967

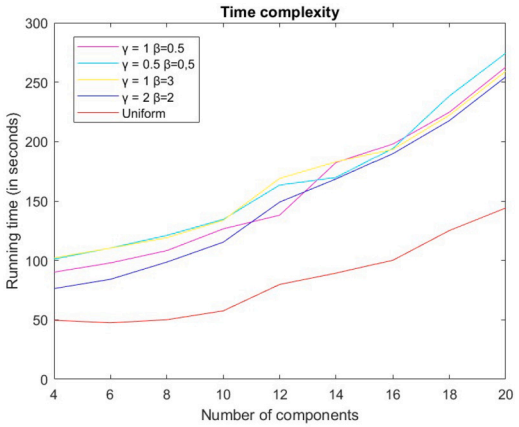

Fig. 1. Runtimes of the algorithm (in seconds).

4. Results and discussion

In this section, we present the results of the study and the related discussion. Consider first the various statistical measures under analysis. To further validate our method, the graphs showing the prediction errors for different statistical indicators include a benchmark which represents the error made by predicting failure times without rational expectations. The benchmark error is obtained as follows. First, we calculate the absolute value of the difference between the average of the failure times of the *catalog* systems and the individual failure times of the *in-vivo* systems. Second, we take the mean of the quantities computed in the first step.

Fig. 2 shows the prediction errors (in absolute value) for the analysis of the variance; we treat all the considered initial weight distributions. Note that the variance is a good estimator for the dispersion of the weights. As can be seen from the graphs in Fig. 2, the errors decreased towards zero in all five cases. The high volatility of the system components is linked to the error reduction in the prediction, and this in turn is due to the decreased uncertainty regarding failure time. The data are widely spread around the mean, and a broader range of values can be captured. The rational expectations model performs better than the benchmark in the variance case for Errors 1 and in most cases for Errors 2 and Errors 3. Only for low values of the variance do, we see an occasional predominance of the benchmark. Thus, our method performs better in systems characterized by high volatility.

This trend holds true for all five initial distributions, with a substantial difference in the magnitude of the errors for the left-skewed Beta distribution ($\gamma = 1$ and $\beta = 3$) and the symmetric Beta distribution centred on the extremes ($\gamma = \beta = 0.5$). Indeed, these cases produce significantly smaller errors than the other initial distributions. Variance performs well for prediction purposes, and the errors stabilize around zero for all initial weight distributions except for the 50^{th} and the 90^{th} percentiles of the Beta distribution with $\gamma = 1$ and $\beta = 3$. In this case, from a certain threshold of variance onwards, we observe a small increase in the errors, which indicates that excessive fluctuations affect the predictability of the variance for medium or very large errors. When we detect a high variance, we pay close attention to data without generalizing the values never seen before. So, for the negative exponential distribution, we observe too high values and excessive turbulence within the observations; this can highlight distortions that lead the model to perform less well in terms of predictions. With low variance values, the model underfits the residual failure times, and it cannot identify the true moment of system failure.

Fig. 2 shows a high level of heterogeneity in the trends, with many spikes in the initial parts of the curves and more stable errors after a given level of variance. The prediction of the failure times from these levels of variance is almost perfect. In these windows, the components are strongly dependent on each other; this allows us to predict how long it will take for the system to fail. Moreover, there is a different trend in Errors 3 compared to Errors 1. The line for the biggest errors decreases faster toward zero, while the line

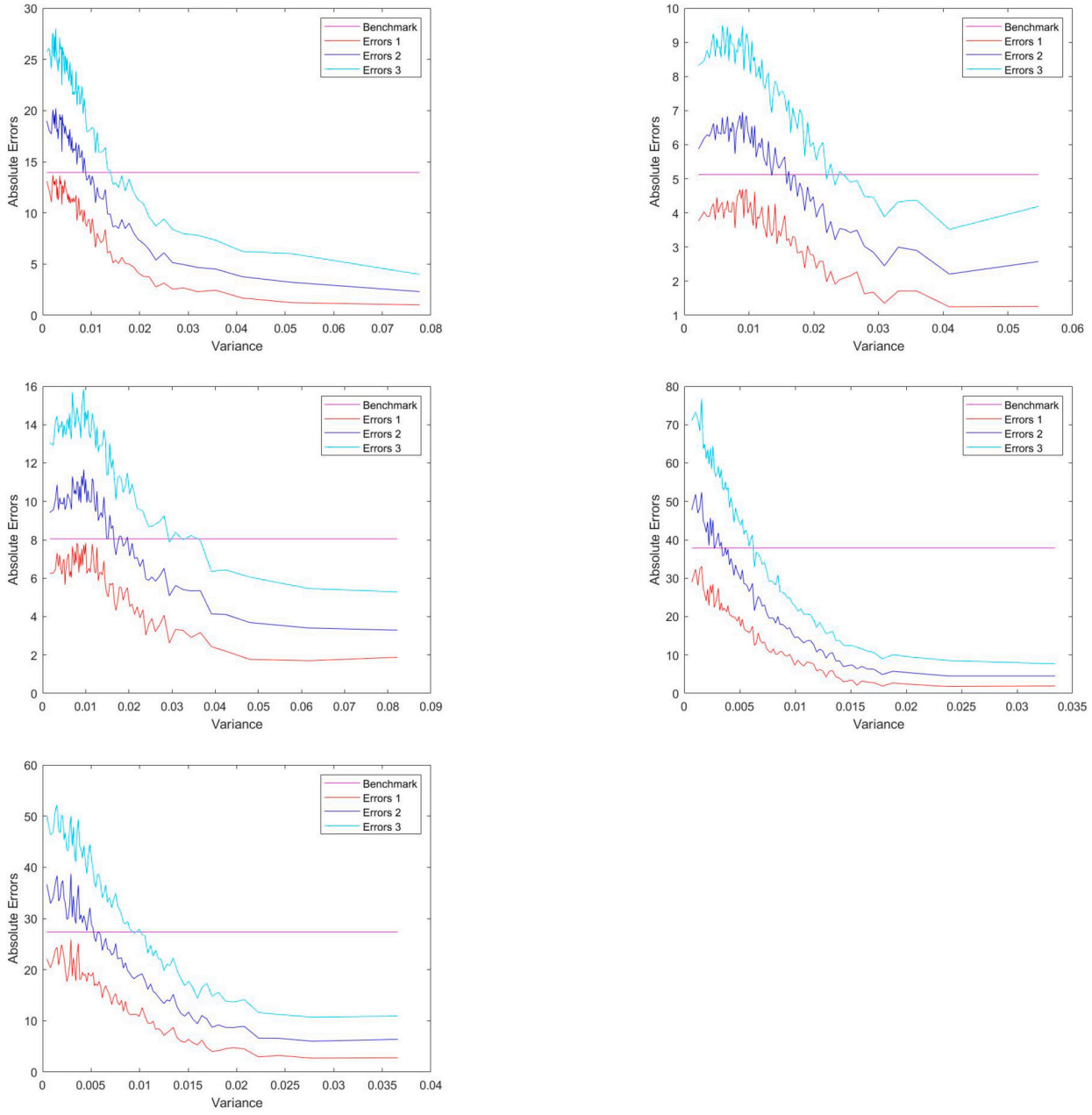

Fig. 2. Prediction errors (in absolute value) for Errors 1 (red line), Errors 2 (blue line), Errors 3 (light blue line), and the benchmark (magenta line) in the case of the variance for each initial weight distribution. The five graphs correspond to the five initial weight distributions. (Top left) Uniform distribution, (top right) Beta distribution with $\gamma = 1$ and $\beta = 3$, (center left) Beta distribution with $\gamma = \beta = 0.5$, (center right) Beta distribution with $\gamma = \beta = 2$, (bottom left) Beta distribution with $\gamma = 1$ and $\beta = 0.5$.

for the smallest errors displays a slight flattening behaviour. This flattening arises because, when we have a very low variance (close to zero), the gain in the prediction for a big error is more evident than what we will see for a small error.

