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An Effective Positivstellensatz over the Rational Numbers for Finite Semialgebraic Sets

Lorenzo Baldi¹ & Teresa Krick² & Bernard Mourrain³

¹ Universität Leipzig & Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany

² Departamento de Matemática & IMAS-Conicet, Universidad de Buenos Aires, Argentina

³ Centre Inria d'Université Côte d'Azur, Sophia Antipolis, France

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Abstract

We study the problem of representing multivariate polynomials with rational coefficients, which are nonnegative and strictly positive on finite semialgebraic sets, using rational sums of squares.

We focus on the case of finite semialgebraic sets S defined by equality constraints, generating a zero-dimensional ideal I , and by nonnegative sign constraints.

First, we obtain existential results. We prove that a strictly positive polynomial f with coefficients in a subfield K of \mathbb{R} has a representation in terms of weighted Sums-of-Squares with coefficients in this field, even if the ideal I is not radical. We generalize this result to the case where f is nonnegative on S and $(f) + (I : f) = 1$. We deduce that nonnegative polynomials with coefficients in K can be represented in terms of Sum-of-Squares of polynomials with coefficients in K , when the ideal is radical.

Second, we obtain degree bounds for such Sums-of-Squares representations, which depend linearly on the regularity of the ideal and the degree of the defining equations, when they form a graded basis.

Finally, we analyze the bit complexity of the Sums-of-Squares representations for polynomials with coefficients in \mathbb{Q} , in the case the ideal is radical. The bitsize bounds are quadratic or cubic in the Bezout bound, and linear in the regularity, generalizing and improving previous results obtained for special zero dimensional ideals.

As an application in the context of polynomial optimization, we retrieve and improve results on the finite convergence and exactness of the moment/Sums-of-Squares hierarchy.

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1 Introduction

The certification of the nonnegativity of a (multivariate) polynomial with real coefficients is a central problem in real algebra. Many different theorems, known as *Positivstellensätze*, provide existential results on the representation of nonnegative polynomials in terms of Sums of Squares (SoS). While not all globally nonnegative polynomials are sums of squares of polynomials, Artin [Art27] answered positively to Hilbert’s 17th problem by showing that a representation as sums of squares of rational functions is always possible. This result was generalized by the Positivstellensätze of Krivine [Kri64] and Stengle [Ste74] to polynomials that are nonnegative on arbitrary semialgebraic sets. Later on, the breakthrough results of Schmüdgen [Sch91] and Putinar [Put93] showed that general denominator-free SoS representations exist for strictly positive polynomials on compact basic semialgebraic sets.

SoS certificates for nonnegative polynomials have many applications: for instance, they can be used to develop constructive mathematics, see e.g. [LQ15], or in theorem checker as positivity witnesses [Har07, Mag14, MDR17].

Another area of application, where SoS representations play an important role, is Polynomial Optimization (POP). Indeed, after the discovery of the connection between SoS and positive semidefinite matrices, due to Shor [Sho87] and Choi, Lam and Reznick [CLR95], SoS representations acquired an important role in optimization and numerical computation. This connection was exploited by Lasserre [Las01] and Parrilo [Par00] to construct the celebrated moment/SoS hierarchies, which provide arbitrarily tight semidefinite relaxations to polynomial optimization problems. The spectrum of applications of polynomial optimization itself is huge (see e.g. [Las09] and references therein).

In all these applications, controlling the degrees and sizes of the SoS representation is a bottleneck for the performance of the related algorithms: this is an active and challenging area of research, which is witnessing recent interesting developments. General degree bounds for Positivstellensätze are challenging to obtain and not proved to be tight: for instance the best known bound for the Krivine-Stengle Positivstellensatz, presented in [LPR20], consists of five towers of exponentials in the degree of the input polynomials and the number of variables, and similar bounds apply for Putinar’s Positivstellensatz (see e.g. [BMP25, BS24] and references therein).

It is therefore natural to analyze simpler special cases, still useful for applications, where we can expect much precise degree estimates. For instance, Lasserre studied in [Las02] the case of strictly positive polynomials over a finite grid of real points, and then Parrilo considered in [Par02] the more general case of nonnegative polynomials over a finite basic semialgebraic set, assuming a radicality assumption in the description of the set.

It is also natural to ask in which field we can take the coefficients of the SoS representation, if the positive polynomial has coefficients in a subfield K of \mathbb{R} , in particular when $K = \mathbb{Q}$. Although the existence of rational SoS representations is known for globally nonnegative, univariate polynomials with rational coefficients [Lan06, Pou71] and for strictly positive polynomials on compact basic semialgebraic sets in the multivariate case (mostly under some additional assumptions [Sch02, Pow11, MSED21, DP22]), this is not always true for globally nonnegative multivariate polynomials: Scheiderer [Sch16] showed that in the multivariate case there exist rational polynomials that are nonnegative on \mathbb{R}^n and admit a real SoS representation, but not a rational one.

The zero-dimensional setting. In this paper, we will consider finite semialgebraic sets, focusing in particular on the existence of exact effective SoS representations. Instead of working with the specific field \mathbb{Q} of rational numbers, we study polynomials with coefficients in a subfield K of \mathbb{R} , that are strictly positive (or nonnegative) on a finite basic closed semialgebraic set $S \subset \mathbb{R}^n$ defined by polynomials with coefficients in K . We aim at analyzing the smallest field in which we can compute a SoS representation and at bounding the degree and the height (when $K = \mathbb{Q}$) of the coefficients of this SoS representation.

The precise problems we consider here are the following. Given a field $K \subset \mathbb{R}$, and

$$\mathbf{g} = \{g_1, \dots, g_r\}, \mathbf{h} = \{h_1, \dots, h_s\} \subset K[\mathbf{x}] := K[x_1, \dots, x_n], \quad (1)$$

two sets of polynomials in n variables with coefficients in K , where the ideal $I = (\mathbf{h}) \subset K[\mathbf{x}]$ generated by \mathbf{h} is zero-dimensional, which then define the *finite* (basic closed) semialgebraic set

$$S = S(\mathbf{g}, \mathbf{h}) = \{\xi \in \mathbb{R}^n : g_i(\xi) \geq 0 \text{ for } 1 \leq i \leq r \text{ and } h_j(\xi) = 0 \text{ for } 1 \leq j \leq s\} \subset \mathbb{R}^n. \quad (2)$$

Given a polynomial $f \in K[\mathbf{x}]$, we study the following problems.

1. If $f(\xi) > 0$ for all $\xi \in S$, how can we represent f as

$$f = \sum_k \omega_{0,k} q_{0,k}^2 + \sum_i \left(\sum_k \omega_{i,k} q_{i,k}^2 \right) g_i + \sum_j p_j h_j$$

with $\omega_{i,k} \in K_{\geq 0}$ and $q_{i,k}, p_j \in K[\mathbf{x}]$, and what are the degrees of $q_{i,k}, p_j$?

2. If $f(\xi) \geq 0$ for all $\xi \in S$, which condition guarantees that f can be represented as an SoS as above, and what are the degrees of $q_{i,k}$ and p_j ?

3. When $K = \mathbb{Q}$, what is the bitsize of the $\omega_{i,k}$ and the coefficients of $q_{i,k}, p_j$? What is the cost to compute such a representation?

Before stating our main results, we recall that an ideal $I \subset K[\mathbf{x}]$ generated by $\mathbf{h} = \{h_1, \dots, h_s\}$ is a zero-dimensional ideal when

$$V_{\mathbb{C}} := \{\zeta \in \mathbb{C}^n : h_j(\zeta) = 0 \text{ for } 1 \leq j \leq s\}$$

is a finite set, or equivalently, when the quotient ring $K[\mathbf{x}]/I$ is a finite dimensional K -vector space. In the sequel we will denote by B a monomial basis of $K[\mathbf{x}]/I$, and by $\langle B \rangle_K$, the vector space spanned by B in $K[\mathbf{x}]$. We also recall that \mathbf{h} is a *graded basis* of the ideal I when for all $p \in I$, there exist $p_j \in K[\mathbf{x}]$ with $\deg(p_j) \leq \deg(p) - \deg(h_j)$, $1 \leq j \leq s$, such that $p = \sum_{j=1}^s p_j h_j$.

We can now state our main theorems.

Theorem A (See Theorems 2.7 and 3.2). *Let $K \subset \mathbb{R}$ be a field, $\mathbf{g}, \mathbf{h} \subset K[\mathbf{x}]$ be as in (1), $I = (\mathbf{h}) \subset K[\mathbf{x}]$ be a zero-dimensional ideal, B be a monomial basis of $K[\mathbf{x}]/I$ with $|B| = D$, and $S \subset \mathbb{R}^n$ be the finite basic closed semialgebraic set defined in (2).*

If $f \in K[\mathbf{x}]$ is such that $f > 0$ on S , or $f \geq 0$ on S and $(I : f) + (f) = (1)$, then there exist

- $\omega_{i,k} \in K_{\geq 0}$ and $q_{i,k} \in \langle B \rangle_K$ for $0 \leq i \leq r$ and $1 \leq k \leq D$,
- $p_j \in K[\mathbf{x}]$ for $1 \leq j \leq s$

such that

$$f = \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 + \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i + \sum_{j=1}^s p_j h_j \quad (3)$$

Furthermore, if \mathbf{h} is a graded basis of I and $\deg(B)$ is an upper bound for the maximum degree of a monomial in B , then for all j

$$\deg(p_j h_j) \leq \max\{\deg(f), \deg(g_i) + 2 \deg(B) : 1 \leq i \leq r\}.$$

Note that the smallest possible degree of a basis B of $K[\mathbf{x}]/I$ is the *Castelnuovo-Mumford regularity* $\text{reg}(K[\mathbf{x}]/I)$ of $K[\mathbf{x}]/I$ (see e.g. [BS87]), which can always be upper bounded by $D = \dim(K[\mathbf{x}]/I)$, by choosing a suitable monomial basis B of $K[\mathbf{x}]/I$, which itself satisfies $D \leq \max_j \{\deg(h_j)\}^n$ by Bézout bound. Moreover, when \mathbf{h} is a graded basis of degree $\leq d$ generating the zero-dimensional ideal I , we have $\text{reg}(K[\mathbf{x}]/I) \leq n(d-1)$ (see [Cha07, Theorem 9.4] and [Sza08, Appendix, Prop. A.9 and Lem. A.10]).

Since the condition $(I : f) + (f) = (1)$ is always satisfied when the ideal I is radical, we can deduce a complete characterization of nonnegativity of polynomials on S defined by a radical zero-dimensional ideal I (see Remark 3.3).

Theorem B. *Let K , I and S be as in Theorem A, and assume that I is a radical ideal. Then for $f \in K[\mathbf{x}]$, $f \geq 0$ on S if and only if f admits a SoS representation as in (3).*

We note that since every Archimedean field K (i.e. a field where for every $x \in K$, there exists $n \in \mathbb{N}$ such that $x < n$) can be embedded as a subfield of \mathbb{R} by Hölder's embedding theorem (see e.g. [KS22, Th. 2.1.10]), Theorem A and Theorem B hold for any Archimedean field K .

When $K = \mathbb{Q}$, we can assume furthermore that the polynomials $q_{i,k}$ in Equation (3) belong to the free \mathbb{Z} -module $\langle B \rangle_{\mathbb{Z}}$ generated by the monomial basis B , by taking a common denominator of their coefficients and dividing the weights $\omega_{i,k}$ by the square of this common denominator. Our next result deals with the size of these coefficients in the particular case of a radical zero-dimensional ideal I .

Theorem C (See Theorems 5.8 and 6.2). *Let $f, g_1, \dots, g_r, h_1, \dots, h_s$ be polynomials in $\mathbb{Z}[\mathbf{x}]$ of degrees bounded by d and heights bounded by τ . Assume that $I = (\mathbf{h}) \subset \mathbb{Q}[\mathbf{x}]$ is a radical zero-dimensional ideal, and let S and B be as in Theorem A, with $\delta := \max\{d, \deg(B)\}$.*

If $f > 0$ on S (resp. $f \geq 0$ on S), then the following holds for a SoS representation as in (3) of f :

- $\omega_{0,k} \in \mathbb{Q}_{\geq 0}$ and $q_{0,k} \in \langle B \rangle_{\mathbb{Z}}$ (resp. $q_{0,k} \in f \cdot \langle B \rangle_{\mathbb{Z}}$) with heights bounded by $\tilde{\mathcal{O}}(d^{3n}) \delta \tau$;
- $\omega_{i,k} \in \mathbb{Q}_{\geq 0}$ and $q_{i,k} \in \langle B \rangle_{\mathbb{Z}}$ (resp. $q_{i,k} \in f \cdot \langle B \rangle_{\mathbb{Z}}$) with heights bounded by $\tilde{\mathcal{O}}(d^{2n}) \delta \tau$ for $1 \leq i \leq r$;

If in addition, $\mathbf{h} = \{h_1, \dots, h_s\}$ is a graded basis of I , then $\deg(q_{i,j}) \leq nd$ (resp. $\deg(q_{i,j}) \leq (n+1)d$), $p_j \in \mathbb{Q}[\mathbf{x}]$ with $\deg(p_j) \leq \hat{d} - \deg(h_j)$ and height bounded by $\tilde{\mathcal{O}}(d^{2n}) \delta \tau + \tilde{\mathcal{O}}(\hat{d}^n) \tau$, for $1 \leq j \leq s$, where $\hat{d} := 2(d + \deg(B)) + 1$.

In Theorem C the (logarithmic) *height* of a polynomial $p \in \mathbb{Z}[\mathbf{x}]$ is the maximum base 2-logarithm of the absolute values of its coefficients, and the height of a rational a polynomial $p \in \mathbb{Q}[\mathbf{x}]$ is the maximum height of the numerator and a common denominator in a primitive representation $p = \frac{1}{\nu} \hat{p}$ with $\hat{p} \in \mathbb{Z}[\mathbf{x}]$ and $\nu \in \mathbb{N}$. The notation $\tilde{\mathcal{O}}(M)$ means to be bounded by a linear function of M up to logarithmic factors of M .

The bounds in Theorem C involve Bézout type bounds derived from [KPS01] for the height of the Chow form of $V_{\mathbb{C}}$. These bounds could be refined using mixed-volume bounds as in [EMT20] or [MS18], at the cost of a genericity assumption: we won't follow this path hereafter.

1.1 Related works and Putinar-type representations

Putinar-type representations over Archimedean fields. In order to better compare our work with the literature, we fix some notation about quadratic modules. For a field $K \subset \mathbb{R}$, we denote by Σ_K^2 the convex cone of weighted *Sums of Squares* (SoS) of polynomials over K :

$$\Sigma_K^2 := \left\{ f \in K[\mathbf{x}] : \exists r \in \mathbb{N}, \omega_i \in K_{\geq 0}, q_i \in K[\mathbf{x}] \text{ s.t. } f = \omega_1 q_1^2 + \dots + \omega_r q_r^2 \right\}$$

Here we choose to take positive *weighted* SoS in Σ_K^2 , i.e. multiplying by the nonnegative weights $\omega_i \in K_{\geq 0}$, and not classical SoS. When $K = \mathbb{Q}$ or $K = \mathbb{R}$, weighted SoS and standard SoS coincide, since in these two fields every nonnegative element is a sum of squares. However, this is not true in general, e.g. $\sqrt{2}$ is not a Sum of Squares in $K = \mathbb{Q}(\sqrt{2})$; see also [KS22, Cor. 1.1.12]. The choice of weighted SoS is the standard choice when working with ordered fields which are not real closed, see e.g. [PD01, Sch02].

The *quadratic module* generated by the polynomials $\mathbf{g} = \{g_1, \dots, g_r\}$ and $\pm \mathbf{h} = \{\pm h_1, \dots, \pm h_s\}$ is

$$Q_K = Q_K(\mathbf{g}, \pm \mathbf{h}) := \Sigma_K^2 + \sum_{i=1}^r \Sigma_K^2 \cdot g_i + \sum_{j=1}^s K[\mathbf{x}] \cdot h_j \subset K[\mathbf{x}] \quad (4)$$

Putinar’s Positivstellensatz [Put93] states that any real polynomial that is strictly positive on a compact semialgebraic set S can be represented in the quadratic module $Q_{\mathbb{R}}$, under the assumption that $Q_{\mathbb{R}}$ is Archimedean, i.e. that there exists $g \in Q_{\mathbb{R}}$ such that $\{\xi \in \mathbb{R} : g(\xi) \geq 0\}$ is compact. While Schmüdgen’s Positivstellensatz (closely related to Putinar’s one) is known to hold over any Archimedean field $K \subset \mathbb{R}$, see e.g. [Sch02], it is not known if the same is true for Putinar’s Positivstellensatz, because it is uncertain if $Q_{\mathbb{R}}$ Archimedean implies that Q_K is Archimedean.

But special results are known over the rational numbers: if S is defined by polynomial with rational coefficients, $Q_{\mathbb{R}}$ is Archimedean and $f \in \mathbb{Q}[\mathbf{x}]$ is strictly positive on S , then $f \in Q_{\mathbb{Q}}$ if:

- $R - x_1^2 - \dots - x_n^2 \in Q_{\mathbb{Q}}$ for some $R \in \mathbb{Q}_{\geq 0}$ (see [Pow11]);
- S satisfies the unisolvent property¹, implied by the non-emptiness of the interior of $S \subset \mathbb{R}^n$ in the Euclidean topology (see [DP22]).

In our setting, Q_K satisfies the Archimedean condition by taking $g = -\sum h_j^2 \in Q_K$, but none of the conditions above are necessarily satisfied.

The paper [MDV23] analyzes the special case of a nonnegative polynomial $f \in \mathbb{Q}[\mathbf{x}]$ over \mathbb{R}^n and proves that when its gradient ideal $I_{\text{grad}} = (\partial_{x_1} f, \dots, \partial_{x_n} f)$ is a radical zero-dimensional ideal and the infimum $f^* = \inf\{f(\xi) : \xi \in \mathbb{R}^n\}$ is attained, then f has a representation as a rational SoS modulo I_{grad} .

For strictly feasible Gram matrix representations, the existence of rational SoS has been proved in [PP08] and an algorithm based on rounding techniques has been proposed. See also [DP22].

The univariate case over the rational numbers has been investigated for instance in [MSEDS19] and [KMS23]. The statement of Theorem A and also its assumption for the nonnegative case are a natural generalization of the latter to finite semialgebraic sets in \mathbb{R}^n .

To the authors’ best knowledge, Theorem A is the first result establishing the existence of a general Putinar-type representation for strictly positive polynomials over the rational numbers, for our very particular setting of finite semialgebraic sets.

Minimality of our assumption in the nonnegative case. As shown in Example 3.1, not all nonnegative polynomials on S have a representation in Q_K . In the case of $K = \mathbb{R}$, Theorem A can be viewed as an extension of [Par02], which holds for nonnegative polynomials on finite semialgebraic sets S defined by a radical zero-dimensional ideal I , to an arbitrary zero-dimensional ideal I for f satisfying $(I : f) + (f) = (1)$. Moreover, Theorem B can be seen as full extension of the same paper’s result to other fields $K \subset \mathbb{R}$, and in particular to the field \mathbb{Q} .

We now discuss the minimality of our assumption $(I : f) + (f) = (1)$. For zero-dimensional ideals, as already mentioned, $Q_{\mathbb{R}}$ is an Archimedean quadratic module (see also [Mar08, Cor. 7.4.4]), and therefore, by [Mar08, 9.1.2]

$$f \in Q_{\mathbb{R}} \iff f \in Q_{\mathbb{R}} + (f^2) = \Sigma_{\mathbb{R}}^2 + \Sigma_{\mathbb{R}}^2 \cdot g_1 + \dots + \Sigma_{\mathbb{R}}^2 \cdot g_r + I_{\mathbb{R}} + (f^2).$$

Since we can readily verify that $(I : f) + (f) = (1)$ is equivalent to $f \in I + (f^2)$, we then see that our necessary assumption is stronger, but in the same spirit, than the necessary and sufficient one.

We also notice that all the results of this paper can be extended to any finitely generated quadratic module Q_K where the ideal $I = Q_K \cap -Q_K$, called *support* of Q_K , is zero-dimensional. In this case the

¹i.e. the polynomials appearing in the decomposition can be uniquely recovered by their values on the semialgebraic set S .

associated semialgebraic set is finite, and the result can be obtained by replacing the h_j by generators of the support of Q_K . This might induce a shift of the degree in the SoS representations, due to the degree of the SoS representations of the generators of the support in Q_K .

Degrees bounds for Putinar-type representations. The degrees of Putinar-type representations for strictly positive polynomials on general S has been investigated in [NS07, MSED21] and more recently in [BM22, BMP25], respectively with exponential bounds and polynomial bounds depending on the minimum f^* of f on S : the closer f is having a zero on S , the bigger is the degree needed for the representation.

In our results, the degree bounds depend on the regularity of $\mathbb{R}[\mathbf{x}]/I$, of order $\mathcal{O}(nd)$, but they are independent of f^* . This can be explained using the concept of *stability* (see for instance [BS24, Sec. 6]) that occurs in the case of zero-dimensional semialgebraic sets. We discuss the degree bounds in more details in Section 1.2, in connection with Polynomial Optimization.

Regarding the optimality of these bounds, it is shown in [Gri01], that the degree of a SoS certificate for deciding the emptiness of a basic semialgebraic set is at least $c d^n$ for some $c \in \mathbb{R}_{>0}$. As discussed in Section 1.2, the bounds on the degree in Theorem C are tight in the case of binary polynomials $h_i = x_i^2 - x_i$ and linear constraints g_i (see also [KLM15]).

Heights for Putinar-type representations over \mathbb{Q} . Obtaining good bitsize bounds in real algebra is a (mostly open) research challenge. For instance, for the (more general) problem of certifying the insatisfiability of sign conditions using the Krivine-Stengle Positivstellensatz, the only known bitsize bound, provided in the recent seminal work [LPR20], reads as:

$$2^{2 \left(2^{\max\{2,d\} 4^n} + m 2^d \max\{2,d\}^{16^n \text{bit}(d)} \right)}$$

where n denotes the number of variables, m the number of input polynomials and d a bound on their degrees.

For a general Putinar-type representation over \mathbb{Q} , the following height bound of similar order, which fixes some estimates in [MSED21], was presented in [MD21]:

$$(r+1) \left(2^{n\tau d^{2n+2}} \right)^{\mathcal{O}(2^{n\tau d^{2n+2}})},$$

where r is the number of inequality constraints and τ is a height bound for the input. A height analysis has also been developed in [DP24], involving the condition number of a certain Hankel matrix or the distance to the boundary of a cone, while the (worst-case) bounds in Theorem C only depend on the number of variables, the height and the degree of the input.

In [MDV23], bitsize bounds for the coefficients in the SOS representation modulo the gradient ideal, which is radical zero-dimensional, are of order $\mathcal{O}((n+d+\tau)d^{5n+2})$ for the coefficients of $\omega_{0,k}$ and $q_{0,k}$. The polynomial coefficients p_j are given in terms of the lexicographic Gröbner basis of I , under a Shape Lemma assumption, and have heights bounded by $\tilde{\mathcal{O}}(n(\tau+n+d)d^{3n+1})$.

The height bounds obtained in Theorem C, which are polynomial in terms of the maximum degree d of the input integer polynomials, linear in the maximum height τ of the coefficients and the Castelnuovo-Mumford regularity, and simply exponential in terms of the number of variables n , are to our knowledge the first simply exponential bounds for the heights in such SoS representations for arbitrary finite basic closed semialgebraic sets.

