



HAL
open science

Tensor calculus for physicists

Loris Delafosse

► **To cite this version:**

Loris Delafosse. Tensor calculus for physicists. Students seminars at the Faculty of Physics and Engineering of Strasbourg, Sep 2024, Strasbourg (Université), France. ⟨hal-04722859⟩

HAL Id: hal-04722859

<https://hal.science/hal-04722859v1>

Submitted on 10 Oct 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY-SA 4.0 - Attribution - ShareAlike - International License

Tensor calculus for physicists

Loris DELAFOSSE¹

¹*Faculty of Physics and Engineering
University of Strasbourg*

(Dated: 27.09.2024)

Tensors are quite general mathematical objects, used in almost all areas of physics, from continuum mechanics to general relativity and quantum physics. Beginning with a reminder of some useful concepts of linear algebra, this seminar introduces the notion of tensor, discusses its relation with metrics, covariance and contravariance in Euclidean and Minkowski spaces, and eventually presents tensor products in the context of quantum mechanics.

I. BASICS OF LINEAR ALGEBRA

A. Vector spaces

A *scalar* is an element of a given field, typically \mathbb{R} or \mathbb{C} . A *vector space* is then a set E endowed with two operations: an internal addition (if $x, y \in E$, then $x + y$ exists and belongs to E) and a product by a scalar (if λ is a scalar and $x \in E$, then λx exists and belongs to E). These operations satisfy a number of simple axioms (distributivity, commutativity...) which can be found in any introductory textbook. The elements of E are called *vectors*.

A (finite) *basis* of E is a family of n vectors (e_1, e_2, \dots, e_n) such that each vector $x \in E$ can be decomposed as a linear combination of basis vectors, *i.e.* there exists a family of scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that:

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n \quad (1)$$

When they exist, all bases of a given vector space E contain the same number n of vectors. We say the dimension of E is n . If there exists no finite basis, we say E is infinite-dimensional.

Let λ, μ be scalars, let x, y be vectors. A *linear map* is a function φ defined on E such that $\varphi(\lambda x + \mu y) = \lambda \varphi(x) + \mu \varphi(y)$. Each linear map is completely characterized by its action on a basis. Indeed, to compute $\varphi(x)$, we can decompose x according to Eq. (1) and reduce by linearity to the computation of $\varphi(e_1), \dots, \varphi(e_n)$. An *endomorphism* is a linear map from E to itself. A *linear form* is a linear map that returns a scalar. The *dual space* of E , denoted E^* , is the set of all linear forms on E .

B. Euclidean spaces

A *Euclidean inner product* $\langle \cdot, \cdot \rangle$ on a real vector space E is a symmetric, positive definite, bilinear form on E . The inner product allows us to define useful concepts such as orthogonality (two vectors x, y are orthogonal if $\langle x, y \rangle = 0$), the norm $\|x\| = \sqrt{\langle x, x \rangle}$, and the distance $d(x, y) = \|x - y\|$. Introducing an inner product thus enables us to *measure* distances within the vector space.

To every vector $u \in E$, we can associate a linear form $x \mapsto \langle u, x \rangle$. Conversely, every linear form can be written as an inner product with a given vector. Thus, there exists a one-to-one correspondence (a bijection, and even an isomorphism) between E and its dual space [1], and for this reason, linear forms are also called *covectors*.

C. Matrix representations

Once we have a basis $e = (e_1, e_2, \dots, e_n)$, we also have a matrix representation of the objects we have introduced. Consider for example the vector x which admits the decomposition of Eq. (1). The scalar coefficients $\lambda_1, \lambda_2, \dots, \lambda_n$ are called the *coordinates* of x in the basis e , and x can be represented as a column matrix $[x]_e$:

$$[x]_e = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \quad (2)$$

A linear map φ is represented by a matrix $[\varphi]_{ef}$ whose columns are the representations of $\varphi(e_1), \dots, \varphi(e_n)$ in a basis f of the target space:

$$[\varphi(x)]_f = [\varphi]_{ef}[x]_e \quad (3)$$

Thus linear forms (covectors) are written as line vectors.