The different graphs also display the dependence on the initial distribution of components. According to the existing literature, the ideal shape of the weight distribution to minimize prediction errors is the negative exponential one (see [4], [13], [25], and [37]). In our study, this is the Beta distribution with $\gamma = 1$ and $\beta = 3$. In fact, the best predictive efficacy belongs to the asymmetric case with $\gamma = 1$ and $\beta = 3$ and the symmetric one with $\gamma = \beta = 0.5$. In the first case, the predictive gain is very high, although there is a worsening in the final trend, which corresponds to very high levels of variance. However, in the other cases, rational expectations always give excellent results for increasing levels of variance. In fact, we obtain a decrease to near zero. The symmetric Beta distribution with $\gamma = \beta = 2$ and the asymmetric Beta distribution with $\gamma = 1$ and $\beta = 0.5$ are the cases that perform worst as regards absolute error, but for the appropriate variance values, they reach errors close to zero and tend to flatten out for larger values.

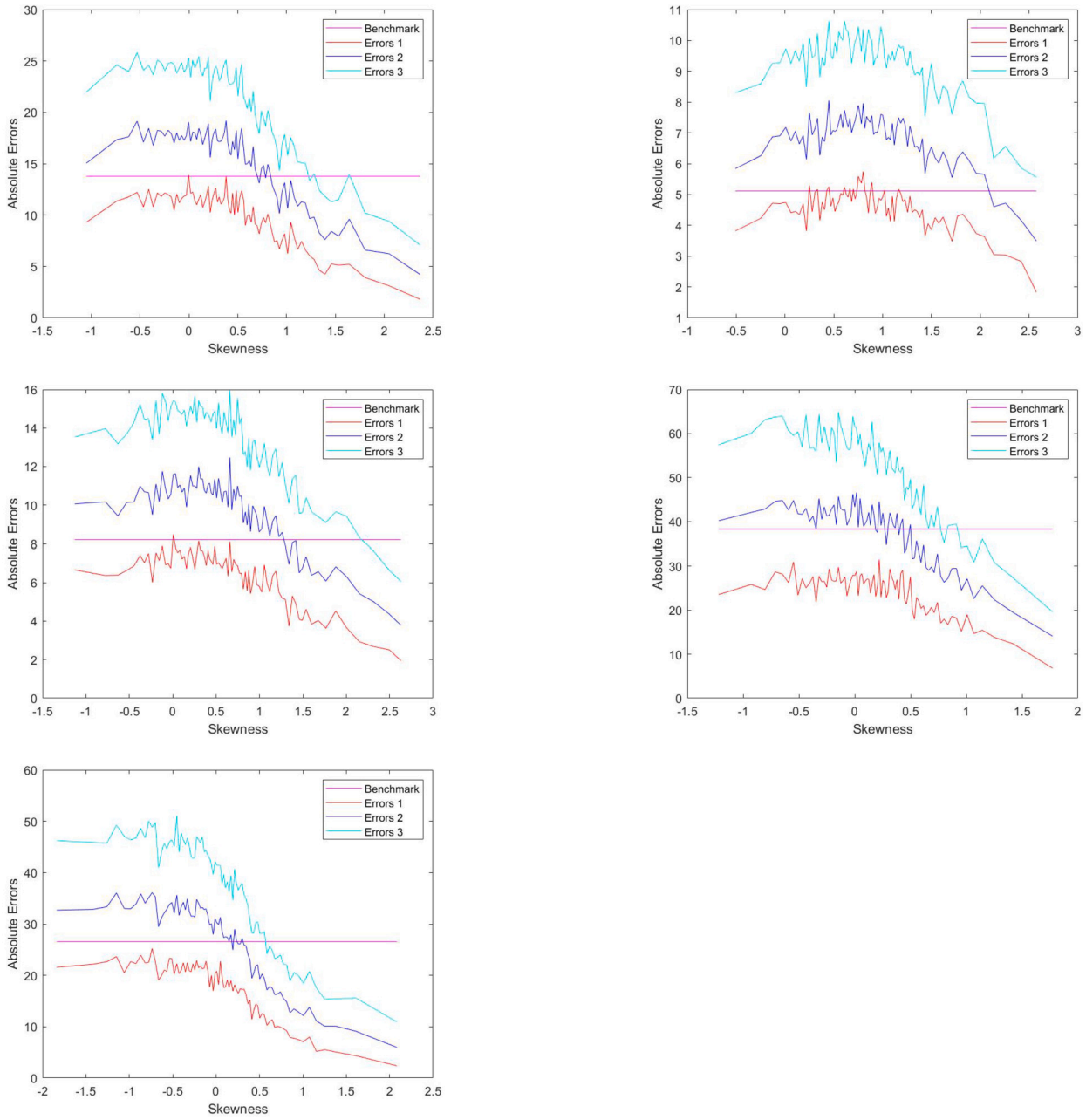

Fig. 3. Prediction errors for the skewness including benchmark errors. The five graphs correspond to the five initial weight distributions: (top left) uniform distribution, (top right) Beta distribution with $\gamma = 1$ and $\beta = 3$, (center left) Beta distribution with $\gamma = \beta = 0.5$, (center right) Beta distribution with $\gamma = \beta = 2$, (bottom left) Beta distribution with $\gamma = 1$ and $\beta = 0.5$.

We can deduce from this analysis the levels of variance that provide an optimal balance without overfitting or underfitting the residual failure times. This specific application suggests that the study of the variance can be a powerful tool to predict the residual failure times of a stochastic system for high values of the variance, which are crucial for understanding the behaviour of prediction models. Jiang et al. [16] investigated software fault prediction models, showing that the lower the variance, the more reliable the system is. Twomey and Smith [41], who used error estimator methods to evaluate prediction models, confirmed with their studies that good performance is linked to low variance. In contrast, our study shows that high variance values are more informative and allow better prediction than lower values, with fewer fluctuations.

Fig. 3 shows the different error trends when skewness is used as the statistical indicator. Through the skewness of the components' weights, we investigate the nature and dynamics of the shape of the distribution. Skewness measures the degree of symmetry or lack of symmetry in a data distribution. Skewness has been used in predicting future market returns with excellent results (see [17]).