Bitsize lower bounds for rational SoS representations. Determining lower bounds for the bitsize (and also the degrees) of SoS representations is a mostly open challenge with important applications. For instance, in Theoretical Computer Science, there is interest in studying the question of certifying emptiness of a given (finite) semialgebraic set S , by providing a SoS representation for -1 , and the bitsize of such SoS representation governs the complexity of the problem. It has also been observed in [O'D17, RW17] that the bitsize of Positivstellensätze representations can grow exponentially in the number of variables. Lower bounds on the bitsize for low degree representations are also provided in these papers, showing in a particular example where $d = 2$ that any degree 2 SoS representations of a nonnegative polynomial must have bitsize in $\Theta(d^n)$. Unfortunately, this lower bound does not apply in our case, since the given generators of the ideal of this example do not form a graded basis. We also notice that in [RW17, Th. 1], the existence of a SoS representation over \mathbb{Q} is assumed, whereas Theorems A and B provide existential results for such representations. Finally, we mention that the tradeoff between bitsizes and degrees of representations in the Positivstellensätze has been investigated in [AH19, Hak21].

1.2 Applications to discrete polynomial optimization

An important area of applications of SoS representation is Polynomial Optimization, which relies on the celebrated moment/SoS hierarchy [Las01, Par00], that we recall briefly.

To define this hierarchy, we need the *truncated quadratic module* generated, in degree $\leq d$, by the polynomials \mathbf{g} and $\pm \mathbf{h}$:

$$Q_{K,d} = Q_{K,d}(\mathbf{g}, \pm \mathbf{h}) := \Sigma_{K,d}^2 + \sum_{i=1}^r \Sigma_{K,d-\deg(g_i)}^2 \cdot g_i + \sum_{j=1}^s K[\mathbf{x}]_{d-\deg(h_j)} \cdot h_j \subset K[\mathbf{x}]_d$$

where

$$\Sigma_{K,k}^2 := \Sigma_K^2 \cap K[\mathbf{x}]_k = \{ \omega_1 q_1^2 + \dots + \omega_r q_r^2 : r \in \mathbb{N}, \omega_i \in K_{\geq 0}, q_i \in K[\mathbf{x}]_{\lfloor k/2 \rfloor} \} \subset K[\mathbf{x}]_k$$

denotes SoS of degree $\leq k$.

Consider the problem of the minimization of a polynomial map f on a finite basic closed semialgebraic set $S = S(\mathbf{g}, \mathbf{h}) \subset \mathbb{R}^n$ as in (2), and denote $f^* = \inf_{x \in S} f(x)$, the minimum of f on S .

Given any integer $\ell \in \mathbb{N}$ we can define a lower bound for f^* as:

$$f_{(\ell)} := \sup \left\{ m \in \mathbb{R} : f - m \in Q_{\mathbb{R},2\ell} \right\} \leq f^*$$

which is the order ℓ of the so-called *SoS hierarchy*. By definition we have $f_{(\ell)} \leq f_{(\ell+1)} \leq f^*$ for all $\ell \in \mathbb{N}$. The hierarchy of dual convex optimization problems

$$f^{(\ell)} := \inf \left\{ \lambda(f) : \lambda \in \mathbb{R}[\mathbf{x}]_{2\ell}^*, \lambda(1) = 1, \lambda(q) \geq 0 \text{ for all } q \in Q_{\mathbb{R},2\ell} \right\} \leq f^*$$

is called the *moment hierarchy*. We have $f^{(\ell)} \leq f^{(\ell+1)} \leq f^*$ and $f_{(\ell)} \leq f^{(\ell)}$.

A natural question is about the *finite convergence* of such hierarchy, namely: does there exist $\ell \in \mathbb{N}$ such that $f_{(\ell)} = f^{(\ell)} = f^*$? In this case, can we certify this convergence, providing an explicit representation $f - f^* = \sigma_0 + \sum_{i=1}^r \sigma_i g_i + \sum_{j=1}^s p_j h_j \in Q_{\mathbb{R}}$ or $Q_{\mathbb{Q}}$?

Theorem A provides an answer to these questions, when applied to the polynomial $f - f^*$.

Corollary 1.1. *Let $\mathbf{g}, \mathbf{h} \subset \mathbb{Q}[\mathbf{x}]$ be as in (1), where \mathbf{h} is a graded basis of the zero-dimensional ideal $I \subset \mathbb{Q}[\mathbf{x}]$, B be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$, and $S \subset \mathbb{R}^n$ be the finite basic closed semialgebraic set defined in (2).*

Let $f \in \mathbb{Q}[\mathbf{x}]$ and f^ be the minimum of f on S . If $(I: f - f^*) + (f - f^*) = (1)$, then $f_{(r)} = f^*$ for*

$$r \geq \frac{1}{2} \max\{\deg(f), 2 \deg(B) + \deg(g_1), \dots, 2 \deg(B) + \deg(g_r)\}.$$

Furthermore, $f - f^$ has a certificate of nonnegativity as in Equation (3) using polynomials of degree $\leq 2r$, which belong to $\mathbb{Q}[\mathbf{x}]$ when $f^* \in \mathbb{Q}$.*

The hypotheses of Corollary 1.1 might at first seem difficult to verify, but they are satisfied for most of the concrete instances of optimization problems on finite sets. The degree bounds are similar to those that can be deduced implicitly in [Par02] for SoS representations over \mathbb{R} , in the case of a zero-dimensional ideal I .

The most prominent example of this is 0/1 optimization. Indeed,

$$\{0, 1\}^n = V(x_1^2 - x_1, \dots, x_n^2 - x_n),$$

where for $h_i = x_i^2 - x_i$, $\mathbf{h} = \{h_1, \dots, h_n\}$ is a graded basis of the zero-dimensional radical ideal $I = (\mathbf{h})$, and $B = \{x_1^{e_1} \cdots x_n^{e_n} : e_i \in \{0, 1\}\}$ has degree n . Moreover, since $\{0, 1\}^n \subset \mathbb{Q}^n$ we have $f^* \in \mathbb{Q}$ and therefore Corollary 1.1 fully applies.

The finite convergence of different families of convex relaxations for 0/1 programs has been studied since a long time, see e.g. [SA90, LS91]. The comparison of these techniques with the more recent moment/SoS hierarchy has been performed in [Lau03], demonstrating the strength of this new approach. In [Lau06], the same degree bound of Corollary 1.1 is provided for the finite convergence of the Moment hierarchy (i.e. $f^{(\ell)}$) and the SoS hierarchy (i.e. $f_{(\ell)}$), without deducing degree bounds for the SoS representation (when it exists). Similarly, sparse-type bounds $2\ell \leq n + \deg(f) - 1$ for the order ℓ of moment-like relaxations in the quotient algebra where $h_i = x_i^2 - 1$ and $V_{\mathbb{R}} = \{-1, 1\}^n$ have been obtained in [STKI17] (see also [FSP15]), without bounding the degree of SoS representations.

The bound of Corollary 1.1 is tight in general, see for instance [KLM15] and references therein: for binary optimization problems with $h_i = x_i^2 - x_i$, finite convergence always happens at order $r = n$ when there are no inequalities, and at order $r = n + 1$ when the g_i 's are linear. Applying Corollary 1.1 (and assuming that $\deg(f) \leq n$, which can be done by reducing f modulo \mathbf{h}) we get the same bounds, and we furthermore show that a rational certificate of nonnegativity for $f - f^*$ exists. The cases when $f_{(n-1)} < f^*$ (resp. $f_{(n)} < f^*$) are identified and characterized in [KLM15], showing that SoS representations may not exist in degree $< 2\delta$ (resp. $< 2\delta + \max_i \deg(g_i)$) and that the degree bounds of Theorem A (and Corollary 1.1) are tight.

More generally, the results of Corollary 1.1 can be directly applied when optimizing over any grid of points, see for instance [Las02]. We also notice that Corollary 1.1 can be used to provide degree bounds and rational certificates for the exactness of θ -body approximations of polynomial ideals, see e.g. [GPT10]. These approximations are, under the assumption that \mathbf{h} is a graded basis, essentially obtained using the moment/SoS hierarchy, see for instance [NP23, Rem. 3.25].

Finally, we remark that the zero-dimensional setting is important also for continuous polynomial optimization problems, as the latter can be reduced to the former by adding gradient [NDS06, MDV23] or KKT [DNP07] constraints.

1.3 Proof strategy and structure of the manuscript

The proof of Theorem A is developed in Sections 2 and 3. We first consider in Section 2.2 the case when the ideal I is radical and f is strictly positive on the whole $V_{\mathbb{R}}(I)$, obtaining in Theorem 2.1. This is a refined version of Theorem A in the radical case, with an additional non-vanishing condition that will allow us to lift our construction to a non-radical ideal I in Section 2.3. Then, we consider in Section 2.4 the case when $f > 0$ on an arbitrary finite basic closed semialgebraic set $S \subset \mathbb{R}^n$ as in (2), by perturbing the original polynomial f , which is positive on S , to an appropriate polynomial positive on all $V_{\mathbb{R}}(I)$. Finally, we study in Section 3 the case where $f \geq 0$ on S with $1 \in (I : f) + (f)$. In this case we show that we can replace our original polynomial f , which is nonnegative on S , by another related polynomial which is strictly positive on S , and conclude by applying the previous results.

Sections 4 to 6 analyze the height bounds for the case $K = \mathbb{Q}$ and I radical: the results are summarized in Theorem C. In Section 4 we derive, from an arithmetic Bézout theorem, height bounds for different polynomials in the quotient ring $\mathbb{Q}[\mathbf{x}]/I$ that we need in the sequel. The cases of f positive on S and f nonnegative on S are then developed in Section 5 and 6 respectively.

Finally Section 7 presents an algorithm to obtained sums of squares representations of the kind previously studied, and some examples that illustrate our constructions.

2 SoS representation for f strictly positive on S

In this section we consider Problem 1 stated in Section 1, whose solution is summarized in Theorem A, under the assumption that f is strictly positive on S . Without loss of generality, we give all our proofs for $K = \mathbb{Q}$ (the smallest field included in \mathbb{R}) since they mutatis-mutandis hold for any field $K \subset \mathbb{R}$, using the fact that K is dense in \mathbb{R} .

2.1 Notation and preliminaries

We fix the following notations that we will use in all our text.

- $\mathbf{g} = \{g_1, \dots, g_r\}$, $\mathbf{h} = \{h_1, \dots, h_s\} \subset \mathbb{Q}[\mathbf{x}]$ are as in (1) and $S = S(\mathbf{g}, \mathbf{h}) \subset \mathbb{R}^n$ is the basic closed semialgebraic set defined in (2).
- $I = (\mathbf{h}) \subset \mathbb{Q}[\mathbf{x}]$ is the zero-dimensional ideal generated by \mathbf{h} , and for $d \in \mathbb{N}$, $\mathbb{Q}[\mathbf{x}]_d = \{p \in \mathbb{Q}[\mathbf{x}] : \deg(p) \leq d\}$ denotes the polynomials of degree $\leq d$, and $I_d = I \cap \mathbb{Q}[\mathbf{x}]_d$.
- $I_{\mathbb{C}}$ denotes the ideal generated by (\mathbf{h}) in $\mathbb{C}[\mathbf{x}]$ and $I_{\mathbb{R}}$ denotes the ideal it generates in $\mathbb{R}[\mathbf{x}]$. Analogously, $V_{\mathbb{C}} := V_{\mathbb{C}}(I)$ is the affine variety defined by \mathbf{h} in \mathbb{C}^n and $V_{\mathbb{R}} := V_{\mathbb{R}}(I)$ is the affine variety defined by \mathbf{h} in \mathbb{R}^n .
- $D := \dim(\mathbb{Q}[\mathbf{x}]/I) = \#(V_{\mathbb{C}}(I))$, where the roots are counted with multiplicities, which satisfies $D \leq \max_j \{\deg(h_j)\}^n$ by Bézout's theorem.
- $S^D(\mathbb{R})$ (resp. $S^D(\mathbb{Q})$) denotes the set of symmetric matrices in $\mathbb{R}^{D \times D}$ (resp. $\mathbb{Q}^{D \times D}$).
- B denotes a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ associated to the basis \mathbf{h} , with first element 1 and then constructed by adding in the basis monomials obtained by multiplying with the variables x_1, \dots, x_n . We denote $\deg(B) := \max_{b \in B} \deg(b)$, which implies that $\deg(B) < D$. We also

denote by $\langle B \rangle_{\mathbb{Q}}$ (resp. $\langle B \rangle_{\mathbb{R}}, \langle B \rangle_{\mathbb{C}}$) the \mathbb{Q} -vector space spanned by B in $\mathbb{Q}[\mathbf{x}]$ (resp. \mathbb{R} -vector and \mathbb{C} -vector space space in $\mathbb{R}[\mathbf{x}]$ and $\mathbb{C}[\mathbf{x}]$). We also observe that any polynomial $p \in \mathbb{Q}[\mathbf{x}]$ can be decomposed as

$$p = \sum_{j=1}^s p_j h_j + \mathcal{N}(p)$$

for a unique $\mathcal{N}(p) \in \langle B \rangle_{\mathbb{Q}}$ (the *normal form* of p) and some $p_j \in \mathbb{Q}[\mathbf{x}]$ for $1 \leq j \leq s$. Moreover, when \mathbf{h} is a *graded basis* of I , i.e. when for any $d \in \mathbb{N}$ we have

$$I_d = \sum_{j=1}^s \mathbb{Q}[\mathbf{x}]_{d-\deg(h_j)} \cdot h_j,$$

we can take $p_j \in \mathbb{Q}[\mathbf{x}]_{\deg(p)-\deg(h_j)}$ for $1 \leq j \leq s$.

- We will make use of the idempotents $u_{\zeta} \in \langle B \rangle_{\mathbb{C}}$ for $\zeta \in V_{\mathbb{C}}(I) = V_{\mathbb{C}}$, which satisfy for $\zeta, \xi \in V_{\mathbb{C}}$:
 - $u_{\zeta}(\zeta) = 1$ and $u_{\zeta}(\xi) = 0$ for $\xi \neq \zeta$;
 - $u_{\bar{\zeta}} = \overline{u_{\zeta}}$, and in particular $u_{\xi} \in \mathbb{R}[\mathbf{x}]$ if $\xi \in V_{\mathbb{R}}$;
 - when I is a radical ideal (which implies that if $p \in \mathbb{C}[\mathbf{x}]$ satisfies that $p(\zeta) = 0$ for all $\zeta \in V_{\mathbb{C}}$, then $p \in I$) we have $u_{\zeta}^2 \equiv u_{\zeta} \pmod{I_{\mathbb{C}}}$ and $u_{\zeta} u_{\xi} \equiv 0 \pmod{I_{\mathbb{C}}}$ if $\zeta \neq \xi$.

2.2 The case when $S = V_{\mathbb{R}}(I)$ for a radical zero-dimensional ideal I

We assume in this section that I is a radical zero-dimensional ideal, i.e. in addition to be zero-dimensional, we assume that $\sqrt{I} = I$, so that all the points $\zeta \in V_{\mathbb{C}}$ have multiplicity 1. We also assume that $\mathbf{g} = \emptyset$, so that S coincides with the finite real algebraic variety $V_{\mathbb{R}} = V_{\mathbb{R}}(I)$.

Our goal is to extend the univariate case treated in [KMS23, Proposition 2.2] to the multivariate setting. The final result of the section is the following theorem.

Theorem 2.1. *Let $I \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional radical ideal and let $f \in \mathbb{Q}[\mathbf{x}]$ be such that $f(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}(I)$. Let B be a basis of $\mathbb{Q}[\mathbf{x}]/I$. Then, for all $\zeta \in V_{\mathbb{C}}(I)$, there exist $\omega_{\zeta} \in \mathbb{Q}_{>0}$ and $\theta_{\zeta} \in \langle B \rangle_{\mathbb{Q}}$ such that*

$$f \equiv \sum_{\zeta \in V_{\mathbb{C}}} \omega_{\zeta} \theta_{\zeta}^2 \pmod{I}.$$

Moreover, when $V_{\mathbb{C}} \neq \emptyset$ there exists $\zeta_0 \in V_{\mathbb{C}}$ such that $\theta_{\zeta_0}(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$.

We begin by considering in next lemma the case where $V_{\mathbb{R}}(I) \neq \emptyset$.

Lemma 2.2. *Let $I \subset \mathbb{R}[\mathbf{x}]$ be a zero-dimensional radical ideal such that that $V_{\mathbb{R}}(I) \neq \emptyset$, and let $f \in \mathbb{R}[\mathbf{x}]$ be such that $f(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}$. Let B be a basis of $\mathbb{R}[\mathbf{x}]/I$. Then there exist $\theta_{\zeta} \in \langle B \rangle_{\mathbb{R}}$, $\zeta \in V_{\mathbb{C}}$, which form a basis of $\mathbb{R}[\mathbf{x}]/I$, such that*

$$f \equiv \sum_{\zeta \in V_{\mathbb{C}}} \theta_{\zeta}^2 \pmod{I_{\mathbb{R}}}.$$

Moreover, there exists $\xi_0 \in V_{\mathbb{R}}$ such that $\theta_{\xi_0}(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$.

Proof. The condition $\theta_{\xi_0}(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$ in the statement requires a change in the proof of [KMS23, Proposition 2.2]: we will take a suitable linear combination of the u_i 's to guarantee it. We choose $\xi_0 \in V_{\mathbb{R}}$ and given $\varepsilon \in \mathbb{R}$ for $\xi \in V_{\mathbb{R}} \setminus \{\xi_0\}$, we define

$$v_{\xi_0} = u_{\xi} + \varepsilon \sum_{\xi \in V_{\mathbb{R}}, \xi \neq \xi_0} u_{\xi} + \sum_{\zeta \in V_{\mathbb{C}} \setminus V_{\mathbb{R}}} u_{\zeta} \in \langle B \rangle_{\mathbb{R}} \quad (5)$$

which is a real polynomial since for $\zeta \in V_{\mathbb{C}} \setminus V_{\mathbb{R}}$, $u_{\bar{\zeta}} = \bar{u}_{\zeta}$. This polynomial satisfies $v_{\xi_0}(\xi_0) = 1$, $v_{\xi_0}(\xi) = \varepsilon$ for all $\xi \in V_{\mathbb{R}}$, $\xi \neq \xi_0$, and $v_{\xi_0}(\zeta) = 1$ for all $\zeta \in V_{\mathbb{C}} \setminus V_{\mathbb{R}}$. Set Z to be a minimal set of complex non-real roots in $V_{\mathbb{C}}$ such that $V_{\mathbb{C}} \setminus V_{\mathbb{R}} = Z \cup \bar{Z}$. Then, given $\lambda_{\zeta} \in \mathbb{R}$ for $\zeta \in V_{\mathbb{C}} \setminus V_{\mathbb{R}}$ we can verify that

$$\begin{aligned} f &\equiv f(\xi_0) v_{\xi_0}^2 + \sum_{\xi \in V_{\mathbb{R}}, \xi \neq \xi_0} (f(\xi) - \varepsilon^2 f(\xi_0)) u_{\xi}^2 \\ &\quad + \sum_{\zeta \in Z} ((f(\zeta) - f(\xi_0)) u_{\zeta}^2 + (f(\bar{\zeta}) - f(\xi_0)) \bar{u}_{\zeta}^2 + 2\lambda_{\zeta} u_{\zeta} \bar{u}_{\zeta}) \pmod{I_{\mathbb{R}}} \end{aligned}$$

since the expression in the right-hand side is a real polynomial that coincides with f on all $\zeta \in V_{\mathbb{C}}$, and therefore, by the Nullstellensatz, their difference belongs to $\sqrt{I} = I$ by assumption.

We choose $\varepsilon \in \mathbb{R}_{>0}$ small enough such that $f(\xi) - \varepsilon^2 f(\xi_0) > 0$ for all $\xi \in V_{\mathbb{R}} \setminus \{\xi_0\}$, and for $\zeta \in Z$, we will apply the following identity to

$$(f(\zeta) - f(\xi_0)) u_{\zeta}^2 + (f(\bar{\zeta}) - f(\xi_0)) \bar{u}_{\zeta}^2 + 2\lambda_{\zeta} u_{\zeta} \bar{u}_{\zeta} \quad (6)$$

for a suitable λ : Given $a, b, \lambda \in \mathbb{R}$ with $a + \lambda \neq 0$ and $u, v \in \mathbb{R}[\mathbf{x}]$:

$$\begin{aligned} &(a + \mathbf{i}b)(u + \mathbf{i}v)^2 + (a - \mathbf{i}b)(u - \mathbf{i}v)^2 + 2\lambda|u + \mathbf{i}v|^2 \\ &= 2((\lambda + a)u^2 - 2buv + (\lambda - a)v^2) \\ &= 2(\lambda + a)\left(u - \frac{b}{\lambda + a}v\right)^2 + 2\frac{\lambda^2 - |a + \mathbf{i}b|^2}{\lambda + a}v^2. \end{aligned} \quad (7)$$

We observe that when $\lambda > |a + \mathbf{i}b|$, then $\lambda + a > 0$ and $\lambda^2 - |a + \mathbf{i}b|^2 > 0$. Identity (7) thus shows that when $\lambda > |f(\zeta) - f(\xi_0)|$, (6) is a sum of two positive-weighted squares of real polynomials.

Therefore, by taking $\lambda \in \mathbb{R}_{>0}$ such that $\lambda > |f(\zeta) - f(\xi_0)|$ for all $\zeta \in V_{\mathbb{C}} \setminus V_{\mathbb{R}}$, we obtain that

$$f \equiv \sum_{\xi \in V_{\mathbb{C}}} \theta_{\xi}^2 \pmod{I_{\mathbb{R}}} \quad (8)$$

where, denoting respectively \Re and \Im the real and imaginary part:

- $\theta_{\xi_0} = \sqrt{f(\xi_0)} v_{\xi_0} \in \langle B \rangle_{\mathbb{R}}$ where \tilde{u}_{ξ_0} is defined in (5);
- $\theta_{\xi} = \sqrt{f(\xi) - \varepsilon^2 f(\xi_0)} u_{\xi} \in \langle B \rangle_{\mathbb{R}}$, for all $\xi \in V_{\mathbb{R}} \setminus \{\xi_0\}$;
- $\theta_{\zeta} = \sqrt{2(\lambda + \Re(f(\zeta)) - f(\xi_0))} (\Re(u_{\zeta}) - \frac{\Im(f(\zeta))}{\lambda_{\zeta} + \Re(f(\zeta)) - f(\xi_0)} \Im(u_{\zeta})) \in \langle B \rangle_{\mathbb{R}}$, for all $\zeta \in Z$;
- $\theta_{\bar{\zeta}} = \sqrt{2\frac{\lambda_{\zeta}^2 - |f(\zeta) - f(\xi_0)|^2}{\lambda_{\zeta} + \Re(f(\zeta)) - f(\xi_0)}} \Im(u_{\zeta}) \in \langle B \rangle_{\mathbb{R}}$, for all $\zeta \in Z$.

