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# GEHRING'S LEMMA FOR KINETIC FOKKER-PLANCK EQUATIONS

JESSICA GUERAND, CYRIL IMBERT, AND CLÉMENT MOUHOT

ABSTRACT. In this article, we establish a “Gehring lemma” for a real function satisfying a reverse Hölder inequality on all “kinetic cylinders” contained in a large one: it asserts that the integrability degree of the function improves under such an assumption. The kinetic cylinders are derived from the non-commutative group of invariances of the Kolmogorov equation. Our contributions here are (1) the extension of Gehring’s Lemma to this kinetic (hypoelliptic) scaling used to generate the cylinders, (2) the localisation of the lemma in this hypoelliptic context (using ideas from the elliptic theory), (3) the streamlining of a short and quantitative proof. We then use this lemma to establish that the velocity gradient of weak solutions to linear kinetic equations of Fokker-Planck type with rough coefficients have Lebesgue integrability strictly greater than two, while the natural energy estimate merely ensures that it is square integrable. Our argument here is new but relies on Poincaré-type inequalities established in previous works.

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## 1. INTRODUCTION

In this note, we are interested in establishing a version of Gehring’s Lemma in the context of kinetic equations of Fokker-Planck type. Such equations write

$$(1.1) \quad (\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot (\mathcal{A} \nabla_v f) + \mathcal{B} \cdot \nabla_v f + S, \quad (t, x, v) \in Q$$

for  $0 < \lambda < \Lambda$ , some function  $\mathcal{A}$  valued in the set of real symmetric matrices whose eigenvalues lie in a given interval  $[\lambda, \Lambda]$ , some real-valued vector field  $\mathcal{B}$  bounded in  $Q$  (i.e.  $\|\mathcal{B}\|_{L^\infty(Q)} \leq \Lambda$ ) and some real-valued source term  $S \in L^2(Q)$ . The unknown function  $f$  is also real-valued and the equation is posed in some cylinder  $Q \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ .

A “Gehring Lemma” roughly says that a function  $f \in L^q(B)$  for some ball  $B$  enjoys in fact better Lebesgue integrability as soon as the averaged  $L^q$ -norm of  $f$  in any ball  $B' \subset B$  is controlled by the averaged  $L^1$ -norm of  $f$  in  $B'$ , with uniform constant. It was first proved by Gehring [Geh73] motivated by open problems related to quasiconformal mappings, and then adapted and used by various authors to study gain integrability on the gradient of solutions of elliptic equations, see for instance the following early contributions: [ME75, GM79]. In order to establish a similar result for solutions of parabolic equations [GS82, Ark97, AL99], it is necessary to extend Gehring’s Lemma by replacing Euclidian balls with parabolic cylinders:  $Q_r(t, v) = (t-r^2, t] \times B_r(v)$  which do respect the invariances of such parabolic equations: the translations in  $(t, v)$  and the parabolic scalings  $(t, v) \mapsto (r^2 t, rv)$  for  $r > 0$ .

The class of kinetic Fokker-Planck equations (1.1) is invariant by the slightly more complicated scalings  $r \mapsto (r^2 t, r^3 x, r v)$  for  $r > 0$ . It is preserved by translations in  $(t, v)$  but not in  $x$ , instead it enjoys Galilean invariance: given  $z_0 = (t_0, x_0, v_0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  and  $z = (t, x, v)$ , we define

$$z_0 \circ z = (t_0 + t, x_0 + x + t v_0, v_0 + v).$$

If  $f$  solves (1.1), then  $f(z_0 \circ z)$  solves an equation of the same form (in particular, the eigenvalues of the diffusion matrices still lie in  $[\lambda, \Lambda]$ ). Because of these two invariances, it is natural to define the so-called *kinetic cylinders* by  $Q_r(z_0) = z_0 \circ Q_r$  with  $Q_r = (-r^2, 0] \times B_{r^3} \times B_r$ . More explicitly,

$$Q_r(z_0) = \{z = (t, x, v) : -r^2 < t - t_0 \leq 0, |x - x_0 - (t - t_0)v_0| < r^3, |v - v_0| < r\}.$$

Our first main result is a Gehring Lemma extended to this setting. For a kinetic cylinder  $Q$  and a function  $F \in L^1(Q)$ , we let  $\int_Q F$  denote  $|Q|^{-1} \int_Q F$  where  $|Q|$  stands for the volume of  $Q$ .