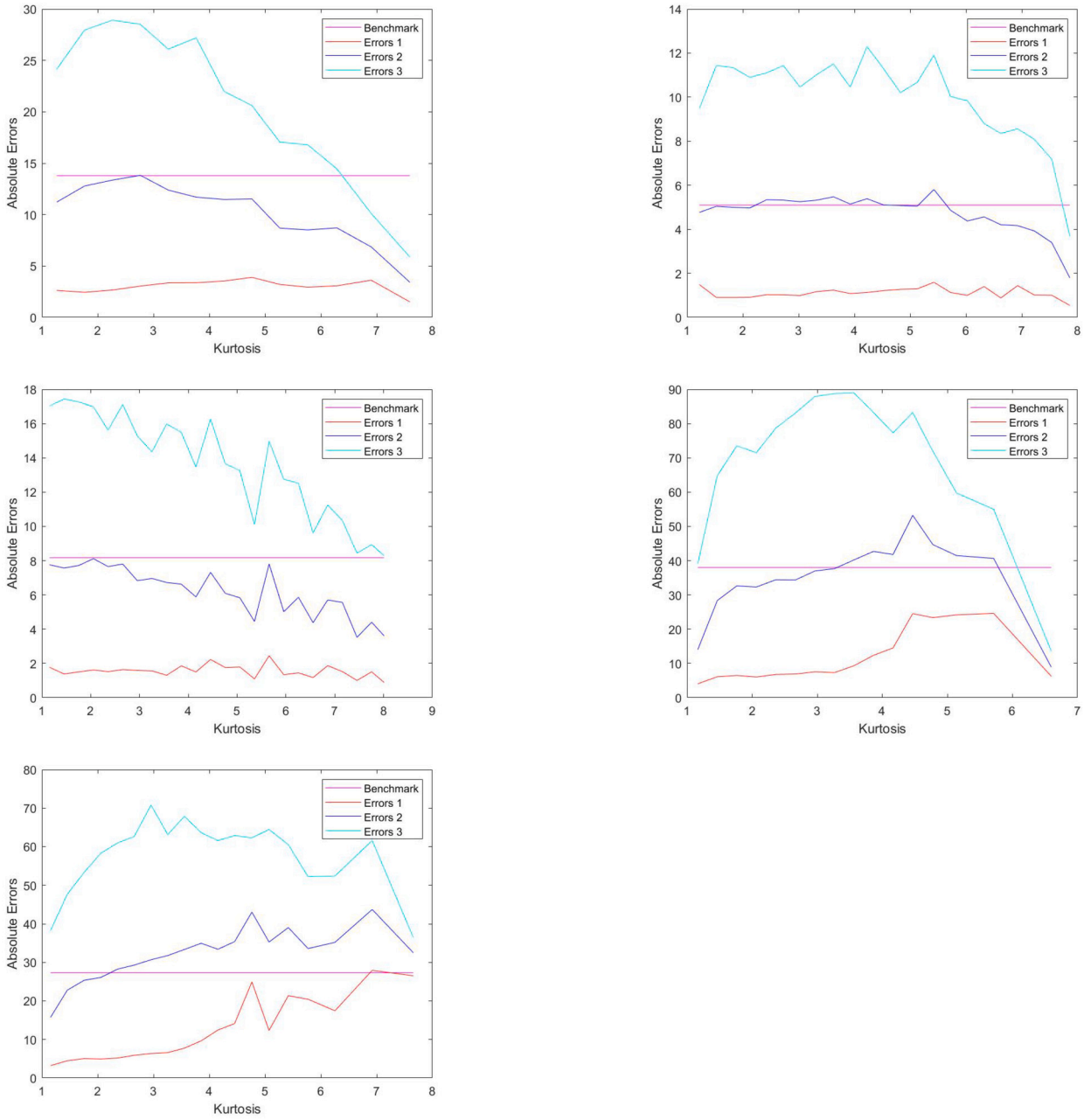

Fig. 4. Prediction errors for the kurtosis including benchmark errors. The five graphs correspond to the five initial weight distributions: (top left) uniform distribution, (top right) Beta distribution with $\gamma = 1$ and $\beta = 3$, (center left) Beta distribution with $\gamma = \beta = 0.5$, (center right) Beta distribution with $\gamma = \beta = 2$, (bottom left) Beta distribution with $\gamma = 1$ and $\beta = 0.5$.

Likewise, in our study, it can be a good predictor depending on the values assumed by the initial distribution. Compared to the variance, the skewness fluctuates more in the central part of the error curves shown in the graphs of Fig. 3. It exhibits large spikes, especially at values close to zero where the data are fairly symmetric. The highest errors are achieved in conditions of symmetry, negative asymmetry, or slight positive asymmetry of the distribution. Regardless of the initial distribution, a forecast bias is evident around a skewness range from -1.5 to +2.5. There are no remarkable differences between the various curves. The best accuracy is obtained by applying our rational expectations-based model to the Beta distribution with $\gamma = \beta = 0.5$ and the Beta distribution with $\gamma = 1$ and $\beta = 3$. The least predictive gain and very high maximum error levels are reached for the Beta distribution with $\gamma = \beta = 2$ (as in the study of the variance). While highly skewed data generally affect the accuracy of the predictive model (see [22]), the results obtained here provide evidence that extremely positively skewed weights pick up the best information with good predictive gain. Comparing our model with the benchmark, the outcomes obtained for the variance are substantially confirmed, with a predominance

