



HAL
open science

Minimality of vortex solutions to Ginzburg-Landau type systems for gradient fields in the unit ball in dimension

$$N \geq 4$$

Radu Ignat, Mickaël Nahon, Luc Nguyen

► To cite this version:

Radu Ignat, Mickaël Nahon, Luc Nguyen. Minimality of vortex solutions to Ginzburg-Landau type systems for gradient fields in the unit ball in dimension $N \geq 4$. 2024. hal-04722729

HAL Id: hal-04722729

<https://hal.science/hal-04722729v1>

Preprint submitted on 5 Oct 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Minimality of vortex solutions to Ginzburg–Landau type systems for gradient fields in the unit ball in dimension $N \geq 4$

Radu Ignat*, Mickael Nahon† and Luc Nguyen‡

Abstract

We prove that the degree-one vortex solution is the unique minimizer for the Ginzburg–Landau functional for gradient fields (that is, the Aviles–Giga model) in the unit ball B^N in dimension $N \geq 4$ and with respect to its boundary value. A similar result is also proved **in a model for \mathbb{S}^N -valued maps arising in** the theory of micromagnetics. Two methods are presented. The first method is an extension of the analogous technique previously used to treat the unconstrained Ginzburg–Landau functional in dimension $N \geq 7$. The second method uses a symmetrization procedure for gradient fields such that the L^2 -norm is invariant while the L^p -norm, $2 < p < \infty$, and the H^1 -norm are lowered.

Keywords: Minimality, vortex solutions, gradient fields, Ginzburg–Landau, Aviles–Giga, Hardy’s inequality, symmetrization.

MSC 2020: 26D10, 35J20, 35J50, 35Q56.

Contents

1	Introduction	2
1.1	The Aviles–Giga model	3
1.2	The \mathbb{S}^N -valued Ginzburg–Landau model	4
1.3	The extended model	6
1.4	Ideas of the proofs	9

*Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Radu.Ignat@math.univ-toulouse.fr

†Laboratoire Jean Kuntzmann, UMR 5224, Université Grenoble Alpes, 700 avenue centrale, 38401 Saint Martin d’Hères. Email: mickael.nahon@univ-grenoble-alpes.fr

‡Mathematical Institute and St Edmund Hall, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom. Email: luc.nguyen@maths.ox.ac.uk

2	First proof of main results	13
2.1	Proof of Theorem 3	13
2.2	Proof of Theorems 1 and 2	20
3	Symmetrization and second proof of main results in dimension $N \geq 5$	21
3.1	A symmetrization of scalar functions	21
3.2	A symmetrization of gradient fields and proof of Theorem 5	24
3.3	Second proof of Theorems 1, 2 and 3 in dimension $N \geq 5$	29
3.4	Symmetry for solutions to a nonlinear eigenvalue problem	30
A	The negativity of F_ϵ in dimension $N \in \{2, 3\}$	32

1 Introduction

Let B^N be the unit ball in \mathbb{R}^N . Consider the Ginzburg–Landau (GL) functional

$$E_\epsilon^{GL}[U] = \int_{B^N} \left[\frac{1}{2} |\nabla U|^2 + \frac{1}{2\epsilon^2} W(1 - |U|^2) \right] dx,$$

where $\epsilon > 0$, $W(t) = \frac{t^2}{2}$ and U belongs to the set

$$\mathcal{A}^{GL} = \{U \in H^1(B^N, \mathbb{R}^N) : U(x) = x \text{ on } \partial B^N\}.$$

The functional E_ϵ^{GL} has a unique radially symmetric critical point in \mathcal{A}^{GL} of the form

$$U_\epsilon(x) = f_\epsilon(r) \frac{x}{r} \in \mathcal{A}^{GL}, \quad r = |x|, \tag{1.1}$$

where the profile f_ϵ is the unique solution to the ODE (see e.g. [31, 36])

$$\begin{cases} -f_\epsilon''(r) - \frac{N-1}{r} f_\epsilon'(r) + \frac{N-1}{r^2} f_\epsilon(r) = \frac{1}{\epsilon^2} f_\epsilon(r) W'(1 - f_\epsilon(r)^2), \\ f_\epsilon(0) = 0, \quad f_\epsilon(1) = 1. \end{cases} \tag{1.2}$$

Moreover $f_\epsilon > 0$ and $f_\epsilon' > 0$ in $(0, 1)$.

The map U_ϵ in (1.1), called the (\mathbb{R}^N -valued) *Ginzburg–Landau vortex solution* of topological degree one, can be considered as a regularization of the singular harmonic map $n : B^N \rightarrow \mathbb{S}^{N-1}$ given by $n(x) = \frac{x}{|x|}$ for every $x \in B^N$, which is the unique minimizing \mathbb{S}^{N-1} -valued harmonic map for $N \geq 3$ with respect to the boundary condition $n(x) = x$ on ∂B^N (see Brezis, Coron and Lieb [11] and Lin [48]). The question about the minimality of U_ϵ for any $\epsilon > 0$ was raised in dimension $N = 2$ in Bethuel, Brezis and Hélein [8, Problem 10, page 139], and in higher dimensions in Brezis [10, Section 2]. It is

not hard to see that, when ϵ is sufficiently large, E_ϵ^{GL} is strictly convex and so U_ϵ is the unique bounded critical point of E_ϵ^{GL} in \mathcal{A}^{GL} for every $N \geq 2$ (see e.g. [8] or [40, Remark 3.3]). In dimension $N = 2$, Pacard and Rivière showed in [57] that, for small $\epsilon > 0$, U_ϵ is the unique critical point of E_ϵ^{GL} in \mathcal{A}^{GL} ; however, whether u_ϵ is the unique minimizer of E_ϵ^{GL} for all $\epsilon > 0$ remains an open question. In dimensions $N \geq 7$, this question was answered positively in recent works of Ignat, Nguyen, Slastikov and Zarnescu [39, 40]: U_ϵ is the unique minimizer of E_ϵ^{GL} in \mathcal{A}^{GL} for every $\epsilon > 0$. It is not known whether U_ϵ minimizes E_ϵ^{GL} in \mathcal{A}^{GL} in dimensions $3 \leq N \leq 6$ when ϵ is small. However, it is known that for every $\epsilon > 0$, U_ϵ is a local minimizer of E_ϵ^{GL} in \mathcal{A}^{GL} – for dimension $N = 2$, see Mironescu [52] and also Lieb and Loss [47]; for dimension $3 \leq N \leq 6$, see Ignat and Nguyen [34].

Minimizers of the Ginzburg–Landau energy E_ϵ^{GL} are sometimes referred to as approximate \mathbb{S}^{N-1} -valued harmonic maps (e.g. [13]) in the sense that they are smooth and approximate a minimizing harmonic map with or without singularity. In this context, the uniqueness of the vortex solution is of special relevance in dimension $N = 3, 4$. Indeed, in these dimensions, by [61, Theorem 2.7], a minimizing harmonic map from $B^N \rightarrow \mathbb{S}^{N-1}$ is regular away from a set of isolated points, and by [11, Theorem 1.2] in dimension $N = 3$ and [54, Theorems 1.2 and 2.1] in dimension $N = 4$, near each such singular point, the harmonic map has degree ± 1 and its tangent map is the vortex map $\frac{x}{|x|}$ up to a rigid rotation. The uniqueness of the vortex solution, if it is true, would suggest that each approximate harmonic map would look like the vortex solution near each singular point of the true harmonic map.

Strong evidence to support the uniqueness of the vortex solution in any dimension $N \geq 2$ is available in the blown-up limit: When the domain is the whole space \mathbb{R}^N , the minimality (in the sense of De Giorgi) of the vortex solution was proved in dimension $N = 2$ by Mironescu [53], dimension $N = 3$ by Millot and Pisante [51] and dimension $N \geq 4$ by Pisante [58]. See also [16, 28, 29, 56] for studies on stability issues.

The main aim of this paper is to show that in dimensions $4 \leq N \leq 6$ and for every $\epsilon > 0$, U_ϵ is the unique minimizer of E_ϵ^{GL} relative to the set of **gradient field configurations** in \mathcal{A}^{GL} (this is often referred to as the Aviles–Giga model).

1.1 The Aviles–Giga model

Consider a general non-negative convex C^2 potential $W : (-\infty, 1] \rightarrow [0, \infty)$ such that $W(0) = 0$ and for every $\epsilon > 0$, the Ginzburg–Landau energy $E_\epsilon^{GL}(U)$ restricted to gradient fields

$$U = \nabla u \in H^1(B^N, \mathbb{R}^N) \quad \text{such that} \quad U|_{\partial B^N} = Id.$$

Within a suitable rescaling (i.e., $\epsilon E_\epsilon^{GL}(\nabla u)$), this is the so-called Aviles–Giga model (introduced with the standard potential $W(t) = t^2/2$).

This was introduced by Aviles and Giga in [3] to account for codimension 1 and 2 defects in 3D Smectic-A liquid crystals. In dimension $N = 2$ (with $W(t) = t^2$) it is conjectured that, on the set of gradient fields, ϵE_ϵ^{GL} Γ -converges to an energy that penalizes line defects, and furthermore, when restricted to limit stream functions $u \in W^{1,1}$ such that $|\nabla u| = 1$, $\nabla u \in BV$, the limit energy is given by $E_0(\nabla u) = \frac{1}{6} \int_{J_{\nabla u}} |[\nabla u]|^3 d\mathcal{H}^1$, where $[\nabla u]$ is the jump of ∇u across the jump set $J_{\nabla u}$. Many results point to this conjecture being true in dimension $N = 2$ (see for instance [1, 14, 18, 19, 43, 59]). In particular, every optimal transition layer across a line-defect is one-dimensional (so, no microstructure exists) and gives the cubic cost $|[\nabla u]|^3$ in E_0 , see [33]. However, in dimension $N \geq 3$, the situation is very different as was pointed out in [17]: one-dimensional transition layers across a one-codimensional defect are no longer optimal as microstructures with strictly lower energy exist. One of the motivation for studying the Aviles–Giga model in our setting (in the ball with the boundary condition $\nabla u(x) = x$) is to understand in the easiest case of a point defect if there exists symmetry-breaking counterexample to the natural vortex solution.

Note that the (\mathbb{R}^N -valued) Ginzburg–Landau vortex solution U_ϵ introduced in (1.1) is a gradient field $U_\epsilon = \nabla u_\epsilon$ for some radial function $u_\epsilon = u_\epsilon(r)$ determined (up to a constant) by $u'_\epsilon = f_\epsilon$ in $(0, 1)$ where f_ϵ is the unique solution in (1.2).

We prove the following result:

Theorem 1. *Assume that $4 \leq N \leq 6$ and $W : (-\infty, 1] \rightarrow [0, \infty)$ is a C^2 non-negative convex function such that $W(0) = 0$. For every $\epsilon > 0$, the radially symmetric vortex solution U_ϵ in (1.1) is the unique minimizer of E_ϵ^{GL} over the set of gradient fields $\{U = \nabla u \in \mathcal{A}^{GL}\}$.*

Note that the above result holds in dimension $N \geq 7$ as a consequence of [39, 40]. We expect the result holds also in dimension $N \in \{2, 3\}$ and more generally for non-gradient fields as pointed out previously, but our methods do not prove it; see the discussion in Subsection 1.4. We mention here the work Lorent [49, 50] and Lamy and Marconi [44] on stability of the vortex solution in dimension $N = 2$ and in the limit $\epsilon \rightarrow 0$ (for the Aviles–Giga model as well as other micromagnetic models).

1.2 The \mathbb{S}^N -valued Ginzburg–Landau model

We consider the following model:

$$E_\eta^{MM}[M] = \int_{B^N} \left[\frac{1}{2} |\nabla M|^2 + \frac{1}{2\eta^2} \tilde{W}(M_{N+1}^2) \right] dx$$

where $\eta > 0$ and $M = (\nabla m, M_{N+1})$ is a unit-length vector field that is a gradient field in the first N components belonging to

$$\mathcal{A}^{MM} = \{M = (\nabla m, M_{N+1}) \in H^1(B^N, \mathbb{S}^N) : M(x) = (x, 0) \text{ on } \partial B^N\}.$$

The non-negative potential $\tilde{W} : [0, \infty) \rightarrow [0, \infty)$ is a C^2 convex function such that $\tilde{W}(0) = 0$.

This model comes from micromagnetics where the order parameter M stands for the magnetization in ferromagnetic materials (see [27])¹, and also the Oseen-Frank theory for nematic liquid crystals (see [2]). Considering radially symmetric critical points of E_η^{MM} in \mathcal{A}^{MM} , one is led to

$$M_\eta(x) = (\tilde{f}_\eta(r) \frac{x}{r}, g_\eta(r)) \in \mathcal{A}^{MM} \quad (1.3)$$

where the radial profiles \tilde{f}_η and g_η satisfy

$$\tilde{f}_\eta^2 + g_\eta^2 = 1 \quad \text{in } (0, 1), \quad (1.4)$$

and the system of ODEs:

$$-\tilde{f}_\eta'' - \frac{N-1}{r} \tilde{f}_\eta' + \frac{N-1}{r^2} \tilde{f}_\eta = \lambda(r) \tilde{f}_\eta \quad \text{in } (0, 1), \quad (1.5)$$

$$-g_\eta'' - \frac{N-1}{r} g_\eta' = -\frac{1}{\eta^2} \tilde{W}'(g_\eta^2) g_\eta + \lambda(r) g_\eta \quad \text{in } (0, 1), \quad (1.6)$$

$$\tilde{f}_\eta(1) = 1 \text{ and } g_\eta(1) = 0, \quad (1.7)$$

where

$$\lambda(r) = (\tilde{f}_\eta')^2 + \frac{N-1}{r^2} \tilde{f}_\eta^2 + (g_\eta')^2 + \frac{1}{\eta^2} \tilde{W}'(g_\eta^2) g_\eta^2 \quad (1.8)$$

is the Lagrange multiplier due to the unit length constraint in \mathcal{A}^{MM} . Note that indeed the vortex solution M_η in (1.3) is of the form $M_\eta = (\nabla m_\eta, M_{\eta, N+1}) \in \mathcal{A}^{MM}$ for some radial function $m_\eta = m_\eta(r)$ determined (up to a constant) by $m_\eta' = \tilde{f}_\eta$ in $(0, 1)$.

As proved in [34], the solutions to (1.3)–(1.7) satisfy the dichotomy: either $\tilde{f}_\eta(0) = 0$ or $\tilde{f}_\eta(0) = 1$. Furthermore, in the latter case, it holds that $N \geq 3$ and $(\tilde{f}_\eta = 1, g_\eta = 0)$ in $(0, 1)$, which corresponds to the *equator map*

$$\bar{M}(x) := \left(\frac{x}{r}, 0\right).$$

¹In dimension $N = 2$, E_η^{MM} is the reduced energy functional in a certain thin-film ferromagnetic regime (see e.g. [20, Section 4.5] or [32, Section 7]) where, after a rotation by $\frac{\pi}{2}$ in the first two components of M , the condition $\nabla \times (M_1, M_2) = 0$ is imposed in the space of admissible configurations in \mathcal{A}^{MM} .

In dimension $N \geq 7$, \bar{M} is the unique minimizing harmonic map from B^N into \mathbb{S}^N in $H^1(B^N, \mathbb{S}^N)$ with boundary condition $(Id, 0)$ on ∂B^N (Jäger and Kaul [42]; see also Sandier and Shafrir [60] and [40, Example 1.6]); so \bar{M} is the unique minimizer of E_η^{MM} in \mathcal{A}^{MM} for every $\eta > 0$. Therefore, in the following, we focus on dimensions $2 \leq N \leq 6$ and on *escaping* \mathbb{S}^N -valued radially symmetric vortex solutions

$$M_\eta^\pm(x) = (\tilde{f}_\eta(r) \frac{x}{r}, \pm g_\eta(r)) \quad \text{with } g_\eta > 0 \text{ in } (0, 1).$$

It was proved in Hang and Lin [30] in dimension $N = 2$ and [34] in dimension $3 \leq N \leq 6$ that, for any $\eta > 0$, (1.3)–(1.7) has a unique escaping solution (\tilde{f}_η, g_η) with $g_\eta > 0$ and M_η^\pm are locally minimizers for E_η^{MM} . Moreover, $\tilde{f}_\eta(0) = 0$, $\tilde{f}_\eta > 0$, $\tilde{f}_\eta' > 0$ and $g_\eta' < 0$ in $(0, 1)$. (See also [46] for a related work in the context of micromagnetic skyrmions in \mathbb{R}^2 .)