Since the polynomials $\{u_\zeta : \zeta \in V_{\mathbb{C}}\}$ are linearly independent over \mathbb{C} , the polynomials

$$\{u_\xi, \Re(u_\zeta), \Im(u_\zeta) : \xi \in V_{\mathbb{R}}, \zeta \in Z\}$$

are also linearly independent over \mathbb{R} , and therefore a basis of $\mathbb{R}[\mathbf{x}]/I$ because of its cardinality. This implies that the polynomials $\{\theta_\zeta : \zeta \in V_{\mathbb{C}}\}$, are a basis of $\mathbb{R}[\mathbf{x}]/I$ and by construction $\theta_{\zeta_0}(\zeta) \neq 0$, $\forall \zeta \in V_{\mathbb{C}}$. This concludes the proof. \square

We now consider the other case, when $V_{\mathbb{R}} = \emptyset$.

Lemma 2.3. *Let $I \subset \mathbb{R}[\mathbf{x}]$ be a zero-dimensional radical ideal such that $V_{\mathbb{R}}(I) = \emptyset$. Let B be a basis of $\mathbb{R}[\mathbf{x}]/I$ and $f \in \mathbb{R}[\mathbf{x}]$. Then there exist $\theta_\zeta \in \langle B \rangle_{\mathbb{R}}$, $\zeta \in V_{\mathbb{C}}$, which form a basis of $\mathbb{R}[\mathbf{x}]/I$, such that*

$$f \equiv \sum_{\zeta \in V_{\mathbb{C}}} \theta_\zeta^2 \pmod{I_{\mathbb{R}}}.$$

Moreover, there exists $\zeta_0 \in V_{\mathbb{C}}$ such that $\theta_{\zeta_0}(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$.

Proof. We choose $\zeta_0 \in V_{\mathbb{C}}$ and for $\varepsilon \in \mathbb{R}$ define

$$v_{\zeta_0} = u_{\zeta_0} + \varepsilon u_{\bar{\zeta}_0} + \sum_{\substack{\zeta \in V_{\mathbb{C}} \\ \zeta \neq \zeta_0, \zeta \neq \bar{\zeta}_0}} u_\zeta \quad \text{and} \quad v_{\bar{\zeta}_0} = \bar{v}_{\zeta_0}$$

which satisfy $v_{\zeta_0}(\zeta_0) = 1$, $v_{\zeta_0}(\bar{\zeta}_0) = \varepsilon$ and $v_{\zeta_0}(\zeta) = 1$, $\forall \zeta \in V_{\mathbb{C}}$ with $\zeta \neq \zeta_0, \zeta \neq \bar{\zeta}_0$. We note that the set $\{v_{\zeta_0}, v_{\bar{\zeta}_0}, u_\zeta : \zeta \in V_{\mathbb{C}}, \zeta \neq \zeta_0, \zeta \neq \bar{\zeta}_0\}$ is linearly independent over \mathbb{C} for $\varepsilon \neq 1$, and therefore a basis of $\mathbb{C}[\mathbf{x}]/I$.

Let $Z \subset V_{\mathbb{C}}$ be a minimal set such that $V_{\mathbb{C}} = Z \cup \bar{Z}$, and let $f \in \mathbb{R}[\mathbf{x}]$. Given $\lambda_0, \lambda \in \mathbb{R}$, we observe that

$$\begin{aligned} f \equiv & \left(\frac{f(\zeta_0) - \varepsilon^2 f(\bar{\zeta}_0)}{1 - \varepsilon^4} - \varepsilon \lambda_0 \right) v_{\zeta_0}^2 + \left(\frac{f(\bar{\zeta}_0) - \varepsilon^2 f(\zeta_0)}{1 - \varepsilon^4} - \varepsilon \lambda_0 \right) v_{\bar{\zeta}_0}^2 \\ & + (1 + \varepsilon^2) \lambda_0 v_{\zeta_0} v_{\bar{\zeta}_0} + \sum_{\zeta \in Z, \zeta \neq \zeta_0} (\rho_\zeta u_\zeta^2 + \bar{\rho}_\zeta u_{\bar{\zeta}}^2 + 2\lambda u_\zeta u_{\bar{\zeta}}) \pmod{I_{\mathbb{R}}} \end{aligned} \quad (9)$$

where

$$\rho_\zeta := f(\zeta) - \left(\frac{f(\zeta_0) + f(\bar{\zeta}_0)}{1 + \varepsilon^2} + (1 - \varepsilon)^2 \lambda_0 \right) = f(\zeta) - \left(\frac{2\Re(f(\zeta_0))}{1 + \varepsilon^2} + (1 - \varepsilon)^2 \lambda_0 \right)$$

since $f(\zeta)$ and the expression in the right-hand side coincide on all $\zeta \in V_{\mathbb{C}}$. In order to apply Identity (7) to the first terms in the right-hand side of Equation (9), we first choose $\varepsilon, \lambda_0 \in \mathbb{R}_{>0}$ such that

$$(1 + \varepsilon^2) \lambda_0 > 2 \left| \frac{f(\zeta_0) - \varepsilon^2 f(\bar{\zeta}_0)}{1 - \varepsilon^4} - \varepsilon \lambda_0 \right|.$$

(we can first choose $\lambda_0 > 2|f(\zeta_0)|$ and then $\varepsilon > 0$ so that the above inequality, which is valid at $\varepsilon = 0$, still holds for $\varepsilon > 0$ small enough by continuity). Identity (7) then gives expressions

$$\theta_{\zeta_0} = \sqrt{2(\lambda + a)} \left(\Re(v_{\zeta_0}) - \frac{b}{\lambda + a} \Im(v_{\zeta_0}) \right), \quad \theta_{\bar{\zeta}_0} = \sqrt{2 \frac{\lambda^2 - |a + \mathbf{i}b|^2}{\lambda + a}} \Im(v_{\zeta_0}) \in \mathbb{R}[\mathbf{x}].$$

Now since

$$\Re(v_{\zeta_0})(\zeta_0) = \Re(v_{\zeta_0})(\bar{\zeta}_0) = \frac{1+\varepsilon}{2}, \quad \Re(v_{\zeta_0})(\zeta) = \Re(v_{\zeta_0})(\bar{\zeta}) = 1 \quad \text{for } \zeta \in Z, \zeta \neq \zeta_0,$$

and

$$\Im(v_{\zeta_0})(\zeta_0) = \frac{(1-\varepsilon)\mathbf{i}}{2}, \quad \Im(v_{\zeta_0})(\bar{\zeta}_0) = -\Im(v_{\zeta_0})(\zeta_0), \quad \Im(v_{\zeta_0})(\zeta) = \Im(v_{\zeta_0})(\bar{\zeta}) = 0 \quad \text{for } \zeta \in Z, \zeta \neq \zeta_0,$$

we have $\theta_{\zeta_0}(\zeta) \neq 0$ for all $\zeta \in Z$.

For the remaining ζ , we take $\lambda \in \mathbb{R}$ such that $\lambda > |\rho_\zeta|$, for all $\zeta \in V_{\mathbb{C}}, \zeta \neq \zeta_0$. This allows to deduce from Identity (7) a decomposition of the form

$$f \equiv \sum_{\zeta \in Z} (\theta_\zeta^2 + \theta_{\bar{\zeta}}^2) \pmod{I},$$

with $\theta_{\zeta_0}(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$, which concludes the proof. \square

With these two lemmas, which provide real representations, we can now prove Theorem 2.1.

Proof of Theorem 2.1. If $V_{\mathbb{C}} = \emptyset$, then $I_{\mathbb{Q}} = (1)$ and we have the decomposition $f \equiv 0 \pmod{(1)}$. We therefore assume that $V_{\mathbb{C}} \neq \emptyset$.

From Lemma 2.2 and Lemma 2.3, we have that

$$f \equiv \sum_{\zeta \in V_{\mathbb{C}}} \tilde{\theta}_\zeta^2 \pmod{I_{\mathbb{R}}} \quad \text{with} \quad \sum_{\zeta \in V_{\mathbb{C}}} \tilde{\theta}_\zeta^2 = B(\tilde{\Theta} \tilde{\Theta}^t)B^t,$$

for a basis $\{\tilde{\theta}_\zeta, \zeta \in V_{\mathbb{C}}\} = (\tilde{\theta}_1, \dots, \tilde{\theta}_D)$ in $\langle B \rangle_{\mathbb{R}}$ of $\mathbb{R}[\mathbf{x}]/I$ where $\tilde{\theta}_D := \tilde{\theta}_{\zeta_0}$ satisfies $\tilde{\theta}_D(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$, and, by setting $B = (b_1, \dots, b_D)$, $\tilde{\Theta} \in \mathbb{R}^{D \times D}$ is the invertible matrix defined by $(\tilde{\theta}_1, \dots, \tilde{\theta}_D) = B \tilde{\Theta}$, which satisfies $((B \tilde{\Theta})(\zeta))_D = \sum_{i=1}^D \tilde{\Theta}_{iD} b_i(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$.

As \mathbb{Q} is dense in \mathbb{R} , we can choose a close invertible approximation $\Theta \in \mathbb{Q}^{D \times D}$ of $\tilde{\Theta}$, so that $\tilde{\Theta} \Theta^{-1} = \text{Id} + E \sim \text{Id}$, i.e. $\tilde{\Theta} = \Theta(\text{Id} + E)$, which still satisfies that $((B \Theta)(\zeta))_D \neq 0$ for all $\zeta \in V_{\mathbb{C}}$, and $\tilde{Q} := (\text{Id} + E)(\text{Id} + E^t) \in S^D(\mathbb{R})$, close to Id , is still definite positive.

Let now $\hat{Q} \in S^D(\mathbb{Q})$ be a close enough positive definite rational approximation of \tilde{Q} , so that the projection $Q \in S^D(\mathbb{Q})$ of \hat{Q} on the linear variety

$$\mathcal{L}_f = \{Y \in S^D(\mathbb{R}) : f \equiv B(\Theta Y \Theta^t)B^t \pmod{I_{\mathbb{R}}}\},$$

which satisfies $f \equiv B(\Theta Q \Theta^t)B^t \pmod{I}$, is still positive definite.

Applying the square-root-free Cholesky decomposition (see Proposition A.5) to Q , we obtain

$$Q = L \Delta L^t$$

where $L \in \mathbb{Q}^{D \times D}$ is invertible and lower triangular and Δ is a diagonal matrix with rational positive entries. Moreover, we observe that the last column of ΘL satisfies that $(\Theta L)_{i,D} = L_{D,D} \Theta_{i,D}$ for $1 \leq i \leq D$, and therefore, $((B \Theta L)(\zeta))_D = L_{D,D} ((B \Theta)(\zeta))_D \neq 0$ for all $\zeta \in V(\mathbb{C})$.

We conclude by observing that

$$B(\Theta Q \Theta^t)B^t = (B \Theta L) \Delta (B \Theta L)^t = \sum_{\zeta \in V_{\mathbb{C}}} \omega_\zeta \theta_\zeta^2$$

with $\omega_\zeta \in \mathbb{Q}_{>0}$ (the coefficients of Δ) and $\theta_\zeta = B \Theta L \in \langle B \rangle_{\mathbb{Q}}$ which satisfies $\theta_{\zeta_0}(\zeta) = ((B \Theta L)(\zeta))_D \neq 0$, for all $\zeta \in V_{\mathbb{C}}$. \square

2.3 The case $S = V_{\mathbb{R}}(I)$ for an arbitrary zero-dimensional ideal I

Here we consider the case where $I = (\mathbf{h}) \in \mathbb{Q}[\mathbf{x}]$ is an arbitrary zero-dimensional ideal defining the roots $\zeta \in V_{\mathbb{C}}$ with some multiplicities. We will extend Theorem 2.1 to this setting by Hensel lifting. We need the following elementary lemma.

Lemma 2.4. *Let $J \in \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional ideal and $\theta \in \mathbb{Q}[\mathbf{x}]$ such that $\theta(\zeta) \neq 0, \forall \zeta \in V_{\mathbb{C}}(J)$. Then θ has an inverse σ modulo J in $\mathbb{Q}[\mathbf{x}]$.*

Proof. Since $\theta(\zeta) \neq 0, \forall \zeta \in V_{\mathbb{C}}, V_{\mathbb{C}}(J + (\theta)) = \emptyset$ and therefore, by the Nullstellensatz, $1 \in J + (\theta) \subset \mathbb{Q}[\mathbf{x}]$. Therefore, there exists $\sigma \in \mathbb{Q}[\mathbf{x}]$ such that $\theta \sigma \equiv 1 \pmod{J}$. \square

We now lift a sum of squares representation modulo a radical ideal J to its powers by Hensel lifting, applying Newton iterations for computing square-roots modulo an ideal (see e.g. [vG13, Chap. 9], [KMS23, Lemma 2.10]).

Proposition 2.5. *Let $J \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional ideal. Let $\theta \in \mathbb{Q}[\mathbf{x}]$ and assume that there exists $\theta_0 \in \mathbb{Q}[\mathbf{x}]$ with $\theta_0(\zeta) \neq 0, \forall \zeta \in V_{\mathbb{C}}(J)$, such that $\theta_0^2 \equiv \theta \pmod{J}$. Then for all $k \in \mathbb{N}$, there exists $\theta_k \in \mathbb{Q}[\mathbf{x}]$ such that*

$$\theta_k^2 \equiv \theta \pmod{J^{2^k}}.$$

Proof. We prove the proposition by induction, applying Newton iterations to the equation $x^2 - \theta = 0$. The property is true for $k = 0$ by hypothesis. Assume that it is true for $k \in \mathbb{N}$. There exists $\theta_k \in \mathbb{Q}[\mathbf{x}]$ such that $\theta_k^2 \equiv \theta \pmod{J^{2^k}}$ or equivalently $\theta_k^2 - \theta \in J^{2^k}$. Then, since $\theta_k^2 \equiv \theta_0^2 \pmod{J}$, one has that for all $\zeta \in V_{\mathbb{C}}, \theta_k^2(\zeta) \neq 0$, and by Lemma 2.4, θ_k is invertible modulo the zero-dimensional ideal $J^{2^{k+1}}$.

Let $\sigma_k \in \mathbb{Q}[\mathbf{x}]$ be such that $\theta_k \sigma_k \equiv 1 \pmod{J^{2^{k+1}}}$. We define $\theta_{k+1} = \frac{1}{2}(\theta_k + \theta \sigma_k)$. Then, for $p_k := \theta_k^2 - \theta$, we have

$$\begin{aligned} \theta_{k+1} - \theta_k &= \frac{1}{2}(\theta_k + \theta \sigma_k) - \theta_k = \frac{1}{2}(-\theta_k + \theta \sigma_k) \\ &\equiv \frac{1}{2}\sigma_k(-\theta_k^2 + \theta) \pmod{J^{2^{k+1}}} \\ &\equiv -\frac{1}{2}\sigma_k p_k \pmod{J^{2^{k+1}}}. \end{aligned}$$

We deduce that

$$\begin{aligned} \theta_{k+1}^2 &= (\theta_k + \theta_{k+1} - \theta_k)^2 \equiv (\theta_k - \frac{1}{2}\sigma_k p_k)^2 \pmod{J^{2^{k+1}}} \\ &\equiv \theta_k^2 - \theta_k \sigma_k p_k + \frac{1}{4}\sigma_k^2 p_k^2 \pmod{J^{2^{k+1}}} \equiv \theta_k^2 - \theta_k \sigma_k p_k \pmod{J^{2^{k+1}}} \\ &\equiv p_k + \theta - p_k \pmod{J^{2^{k+1}}} \equiv \theta \pmod{J^{2^{k+1}}}, \end{aligned}$$

since $p_k^2 \in (J^{2^k})^2 = J^{2^{k+1}}$ and $\theta_k \sigma_k \equiv 1 \pmod{J^{2^{k+1}}}$.

This completes the proof by induction. \square

Proposition 2.5 allows us to extend Theorem 2.1 to the case when I is an arbitrary zero-dimensional ideal, not necessarily radical.

Theorem 2.6. *Let $I \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional ideal and $f \in \mathbb{Q}[\mathbf{x}]$ be such that $f(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}$. Let B be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $|B| = D$. Then, there exist $\omega_k \in \mathbb{Q}_{\geq 0}$ and $q_k \in \langle B \rangle_{\mathbb{Q}}$, $1 \leq k \leq D$, such that*

$$f \equiv \sum_{k=1}^D \omega_k q_k^2 \pmod{I}.$$

Proof. Define $J := \sqrt{I}$. Since $\mathbb{Q}[\mathbf{x}]/J \simeq (\mathbb{Q}[\mathbf{x}]/I)/(J/I)$, we can assume that a monomial basis of $\mathbb{Q}[\mathbf{x}]/J$ is included in the monomial basis B of $\mathbb{Q}[\mathbf{x}]/I$. Then, by Theorem 2.1, since $\#(V_{\mathbb{C}}(J)) = D$, there exists $\omega_k \in \mathbb{Q}_{>0}$, $\theta_k \in \langle B \rangle_{\mathbb{Q}}$, $1 \leq k \leq D$, where we can assume $\theta_1(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$, such that

$$f \equiv \sum_{k=1}^D \omega_k \theta_k^2 \pmod{J}.$$

Let

$$\theta := \frac{1}{\omega_1} \left(f - \sum_{k=2}^D \omega_k \theta_k^2 \right) \in \mathbb{Q}[\mathbf{x}],$$

so that $\theta_1^2 \equiv \theta \pmod{J}$ with $\theta_1(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$. By Proposition 2.5 applied to $k \in \mathbb{N}$ such that $J^{2^k} \subset I$, there exists $q_1 \in \mathbb{Q}[\mathbf{x}]$ such that $q_1^2 \equiv \theta \pmod{I}$. Moreover, by reducing q_1 modulo I we can assume that $q_1 \in \langle B \rangle_{\mathbb{Q}}$. Setting $q_k := \theta_k$ for $2 \leq k \leq D$, we deduce that

$$f \equiv \sum_{k=1}^D \omega_k q_k^2 \pmod{I}$$

with $q_k \in \langle B \rangle_{\mathbb{Q}}$ for $1 \leq k \leq D$. □

2.4 Extension to a basic closed semialgebraic set S

We show in this section how to extend Theorem 2.6 to the case when S is an arbitrary basic closed semialgebraic set defined as in (2) by a set of polynomials $\mathbf{g} = \{g_1, \dots, g_r\}$ and $\mathbf{h} = \{h_1, \dots, h_s\}$ with the ideal $I = (\mathbf{h})$ non-necessarily radical. More precisely we obtain the following result.

Theorem 2.7. *Let $\mathbf{g}, \mathbf{h} \subset \mathbb{Q}[\mathbf{x}]$ be as in Equation (1), $I = (\mathbf{h}) \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional ideal, B be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $|B| = D$, and $S \subset \mathbb{R}^n$ be the finite basic closed semialgebraic set defined as in (2).*

If $f \in \mathbb{Q}[\mathbf{x}]$ is such that $f > 0$ on S , then there exist $\omega_{i,k} \in \mathbb{Q}_{\geq 0}$, $q_{i,k} \in \langle B \rangle_{\mathbb{Q}}$ and $p_j \in \mathbb{Q}[\mathbf{x}]$ for $0 \leq i \leq r$, $1 \leq k \leq D$ and $1 \leq j \leq s$ such that

$$f = \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 + \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i + \sum_{j=1}^s p_j h_j.$$

Furthermore, if \mathbf{h} is a graded basis of I and $\deg(B)$ is an upper bound for the maximum degree of a monomial in B , then for all j

$$\deg(p_j h_j) \leq \max\{\deg(f), \deg(g_i) + 2 \deg(B) : 1 \leq i \leq r\}.$$

First, in the next lemma we proceed as in [Par02, Algorithm 2], and reduce from the condition $f(\xi) > 0$ for all $\xi \in S$ (where S is the closed basic semialgebraic set described in (2)) to the condition $\tilde{f}(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}(I)$, where $I = (\mathbf{h})$ and \tilde{f} is an appropriate perturbation of f .

Lemma 2.8. *Let S be a basic closed semialgebraic set as in (2), and assume that $I = (\mathbf{h})$ is zero-dimensional. Let $f \in \mathbb{Q}[\mathbf{x}]$ be such that $f(\xi) > 0$ for all $\xi \in S$. Then there exist $\omega_{i,k} \in \mathbb{Q}_{\geq 0}$ and $q_{i,k} \in \langle B \rangle$, $1 \leq i \leq r$, $1 \leq k \leq D$, such that*

$$\varphi = \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i$$

satisfies

$$(f - \varphi)(\xi) > 0 \text{ for all } \xi \in V_{\mathbb{R}}.$$

Proof. For every $\xi \in V_{\mathbb{R}} \setminus S$, there exists i_{ξ} with $1 \leq i_{\xi} \leq r$ such that $g_{i_{\xi}}(\xi) < 0$. Let $\rho_{\xi} \in \mathbb{R}_+$ be such that $f(\xi) - \rho_{\xi} g_{i_{\xi}}(\xi) > 0$. We define

$$\tilde{\varphi} := \sum_{\xi \in V_{\mathbb{R}} \setminus S} \rho_{\xi} u_{\xi}^2 g_{i_{\xi}}$$

where $u_{\xi} \in \langle B \rangle_{\mathbb{R}}$ are the idempotents associated to $\xi \in V_{\mathbb{R}} \setminus S$. Then

- $(f - \tilde{\varphi})(\xi) = f(\xi) > 0$ for $\xi \in S$,
- $(f - \tilde{\varphi})(\xi) = f(\xi) - \rho_{\xi} g_{i_{\xi}}(\xi) > 0$ for $\xi \in V_{\mathbb{R}} \setminus S$.

Now, since \mathbb{Q} is dense in \mathbb{R} we can approximate ρ_{ξ} by some rational value and u_{ξ} by some rational polynomial of the same degree, for all $\xi \in V_{\mathbb{R}} \setminus S$, to obtain a polynomial $\varphi \in \mathbb{Q}[\mathbf{x}]$ as stated such that $(f - \varphi)(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}$ still hold. \square

Proof of Theorem 2.7. Let φ be the polynomial defined in Lemma 2.8 and let $\tilde{f} = f - \varphi$. As $\tilde{f}(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}$, by Theorem 2.6 there exists $\omega_{0,k} \in \mathbb{Q}_{\geq 0}$, $q_{0,k} \in \langle B \rangle_{\mathbb{Q}}$, $1 \leq k \leq D$, such that

$$\tilde{f} \equiv \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 \pmod{I}.$$

Thus $f \equiv \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 + \varphi \pmod{I}$, i.e. $f - \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 - \varphi \in I$, and since

$$\deg\left(f - \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 - \varphi\right) \leq \max\{\deg(f), 2 \deg(B) + \max_i \{\deg(g_i)\}\},$$

when \mathbf{h} is a graded basis of I , we can take

$$f - \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 - \varphi = \sum_{j=1}^s p_j h_j$$

with

$$\deg(p_j h_j) \leq \max\{\deg(f), 2 \deg(B) + \max_i \{\deg(g_i)\}\}. \quad \square$$

3 SoS representation for f nonnegative on S

Here we consider Problem 2 stated in Section 1, whose answer is summarized in Theorem A, when f is nonnegative on S and $(I : f) + (f) = (1)$.

We first observe that if K is a subfield of \mathbb{R} and $f \in K[\mathbf{x}]$ is a polynomial such that $f(\xi) \geq 0$ for all $\xi \in S$, it might happen that f does not belong to the associate quadratic module Q_K , as shown by the following example (see also [Par02, Remark 1]).