The inner product also admits a matrix representation, which will simply be the identity matrix for the usual inner product in an orthonormal basis. Indeed, if we want to calculate the inner product of $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, we have:

$$(x_1 \ y_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1x_2 + y_1y_2 \quad (4)$$

Matrices have another use: they allow for basis changes. Let us consider e and f two bases of the same space E . There must be a linear map sending, for each i , the vector e_i onto the vector f_i , and this linear map is represented in the e basis by:

$$P = ([f_1]_e \ [f_2]_e \ \dots \ [f_n]_e) \quad (5)$$

We then have, for all i , $[f_i]_e = P[e_i]_e$. P is called the basis change matrix from basis e to basis f . It is invertible by construction because the vectors $[f_1]_e, [f_2]_e, \dots, [f_n]_e$ are linearly independent, hence $\det P \neq 0$. Consequently: $[e_i]_f = P^{-1}[f_i]_f$.

Consider a vector x admitting the decomposition of Eq. (1) and project it onto the basis f :

$$[x]_f = P^{-1} \sum_{i=1}^n \lambda_i [f_i]_f = P^{-1} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = P^{-1}[x]_e \quad (6)$$

Endomorphisms also admit basis changes. Consider an endomorphism φ , which maps x to y :

$$[\varphi]_f \underbrace{[x]_f}_{=P^{-1}[x]_e} = [y]_f = P^{-1}[y]_e \quad (7)$$

$$[\varphi]_f P^{-1}[x]_e = P^{-1}[\varphi]_e [x]_e \quad \text{vrai quel que soit } x \in E \text{ donc :} \quad (8)$$

$$[\varphi]_f P^{-1} = P^{-1}[\varphi]_e \quad \text{d'où :} \quad (9)$$

$$[\varphi]_f = P^{-1}[\varphi]_e P \quad (10)$$

II. TENSORS AND METRICS

Throughout this section, we work in a n -dimensional vector space E .

A. Notion and operations

The term *tensor* is a generic term encompassing:

- scalars;
- vectors;
- covectors;
- objects formed by tensor products of vectors and covectors, and by linear combinations of these tensor products.

It seems only natural to classify tensors by their *rank* (*ordre* in French), which characterizes, in a way, their “size”. Scalars are thus tensors of rank 0, vectors and covectors (represented by row and column vectors) are tensors of rank 1, endomorphisms (represented by square matrices) are tensors of rank 2. By extension, a tensor of rank 3 is an object that can be represented by a “cubic matrix” in 3D, and so on. The notion of rank is more rigorously defined through the tensor product. We denote a tensor of rank p using p superscript bars.

Let \bar{a} and \bar{b} be two vectors. Their *tensor product* $\bar{a} \otimes \bar{b}$ is an endomorphism defined by its action on a vector \bar{x} :

$$(\bar{a} \otimes \bar{b})\bar{x} \stackrel{\text{def}}{=} \langle \bar{b}, \bar{x} \rangle \bar{a} \quad (11)$$

Let us compute the matrix elements of the endomorphism $\bar{C} = \bar{a} \otimes \bar{b}$ in the e basis:

$$C_{ij} = \langle \bar{e}_i, \bar{C}\bar{e}_j \rangle = \langle \bar{e}_i, (\bar{a} \otimes \bar{b})\bar{e}_j \rangle = \langle \bar{e}_i, \bar{a} \rangle \langle \bar{b}, \bar{e}_j \rangle = a_i b_j \quad (12)$$

In other words, the tensor product of two vectors is the the matrix product:

$$[\bar{a} \otimes \bar{b}]_e = [\bar{a}]_e [\bar{b}]_e^\top \quad (13)$$

Not all endomorphisms are tensor products of vectors. However, by taking all tensor products of the basis vectors, we obtain a basis for the space of endomorphisms. Consequently, all endomorphisms are linear combinations of tensor products of vectors.

Similarly, we can define tensor products between p vectors, which allows us to obtain tensors of rank p . The rules of calculation and the interpretation of such objects will be specified in Section IID, after the introduction of Ricci calculus. More generally, the tensor product of a tensor of rank p and a tensor of rank q is a tensor of rank $p + q$.