We prove the following result:

Theorem 2. *Assume $4 \leq N \leq 6$ and $\tilde{W} : [0, \infty) \rightarrow [0, \infty)$ is a C^2 non-negative convex function such that $\tilde{W}(0) = 0$. For every $\eta > 0$, E_η^{MM} has exactly two minimizers over the set $\{(\nabla m, M_{N+1}) \in \mathcal{A}^{MM}\}$ and they are given by the escaping vortex solutions $M_\eta^\pm(x) = (\tilde{f}_\eta(r) \frac{x}{r}, \pm g_\eta(r))$ with $g_\eta > 0$ in $(0, 1)$. In particular, minimizers of E_η^{MM} in \mathcal{A}^{MM} are radially symmetric for every $\eta > 0$.*

As in the case of the Aviles–Giga model, we expect the above result holds also in dimension $N \in \{2, 3\}$.

1.3 The extended model

More generally, we consider a family of extended energy functionals $E_{\epsilon, \eta}$ depending on two positive parameters ϵ, η of which E_ϵ^{GL} and E_η^{MM} are limiting cases when $\eta \rightarrow 0$ and $\epsilon \rightarrow 0$, respectively:

$$E_{\epsilon, \eta}[U] = \int_{B^N} \left[\frac{1}{2} |\nabla U|^2 + \frac{1}{2\epsilon^2} W(1 - |U|^2) + \frac{1}{2\eta^2} \tilde{W}(U_{N+1}^2) \right] dx, \quad \epsilon, \eta > 0, \quad (1.9)$$

where $U = (\nabla u, U_{N+1}) : B^N \rightarrow \mathbb{R}^{N+1}$ is a gradient field in the first N components and belongs to

$$\mathcal{A} = \{U = (\nabla u, U_{N+1}) \in H^1(B^N, \mathbb{R}^{N+1}) : U(x) = (x, 0) \text{ on } \partial B^N\}.$$

Here, $W : (-\infty, 1] \rightarrow [0, \infty)$ and $\tilde{W} : [0, \infty) \rightarrow [0, \infty)$ are non-negative C^2 convex potentials such that $W(0) = \tilde{W}(0) = 0$. We point out that these imply that $W'(0) = 0$, $tW'(t) \geq 0$ in $(-\infty, 1] \setminus \{0\}$, and $\tilde{W}'(t) \geq 0$ in $[0, \infty)$. However, we allow the possibility that W or \tilde{W} can be zero in a neighborhood of the origin.

Radially symmetric critical points of $E_{\epsilon,\eta}$ in \mathcal{A} take the form

$$U_{\epsilon,\eta} = (f_{\epsilon,\eta}(r)\frac{x}{r}, g_{\epsilon,\eta}(r)) \in \mathcal{A}, \quad (1.10)$$

where $(f_{\epsilon,\eta}, g_{\epsilon,\eta})$ satisfies the system of ODEs

$$-f_{\epsilon,\eta}'' - \frac{N-1}{r}f_{\epsilon,\eta}' + \frac{N-1}{r^2}f_{\epsilon,\eta} = \frac{1}{\epsilon^2}W'(1-f_{\epsilon,\eta}^2-g_{\epsilon,\eta}^2)f_{\epsilon,\eta}, \quad (1.11)$$

$$-g_{\epsilon,\eta}'' - \frac{N-1}{r}g_{\epsilon,\eta}' = \frac{1}{\epsilon^2}W'(1-f_{\epsilon,\eta}^2-g_{\epsilon,\eta}^2)g_{\epsilon,\eta} - \frac{1}{\eta^2}\tilde{W}'(g_{\epsilon,\eta}^2)g_{\epsilon,\eta}, \quad (1.12)$$

$$f_{\epsilon,\eta}(1) = 1 \text{ and } g_{\epsilon,\eta}(1) = 0. \quad (1.13)$$

Note that the above implies $f_{\epsilon,\eta}(0) = 0$ and $g_{\epsilon,\eta}'(0) = 0$ (see [34, Lemma A.5]). Also, note that the first N components of $U_{\epsilon,\eta}(r)$ is a gradient field $\nabla\varphi_{\epsilon,\eta}$ for some radial function $\varphi_{\epsilon,\eta}(r)$ determined (up to a constant) by $\varphi_{\epsilon,\eta}' = f_{\epsilon,\eta}$ in $(0, 1)$.

In dimensions $N \geq 7$, it follows from [39, 40] that the *non-escaping* vortex solution

$$\bar{U}_\epsilon(x) = (f_\epsilon(r)\frac{x}{r}, 0)$$

is the unique global minimizer of $E_{\epsilon,\eta}$ in \mathcal{A} for every $\epsilon > 0$ and $\eta > 0$. Therefore, in the following, we focus on dimensions $2 \leq N \leq 6$; we will analyse *escaping* radially symmetric vortex solutions

$$U_{\epsilon,\eta}^\pm = (f_{\epsilon,\eta}(r)\frac{x}{r}, \pm g_{\epsilon,\eta}(r)), \quad g_{\epsilon,\eta} > 0 \text{ in } (0, 1).$$

It is shown by [34] that such an escaping radially symmetric critical point $U_{\epsilon,\eta}$ with $g_{\epsilon,\eta} > 0$ exists if and only if $2 \leq N \leq 6$, $W'(1) > 0$, $0 < \epsilon < \epsilon_0$ and $\eta > \eta_0(\epsilon)$ for some $\epsilon_0 \in (0, \infty)$ and a continuous non-decreasing function $\eta_0 : [0, \epsilon_0) \rightarrow [0, \infty)$ with $\eta_0(0) = 0$. In this case, it is the unique escaping solution of (1.10)–(1.13) with $g_{\epsilon,\eta} > 0$ in $(0, 1)$; moreover, we have $f_{\epsilon,\eta}(0) = 0$, $f_{\epsilon,\eta}^2 + g_{\epsilon,\eta}^2 < 1$, $f_{\epsilon,\eta} > 0$, $f_{\epsilon,\eta}' > 0$, $g_{\epsilon,\eta}' < 0$ in $(0, 1)$. See Section 1.4 and Figure 1 for more information.

We prove the following theorem:

Theorem 3. *Suppose $4 \leq N \leq 6$ and $W : (-\infty, 1] \rightarrow [0, \infty)$ and $\tilde{W} : [0, \infty) \rightarrow [0, \infty)$ are C^2 non-negative convex functions satisfying $W(0) = \tilde{W}(0) = 0$. For every $\epsilon > 0, \eta > 0$, we have the following dichotomy:*

- *Either the escaping radially symmetric vortex solutions $U_{\epsilon,\eta}^\pm$ exist and they are the only two minimizers of $E_{\epsilon,\eta}$ in \mathcal{A} ,*

- Or the escaping radially symmetric vortex solutions $U_{\epsilon,\eta}^\pm$ do not exist and the non-escaping vortex solution \bar{U}_ϵ is the unique minimizer of $E_{\epsilon,\eta}$ in \mathcal{A} .

In particular, minimizers of $E_{\epsilon,\eta}$ in \mathcal{A} are always radially symmetric for every $\epsilon, \eta > 0$.

To complete the picture, we recall facts from [34] on the escaping vs. non-escaping phenomena. The escaping phenomenon is related to the loss of stability of the non-escaping vortex solution \bar{U}_ϵ . More precisely, consider the stability operator $\frac{\delta^2 E_{\epsilon,\eta}}{\delta U_{N+1}^2}$ at \bar{U}_ϵ along the $N + 1$ direction:

$$\bar{T}_{\epsilon,\eta} = -\Delta - \frac{1}{\epsilon^2} W'(1 - f_\epsilon^2) + \frac{1}{\eta^2} \tilde{W}'(0).$$

The first eigenvalue of $\bar{T}_{\epsilon,\eta}$ on $H_0^1(B^N, \mathbb{R})$ takes the form $\ell(\epsilon) + \frac{1}{\eta^2} \tilde{W}'(0)$ where $\ell(\epsilon)$ is the first eigenvalue of

$$L_\epsilon := -\Delta - \frac{1}{\epsilon^2} W'(1 - f_\epsilon^2). \quad (1.14)$$

Then the

escaping vortex solutions $U_{\epsilon,\eta}^\pm$ with $g_{\epsilon,\eta} > 0$ exists if and only if $\ell(\epsilon) + \frac{1}{\eta^2} \tilde{W}'(0) < 0$.

When $N \geq 7$ or $W'(1) = 0$, it holds always that $\ell(\epsilon) > 0$, hence escaping vortex solutions do not exist. When $2 \leq N \leq 6$ and $W'(1) > 0$,

there exists $\epsilon_0 > 0$ such that $\ell(\epsilon) > 0$ for $\epsilon > \epsilon_0$ and $\ell(\epsilon) < 0$ for $0 < \epsilon < \epsilon_0$.

Thus, in this case, the function $\eta_0(\epsilon)$ mentioned above (so that escaping vortex solutions exist if and only if $0 < \epsilon < \epsilon_0$ and $\eta > \eta_0(\epsilon)$) is given by

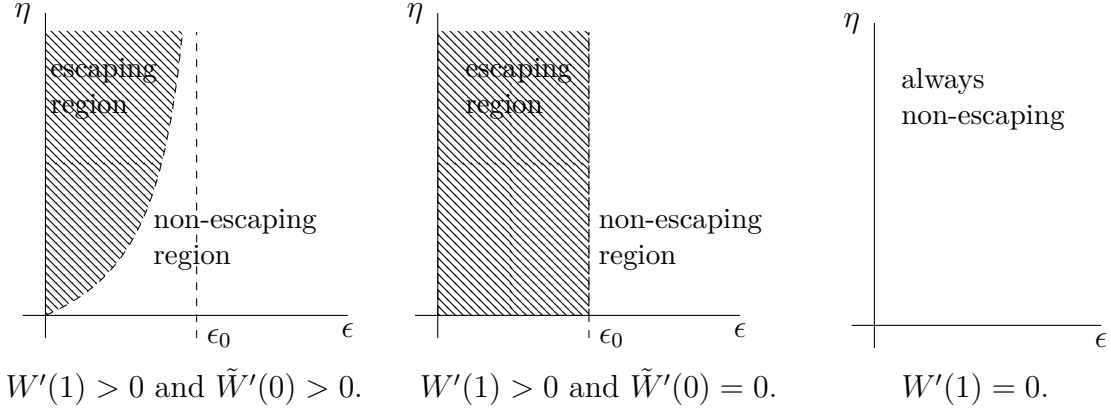
$$\eta_0(\epsilon) = \sqrt{\frac{\tilde{W}'(0)}{|\ell(\epsilon)|}} \text{ for } 0 < \epsilon < \epsilon_0.$$

In Figure 1, we describe the dichotomy of escaping and non-escaping phenomena for minimizers² of $E_{\epsilon,\eta}$ in radial symmetry in dimension $2 \leq N \leq 6$. Theorem 3 asserts that, in dimension $4 \leq N \leq 6$, this picture remains valid in the larger set \mathcal{A} of gradient field configurations in the first N components.

For the case $\eta = \infty$ (that is the \mathbb{R}^{N+1} -valued Ginzburg–Landau model), we refer the reader to the recent article Ignat and Rus [41]. For a similar bifurcation from non-escaping to escaping phenomenon, see Bethuel, Brezis, Coleman and Hélein [7].

²Recall from [34] that solutions of (1.11)–(1.13) satisfying $g_{\epsilon,\eta} > 0$, when they exist, are minimizing for $E_{\epsilon,\eta}$ relatively to the set of radially symmetric configurations.

Figure 1: Escaping vs. Non-escaping phenomenon in dimension $2 \leq N \leq 6$.



1.4 Ideas of the proofs

Theorems 1 and 2 will be obtained from Theorem 3 by taking the limits $\eta \rightarrow 0$ or $\epsilon \rightarrow 0$, respectively. For simplicity, instead of describing the proof of Theorem 3 (which is the main result), we explain instead the strategy of the proof in the case $\eta = 0$, i.e. Theorem 1 for the Aviles–Giga model. We will present two methods of proof. Roughly speaking, the first method follows the strategy in [39, 40] adapted to the situation of gradient field configurations, and the second method uses a symmetrization procedure.

Method 1

It was observed in [39, 40] that the convexity of the potential W implies the inequality

$$E_\epsilon^{GL}[U_\epsilon + V] - E_\epsilon^{GL}[U_\epsilon] \geq \frac{1}{2} \int_{B^N} L_\epsilon V \cdot V \, dx =: \frac{1}{2} F_\epsilon[V] \text{ for } V \in H_0^1(B^N, \mathbb{R}^N), \quad (1.15)$$

where L_ϵ is the operator defined in (1.14). Recall that in dimension $N \geq 7$, the first eigenvalue $\ell(\epsilon)$ of L_ϵ is positive, $F_\epsilon[V] > 0$ for $V \in H_0^1(B^N, \mathbb{R}^N) \setminus \{0\}$, and hence, U_ϵ is the unique minimizer of E_ϵ^{GL} in \mathcal{A}^{GL} . In dimension $2 \leq N \leq 6$, one has $\ell(\epsilon) < 0$ for $\epsilon < \epsilon_0$, and so it is not clear from the above argument if U_ϵ is a minimizer of E_ϵ^{GL} in \mathcal{A}^{GL} . However, in the current case of gradient field configurations (i.e. $V = \nabla v$), we are able to conclude in dimension $N \geq 4$.

To appreciate the idea, consider the limit $\epsilon \rightarrow 0$ where $L_\epsilon \rightarrow -\Delta - \frac{N-1}{r^2} =: L_*$ as bounded linear operators from $H_0^1(B^N, \mathbb{R}^N)$ into $H^{-1}(B^N, \mathbb{R}^N)$. Although L_* is not positive definite when $N \leq 6$, we have the following inequality³ for gradient fields in

³Here $H_0^2(B^N, \mathbb{R})$ is the closure of $\mathcal{C}_c^\infty(B^N, \mathbb{R})$ in $H^2(B^N, \mathbb{R})$. In particular, if $v \in H_0^2(B^N, \mathbb{R})$, then v and ∇v have zero trace on the boundary.

dimension $N \geq 4$:

$$\int_{B^N} L_*(\nabla v) \cdot (\nabla v) dx = \int_{B^N} \left((\Delta v)^2 - \frac{N-1}{r^2} |\nabla v|^2 \right) dx \geq 0 \text{ for } v \in H_0^2(B^N, \mathbb{R}),$$

i.e. L_* is positive definite on the subspace of gradient fields in $H_0^1(B^N, \mathbb{R}^N)$. This is a consequence of the sharp Hardy inequality for gradient fields (see e.g. [4, 12, 26, 63]):

$$\int_{\mathbb{R}^N} (\Delta v)^2 dx \geq c_N \int_{\mathbb{R}^N} \frac{|\nabla v|^2}{r^2} dx, \text{ for } v \in H_0^2(B^N, \mathbb{R}) \text{ where } c_N := \begin{cases} \frac{N^2}{4} & \text{if } N \geq 5 \\ 3 & \text{if } N = 4 \\ \frac{25}{36} & \text{if } N = 3, \\ 0 & \text{if } N = 2. \end{cases} \quad (1.16)$$

For the general case $\epsilon > 0$, we combine the above idea with the machineries in [39, 40] and [34], based on the Hardy decomposition method.