Example 3.1. Let $I = (x^2) \subset K[x]$ and $f = x$. We have $f(\xi) \geq 0$ for $\xi \in S = \{0\}$. But $f \notin Q_K = Q_K(\pm x^2)$ (see definition (4)). Otherwise, there would exist $\sigma_0 \in \Sigma_K^2$ and $p \in K[\mathbf{x}]$ such that

$$x = \sigma_0 + p \cdot x^2.$$

This implies that σ_0 , which is a sum of squares of polynomials which can't have a constant term, is divisible by x^2 . This contradicts the possible SoS decomposition and shows that $x \notin Q_K$. We observe that the same argument can be used to show that $x \notin Q_K(\pm x^k)$ for $k \geq 2$.

However we can give a positive answer under a condition of separability of the zeros of f , which is discussed in detail in Section 1.1.

Theorem 3.2. Let $\mathbf{g}, \mathbf{h} \subset \mathbb{Q}[\mathbf{x}]$ be as in (1), $I = (\mathbf{h}) \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional ideal, B be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $|B| = D$, and $S \subset \mathbb{R}^n$ be the finite basic closed semialgebraic set defined in (2).

If $f \in \mathbb{Q}[\mathbf{x}]$ is such that $f \geq 0$ on S and $(I : f) + (f) = (1)$, then there exist $\omega_{i,k} \in \mathbb{Q}_{\geq 0}$, $q_{i,k} \in \langle B \rangle_{\mathbb{Q}}$ and $p_j \in \mathbb{Q}[\mathbf{x}]$ for $0 \leq i \leq r$, $1 \leq k \leq D$ and $1 \leq j \leq s$ such that

$$f = \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 + \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i + \sum_{j=1}^s p_j h_j.$$

Furthermore, if \mathbf{h} is a graded basis of I and $\deg(B)$ is an upper bound for the maximum degree of a monomial in B , then for all j

$$\deg(p_j h_j) \leq \max\{\deg(f), \deg(g_i) + 2 \deg(B) : 1 \leq i \leq r\}.$$

Proof. Since $(I : f) + (f) = (1)$, we can find $a \in \mathbb{Q}[\mathbf{x}]$ and $b \in (I : f)$ such that $b + a f = 1$.

We reduce a and b modulo I and obtain $\tilde{a}, \tilde{b} \in \langle B \rangle$ that satisfy $\tilde{b} \in (I : f)$ and $\tilde{b} + \tilde{a} f \equiv 1 \pmod{I}$. Set $S^* := \{\xi \in S : f(\xi) = 0\} \subset V_{\mathbb{R}}(f)$, let $\rho \in \mathbb{Q}_+$ be such that $\rho + \tilde{a}(\xi) > 0$ for all $\xi \in S^*$, and define $\bar{a} := \tilde{a} + \rho \tilde{b} \in \mathbb{Q}[\mathbf{x}]$. Then, since $\tilde{b} f \in I$ one has

$$\tilde{b} + \bar{a} f \equiv 1 \pmod{I}. \tag{10}$$

We observe that $\bar{a}(\xi) > 0$ for all $\xi \in S$. Indeed, for $\xi \in S^*$ Identity (10) implies that $\tilde{b}(\xi) = 1$ when $f(\xi) = 0$, and then

$$\bar{a}(\xi) = \tilde{a}(\xi) + \rho \tilde{b}(\xi) = \tilde{a}(\xi) + \rho > 0,$$

while for $\xi \in S \setminus S^*$, $\tilde{b} f \in I$ implies that $\tilde{b}(\xi) = 0$ since $f(\xi) \neq 0$, and then

$$\bar{a}(\xi) = 1/f(\xi) > 0.$$

We deduce from Theorem 2.7 that

$$\bar{a} \equiv \sum_{k=1}^D \omega_{0,k} \bar{q}_{0,k}^2 + \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} \bar{q}_{i,k}^2 \right) g_i \pmod{I}$$

for some $\omega_{i,k} \in \mathbb{Q}_+$ and $\bar{q}_{i,k} \in \mathbb{Q}[\mathbf{x}]$. Moreover, Identity (10) and $\tilde{b}f \in I$ imply that

$$f \equiv \bar{a} f^2 = \sum_{k=1}^D \omega_{0,k} (\bar{q}_{0,k} f)^2 + \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} (\bar{q}_{i,k} f)^2 \right) g_i \pmod{I}.$$

Now let $\bar{q}_{i,k} f \equiv q_{i,k} \pmod{I}$ with $q_{i,k} \in \langle B \rangle$. Then

$$f \equiv \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 + \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i \pmod{I}$$

with $\deg(q_{i,k}) \leq \delta$.

Finally, when \mathbf{h} is a graded basis of I we deduce the degree bounds as in the proof of Theorem 2.7. \square

In [KMS23] the authors showed that for the univariate case, the condition $h/\gcd(f, h)$ and f being relatively prime is a sufficient condition in order to guarantee that if f is a polynomial such that $f(\xi) \geq 0$ for all $\xi \in V_{\mathbb{R}}(I)$, then $f \in Q_{\mathbb{Q}}$ (here $I = (h)$). The condition $(I : f) + (f) = (1)$ of Theorem 3.2 is equivalent to that condition in the univariate case.

Remark 3.3. *A direct corollary of this theorem is the characterisation of positive polynomials on S , given in Theorem B. Indeed if I is radical, we have $(I : f) + (f) = (1)$ since $V_{\mathbb{C}}(I : f) \cap V_{\mathbb{C}}(f) = \emptyset$, and by Theorem 3.2 any polynomial nonnegative on S has the desired SoS representation.*

If I is not radical the result may not hold, as previously shown in Example 3.1.

4 Height bounds in the quotient algebra $\mathbb{Q}[\mathbf{x}]/I$ for I radical

Sections 5 and 6 below deal with the proof of (C), where we compute height bounds for all the expressions appearing in the SoS representation Equation (3) presented in Theorem A in the case when $K = \mathbb{Q}$ and I is a zero-dimensional *radical* ideal in $\mathbb{Q}[\mathbf{x}]$. Without loss of generality, we assume in the sequel that $f, \mathbf{g} = \{g_1, \dots, g_r\}$ and $\mathbf{h} = \{h_1, \dots, h_s\}$ have all *integer* coefficients.

As a preparation for the proof of Theorem C, in this section we derive some general bounds for the heights of polynomials in the quotient algebra $\mathbb{Q}[\mathbf{x}]/I$ when I is a zero-dimensional radical ideal. We add the following notations to the ones already introduced in Section 2.1.

- For $x \in \mathbb{C}$, we denote $\mathfrak{h}(x) := \log(|x|)$ with $\mathfrak{h}(0) = -\infty$, where \log denotes logarithm in base 2 (which coincides with the usual (logarithmic) *height* of x , essentially its number of binary digits, when $x \in \mathbb{Z}$).
- Given $p = \sum_{\alpha} a_{\alpha} \mathbf{x}^{\alpha} \in \mathbb{C}[\mathbf{x}]$ or $\mathbf{A} = (A_{i,j})_{i,j} \in \mathbb{C}^{N \times M}$, we denote

$$\mathfrak{h}(p) := \max_{\alpha} \{\mathfrak{h}(a_{\alpha})\} \quad \text{and} \quad \mathfrak{h}(\mathbf{A}) := \max_{i,j} \{\mathfrak{h}(A_{i,j})\}.$$

Note that if $p \in \mathbb{Z}[\mathbf{x}]$, $\mathfrak{h}(p)$ coincides with the (logarithmic) height $h(p)$ of the polynomial p , and the same holds for a matrix $\mathbf{A} \in \mathbb{Z}[\mathbf{x}]$. However, when $p \in \mathbb{Q}[\mathbf{x}]$ (or $\mathbf{A} \in \mathbb{Q}^{N \times M}$), these two quantities differ, as $h(p) = \max\{h(\widehat{p}), h(\nu)\}$, where $p = \widehat{p}/\nu$ with $\widehat{p} \in \mathbb{Z}[\mathbf{x}]$ a primitive polynomial associated to p and $\nu \in \mathbb{N}$ (analogously for \mathbf{A}). To avoid misunderstandings, we will not use in the sequel the notation \mathfrak{h} for a rational non-integer polynomial or matrix, but instead refer to \mathfrak{h} of integer numerators and a common denominator.

- Finally, given $n, d \in \mathbb{N}$ we denote

$$\mathcal{C}(n) := cn \log(n+1) \quad \text{and} \quad \mathcal{C}(n; d) := cn \log(d(n+1))$$

for some suitable positive computable constant $c \in \mathbb{R}_{>0}$ (in particular, $\mathcal{C}(1) = c$).

4.1 Bounds on the values of a polynomial at the roots

The following lemma applies an arithmetic Bézout theorem for the height of a variety in terms of the heights of its defining polynomials that is developed in [KPS01].

Lemma 4.1. *Let $J = (h_1, \dots, h_{s+1})$ be a zero-dimensional ideal defined by polynomials $h_1, \dots, h_{s+1} \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$ for $1 \leq j \leq s$, $e = \max\{d, \deg(h_{s+1})\}$ and $\mathfrak{h}(h_j) \leq \tau$ for $1 \leq j \leq s+1$. Then*

$$\sum_{\zeta \in V_{\mathbb{C}}(J)} \mathfrak{h}(\|(1, \zeta)\|_2) \leq \mathcal{C}(n) d^{n-2} e(d + \tau).$$

Proof. The arithmetic Bézout theorem in [KPS01, Cor.2.11], more precisely the comment after it applied to the polynomials $h_1, \dots, h_s \in \mathbb{Q}[\mathbf{x}]$ which define the variety V , implies that the (logarithmic) global height $h(V)$ of V satisfies

$$h(V_{\mathbb{C}}(J)) \leq 2n \log(n+1) d^{n-1} e + d^{n-2} e \tau \leq \mathcal{C}(n) d^{n-2} e(d + \tau).$$

Now, the global height $h(V)$ is defined as the sum of its local heights, see [KPS01, Sec.1.2.4], which implies that in particular the local height h_{∞} corresponding to the ordinary complex absolute value, satisfies

$$\sum_{\zeta \in V_{\mathbb{C}}(J)} h_{\infty}(\zeta) = h_{\infty}(V_{\mathbb{C}}(J)) \leq h(V_{\mathbb{C}}(J)),$$

while by [KPS01, Sec.1.2.3], $h_{\infty}(\zeta) = \mathfrak{h}(\|(1, \zeta)\|_2)$. □

We can now bound the value of a polynomial at the roots $\zeta \in V_{\mathbb{C}}$.

Lemma 4.2. *Let $I = (\mathbf{h})$ be a zero-dimensional ideal defined by polynomials $h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$ and $\mathfrak{h}(h_j) \leq \tau$. Let $p \in \mathbb{Z}[\mathbf{x}]$ be such that $\mathfrak{h}(p) \leq \tau$ and let $d_p = \max\{d, \deg(p)\}$. Then, for any $Z \subset V_{\mathbb{C}}$ such that $p(\zeta) \neq 0, \forall \zeta \in Z$, we have*

$$-\mathcal{C}(n) d^{n-1} d_p(d + \tau) \leq \sum_{\zeta \in Z} \mathfrak{h}(p(\zeta)) \leq \mathcal{C}(n) d^{n-1} d_p(d + \tau)$$

Proof. We apply Lemma 4.1 to the roots of the zero-dimensional ideals of $\mathbb{Z}[\mathbf{x}, x_{n+1}]$:

$$\begin{aligned} I_1 &= I + (x_{n+1} - p(\mathbf{x})) \subset \mathbb{Z}[\mathbf{x}, x_{n+1}], \text{ with } V_{\mathbb{C}}(I_1) = \{(\zeta, p(\zeta)) : \zeta \in V_{\mathbb{C}}\}, \\ I_2 &= I + (1 - x_{n+1}p(\mathbf{x})) \subset \mathbb{Z}[\mathbf{x}, x_{n+1}], \text{ with } V_{\mathbb{C}}(I_2) = \{(\zeta, 1/p(\zeta)) : \zeta \in V_{\mathbb{C}}, p(\zeta) \neq 0\}. \end{aligned}$$

For I_1 this gives by (4.1) that

$$\begin{aligned} \sum_{\zeta \in \mathbb{Z}} \mathfrak{h}(p(\zeta)) &\leq \sum_{\zeta \in V_{\mathbb{C}}} \mathfrak{h}(\|(1, \zeta, p(\zeta))\|_2) \\ &\leq \mathcal{C}(n) d^{n-1} d_p (d + \tau). \end{aligned}$$

Similarly, for I_2 ,

$$\sum_{\zeta \in \mathbb{Z}} \mathfrak{h}\left(\frac{1}{p(\zeta)}\right) \leq \mathcal{C}(n) d^{n-1} (d_p + 1)(d + \tau) \leq \mathcal{C}(n) d^{n-1} d_p (d + \tau)$$

since $d_p \geq d \geq 1$. □

4.2 Reduction in the quotient algebra $\mathbb{Q}[\mathbf{x}]/I$

The goal of this section is to compute bounds for the height of an integer polynomial $p \in \mathbb{Z}[\mathbf{x}]$ when reducing it modulo a zero-dimensional *radical* ideal I . We will achieve this by passing through the description of the quotient algebra $\mathbb{Q}[\mathbf{x}]/I$ via its rational univariate representation (also known as geometric resolution), in order to obtain better bounds than if we were working directly with multivariate reduction by the graded basis \mathbf{h} of the ideal I .

Lemma 4.3. *Let $J = (h_1, \dots, h_s, h_{s+1})$ be a zero-dimensional radical ideal defined by polynomials $h_1, \dots, h_s, h_{s+1} \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$ for $1 \leq j \leq s$, $e = \max\{d, \deg(h_{s+1})\}$ and $\mathfrak{h}(h_j) \leq \tau$ for $1 \leq j \leq s+1$. Then there exists $w_0, w_1, \dots, w_n \in \mathbb{Z}[t]$ with $\deg(w_0) = D := \dim \mathbb{Q}[\mathbf{x}]/J$, $\deg(w_i) < D$ for $1 \leq i \leq n$ such that*

$$\begin{aligned} \varphi_J : \mathbb{Q}[\mathbf{x}]/J &\longrightarrow \mathbb{Q}[t]/(w_0) \\ x_i &\longmapsto w_i(w_0')^{-1} \end{aligned}$$

is an isomorphism of algebras and

$$\mathfrak{h}(w_i) \leq \mathcal{C}(n; d) d^{n-2} e (d + \tau), \quad 0 \leq i \leq n.$$

Proof. Let $L(U, \mathbf{x}) = U_1 x_1 + \dots + U_n x_n \in \mathbb{Q}[U, \mathbf{x}]$ be a generic linear form, and consider the polynomial

$$W_0(t, U) = \text{Ch}_V(t, -U_1, \dots, -U_n) = a \prod_{\zeta \in V} (t - L(U, \zeta)) \in \mathbb{Z}[t][U],$$

where $V := V_{\mathbb{C}}(J)$ and Ch_V is the Chow form of V , with $a \in \mathbb{N}$ such that W_0 is a primitive polynomial (this is possible since V is defined over \mathbb{Q}).

Given $\zeta \in V$, $W_0(L(U, \zeta), U) = 0$ implies that for all i we have

$$\frac{\partial W_0}{\partial t}(L(U, \zeta), U) \zeta_i + \frac{\partial W_0}{\partial U_i}(L(U, \zeta), U) = 0.$$

Therefore, by choosing $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$ such that $\ell(\mathbf{x}) := L(u, \mathbf{x})$ satisfies $\ell(\zeta) \neq \ell(\xi)$ for all $\zeta \neq \xi \in V$, the polynomials

$$w_0(t) = W_0(t, u), \quad w_i(t) = -\frac{\partial W_0}{\partial U_i}(t, u) \in \mathbb{Z}[t]$$

give the stated isomorphism of algebras, where we observe that the map φ_J is well-defined since J is radical so that w_0 has simple roots in \mathbb{C} and w'_0 is invertible modulo w_0 .

In order to choose $\ell(\mathbf{x}) = u_1 x_1 + \dots + u_n x_n \in \mathbb{Z}[\mathbf{x}]$ that separates the points in V , we can observe that the non-zero polynomial

$$P(U) := \prod_{(\zeta, \xi) \in V \times V, \zeta \neq \xi} (L(U, \zeta) - L(U, \xi)) \in \mathbb{C}[U]$$

has degree $D' := D(D-1)/2$ and therefore, there exists an element u of the grid

$$\{(k_1, \dots, k_n) \in \mathbb{Z}^n : 0 \leq k_i \leq D'\}$$

where P does not vanish.

By [KPS01, Section 1.2.4, Inequalities (1.1) and (1,2)] and Lemma 4.1 we have

$$\mathfrak{h}(W_0) = \mathfrak{h}(\text{Ch}_V) \leq \mathfrak{h}(V) + \log(n+1) \deg(V) \leq \mathcal{C}(n) d^{n-2} e(d+\tau).$$

We conclude by observing that the polynomial

$$W_0(t, U) = \sum_{\alpha, i: |\alpha|+i=D} a_{\alpha, i} t^i U^\alpha = \sum_{i=0}^D t^i \left(\sum_{|\alpha|=D-i} a_{\alpha, i} U^\alpha \right) \in \mathbb{Z}[t, U]$$

is homogeneous of degree D and $-\frac{\partial W_0(t, U)}{\partial U_i} \in \mathbb{Z}[t, U]$ is homogeneous of degree $D-1$, and therefore each of the (integer) coefficients of $w_0(t)$ has absolute value bounded by $2^{n+D} 2^{\mathfrak{b}(W_0)} D'^D$, while each of the integer coefficients of $w_i(t)$, $1 \leq i \leq n$, has absolute value bounded by $2^{n+D-1} 2^{\mathfrak{b}(\frac{\partial W_0(t, U)}{\partial U_i})} D'^{D-1}$. This adds a factor $2n d^n \log(d)$ that we can summarize in

$$\mathfrak{h}(w_i) \leq \mathcal{C}(n; d) d^{n-2} e(d+\tau), \quad 0 \leq i \leq n. \quad \square$$

Consider now a zero-dimensional (and radical) ideal $I = (h_1, \dots, h_s)$ defined by $h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$, $\mathfrak{h}(h_j) \leq \tau$. We define the following crucial map \mathcal{U} , which essentially computes a univariate normal form for any polynomial $p \in \mathbb{Q}[\mathbf{x}]$:

$$\begin{aligned} \mathcal{U} : \mathbb{Q}[\mathbf{x}] &\longrightarrow \mathbb{Q}[t]/(w_0) \simeq \langle 1, \dots, t^{D-1} \rangle_{\mathbb{Q}} \\ p &\longmapsto w'_0 \varphi_I(\pi_I(p)) \end{aligned} \quad (11)$$

where $\pi_I : \mathbb{Q}[\mathbf{x}] \rightarrow \mathbb{Q}[\mathbf{x}]/I$ is the projection. Since φ_I is an isomorphism and w'_0 is invertible in $\mathbb{Q}[t]/(w_0)$, the map \mathcal{U} is surjective since π_I is and satisfies $\ker \mathcal{U} = I$.

Lemma 4.4. *Let $I = (\mathbf{h})$ be a zero-dimensional radical ideal defined by polynomials $h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$ and $\mathfrak{h}(h_j) \leq \tau$. Let $p \in \mathbb{Z}[\mathbf{x}]$ be such that $\mathfrak{h}(p) \leq \tau$ and let $d_p = \max\{d, \deg(p)\}$. Then, $\mathcal{U}(p) \in \langle 1, \dots, t^{D-1} \rangle_{\mathbb{Z}}$ and*

$$\mathfrak{h}(\mathcal{U}(p)) \leq \mathcal{C}(n; d) d^{n-1} d_p (d+\tau).$$

Proof. Let $\varphi_I : \mathbb{Q}[\mathbf{x}]/I \rightarrow \mathbb{Q}[t]/(w_0)$, $x_i \mapsto w_i(w_0')^{-1}$, be the isomorphism of Lemma 4.3. We observe that the linear form $\ell(\mathbf{x}) = L(u, \mathbf{x})$ that defines the polynomial $w_0(t) = W_0(u, t)$ in the proof of that lemma is also a separating form for the vanishing set $V_{\mathbb{C}}(J) = \{(\zeta, p(\zeta)) : \zeta \in V_{\mathbb{C}}(I)\}$ of the ideal $J := (h_1(\mathbf{x}), \dots, h_s(\mathbf{x}), x_{n+1} - p(\mathbf{x})) \subset \mathbb{Q}[\mathbf{x}, x_{n+1}]$. Moreover, it is straightforward to verify that under this isomorphism, we have $\varphi_J(x_i) = w_i(w_0')^{-1}$ for $1 \leq i \leq n$. Let $w_{n+1} \in \mathbb{Z}[t]$ be such that $\varphi_J(x_{n+1}) = w_{n+1}(w_0')^{-1}$. Then

$$\mathcal{U}(p) = w_0' \varphi_I(\pi_I(p)) = w_0' \varphi_J(\pi_J(p)) = w_0' \varphi_J(x_{n+1}) = w_{n+1}$$

since $p \in \mathbb{Z}[\mathbf{x}]$ and $x_{n+1} - p(\mathbf{x}) \in J$. This implies that $\mathcal{U}(p) \in \langle 1, \dots, t^{D-1} \rangle_{\mathbb{Z}}$ and the stated bound by application of Lemma 4.3 to $J \subset \mathbb{Q}[\mathbf{x}, x_{n+1}]$. \square

Corollary 4.5. *Let $I = (\mathbf{h})$ be a zero-dimensional radical ideal defined by polynomials $h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$ and $\mathfrak{h}(h_j) \leq \tau$. Let $p \in \mathbb{Z}[\mathbf{x}]$ be such that $\mathfrak{h}(p) \leq \tau_p$ and let $d_p = \max\{d, \deg(p)\}$. Then, $\mathcal{U}(p) \in \langle 1, \dots, t^{D-1} \rangle_{\mathbb{Z}}$ and*

$$\mathfrak{h}(\mathcal{U}(p)) \leq \mathcal{C}(n; d) d^{n-1} d_p (d + \tau) + \tau_p.$$

Proof. Let $p = \sum_{|\alpha| \leq d_p} p_{\alpha} \mathbf{x}^{\alpha}$ with $p_{\alpha} \in \mathbb{Z}$ and $\mathfrak{h}(p_{\alpha}) \leq \tau_p$. Then $\mathcal{U}(p) = \sum_{|\alpha| \leq d_p} p_{\alpha} \mathcal{U}(\mathbf{x}^{\alpha})$, where by Lemma 4.4, $\mathfrak{h}(\mathcal{U}(\mathbf{x}^{\alpha})) \leq \mathcal{C}(n; d) d^{n-1} d_p (d + \tau)$. Since the number of coefficients of p is bounded by $(d_p + 1)^n$, we have

$$\mathfrak{h}(\mathcal{U}(p)) \leq \mathcal{C}(n; d) d^{n-1} d_p (d + \tau) + \mathcal{C}(n; d_p) + \tau_p \leq \mathcal{C}(n; d) d^{n-1} d_p (d + \tau) + \tau_p,$$

by adjusting the term $\mathcal{C}(n; d)$. \square

We will now bound the height of the *normal form* $\mathcal{N}(p) \in \langle B \rangle_{\mathbb{Q}}$ of a polynomial $p \in \mathbb{Z}[\mathbf{x}]$, which is the *unique* polynomial $\mathcal{N}(p)$ in $\langle B \rangle_{\mathbb{Q}}$ such that

$$p = p_1 h_1 + \dots + p_s h_s + \mathcal{N}(p) \tag{12}$$

for some $p_j \in \mathbb{Q}[\mathbf{x}]$, $1 \leq j \leq s$ (we recall that here that $B \subset \mathbb{Z}[\mathbf{x}]$ is a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $\deg(B) = \max\{\deg(b) : b \in B\}$). To do that, we are going to exploit the properties of \mathcal{U} , defined in (11).