Conversely, it is possible to *contract* two tensors. The scalar product of two vectors, as well as the application of an endomorphism to a vector, are actually special cases of contraction. In these examples, the contraction of a tensor of rank p and a tensor of rank q is a tensor of rank $|p - q|$, but this is not true for all contractions. No contraction increases the rank. This will also be detailed in Section IID.

B. Basis change in a Euclidean space Covariance and contravariance

In this section, we explain how basis change is performed on tensors. Let us begin with the inner product matrix. Since we are dealing with a matrix, it is tempting to apply Eq. (10) starting from the identity matrix. It then appears that the matrix of the inner product should be the identity in all bases, but this is absurd: if we consider a *non-orthonormal* basis $e = (\bar{e}_1, \bar{e}_2)$ and compute the inner product $\langle \bar{e}_1, \bar{e}_2 \rangle$ in this basis, we would conclude that \bar{e}_1 and \bar{e}_2 are orthogonal because

$$(1 \ 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad (14)$$

Let $[\bar{g}]_e$ be the matrix of an arbitrary inner product in the basis e . The invariance of the inner product under a change of basis yields:

$$[\bar{x}]_f^\top [\bar{g}]_f [\bar{y}]_f = [\bar{x}]_e^\top [\bar{g}]_e [\bar{y}]_e \quad (15)$$

$$[\bar{x}]_f^\top [\bar{g}]_f [\bar{y}]_f = [\bar{x}]_f^\top P^\top [\bar{g}]_e P [\bar{y}]_f \quad \text{true for any } x, y \in E \text{ so :} \quad (16)$$

$$[\bar{g}]_f = P^\top [\bar{g}]_e P \quad (17)$$

Note that if we only consider orthonormal bases in a Euclidean space, then the basis change matrices are rotation matrices ($P^\top = P^{-1}$) and there is no difference between inner products and endomorphisms.

Similarly, for a covector \bar{a} (represented by a row matrix):

$$[\bar{a}]_f = [\bar{a}]_e P \quad (18)$$

Thus, some objects transform with the direct basis change matrix (or its transpose, it doesn't matter, it is the same coefficients in a different order) while others transform with the inverse basis change matrix (or its transpose).

An object that transforms with the direct basis change matrix is called *covariant*. An object that transforms with the inverse matrix is called *contravariant*. The number of basis change matrices to be used to transform a tensor increases with the rank of the tensor: a vector is multiplied by one matrix (cf. Eq. (6)), an endomorphism is multiplied by two matrices (cf. Eq. (10))... More generally, a tensor of rank m must be multiplied by m matrices in total during a change of basis. Among these m matrices, if there are p inverse change of basis matrices and $q = m - p$ direct matrices, the tensor is said to be p times contravariant and q times covariant. It is also said to be a (p, q) -tensor, or a tensor of *type* (p, q) .

Thus:

- scalars are (0,0)-tensors;
- vectors are (1,0)-tensors;
- covectors are (0,1)-tensors;
- endomorphisms are (1,1)-tensors;
- inner products are (0,2)-tensors.

From now on, we abandon the classical notation with superscript bars, which does not distinguish between covariant and contravariant components. This notation is used in classical mechanics, where the distinction is unnecessary since one works in a Euclidean space with orthonormal bases.

C. Metric tensor and pseudo-Euclidean spaces

It appears that a simple way to define the notion of distance is to introduce a symmetric, positive definite (0,2)-tensor. This tensor, when applied to two vectors, returns their inner product.

However, it is sometimes useful to have spaces where the notion of distance is non intuitive, spaces in which the distance between two different vectors can be zero. For this purpose, we can introduce an operation similar to the inner product (a symmetric bilinear form) but which is not positive definite: we call this operation a pseudo-inner product. We then no longer work in a Euclidean space but in a pseudo-Euclidean space.

One possibility for defining this pseudo-inner product is to introduce a symmetric (0,2)-tensor that is not positive definite, for example:

$$[\eta] \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (19)$$

This tensor is called the *metric* of the space. In a space equipped with this metric [2], the (pseudo-)inner product between two vectors becomes:

$$(x_1 \ y_1) [\eta] \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1 x_2 - y_1 y_2 \quad (20)$$

It is then easy to verify, for example, that the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has “norm” zero.