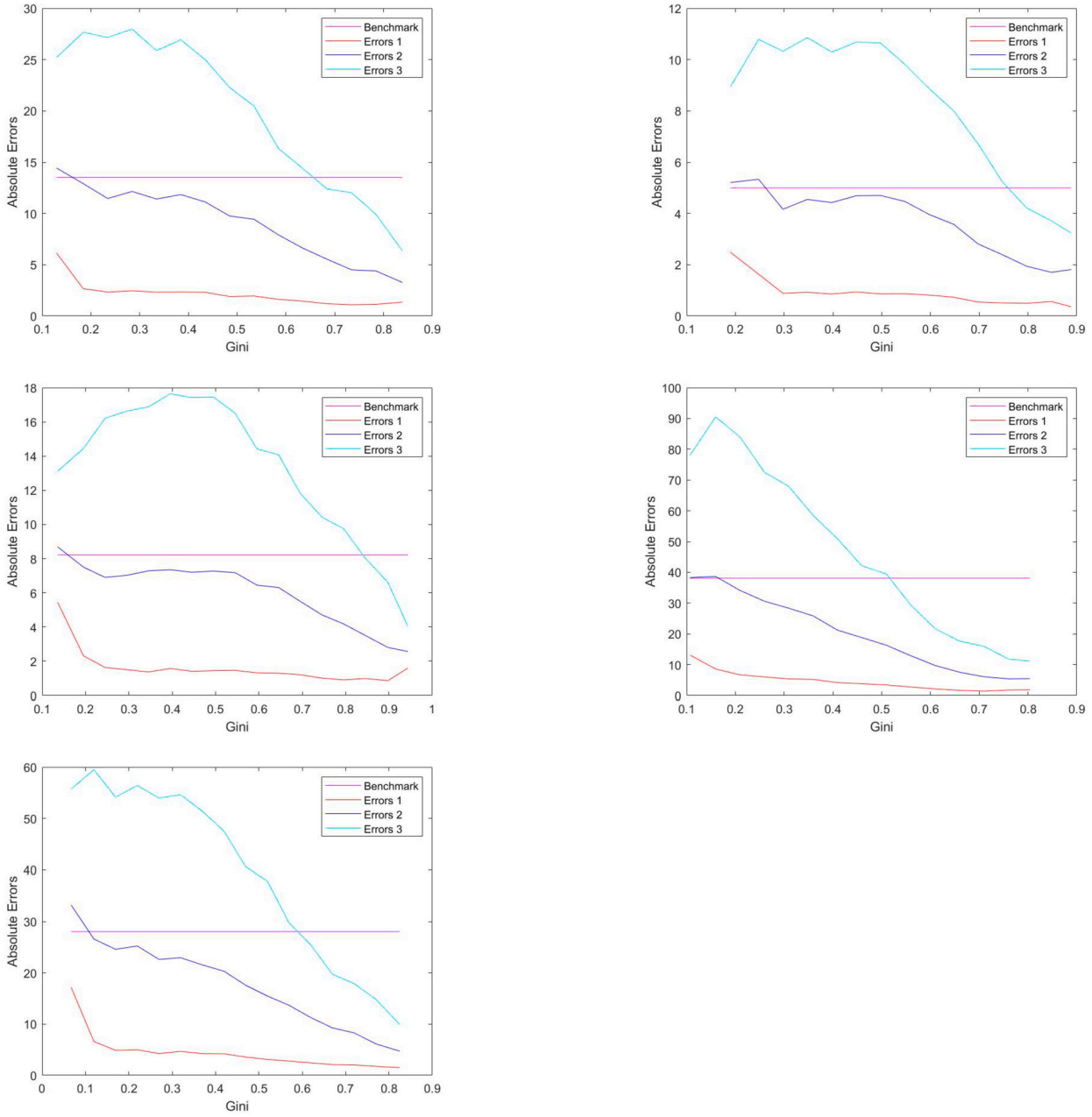

Fig. 5. Prediction errors for the Gini coefficient including benchmark errors. The five graphs correspond to the five initial weight distributions: (top left) uniform distribution, (top right) Beta distribution with $\gamma = 1$ and $\beta = 3$, (center left) Beta distribution with $\gamma = \beta = 0.5$, (center right) Beta distribution with $\gamma = \beta = 2$, (bottom left) Beta distribution with $\gamma = 1$ and $\beta = 0.5$.

of the rational expectations-based approach for Errors 1 and for large values of the skewness. The magnitude of the errors also varies according to the considered weight distribution in this case.

Regarding the use of statistical moments in forecasting models, [1], [19], [34], and [35] show that the lowest moments of the distributions (the second moment in our case, i.e., the variance) are more efficient than the highest moments (the third and fourth moments, or the skewness and kurtosis, respectively), which are attested to be less stable and less reliable. With our analysis, we confirm these conclusions. In fact, the variance is the moment that comes closest to zero errors, with some uncertainty in the case of very low variability (components very similar to each other). The situation worsens for the skewness. The results for the third statistical moment are characterized by less regularity and mixed trends, but with the achievement of very low error values.

The situation worsens still further for the kurtosis, as we will now see. Fig. 4 shows the prediction errors when the kurtosis is used as statistical indicator. There is no straightforward behavior in this case. The kurtosis is a measure of the "tailedness" of the distribution, describing the tails and identifying the outliers. Fig. 4 shows the prediction errors when the kurtosis is used as statistical

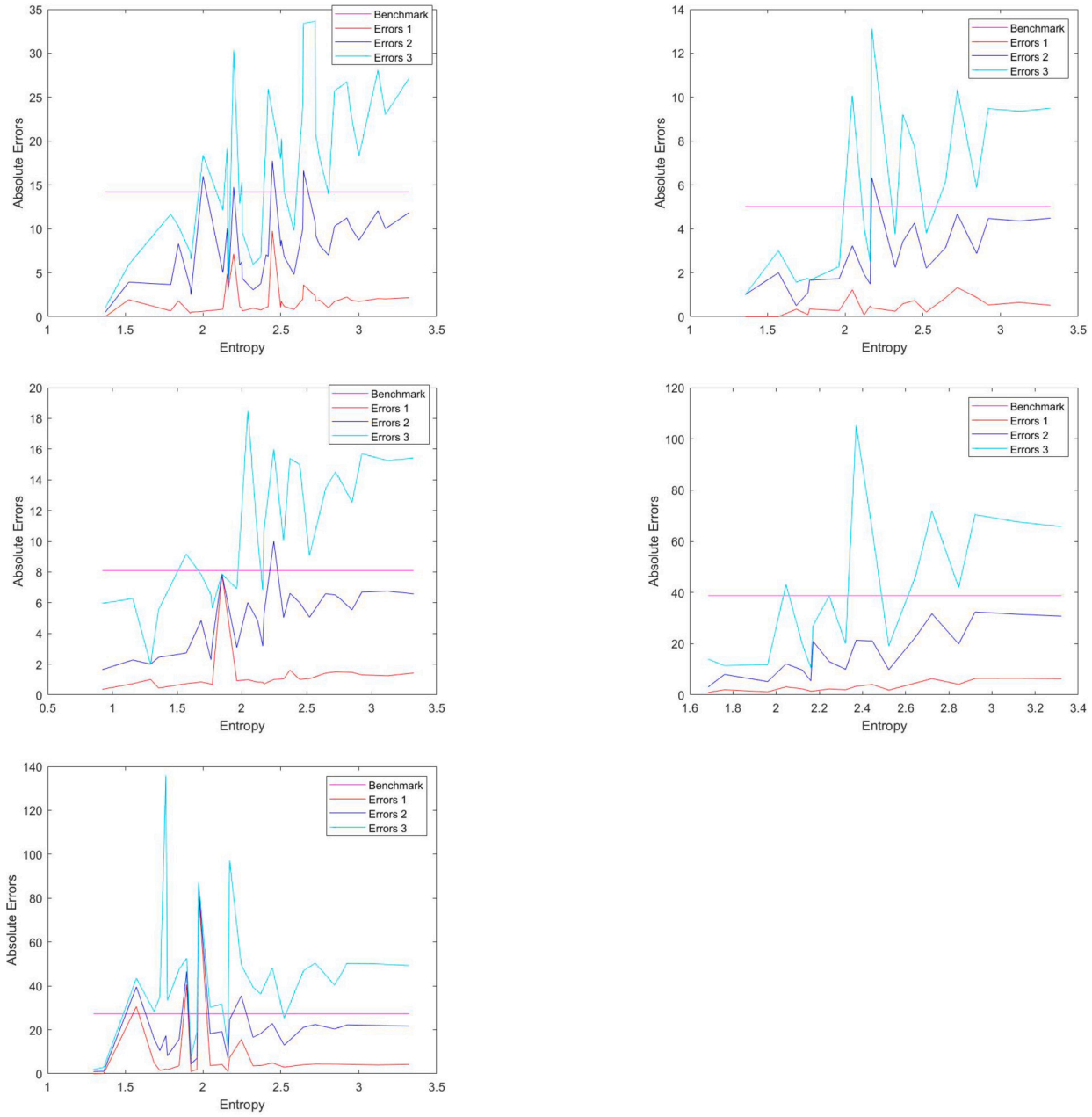

Fig. 6. Prediction errors for the Shannon entropy including benchmark errors. The five graphs correspond to the five initial weight distributions: (top left) uniform distribution, (top right) Beta distribution with $\gamma = 1$ and $\beta = 3$, (center left) Beta distribution with $\gamma = \beta = 0.5$, (center right) Beta distribution with $\gamma = \beta = 2$, (bottom left) Beta distribution with $\gamma = 1$ and $\beta = 0.5$.

indicator. There is no straightforward behavior in this case. As in the study of variance, the two distributions that perform better and provide lower and decreasing errors are the Beta distribution with $\gamma = \beta = 0.5$ and the Beta distribution with $\gamma = 1$ and $\beta = 3$. At a glance, the latter analysis performs best. In the first three graphs, we observe a different error trend for Errors 3 and Errors 1. The trend for the smallest errors is almost linear, with almost constant errors for all the kurtosis values recorded in the simulations. In contrast, the largest errors follow a path that reaches a maximum for low kurtosis values, hence when there is a platykurtic weight distribution. The kurtosis loses predictive power when the distribution has a lower likelihood of extreme weights than a normal distribution. When the weights are distributed around their average and we observe thinner tails, the use of rational expectations proves to be ineffective. In the case of low kurtosis, the rational expectations-based approach is also better than the benchmark at the 50^{th} percentile for all the initial weight distributions, but most clearly for the Beta distributions with $\gamma = \beta = 0.5$ and $\gamma = 1$ and $\beta = 3$. Our model becomes successful when there is a positive excess of kurtosis and the data distribution is fat-tailed. If we consider even the most extreme values of the data, we obtain an increasingly inclusive range of cataloged weights. The application of rational