Proposition 4.6. *Let $I = (\mathbf{h})$ be a zero-dimensional radical ideal defined by polynomials $h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$, $\mathfrak{h}(h_j) \leq \tau$, let $p \in \mathbb{Z}[\mathbf{x}]$ with $d_p = \max\{\deg(p), d\}$ and $\mathfrak{h}(p) \leq \tau_p$, and set $\delta := \max\{d, \deg(B)\}$. Then, $\mathcal{N}(p) = \widehat{\mathcal{N}}(p)/\nu$ where $\nu \in \mathbb{N}$ and $\widehat{\mathcal{N}}(p) \in \langle B \rangle_{\mathbb{Z}}$ satisfy*

$$\mathfrak{h}(\nu) \leq \mathcal{C}(n; d) d^{2n-1} \delta (d + \tau) \quad \text{and} \quad \mathfrak{h}(\widehat{\mathcal{N}}(p)) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p) (d + \tau) + \tau_p.$$

Proof. Let $\mathcal{N}(p) = \sum_{i=1}^D c_i b_i$ where $B = \{b_1, \dots, b_D\}$ and $c_i \in \mathbb{Q}$, $1 \leq i \leq D$. Then we have $\mathcal{U}(p) = \mathcal{U}(\mathcal{N}(p))$ since $p \equiv \mathcal{N}(p) \pmod{I}$. Therefore

$$\mathcal{U}(p) = \sum_i c_i \mathcal{U}(b_i)$$

where $\mathbf{c} = (c_1, \dots, c_D)$ is the solution of the system $\mathbf{A}\mathbf{c} = \mathbf{b}$, with $\mathbf{A} \in \mathbb{Z}^{D \times D}$ the invertible matrix of coefficients of $\mathcal{U}(b_i) \in \langle 1, \dots, t^{D-1} \rangle_{\mathbb{Z}}$ and $\mathbf{b} \in \mathbb{Z}^D$ the coefficient vector of $\mathcal{U}(p)$.

By Corollary 4.5, we have

$$\mathfrak{h}(\mathbf{A}) \leq \mathcal{C}(n; d) d^{n-1} \delta(d + \tau), \quad \mathfrak{h}(\mathbf{b}) \leq \mathcal{C}(n; d) d^{n-1} d_p(d + \tau) + \tau_p.$$

By Lemma A.1, we deduce that $\mathbf{c} = \widehat{\mathcal{N}}(p)/\nu$ with $\nu \in \mathbb{N}$, $\widehat{\mathcal{N}}(p) \in \langle 1, \dots, t^{D-1} \rangle_{\mathbb{Z}}$, and

$$\begin{aligned} \mathfrak{h}(\nu) &\leq D(\log(D) + \mathcal{C}(n; d) d^{n-1} \delta(d + \tau)) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau), \\ \mathfrak{h}(\widehat{\mathcal{N}}(p)) &\leq (D-1)(\log(D) + \mathcal{C}(n; d) d^{n-1} \delta(d + \tau)) + \mathcal{C}(n; d) d^{n-1} d_p(d + \tau) + \tau_p \\ &\leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \tau_p \end{aligned}$$

by adjusting $\mathcal{C}(n; d)$, which concludes the proof. \square

Remark 4.7. Proposition 4.6 remains true for a polynomial $p \in \mathbb{C}[\mathbf{x}]$. We get $\mathcal{N}(p) \in \langle B \rangle_{\mathbb{C}}$ with

$$\mathfrak{h}(\mathcal{N}(p)) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \tau_p.$$

This is because in the proof, when solving the linear system $\mathbf{A}\mathbf{y} = \mathbf{b}$ where $\mathbf{A} \in \mathbb{Z}^{D \times D}$ and $\mathbf{b} \in \mathbb{C}^D$, $\nu \in \mathbb{N}$ satisfies $\nu \geq 1$.

We will also need to bound the height of the polynomials $p_1, \dots, p_s \in \mathbb{Q}[\mathbf{x}]$ in a decomposition as in (12), assuming that $\mathbf{h} = \{h_1, \dots, h_s\}$ is a *graded basis* of I . The bound we obtain for $\mathfrak{h}(\mathcal{N}(p))$ has a worse dependence on the degree of p than in Proposition 4.6, and in the sequel we will not apply it, but this is the only bound we get for the polynomial coefficients p_j , $1 \leq j \leq s$.

Proposition 4.8. Let h_1, \dots, h_s be a graded basis of I with $\deg(h_j) \leq d$ and $\mathfrak{h}(h_j) \leq \tau$, $1 \leq j \leq s$, and let $B = \{b_1, \dots, b_D\}$ be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$. Let $p \in \mathbb{Z}[\mathbf{x}]$ and set $d_p := \deg(p) + 1$ and $\mathfrak{h}(p) \leq \tau_p$. Then there exist $\nu \in \mathbb{N}$, $\lambda_j \in \mathbb{Z}$, $1 \leq i \leq D$, and $p_j \in \mathbb{Z}[\mathbf{x}]$ with $\deg(p_j) \leq \deg(p) - \deg(h_j)$, $1 \leq j \leq s$, such that

$$p = \frac{1}{\nu} \left(\sum_{j=1}^s p_j h_j + \sum_{i=1}^D \lambda_i b_i \right)$$

where $\mathcal{N}(p) = \frac{1}{\nu} (\sum_{i=1}^D \lambda_i b_i)$, satisfying

$$\mathfrak{h}(\nu) \leq \mathcal{C}(n; d_p) d_p^n \tau \quad \text{and} \quad \mathfrak{h}(\lambda_i), \mathfrak{h}(p_j) \leq \mathcal{C}(n; d_p) d_p^n \tau + \tau_p \quad \text{for } 1 \leq i \leq D, 1 \leq j \leq s.$$

Proof. To compute the vector $(\lambda_1, \dots, \lambda_D)$ and all coefficients of all polynomials p_j , we solve a linear system $\mathbf{A}\mathbf{y} = \mathbf{b}$ where $\mathbf{A} \in \mathbb{Z}^{M \times N}$ which non-zero entries are coefficients of the b_i 's and the polynomials h_j , so that $\mathfrak{h}(\mathbf{A}) \leq \tau$, and $\mathbf{b} \in \mathbb{Z}^M$ is the vector of coefficients of p , so that $\mathfrak{h}(\mathbf{b}) \leq \tau_p$. Since the number of coefficients of p is at most d_p^n and we can assume that \mathbf{A} is of full rank M , we have $M \leq d_p^n$. By applying Lemma A.1 we obtain $\nu \in \mathbb{Z}$ and the $\lambda_i \in \mathbb{Z}$ and the coefficients of the m polynomials $p_k \in \mathbb{Z}[\mathbf{x}]$ with

$$\begin{aligned} \mathfrak{h}(\nu) &\leq \frac{M}{2} \log(M) + M\tau \leq \mathcal{C}(n; d_p) d_p^n \tau, \\ \mathfrak{h}(\lambda_i), \mathfrak{h}(p_j) &\leq \frac{M}{2} \log(M) + (M-1)\tau + \tau_p \leq \mathcal{C}(n; d_p) d_p^n \tau + \tau_p. \end{aligned} \quad \square$$

4.3 Bounds on the singular values of the Vandermonde matrix

Let I be a zero-dimensional radical ideal and let $V = V_{B, \zeta} \in \mathbb{C}^{D \times D}$ be the Vandermonde matrix of the roots $V_{\mathbb{C}}(I) = \{\zeta_1, \dots, \zeta_D\}$ in the monomial basis B of $\mathbb{Q}[\mathbf{x}]/I$, with $|B| = D$, and let $U = (u_{\zeta_1}, \dots, u_{\zeta_D}) \in \mathbb{C}^{D \times D}$ be the coefficient matrix of the interpolation polynomials $u_{\zeta_i} \in \mathbb{C}[\mathbf{x}]$ in the basis B . By definition, $V^T U = \text{Id}_D$. We denote by $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ respectively the maximal and minimal singular values of a matrix.

Lemma 4.9. *Let $I = (\mathbf{h})$ be a zero-dimensional radical ideal defined by polynomials $h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$, $\mathfrak{h}(h_j) \leq \tau$, and set $\delta := \max\{d, \deg(B)\}$. Then*

$$\mathfrak{h}(\sigma_{\max}(V)) \leq \mathcal{C}(n) d^{n-1} \delta(d + \tau) \quad \text{and} \quad \mathfrak{h}(\sigma_{\min}(U)) \geq -\mathcal{C}(n) d^{n-1} \delta(d + \tau).$$

Proof. We have the well-known inequalities

$$\sigma_{\max}(V) \leq \|V\|_F \leq D \max\{|\zeta^\alpha| : \mathbf{x}^\alpha \in B, \zeta \in V_{\mathbb{C}}\}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of matrix V . We conclude by applying Lemma 4.2 to the polynomials $\mathbf{x}^\alpha \in \mathbb{Z}[\mathbf{x}]$ which satisfy $\deg(\mathbf{x}^\alpha) \leq \delta$ and $\mathfrak{h}(\mathbf{x}^\alpha) = 0 \leq \tau$:

$$\mathfrak{h}(\sigma_{\max}(V)) \leq \log(D) + \mathcal{C}(n) d^{n-1} \delta(d + \tau) \leq \mathcal{C}(n) d^{n-1} \delta(d + \tau)$$

since $D \leq d^n$.

As $U^T V = \text{Id}$, we deduce the lower bound on $\sigma_{\min}(U) = (\sigma_{\max}(V))^{-1}$. \square

We now give an upper bound for the height of the idempotents u_ζ , $\zeta \in V_{\mathbb{C}}(I)$. For that purpose, we use the following lemma.

Lemma 4.10. *Let $I = (\mathbf{h})$ be a zero-dimensional ideal defined by polynomials $h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(h_j) \leq d$ and $\mathfrak{h}(h_j) \leq \tau$. Then, for any i , $1 \leq i \leq n$, and $Z \subset V_{\mathbb{C}}^2$ with $\zeta_i \neq \xi_i$ for all $(\zeta, \xi) \in Z$ with $\zeta = (\zeta_1, \dots, \zeta_n)$ and $\xi = (\xi_1, \dots, \xi_n)$, one has*

$$\sum_{(\zeta, \xi) \in Z} \mathfrak{h}(\zeta_i - \xi_i) \geq -\mathcal{C}(n) d^{2n} (d + \tau).$$

Proof. We apply lemma 4.2 to the zero-dimensional ideal $J = (\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{y})) \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ which satisfies $V_{\mathbb{C}}(J) = V_{\mathbb{C}}^2$, and the polynomials $p_i(\mathbf{x}, \mathbf{y}) = x_i - y_i$, $1 \leq i \leq n$, of degree 1 and height 0. \square

Definition 4.11 (Interpolation polynomials). For each $\zeta \in V_{\mathbb{C}}$, let $\varphi_\zeta \in \mathbb{C}[\mathbf{x}]$ be defined in the following way: For each $\xi \neq \zeta$ pick a coordinate ξ_{i_ξ} with $\xi_{i_\xi} \neq \zeta_{i_\xi}$ and let

$$\varphi_\zeta = \prod_{\xi \neq \zeta} \frac{x_{i_\xi} - \xi_{i_\xi}}{\zeta_{i_\xi} - \xi_{i_\xi}} \in \mathbb{C}[\mathbf{x}].$$

We verify that $\varphi_\zeta(\zeta) = 1$ and $\varphi_\zeta(\xi) = 0$ for $\xi \neq \zeta$.

Lemma 4.12. *Let $(\varphi_\zeta)_{\zeta \in V_{\mathbb{C}}(I)}$ be the interpolation polynomials of Definition 4.11. Then*

$$\deg(\varphi_\zeta) = D - 1 \quad \text{and} \quad \mathfrak{h}(\varphi_\zeta) \leq \mathcal{C}(n) d^{2n} (d + \tau).$$

Proof. The degree bound is obvious by the definition.

For the height bound, the denominator $\prod_{\xi \neq \zeta} (\zeta_{i_\xi} - \xi_{i_\xi})$ of φ_ζ is composed by at most n products of terms of the form

$$\prod_{(\zeta, \xi) \in Z_i} (\zeta_i - \xi_i),$$

where $Z_i \subset V_{\mathbb{C}}^2$ is the subset of $\xi \neq \zeta$ that we chose such that $i_\xi = i$, $1 \leq i \leq n$. Each of these terms satisfies the lower bound of Lemma 4.10. This implies the bound

$$\sum_{\xi \neq \zeta} \mathfrak{h}(\zeta_{i_\xi} - \xi_{i_\xi}) \geq -\mathcal{C}(n) d^{2n}(d + \tau).$$

The coefficients of the numerator of φ_ζ are the elementary symmetric polynomials s_α on the ξ_{i_ξ} , which satisfy

$$|s_\alpha| \leq 2^{D-1} \max\{1, |\zeta_i| : \zeta \in V_{\mathbb{C}}\}^{D-1}$$

and therefore

$$\mathfrak{h}(s_\alpha) \leq (D-1)(1 + \max\{\mathfrak{h}(\zeta_i) : \zeta \in V_{\mathbb{C}}\}) \leq \mathcal{C}(n) d^{2n-1}(d + \tau)$$

by Lemma 4.1. This implies that each of the interpolant φ_ζ of degree $D-1$ satisfies

$$\mathfrak{h}(\varphi_\zeta) \leq \mathcal{C}(n) d^{2n}(d + \tau). \quad \square$$

Corollary 4.13. *Let $u_\zeta \in \langle B \rangle_{\mathbb{C}}$, $\zeta \in V_{\mathbb{C}}$, be the idempotents corresponding to $V_{\mathbb{C}}$. Then*

$$\mathfrak{h}(u_\zeta) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau), \quad \forall \zeta \in V_{\mathbb{C}}.$$

Proof. This is because $u_\zeta = \mathcal{N}(\varphi_\zeta)$ with $\deg(\varphi_\zeta) = D-1$ and $\mathfrak{h}(\varphi_\zeta) \leq \mathcal{C}(n; d) d^{2n}(d + \tau)$. We apply Remark 4.7 and obtain

$$\mathfrak{h}(u_\zeta) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau) + \mathcal{C}(n; d) d^{n-1} D(d + \tau) + \mathcal{C}(n) d^{2n}(d + \tau) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau)$$

by adjusting the constant $\mathcal{C}(n; d)$, since $\delta \geq d$. \square

5 Height bounds for I radical and f strictly positive on S

In this section, we study height bounds for the SoS representation (3) of a polynomial $f \in \mathbb{Z}[\mathbf{x}]$ strictly positive on $S = S(\mathbf{g}, \mathbf{h})$ as in (2), for the case when $I = (\mathbf{h})$ is a radical zero-dimensional ideal, and $\mathbf{g}, \mathbf{h} \subset \mathbb{Z}[\mathbf{x}]$ with $\deg(g_i), \deg(h_j) \leq d$ and $\mathfrak{h}(g_i), \mathfrak{h}(h_j) \leq \tau$ for $1 \leq i \leq r$, $1 \leq j \leq s$. We keep the notation that B is a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $D := |B|$ and $\delta := \max\{d, \deg(B)\}$.

5.1 The case when $S = V_{\mathbb{R}}(I)$

Here we present height bounds for an expression (3) for an arbitrary polynomial $p \in \mathbb{Z}[\mathbf{x}]$ in the assumptions of Theorem 2.1, i.e. besides being I a radical zero-dimensional ideal, $p > 0$ on the whole $V_{\mathbb{R}}(I)$. We observe that from Lemmas 2.2 and 2.3 one has

$$p \equiv \sum_{\zeta \in V_{\mathbb{C}}} \theta_\zeta^2 \pmod{I_{\mathbb{R}}} \quad \text{with} \quad \sum_{\zeta \in V_{\mathbb{C}}} \theta_\zeta^2 = B(\Theta \Theta^t)B^t, \quad (13)$$

where $\Theta \in \mathbb{R}^{D \times D}$ is the coefficient matrix of the basis $(\theta_\zeta)_{\zeta \in V_{\mathbb{C}}}$ in the basis B of $\mathbb{Q}[\mathbf{x}]/I$.

However, since we only deal with *radical* zero-dimensional ideals, and therefore we will not perform the Hensel lifting described in Proposition 2.5, we do not need the assumption that there exists $\zeta_0 \in V_{\mathbb{C}}$ such that $\theta_{\zeta_0}(\zeta) \neq 0$ for all $\zeta \in V_{\mathbb{C}}$ described in the output of Theorem 2.1. This simplifies the description of the polynomials θ_ζ described in Lemmas 2.2 and 2.3. Indeed in this case, if $V_{\mathbb{C}} = V_{\mathbb{R}} \cup Z \cup \overline{Z}$, where Z is a representative set of non-conjugate non-real roots on $V_{\mathbb{C}}$, it is enough to take

- $\theta_\xi = \sqrt{\omega_\xi} u_\xi \in \langle B \rangle_{\mathbb{R}}$ where $\omega_\xi := p(\xi)$ for all $\xi \in V_{\mathbb{R}}$,
 - $\theta_\zeta = \sqrt{\omega_\zeta} \left(\Re(u_\zeta) - \frac{\Im(p(\zeta))}{\lambda_\zeta + \Re(p(\zeta))} \Im(u_\zeta) \right) \in \langle B \rangle_{\mathbb{R}}$ where $\omega_\zeta := 2(\lambda_\zeta + \Re(p(\zeta)))$ for all $\zeta \in Z$,
 - $\theta_{\overline{\zeta}} = \sqrt{\omega_{\overline{\zeta}}} \Im(u_\zeta) \in \langle B \rangle_{\mathbb{R}}$ where $\omega_{\overline{\zeta}} := 2 \frac{\lambda_\zeta^2 - |p(\zeta)|^2}{\lambda_\zeta + \Re(p(\zeta))}$ for all $\zeta \in Z$,
- (14)

with $\lambda_\zeta > |p(\zeta)|$ for all $\zeta \in Z$.

Lemma 5.1. *Let $p \in \mathbb{Z}[\mathbf{x}]$ be such that $p(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}$ and $|\mathfrak{h}(p(\zeta))| \leq \eta_p$ for all $\zeta \in V_{\mathbb{C}}$ such that $p(\zeta) \neq 0$. Given $\zeta \in V_{\mathbb{C}}$, let $\omega_\zeta \in \mathbb{R}_{>0}$ be defined as in Identities (14) and given $\zeta \in Z$, set $\mu_\zeta := -\frac{\Im(p(\zeta))}{\lambda_\zeta + \Re(p(\zeta)) - p(\xi_0)}$. Then*

$$|\mathfrak{h}(\omega_\zeta)| \leq \mathcal{C}(1) \eta_p, \quad \forall \zeta \in V_{\mathbb{C}}, \quad \text{and} \quad \mathfrak{h}(\mu_\zeta) \leq \mathcal{C}(1) \eta_p, \quad \forall \zeta \in Z.$$

Proof. By hypothesis, for all $\xi \in V_{\mathbb{R}}$, $|\mathfrak{h}(\omega_\xi)| = |\mathfrak{h}(p(\xi))| \leq \eta_p$.

For $\zeta \in Z$, choosing λ_ζ such that

$$|p(\zeta)| + 1 < \lambda_\zeta < |p(\zeta)| + 2, \tag{15}$$

we have

$$1 < \lambda_\zeta - |\Re(p(\zeta))| < \lambda_\zeta + \Re(p(\zeta)) < 2|p(\zeta)| + 2$$

which implies that $|\mathfrak{h}(\omega_\zeta)| \leq \mathcal{C}(1) \eta_p$. Similarly,

$$2 \frac{\lambda_\zeta^2 - |p(\zeta)|^2}{\lambda_\zeta + \Re(p(\zeta))} \geq \frac{\lambda_\zeta + |p(\zeta)|}{|p(\zeta)| + 1} \geq \frac{1}{|p(\zeta)| + 1}$$

and

$$2 \frac{\lambda_\zeta^2 - |p(\zeta)|^2}{\lambda_\zeta + \Re(p(\zeta))} \leq 4(\lambda_\zeta + |p(\zeta)|) \leq 8(|p(\zeta)| + 1).$$

We deduce that for all $\zeta \in Z$, $|\mathfrak{h}(\omega_{\overline{\zeta}})| \leq \mathcal{C}(1) \eta_p$.

The bound for μ_ζ for $\zeta \in Z$ follows from the fact that by Equation (15) $|\mu_\zeta| \leq |p(\zeta)|$, which implies that $\mathfrak{h}(\mu_\zeta) \leq \mathcal{C}(1) \eta_p$. \square

Lemma 5.2. *Let $p \in \mathbb{Z}[\mathbf{x}]$ be such that $p(\xi) > 0$ for all $\xi \in V_{\mathbb{R}}$ and $|\mathfrak{h}(p(\zeta))| \leq \eta_p$ for all $\zeta \in V_{\mathbb{C}}$ such that $p(\zeta) \neq 0$. Let $\tilde{Q} := \Theta \Theta^t \in S^D(\mathbb{R})$ with Θ as in Equation (13) be such that*

$$p \equiv B \tilde{Q} B^t \pmod{I_{\mathbb{R}}}.$$

Then

$$\mathfrak{h}(\tilde{Q}) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau) + \mathcal{C}(1) \eta_p \quad \text{and} \quad \mathfrak{h}(\sigma_{\min}(\tilde{Q})) \geq -(\mathcal{C}(n) d^{n-1} \delta(d + \tau) + \mathcal{C}(1) \eta_p).$$

Proposition 5.3. Let $p \in \mathbb{Z}[\mathbf{x}]$ with $d_p = \max\{d, \deg(p)\}$ and $\mathfrak{h}(p) \leq \tau_p$. Define the linear variety

$$\mathcal{L}_p = \{Y \in S^D(\mathbb{R}) : BYB^t \equiv p \pmod{I_{\mathbb{R}}}\}.$$

Let Q/ν , with $Q \in S^D(\mathbb{Z})$ and $\nu \in \mathbb{N}$, be a positive definite matrix with $\mathfrak{h}(Q), \mathfrak{h}(\nu) \leq \tau_Q$, which satisfies

$$\text{dist}\left(\frac{Q}{\nu}, \mathcal{L}_p\right) < \text{dist}\left(\frac{Q}{\nu}, \Sigma\right),$$

where Σ is the set of singular matrices in $\mathbb{R}^{D \times D}$. Then the orthogonal projection Q_0/ν_0 of Q/ν on \mathcal{L}_p , with $Q_0 \in S^D(\mathbb{Z})$ and $\nu_0 \in \mathbb{N}$, which is a positive definite matrix, satisfies that

$$\mathfrak{h}(Q_0), \mathfrak{h}(\nu_0) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \tau_p + \tau_Q.$$

Proof. Set $Y = (y_{i,j}) \in S^D(\mathbb{R})$. We have

$$Y \in \mathcal{L}_p \iff \mathcal{U}(p - BYB^t) = 0 \iff \sum_{i,j} \mathcal{U}(b_i b_j) y_{i,j} = \mathcal{U}(p),$$

where $B = \{b_1, \dots, b_D\}$ is the basis of $\mathbb{Q}[\mathbf{x}]/I$, and \mathcal{U} is the map defined in (11). Thus, \mathcal{L}_p is defined by a linear system of equations of the form $\mathbf{A}\mathbf{y} = \mathbf{b}$ where \mathbf{y} is the vector of $N = D(D+1)/2$ unknown coefficients of $Y \in S^D(\mathbb{R})$, $\mathbf{A} \in \mathbb{Z}^{M \times N}$ is given by the coefficients of $\mathcal{U}(b_i b_j)$ in the basis $\{1, t, \dots, t^{D-1}\}$ for $b_i, b_j \in B$ and $\mathbf{b} \in \mathbb{Z}^M$ is given by the coefficients of $\mathcal{U}(p)$ (here we can assume that \mathbf{A} is of maximal rank $M \leq D$ by deleting its superfluous linear dependent rows). Since by Lemma 4.4 and Corollary 4.5 we have

$$\mathfrak{h}(\mathbf{A}) \leq \mathcal{C}(n; d) d^{n-1} \delta (d + \tau) \quad \text{and} \quad \mathfrak{h}(\mathbf{b}) \leq \mathcal{C}(n; d) d^{n-1} d_p (d + \tau) + \tau_p,$$

we conclude by applying Lemma A.2 that the projection Q_0/ν_0 of Q/ν on \mathcal{L}_p satisfies

$$\mathfrak{h}(Q_0), \mathfrak{h}(\nu_0) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \tau_p + \tau_Q. \quad \square$$

In order to round real polynomials to obtain polynomials in $\mathbb{Q}[\mathbf{x}]$, we will need the following rounding remark.