**Theorem 1** (Localised Gehring's Lemma with the kinetic scalings). *Let  $\gamma > 1$ . Consider non-negative functions  $g \in L^q(Q_\gamma)$  and  $h \in L^\sigma(Q_\gamma)$  for some  $\sigma > q > 1$ . Assume that there is  $b > 1$  and  $\theta \in (0, 1)$  so that for all  $R > 0$  and  $z_0 \in Q_\gamma$  such that  $Q_{\gamma R}(z_0) \subset Q_\gamma$*

$$(1.2) \quad \int_{Q_R(z_0)} g^q \leq b \left\{ \left( \int_{Q_{\gamma R}(z_0)} g \right)^q + \int_{Q_{\gamma R}(z_0)} h^q \right\} + \theta \int_{Q_{\gamma R}(z_0)} g^q.$$

If  $\theta < \theta_0 := [2(75)^{4d+2} 4^q]^{-1}$  and  $q < p^* := \min\left(\sigma, \frac{bq - \theta_0}{b - \theta_0}\right)$  then  $g \in L^p(Q)$  for  $p \in [q, p^*]$ , and this norm is controlled by

$$(1.3) \quad \left( \int_{Q_1} g^p \right)^{1/p} \leq C_G \left( \left( \int_{Q_\gamma} g^q \right)^{1/q} + \left( \int_{Q_\gamma} h^p \right)^{1/p} \right)$$

where  $C_G$  depends only on  $b, \theta, p, q, \gamma, d$ .

Our second main result is an application of the previous lemma to kinetic equations of Fokker-Planck type, and proves the gain of integrability on the velocity gradient of their solutions.

**Theorem 2** (Gain of integrability of the velocity gradient). *Let  $f$  be a weak solution to (1.1) in  $Q$  with  $S \in L^{2+\varepsilon_0}(Q)$  for some  $\varepsilon_0 > 0$ . Let  $\gamma > 1$  and  $R > 0$ . There exist two constants  $\varepsilon \in (0, \varepsilon_0)$  and  $C > 0$ , both depending on  $d, \lambda, \Lambda, \gamma, R, \varepsilon_0$ , such that, for all  $z \in Q$  such that  $Q_{\gamma R}(z) \subset Q$ , we have,*

$$\|\nabla_v f\|_{L^{2+\varepsilon}(Q_R(z))} \leq C \left( \|\nabla_v f\|_{L^2(Q_{\gamma R}(z))} + \|S\|_{L^{2+\varepsilon}(Q_{\gamma R}(z))} \right).$$

*Remark 3.* Weak solutions of (1.1) are defined in Section 2 (see Definition 7).

**1.1. Comments on the main results and their proofs.** Theorem 1 asserts that an  $L^q$  function in a kinetic cylinder  $Q_\gamma$  with  $\gamma > 1$  is in fact in  $L^p$  with  $p > q$  in the smaller kinetic cylinder  $Q_1$  as soon as a family of reverse Hölder inequalities hold true at every scales, where the scales used stem from the invariances of the Kolomogorov equation. It is an *interior* regularity estimate since, on the one hand, reverse Hölder inequalities relate Lebesgue norms in interior cylinders  $Q_R(z)$  with norms in larger cylinders  $Q_{\gamma R}(z)$ , and on the other hand the improvement only holds in  $Q_1 \subset Q_\gamma$ .

The global strategy of the original proof in [Geh73] is to *transfer* reverse Hölder inequalities from cylinders, whose geometry is independent of the function  $g$ , onto the superlevel sets of  $g$ . This reduces then the proof to the case of dimension  $d = 1$ , and one concludes by explicit calculations. However, this strategy works in the whole space, and the first step of the proof is to reduce to the case where the function  $g$  is defined on the whole space  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ . We do so thanks to a *localization procedure*, following an idea appearing in [Iwa98] in the case of Gehring's Lemma for elliptic equations, i.e. for the standard Euclidean scalings.

In order to transfer the reverse Hölder inequalities from the function to its superlevel sets, we use a standard covering argument. This requires an extension of the classical Vitali lemma and a differentiation theorem à

la Lebesgue for kinetic cylinders. Both results are already proved in [IS20, Lemma 10.5 & Theorem 10.3]: we recall their statements in Section 2 and refer to [IS20] for their proofs.