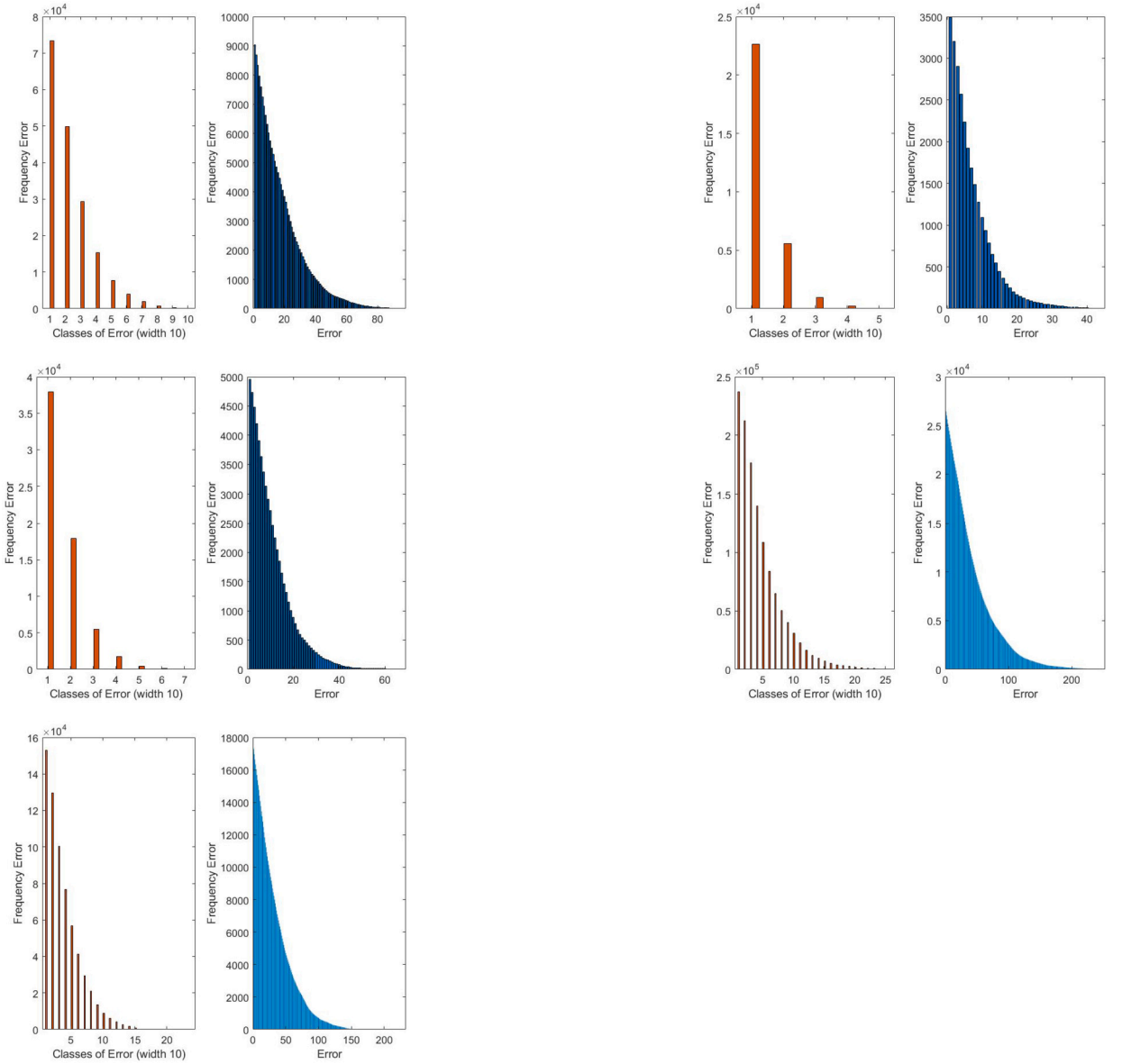

Fig. 7. Histograms of the frequency of the prediction errors for the analysis of variance for each of the initial weight distributions. (Top left) uniform distribution; (top right) Beta distribution with $\gamma = 1$ and $\beta = 3$; (center left) Beta distribution with $\gamma = \beta = 0.5$; (center right) Beta distribution with $\gamma = \beta = 2$; (bottom left) Beta distribution with $\gamma = 1$ and $\beta = 0.5$. In each pair of histograms, the one on the left shows the frequency for error classes of width 10 (each bar contains 10 error values in ascending order, except the last class which groups the last remaining error values); the one on the right shows the error frequency of each individual value excluding zero (the case of no errors in the prediction).

expectations turns out to be an excellent method for forecasting residual failure times even taking into account values at a greater distance from the mean.

The situation is completely reversed if we start from a symmetric weight distribution centred on 0.5 (the beta distribution with $\gamma = \beta = 2$). For distributions with non-fat tails, kurtosis does not work well. This case is an exception, showing anomalous behaviour. It yields very high error levels. It does not work because the initial values assigned to the weights extracted from this distribution are concentrated at 0.5. It underestimates the extreme values close to 0 and 1, which are taken into account in the other distributions. The minimum errors made in predicting failure times using rational expectations are those at kurtosis levels around 8. It can only be used if we have distributions with extreme values. On the other hand, if the weights are distributed with an asymmetry to the right (the beta distribution with $\gamma = 1$ and $\beta = 0.5$), kurtosis is a good predictor for thin-tailed distributions with continuously rising curves as the values increase.

Fig. 5 displays the prediction errors (in absolute value) for the Gini coefficient for each of the five different initial weight distributions. The Gini coefficient can be used to measure the accuracy of our model for predicting residual failure times. An increase in