Remark 5.4. Let $\tilde{\xi} = \sum_{k \leq M-1} a_k 2^k \in \mathbb{R}_{>0}$ for some $M \in \mathbb{N}$ with $a_k \in \{0, 1\}$, $\forall k$, be the binary expansion of a positive real number with $\mathfrak{h}(\xi) \leq M$, and let

$$\frac{\xi}{2^N} := \sum_{-N \leq k \leq M-1} a_k 2^k \in \mathbb{Q} \quad \text{with} \quad \xi := \sum_{0 \leq k \leq N+M-1} a_{k-N} 2^k \in \mathbb{Z}$$

be the truncation of $\tilde{\xi}$ to N digits after the comma. Then

$$2^N \tilde{\xi} - \xi < 1 \quad \text{and} \quad \mathfrak{h}(\xi) \leq N + M.$$

In this case, we say that we round the real number $\tilde{\xi}$ to a rational number $\xi/2^N$, with $\hat{\xi} \in \mathbb{N}$, at precision 2^{-N} .

All this preparation allows us to prove our main theorem for the case when $I \subset \mathbb{Q}[\mathbf{x}]$ is a radical zero-dimensional ideal and $p \in \mathbb{Z}[\mathbf{x}]$ satisfies $p > 0$ on $V_{\mathbb{R}}(I)$. We recall that I is generated by polynomials of degree $\leq d$ and height $\leq \tau_p$, and that B is the monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ and $\delta = \max\{d, \deg(B)\}$.

Theorem 5.5. *Let $p \in \mathbb{Z}[\mathbf{x}]$ be such that $p > 0$ on $V_{\mathbb{R}}(I)$. Let $d_p := \max\{d, \deg(p)\}$, $\mathfrak{h}(p) \leq \tau_p$ and $|\mathfrak{h}(p(\zeta))| \leq \eta_p$ for all $\zeta \in V_{\mathbb{C}}$ such that $p(\zeta) \neq 0$. Then there exists a positive definite matrix $Q_0 \in S^D(\mathbb{Z})$ and $\nu_0 \in \mathbb{N}$ such that*

$$p \equiv \frac{1}{\nu_0} B Q_0 B^t \pmod{I}$$

where

$$\mathfrak{h}(Q_0), \mathfrak{h}(\nu_0) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \mathcal{C}(1) \eta_p + \tau_p.$$

Proof. Let $\tilde{Q} = \Theta \Theta^t \in S^D(\mathbb{R})$ be the positive definite matrix constructed in Lemma 5.2, which satisfies that

$$p \equiv B \tilde{Q} B^t \pmod{I_{\mathbb{R}}},$$

with

$$\mathfrak{h}(\tilde{Q}) \leq \mathcal{C}(n; d) d^{2n-1} \delta (d + \tau) + \mathcal{C}(1) \eta_p \quad \text{and} \quad \mathfrak{h}(\sigma_{\min}(\tilde{Q})) \geq -(\mathcal{C}(n) d^{n-1} \delta (d + \tau) + \mathcal{C}(1) \eta_p).$$

By Remark 5.4 we now round $\tilde{Q} \in S^D(\mathbb{R})$ to $2^{-N} Q$ with $Q \in S^D(\mathbb{Z})$ and $N \in \mathbb{N}$ with $\mathfrak{h}(N) = \mathcal{C}(n) d^{n-1} \delta (d + \tau) + \mathcal{C}(1) \eta_p$ so that $\text{dist}(\tilde{Q}, 2^{-N} Q) < \sigma_{\min}(\tilde{Q})$ and

$$\mathfrak{h}(Q) \leq \mathcal{C}(n; d) d^{2n-1} \delta (d + \tau) + \mathcal{C}(1) \eta_p.$$

By Proposition 5.3, the projection Q_0/ν_0 with $Q_0 \in S^D(\mathbb{Z})$ and $\nu_0 \in \mathbb{N}$ of Q/ν on the linear variety

$$\mathcal{L}_p = \{Y \in S^D(\mathbb{R}) : f \equiv B Y B^t \pmod{I_{\mathbb{R}}}\}$$

is a positive definite matrix which satisfies

$$\begin{aligned} \mathfrak{h}(Q_0), \mathfrak{h}(\nu_0) &\leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \tau_p + \mathcal{C}(n; d) d^{2n-1} \delta (d + \tau) + \mathcal{C}(1) \eta_p \\ &\leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \mathcal{C}(1) \eta_p + \tau_p, \end{aligned}$$

by adjusting the term $\mathcal{C}(n; d)$. □

5.2 The case when $f > 0$ on S for $S \subset V$

Lemma 5.6. *Let S be as in (2) and let $p \in \mathbb{Z}[\mathbf{x}]$ with $\mathfrak{h}(p) \leq \tau_p$ be such that $p(\xi) > 0$ for all $\xi \in S$ and $|\mathfrak{h}(p(\zeta))| \leq \eta_p$ for all $\zeta \in V_{\mathbb{C}}$ such that $p(\zeta) \neq 0$. Set $d_p := \max\{d, \deg(p)\}$. Then there exist $\nu_1 \in \mathbb{N}$ and for all $\xi \in V_{\mathbb{R}} \setminus S$, there exist $\rho_{\xi} \in \mathbb{N}$, $\hat{u}_{\xi} \in \langle B \rangle_{\mathbb{Z}}$ and $0 \leq i_{\xi} \leq r$ such that*

$$\hat{p} = \nu_1 p - \sum_{\xi \in V_{\mathbb{R}} \setminus S} \rho_{\xi} \hat{u}_{\xi}^2 g_{i_{\xi}} \in \mathbb{Z}[\mathbf{x}]$$

satisfies

- $\hat{p}(\zeta) > 0$ for all $\zeta \in V_{\mathbb{R}}$,

- $h(\nu_1) \leq \mathcal{C}(n) d^{n-1} \delta(d + \tau) + \eta_p$ and $\mathfrak{h}(\rho_\xi) \leq \mathcal{C}(n) d^n(d + \tau) + \eta_p, \forall \xi \in V_{\mathbb{R}} \setminus S,$
- $\mathfrak{h}(\widehat{u}_\xi) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau) + \eta_p, \forall \xi \in V_{\mathbb{R}} \setminus S,$
- $|\mathfrak{h}(\widehat{p}(\zeta))| \leq \widehat{\eta}_p := \mathcal{C}(n) d^{n-1} \delta(d + \tau) + \eta_p$ for all $\zeta \in V_{\mathbb{C}}$ such that $p(\zeta) \neq 0,$
- $\deg(\widehat{p}) \leq \widehat{d}_p := \max\{d_p, d + 2\delta\}$ and $\mathfrak{h}(\widehat{p}) \leq \widehat{\tau}_p := \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau) + 2\eta_p + \tau_p.$

Proof. We first note that by Lemma 4.2, we have that for all $\zeta \in V_{\mathbb{C}}$ such that $g_i(\zeta) \neq 0,$

$$|\mathfrak{h}(g_i(\zeta))| \leq \mathcal{C}(n) d^n(d + \tau). \quad (16)$$

We proceed as in the proof of Theorem 2.7, which applies Lemma 2.8. There we defined \widetilde{p} as

$$\widetilde{p} := p - \sum_{\xi \in V_{\mathbb{R}} \setminus S} \rho_\xi u_\xi^2 g_{i_\xi} \in \mathbb{R}[\mathbf{x}]$$

where for $\xi \in V_{\mathbb{R}} \setminus S,$ i_ξ is such that $g_{i_\xi}(\xi) < 0,$ $u_\xi \in \langle B \rangle_{\mathbb{R}}$ is the idempotent associated to ξ and ρ_ξ is chosen such that $p(\xi) - \rho_\xi g_{i_\xi}(\xi) > 0,$ in the following way.

If $p(\xi) > 0$ we can take $\rho_\xi = 0,$ and if $p(\xi) = 0$ we can take $\rho_\xi = 1.$ Now if $p(\xi) < 0$ we need to choose ρ_ξ such that $\rho_\xi > \frac{p(\xi)}{g_{i_\xi}(\xi)}$. By eq. (16), we can take $\rho_\xi \in \mathbb{N}$ with $\mathfrak{h}(\rho_\xi) = \mathcal{C}(n) d^n(d + \tau) + \eta_p$ for

some adjusted $\mathcal{C}(n)$ so that $\rho_\xi > 2 \frac{p(\xi)}{g_{i_\xi}(\xi)}$ and therefore

$$p(\xi) - \rho_\xi g_{i_\xi}(\xi) > -p(\xi) \quad \text{and} \quad \mathfrak{h}(p(\xi) - \rho_\xi g_{i_\xi}(\xi)) \geq -\eta_p.$$

Therefore, for all $\zeta \in V_{\mathbb{R}},$ $\widetilde{p}(\zeta) > 0$ since for $\zeta \in S,$ $\widetilde{p}(\zeta) = p(\zeta)$ while for $\zeta \in V_{\mathbb{R}} \setminus S,$

$$\widetilde{p}(\zeta) = p(\zeta) - \rho_\zeta g_{i_\zeta}(\zeta).$$

Moreover, for all $\zeta \in V_{\mathbb{R}}$ we have $\mathfrak{h}(\widetilde{p}(\zeta)) \geq -\eta_p,$ and

$$\mathfrak{h}(\widetilde{p}(\zeta)) = \mathfrak{h}(p(\zeta) - \rho_\zeta g_{i_\zeta}(\zeta)) \leq \log(2) + \max\{\mathfrak{h}(p(\zeta)), \mathfrak{h}(\rho_\zeta) + \mathfrak{h}(g_{i_\zeta}(\zeta))\} \leq \mathcal{C}(n) d^n(d + \tau) + \eta_p.$$

We deduce more generally that for all $\zeta \in V_{\mathbb{C}}$ such that $p(\zeta) \neq 0$ we have

$$|\mathfrak{h}(\widetilde{p}(\zeta))| \leq \mathcal{C}(n) d^n(d + \tau) + \eta_p.$$

Now, for all $\xi \in V_{\mathbb{R}} \setminus S,$ we round the δ coefficients $u_{\xi, \alpha}$ of $u_\xi = \sum_{\alpha} u_{\xi, \alpha} \mathbf{x}^\alpha \in \langle B \rangle_{\mathbb{R}}$ to $\widehat{u}_{\xi, \alpha}/2^N$ with $\widehat{u}_{\xi, \alpha} \in \mathbb{Z}$ up to precision $2^{-N},$ for some $N \in \mathbb{N},$ so that $|2^N u_{\xi, \alpha} - \widehat{u}_{\xi, \alpha}| < 1$ and $\mathfrak{h}(\widehat{u}_{\xi, \alpha}) \leq N + \mathfrak{h}(u_{\xi, \alpha})$ as in the Remark 5.4, i.e. $\mathfrak{h}(\widehat{u}_\xi) \leq N + \mathfrak{h}(u_\xi).$

We observe that for all $\zeta \in V_{\mathbb{C}},$ as in the proof of lemma 4.9, we have

$$\mathfrak{h}((2^N u_\xi - \widehat{u}_\xi)(\zeta)) \leq \mathfrak{h}\left(\sum_{\alpha} |2^N u_{\xi, \alpha} - \widehat{u}_{\xi, \alpha}| |\zeta|^\alpha\right) \leq \log(D) + \mathcal{C}(n) d^{n-1} \delta(d + \tau) \leq \mathcal{C}(n) d^{n-1} \delta(d + \tau)$$

Furthermore,

$$|(2^N u_\xi + \widehat{u}_\xi)(\zeta)| \leq 2^{N+1} |u_\xi(\zeta)| + |(\widehat{u}_\xi - 2^N u_\xi)(\zeta)| \leq 2^{N+1} + |(\widehat{u}_\xi - 2^N u_\xi)(\zeta)|$$

implies that

$$\begin{aligned}
\mathfrak{h}\left(\left((2^N u_\xi)^2 - \widehat{u}_\xi^2\right)(\zeta)\right) &\leq \log\left(2^{N+1}|u_\xi(\zeta)| + |(\widehat{u}_\xi - 2^N u_\xi)(\zeta)|\right) + \mathfrak{h}\left((2^N u_\xi - \widehat{u}_\xi)(\zeta)\right) \\
&\leq \log\left(2^{N+1} + |(\widehat{u}_\xi - 2^N u_\xi)(\zeta)|\right) + \mathfrak{h}\left((2^N u_\xi - \widehat{u}_\xi)(\zeta)\right) \\
&\leq N + 2 + \mathcal{C}(n) d^{n-1} \delta(d + \tau)
\end{aligned}$$

if we take $N + 1 \geq \mathcal{C}(n) d^{n-1} \delta(d + \tau) \geq \mathfrak{h}\left((2^N u_\xi - \widehat{u}_\xi)(\zeta)\right)$. We then define

$$\nu_1 := 2^{2N}, \quad \widehat{p} := \nu_1 p - \sum_{\xi \in V_{\mathbb{R}} - S} \rho_\xi \widehat{u}_\xi^2 g_{i_\xi} \in \mathbb{Z}[\mathbf{x}].$$

We have that for all $\zeta \in V_{\mathbb{R}}$,

$$\begin{aligned}
\widehat{p}(\zeta) &\geq \left(\nu_1 p - \sum_{\xi \in V_{\mathbb{R}} - S} \rho_\xi (2^N u_\xi)^2 g_{i_\xi}\right)(\zeta) - \left|\left(\sum_{\xi \in V_{\mathbb{R}} \setminus S} \rho_\xi \left((2^N u_\xi)^2 - \widehat{u}_\xi^2\right) g_{i_\xi}\right)(\zeta)\right| \\
&\geq 2^{2N} \widetilde{p}(\zeta) - \sum_{\xi \in V_{\mathbb{R}} \setminus S} \rho_\xi \left|\left((2^N u_\xi)^2 - \widehat{u}_\xi^2\right)(\zeta)\right| |g_{i_\xi}(\zeta)| \\
&\geq 2^{2N - \mathcal{C}(n) d^n (d + \tau) - \eta_p} - D 2^{N + 2 + 3\mathcal{C}(n) d^{n-1} \delta(d + \tau)}.
\end{aligned}$$

Therefore, by taking $\mathfrak{h}(\nu_1) = 2N = \mathcal{C}(n) d^{n-1} (d + \delta)(d + \tau) + \eta_p$ for some adjusted $\mathcal{C}(n)$, we conclude that for all $\zeta \in V_{\mathbb{R}}$, $\widehat{p}(\zeta) > 0$, and moreover, for all $\zeta \in V_{\mathbb{C}}$ such that $p(\zeta) \neq 0$ we can also show that

$$|\mathfrak{h}(\widehat{p}(\zeta))| \leq \mathcal{C}(n) d^{n-1} \delta(d + \tau) + \eta_p$$

proceeding in the same way than before for the lower bound and using that

$$|\widehat{p}(\zeta)| \leq 2^{2N} |\widetilde{p}(\zeta)| + \sum_{\xi \in V_{\mathbb{R}} \setminus S} \rho_\xi \left|\left((2^N u_\xi)^2 - \widehat{u}_\xi^2\right)(\zeta)\right| |g_{i_\xi}(\zeta)|$$

for the upper bound.

We finally observe that $\deg(\widehat{p}) \leq \max\{d_p, d + 2\delta\}$ and, by application of Corollary 4.13, which implies that

$$\mathfrak{h}(\widehat{u}_\xi) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau) + \eta_p,$$

and therefore

$$\mathfrak{h}(\widehat{p}) \leq \mathcal{C}(n; d) d^{2n-1} \delta(d + \tau) + 2\eta_p + \tau_p$$

(by adjusting $\mathcal{C}(n; d)$ again) since \widehat{u}_ξ has at most $D \leq d^n$ monomials. \square

Theorem 5.7. *Let $I = (\mathbf{h}) \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional radical ideal and let B be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $\delta := \max\{d, \deg(B)\}$. Let $S = S(\mathbf{g}, \mathbf{h})$ be as in Equation (2) where $g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(g_i), \deg(h_j) \leq d$ and $\mathfrak{h}(g_i), \mathfrak{h}(h_j) \leq \tau$, $1 \leq i \leq r$, $1 \leq j \leq s$. Let $p \in \mathbb{Z}[\mathbf{x}]$ be such that $p > 0$ on S with $d_p := \max\{d, \deg(p)\}$, $\mathfrak{h}(p) \leq \tau_p$ and $|\mathfrak{h}(p(\zeta))| \leq \eta_p$ for all $\zeta \in V_{\mathbb{C}}(I)$ such that*

$p(\zeta) \neq 0$. Then there exist $\nu_0, \nu_1 \in \mathbb{N}$, a positive definite matrix $Q_0 \in S^D(\mathbb{Z})$, and $\omega_{i,k} \in \mathbb{N}$ and $q_{i,k} \in \langle B \rangle_{\mathbb{Z}}$ for $1 \leq i \leq r$, $1 \leq k \leq D$, such that

$$p \equiv \frac{1}{\nu_0} B^t Q_0 B + \frac{1}{\nu_1} \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i \pmod{I}$$

where

- $\mathfrak{h}(Q_0), \mathfrak{h}(\nu_0) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \mathcal{C}(1) \eta_p + \tau_p$;
- $\mathfrak{h}(\nu_1) \leq \mathcal{C}(n) d^{n-1} \delta (d + \tau) + \eta_p$;
- $\mathfrak{h}(\omega_{i,k}) \leq \mathcal{C}(n) d^n (d + \tau) + \eta_p$;
- $\mathfrak{h}(q_{i,k}) \leq \mathcal{C}(n; d) d^{2n-1} \delta (d + \tau) + \eta_p$.

Proof. We apply Theorem 5.5 to \widehat{p}/ν_1 , where \widehat{p} is the polynomial obtained in Lemma 5.6. The bounds for ν_1 , $\omega_{i,k}$ and $q_{i,k}$ are obtained directly from this lemma. Since for \widehat{p} we have

$$\widehat{d}_p = \max\{d_p, d + 2\delta\}, \widehat{\eta}_p = \mathcal{C}(n) d^{n-1} \delta (d + \tau) + \eta_p \quad \text{and} \quad \widehat{\tau}_p = \mathcal{C}(n; d) d^{2n-1} \delta (d + \tau) + 2\eta_p + \tau_p,$$

Theorem 5.5 implies that

$$\begin{aligned} \mathfrak{h}(Q_0), \mathfrak{h}(\nu_0) &\leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + \widehat{d}_p)(d + \tau) + \mathcal{C}(1) \widehat{\eta}_p + \widehat{\tau}_p \\ &\leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_p)(d + \tau) + \mathcal{C}(1) \eta_p + \tau_p, \end{aligned}$$

by adjusting again the constant $\mathcal{C}(n; d)$. □

As a consequence of the previous result, we obtain the following complete SoS representation Equation (3) which includes degrees and height bounds for a polynomial f strictly positive on S when I is a radical zero-dimensional ideal.

Theorem 5.8. *Let $I = (\mathbf{h}) \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional radical ideal and let B be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $D := |B|$ and $\delta := \max\{d, \deg(B)\}$. Let $S = S(\mathbf{g}, \mathbf{h})$ be as in Equation (2) where $g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(g_i), \deg(h_j) \leq d$ and $\mathfrak{h}(g_i), \mathfrak{h}(h_j) \leq \tau$, $1 \leq i \leq r$, $1 \leq j \leq s$. Let $f \in \mathbb{Z}[\mathbf{x}]$ be such that $f > 0$ on S with $d_f := \max\{d, \deg(f)\}$ and $\mathfrak{h}(f) \leq \tau$. Then there exist $\nu_{0,k}, \nu_1, \nu_2 \in \mathbb{N}$, $\omega_{i,k} \in \mathbb{N}$, $q_{i,k} \in \langle B \rangle_{\mathbb{Z}}$ and $p_j \in \mathbb{Z}[\mathbf{x}]$ for $1 \leq i \leq r$, $1 \leq k \leq D$ and $1 \leq j \leq s$, such that*

$$f = \sum_{k=1}^D \frac{1}{\nu_{0,k}} q_{0,k}^2 + \frac{1}{\nu_1} \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i + \frac{1}{\nu_2} \sum_{j=1}^s p_j h_j$$

where

- $\mathfrak{h}(\nu_{0,k}) \leq \mathcal{C}(n; d) d^{2n-1} (d^n \delta + d_f)(d + \tau)$ for $1 \leq k \leq D$;
- $\mathfrak{h}(\nu_1) \leq \mathcal{C}(n) d^{n-1} (\delta + d_f)(d + \tau)$;
- $\mathfrak{h}(\omega_{i,k}) \leq \mathcal{C}(n) d^{n-1} d_f (d + \tau)$ for $1 \leq i \leq r$, $1 \leq k \leq D$;
- $\mathfrak{h}(q_{i,k}) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_f)(d + \tau)$ for $0 \leq i \leq r$, $1 \leq k \leq D$.

Furthermore, if \mathbf{h} is a graded basis of I , then for all j

$$\deg(p_j) < \widehat{d} - \deg(h_j) \quad \text{and} \quad \mathfrak{h}(\nu_2), \mathfrak{h}(p_j) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_f)(d + \tau) + \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau,$$

for $\widehat{d} := \max\{d_f, d + 2 \deg(B)\} + 1$.