The Gehring lemma is often used to obtain a gain of integrability on the gradients of solutions to elliptic or parabolic equations (see below). It naturally starts with an estimate of Cacciopoli type, *i.e.* the basic local energy estimate for such equations. It relates the gradient of the solution with its  $L^2$  norm, say. In order to derive a local reverse Hölder inequality for the gradient, it is thus necessary to estimate from above the  $L^2$  norm of the solution with the  $L^q$  norm of the gradient for some  $q \in [1, 2)$ . In order to do so, one typically uses a gain of integrability on the solutions (not their gradient). In the simplest case of elliptic equations, the gain of integration on solutions follows from the energy estimate and the Sobolev embedding, and the control of the  $L^q$  norm of the solutions by that of their gradient follows from the Poincaré-Wirtinger inequality.

Several ingredients are needed in order to extend this strategy of proof from the elliptic setting to kinetic equations of Fokker-Planck type. Cacciopoli estimate and the gain of integrability were already proved in [PP04]. They play a key role in the proof of the Harnack inequality in [GIMV19]. The other ingredient we need for applying Gehring’s Lemma is an inequality of Poincaré type in  $L^q$  with  $q \in (1, 2)$ . The difficulty lies in the fact that only the gradient in the velocity variable should appear in the right hand side of the desired inequality. Several Poincaré inequalities were proved in the context of kinetic equations. Such an estimate is key in the derivation of the local Hölder estimate in [WZ09]. A more accurate Poincaré inequality was derived in [AAMN21]. We largely follow [AAMN21] in order to derive the Poincaré inequality we need. The reader is also referred to [GM22, GI23] for other Poincaré inequalities in the context of kinetic Fokker-Planck equations, and [DHHM23] for their extensions in the case of boundary conditions.

We note that the gain of integrability of the velocity gradient of solutions to kinetic Fokker-Planck equations was announced in [GIMV19]. However it was based on a Gehring Lemma stated with other type of cylinders and, as a consequence, the proof for the estimate on the velocity gradient was incomplete.

**1.2. Short review of literature.** Gehring first discovered in [Geh73] the fact that the integrability of a function improves when it satisfies a family of reverse Hölder inequalities. It allowed him to unlock the study of regularity of so-called quasiconformal mappings for which reverse Hölder inequalities were known to hold true. This finding turned out to be a powerful tool in the study of regularity of both (quasi)-minimizers of functionals and solutions to partial differential equations. Many authors established variants of the original Gehring lemma in different contexts. In [GM79], a local version of the Gehring lemma was obtained. In [Mod85, Kin94], the authors added to the Lebesgue measure a weight that satisfies a doubling condition. In [Iwa98], the Lebesgue spaces are replaced by Orlicz spaces (see also [GZ12, NW18]).

The other direction in which Gehring’s Lemma can be extended, which is the focus of this paper, is when the reverse Hölder inequalities are assumed in cylinders that are not necessarily Euclidian balls. Motivated by the study of hypoelliptic equations involving square Hörmander operators of type A, *i.e.* of the “sum of squares” form  $\sum_i X_i X_i^*$ , a lemma of Gehring type is proved in this direction in [Gia94] proved a Gehring lemma when the reversed Hölder inequalities are assumed in “balls” defined by the pseudo-distance created by the vector fields  $X_i$ . We note that, in fact, instead of considering a family of reverse Hölder inequalities, the sufficient condition for getting an improvement of integrability can be expressed in terms of two maximal functions associated with those balls. Even more general operators can be handled by considering reverse Hölder inequalities on so-called metric balls [FSC96] (see also [ZG05]). However this class of operators does not include (1.1), which is in fact the prototypical case of Hörmander operators of type B, due to the presence of the first order transport operator. Such kinetic equation is moreover similar to the Kolmogorov equation [Kol34] that was the initial motivation of Hörmander’s seminal paper [H67].

In [Ark97, AL99], the authors are interested in the regularity of solutions to parabolic quasilinear systems of equations. They show that it is possible to derive reverse Hölder inequalities for the gradients of such solutions if Euclidian balls are replaced by parabolic cylinders. They show that this family of inequalities yields again an improvement of the integrability of these gradient functions.