Unfortunately, the above strategy does not work in dimension $N = \{2, 3\}$ (for the proof, see Appendix A):

Proposition 4. *In dimension $N \in \{2, 3\}$, there exists a function $v \in \mathcal{C}_c^2(B^N \setminus \{0\}) \subset H_0^2(B^N)$ such that $F_\epsilon(\nabla v) < 0$ when ϵ is sufficiently small.*

Method 2

As mentioned above, the second method of proof uses a symmetrization procedure. For that, we use the spherical coordinates: for every $x \in B^N$, we write $x = r\theta$ with $r = |x|$ and $\theta \in \mathbb{S}^{N-1}$. For $v \in H^1(B^N, \mathbb{R})$, we associate the radial function $\check{v} = \check{v}(r)$ given by the formula

$$\check{v}(r) = - \int_r^1 \left(\int_{\mathbb{S}^{N-1}} |\nabla v(s\theta)|^2 d\sigma(\theta) \right)^{1/2} ds \leq 0, \quad r \in (0, 1). \quad (1.17)$$

One can think of this as a kind of rearrangement in the spherical harmonic decomposition of v (see Section 3.2 for more detailed discussion).

We prove the following.

Theorem 5. *Let $N \geq 2$, $v \in H^1(B^N, \mathbb{R})$ and \check{v} be associated to v by (1.17). The following conclusions hold.*

- (i) *The map $v \mapsto \check{v}$ is a Lipschitz continuous map from $H^1(B^N, \mathbb{R})$ into $H_0^1(B^N, \mathbb{R})$. Moreover,*

$$\int_{\mathbb{S}^{N-1}} |\nabla \check{v}(r\theta)|^2 d\sigma(\theta) = \int_{\mathbb{S}^{N-1}} |\nabla v(r\theta)|^2 d\sigma(\theta) \text{ for a.e. } r \in (0, 1). \quad (1.18)$$

(ii) Let $G : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be continuous. If G is convex in the second variable, then

$$\int_{B^N} G(r, |\nabla \check{v}|^2) dx \leq \int_{B^N} G(r, |\nabla v|^2) dx. \quad (1.19)$$

In particular, for any $2 < p < \infty$,

$$\int_{B^N} |\nabla \check{v}|^p dx \leq \int_{B^N} |\nabla v|^p dx. \quad (1.20)$$

(iii) If $v \in H_0^1(B^N)$, i.e. if $v = 0$ on ∂B^N and $1 \leq p \leq 2$, then

$$\int_{\mathbb{S}^{N-1}} |\check{v}(r\theta)|^p d\sigma(\theta) \geq \int_{\mathbb{S}^{N-1}} |v(r\theta)|^p d\sigma(\theta) \text{ for a.e. } r \in (0, 1). \quad (1.21)$$

(iv) Assume in addition that $v \in H^2(B^N, \mathbb{R})$ with the boundary condition $\nabla v(x) = cx$ on ∂B^N (in the $H^{1/2}(\partial B)$ sense) for some constant $c \in \mathbb{R}$. Then $\check{v} \in H^2(B^N, \mathbb{R})$ and $\nabla \check{v}(x) = |c|x$ on ∂B^N . If $N \geq 5$, then

$$\int_{B^N} (\Delta \check{v})^2 dx \leq \int_{B^N} (\Delta v)^2 dx. \quad (1.22)$$

If $N \in \{2, 3, 4\}$, (1.22) continues to hold provided that $\int_{\mathbb{S}^{N-1}} v(r\theta)\theta d\sigma(\theta) = 0$ for a.e. $r \in (0, 1)$. In either case, equality is attained if and only if v is radially symmetric and $|v'| = \check{v}'$ in $(0, 1)$.

To apply Theorem 5 to prove Theorem 1, we only need to note that for $\nabla u \in \mathcal{A}^{GL}$, by integrating by parts using $\nabla u(x) = x$ on ∂B^N ,

$$\begin{aligned} \int_{B^N} |\nabla^2 u|^2 dx &= \int_{B^N} (\Delta u)^2 dx - \underbrace{\int_{\mathbb{S}^{N-1}} \sum_{i,j=1}^N \partial_i \partial_j u(\theta) (\delta_{ij} - \theta_i \theta_j) d\sigma(\theta)}_{\nabla^2 u : (I_N - \theta \otimes \theta)} \\ &= \int_{B^N} (\Delta u)^2 dx - \int_{\mathbb{S}^{N-1}} [(N-1)\partial_r u(\theta) + \Delta_{\mathbb{S}^{N-1}} u(\theta)] d\sigma(\theta) \\ &= \int_{B^N} (\Delta u)^2 dx - (N-1)|\mathbb{S}^{N-1}|. \end{aligned} \quad (1.23)$$

(Here we have used the fact that $I_N - \theta \otimes \theta$ is the projection onto the tangent hyperplane $T_\theta \mathbb{S}^{N-1}$.) Therefore, in dimension $N \geq 5$, Theorem 5 immediately implies that minimizers of E_ϵ^{GL} in $\{U = \nabla u \in \mathcal{A}^{GL}\}$ are radially symmetric. Thanks to the characterization

of radially symmetric critical points in [34], Theorem 1 follows. Theorem 2 also follows in a similar manner. For Theorem 3, we need an extra symmetrization for the U_{N+1} component; see Section 3.1.

In Theorem 5(iv), the requirement $\int_{\mathbb{S}^{N-1}} v(r\theta)\theta d\sigma(\theta) = 0$ in dimension $N \in \{2, 3, 4\}$ cannot simply be dropped due to existence of counter-examples. (For examples of symmetry breaking phenomena in the context of Hardy's inequality for gradient fields in dimension $N \in \{3, 4\}$, see e.g. [12].)

Our rearrangement is related to a vectorial rearrangement in Lieb and Loss [47]. For $V \in H^1(B^N, \mathbb{R}^N)$, one associates the radially symmetric vector field \check{V} defined by

$$\check{V}(x) = \left(\int_{\mathbb{S}^{N-1}} |V(r\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} \frac{x}{r}. \quad (1.24)$$

It was shown in [47] that, provided $V(x) = x$ on ∂B^N and $\int_{\mathbb{S}^{N-1}} V(r\theta) d\sigma(\theta) = 0$ for a.e. $r \in (0, 1)$,

$$\int_{B^N} |\nabla \check{V}|^2 dx \leq \int_{B^N} |\nabla V|^2 dx.$$

It is readily seen that if $V = \nabla v$, then $\check{V} = \nabla \check{v}$. Thus, when $N \in \{2, 3, 4\}$, the conclusion (1.22) in Theorem 5 can be deduced from the above result in [47].

Fewer rearrangement methods are known to prove symmetry of solutions of higher order elliptic equations than for second order ones. This can be partly explained by the absence of a maximum principle for higher order elliptic equations or systems, which makes Schwarz symmetrization methods inapplicable in general. There are some exceptions, see for instance the two papers of Nadirashvili [55] and Talenti [62] where it is shown by rearrangement arguments that minimizers of $\frac{|\{u \neq 0\}|^2 \int_{\mathbb{R}^2} (\Delta u)^2}{\int_{\mathbb{R}^2} u^2}$ are radially symmetric.⁴

More recently, a rearrangement principle developed in Lenzmann and Sok [45] deals with the radial symmetry of optimizers of Gagliardo–Nirenberg type inequalities of arbitrarily high orders, as well as ground states of higher order non-linear Schrödinger equations of the form

$$Lv + \omega v = v|v|^{p-2} \text{ in } \mathbb{R}^N$$

where L is a certain pseudodifferential operator, $\omega > 0$ and $p \in (2, p^*)$ for some critical exponent $p^* > 2$ depending only on the dimension N and the operator L . The rearrangement principle here is based on Schwarz rearrangement of the Fourier transform: any function $v : \mathbb{R}^N \rightarrow \mathbb{C}$ is symmetrized as $v^\sharp = \mathcal{F}^{-1} [|\mathcal{F}[v]|^*]$, where \mathcal{F} is the Fourier transform and w^* designates the radially decreasing Schwarz rearrangement of w .

⁴For other results on symmetry of solutions of higher order elliptic equations or systems which do not use rearrangement inequalities, see e.g. [5, 15, 22, 25] and the references therein.

We make a comparison between the rearrangement \check{v} in Theorem 5 and the rearrangement v^\sharp of [45] in the following table.

	\check{v} on B^N	v^\sharp on \mathbb{R}^N
L^2 -norm	$\ \check{v}\ _{L^2} \geq \ v\ _{L^2}$	$\ v^\sharp\ _{L^2} = \ v\ _{L^2}$
L^p -norm, $1 \leq p < 2$	$\ \check{v}\ _{L^p} \geq \ v\ _{L^p}$?
L^p -norm, even integer $p > 2$?	$\ v^\sharp\ _{L^p} \geq \ v\ _{L^p}$
\dot{H}^1 -norm	$\ \nabla\check{v}\ _{L^2} = \ \nabla v\ _{L^2}$	$\ \nabla v^\sharp\ _{L^2} \leq \ \nabla v\ _{L^2}$
$\dot{W}^{1,p}$ -norm, $p > 2$	$\ \nabla\check{v}\ _{L^p} \leq \ \nabla v\ _{L^p}$?
\dot{H}^2 -norm	$\ \Delta\check{v}\ _{L^2} \leq \ \Delta v\ _{L^2}$	$\ \Delta v^\sharp\ _{L^2} \leq \ \Delta v\ _{L^2}$
\dot{H}^s -norm, $s > 0$?	$\ v^\sharp\ _{\dot{H}^s} \leq \ v\ _{\dot{H}^s}$

As an application Theorem 5, we consider the radial symmetry of ‘ground state’ solutions to the nonlinear eigenvalue problem

$$\begin{cases} \Delta^2 v = \lambda v + |v|^{p-2}v \text{ on } B^N, \\ v = \partial_r v = 0 \text{ on } \partial B^N, \end{cases} \quad (1.25)$$

where $1 \leq p < 2$. See Section 3.4.

Acknowledgments. R.I. is partially supported by the ANR projects ANR-21-CE40-0004 and ANR-22-CE40-0006-01. L.N. was partially supported by the ANR LabEx CIMI (grant ANR-11-LABX-0040) within the French State Programme “Investissements d’Avenir” during his visit to Université Toulouse III – Paul Sabatier. Part of this work was done while all authors visited the Hausdorff Research Institute for Mathematics in Bonn, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2047/1 – 390685813, as part of the Trimester Program on “Mathematics of Complex Materials”.

2 First proof of main results

2.1 Proof of Theorem 3

In this section we give the first proof of Theorem 3 based on the strategy in [39, 40] and exploiting the additional structure of a gradient field for the first N -components of the current admissible configurations.

Recall that $\ell(\epsilon)$ is the first eigenvalue of the operator $L_\epsilon = -\Delta - \frac{1}{\epsilon^2}W'(1 - f_\epsilon^2)$ and that the escaping radially symmetric critical points $U_{\epsilon,\eta}^\pm$ with $g_{\epsilon,\eta} > 0$ exist if and only

if $W'(1) > 0$ and $\ell(\epsilon) + \frac{1}{\eta^2}\tilde{W}'(0) < 0$ (equivalently $0 < \epsilon < \epsilon_0$ and $\eta > \eta_0(\epsilon)$). For fixed $\epsilon > 0, \eta > 0$, we let

$$\Phi = \begin{cases} U_{\epsilon,\eta}^+ & \text{if } W'(1) > 0 \text{ and } \ell(\epsilon) + \frac{1}{\eta^2}\tilde{W}'(0) < 0 \text{ (i.e. there is an escaping solution),} \\ \bar{U}_\epsilon & \text{otherwise (i.e. there is no escaping solution),} \end{cases}$$

and

$$(f, g) = \begin{cases} (f_{\epsilon,\eta}, g_{\epsilon,\eta}) & \text{if } W'(1) > 0 \text{ and } \ell(\epsilon) + \frac{1}{\eta^2}\tilde{W}'(0) < 0, \\ (f_\epsilon, 0) & \text{otherwise,} \end{cases}$$

so that $\Phi(x) = (f(r)\frac{x}{r}, g(r))$.

We consider the differential operators $L_{\epsilon,\eta}$ and $T_{\epsilon,\eta}$:

$$L_{\epsilon,\eta} = -\Delta - \frac{1}{\epsilon^2}W'(1 - f^2 - g^2), \quad T_{\epsilon,\eta} = -\Delta - \frac{1}{\epsilon^2}W'(1 - f^2 - g^2) + \frac{1}{\eta^2}\tilde{W}'(g^2).$$

Then the criticality condition of $\Phi(x) = (f(r)\frac{x}{r}, g(r))$ in B^N with respect to $E_{\epsilon,\eta}$ is exactly $L_{\epsilon,\eta}f = 0, T_{\epsilon,\eta}g = 0$.

For any $v \in H_0^2(B^N, \mathbb{R})$, we let

$$F_{\epsilon,\eta}[\nabla v] = \int_{B^N} \left((\Delta v)^2 - \frac{1}{\epsilon^2}W'(1 - f^2 - g^2)|\nabla v|^2 \right) dx = \int_{B^N} L_{\epsilon,\eta}(\nabla v) : (\nabla v) dx. \quad (2.1)$$

Note that $\int_{B^N} |\nabla^2 v|^2 dx = \int_{B^N} (\Delta v)^2 dx$ since $v \in H_0^2(B^N)$. Note also that, in the non-escaping case, $\Phi = \bar{U}_\epsilon$, $L_{\epsilon,\eta} = L_\epsilon$, $T_{\epsilon,\eta} = \bar{T}_{\epsilon,\eta}$ and $F_{\epsilon,\eta} = F_\epsilon$ introduced in sections 1.3 and 1.4.

As in [39, 40], the starting point of the proof is the following consequence of the convexity of W and \tilde{W} :

Lemma 6. For any $v \in H_0^2(B^N, \mathbb{R})$ and $p \in H_0^1(B^N, \mathbb{R})$,

$$E_{\epsilon,\eta}[\Phi + (\nabla v, p)] - E_{\epsilon,\eta}[\Phi] \geq \frac{1}{2}F_{\epsilon,\eta}[\nabla v] + \frac{1}{2} \int_{B^N} T_{\epsilon,\eta}p \cdot p dx.$$

Proof. We have

$$\begin{aligned} E_{\epsilon,\eta}[\Phi + (\nabla v, p)] - E_{\epsilon,\eta}[\Phi] &= \frac{1}{2} \int_{B^N} \left\{ 2\nabla\Phi : \nabla(\nabla v, p) + |\nabla^2 v|^2 + |\nabla p|^2 \right. \\ &\quad + \frac{1}{\epsilon^2}[W(1 - |\Phi + (\nabla v, p)|^2) - W(1 - |\Phi|^2)] \\ &\quad \left. + \frac{1}{\eta^2}[\tilde{W}((g + p)^2) - \tilde{W}(g^2)] \right\} dx. \end{aligned} \quad (2.2)$$

By the convexity of W and \tilde{W} , we have

$$\begin{aligned} W(1 - |\Phi + (\nabla v, p)|^2) - W(1 - |\Phi|^2) &\geq W'(1 - |\Phi|^2) (|\Phi|^2 - |\Phi + (\nabla v, p)|^2) \\ &= -W'(1 - |\Phi|^2) (2\Phi \cdot (\nabla v, p) + |\nabla v|^2 + p^2) \\ \tilde{W}((g + p)^2) - \tilde{W}(g^2) &\geq \tilde{W}'(g^2) ((g + p)^2 - g^2) \\ &= \tilde{W}'(g^2) (2gp + p^2). \end{aligned}$$

Since Φ is a critical point of $E_{\epsilon, \eta}$, we also have

$$\int_{B^N} \left\{ \nabla \Phi : \nabla (\nabla v, p) - \frac{1}{\epsilon^2} W'(1 - |\Phi|^2) \Phi \cdot (\nabla v, p) + \frac{1}{\eta^2} \tilde{W}'(g^2) gp \right\} dx = 0.$$

Inserting the last two estimates into (2.2) we arrive at

$$\begin{aligned} E_{\epsilon, \eta}[\Phi + (\nabla v, p)] - E_{\epsilon, \eta}[\Phi] \\ \geq \frac{1}{2} \int_{B^N} \left(|\nabla^2 v|^2 + |\nabla p|^2 - \frac{1}{\epsilon^2} W'(1 - f^2 - g^2) (|\nabla v|^2 + p^2) + \frac{1}{\eta^2} \tilde{W}'(g^2) p^2 \right) dx, \end{aligned}$$

which is precisely the conclusion. \square

We will frequently make use of the following Hardy decomposition:

Lemma 7 ([37, Lemma A.1]). *Let $A : B^N \rightarrow \mathbb{R}^{N \times N}$ be a C^1 non-negative semi-definite symmetric form, i.e. $A(x)\xi \cdot \xi \geq 0$ for every $x \in B^N$ and $\xi \in \mathbb{R}^N$. We define the operator*

$$L := -\nabla \cdot (A\nabla)$$

and consider a smooth positive function $\psi : B^N \rightarrow \mathbb{R}$. Then for every $u \in C_c^\infty(B^N, \mathbb{R})$, we have the following Hardy decomposition:

$$\int_{B^N} Lu \cdot u \, dx = \int_{B^N} \psi^2 A(x) \nabla \left(\frac{u}{\psi} \right) \cdot \nabla \left(\frac{u}{\psi} \right) \, dx + \int_{B^N} \frac{u^2}{\psi^2} L\psi \cdot \psi \, dx.$$

Before moving on with the proof, let us make a simple observation on the non-negativity of $T_{\epsilon, \eta}$.