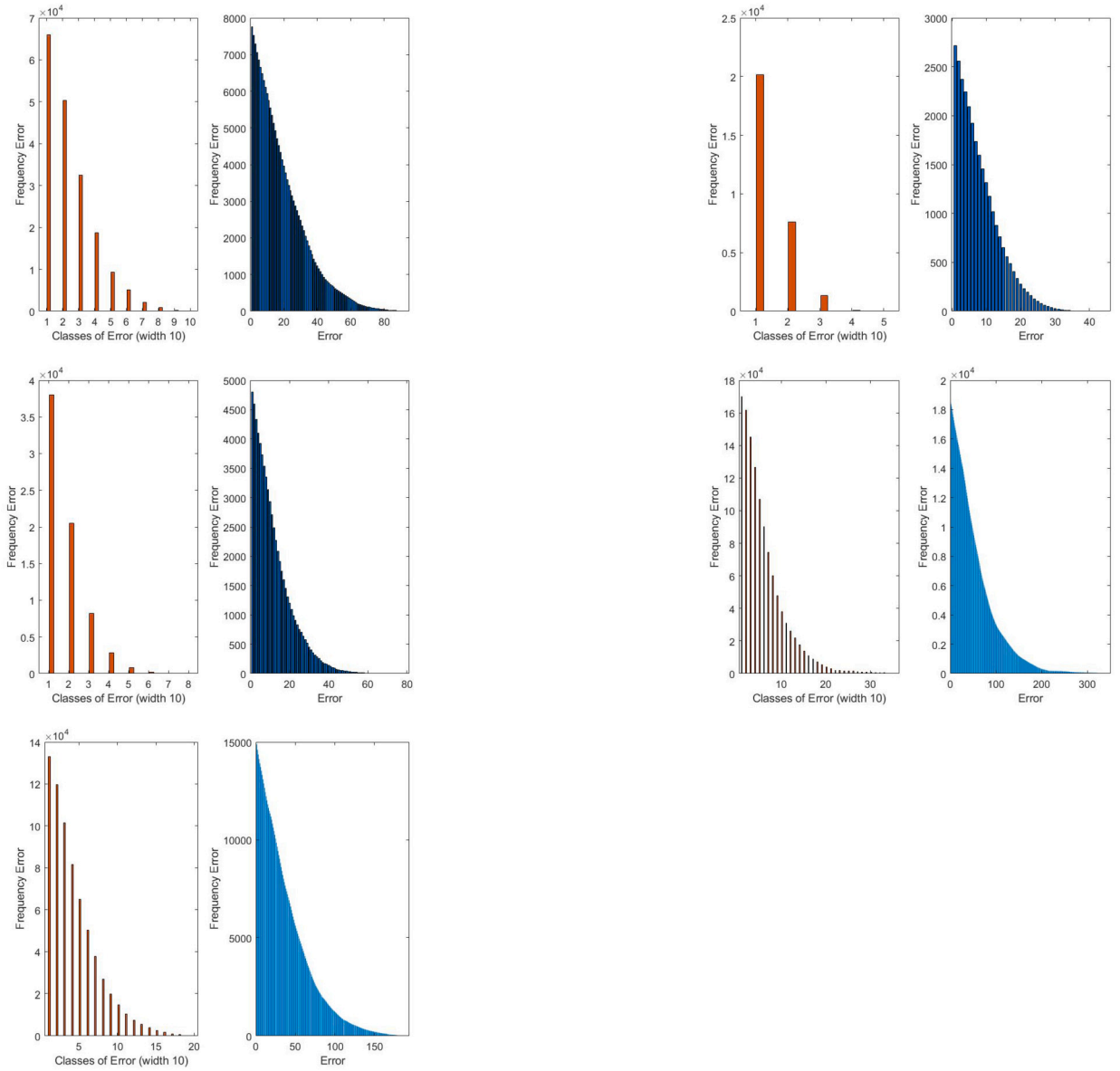

Fig. 8. Histograms of the frequency of the prediction errors for the analysis of skewness for each of the initial weight distributions. See Fig. 7 for full explanation of the histograms.

the Gini coefficient corresponds to an increase in inequality among the values taken into consideration in the analysis. Values close to 1 indicate that the range of possible weights cataloged on which the study is based is very wide. Few researchers have used the Gini coefficient as a prediction tool, although some authors have argued that it could be useful when dealing with failure prediction models (see [31]). The error behavior is similar to what was observed for kurtosis in some scenarios, even though there are differences to note. The comments made about the different trends of the Errors 1, Errors 2, and Errors 3 curves in the case of kurtosis can be extended to the present case. First of all, the 10^{th} percentile shows a linear trend, but for values of the Gini coefficient near zero (the initial trend), we observe a peak that decreases rapidly, settling at error levels close to zero. Secondly, the symmetric Beta distribution with $\gamma = \beta = 2$ and the asymmetric Beta distribution with $\gamma = 1$ and $\beta = 0.5$ behave oppositely to what was observed for kurtosis. Although the level of initial errors is very high, there is a predictive gain that grows exponentially as the values of the Gini coefficient increase. The error analysis shows an improvement in the accuracy of the predictions for increasing coefficient values up to the attainment of prediction errors tending to zero. Near the maximum polarization of the components, excellent levels of prediction are reached. For very small Gini values, we can deduce that we are in the initial time of the simulations when no nodes or few nodes have yet failed. The values of the weights are more equidistributed, and the equidistribution will decrease as the components fail and the weights of the failed nodes are proportionally redistributed among the components still active. Low coefficient values correspond to less cataloged information and greater uncertainty associated with forecasts using rational expectations. For all initial

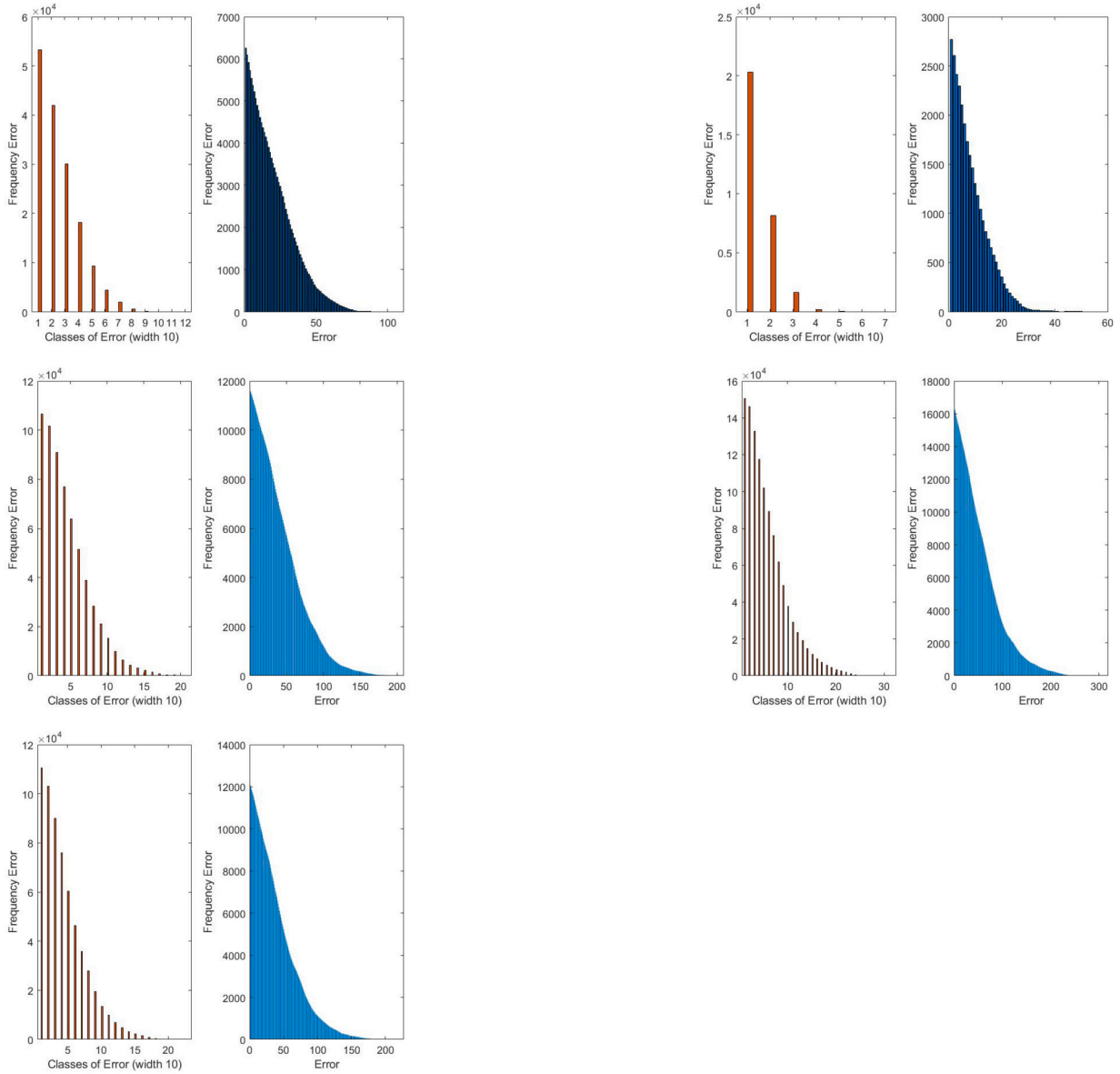

Fig. 9. Histograms of the frequency of the prediction errors for the analysis of kurtosis for each of the initial weight distributions. See Fig. 7 for full explanation of the histograms.

weight distributions, the benchmark method is less satisfactory than the rational expectation method at the Errors 2 level. Moreover, rational expectations-based predictions are more accurate when the distributions take on values that are relatively inhomogeneous and more polarized.