Proof. When we apply Theorem 5.7 to the polynomial f , taking into account that by Lemma 4.2,

$$|\mathfrak{h}(f(\zeta))| \leq \mathcal{C}(n) d^{n-1} d_f (d + \tau)$$

for all $\zeta \in V_{\mathbb{C}}$ such that $f(\zeta) \neq 0$, we obtain

$$f \equiv \frac{1}{\nu_0} B Q_0 B^t + \frac{1}{\nu_1} \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i \pmod{I}$$

where $Q_0 \in S^D(\mathbb{Z})$, $\nu_0 \in \mathbb{N}$ with

$$\mathfrak{h}(Q_0), \mathfrak{h}(\nu_0) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_f)(d + \tau), \quad (17)$$

and $\nu_1 \in \mathbb{N}$, $\omega_{i,k} \in \mathbb{N}$, $q_{i,k} \in \langle B \rangle_{\mathbb{Z}}$ with

$$\mathfrak{h}(\nu_1) \leq \mathcal{C}(n) d^{n-1} (d_f + \delta)(d + \tau), \mathfrak{h}(\omega_{i,k}) \leq \mathcal{C}(n) d^{n-1} d_f (d + \tau), \mathfrak{h}(q_{i,k}) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_f)(d + \tau).$$

Then, we apply Proposition A.5 to the positive definite matrix $Q_0 \in S^D(\mathbb{Z})$ and obtain that

$$B^t Q_0 B = \sum \frac{1}{\widehat{\nu}_{0,k}} q_{0,k}^2$$

where by Equation (17)

$$h(\widehat{\nu}_{0,k}), \mathfrak{h}(q_{0,k}) \leq 2D(\log(D) + \tau_Q) \leq \mathcal{C}(n; d) d^{2n-1} (d^n \delta + d_f)(d + \tau)$$

for $1 \leq k \leq D$. We define $\nu_{0,k} := \nu_0 \widehat{\nu}_{0,k}$, which satisfies the same height bound as $\widehat{\nu}_{0,k}$ to get

$$\frac{1}{\nu_0} B Q_0 B^t = \sum_{k=1}^D \frac{1}{\nu_{0,k}} q_{0,k}^2.$$

This shows the existence of an SoS for f with the stated height bounds for $\nu_{0,k}$, ν_1 , $\omega_{i,k}$ and $q_{i,k}$.

In the case when \mathbf{h} is a graded basis of I , in order to obtain the height bounds for $\nu_2 \in \mathbb{Z}$ and the polynomial coefficients $p_j \in \mathbb{Z}[\mathbf{x}]$ for $1 \leq j \leq s$, we observe that the polynomial

$$\widehat{f} := \nu_0 \nu_1 f - \nu_1 B Q_0 B^t - \nu_0 \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i \in I \cap \mathbb{Z}[\mathbf{x}]$$

satisfies

$$\deg(\widehat{f}) < \widehat{d} := \max\{d_f, d + 2 \deg(B)\} + 1 \quad \text{and} \quad \mathfrak{h}(\widehat{f}) \leq \widehat{\tau} := \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_f)(d + \tau).$$

By Proposition 4.8, we have $\widehat{f} = \frac{1}{\widehat{d}} \sum_{j=1}^s p_j h_j$ with

$$\mathfrak{h}(\widehat{d}) \leq \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau, \quad \mathfrak{h}(p_j) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_f)(d + \tau) + \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau$$

Finally, we define $\nu_2 := \nu_0 \nu_1 \widehat{d}$ which also satisfies

$$\mathfrak{h}(\nu_2) \leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_f)(d + \tau) + \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau. \quad \square$$

Remark 5.9. When $\deg(f), \deg(g_i), \deg(h_i) \leq d$ we get Theorem C, since $d + \tau \leq d\tau$ for $d, \tau, \delta \geq 1$.

6 Height bounds for I radical and f nonnegative on S

We consider now the case where $f \in \mathbb{Z}[\mathbf{x}]$ is nonnegative on $S = S(\mathbf{g}, \mathbf{h})$ as in Equation (2), for the case when $I = (\mathbf{h})$ is a radical zero-dimensional ideal, and $\mathbf{g}, \mathbf{h} \subset \mathbb{Z}[\mathbf{x}]$ with $\deg(g_i), \deg(h_j) \leq d$ and $\mathfrak{h}(g_i), \mathfrak{h}(h_j) \leq \tau$ for $1 \leq i \leq r, 1 \leq j \leq s$. Again, B is a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $D := |B|$ and $\delta := \max\{d, \deg(B)\}$.

Since I is radical, we have $(I : f) + (f) = (1)$. Hereafter we use this identity to reduce to the cases treated in Section 5.

Lemma 6.1. *Let $I \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional and radical ideal. Let $f \in \mathbb{Z}[\mathbf{x}]$ be such that $f \geq 0$ on S and set $d_f := \max\{d, \deg(f)\}$ and $\mathfrak{h}(f) \leq \tau$. Then, there exists $a, b \in \langle B \rangle_{\mathbb{Z}}$ and $\gamma \in \mathbb{N}$ such that*

- $a f + b = \gamma \pmod{I}$,
- $a > 0$ on S ,
- $\mathfrak{h}(a), \mathfrak{h}(b), \mathfrak{h}(\gamma) \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau)$,
- $|\mathfrak{h}(a(\zeta))| \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau)$ for any $\zeta \in V_{\mathbb{C}}(I)$ such that $a(\zeta) \neq 0$.

Proof. Let $B = \{b_1, \dots, b_D\}$ be the basis of $\mathbb{Q}[\mathbf{x}]/I$. We first analyze polynomials $\tilde{a} \in \langle B \rangle_{\mathbb{Q}}$ and $\tilde{b} \in (I : f) \cap \langle B \rangle_{\mathbb{Q}}$ such that $\tilde{a} f + \tilde{b} \equiv 1 \pmod{I}$ (we note that such \tilde{a} and \tilde{b} exist since $(I : f) + (f) = (1)$).

Let $\tilde{b} = \sum_i \lambda_i b_i$ for some $\lambda_i \in \mathbb{Q}$ and $\tilde{a} = \sum_i \mu_i b_i$ for some $\mu_i \in \mathbb{Q}, 1 \leq i \leq D$. Since I is radical, we can apply the results of Section 4.2: we have

$$\tilde{b} \in (I : f) \iff \tilde{b} f \in I \iff \mathcal{U}(\tilde{b} f) = 0 \iff \sum_{1 \leq i \leq D} \mathcal{U}(f b_i) \lambda_i = 0,$$

where \mathcal{U} is the normal form map defined in (11). Analogously,

$$\tilde{a} f + \tilde{b} \equiv 1 \pmod{I} \iff \sum_{1 \leq i \leq D} (\mathcal{U}(f b_i) \mu_i + \mathcal{U}(b_i) \lambda_i) = \mathcal{U}(1) = 1.$$

Therefore $(\lambda_1, \dots, \lambda_D, \mu_1, \dots, \mu_D)$ is a solution of the linear system $\mathbf{A} \mathbf{y} = \mathbf{b}$ where $\mathbf{A} \in \mathbb{Z}^{2D \times 2D}$ is the matrix of coefficients of $(\mathcal{U}(f b_i), \mathcal{U}(b_i)) \in \langle 1, \dots, t^{D-1} \rangle \times \langle 1, \dots, t^{D-1} \rangle$ and $\mathbf{b} = (0, \dots, 0; 1, \dots, 0)$.

By Lemma 4.4, $\mathfrak{h}(\mathbf{A}) \leq \mathcal{C}(n; d) d^{n-1}(\delta + d_f)(d + \tau)$, and we deduce from Lemma A.1, that there is a solution \mathbf{c}/γ with $\mathbf{c} \in \mathbb{Z}^{2D}$ and $\gamma \in \mathbb{N}$ with

$$\mathfrak{h}(\gamma), \mathfrak{h}(\mathbf{c}) \leq \mathcal{C}(n; d) d^{2n-1}(\delta + d_f)(d + \tau)$$

since $D \leq d^n$. This implies that $\sum_{i=1}^D \mu_i b_i = \widehat{a}/\gamma$ with $\widehat{a} \in \langle B \rangle_{\mathbb{Z}}$ and $\sum_{i=1}^D \lambda_i b_i = b/\gamma$ with $b \in \langle B \rangle_{\mathbb{Z}}$ and $\gamma \in \mathbb{N}$, which satisfy

$$\widehat{a}f + b = \gamma \pmod{I} \quad \text{and} \quad \mathfrak{h}(\gamma), \mathfrak{h}(\widehat{a}), \mathfrak{h}(b) \leq \mathcal{C}(n; d) d^{2n-1}(\delta + d_f)(d + \tau). \quad (18)$$

Moreover, by Lemma 4.2, for $\zeta \in V_{\mathbb{C}}(I)$ we have that $|\mathfrak{h}(b_i(\zeta))| \leq \mathcal{C}(n; d) d^{n-1}\delta(d + \tau)$ which implies that

$$\begin{aligned} \mathfrak{h}(\widehat{a}(\zeta)), \mathfrak{h}(b(\zeta)) &\leq \log(D) + \mathcal{C}(n; d) d^{2n-1}(\delta + d_f)(d + \tau) + \mathcal{C}(n; d) d^{n-1}\delta(d + \tau) \\ &\leq \mathcal{C}(n; d) d^{2n-1}(\delta + d_f)(d + \tau). \end{aligned} \quad (19)$$

We now modify \widehat{a} to ensure that it will be strictly positive on all $\xi \in S$ knowing that $f(\xi) \geq 0$ for all $\xi \in S$. Note that for all $\zeta \in V_{\mathbb{C}}$, we have $\widehat{a}(\zeta)f(\zeta) + b(\zeta) = \gamma$.

- Let $\xi \in V_{\mathbb{R}}$. If $f(\xi) = 0$ then $b(\xi) = \gamma \geq 1$ and for $\rho \geq |\widehat{a}(\xi)| + 1$ we have

$$\widehat{a}(\xi) + \rho b(\xi) \geq \widehat{a}(\xi) + |\widehat{a}(\xi)|\gamma + \gamma \geq \gamma \geq 1,$$

and if $f(\xi) > 0$ then $b(\xi) = 0$ since $b f \in I$. Thus, $\widehat{a}(\xi) = \frac{\gamma}{f(\xi)} > 0$ and $\widehat{a}(\xi) + \rho b(\xi) > 0$ for any $\rho \in \mathbb{R}$.

- More generally, if $\zeta \in V_{\mathbb{C}}$ is such that $f(\zeta) = 0$, then for $\rho \geq |\widehat{a}(\zeta)| + 1$, $|\widehat{a}(\zeta) + \rho b(\zeta)| \geq 1$, and if $f(\zeta) \neq 0$, then $|\widehat{a}(\zeta) + \rho b(\zeta)| = \frac{\gamma}{|f(\zeta)|}$.

Now, let $\rho \in \mathbb{N}$ be such that $\rho \geq \max\{|\widehat{a}(\zeta)| + 1 : \zeta \in V_{\mathbb{C}} \text{ s.t. } f(\zeta) = 0\}$ and define $a := \widehat{a} + \rho b$. Then $a \in \langle B \rangle_{\mathbb{Z}}$ and $a f + b \equiv \gamma \pmod{I}$ since $b f \in I$. Moreover, since by Equation (19), we can take

$$\rho \leq \mathcal{C}(n; d) d^{2n-1}(\delta + d_f)(d + \tau),$$

we deduce that

$$\mathfrak{h}(a) = \mathfrak{h}(\widehat{a} + \rho b) \leq \mathcal{C}(n; d) d^{2n-1}(\delta + d_f)(d + \tau).$$

Finally, Equation (19) also implies that

$$\mathfrak{h}(a(\zeta)) \leq \mathcal{C}(n; d) d^{2n-1}(\delta + d_f)(d + \tau)$$

and for any $\zeta \in V_{\mathbb{C}}$ such that $a(\zeta) \neq 0$, we have $\mathfrak{h}(a(\zeta)) \geq -\mathcal{C}(n; d) d^{n-1}(\delta + d_f)(d + \tau)$ by Lemma 4.2. \square

We can now deduce the bounds on the SoS representation of f .

Theorem 6.2. Let $I = (\mathbf{h}) \subset \mathbb{Q}[\mathbf{x}]$ be a zero-dimensional radical ideal and let B be a monomial basis of $\mathbb{Q}[\mathbf{x}]/I$ with $D := |D|$ and $\delta := \max\{d, \deg(B)\}$. Let $S = S(\mathbf{g}, \mathbf{h})$ be as in Equation (2) where $g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{Z}[\mathbf{x}]$ with $\deg(g_i), \deg(h_j) \leq d$ and $\mathfrak{h}(g_i), \mathfrak{h}(h_j) \leq \tau$, $1 \leq i \leq r$, $1 \leq j \leq s$. Let $f \in \mathbb{Z}[\mathbf{x}]$ be such that $f \geq 0$ on S with $d_f := \max\{d, \deg(f)\}$ and $\mathfrak{h}(f) \leq \tau$. Then there exist $\nu_{0,k}, \nu_1, \nu_2 \in \mathbb{N}$, $\omega_{i,k} \in \mathbb{N}$, $q_{i,k} \in f \cdot \langle B \rangle_{\mathbb{Z}}$ and $p_j \in \mathbb{Z}[\mathbf{x}]$ for $1 \leq i \leq r$, $1 \leq k \leq D$ and $1 \leq j \leq s$, such that

$$f = \sum_{k=1}^D \frac{1}{\nu_{0,k}} q_{0,k}^2 + \frac{1}{\nu_1} \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i + \frac{1}{\nu_2} \sum_{j=1}^s p_j h_j$$

where

- $\mathfrak{h}(\nu_{0,k}), \mathfrak{h}(q_{0,k}) \leq \mathcal{C}(n; d) d^{3n-1} (\delta + d_f)(d + \tau)$ for $1 \leq k \leq D$;
- $\mathfrak{h}(\nu_1), \mathfrak{h}(\omega_{i,k}), \mathfrak{h}(q_{i,k}) \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau)$ for $1 \leq i \leq r$, $1 \leq k \leq D$.

Furthermore, if \mathbf{h} is a graded basis of I , then for all j ,

$$\deg(p_j) < \widehat{d} - \deg(h_j) \quad \text{and} \quad \mathfrak{h}(\nu_2), \mathfrak{h}(p_j) \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau) + \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau$$

for $\widehat{d} := 2(d_f + \deg(B)) + 1$.

Proof. We apply Theorem 5.7 to $a > 0$ on S defined in Lemma 6.1 such that $d_a = \max\{d, \delta\}$ and $\tau_a = \eta_a = \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau)$. We get

$$a \equiv \frac{1}{\nu_0} B Q_0 B^t + \frac{1}{\widehat{\nu}_1} \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} \widehat{q}_{i,k}^2 \right) g_i \pmod{I}$$

where $\nu_0, \widehat{\nu}_1 \in \mathbb{N}$, $Q_0 \in S^D(\mathbb{Z})$ is a positive definite matrix, and $\omega_{i,k} \in \mathbb{N}$, $\widehat{q}_{i,k} \in \langle B \rangle_{\mathbb{Z}}$ satisfy

$$\begin{aligned} \mathfrak{h}(\nu_0), \mathfrak{h}(Q_0) &\leq \mathcal{C}(n; d) d^{n-1} (d^n \delta + d_a)(d + \tau) + \mathcal{C}(1) \eta_a + \tau_a \\ &\leq \mathcal{C}(n; d) d^{2n-1} \delta (d + \tau) + \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau) \\ &\leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau). \end{aligned}$$

Analogously,

$$\mathfrak{h}(\widehat{\nu}_1), \mathfrak{h}(\omega_{i,k}), \mathfrak{h}(\widehat{q}_{i,k}) \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau) \quad \text{for } 1 \leq i \leq r, 1 \leq k \leq D.$$

Since from Lemma 6.1 we have $af + b \equiv \gamma \pmod{I}$ and $bf \equiv 0 \pmod{I}$ we deduce that

$$\begin{aligned} f &\equiv \frac{1}{\gamma} a f^2 \pmod{I} \\ &\equiv \frac{1}{\gamma \nu_0} f^2 B Q_0 B^t + \frac{1}{\gamma \widehat{\nu}_1} \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} (f \widehat{q}_{i,k})^2 \right) g_i \pmod{I}, \end{aligned}$$

where $\gamma \in \mathbb{N}$ is s.t. $\mathfrak{h}(\gamma) \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau)$.

Then, we apply Proposition A.6 to the positive definite matrix $Q_0 \in S^D(\mathbb{Z})$ and obtain that

$$B Q_0 B^t = \sum \frac{1}{\widehat{\nu}_{0,k}} \widehat{q}_{0,k}^2$$

where for $1 \leq k \leq D$, $\widehat{\nu}_{0,k} \in \mathbb{N}$ and $\widehat{q}_{0,k} \in \langle B \rangle_{\mathbb{Z}}$ satisfy

$$h(\widehat{\nu}_{0,k}), \mathfrak{h}(\widehat{q}_{0,k}) \leq \mathcal{C}(n; d) d^{3n-1} (\delta + d_f)(d + \tau).$$

Therefore, defining $\nu_{0,k} := \gamma \nu_0 \widehat{\nu}_{0,k}$, $\nu_1 := \gamma \widehat{\nu}_1$ and $q_{i,k} := f \widehat{q}_{i,k}$ we obtain $\nu_{0,k}, \nu_1 \in \mathbb{N}$ and $q_{i,k} \in f \cdot \langle B \rangle_{\mathbb{Z}}$ which satisfy

$$f \equiv \sum_{k=1}^D \frac{1}{\nu_{0,k}} q_{0,k}^2 + \frac{1}{\nu_1} \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) \pmod{I}$$

with

$$\begin{aligned} \mathfrak{h}(\nu_{0,k}), \mathfrak{h}(q_{0,k}) &\leq \mathcal{C}(n; d) d^{3n-1} (\delta + d_f)(d + \tau) \quad \text{for } 1 \leq k \leq D, \\ \mathfrak{h}(\nu_1), \mathfrak{h}(q_{i,k}) &\leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau) \quad \text{for } 1 \leq i \leq r, 1 \leq k \leq D. \end{aligned}$$

This shows the first part of the statement.

When \mathbf{h} is a graded basis of I , we conclude as in the proof of Theorem 5.8 by observing that the polynomial

$$\widehat{f} := \gamma \nu_0 \nu_1 f - \nu_1 f^2 B Q_0 B^t - \nu_0 \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i \in I \cap \mathbb{Z}[\mathbf{x}]$$

satisfies

$$\deg(\widehat{f}) < \widehat{d} := 2(d_f + \deg(B)) + 1 \quad \text{and} \quad \mathfrak{h}(\widehat{f}) \leq \widehat{\tau} := \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau).$$

By Proposition 4.8, we have $\widehat{f} = \frac{1}{\widehat{\nu}} \sum_{j=1}^s p_j h_j$ with

$$\mathfrak{h}(\widehat{\nu}) \leq \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau, \quad \mathfrak{h}(p_j) \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau) + \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau$$

Finally, we obtain the required SoS representation by defining $\nu_2 := \gamma \nu_0 \nu_1 \widehat{\nu}$, which also satisfies

$$\mathfrak{h}(\nu_2) \leq \mathcal{C}(n; d) d^{2n-1} (\delta + d_f)(d + \tau) + \mathcal{C}(n; \widehat{d}) \widehat{d}^n \tau. \quad \square$$

7 Algorithm and examples

The approach developed in the manuscript leads naturally to an algorithm to compute the SoS decomposition of a strictly positive polynomial, using Semi-Definite Programming (SDP), which we draft below.

For $\ell \in \mathbb{N}$, let \mathbf{m}_ℓ be the row vector of all monomials of degree $\leq \ell$. It is a well-known fact that $p \in \mathbb{R}[\mathbf{x}]_{2\ell}$ is a sum of squares iff there exist $Q \succcurlyeq 0$ (a positive semi-definite symmetric matrix) such that $p = \mathbf{m}_\ell Q \mathbf{m}_\ell^t$. Therefore, computing SoS representations of a certain degree boils down to computing positive semi-definite matrices that represent the sums of squares. This task can be performed efficiently, using existing SDP solvers, provided the degree of the polynomials and the size of the SDP matrices is not too big, or equivalently that ℓ is small enough.

The SoS representation of a strictly positive polynomial

$$f \equiv \sum_{k=1}^D \omega_{0,k} q_{0,k}^2 + \sum_{i=1}^r \left(\sum_{k=1}^D \omega_{i,k} q_{i,k}^2 \right) g_i \pmod{I}$$

given in Theorem A can be rewritten as

$$f \equiv B\tilde{Q}_0 B^t + \sum_i (B\tilde{Q}_i B^t) g_i \pmod{I}$$

with $\tilde{Q}_i \succcurlyeq 0$ and $\tilde{Q}_0 \succ 0$ (i.e. respectively \tilde{Q}_i positive semidefinite and \tilde{Q}_0 positive definite) when $f > 0$ on S , see also Theorem 2.7.

This can be reformulated into a classical SDP decomposition using monomials of bounded degree, as shown in the next lemma:

Lemma 7.1. *If f admits a representation of the form (7) with $\tilde{Q}_i \succcurlyeq 0$ and $\tilde{Q}_0 \succ 0$ and \mathbf{h} is a graded basis, then for $\ell_i \geq \deg(B)$,*

$$f = \mathbf{m}_{\ell_0} Q_0 \mathbf{m}_{\ell_0}^t + \sum_i (\mathbf{m}_{\ell_i} Q_i \mathbf{m}_{\ell_i}^t) g_i + \sum_{j=1}^s p_j h_j \quad (20)$$

with $Q_0 \succ 0$, $Q_i \succcurlyeq 0$ for $i = 1, \dots, r$, \mathbf{m}_{ℓ} is the vector of all monomials of degree $\leq \ell$, $\ell_i \geq \deg(B)$ and $p_j \in \mathbb{R}[\mathbf{x}]$ satisfies $\deg(p_j) \leq \max\{\deg(f), 2\ell_0, \deg(g_i) + 2\ell_i\} - \deg(h_j)$.

Proof. Since B is a monomial basis of $\mathbb{R}[\mathbf{x}]/I$ and \mathbf{h} is a graded basis of I , we can decompose $\mathbb{R}[\mathbf{x}]_{\ell} = B \oplus I_{\ell}$ with $I_{\ell} = I \cap \mathbb{R}[\mathbf{x}]_{\ell} = \{\sum_{j=1}^s p_j h_j \mid \deg(p_j) \leq \ell - \deg(h_j), j = 1, \dots, s\}$. Choosing a basis C of I_{ℓ} so that $B \cup C$ is basis of $\mathbb{R}[\mathbf{x}]_{\ell}$, we take Q_i to be a block diagonal matrix in the basis $B \cup C$ with the upper block equal to \tilde{Q}_i and the identity for the lower block. Then we have $Q_0 \succ 0$, $Q_i \succcurlyeq 0$ and $\mathbf{m}_{\ell_i} Q_i \mathbf{m}_{\ell_i}^t \equiv B\tilde{Q}_i B^t \pmod{I}$. We deduce that

$$f - \left(\mathbf{m}_{\ell_0} Q_0 \mathbf{m}_{\ell_0}^t + \sum_i (\mathbf{m}_{\ell_i} Q_i \mathbf{m}_{\ell_i}^t) g_i \right) \in I_{\max\{\deg(f), 2\ell_0, \deg(g_i) + 2\ell_i\}},$$

which yields the decomposition (20). □

To find such a decomposition, we solve a Semi-Definite Program that maximizes the smallest eigenvalue of Q_0 in (20). ?? implies that, when considering monomials of sufficiently high degree, this smallest eigenvalue will be strictly positive. Then we round, to a correct precision and taking into account the smallest eigenvalue of Q_0^* , the approximate optimal matrices Q_i^* and polynomials p_j^* , to obtain an exact SoS representation for f .

Algorithm 1: Computing a rational decomposition of $f > 0$ on S

Input: $f, g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{Q}[\mathbf{x}]$ and $\ell_0, \dots, \ell_r \in \mathbb{N}$ such that $\ell_i \geq \delta$, where δ is the degree of a finite basis B of $\mathbb{Q}[\mathbf{x}]/I$, and (h_1, \dots, h_s) is a graded basis of I .