Lemma 8. *The first eigenvalue of $T_{\epsilon, \eta}$ on $H_0^1(B^N, \mathbb{R})$ is $\left(\ell(\epsilon) + \frac{1}{\eta^2} \tilde{W}'(0) \right)_+$ and the corresponding first eigenspace of $T_{\epsilon, \eta}$*

- *coincides with the first eigenspace of L_ϵ when $\ell(\epsilon) + \frac{1}{\eta^2} \tilde{W}'(0) \geq 0$ (i.e. when $g \equiv 0$), and*

- is generated by g when $\ell(\epsilon) + \frac{1}{\eta^2}\tilde{W}'(0) < 0$ (i.e. when $g > 0$).

In particular $T_{\epsilon,\eta}$ is non-negative semi-definite on $H_0^1(B^N, \mathbb{R})$ and

$$\int_{B^N} T_{\epsilon,\eta} p \cdot p \, dx \geq \int_{B^N} \left[h^2 \left| \nabla \left(\frac{p}{h} \right) \right|^2 + \left(\ell(\epsilon) + \frac{1}{\eta^2} \tilde{W}'(0) \right)_+ p^2 \right] dx \geq 0,$$

where h is any first eigenfunction of $T_{\epsilon,\eta}$.

Proof. Recall that, by [34, Theorem 2.4] on escaping and non-escaping critical points of $E_{\epsilon,\eta}$, when $g \equiv 0$, we have $\ell(\epsilon) + \frac{1}{\eta^2}\tilde{W}'(0) \geq 0$, while, when $g > 0$, $\ell(\epsilon) + \frac{1}{\eta^2}\tilde{W}'(0) < 0$. The first bullet point is then clear as $T_{\epsilon,\eta} = L_\epsilon + \frac{1}{\eta^2}\tilde{W}'(0)$ and the first eigenvalue of L_ϵ is $\ell(\epsilon)$. Next, note that since $\Phi(x) = (f(r)\frac{x}{r}, g(r))$ is a critical point of $E_{\epsilon,\eta}$, we have $T_{\epsilon,\eta}g = 0$. Thus, when $g > 0$, g must be a first eigenfunction of $T_{\epsilon,\eta}$ and the first eigenvalue of $T_{\epsilon,\eta}$ must be zero. The second bullet point follows. The last assertion follows from the Hardy decomposition Lemma 7 with the decomposition $p = h\frac{p}{h}$. \square

The last ingredient for the proof of Theorem 3 is:

Proposition 9. *Suppose $N \geq 4$. For any $v \in H_0^2(B^N, \mathbb{R})$ we have*

$$F_{\epsilon,\eta}[\nabla v] \geq \frac{(N-2)^2}{4} \int_{B^N} \frac{(\partial_r v)^2}{r^2} \, dx + \left(\frac{N^2}{2} - 2N \right) \int_{B^N} \frac{|\nabla v|^2 - (\partial_r v)^2}{r^2} \, dx \geq 0.$$

Remark 10. *Note that for general $V \in H_0^1(B^N, \mathbb{R}^N)$ which is not necessarily a gradient field, it was shown in [39, 40] in dimension $N \geq 7$ that $F_{\epsilon,\eta}[V] = F_\epsilon[V] \geq 0$.*

Before giving the proof of the above proposition, let us prove Theorem 3.

First proof of Theorem 3. Indeed, as $N \geq 4$, we have by Proposition 9 that $F_{\epsilon,\eta}[\nabla v] \geq 0$ for every $v \in H_0^2(B^N, \mathbb{R})$ with equality if and only if $\partial_r v = 0$ a.e., which implies $v = 0$. Therefore, by Lemmas 6 and 8, Φ is a minimizer of our problem. If $\tilde{\Phi}$ is another minimizer of $E_{\epsilon,\eta}$, then $E_{\epsilon,\eta}[\tilde{\Phi}] = E_{\epsilon,\eta}[\Phi]$. By Lemmas 6, 8 and Proposition 9, this is possible only if $\tilde{\Phi} - \Phi = (0, h)$ for some **radially symmetric** h in the first eigenspace of $T_{\epsilon,\eta}$ (because $T_{\epsilon,\eta}$ is **rotationally invariant**). We thus have that $\tilde{\Phi}$ is a radially symmetric minimizer of $E_{\epsilon,\eta}$. [34, Theorem 2.4] then gives the desired uniqueness for minimizer(s). \square

Proof of Proposition 9. It is enough to prove the estimate for $v \in \mathcal{C}_c^\infty(B^N \setminus \{0\}, \mathbb{R})$. The general case follows from Fatou's lemma and the density of $\mathcal{C}_c^\infty(B^N \setminus \{0\}, \mathbb{R})$ in $H_0^2(B^N, \mathbb{R})$ (note $N \geq 4$).

We denote by $(\phi_k)_{k \in \mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{S}^{N-1})$ given by eigenfunctions of the Laplace-Beltrami operator on the unit sphere, meaning that for any $k \in \mathbb{N}$ we have

$$-\Delta_{\mathbb{S}^{N-1}} \phi_k = \lambda_k \phi_k$$

where $0 = \lambda_0 < N - 1 = \lambda_1 = \dots = \lambda_N < 2N = \lambda_{N+1} \leq \dots \rightarrow +\infty$. In particular we have

$$\int_{\mathbb{S}^{N-1}} \phi_k \phi_l d\sigma(\theta) = \delta_{kl} \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} \nabla_{\mathbb{S}^{N-1}} \phi_k \cdot \nabla_{\mathbb{S}^{N-1}} \phi_l d\sigma(\theta) = \lambda_k \delta_{kl}. \quad (2.3)$$

Consider the decomposition of v in spherical harmonics: we write

$$v(r\theta) = \sum_{k \geq 0} v_k(r) \phi_k(\theta) \quad \text{for } r \in (0, 1), \theta \in \mathbb{S}^{N-1}$$

where $v_k \in \mathcal{C}_c^\infty((0, 1), \mathbb{R})$. We have

$$\nabla v = \sum_{k \geq 0} \left(v'_k \phi_k \frac{x}{r} + \frac{1}{r} v_k \nabla_{\mathbb{S}^{N-1}} \phi_k \right), \quad \Delta v = \sum_{k \geq 0} \left(v''_k + \frac{N-1}{r} v'_k - \frac{\lambda_k}{r^2} v_k \right) \phi_k.$$

Using the orthogonality relations (2.3) and the identities

$$\begin{aligned} \int_0^1 r^{N-2} v''_k v'_k dr &= -\frac{N-2}{2} \int_0^1 r^{N-3} (v'_k)^2 dr, \\ \int_0^1 r^{N-4} v'_k v_k dr &= -\frac{N-4}{2} \int_0^1 r^{N-5} v_k^2 dr \quad \text{for } k \geq 1, \\ \int_0^1 r^{N-3} v''_k v_k dr &= \int_0^1 \left[-r^{N-3} (v'_k)^2 + \frac{(N-3)(N-4)}{2} r^{N-5} v_k^2 \right] dr \quad \text{for } k \geq 1, \end{aligned}$$

we get

$$\begin{aligned} \int_{B^N} (\Delta v)^2 dx &= \sum_{k \geq 0} \int_{B^N} \left(v''_k + \frac{N-1}{r} v'_k - \frac{\lambda_k}{r^2} v_k \right)^2 \phi_k^2 dx \\ &= \sum_{k \geq 0} \int_0^1 \left(r^{N-1} (v''_k)^2 + (N-1 + 2\lambda_k) r^{N-3} (v'_k)^2 + \lambda_k (\lambda_k + 2N-8) r^{N-5} v_k^2 \right) dr, \end{aligned} \quad (2.4)$$

and

$$\int_{B^N} W'(1 - f^2 - g^2) |\nabla v|^2 dx = \sum_{k \geq 0} \int_0^1 W'(1 - f^2 - g^2) \left(r^{N-1} (v'_k)^2 + \lambda_k r^{N-3} v_k^2 \right) dr.$$

Inserting these into (2.1), we split $F_{\epsilon,\eta}$ into three terms as follows:

$$\begin{aligned}
F_{\epsilon,\eta}[\nabla v] = \sum_{k \geq 0} \left\{ \underbrace{\int_0^1 r^{N-1} \left((v_k'')^2 - \frac{1}{\epsilon^2} W'(1-f^2-g^2)(v_k')^2 \right) dr}_{\text{I}_k} \right. \\
+ \underbrace{\int_0^1 \lambda_k r^{N-1} \left(r^{-2}(v_k')^2 - \frac{1}{\epsilon^2} W'(1-f^2-g^2)r^{-2}v_k^2 \right) dr}_{\text{II}_k} \\
\left. + \underbrace{\int_0^1 \left((N-1+\lambda_k)r^{N-3}(v_k')^2 + \lambda_k(\lambda_k+2N-8)r^{N-5}v_k^2 \right) dr}_{\text{III}_k} \right\}.
\end{aligned}$$

For terms I_k and II_k we will apply the Hardy decomposition Lemma 7 using

$$L_{\epsilon,\eta}f = -\frac{N-1}{r^2}f.$$

More precisely, for any function $w \in \mathcal{C}_c^\infty(B^N, \mathbb{R})$ we have the identity

$$\begin{aligned}
\int_{B^N} L_{\epsilon,\eta}(fw) \cdot (fw) dx &= \int_{B^N} (f^2|\nabla w|^2 + w^2L_{\epsilon,\eta}f \cdot f) dx \\
&= \int_{B^N} f^2 \left(|\nabla w|^2 - \frac{N-1}{r^2}w^2 \right) dx. \tag{2.5}
\end{aligned}$$

- **Estimate of I_k :** For the first term we use the decomposition $v_k' = f \frac{v_k'}{f}$, i.e. $w = \frac{v_k'}{f} \in \mathcal{C}_c^\infty(B^N \setminus \{0\}, \mathbb{R})$ in (2.5):

$$\text{I}_k = \int_0^1 r^{N-1} L_{\epsilon,\eta}(v_k') \cdot (v_k') dr = \int_0^1 \left[r^{N-1} f^2 \left| \left(\frac{v_k'}{f} \right)' \right|^2 - (N-1)r^{N-3}(v_k')^2 \right] dr.$$

We let $\zeta(r) = r^{-\frac{N-2}{2}}$ so that, when seen as a radial function in $\mathbb{R}^N \setminus \{0\}$, ζ verifies

$$-\nabla \cdot (f^2 \nabla \zeta) = -f^2 \Delta \zeta - 2ff'\zeta' = \frac{(N-2)^2}{4r^2} f^2 \zeta - 2ff'\zeta' \geq \frac{(N-2)^2}{4r^2} f^2 \zeta,$$

since $\zeta' < 0$ and $f, f' > 0$ in $(0, 1)$. By the Hardy decomposition Lemma 7 for the operator $\nabla \cdot (f^2 \nabla)$ and the decomposition $\frac{v_k'}{f} = \zeta \frac{v_k'}{f\zeta}$, we thus have

$$\text{I}_k \geq \int_0^1 r^{N-1} \left(f^2 \zeta^2 \left| \left(\frac{v_k'}{f\zeta} \right)' \right|^2 + \left(\frac{(N-2)^2}{4} - (N-1) \right) r^{N-3} (v_k')^2 \right) dr. \tag{2.6}$$

- **Estimate of II_k :** First notice the elementary identity

$$\begin{aligned} \int_0^1 r^{N-3} (v'_k)^2 dr &= \int_0^1 \left(r^{N-1} \left(\left(\frac{v_k}{r} \right)' \right)^2 + 2r^{N-4} v_k v'_k - r^{N-5} v_k^2 \right) dr \\ &= \int_0^1 \left(r^{N-1} \left(\left(\frac{v_k}{r} \right)' \right)^2 - (N-3) r^{N-5} v_k^2 \right) dr \end{aligned}$$

so

$$\text{II}_k = \lambda_k \int_0^1 \left(r^{N-1} L_{\epsilon, \eta} \left(\frac{v_k}{r} \right) \cdot \left(\frac{v_k}{r} \right) - (N-3) r^{N-5} v_k^2 \right) dr.$$

This time we use the decomposition $\frac{v_k}{r} = f \frac{v_k}{rf}$ (i.e. $w = \frac{v_k}{rf}$ in (2.5)) to obtain

$$\text{II}_k = \lambda_k \int_0^1 \left(r^{N-1} f^2 \left| \left(\frac{v_k}{rf} \right)' \right|^2 - 2(N-2) r^{N-5} v_k^2 \right) dr.$$

By the Hardy decomposition Lemma 7 for the operator $\nabla \cdot (f^2 \nabla)$ and the decomposition $\frac{v_k}{rf} = \zeta \frac{v_k}{rf\zeta}$ as above we get the estimate

$$\text{II}_k \geq \lambda_k \int_0^1 \left(r^{N-1} f^2 \zeta^2 \left| \left(\frac{v_k}{rf\zeta} \right)' \right|^2 + \left(\frac{(N-2)^2}{4} - 2(N-2) \right) r^{N-5} v_k^2 \right) dr. \quad (2.7)$$

- **Estimate of III_k :** For the last term we simply apply the Hardy inequality once: for any $v \in \mathcal{C}_c^\infty((0, 1), \mathbb{R})$, $\int_0^1 r^{N-3} (v')^2 dr \geq \frac{(N-4)^2}{4} \int_0^1 r^{N-5} v^2 dr$. This gives

$$\text{III}_k \geq \int_0^1 \left((N-1) r^{N-3} (v'_k)^2 + \lambda_k \left(\lambda_k + 2N - 8 + \frac{(N-4)^2}{4} \right) r^{N-5} v_k^2 \right) dr. \quad (2.8)$$

Summing the estimates (2.6), (2.7), (2.8) we get

$$\begin{aligned} F_{\epsilon, \eta}[\nabla v] &\geq \sum_{k \geq 0} \int_0^1 \left(\frac{(N-2)^2}{4} r^{N-3} (v'_k)^2 + \lambda_k \left(\frac{N^2}{2} - 3N + 1 + \lambda_k \right) r^{N-5} v_k^2 \right) dr \\ &\geq \sum_{k \geq 0} \int_0^1 \left(\frac{(N-2)^2}{4} r^{N-3} (v'_k)^2 + \lambda_k \left(\frac{N^2}{2} - 2N \right) r^{N-5} v_k^2 \right) dr \\ &= \frac{(N-2)^2}{4} \int_{B^N} \frac{(\partial_r v)^2}{r^2} dx + \left(\frac{N^2}{2} - 2N \right) \int_{B^N} \frac{|\nabla v|^2 - (\partial_r v)^2}{r^2} dx. \end{aligned}$$

where the second inequality follows from $\lambda_k^2 \geq (N-1)\lambda_k$. The result is proved. \square

2.2 Proof of Theorems 1 and 2

Theorem 1 for the Aviles–Giga model is a simple consequence of Theorem 3 for the extended model.