We now focus on the results for the Shannon entropy illustrated in Fig. 6. There is no discussion in the literature about the use of entropy as a prediction tool. Our analysis aims to fill this gap. The results show highly irregular error trends, characterized by many alternating spikes over a wide range of values. Any improvement is completely random, and rational expectations are clearly ineffective as a prediction tool. Comparing the values with the analyses for the other statistical indicators, the errors obtained are systematically worse for each initial weight distribution, and the possibility of distortions occurring by applying our model to entropy is evident. The Shannon entropy is therefore the least informative and least useful statistical indicator in this forecasting model. There is no regularity in the trends, and we fail to capture any information about which entropy values are best suited for better predicting failure times.

We now discuss a series of histograms representing the absolute frequencies of the prediction errors. The aim is to better understand the extent of the errors for each of the statistical indicators and initial weight distributions. The figure contains five pairs of histograms. Regarding the histograms on the left in each pair, the error values are divided into classes of width 10, with the residual values included in the last class whose size is variable; regarding the histograms on the right, we eliminate the zero value (no errors made in the

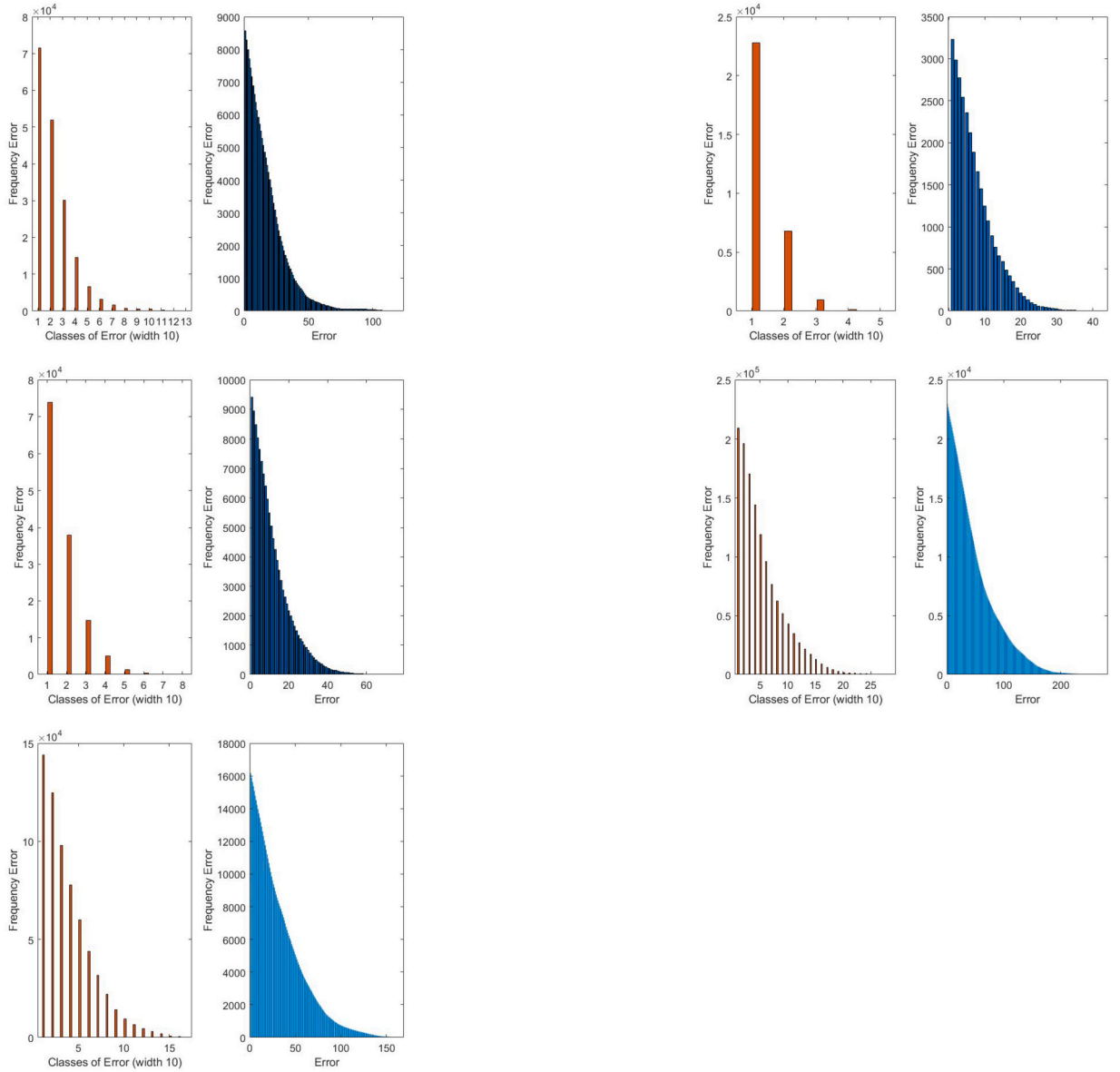

Fig. 10. Histograms of the frequency of the prediction errors for the analysis of Gini coefficient for each of the initial weight distributions. See Fig. 7 for full explanation of the histograms.

prediction) so that we can represent the frequency of each error value. Figs. 7–11 show this analysis for the variance, the skewness, the kurtosis, the Gini coefficient, and the Shannon entropy, respectively.

Some comments apply to all the statistical indicators. In fact, the trend in the error frequencies is very similar in the various cases. The variance, kurtosis, and Gini coefficient, which are the best prediction tools, show more linear and regular trends, while the skewness and Shannon entropy show deviations that can be associated with their debatable role in forecasting failure times. This confirms the first part of the analysis. In general, the frequencies of the prediction errors reach a very high level for errors located in the lower classes and then decrease in a strictly monotonic manner as the error values increase (higher classes). Predictive ability is not a trivial matter because the error values with higher frequency are found when the error is close to zero, and the frequencies of the errors fall very quickly as the errors increase in size. The errors corresponding to 0 and 1 (zero or almost zero errors) are the most frequent. This confirms the effectiveness of our analysis, which is consistently reliable with many correct predictions. The frequencies decrease dramatically as the prediction error increases.