More recently, non-local versions of the Gehring lemma were derived [KMS14, KMS15]. Local reverse Hölder inequalities are replaced by estimates of Cacciopoli type for linear integro-differential equations with kernels comparable with the one of the fractional Laplacian. A short proof using complex interpolation theory were recently given in [ABES20]. We note that *discrete* versions of the Gehring lemma have also been devised, see for instance [SKP19] and the references therein.

We conclude this short review of literature by mentioning the existence of the nice survey article [Iwa98] about the Gehring lemma. As mentioned above, we borrowed from this work the idea of using a cut-off function in the localization process in the proof of Theorem 1 (see the function  $\zeta$  from Subsection 3.1).

**1.3. Organization and notation.** We gather in Section 2 known results that are needed in the proofs of the main results. Section 3 is devoted to the proof of Gehring's Lemma (Theorem 1) while Section 4 contains the proof of the gain of integrability for the velocity gradient (Theorem 2). The variable  $z$  denotes  $(t, x, v)$ . The open Euclidian ball centered at the origin of radius  $r > 0$  is denoted by  $B_r$ . If the center is  $v_0$ , then it is denoted by  $B_r(v_0)$ . Cylinders centered at the origin are denoted by  $Q_r := (-r^2, 0] \times B_{r^3} \times B_r$ , and  $Q_r(z_0)$  denotes  $z_0 \circ Q_r$ .

## 2. PRELIMINARIES

**2.1. The Vitali lemma.** We first state and prove a Vitali lemma for kinetic cylinders similar to the one in [IS20, Lemma 10.5] but whose definition is slightly different. To state this covering result, we need to introduce those kinetic cylinders that are scaled by a factor 5 and shifted in time (note that this definition is *not* simply the hypoelliptic scaling by a factor 5)

$$(2.1) \quad 5Q_R(z_0) = Q_{5R} \left( z_0 \circ \left( \tau_{t_0, R}, 0, 0 \right) \right) \quad \text{with } \tau_{t_0, R} := \min(-t_0, 12R^2).$$

More explicitly, we have

$$5Q_R(z_0) = \left\{ (t, x, v) : -25R^2 + \tau_{t_0, R} < t - t_0 \leq \tau_{t_0, R}, |x - x_0 - (t - t_0)v_0| < (5R)^3, |v - v_0| < 5R \right\}.$$

The shift in time ensures that  $z_0$  lies in the interior of  $5Q_R(z_0)$  unless  $t_0 = 0$ . It is necessary to consider such a shift in order to get a covering à la Vitali.

**Lemma 4 (Vitali lemma).** *Let  $(Q_j)_{j \in J}$  be a collection of kinetic cylinders in  $(-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$  with bounded radii. Then there exists a disjoint countable subcollection  $(Q_i)_{i \in I}$  where  $I \subset J$  such that*

$$\bigcup_{j \in J} Q_j \subset \bigcup_{i \in I} 5Q_i,$$

where  $5Q_i$  is defined in (2.1).

*Proof.* Vitali lemma applies thanks to the following property for  $z_1 = (t_1, x_1, v_1)$  and  $z_2 = (t_2, x_2, v_2)$  with  $t_1, t_2 \leq 0$  and  $r_1, r_2 > 0$

$$Q_{r_1}(z_1) \cap Q_{r_2}(z_2) \neq \emptyset \quad \text{and } r_1 \leq 2r_2 \quad \Rightarrow \quad Q_{r_1}(z_1) \subset Q_{5r_2}(z_2 \circ (\tau_{t_2, r_2}, 0, 0)).$$

Take  $z_0 = (t_0, x_0, v_0)$  in the intersection and  $z = (t, x, v) \in Q_{r_1}(z_1)$ . We justify that  $z \in Q_{5r_2}(z_2 \circ (\tau_{t_2, r_2}, 0, 0))$  thanks to the following inequalities. Inequality  $|v - v_2| \leq 5r_2$  comes naturally. Then since  $t \leq 0$ ,  $t - t_2 \leq -t_2$  and  $t - t_2 = (t - t_1) + (t_1 - t_0) + (t_0 - t_2) \leq r_1^2 \leq 4r_2^2$  so that  $t - t_2 \leq \tau_{t_2, r_2}$ . Moreover,  $t - t_2 \geq -r_1^2 - r_2^2 \geq -5r_2^2 \geq -25r_2^2 + \tau_{t_2, r_2}$ . Finally

$$\begin{aligned} |x - x_2 - (t - t_2)v_2| &\leq |x - x_1 - (t - t_1)v_1| + |x_1 - x_0 + (t_0 - t_1)v_1| + |x_0 - x_2 - (t_0 - t_2)v_2| \\ &\quad + (|t - t_1| + |t_1 - t_0|)(|v_1 - v_0| + |v_0 - v_2|) \leq 41r_2^3 \leq (5r_2)^3, \end{aligned}$$

which concludes the proof. □

**2.2. Lebesgue's differentiation.** Then we recall Lebesgue's differentiation theorem in the context of kinetic cylinders, see [IS20, Theorem 10.3].