Proof of Theorem 1. Fix $\epsilon > 0$. Pick any convex C^2 function $\tilde{W} : [0, \infty) \rightarrow [0, \infty)$ with $\tilde{W}(0) = 0$ and $\tilde{W}'(0) > 0$, e.g. $\tilde{W}(t) = t$. By [34, Theorem 2.4], there exists a small $\eta > 0$ such that $E_{\epsilon, \eta}$ has no escaping radially symmetric critical points. By Theorem 3, $\bar{U}_\epsilon = (U_\epsilon, 0)$ is the unique minimizer of $E_{\epsilon, \eta}$ in \mathcal{A} . It follows that

$$E_\epsilon^{GL}[\nabla u] = E_{\epsilon, \eta}[(\nabla u, 0)] \geq E_{\epsilon, \eta}[\bar{U}_\epsilon] = E_\epsilon[U_\epsilon] \text{ for all } \nabla u \in \mathcal{A}^{GL}.$$

This means that U_ϵ is a minimizer of E_ϵ^{GL} in $\{\nabla u \in \mathcal{A}^{GL}\}$. Conversely, if $\nabla \tilde{u}$ is a minimizer of E_ϵ^{GL} in $\{\nabla u \in \mathcal{A}^{GL}\}$, then

$$E_{\epsilon, \eta}[(\nabla \tilde{u}, 0)] = E_\epsilon^{GL}[\nabla \tilde{u}] = E_\epsilon^{GL}[U_\epsilon] = E_{\epsilon, \eta}[\bar{U}_\epsilon],$$

i.e. $(\nabla \tilde{u}, 0)$ is also a minimizer of $E_{\epsilon, \eta}$ in \mathcal{A} . By Theorem 3, $\nabla \tilde{u} = U_\epsilon$ as desired. \square

We next prove Theorem 2 for the \mathbb{S}^N -valued Ginzburg–Landau model.

Proof of Theorem 2. Set $W(t) = t^2$ and fix some $\eta > 0$. As $4 \leq N \leq 6$ and $W'(1) > 0$, we know by [34, Theorem 2.4] that for $\epsilon > 0$ small enough, there exists a unique escaping radially symmetric critical point of the form

$$U_{\epsilon, \eta} = (f_{\epsilon, \eta}(r) \frac{x}{r}, g_{\epsilon, \eta}(r)) \in \mathcal{A}, \quad g_{\epsilon, \eta} > 0 \text{ in } (0, 1)$$

of the energy $E_{\epsilon, \eta}$. Pick an arbitrary $M = (\nabla m, M_{N+1}) \in \mathcal{A}^{MM}$ (in particular, $|M| = 1$) and set

$$(\nabla v_{\epsilon, \eta}, p_{\epsilon, \eta}) := M - U_{\epsilon, \eta}.$$

Then by Section 2, we know that

$$\begin{aligned} E_\eta^{MM}[M] &= E_{\epsilon, \eta}[U_{\epsilon, \eta} + (\nabla v_{\epsilon, \eta}, p_{\epsilon, \eta})] \\ &\geq E_{\epsilon, \eta}[U_{\epsilon, \eta}] + \frac{1}{2} F_{\epsilon, \eta}[\nabla v_{\epsilon, \eta}] + \frac{1}{2} \int_{B^N} T_{\epsilon, \eta} p_{\epsilon, \eta} \cdot p_{\epsilon, \eta} dx \end{aligned}$$

with

$$\begin{aligned} F_{\epsilon, \eta}[\nabla v_{\epsilon, \eta}] &\geq \frac{(N-2)^2}{4} \int_{B^N} \frac{(\partial_r v_{\epsilon, \eta})^2}{r^2} dx + \left(\frac{N^2}{2} - 2N \right) \int_{B^N} \frac{|\nabla_{\mathbb{S}^{N-1}} v_{\epsilon, \eta}|^2}{r^4} dx, \\ \int_{B^N} T_{\epsilon, \eta} p_{\epsilon, \eta} \cdot p_{\epsilon, \eta} dx &\geq \int_{B^N} g_{\epsilon, \eta}^2 \left| \nabla \left(\frac{p_{\epsilon, \eta}}{g_{\epsilon, \eta}} \right) \right|^2 dx. \end{aligned}$$

By [34, Remark 2.17], for a subsequence $\epsilon \rightarrow 0$, we have that $U_{\epsilon,\eta} \rightarrow M_\eta^+$ in $H^1(B^N)$ (in particular, $\nabla U_{\epsilon,\eta} \rightarrow \nabla M_\eta^+$ and $U_{\epsilon,\eta} \rightarrow M_\eta^+$ a.e. in B^N) and $E_{\epsilon,\eta}(U_{\epsilon,\eta}) \rightarrow E_\eta^{MM}[M_\eta^+]$ where $M_\eta^+ = (\tilde{f}_\eta \frac{x}{r}, g_\eta)$ is the unique escaping radially symmetric critical point of E_η^{MM} with $g_\eta > 0$ in $(0, 1)$. Therefore,

$$(\nabla v_{\epsilon,\eta}, p_{\epsilon,\eta}) \rightarrow M - M_\eta^+ =: (\nabla \tilde{v}_\eta, \tilde{p}_\eta)$$

in $H^1(B^N)$ and a.e. in B^N as well as $\nabla(\nabla v_{\epsilon,\eta}, p_{\epsilon,\eta}) \rightarrow \nabla(\nabla \tilde{v}_\eta, \tilde{p}_\eta)$ a.e. in B^N for a subsequence $\epsilon \rightarrow 0$. **Putting the above inequalities together and using Fatou's lemma**, it follows for a subsequence $\epsilon \rightarrow 0$:

$$\begin{aligned} E_\eta^{MM}[M] &= E_\eta^{MM}[M_\eta^+ + (\nabla \tilde{v}_\eta, \tilde{p}_\eta)] \\ &\geq E_\eta^{MM}[M_\eta^+] + \frac{1}{2} \int_{B^N} g_\eta^2 \left| \nabla \left(\frac{\tilde{p}_\eta}{g_\eta} \right) \right|^2 dx \\ &\quad + \frac{(N-2)^2}{8} \int_{B^N} \frac{(\partial_r \tilde{v}_\eta)^2}{r^2} + \frac{1}{2} \left(\frac{N^2}{2} - 2N \right) \int_{B^N} \frac{|\nabla_{\mathbb{S}^{N-1}} \tilde{v}_\eta|^2}{r^4}. \end{aligned}$$

We conclude to the minimality of M_η^+ . If M is another minimizer, within the above notations, then $E_\eta^{MM}[M] = E_\eta^{MM}[M_\eta^+]$ and so $\partial_r \tilde{v}_\eta = 0$ in B^N yielding $\tilde{v}_\eta = 0$ (as $\tilde{v}_\eta = 0$ on ∂B^N); also, $\tilde{p}_\eta = \alpha g_\eta$ for some constant $\alpha \in \mathbb{R}$. Since $|M| = 1$ and $M = (0, \tilde{p}_\eta) + M_\eta$, we deduce that $(\tilde{p}_\eta + g_\eta)^2 = g_\eta^2$ yielding $\alpha = 0$ or -2 , i.e. $M = M_\eta^+$ or $M = M_\eta^-$. \square

3 Symmetrization and second proof of main results in dimension $N \geq 5$

3.1 A symmetrization of scalar functions

In this section, we consider a spherical average rearrangement which is probably known to the experts. See e.g. [64, Chapter 1, Section 9] for a similar rearrangement in the context of the Laplace operator. Let $1 \leq q < \infty$. For a function $g \in L^q(B^N, \mathbb{R})$, define a radial symmetrization g^\flat of g by

$$g^\flat(r) = \left\{ \int_{\mathbb{S}^{N-1}} |g(r\theta)|^q d\sigma(\theta) \right\}^{1/q} \geq 0, \quad r \in (0, 1). \quad (3.1)$$

This symmetrization depends on the parameter q . When $q = 2$, we can also think of this as a rearrangement in the spherical harmonic decomposition of g .

Theorem 11. *Let $N \geq 2$, $1 \leq q < \infty$, $g \in L^q(B^N, \mathbb{R})$ and g^\flat be associated to g by (3.1). We have the following conclusions.*

(i) The map $g \mapsto g^\flat$ is a 1-Lipschitz continuous map from $L^q(B^N, \mathbb{R})$ into itself:

$$\|g^\flat - h^\flat\|_{L^q(B^N, \mathbb{R})} \leq \|g - h\|_{L^q(B^N, \mathbb{R})}.$$

Moreover, $\int_{\mathbb{S}^{N-1}} |g^\flat(r\theta)|^q d\sigma(\theta) = \int_{\mathbb{S}^{N-1}} |g(r\theta)|^q d\sigma(\theta)$ for a.e. $r \in (0, 1)$.

(ii) Let $G : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be continuous. If G is convex in the second variable, then

$$\int_{B^N} G(r, |g^\flat(x)|^q) dx \leq \int_{B^N} G(r, |g(x)|^q) dx.$$

In particular, for any $q < p < \infty$,

$$\int_{B^N} |g^\flat|^p dx \leq \int_{B^N} |g|^p dx.$$

(iii) Assume in addition that $g \in W^{1,q}(B^N, \mathbb{R})$. Then $g^\flat \in W^{1,q}(B^N, \mathbb{R})$ and

$$\int_{B^N} |\nabla g^\flat|^q dx \leq \int_{B^N} |\nabla g|^q dx. \quad (3.2)$$

Equality is attained if and only if g is radially symmetric and $|g| = g^\flat$ in $(0, 1)$.

Proof. Proof of (i): From the definition of the radial function $g^\flat(x) = g^\flat(r)$ we have

$$\int_{\mathbb{S}^{N-1}} |g^\flat(r\theta)|^q d\sigma(\theta) = \int_{\mathbb{S}^{N-1}} |g(r\theta)|^q d\sigma(\theta) \text{ for a.e. } r \in (0, 1),$$

which implies $g^\flat \in L^q(B^N)$. Also, by the reverse triangle inequality, we have for $g, h \in L^q(B^N)$ that

$$\begin{aligned} \|g^\flat - h^\flat\|_{L^q(B^N)}^q &= |\mathbb{S}^{N-1}| \int_0^1 |g^\flat(r) - h^\flat(r)|^q r^{N-1} dr \\ &= \int_0^1 \left| \|g(r\cdot)\|_{L^q(\mathbb{S}^{N-1})} - \|h(r\cdot)\|_{L^q(\mathbb{S}^{N-1})} \right|^q r^{N-1} dr \\ &\leq \int_0^1 \|g(r\cdot) - h(r\cdot)\|_{L^q(\mathbb{S}^{N-1})}^q r^{N-1} dr = \|g - h\|_{L^q(B^N)}^q. \end{aligned}$$

Therefore $g \mapsto g^\flat$ is a 1-Lipschitz continuous map on $L^q(B^N)$.

Proof of (ii): By Jensen inequality,

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} G(r, |g^b(r\theta)|^q) d\sigma(\theta) &= G\left(r, \int_{\mathbb{S}^{N-1}} |g^b(r\theta)|^q d\sigma(\theta)\right) \\ &= G\left(r, \int_{\mathbb{S}^{N-1}} |g(r\theta)|^q d\sigma(\theta)\right) \leq \int_{\mathbb{S}^{N-1}} G(r, |g(r\theta)|^q) d\sigma(\theta). \end{aligned}$$

Integrating in r gives **the desired inequality**. In particular, with $G(r, s) = s^{p/q}$ with $p > q$, we see that the L^p -norm of g^b is no more than that of g .

Proof of (iii): Consider first the case g belongs to $\mathcal{C}^\infty(\bar{B}^N)$, which is a dense subset of $W^{1,q}(B^N)$. For technical reasons, we introduce, for $\mu > 0$,

$$g_\mu^b(r) = \left\{ \int_{\mathbb{S}^{N-1}} (g(r\theta)^2 + \mu)^{q/2} d\sigma(\theta) \right\}^{1/q} \geq \mu^{1/2}, \quad r \in (0, 1).$$

Note that $g_\mu^b \rightarrow g^b$ in $L^q(B^N)$ as $\mu \rightarrow 0$. We have, by Hölder's inequality,

$$\begin{aligned} |g_\mu^b(r)|^{q-1} |(g_\mu^b)'(r)| &\leq \int_{\mathbb{S}^{n-1}} (g(r\theta)^2 + \mu)^{(q-1)/2} |\partial_r g(r\theta)| d\sigma(\theta) \\ &\leq |g_\mu^b(r)|^{q-1} \left\{ \int_{\mathbb{S}^{n-1}} |\partial_r g(r\theta)|^q d\sigma(\theta) \right\}^{1/q}. \end{aligned}$$

As $g_\mu^b \geq \mu^{1/2} > 0$, this implies

$$\int_{\mathbb{S}^{N-1}} |\nabla g_\mu^b(r\theta)|^q d\sigma(\theta) = |(g_\mu^b)'(r)|^q \leq \int_{\mathbb{S}^{N-1}} |\partial_r g(r\theta)|^q d\sigma(\theta).$$

Integrating over $r \in (0, 1)$ gives

$$\int_{B^N} |\nabla g_\mu^b|^q dx \leq \int_{B^N} |\partial_r g|^q dx \leq \int_{B^N} |\nabla g|^q dx.$$

This implies g_μ^b is bounded in $W^{1,q}(B^N)$ and hence converges weakly to g^b in $W^{1,q}(B^N)$ as $\mu \rightarrow 0$. Hence

$$\int_{B^N} |\nabla g^b|^q dx \leq \int_{B^N} |\partial_r g|^q dx \leq \int_{B^N} |\nabla g|^q dx, \quad (3.3)$$

which proves (3.2) for $g \in \mathcal{C}^\infty(\bar{B}^N)$.

Suppose now $g \in W^{1,q}(B^N)$. Pick $\{g_{(j)}\} \subset \mathcal{C}^\infty(\bar{B}^N)$ such that $g_{(j)} \rightarrow g$ in $W^{1,q}(B^N)$. By (i), $g_{(j)}^b \rightarrow g^b$ in $L^q(B^N)$. Also, by (3.3),

$$\int_{B^N} |\nabla g_{(j)}^b|^q dx \leq \int_{B^N} |\partial_r g_{(j)}|^q dx \leq \int_{B^N} |\nabla g_{(j)}|^q dx. \quad (3.4)$$

This implies that $g_{(j)}^b$ is bounded in $W^{1,q}(B^N)$ and hence converges weakly in $W^{1,q}(B^N)$ to g^b . Sending $j \rightarrow \infty$ we see that (3.3) remains valid for $g \in W^{1,q}(B^N)$, which proves (3.2). Moreover, equality holds in (3.2) if and only if $|\nabla g| = |\partial_r g|$ a.e., i.e. g is radially symmetric. \square

3.2 A symmetrization of gradient fields and proof of Theorem 5

Recall the symmetrization \check{v} for a function $v \in H^1(B^N, \mathbb{R})$ is given by the formula (1.17):

$$\check{v}(r) = - \int_r^1 \left\{ \int_{\mathbb{S}^{N-1}} |\nabla v(s\theta)|^2 d\sigma(\theta) \right\}^{1/2} ds \leq 0, \quad r \in (0, 1).$$

We will use the following density result.