D. Ricci calculus

All operations on tensors (sums, tensor products, contractions, etc.) can be written in terms of coordinates. For example, as shown in Eq. (12), the tensor product c of two vectors a and b can be written as:

$$c_{ij} = a_i b_j \quad (21)$$

(where a_1, \dots, a_n are the components of a in the chosen basis, and similarly for b and c). The matrix product c of two endomorphisms a and b represented by square matrices is:

$$c_{ik} = \sum_{j=1}^n a_{ij} b_{jk} \quad (22)$$

The inner product (with metric g) between two vectors x and y is:

$$\sum_{i=1}^n \sum_{j=1}^n x_i g_{ij} y_j \quad (23)$$

We observe that whenever there is a summation, it occurs over an index that is repeated twice. Moreover, one of the two indices refers to a contravariant component, while the other refers to a covariant component.

Ricci's notation takes into account these very general observations to simplify the writing of tensors while making their type evident.

Tensors are represented by their components in a given basis. Thus, x^μ is the notation of a vector, where μ is an index that takes all values between 1 and n . Upper indices represent contravariant components, while lower indices represent covariant components. Therefore, x^μ is a vector, but y_μ is a covector. Endomorphisms, which are (1,1)-tensors, are denoted as $\varphi^\mu{}_\nu$.

The indices *repeated within the same term* are summation indices (dummy indices), according to Einstein's convention. Consequently, $x^\mu y_\mu$ actually denotes the sum $\sum_{\mu=1}^n x^\mu y_\mu$. On the other hand, $x^\mu + y^\mu$ is not an implicit summation, because the index μ is repeated in two different terms (it is simply the sum of two vectors). When an index μ is not repeated, it simply denotes the μ -th component of the tensor (it is a free index). In $x^\mu + y^\mu$, μ is always a free index, which can take any value between 1 and n .

The number of free indices associated with a tensor gives its rank. The number of upper and lower indices gives its type. This notation clearly shows the nature of the objects and which operations are possible or forbidden, without introducing special symbols for the operations! Thus, the expression $x^\mu + y_\mu$ makes no sense because the same index is in two different positions in two different terms (we are trying to add two objects of different nature). Similarly, the expression $\lambda_\mu + x^\mu y_\mu$ is forbidden because the first term is a covector while the second is a scalar (we are summing over the dummy index μ , so the object $x^\mu y_\mu$ is a tensor of rank 0). Finally, the formula $x^\mu y^\mu$ makes no sense either because the index μ is twice in the upper position.

$\varphi^\mu{}_\nu$ denotes an endomorphism. $\varphi^\mu{}_\mu$ is a scalar since there are no free indices (it is the trace of φ , in this case).

The tensor product between two arbitrary tensors is a simple component-wise product. We can write *e.g.* $T^{\mu\nu}{}_{\sigma\rho} = x^\mu A^\nu{}_{\sigma\rho}$ for the tensor product between a vector x and a (2,1)-tensor A .

Contractions are implicit summations. Thus, applying the endomorphism φ to the vector x to obtain a vector y is written as $y^\mu = \varphi^\mu{}_\nu x^\nu$. μ is a free index and ν is a dummy index.

The composition c of two endomorphisms a and b appears to be a one-index contraction: $c^\mu{}_\nu = a^\mu{}_\kappa b^\kappa{}_\nu$.

Finally, the metric is written $g_{\mu\nu}$ because it is a (0,2)-tensor. The two lower indices show that it takes two vectors as arguments to return a scalar $x^\mu g_{\mu\nu} y^\nu$. However, Ricci's notation highlights another interpretation of the metric: it is a map taking a vector as an argument and returning a covector: $g_{\mu\nu} x^\nu \stackrel{\text{def}}{=} x_\mu$. Thus, the metric is also the function that realizes the correspondence between a vector and its associated covector.

For this reason, the coordinates x_μ of the associated linear form are sometimes called the *covariant coordinates* of the vector x . This is an alternative way to locate a point in space, as Fig. 1 illustrates.

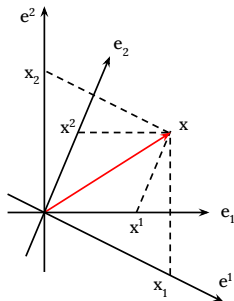


FIG. 1. Contravariant coordinates (parallel projections) et covariant coordinates (orthogonal projections) of a vector x .