**Lemma 5** (Lebesgue differentiation). *Let  $f \in L^1(\Omega)$  where  $\Omega$  is an open set of  $\mathbb{R}^{2d+1}$ . Then for almost every  $z = (t, x, v) \in \Omega$ ,*

$$\lim_{r \rightarrow 0^+} \int_{Q_r(z_r)} |f(z') - f(z)| dt' dx' dv' = 0$$

where the sequences  $z_r \in \Omega$  and  $r > 0$  are so that  $z \in Q_r(z_r) \subset \Omega$  for all  $r > 0$ .

*Remark 6.* In [IS20, Theorem 10.3], the result is stated for  $z_r = z$ . The proof follows the classical argument: with a Vitali covering lemma at hand, prove first a weak  $L^1$  inequality for a maximal function and then argue by approximation with continuous functions. This proof does not use the fact that  $z_r = z$ .

**2.3. Weak solutions.** There are several notions of weak solutions for kinetic Fokker-Planck equations (1.1). The weakest notion appears in [AIN24], it supersedes for instance the notion introduced in [GIMV19, GM22, GI23]. It is stated for  $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$  but it can be localized by multiplying by a cut-off function. We also give the definition in the inhomogeneous setting ( $L^2_{t,x} H^1_v$  instead of  $L^2_{t,x} \dot{H}^1_v$ ) in order to remove the restriction  $1 < d/2$  – see [AIN24, § 7.2].

**Definition 7** (Weak solutions – [AIN24]). Let  $\mathcal{U} = (a, b) \times \Omega_x \times \Omega_v$  with  $-\infty < a < b \leq +\infty$  and  $\Omega_x, \Omega_v$  two open sets of  $\mathbb{R}^d$ . A function  $f : \mathcal{U} \rightarrow \mathbb{R}$  is a *weak solution* of (1.1) if  $f \in L^2((a, b) \times \Omega_x, H^1(\Omega_v))$  and if  $f$  satisfies (1.1) in the sense of distributions in  $\mathcal{U}$ .

**2.4. Energy estimate.** We next recall the natural energy estimate associated with Eq. (1.1). The reader is referred to [PP04, GIMV19] or [GM22, Proposition 9] for a proof.

**Proposition 8** (Energy estimate). *Let  $f$  be a weak sub-solution of (1.1) in  $Q$ . Then for all cylinders  $Q_r(z_0)$  and  $Q_R(z_0)$  with  $0 < r < R$  such that  $Q_R(z_0) \subset Q$ ,*

$$\int_{Q_r(z_0)} |\nabla_v f|^2 \leq C \left( \bar{C} \int_{Q_R(z_0)} f^2 + \int_{Q_R(z_0)} S^2 \right)$$

where  $Q_r^t(z_0) = \{(x, v) \in \mathbb{R}^2 : (t, x, v) \in Q_r(z_0)\}$  and  $C$  only depends on the dimension  $d$ ,  $\lambda$  and  $\Lambda$ , and  $\bar{C}$  only depends on  $R$  and  $r$ .

**2.5. Gain of integrability of the solution.** In this subsection, we recall a result of gain of integrability of the solution [GM22, Proposition 11] with  $p \in [2, 2 + 1/d)$  which is a tool for the proof of Theorem 2.

**Proposition 9** (Gain of integrability of the solution). *Let  $f$  be a weak solution of (1.1) in  $Q$ . Then for all cylinders  $Q_r(z_0)$  and  $Q_R(z_0)$  with  $0 < r < R$ ,  $z_0 \in Q$  such that  $Q_R(z_0) \subset Q$ ,  $f$  satisfies*

$$\|f\|_{L^p(Q_r(z_0))}^2 \leq C \left( \bar{C}^2 \|f\|_{L^2(Q_R(z_0))}^2 + \bar{C} \|S\|_{L^2(Q_R(z_0))}^2 \right),$$

where  $p = 2 + \frac{1}{d} > 2$ ,  $C$  only depends on the  $d$ ,  $p$ ,  $\lambda$ ,  $\Lambda$  and  $\bar{C}$  only depends on  $R, r$  and  $v_0$ .