Lemma 12. *For $N \geq 2$, the set \mathcal{S} of functions in $\mathcal{C}^\infty(\bar{B}^N)$ which are constant in a neighborhood of the origin is dense in $H^2(B^N)$. Moreover, if $v \in H^2(B^N)$ verifies $\int_{\mathbb{S}^{N-1}} v(r\theta)\theta d\sigma(\theta) = 0$ for almost every $r \in (0, 1)$, then its approximation sequence in \mathcal{S} may be chosen with the same property.*

Proof. It is well known that $\mathcal{C}^\infty(\bar{B}^N)$ is dense in $H^2(B^N)$. Thus, to show that \mathcal{S} is dense in $H^2(B^N)$, we only need to show that a given $v \in \mathcal{C}^\infty(\bar{B}^N)$ can be approximated by a sequence of functions in \mathcal{S} . In the proof, C denotes a constant that can change between lines but depends only on the dimension N . Pick a cut-off function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ with $\varphi \equiv 1$ in $(-\infty, 1/2]$, $\varphi \equiv 0$ in $[1, \infty)$. For $j \geq 10$ and $x \in B^N$, let

$$\varphi_{(j)}(x) = \begin{cases} \varphi(j|x|) & \text{if } N \geq 3, \\ 1 - \varphi\left(\frac{\ln \ln \frac{1}{|x|}}{2 \ln \ln j}\right) & \text{if } N = 2. \end{cases}$$

Note that $\varphi_{(j)}(x) = 0$ for $|x| \geq \frac{1}{j}$ and $\varphi_{(j)}(x) = 1$ when $|x|$ is small enough. Define

$$v_{(j)} = v(0)\varphi_{(j)} + v(1 - \varphi_{(j)}) = v - (v - v_0)\varphi_{(j)} \in \mathcal{S}, \quad j \geq 1.$$

We estimate

$$|v(x) - v(0)| \leq \|\nabla v\|_{L^\infty(B^N)}|x|,$$

$$\|\varphi_{(j)}\|_{L^2(B^N)} \leq Cj^{-N/2},$$

$$\|\nabla \varphi_{(j)}\|_{L^2(B^N)} + \|r\nabla^2 \varphi_{(j)}\|_{L^2(B^N)} \leq C\omega_N(j) \text{ with } \omega_N(j) = \begin{cases} Cj^{-(N-2)/2} & \text{if } N \geq 3, \\ \frac{C}{(\ln j \ln \ln j)^{1/2}} & \text{if } N = 2. \end{cases}$$

We thus have

$$\begin{aligned} \|(v - v(0))\varphi_{(j)}\|_{L^2(B^N)} &\leq Cj^{-N/2}\|v\|_{L^\infty(B^N)}, \\ \|\nabla[(v - v(0))\varphi_{(j)}]\|_{L^2(B^N)} &\leq Cj^{-N/2}\|\nabla v\|_{L^\infty(B^N)} + C\omega_N(j)\|v\|_{L^\infty(B^N)}, \\ \|\nabla^2[(v - v(0))\varphi_{(j)}]\|_{L^2(B^N)} &\leq Cj^{-N/2}\|\nabla^2 v\|_{L^\infty(B^N)} + C\omega_N(j)\|\nabla v\|_{L^\infty(B^N)}. \end{aligned}$$

Clearly, these estimates imply that $v_{(j)} \rightarrow v$ in $H^2(B^N)$. We have proved that \mathcal{S} is dense in $H^2(B^N)$.

Now suppose $v \in H^2(B^N)$ and $\int_{\mathbb{S}^{N-1}} v(r\theta)\theta d\sigma(\theta) = 0$. Let $v_{(j)} \in \mathcal{S}$ be such that $v_{(j)} \rightarrow v$ in $H^2(B^N)$. Define $\tilde{v}_{(j)}(r\theta) = v_{(j)}(r\theta) - \sum_{k=1}^N v_{(j),k}(r)\phi_k(\theta)$ where $v_{(j),k}(r) = \int_{\mathbb{S}^{N-1}} v_{(j)}(r\theta)\phi_k(\theta) d\sigma(\theta)$. It is clear that $\int_{\mathbb{S}^{N-1}} \tilde{v}_{(j)}(r\theta)\theta d\sigma(\theta) = 0$. Since $v_{(j)}$ is constant near 0, $v_{(j),k}$ is supported away from 0, and so $\tilde{v}_{(j)} \in \mathcal{S}$. Finally, since the map $w \in H^2(B^N) \mapsto (r\theta \mapsto w_k(r)\phi_k(\theta))$ is continuous in $H^2(B^N)$ and $v_k \equiv 0$ for $k = 1, \dots, N$ we have

$$\lim_{j \rightarrow \infty} \|\tilde{v}_{(j)} - v\|_{H^2(B^N)} \leq \lim_{j \rightarrow \infty} \|v_{(j)} - v\|_{H^2(B^N)} + \lim_{j \rightarrow \infty} \left\| \sum_{k=1}^N v_{(j),k}(r)\phi_k(\theta) \right\|_{H^2(B^N)} = 0.$$

The proof is complete. \square

Proof of Theorem 5. Proof of (i): By Cauchy-Schwarz' inequality,

$$\begin{aligned} \check{v}(r)^2 &= \left\{ \int_r^1 \left[\int_{\mathbb{S}^{N-1}} |\nabla v(s\theta)|^2 d\sigma(\theta) \right]^{1/2} ds \right\}^2 \\ &\leq \left\{ \int_r^1 s^{1-N} ds \right\} \left\{ \int_r^1 s^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(s\theta)|^2 d\sigma(\theta) ds \right\}. \end{aligned}$$

Hence $\check{v}(r)$ is well-defined and finite in $(0, 1)$; in fact, $|\check{v}(r)| \leq C_N r^{-\frac{N-2}{2}} \|\nabla v\|_{L^2(B^N)}$ for $N \geq 3$ (resp. $|\check{v}(r)| \leq C\sqrt{\log(1/r)}\|\nabla v\|_{L^2(B^2)}$ when $N = 2$). In particular, $\check{v} \in L^2(B^N)$. Moreover, by the definition of \check{v} we have $\int_{\mathbb{S}^{N-1}} |\nabla \check{v}(r, \theta)|^2 d\sigma(\theta) = \int_{\mathbb{S}^{N-1}} |\nabla v(r\theta)|^2 d\sigma(\theta)$ for a.e. $r \in (0, 1)$. As $\check{v}(1) = 0$, these imply that $\check{v} \in H_0^1(B^N)$.

As in the proof of (i) in Theorem 11, the map $\nabla v \mapsto \nabla \check{v}$ is a 1-Lipschitz continuous map from $L^2(B^N, \mathbb{R}^N)$ into itself. By Poincaré's inequality, the map $v \mapsto \check{v}$ is a Lipschitz continuous map from $H^1(B^N)$ into $H_0^1(B^N)$.

Proof of (ii): This is similar to that in the proof of Theorem 11 and is omitted.

Proof of (iii): By density and (i), it suffices to consider $v \in \mathcal{C}_c^\infty(B^N)$.

Let $A(r) = \int_{\mathbb{S}^{N-1}} |v(r\theta)|^p d\sigma(\theta)$. We have, by Hölder's inequality

$$\begin{aligned} |A'(r)| &\leq p \int_{\mathbb{S}^{N-1}} |v(r\theta)|^{p-1} |\partial_r v(r\theta)| d\sigma \leq p \left\{ \int_{\mathbb{S}^{N-1}} |v(r\theta)|^{2(p-1)} d\sigma \right\}^{1/2} \check{v}'(r) \\ &\leq p A(r)^{\frac{p-1}{p}} \check{v}'(r), \end{aligned}$$

where we have used $2(p-1) \leq p$ when $1 \leq p \leq 2$.

Fix some $\mu > 0$. Then $|\frac{d}{dr}(\mu + A(r))^{1/p}| \leq \check{v}'(r)$. This together with $A(1) = 0$ (since $v = 0$ on ∂B^N) implies

$$(\mu + A(r))^{1/p} \leq \mu^{1/p} + \int_r^1 \check{v}'(r) dr = \mu^{1/p} - \check{v}(r) = \mu^{1/p} + |\check{v}(r)|.$$

Sending $\mu \rightarrow 0$, we get the conclusion.

Proof of (iv): Note that on ∂B^N , $\nabla v(x) = cx$ is normal to ∂B^N and so the tangent derivatives of v along ∂B^N are zero. Hence v is constant on ∂B^N . Without loss of generality, we can assume that $v = 0$ on ∂B^N . Let $(\phi_k)_{k=0}^\infty$ be an orthonormal basis of $L^2(\mathbb{S}^{N-1})$ consisting of eigenfunctions of the Laplace-Beltrami operator on \mathbb{S}^{N-1} corresponding to eigenvalues $0 = \lambda_0 < N-1 = \lambda_1 = \dots = \lambda_N < 2N = \lambda_{N+1} \leq \dots \rightarrow \infty$. We decompose

$$v(r\theta) = \sum_{k=0}^\infty v_k(r)\phi_k(\theta) \text{ where } v_k(r) = \int_{\mathbb{S}^{N-1}} v(r\theta)\phi_k(\theta) d\sigma(\theta).$$

Note that $v_k \in H_{\text{loc}}^2(0, 1)$, $\check{v} = \check{v}_0\phi_0$ and

$$(\check{v}'_0)^2 = \sum_{k=0}^\infty \left[(v'_k)^2 + \frac{\lambda_k}{r^2} v_k^2 \right], \quad (3.5)$$

$$\int_{B^N} (\Delta v)^2 dx = \sum_{k=0}^\infty \int_0^1 r^{N-1} \left(v''_k + \frac{N-1}{r} v'_k - \frac{\lambda_k}{r^2} v_k \right)^2 dr. \quad (3.6)$$

Note also that our hypotheses give in the case $N \in \{3, 4\}$ that $v_1 = \dots = v_N = 0$.

We first prove inequality (1.22) when v belongs to the set \mathcal{S} defined in Lemma 12. Then $v_0 \in \mathcal{C}^\infty([0, 1])$, v_0 is constant near 0, $v_k \in \mathcal{C}_c^\infty((0, 1])$ for $k \geq 1$,

$$v_0(1) = 0, v'_0(1) = \frac{c}{\phi_0} \text{ and } v_k(1) = v'_k(1) = 0 \text{ for } k \geq 1.$$

This implies $\nabla \check{v}(x) = |c|x$ on ∂B^N (recall that, by definition, $\check{v}' \geq 0$ in $(0, 1)$). Also,

$$\begin{aligned} \int_0^1 r^{N-2} v''_k v'_k dr &= -\frac{N-2}{2} \int_0^1 r^{N-3} (v'_k)^2 dr + \begin{cases} \frac{c^2}{2\phi_0^2} & \text{if } k=0, \\ 0 & \text{if } k \geq 1, \end{cases} \\ \int_0^1 r^{N-4} v'_k v_k dr &= -\frac{N-4}{2} \int_0^1 r^{N-5} v_k^2 dr \text{ for } k \geq 1, \\ \int_0^1 r^{N-3} v''_k v_k dr &= \int_0^1 \left[-r^{N-3} (v'_k)^2 + \frac{(N-3)(N-4)}{2} r^{N-5} v_k^2 \right] dr \text{ for } k \geq 1. \end{aligned}$$

Inserting the above identities in (3.6), we obtain

$$\int_{B^N} (\Delta v)^2 dx = (N-1) \frac{c^2}{\phi_0^2} + \sum_{k=0}^{\infty} \int_0^1 r^{N-1} \left[(v_k'')^2 + \frac{2\lambda_k + N - 1}{r^2} (v_k')^2 + \frac{\lambda_k(\lambda_k + 2(N-4))}{r^4} v_k^2 \right] dr. \quad (3.7)$$

Next, note that, when $v \in \mathcal{S}$, the right hand side of (3.5) is a smooth non-negative function and so \check{v}' is Lipschitz continuous. Applying (3.7) to \check{v} , we get

$$\int_{B^N} (\Delta \check{v})^2 dx = (N-1) \frac{c^2}{\phi_0^2} + \int_0^1 r^{N-1} \left[(\check{v}_0'')^2 + \frac{N-1}{r^2} (\check{v}_0')^2 \right] dr. \quad (3.8)$$

To continue, we need to estimate \check{v}_0'' . For technical reasons, we consider for small $\mu > 0$ a regularized version of \check{v}_0 :

$$\check{v}'_{\mu} = \left\{ \mu + \sum_{k=0}^{\infty} \left[(v_k')^2 + \frac{\lambda_k}{r^2} v_k^2 \right] \right\}^{1/2} \geq \mu^{1/2}.$$

Clearly \check{v}'_{μ} is smooth and $\check{v}'_{\mu} \rightarrow \check{v}'_0$ pointwise in $(0, 1)$ as $\mu \rightarrow 0$. Now, for some $t_k \in \mathbb{R}$ to be chosen later, we have by (3.5) that

$$\begin{aligned} |\check{v}'_{\mu}| |\check{v}''_{\mu}| &= \left| \sum_{k=0}^{\infty} \left[v_k' v_k'' + \frac{\lambda_k}{r^2} v_k v_k' - \frac{\lambda_k}{r^3} v_k^2 \right] \right| \\ &\leq |v_0'| |v_0''| + \left| \sum_{k=1}^{\infty} \left[v_k' (v_k'' + \frac{t_k}{r^2} v_k) + \frac{1}{r} v_k \left(\frac{\lambda_k - t_k}{r} v_k' - \frac{\lambda_k}{r^2} v_k \right) \right] \right| \\ &\leq |v_0'| |v_0''| + \sum_{k=1}^{\infty} \left[(v_k')^2 + \frac{\lambda_k}{r^2} v_k^2 \right]^{1/2} \left[(v_k'' + \frac{t_k}{r^2} v_k)^2 + \frac{1}{\lambda_k} \left(\frac{\lambda_k - t_k}{r} v_k' - \frac{\lambda_k}{r^2} v_k \right)^2 \right]^{1/2} \\ &\leq |\check{v}'_{\mu}| \left\{ |v_0''|^2 + \sum_{k=1}^{\infty} \left[(v_k'' + \frac{t_k}{r^2} v_k)^2 + \frac{1}{\lambda_k} \left(\frac{\lambda_k - t_k}{r} v_k' - \frac{\lambda_k}{r^2} v_k \right)^2 \right] \right\}^{1/2}. \end{aligned}$$

Since $\check{v}'_{\mu} \geq \mu^{1/2} > 0$, this implies

$$|\check{v}''_{\mu}| \leq \left\{ |v_0''|^2 + \sum_{k=1}^{\infty} \left[(v_k'' + \frac{t_k}{r^2} v_k)^2 + \frac{1}{\lambda_k} \left(\frac{\lambda_k - t_k}{r} v_k' - \frac{\lambda_k}{r^2} v_k \right)^2 \right] \right\}^{1/2}.$$

This implies that $\{\check{v}'_{\mu}\}$ is bounded in $W^{1,\infty}((0, 1))$ and converges weakly* in $W^{1,\infty}((0, 1))$ to \check{v}'_0 as $\mu \rightarrow 0$ (since $\check{v}'_{\mu} \rightarrow \check{v}'_0$ pointwise), and

$$|\check{v}''_0| \leq \left\{ |v_0''|^2 + \sum_{k=1}^{\infty} \left[(v_k'' + \frac{t_k}{r^2} v_k)^2 + \frac{1}{\lambda_k} \left(\frac{\lambda_k - t_k}{r} v_k' - \frac{\lambda_k}{r^2} v_k \right)^2 \right] \right\}^{1/2}.$$

Returning to (3.8), we get

$$\begin{aligned}
\int_{B^N} (\Delta \check{v})^2 dx &\leq (N-1) \frac{c^2}{\phi_0^2} + \int_0^1 r^{N-1} \left[(v_0'')^2 + \frac{N-1}{r^2} (v_0')^2 \right] dr \\
&+ \sum_{k=1}^{\infty} \int_0^1 r^{N-1} \left[(v_k'')^2 + \frac{t_k}{r^2} v_k^2 + \frac{1}{\lambda_k} \left(\frac{\lambda_k - t_k}{r} v_k' - \frac{\lambda_k}{r^2} v_k \right)^2 + \frac{N-1}{r^2} (v_k')^2 + \frac{(N-1)\lambda_k}{r^4} v_k^2 \right] dr \\
&= (N-1) \frac{c^2}{\phi_0^2} + \int_0^1 r^{N-1} \left[(v_0'')^2 + \frac{N-1}{r^2} (v_0')^2 \right] dr \\
&+ \sum_{k=1}^{\infty} \int_0^1 r^{N-1} \left[(v_k'')^2 + \frac{\lambda_k^{-1}(\lambda_k - t_k)^2 - 2t_k + N - 1}{r^2} (v_k')^2 \right. \\
&\quad \left. + \frac{2\lambda_k(N-2) + t_k^2 + t_k(N-4)^2}{r^4} v_k^2 \right] dr.
\end{aligned}$$

Recalling (3.7), we get

$$\begin{aligned}
\int_{B^N} (\Delta v)^2 dx - \int_{B^N} (\Delta \check{v})^2 dx &\geq \sum_{k=1}^{\infty} \int_0^1 r^{N-1} \left[\frac{\lambda_k - \lambda_k^{-1} t_k^2 + 4t_k}{r^2} (v_k')^2 \right. \\
&\quad \left. + \frac{\lambda_k^2 - 4\lambda_k - t_k^2 - t_k(N-4)^2}{r^4} v_k^2 \right] dr. \tag{3.9}
\end{aligned}$$

Case 1: If $N \geq 5$, we choose $t_k = 0$, and using the sharp Hardy inequality $\int_0^1 r^{N-3} (v_k')^2 dr \geq \frac{(N-4)^2}{4} \int_0^1 r^{N-5} v_k^2 dr$ to obtain from (3.9) the inequality

$$\int_{B^N} (\Delta v)^2 dx \geq \int_{B^N} (\Delta \check{v})^2 dx + \sum_{k=1}^{\infty} \lambda_k s_k \int_0^1 r^{N-5} v_k^2 dr, \tag{3.10}$$

where, for $k \geq 1$,

$$s_k = \lambda_k + \frac{(N-4)^2}{4} - 4 > 0 \quad (\text{since } \lambda_k \geq N-1 \geq 4).$$

Inequality (1.22) thus follows.