III. TENSORS IN QUANTUM MECHANICS

A. Tensor product of vector spaces

Here, we introduce a more rigorous definition of the tensor product. Let E and F be two vector spaces. The *tensor product* of E and F , denoted $E \otimes F$, is defined as a vector space generated by pairs of elements from E and F : if $x \in E$ and $y \in F$, then $x \otimes y$ is, by definition, an element of $E \otimes F$.

The difference with the Cartesian product, denoted $E \times F$, lies in the definition of the vector space operations (internal addition and scalar multiplication). Let x_1, x_2 be two vectors in E and y_1, y_2 be two vectors in F . The pairs (x_1, y_1) and (x_2, y_2) are then elements of the Cartesian product $E \times F$. Their sum is defined component-wise: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, and scalar multiplication distributes over the components: $\lambda(x_1, y_1) = (\lambda x_1, \lambda y_1)$. In contrast, the tensor product is distributive over addition: $x_1 \otimes y_1 + x_1 \otimes y_2 = x_1 \otimes (y_1 + y_2)$, but $x_1 \otimes y_1 + x_2 \otimes y_2$ cannot be simplified in general. As for scalar multiplication, it applies to either of the factors: $\lambda(x_1 \otimes y_1) = (\lambda x_1) \otimes y_1 = x_1 \otimes (\lambda y_1)$.

In a Cartesian product, the dimensions add up, thus $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is of dimension 2. In a tensor product, the dimensions multiply, thus the space $E \otimes E^*$ of endomorphisms on a vector space E is of dimension n^2 if E is n -dimensional (think of square matrices!).

The tensor product space consists of all tensor products between vectors of E and F , and all linear combinations of such products. All the tensors considered in the previous section are elements of a tensor product space of E by itself and by its dual. More precisely, a tensor that is p times contravariant and q times covariant is, by definition, an element of the space:

$$\underbrace{E \otimes \dots \otimes E}_p \otimes \underbrace{E^* \otimes \dots \otimes E^*}_q \quad (24)$$

up to the order of the tensor products.

In quantum mechanics, the state space is constructed through successive tensor products. For example, a three-dimensional quantum system admits a state space $E = E_x \otimes E_y \otimes E_z$, where each of the spaces E_x, E_y, E_z contains the states of a one-dimensional quantum system.

B. Dirac notation

Dirac notation is also a practical notation for immediately visualizing the nature of different tensors and performing operations without having to introduce special symbols.

The space we base ourselves on is the state space. Its elements, the state vectors (or kets), are denoted by $|\cdot\rangle$. The covectors (or bras) are denoted by $\langle\cdot|$. The direction of the bracket in Dirac notation plays the role of the index position in Ricci calculus: distinguishing between covariant and contravariant objects. This time, the need to distinguish does not come from the choice of non-orthonormal bases as in Euclidean spaces, nor from the pseudo-inner product as in Minkowski space. It comes from the fact that the state space is a *complex* vector space. Consider indeed an orthonormal basis $\{|\alpha\rangle\}$. To the ket $|\psi\rangle$, we associate the bra $\langle\psi|$, whose coordinates $\langle\psi|\alpha\rangle$ are the complex conjugates of the coordinates $\langle\alpha|\psi\rangle$ of the ket, due to the sesquilinearity of the Hermitian inner product.

Tensor products are obtained by juxtaposing kets (or bras) belonging to different spaces: thus, $|\psi_1\rangle|\psi_2\rangle$ is a tensor of type (2,0). $|\psi_1\rangle\langle\psi_2|$ is an endomorphism. Juxtaposing two tensors on the vertical side of a ket or a bra defines a contraction: $\langle\psi_1|\psi_2\rangle$ is an inner product, for example. Applying the endomorphism $|\psi_1\rangle\langle\psi_2|$ to the vector $|\psi_3\rangle$ is written $|\psi_1\rangle\langle\psi_2|\psi_3\rangle$ (compare with the definition of the tensor product in Eq. (11)).

Thus, if $\{|x\rangle\}$ is a basis of E_x , $\{|y\rangle\}$ a basis of E_y , and $\{|z\rangle\}$ a basis of E_z , then $\{|x\rangle|y\rangle|z\rangle\}$ is a basis of E .