These graphs give us an overview of the order of magnitude of this situation. Indeed, there is an inverse proportionality between the frequency of errors (ordinate) and the error class (abscissa). The densest and most excessive errors are those concerning the symmetric Beta distributions with $\gamma = \beta = 2$ and the asymmetric ones with $\gamma = 1$ and $\beta = 0.5$. In these cases, the frequency of the lowest errors is much higher than in the other scenarios. Very often, our predictions are correct. However, the large number of error

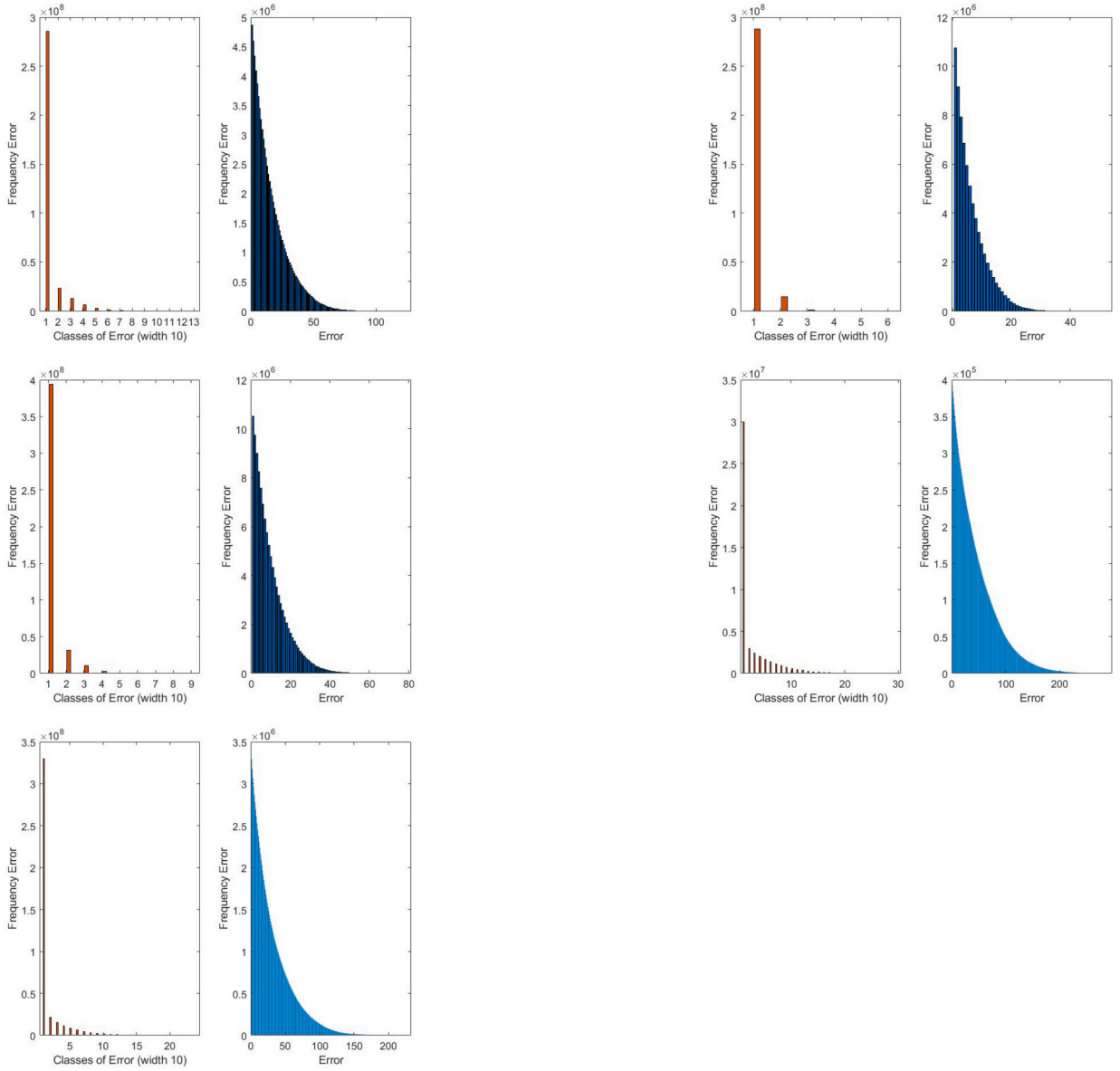

Fig. 11. Histograms of the frequency of the prediction errors for the analysis of Shannon entropy for each of the initial weight distributions. See Fig. 7 for full explanation of the histograms.

classes makes it seem as though there is a high probability of making very large errors when the predictions are wrong. The best predictive gain occurs for the Beta distribution with $\gamma = 1$ and $\beta = 3$. This confirms that the negative exponential distribution is the most suitable for forecasting failure times. This analysis is thus consistent with our earlier discussion.

5. Conclusions

In the context of reliability systems, our goal is to improve prediction performance using rational expectations by conditioning the residual failure times of simulated stochastic systems on the memory of past events. The key to our approach is the information stored over time and the use of rational expectations with a particular focus on the effect of different statistical indicators on prediction results. We carry out a transversal analysis over time: we emphasize the contribution of the statistical indicator, showing how the error varies as a function of these measures. We examine how the error trends are influenced by several indicators known to be important for their informative content, the values given by these indicators, and also the initial weight distribution used in the study.

To provide an in-depth investigation of our predictions, we take into account three different quantiles of the error distribution at 10%, 50%, and 90%. We thus have a comparison between small, medium, and large errors. We also explore how the results depend on the indicator considered for the analysis and the initial weight distributions. Here, we use the Beta distribution, which is a highly

flexible two-parameter law that can describe a wide range of situations. Indeed, the selection of the parameters leads to specific shapes of the distribution, including the uniform case, the presence of bimodality, and left- and right-skewed distributions. In the paper, we consider several different cases. We observe that the proposed methodological framework is flexible, and there are no constraints in applying the method/algorithm to other distributions or to any real-life situation with data described only by empirical distributions.

Some of our results agree with those to be found in the literature, e.g., ideal initial distributions in the predictor field, where it is found that the negative exponential distribution performs best. In contrast, other results offer new and original perspectives. Note that time is not an explicit variable in our prediction model. However, time does play a key role in determining the dynamics of the lifetime of these systems. Indeed, we treat the system components as failing according to a stepwise process, in which the weight of any broken node is reallocated to the other nodes in proportion to their weights. This drives the dynamics/interaction in the model, and consequently, our results only apply to this specific kind of dynamics/interaction. Naturally, our approach can be generalized to any type of dynamics/interaction, based on the considered reliability system, using other reallocation rules. One can also introduce some time-dependent factors affecting the failure rates of the system components. A completely new study would be needed to consider what happens for each new type of dynamics/interaction.

Moreover, the method proposed in this paper is illustrated via simulated data and it would be interesting to find practical validations of the general findings presented here. However, we point out that the construction of a catalog would require a large number of observations of several reliability systems, from their “birth”to failure, and this could be accomplished only in the presence of a large empirical dataset or by implementing a large number of simulations. The present paper works efficiently on the ground of the latter approach. For an empirical-based analysis, we are presently considering the possibility of accessing bank data in the Eurozone as a means to give practical validations of some of our findings. In doing so, we aim to develop a study oriented more toward the economic and financial aspects of reliability theory applications. This task is now on our research agenda.

Interestingly, the question of parameter uncertainty arises when dealing with empirical applications. The exogenous selection of the parameter set performed here is based on the need to describe a wide set of situations in terms of initial weight distributions – by varying γ and β – and the number of components of the systems, denoted by n in this case. Therefore, we do not have to face the problem of parameter uncertainty here, because we are not dealing with empirical data and the consequent possible misrepresentation of the parameters. Parameter uncertainty is thus a question for future explorations of this topic when empirical systems are considered.

CRediT authorship contribution statement

Jessica Riccioni: Writing – review & editing, Writing – original draft, Validation, Supervision, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Jorgen-Vitting Andersen:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Roy Cerqueti:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.ins.2024.121483.

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