Case 2: If $N \in \{2, 3, 4\}$, recall that our hypotheses give $v_1 = \dots = v_N = 0$. We choose $t_k = (2 - \sqrt{5})\lambda_k$ in (3.9) so that the term involving v_k' vanishes, and arrive again at (3.10) but with

$$s_k = \begin{cases} 0 & \text{if } 1 \leq k \leq N, \\ (\sqrt{5} - 2)(4\lambda_k + (N-4)^2) - 4 & \text{if } k \geq N+1. \end{cases}$$

As $\lambda_k \geq 2N$ for $k \geq N+1$, we have

$$s_k \geq (\sqrt{5} - 2)(N^2 + 16) - 4 \geq 20\sqrt{5} - 44 > 0 \text{ for } N \in \{2, 3, 4\}, k \geq N+1.$$

Inequality (1.22) thus follows from (3.10).

Consider now the general case $v \in H^2(B^N)$. By Lemma 12 we can select $\{v_{(j)}\} \subset \mathcal{S}$ such that $v_{(j)} \rightarrow v$ in $H^2(B^N)$ as $j \rightarrow \infty$. Moreover, in case $N \in \{2, 3, 4\}$, it holds also that $\int_{\mathbb{S}^{N-1}} v_{(j)}(r\theta)\theta d\sigma(\theta) = 0$. By Fubini's theorem, after passing to a subsequence, we have $(v_{(j)})_k \rightarrow v_k$ a.e. in $(0, 1)$ for the spherical harmonic coefficients of $v_{(j)}$ and v . Also, by (i), $\nabla \check{v}_{(j)} \rightarrow \nabla \check{v}$ in $L^2(B^N, \mathbb{R}^N)$. Since $v_{(j)} \in \mathcal{S}$, we have by (3.10)

$$\int_{B^N} (\Delta v_{(j)})^2 dx \geq \int_{B^N} (\Delta \check{v}_{(j)})^2 dx + \sum_{k=1}^{\infty} \lambda_k s_k \int_0^1 r^{N-5} (v_{(j)})_k^2 dr. \quad (3.11)$$

This implies that $\{\check{v}_{(j)}\}$ is bounded in $H^2(B^N)$. As $\nabla \check{v}_{(j)} \rightarrow \nabla \check{v}$ in $L^2(B^N, \mathbb{R}^N)$, this implies $\Delta \check{v}_{(j)}$ converges weakly in $L^2(B^N)$ to $\Delta \check{v}$; in particular, $\check{v} \in H^2(B^N)$. Sending $j \rightarrow \infty$ in (3.11), using the convergence of $v_{(j)}$ to v in $H^2(B^N)$ on the left hand side, the weak convergence of $\Delta \check{v}_{(j)}$ to $\Delta \check{v}$ in $L^2(B^N)$ and Fatou's lemma for the infinite sum on the right hand side, we see that (3.10) remains valid for $v \in H^2(B^N)$. This proves (1.22) for $v \in H^2(B^N)$. Also, equality occurs in (1.22) if and only if $v_k = 0$ for all $k \geq 1$, meaning v is radially symmetric and $|v'| = \check{v}'$ in $(0, 1)$. \square

3.3 Second proof of Theorems 1, 2 and 3 in dimension $N \geq 5$

Second proof of Theorem 1 in dimension $N \geq 5$. As $s \mapsto W(1-s)$ is convex, we deduce from Theorem 5 and (1.23) that

$$E_\epsilon^{GL}[\nabla u] \geq E_\epsilon^{GL}[\nabla \check{u}] \text{ for all } \nabla u \in \mathcal{A}^{GL},$$

where equality holds if and only if u is radially symmetric. In particular, if $\nabla u \in \mathcal{A}^{GL}$ is a minimizer of E_ϵ^{GL} among gradient field configurations in \mathcal{A}^{GL} , then so is $\nabla \check{u}$ with $E_\epsilon^{GL}[\nabla u] = E_\epsilon^{GL}[\nabla \check{u}]$ and hence u is radially symmetric. The conclusion then follows from [34, Theorem 2.1] on the uniqueness of radially symmetric critical point of E_ϵ^{GL} in \mathcal{A}^{GL} . \square

Second proof of Theorem 2 in dimension $N \geq 5$. Observe that if $(\nabla m, M_{N+1}) \in \mathcal{A}^{MM}$ and if \check{m} denotes the symmetrization of m by (1.17) and M_{N+1}^b denotes the symmetrization of M_{N+1} by (3.1) with the parameter $q = 2$, then $(\nabla \check{m}, M_{N+1}^b) \in \mathcal{A}^{MM}$ because

$$|\nabla \check{m}|^2(r) + (M_{N+1}^b(r))^2 = \int_{\mathbb{S}^{N-1}} (|\nabla m|^2(r\theta) + M_{N+1}^2(r\theta)) d\sigma(\theta) = 1.$$

Thus, by Theorems 5 and 11 as well as (1.23), if $(\nabla m, M_{N+1}) \in \mathcal{A}^{MM}$ is a minimizer of E_η^{MM} in \mathcal{A}^{MM} , then $(\nabla \check{m}, M_{N+1}^b)$ is also a minimizer of E_η^{MM} in \mathcal{A}^{MM} and $(\nabla m, M_{N+1})$ is radially symmetric. The conclusion then follows from [34, Theorem 2.6] on the classification of radially symmetric minimizers of E_η^{MM} . \square

Second proof of Theorem 3 in dimension $N \geq 5$. Let $U = (\nabla v, g) \in \mathcal{A}$ be a minimizer of $E_{\epsilon, \eta}$ in \mathcal{A} . Define the symmetrization \check{v} and g^b of v and g as in the previous two sections with $q = 2$, and let $\tilde{U} = (\nabla \check{v}, g^b)$. By Theorems 5 and 11, we have

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} |\nabla \check{v}(r\theta)|^2 d\sigma(\theta) &= \int_{\mathbb{S}^{N-1}} |\nabla v(r\theta)|^2 d\sigma(\theta) \text{ for a.e. } r \in (0, 1), \text{ by (1.18)} \\ \int_{\mathbb{S}^{N-1}} g^b(r\theta)^2 d\sigma(\theta) &= \int_{\mathbb{S}^{N-1}} g(r\theta)^2 d\sigma(\theta) \text{ for a.e. } r \in (0, 1), \\ \int_{B^N} \tilde{W}((g^b)^2) dx &\leq \int_{\mathbb{S}^{N-1}} \tilde{W}(g^2) dx, \\ \int_{B^N} (\Delta \check{v})^2 dx &\leq \int_{B^N} (\Delta v)^2 dx, \text{ by (1.22)}. \end{aligned}$$

The first two identities and the convexity of W give

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} W(1 - |\tilde{U}|^2)(r\theta) d\sigma(\theta) &= W\left(\int_{\mathbb{S}^{N-1}} (1 - |\tilde{U}|^2)(r\theta) d\sigma(\theta)\right) \\ &= W\left(\int_{\mathbb{S}^{N-1}} (1 - |U|^2)(r\theta) d\sigma(\theta)\right) \leq \int_{\mathbb{S}^{N-1}} W(1 - |U|^2)(r\theta) d\sigma(\theta). \end{aligned}$$

These estimates together with (1.23) imply that $E_{\epsilon, \eta}[\tilde{U}] \leq E_{\epsilon, \eta}[U]$ and so \tilde{U} is also a minimizer of $E_{\epsilon, \eta}$ in \mathcal{A} with $E_{\epsilon, \eta}[\tilde{U}] = E_{\epsilon, \eta}[U]$. Returning to the equality cases in Theorems 5 and 11, we have that v and g are radially symmetric, i.e. U is a radially symmetric minimizer of $E_{\epsilon, \eta}$. The conclusion follows from [34, Theorem 2.4] on the classification of radially symmetric minimizer of $E_{\epsilon, \eta}$. \square

3.4 Symmetry for solutions to a nonlinear eigenvalue problem

For $d > 0$, $1 \leq p < 2$ and $\lambda \in \mathbb{R}$, consider the energy functional

$$J[v] = \frac{1}{2} \|\Delta v\|_{L^2(B^N)}^2 - \frac{\lambda}{2} \|v\|_{L^2(B^N)}^2$$

on the set

$$S_{p,d} = \left\{ v \in H_0^2(B^N) : \|v\|_{L^p(B^N)} = d \right\}.$$

Let $\lambda_1(\Delta^2)$ denote the first eigenvalue of the bi-Laplacian in $H_0^2(B^N)$. When $\lambda < \lambda_1(\Delta^2)$, after adjusting by a scaling factor to remove the Lagrange multiplier, minimizers of J on $S_{p,d}$ correspond to solutions of the elliptic problem (1.25):

$$\begin{cases} \Delta^2 v = \lambda v + |v|^{p-2}v & \text{on } B^N, \\ v = \partial_r v = 0 & \text{on } \partial B^N. \end{cases}$$

(For $\lambda \geq \lambda_1(\Delta^2)$, the partial differential equation is different **due to the different sign of the Lagrange multiplier**, namely

$$\Delta^2 v = \begin{cases} \lambda_1(\Delta^2)v & \text{if } \lambda = \lambda_1(\Delta^2), \\ \lambda v - |v|^{p-2}v & \text{if } \lambda > \lambda_1(\Delta^2), \end{cases}$$

we do not consider these cases here for simplicity.)

Problem (1.25) has been studied by many authors and a summary of known results would go beyond the scope of the present paper. We refer the reader to e.g. [6, 9, 21, 23, 24, 45] and the references therein.

We prove:

Corollary 13. *Let $N \geq 5$ and $1 \leq p < 2$. For $\lambda < \lambda_1(\Delta^2)$, minimizers of J over $S_{p,d}$ are radially symmetric, do not change sign and are either radially non-decreasing or radially non-increasing.*

Proof. Note that as $\lambda < \lambda_1(\Delta^2)$, J is coercive on $H_0^2(B^N)$. By the compactness embedding theorem, J has a minimizer over $S_{p,d}$.

Let v be a minimizer of J over $S_{p,d}$; in particular, $J[v] \geq 0$. By Theorem 5, we have

$$\begin{cases} \|\Delta \check{v}\|_{L^2(B^N)} \leq \|\Delta v\|_{L^2(B^N)}, \\ \|\check{v}\|_{L^p(B^N)} \geq \|v\|_{L^p(B^N)} = d, \\ \|\check{v}\|_{L^2(B^N)} \geq \|v\|_{L^2(B^N)}. \end{cases} \quad (3.12)$$

Let

$$\bar{v} = \mu \check{v} \text{ where } \mu = d \|\check{v}\|_{L^p(B^N)}^{-1} \stackrel{(3.12)}{\leq} 1$$

so that $\bar{v} \in S_{p,d}$. We compute, keeping in mind that $\mu \leq 1$,

$$J[\bar{v}] = \mu^2 J[\check{v}] \stackrel{(3.12)}{\leq} \mu^2 J[v] \leq J[v],$$

where for the last inequality we use the fact that $J[v] \geq 0$. It follows that \bar{v} is also a minimizer of J over $S_{p,d}$, which in turn implies $J[\check{v}] = J[v]$ and all inequalities in (3.12) are saturated. Appealing to the equality case in Theorem 5(iv), we see that v is radially symmetric and $\check{v}' = |v'|$.

It remains to prove that v and $\partial_r v$ do not change sign. Indeed, we have

$$|v(r)| = |v(r) - v(1)| = \left| \int_r^1 v'(s) ds \right| \leq \int_r^1 |v'(s)| ds = \int_r^1 \check{v}'(s) ds = |\check{v}(r)|.$$

As $\|\check{v}\|_{L^2(B^N)} = \|v\|_{L^2(B^N)}$, it follows that equality is attained in the above inequality, i.e. v' does not change sign. As $v(1) = 0$, it follows also that v does not change sign. \square

A The negativity of F_ϵ in dimension $N \in \{2, 3\}$

We now give the proof of Proposition 4 on the negativity of F_ϵ in dimension $N \in \{2, 3\}$.

Proof of Proposition 4. We follow ideas from e.g. the proof of [34, Lemma 2.3], [35, Proposition 4.1], [38, Theorem 1.7]. The main task is to show that there exists $v \in \mathcal{C}_c^2(B^N \setminus \{0\})$ such that⁵

$$F_*[\nabla v] := \int_{B^N} \left[(\Delta v)^2 - \frac{N-1}{r^2} |\nabla v|^2 \right] dx < 0. \quad (\text{A.1})$$

Supposing for the moment that such a v has been found, we proceed to show that $F_\epsilon[\nabla v] < 0$ for this particular v and for sufficiently small $\epsilon > 0$. Indeed, using the Hardy decomposition Lemma 7 with the decomposition $\nabla v = f_\epsilon \frac{\nabla v}{f_\epsilon}$, noting that $\Delta f_\epsilon = \frac{N-1}{r^2} f_\epsilon - \frac{1}{\epsilon^2} W'(1 - f_\epsilon^2) f_\epsilon$, we find

$$F_\epsilon[\nabla v] = \int_{B^N} f_\epsilon^2 \left[\left| \nabla \left(\frac{\nabla v}{f_\epsilon} \right) \right|^2 - \frac{N-1}{r^2} \frac{|\nabla v|^2}{f_\epsilon^2} \right] dx.$$

Since $f_\epsilon \rightarrow 1$ in $\mathcal{C}_{\text{loc}}^1(B^N \setminus \{0\})$ and $v \in \mathcal{C}_c^2(B^N \setminus \{0\})$, we deduce that

$$\lim_{\epsilon \rightarrow 0} F_\epsilon[\nabla v] = \int_{B^N} \left[|\nabla^2 v|^2 - \frac{N-1}{r^2} |\nabla v|^2 \right] dx = F_*[\nabla v] < 0,$$

which gives the conclusion.