C. Action of an operator on a separable ket

Let E and F be two vector spaces. We can construct their tensor product $E \otimes F$. A separable ket in $E \otimes F$ is a ket that can be written as a tensor product of a ket in E and a ket in F . Recall that a ket of the form $|\alpha_1\rangle|\alpha_2\rangle + |\alpha_3\rangle|\alpha_4\rangle$ is not *a priori* separable. The operators on $E \otimes F$ are the linear combinations of tensor products of operators on E and F , just as the vectors in $E \otimes F$ are the linear combinations of tensor products of vectors in E and F . If \hat{A} is an operator on E and \hat{B} is an operator on F , then $\hat{A} \otimes \hat{B}$ is an operator on $E \otimes F$, defined by its action on the separable kets of $E \otimes F$:

$$(\hat{A} \otimes \hat{B})(|x\rangle \otimes |y\rangle) = \hat{A}|x\rangle \hat{B}|y\rangle \quad (25)$$

By convention, if an operator \hat{A} acts on E , we extend it to $E \otimes F$ by identifying it with $\hat{A} \otimes \hat{\mathbb{I}}$, where $\hat{\mathbb{I}}$ denotes the identity operator on F . Thus:

$$\hat{A}(|x\rangle |y\rangle) \stackrel{\text{def}}{=} (\hat{A} \otimes \hat{\mathbb{I}})(|x\rangle \otimes |y\rangle) = (\hat{A}|x\rangle) |y\rangle \quad (26)$$

Similarly, if \hat{B} acts on F :

$$\hat{B}(|x\rangle |y\rangle) = (\hat{\mathbb{I}} \otimes \hat{B})(|x\rangle \otimes |y\rangle) = |x\rangle (\hat{B}|y\rangle) \quad (27)$$

TABLE I. Comparative table of the various tensor notations.

	Classical notation	Ricci notation	Dirac notation
Vectors	\bar{x}	x^μ	$ \psi\rangle$
Covectors	\bar{x}	x_μ	$\langle\psi $
Endomorphisms	$\bar{\bar{A}}$	$A^\mu{}_\nu$	\hat{A}
Inner product of vectors	$\bar{a} \cdot \bar{b}$	$a^\mu b_\mu$	$\langle a b\rangle$
Tensor product of vectors	$\bar{a} \otimes \bar{b}$	$a^\mu b_\nu$	$ a\rangle\langle b $
Endomorphism composition	$\bar{\bar{A}}\bar{\bar{B}}$	$A^\mu{}_\kappa B^\kappa{}_\nu$	$\hat{A}\hat{B}^a$

^a If \hat{A} is separable into $|\psi_1\rangle\langle\psi_2|$, and \hat{B} into $|\psi_3\rangle\langle\psi_4|$, then $\hat{A}\hat{B} = |\psi_1\rangle\langle\psi_2| |\psi_3\rangle\langle\psi_4|$.

ACKNOWLEDGMENTS

This work is supported by the Interdisciplinary Thematic Institute QMat, as part of the ITI 2021-2028 program of the University of Strasbourg, CNRS and Inserm, and was supported by IdEx Unistra (ANR-10- IDEX-0002), SFRI STRAT'US project (ANR-20-SFRI- 0012), and EUR QMAT ANR-17-EURE-0024 under the framework of the French Investments for the Future Program.

-
- [1] To be precise, the Fréchet-Riesz representation theorem states that every Hilbert space is isomorphic to its topological dual (the set of *continuous* linear forms). Every Euclidean space is a Hilbert space, and every linear form in finite dimension is continuous. Consequently, every Euclidean space is isomorphic to its dual.
 - [2] Such a space is called a Minkowski plane, by analogy with Minkowski space, which is its 4D equivalent.
 - [3] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Mécanique quantique I* (Hermann, 1977).
 - [4] W. Appel, *Mathématiques pour la physique et les physiciens* (H & K, 2008).
 - [5] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical methods for physicists* (Elsevier Academic Press, 2013).
 - [6] G. Puel and A.-L. Hamon, *Mécanique pour l'ingénieur* (Dunod, 2020).