1. Solve the Semi-Definite Program

$$\begin{aligned} & \sup \quad \lambda \\ & \text{s.t.} \quad Q_0 - \lambda \mathbf{I} \succcurlyeq 0 \\ & \quad f = \mathbf{m}_{\ell_0} Q_0 \mathbf{m}_{\ell_0}^t + \sum_{i=1}^r (\mathbf{m}_{\ell_i} Q_i \mathbf{m}_{\ell_i}^t) g_i + \sum_{j=1}^s p_j h_j \quad \text{with} \\ & \quad Q_i \succcurlyeq 0, p_j \in \mathbb{R}[\mathbf{x}]_{\max\{\deg(f), \deg(g_i) + 2\ell_i\} - \deg(h_k)} \end{aligned} \tag{21}$$

to get approximate optimal Q_i^* , p_j^* and $\lambda^* > 0$ lower bounding the optimal smallest eigenvalue of Q_0^* .

2. Repeat for $\kappa = \max\{\lceil -\log(\lambda^*) \rceil, 0\} \dots$

- Compute a weighted Cholesky factorization of Q_i^* and round it at precision k to get $\widehat{\omega}_{i,k} \in \mathbb{Q}_{\geq 0}$ and $\widehat{q}_{i,k} \in \langle B \rangle_{\mathbb{Q}}$ such that

$$BQ_i^*B^t \approx \sum_k \widehat{\omega}_{i,k} \widehat{q}_{i,k}^2,$$

- Round p_j^* at precision κ to get \widehat{p}_j ,
- Compute a weighted Cholesky decomposition of

$$\widehat{f} = f - \sum_i \left(\sum_k \widehat{\omega}_{i,k} \widehat{q}_{i,k}^2 \right) g_i - \sum_j \widehat{p}_j h_j$$

$$\text{as } \widehat{f} = \sum_k \widehat{\omega}_{0,k} (\widehat{q}_{0,k})^2,$$

until $\widehat{\omega}_{0,k} > 0$.

Output: $f = \sum_k \widehat{\omega}_{0,k} \widehat{q}_{0,k}^2 + \sum_i \left(\sum_k \widehat{\omega}_{i,k} \widehat{q}_{i,k}^2 \right) g_i + \sum_j \widehat{p}_j h_j$ with $\widehat{\omega}_{0,k} \in \mathbb{Q}_{>0}$, $\widehat{\omega}_{i,k} \in \mathbb{Q}_{\geq 0}$, $\widehat{q}_{i,k} \in \mathbb{Q}[\mathbf{x}]_{\ell_i}$, $\widehat{p}_j \in \mathbb{R}[\mathbf{x}]_{\max\{2\ell, \deg(g_i) + 2\ell_i\} - \deg(h_k)}$.

The output of the algorithm is a certificate of positivity for f on S . It may not be of smallest degree as in Theorem A, and with small bitsize coefficients for the SoS representation, as in Theorem C. To decrease the degrees in the SoS representation, one can simply reduce $\widehat{q}_{i,k}$ by the graded basis \mathbf{h} to get $q_{i,k} \in \langle B \rangle_{\mathbb{Q}}$. It should be noticed that the precision needed in the rounding in this algorithm is less than in Section 5, since we maximize the smallest eigenvalue of Q_0 . In other words, the algorithm is self-adaptive, and we can expect good bitsize bounds on the computed SoS representation.

The complexity of Algorithm 1 is dominated by the complexity of solving the SDP program (21) approximately. By [NN94], the number of arithmetic operations needed to find an approximate solution of Equation (21) is in

$$\mathcal{O}(\sqrt{M} N^2 (\sum_{i=0}^r M_i^2 + 2 \sum_{j=1}^s M_j^2) + \sqrt{M} N (\sum_{i=1}^r M_i^3 + 2 \sum_{j=1}^s M_j^3))$$

where

- $\widehat{d} = \max\{\deg(f), \deg(g_i), \deg(h_j)\} + 2\delta + 1$
- $M_i \leq \widehat{d}^n$ (resp. $M'_j \leq \widehat{d}^n$) (resp. $N \leq \widehat{d}^n$) is the number of monomials of degree $< \widehat{d} - \deg(g_i)$ (resp. $< \widehat{d} - \deg(h_j)$ with $g_0 = 1$) (resp. \widehat{d}),
- $M = \sum_{i=0}^r M_0 + 2 \sum_{j=1}^s M'_j \leq (r + 2s + 1)\widehat{d}^n$.

The precision needed to perform this computation is in $\mathcal{O}(\kappa)$ where $\kappa = \max\{[-\log(\lambda^*)], 0\}$. By Theorem 5.7, when $f > 0$ on S , we have $\kappa = \widetilde{\mathcal{O}}(d^n)\delta\tau = \widetilde{\mathcal{O}}(\widehat{d}^{n+1})\tau$. The weighted Cholesky decompositions involved in step 3 requires $\mathcal{O}(M_i^3) = \mathcal{O}(\widehat{d}^{3n})$ arithmetic operations. Thus the total bit complexity of Algorithm 1 when I is radical and $f \geq 0$ on S is in

$$\widetilde{\mathcal{O}}(\kappa)(r + 2s + 1)\widehat{d}^{4.5n} = \widetilde{\mathcal{O}}(\widehat{d}^{5.5n+1})(r + 2s + 1)\tau.$$

This algorithm is implemented in the JULIA package `MomentPolynomialOpt.jl`². We present hereafter experimentation with this tool.

Example 7.2. Set $g_1 := y$, $h_1 := x^2 - 1$, $h_2 := y^2 - x - 2 \in \mathbb{Q}[x, y]$ and $S = S(g; h_1, h_2) \subset \mathbb{R}^2$, the basic closed semialgebraic set defined by $g_1 \geq 0$, $h_1 = h_2 = 0$.

We have $S = \{(1, \sqrt{3}), (-1, 1)\}$. Moreover, (h_1, h_2) is a graded basis of $I = (h_1, h_2)$ and $B = \{1, x, y, xy\}$ is a reduced basis of $\mathbb{Q}[\mathbf{x}]/I$, of degree $\delta = 2$.

Let $f := x + y + 3 \in \mathbb{Q}[x, y]$, which is strictly positive on S . We apply Algorithm 1 in degree 4 to compute a certificate of (strict) positivity of f on S , using a rounding precision of 1 (decimal) digit:

```
WS, P, v, M = sos_decompose(f, G, H, X, 2; exact = true, round = 1)
```

with $\mathbf{G} = [g_1]$, $\mathbf{H} = [h_1, h_2]$, $\mathbf{X} = [x, y]$ and a relaxation order $r = 2$. We obtain the exact decomposition

$$f = q_0 + q_1 g_1 + p_1 h_1 + p_2 h_2$$

where

$$\begin{aligned} q_0 &= \frac{1}{2}x^4 + \frac{3}{10}x^2y^2 - \frac{1}{10}xy^3 + \frac{2}{5}y^4 + \frac{1}{10}x^3 - \frac{3}{50}xy^2 - \frac{9}{125}y^3 \\ &\quad + \frac{3}{10}x^2 + \frac{7}{25}y^2 - \frac{39}{500}y + \frac{7}{10}, \\ &= \frac{1}{2} \left(x^2 + \frac{1}{5}y^2 + \frac{1}{10}x + \frac{1}{5} \right)^2 + \frac{1}{10} \left(xy - \frac{1}{2}y^2 - \frac{3}{20}y \right)^2 \\ &\quad + \frac{71}{200} \left(y^2 - \frac{5}{71}x - \frac{87}{710}y + \frac{44}{213} \right)^2 + \frac{331}{3550} \left(x - \frac{87}{2648}y - \frac{103}{1986} \right)^2 \\ &\quad + \frac{6804227}{79440000} \left(y - \frac{2396940}{6804227} \right)^2 + \frac{1001198282}{1530951075}, \\ q_1 &= \frac{1}{5}x^2 - \frac{6}{25}xy + \frac{34}{125}y^2 - \frac{17}{25}y + \frac{289}{500}, \\ &= \frac{1}{5} \left(x - \frac{3}{5}y \right)^2 + \frac{1}{5} \left(y - \frac{17}{10} \right)^2, \\ p_1 &= -\frac{1}{2}x^2 - \frac{2}{5}y^2 - \frac{1}{10}y - \frac{7}{10}, \\ p_2 &= \frac{1}{10}x^2 + \frac{1}{10}xy - \frac{2}{5}y^2 - \frac{1}{10}x - \frac{1}{5}y - \frac{4}{5}. \end{aligned}$$

²<https://github.com/AlgebraicGeometricModeling/MomentPolynomialOpt.jl>

This example shows that SoS representations can exist in strictly smaller degree than the upper bound 5 of Theorem 5.7.

Example 7.3. Set $h_1 := x^3 - y^2$, $h_2 := x^2 - 2x + y^2 \in \mathbb{Q}[x, y]$ and $S = V_{\mathbb{R}}(h_1, h_2) \subset \mathbb{R}^2$.

We have $S = \{(0, 0), (1, \pm 1), (2, \pm 2\sqrt{2})\}$ where $(0, 0)$ has multiplicity 2. Moreover, (h_1, h_2) is a graded basis of $I = (h_1, h_2)$, and $B = \{1, x, x^2, y, xy, xy^2\}$ is a reduced basis of $\mathbb{Q}[x, y]/I$ of degree $\delta = 3$.

Let $f = x \in \mathbb{Q}[x, y]$, which is nonnegative on S . We can verify that $(I : f) + (f) = (1)$ and that $ax + b = 2$, for $a = 1 + x$ and $b = 2 - x - x^2$, where $bx = -h_1 - h_2 \in I$.

Applying Algorithm 1 to $a = 1 + x$ in degree 4, using a rounding precision of 1 (decimal) digit, we get

$$a = q_0 + p_1 h_1 + p_2 h_2$$

where

$$\begin{aligned} q_0 &= \left(1 - \frac{3}{10}x - \frac{1}{15}y^2 - \frac{1}{3}x^2\right)^2 + \frac{5}{6}\left(y - \frac{57}{100}xy\right)^2 + \frac{233}{300}\left(x - \frac{57}{466}y^2 - \frac{60}{233}x^2\right)^2 \\ &\quad + \frac{16237}{33552}\left(y^2 - \frac{7832}{81185}x^2\right)^2 + \frac{2117}{4000}(xy)^2 + \frac{432621}{811850}(x^2)^2 \\ p_1 &= -\frac{1}{10} - \frac{2}{5}x \\ p_2 &= -\frac{4}{5} - \frac{3}{10}x - \frac{1}{2}y^2 - \frac{3}{10}x^2 \end{aligned}$$

Since $2x = ax^2 + bx = ax^2 - h_1 - h_2$, we deduce a SoS representation of $f = x$:

$$x = \frac{1}{2}q_0 x^2 + \frac{1}{2}(p_1 x^2 - 1)h_1 + \frac{1}{2}(p_2 x^2 - 1)h_2.$$

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A Appendix: Linear algebra bounds

In this appendix, we recall height bounds for the solutions of basic linear algebra operations on matrices with integer coefficients.

A.1 Solving linear systems

An important operation in our approach is to compute a solution of a linear system of equations. We can bound the size of a rational solution of a linear system with integer coefficients as follows:

Lemma A.1. *Let $\mathbf{A} \in \mathbb{Z}^{M \times N}$ be such that $M \leq N$ and $\mathbf{b} \in \mathbb{Z}^M$ with $\mathfrak{h}(\mathbf{A}) \leq \tau_A$ and $\mathfrak{h}(\mathbf{b}) \leq \tau_b$. If the linear system $\mathbf{A}\mathbf{y} = \mathbf{b}$ admits a solution, then one of these solutions can be written by Cramer's rule as $\mathbf{c} = \widehat{\mathbf{c}}/\nu$ with $\widehat{\mathbf{c}} \in \mathbb{Z}^D$ and $\nu \in \mathbb{N}$ satisfying by Hadamard's inequality*

$$\mathfrak{h}(\nu) \leq \frac{M}{2} \log(M) + M\tau_A \quad \text{and} \quad \mathfrak{h}(\widehat{\mathbf{c}}) \leq \frac{M}{2} \log(M) + (M-1)\tau_A + \tau_b.$$

A.2 Projection on a linear space

Lemma A.2. *Let $M \leq N$, $\mathbf{A} \in \mathbb{Z}^{M \times N}$ of rank M and $\mathbf{b} \in \mathbb{Z}^M$, with $1 \leq \mathfrak{h}(\mathbf{A}) \leq \tau_A$, $1 \leq \mathfrak{h}(\mathbf{b}) \leq \tau_b$. Let \mathcal{L} be the linear variety*

$$\mathcal{L} = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{A}\mathbf{y} = \mathbf{b}\}.$$

Let $\mathbf{c}_0 = \widehat{\mathbf{c}}_0/\nu_0 \in \mathbb{Q}^N$ with $\widehat{\mathbf{c}}_0 \in \mathbb{Z}^N$, $\nu_0 \in \mathbb{N}$, $\mathfrak{h}(\widehat{\mathbf{c}}_0), \mathfrak{h}(\nu_0) \leq \tau_0$. Then, the orthogonal projection $\mathbf{c} = \widehat{\mathbf{c}}/\nu \in \mathbb{Q}^N$ with $\widehat{\mathbf{c}} \in \mathbb{Z}^N$, $\nu \in \mathbb{N}$, of \mathbf{c}_0 on \mathcal{L} satisfies

$$\mathfrak{h}(\widehat{\mathbf{c}}), \mathfrak{h}(\nu) \leq (2M+1)(\log(N) + \tau_A) + \tau_b + \tau_0.$$

Proof. Since the columns of \mathbf{A}^t span $(\ker \mathbf{A})^\perp$, the orthogonal projection \mathbf{c} of \mathbf{c}_0 on \mathcal{L} is $\mathbf{c} = \mathbf{c}_0 - \mathbf{A}^t \mathbf{z}$ for $\mathbf{z} \in \mathbb{Q}^M$ such that

$$\mathbf{A}\mathbf{A}^t \mathbf{z} = \mathbf{A}\mathbf{c}_0 - \mathbf{b} = \frac{\mathbf{A}\widehat{\mathbf{c}}_0 - \nu_0 \mathbf{b}}{\nu_0}.$$

Let $\mathbf{z} = \widetilde{\mathbf{z}}/\nu_0$ with $\widetilde{\mathbf{z}} \in \mathbb{Q}^M$, so that

$$\mathbf{A}\mathbf{A}^t \widetilde{\mathbf{z}} = \mathbf{A}\widehat{\mathbf{c}}_0 - \nu_0 \mathbf{b}.$$

As $\mathfrak{h}(\mathbf{A}\mathbf{A}^t) \leq \log(N) + 2\tau_A$ and $\mathfrak{h}(\mathbf{A}\widehat{\mathbf{c}}_0 - \nu_0 \mathbf{b}) \leq \log(N) + \tau_A + \tau_b + \tau_0$, we deduce by Lemma A.1 that $\widetilde{\mathbf{z}} = \widehat{\mathbf{z}}/\nu_1$ with $\nu_1 \in \mathbb{N}$, $\widehat{\mathbf{z}} \in \mathbb{Z}^M$ such that

$$\mathfrak{h}(\nu_1) \leq 2M(\log(N) + \tau_A) \quad \text{and} \quad \mathfrak{h}(\widehat{\mathbf{z}}) \leq 2M \log(N) + (2M-1)\tau_A + \tau_b + \tau_0.$$

Thus, $\mathbf{c} = \mathbf{c}_0 - \mathbf{A}^t \mathbf{z} = \frac{\nu_1 \widehat{\mathbf{c}}_0 - \mathbf{A}^t \widehat{\mathbf{z}}}{\nu_0 \nu_1} = \frac{\widehat{\mathbf{c}}}{\nu}$ satisfies

$$\mathfrak{h}(\nu), \mathfrak{h}(\widehat{\mathbf{c}}) \leq (2M+1) \log(N) + 2M\tau_A + \tau_b + \tau_0. \quad \square$$

A.3 Weighted square-root-free Cholesky decomposition

Another important ingredient is the decomposition of a definite positive quadratic form as a weighted sum of squares, also known as a weighted Cholevsky factorisation. The decomposition proceeds by induction, removing one variable at a time, using a completion-of-square technique.

Let $Q(X_1, \dots, X_M) = \sum_{1 \leq i, j \leq M} q_{i,j} X_i X_j$ be a form with $q_{i,j} = q_{j,i} \in \mathbb{R}$. We denote by $\Delta_{i,j}^k$ the $(k+1) \times (k+1)$ minor formed by the rows $1, \dots, k, i$ and columns $1, \dots, k, j$ of $\mathbf{Q} = (q_{i,j})$.

Notice that $\Delta_{i,j}^0 = q_{i,j}$ and $\Delta_{i,j}^k = 0$ if $i \leq k$ or $j \leq k$. Let $\Delta_k = \Delta_{k,k}^{k-1}$ be the k^{th} first principal minor. We assume hereafter that $\Delta_k \neq 0$ for $k = 1, \dots, M$.

Completing the square with respect to the variable X_1 , we obtain

$$Q(X_1, \dots, X_M) = \frac{1}{q_{1,1}} \left(\sum_i q_{i,1} X_i \right)^2 + Q'(X_2, \dots, X_M),$$

where $Q' = (q'_{i,j})$ with

$$q'_{i,j} = q_{i,j} - \frac{1}{q_{1,1}} q_{i,1} q_{1,j} = \frac{1}{q_{1,1}} (q_{1,1} q_{i,j} - q_{i,1} q_{1,j}).$$

Then $q'_{i,j} = \frac{1}{\Delta_1} \Delta_{i,j}^1$. By the same computation applied to Q' , we get

$$\begin{aligned} Q(X_1, \dots, X_M) &= \frac{1}{\Delta_1} \left(\sum_i \Delta_{i,1}^0 X_i \right)^2 + \frac{1}{q'_{2,2}} \left(\sum_{i=2}^M q'_{i,2} X_i \right)^2 + Q''(X_3, \dots, X_M) \\ &= \frac{1}{\Delta_1} \left(\sum_i \Delta_{i,1}^0 X_i \right)^2 + \frac{1}{\Delta_2 \Delta_1} \left(\sum_i \Delta_{i,2}^1 X_i \right)^2 + Q''(X_3, \dots, X_M) \end{aligned} \quad (22)$$

where $Q'' = (q''_{i,j})$ with

$$q''_{i,j} = \frac{1}{q'_{2,2}} (q'_{2,2} q'_{i,j} - q'_{i,2} q'_{1,j}) = \frac{1}{\Delta_2 \Delta_1} (\Delta_{2,2}^1 \Delta_{i,j}^1 - \Delta_{i,1}^1 \Delta_{1,j}^1).$$

Using Sylvester identity (see e.g. [Bar68]), we have $\Delta_{2,2}^1 \Delta_{i,j}^1 - \Delta_{i,1}^1 \Delta_{1,j}^1 = \Delta_1 \Delta_{i,j}^2$ and thus $q''_{i,j} = \frac{1}{\Delta_2} \Delta_{i,j}^2$.

Repeating this computation, we obtain an explicit decomposition of Q in terms of squares:

Proposition A.3. *Let $Q(X_1, \dots, X_M) = \sum_{1 \leq i, j \leq M} q_{i,j} X_i X_j$ be a quadratic form such that $\Delta_k \neq 0$ for $k = 1, \dots, M$ and let $\Delta_0 = 1$. Then*

$$Q = \sum_{k=1}^M \frac{1}{\Delta_k \Delta_{k-1}} \left(\sum_i \Delta_{i,k}^{k-1} X_i \right)^2.$$

Proof. We prove by induction that

$$Q(X_1, \dots, X_M) = \frac{1}{\Delta_1} \left(\sum_i \Delta_{i,1}^0 X_i \right)^2 + s + \frac{1}{\Delta_{k-1} \Delta_{k-2}} \left(\sum_i \Delta_{i,k-1}^{k-2} X_i \right)^2 + Q^{(k-1)}(X_k, \dots, X_M) \quad (23)$$

where $Q^{(k-1)}(X_k, \dots, X_M) = (q_{i,j}^{(k-1)})$ with $q_{i,j}^{(k-1)} = \frac{1}{\Delta_{k-1}} \Delta_{i,j}^{(k-1)}$. By the same construction as above applied to $Q^{(k-1)}$ we obtain a decomposition at order k from the decomposition (23) or order $k-1$, with $Q^{(k)} = (q_{i,j}^{(k)})$ such that

$$q_{i,j}^{(k)} = \frac{1}{q_{k,k}^{(k-1)}} (q_{k,k}^{(k-1)} q_{i,j}^{(k-1)} - q_{i,k}^{(k-1)} q_{k,j}^{(k-1)}) = \frac{1}{\Delta_k} \Delta_{i,j}^k.$$

Here we use again Sylvester identity: $\Delta_k \Delta_{i,j}^{k-1} - \Delta_{i,k}^{k-1} \Delta_{k,j}^{k-1} = \Delta_{k-1} \Delta_{i,j}^k$. This proves the induction and the proposition since we have (22). \square

Remark A.4. This shows in particular that if Q is a quadratic form such as $\Delta_k \neq 0$, then its signature is $(M-s, s)$ where s is the number of sign changes of the sequence $[\Delta_0 = 1, \Delta_1, \dots, \Delta_M]$. This is a well-known result, probably due to Sylvester.

We deduce the following proposition:

Proposition A.5. Let K be a number field and $Q \in S^M(K)$ such that $\Delta_k \neq 0$ for $k = 1, \dots, M$ and $\Delta_0 = 1$. Then

$$\mathbf{Q} = \mathbf{L} \mathbf{D} \mathbf{L}^t$$

where

- $\mathbf{L} = (\mathbf{L}_{i,j})_{1 \leq i, j \leq M}$ is lower triangular with $\mathbf{L}_{i,j} = \Delta_{i,j}^{j-1}$.
- \mathbf{D} is diagonal with $\mathbf{D}_{i,i} = \frac{1}{\Delta_i \Delta_{i-1}}$.

Proposition A.6. Let $\mathbf{Q} \in S^M(\mathbb{Z})$ be a positive definite matrix with $\mathfrak{h}(\mathbf{Q}) \leq \tau_Q$. Then

$$\mathbf{Q} = \mathbf{L} \mathbf{D} \mathbf{L}^t$$

where $\mathbf{D} = \text{diag}(\frac{1}{\nu_1}, \dots, \frac{1}{\nu_M})$, $\nu_i \in \mathbb{N}$, $\mathbf{L} \in S^M(\mathbb{Z})$ is lower triangular with

$$\mathfrak{h}(\nu_i), \mathfrak{h}(\mathbf{L}) \leq 2M (\log(M) + \tau_Q)$$

Proof. If $\mathbf{Q} \succ 0$, then its principal minors $\Delta_j > 0$ are strictly positive. By Proposition A.5, $\mathbf{Q} = \mathbf{L} \mathbf{D} \mathbf{L}^t$ with $L_{i,j} = \Delta_{i,j}^{j-1} \in \mathbb{Z}$ and $L_{i,j} = 0$ if $i < j$, and $\nu_j = \Delta_j \Delta_{j-1} \in \mathbb{Q}_+$. By Hadamard identity on $j \times j$ minors, we have $\mathfrak{h}(L_{i,j}) \leq j (\log(j) + \tau_Q)$ and $\mathfrak{h}(\omega_j) \leq 2j (\log(j) + \tau_Q)$, which proves the proposition. \square