It remains to find $v \in \mathcal{C}_c^2(B^N \setminus \{0\})$ satisfying (A.1). The proof of Theorem 5 suggests the ansatz

$$v(x) = a(r) \frac{x_1}{r},$$

so that the restriction of v to each sphere centered at the origin is a first eigenfunction of the Laplace-Beltrami operator on the sphere. Using (3.7) with $\lambda_1 = N - 1$, we are thus led to searching for $a \in \mathcal{C}_c^2((0, 1))$ such that

$$\int_0^1 r^{N-1} \left[(a'')^2 + \frac{2(N-1)}{r^2} (a')^2 + \frac{2(N-1)(N-4)}{r^4} a^2 \right] dr < 0.$$

⁵Note that $\mathcal{C}_c^2(B^N \setminus \{0\})$ is not a dense subspace of $H_0^2(B^N)$ in dimension $N \in \{2, 3\}$, hence the existence of such v does not follow immediately from the sharpness of the Hardy inequality (1.16).

We decompose $a(r) = r^{-\frac{N-4}{2}}b(r)$ and compute

$$\begin{aligned} \int_0^1 r^{N-1}(a'')^2 dr &= \int_0^1 r^3 \left(b'' - \frac{N-4}{r}b' + \frac{(N-2)(N-4)}{4r^2}b \right)^2 dr \\ &= \int_0^1 \left(r^3(b'')^2 + \frac{(N-2)(N-4)}{2}r(b')^2 + \frac{(N-2)^2(N-4)^2}{16r}b^2 \right) dr, \\ \int_0^1 r^{N-3}(a')^2 dr &= \int_0^1 r \left(b' - \frac{N-4}{2r}b \right)^2 dr \\ &= \int_0^1 \left(r(b')^2 + \frac{(N-4)^2}{4r}b^2 \right) dr \end{aligned}$$

We thus need to find $b \in \mathcal{C}_c^2((0, 1))$ such that

$$\int_0^1 \left(r^3(b'')^2 + \frac{N^2 - 2N + 4}{2}r(b')^2 + \frac{(N-4)(N^3 + 12N - 16)}{16r}b^2 \right) dr < 0. \quad (\text{A.2})$$

To this end, we fix a cut-off function $\varphi \in \mathcal{C}_c^\infty([0, \infty))$ with $\varphi \equiv 1$ in $[0, 1/4]$, $\varphi \equiv 0$ in $[1/2, \infty)$. For $j > 20$ large to be fixed, we let

$$b(r) = \begin{cases} \varphi(r) & \text{if } r \geq 1/8, \\ \varphi\left(\frac{\ln \ln \frac{1}{r}}{4 \ln \ln j}\right) & \text{if } 0 < r < 1/8. \end{cases}$$

Then, for some constant C independent of j , we have

$$\begin{aligned} \int_0^1 \frac{1}{r}b^2 dr &\geq \int_{1/j}^{1/4} \frac{dr}{r} = \ln \frac{j}{4}, \\ \int_0^1 \left(r^3(b'')^2 + r(b')^2 \right) dr &\leq C. \end{aligned}$$

Therefore, as $(N-4)(N^3 + 12N - 16) < 0$ for $N \in \{2, 3\}$, we can select a sufficiently large j so that (A.2) is satisfied. \square

References

- [1] AMBROSIO, L, DE LELLIS, C., MANTEGAZZA, C. Line energies for gradient vector fields in the plane. *Calculus of Variations and Partial Differential Equations*, 9.4 (1999): 327-355.
- [2] ANDRÉ, N., AND SHAFRIR, I. On nematics stabilized by a large external field. *Rev. Math. Phys.* 11, 6 (1999), 653–710.

- [3] AVILES, P., GIGA, Y. A mathematical problem related to the physical theory of liquid crystal configurations. *Miniconference on geometry/partial differential equations*, 2 (Vol. 12, pp. 1-17). Australian National University, Mathematical Sciences Institute.
- [4] BECKNER, W. Weighted inequalities and Stein-Weiss potentials. *Forum Math.* 20, 4 (2008), 587–606.
- [5] BERCHIO, E., GAZZOLA, F., AND WETH, T. Radial symmetry of positive solutions to nonlinear polyharmonic Dirichlet problems. *J. Reine Angew. Math.* 620 (2008), 165–183.
- [6] BERNIS, F., GARCÍA AZORERO, J., AND PERAL, I. Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order. *Adv. Differential Equations* 1, 2 (1996), 219–240.
- [7] BETHUEL, F., BREZIS, H., COLEMAN, B. D., AND HÉLEIN, F. Bifurcation analysis of minimizing harmonic maps describing the equilibrium of nematic phases between cylinders. *Arch. Rational Mech. Anal.* 118, 2 (1992), 149–168.
- [8] BETHUEL, F., BREZIS, H., AND HÉLEIN, F. *Ginzburg-Landau vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.
- [9] BONHEURE, D., CASTERAS, J.-B., GOU, T., AND JEANJEAN, L. Normalized solutions to the mixed dispersion nonlinear Schrödinger equation in the mass critical and supercritical regime. *Trans. Amer. Math. Soc.* 372, 3 (2019), 2167–2212.
- [10] BREZIS, H. Symmetry in nonlinear PDE's. In *Differential equations: La Pietra 1996 (Florence)*, vol. 65 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 1–12.
- [11] BREZIS, H., CORON, J.-M., AND LIEB, E. H. Harmonic maps with defects. *Comm. Math. Phys.* 107, 4 (1986), 649–705.
- [12] CAZACU, C. A new proof of the Hardy-Rellich inequality in any dimension. *Proc. Roy. Soc. Edinburgh Sect. A* 150, 6 (2020), 2894–2904.
- [13] LIN, F.-H. AND CHEN, Y. Remarks on approximate harmonic maps. *Commentarii mathematici Helvetici* 70 (1995), 161–170.
- [14] CONTI, S. AND DE LELLIS, C. Sharp upper bounds for a variational problem with singular perturbation. *Mathematische Annalen* 338, 1 (2007), 119–146.

- [15] DE FIGUEIREDO, D. G., AND SIRAKOV, B. Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems. *Math. Ann.* *333*, 2 (2005), 231–260.
- [16] DEL PINO, M., FELMER, P., AND KOWALCZYK, M. Minimality and nondegeneracy of degree-one Ginzburg-Landau vortex as a Hardy’s type inequality. *Int. Math. Res. Not.*, 30 (2004), 1511–1527.
- [17] DE LELLIS, C. An example in the gradient theory of phase transitions. *ESAIM: Control, Optimisation and Calculus of Variations*, 2002, vol. 7, p. 285-289.
- [18] DE LELLIS, C., AND OTTO, F. Structure of entropy solutions to the eikonal equation. *J. Eur. Math. Soc. (JEMS)* *5*, (2003), 107–145.
- [19] DESIMONE, A., MÜLLER, S., KOHN, R. V., OTTO, F. A compactness result in the gradient theory of phase transitions. *Proceedings of the Royal Society of Edinburgh Section A Mathematics*; 2001, vol. 131, no 4, p. 833-84411 (2002), 1408–1460.
- [20] DESIMONE, A., KOHN, R. V., MÜLLER, S., AND OTTO, F. A reduced theory for thin-film micromagnetics. *Comm. Pure Appl. Math.* *55*, 11 (2002), 1408–1460.
- [21] EDMUNDS, D. E., FORTUNATO, D., AND JANNELLI, E. Critical exponents, critical dimensions and the biharmonic operator. *Arch. Rational Mech. Anal.* *112*, 3 (1990), 269–289.
- [22] FERRERO, A., GAZZOLA, F., AND WETH, T. Positivity, symmetry and uniqueness for minimizers of second-order Sobolev inequalities. *Ann. Mat. Pura Appl. (4)* *186*, 4 (2007), 565–578.
- [23] FRANK, R. L., LENZMANN, E., AND SILVESTRE, L. Uniqueness of radial solutions for the fractional Laplacian. *Comm. Pure Appl. Math.* *69*, 9 (2016), 1671–1726.
- [24] GAZZOLA, F., GRUNAU, H.-C., AND SQUASSINA, M. Existence and nonexistence results for critical growth biharmonic elliptic equations. *Calc. Var. Partial Differential Equations* *18*, 2 (2003), 117–143.
- [25] GAZZOLA, F., GRUNAU, H.-C., AND SWEERS, G. *Polyharmonic boundary value problems*, vol. 1991 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. Positivity preserving and nonlinear higher order elliptic equations in bounded domains.
- [26] GHOUSSOUB, N., AND MORADIFAM, A. Bessel pairs and optimal Hardy and Hardy-Rellich inequalities. *Math. Ann.* *349*, 1 (2011), 1–57.

- [27] GIOIA, G., AND JAMES, R. D. Micromagnetics of very thin films. *Proc. Roy. Soc. London A* 453 (1997), 213–223.
- [28] GRAVEJAT, P., PACHERIE, E., AND SMETS, D. On the stability of the Ginzburg-Landau vortex. *Proc. Lond. Math. Soc. (3)* 125, 5 (2022), 1015–1065.
- [29] GUSTAFSON, S. Symmetric solutions of the Ginzburg-Landau equation in all dimensions. *Internat. Math. Res. Notices*, 16 (1997), 807–816.
- [30] HANG, F. B., AND LIN, F. H. Static theory for planar ferromagnets and antiferromagnets. *Acta Math. Sin. (Engl. Ser.)* 17, 4 (2001), 541–580.
- [31] HERVÉ, R.-M., AND HERVÉ, M. Étude qualitative des solutions réelles d’une équation différentielle liée à l’équation de Ginzburg-Landau. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 11, 4 (1994), 427–440.
- [32] IGNAT, R. A survey of some new results in ferromagnetic thin films. In *Séminaire: Équations aux Dérivées Partielles. 2007–2008*, Sémin. Équ. Dériv. Partielles. École Polytech., Palaiseau, 2009, pp. Exp. No. VI, 21.
- [33] IGNAT, R. AND MONTEIL, A. A De Giorgi type conjecture for minimal solutions to a nonlinear Stokes equation. *Comm. Pure Appl. Math.* 73 (2020), 771–854.
- [34] IGNAT, R., AND NGUYEN, L. Local minimality of \mathbb{R}^n -valued and \mathbb{S}^n -valued Ginzburg–Landau vortex solutions in the unit ball B^N . *Ann. Inst. H. Poincaré Anal. Non Linéaire* 41, 1 (2024), 663–724.
- [35] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. Stability of the vortex defect in the Landau-de Gennes theory for nematic liquid crystals. *C. R. Math. Acad. Sci. Paris* 351, 13-14 (2013), 533–537.
- [36] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. Uniqueness results for an ODE related to a generalized Ginzburg-Landau model for liquid crystals. *SIAM J. Math. Anal.* 46, 5 (2014), 3390–3425.
- [37] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. Stability of the melting hedgehog in the Landau-de Gennes theory of nematic liquid crystals. *Arch. Ration. Mech. Anal.* 215, 2 (2015), 633–673.
- [38] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. Instability of point defects in a two-dimensional nematic liquid crystal model. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33, 4 (2016), 1131–1152.

- [39] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. Uniqueness of degree-one Ginzburg-Landau vortex in the unit ball in dimensions $N \geq 7$. *C. R. Math. Acad. Sci. Paris* 356, 9 (2018), 922–926.
- [40] IGNAT, R., NGUYEN, L., SLASTIKOV, V., AND ZARNESCU, A. On the uniqueness of minimisers of Ginzburg-Landau functionals. *Ann. Sci. Éc. Norm. Supér. (4)* 53, 3 (2020), 589–613.
- [41] IGNAT, R., AND RUS, M. Vortex sheet solutions for the Ginzburg–Landau system in cylinders: symmetry and global minimality. *Calc. Var. Partial Differential Equations*. 63, 1 (2024), 22pp.
- [42] JÄGER, W., AND KAUL, H. Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems. *J. Reine Angew. Math.* 343 (1983), 146–161.
- [43] JIN, W., KOHN, R. Singular perturbation and the energy of folds. *Journal of Nonlinear Science*, 2000, vol. 10, p. 355–390.
- [44] LAMY, X., AND MARCONI, E. Stability of the vortex in micromagnetics and related models. *Ann. Sc. Norm. Super. Pisa Cl. Sci. in press*, https://doi.org/10.2422/2036-2145.202209_012 (2023).
- [45] LENZMANN, E., AND SOK, J. A sharp rearrangement principle in Fourier space and symmetry results for PDEs with arbitrary order. *Int. Math. Res. Not. IMRN*, 19 (2021), 15040–15081.
- [46] LI, X., AND MELCHER, C. Stability of axisymmetric chiral skyrmions. *J. Funct. Anal.* 275, 10 (2018), 2817–2844.
- [47] LIEB, E. H., AND LOSS, M. Symmetry of the Ginzburg-Landau minimizer in a disc. In *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1995)*. École Polytech., Palaiseau, 1995, pp. Exp. No. XVIII, 12.
- [48] LIN, F.-H. A remark on the map $x/|x|$. *C. R. Acad. Sci. Paris Sér. I Math.* 305, 12 (1987), 529–531.
- [49] LORENT, A. A simple proof of the characterization of functions of low Aviles Giga energy on a ball *via* regularity. *ESAIM Control Optim. Calc. Var.* 18, 2 (2012), 383–400.
- [50] LORENT, A. A quantitative characterisation of functions with low Aviles Giga energy on convex domains. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 13, 1 (2014), 1–66.

- [51] MILLOT, V., AND PISANTE, A. Symmetry of local minimizers for the three-dimensional Ginzburg-Landau functional. *J. Eur. Math. Soc. (JEMS)* 12, 5 (2010), 1069–1096.
- [52] MIRONESCU, P. On the stability of radial solutions of the Ginzburg-Landau equation. *J. Funct. Anal.* 130, 2 (1995), 334–344.
- [53] MIRONESCU, P. Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symétrie radiale. *C. R. Acad. Sci. Paris Sér. I Math.* 323, 6 (1996), 593–598.
- [54] NAKAJIMA, T. Singular points of harmonic maps from 4-dimensional domains into 3-spheres. *Duke Math. J.* 131 (1) (2006), 531–543.
- [55] NADIRASHVILI, N. S. Rayleigh's conjecture on the principal frequency of the clamped plate. *Archive for rational mechanics and analysis* 129, 1 (1995), 1–10.
- [56] OVCHINNIKOV, Y. N., AND SIGAL, I. M. Ginzburg-Landau equation. I. Static vortices. In *Partial differential equations and their applications (Toronto, ON, 1995)*, vol. 12 of *CRM Proc. Lecture Notes*. Amer. Math. Soc., Providence, RI, 1997, pp. 199–220.
- [57] PACARD, F., AND RIVIÈRE, T. *Linear and nonlinear aspects of vortices*, vol. 39 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2000. The Ginzburg-Landau model.
- [58] PISANTE, A. Two results on the equivariant Ginzburg-Landau vortex in arbitrary dimension. *J. Funct. Anal.* 260, 3 (2011), 892–905.
- [59] POLIAKOVSKY, A. Upper bounds for singular perturbation problems involving gradient fields. *J. Eur. Math. Soc. (JEMS)* 9 (2007), 1–43.
- [60] SANDIER, E., AND SHAFRIR, I. On the uniqueness of minimizing harmonic maps to a closed hemisphere. *Calc. Var. Partial Differential Equations* 2, 1 (1994), 113–122.
- [61] SCHOEN, R., UHLENBECK, K. Regularity of minimizing harmonic maps into the sphere. *Inventiones mathematicae* 78 (1984), 89–100.
- [62] TALENTI, G. On the first eigenvalue of the clamped plate. *Annali di Matematica Pura ed Applicata* 129 (1981), 265–280.
- [63] TERTIKAS, A., AND ZOGRAPHOPOULOS, N. B. Best constants in the Hardy-Rellich inequalities and related improvements. *Adv. Math.* 209, 2 (2007), 407–459.

- [64] TRÈVES, F. *Basic linear partial differential equations*. Pure and Applied Mathematics, Vol. 62. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.