### 3. PROOF OF GEHRING'S LEMMA

**3.1. Localization.** In order to deal with the boundary of  $Q_\gamma$ , we use a localisation function in the spirit of Iwaniec's work [Iwa98] for the elliptic case. We define for  $z = (t, x, v) \in \mathbb{R}^{1+2d}$ ,

$$\zeta(t, x, v) = \frac{1}{2} \min \left( \frac{(\gamma - |v|)_+}{5}, \left( \frac{(\gamma^2 + t)_+}{13} \right)^{1/2}, \left( \frac{(\gamma^3 - |x|)_+}{25\gamma} \right)^{1/2} \right).$$

Let us first give a useful estimate on  $\zeta$  in  $Q_r(z_0)$ .

**Lemma 10** (Estimate on  $\zeta$ ). *If  $z \in Q_r(z_0) \subset Q_\gamma$  for  $r > 0$  and  $z_0 \in Q_\gamma$  then  $|\zeta(z) - \zeta(z_0)| \leq \frac{r}{2}$ .*

*Proof.* The idea guiding the calculation is that  $\zeta$  increases with the distance to the parabolic boundary  $\partial_p Q_\gamma$  of  $Q_\gamma$ . Hence, when  $z_0$  is not too far from  $\partial_p Q_\gamma$  and since  $z$  is close to  $z_0$ , so is  $z$ . Here is the detailed calculation for  $z = (t, x, v)$ :

$$\begin{aligned} \frac{(\gamma - |v|)_+}{5} &\leq \frac{(\gamma - |v_0|)_+}{5} + \frac{r}{5} \\ \left( \frac{(\gamma^2 + t)_+}{13} \right)^{\frac{1}{2}} &\leq \left( \frac{(\gamma^2 + t_0)_+ + r^2}{13} \right)^{\frac{1}{2}} \leq \left( \frac{(\gamma^2 + t_0)_+}{13} \right)^{\frac{1}{2}} + \frac{r}{\sqrt{13}} \end{aligned}$$

and finally

$$\begin{aligned} \left( \frac{(\gamma^3 - |x|)_+}{25\gamma} \right)^{\frac{1}{2}} &\leq \left( \frac{(\gamma^3 - |x_0|)_+}{25\gamma} + \frac{r^3 + |v_0|r^2}{25\gamma} \right)^{\frac{1}{2}} \\ &\leq \left( \frac{(\gamma^3 - |x_0|)_+}{25\gamma} + \frac{2r^2}{25} \right)^{\frac{1}{2}} \quad (\text{we used that } |v_0| \leq \gamma \text{ and } r \leq \gamma) \\ &\leq \left( \frac{(\gamma^3 - |x_0|)_+}{25\gamma} \right)^{\frac{1}{2}} + \frac{\sqrt{2}}{5}r, \end{aligned}$$

which gives,

$$\zeta(z) \leq \zeta(z_0) + \frac{r}{5\sqrt{2}}.$$

A similar computation gives  $\zeta(z) \geq \zeta(z_0) - \frac{r}{5\sqrt{2}}$ . □

**Lemma 11** (Shifted and scaled cylinders). *If  $z_0 = (t_0, x_0, v_0) \in Q_\gamma$  and  $r \leq 2\zeta(z_0)$ , then  $5Q_r(z_0) \subset Q_\gamma$ .*

*Proof.* Let  $r \leq \zeta_0 := 2\zeta(z_0)$ . The condition  $z \in 5Q_r(z_0)$  is equivalent to,

$$\begin{cases} |v - v_0| < 5r, \\ -25r^2 + \tau_{t_0, r} < t - t_0 \leq \tau_{t_0, r}, \\ |x - x_0 - (t - t_0)v_0| < (5r)^3. \end{cases}$$

Then we have first  $|v| < |v_0| + 5r$  and the condition  $r \leq \zeta_0 \leq (\gamma - |v_0|)_+/5$  implies  $|v| < \gamma$ . Second we have  $t \leq t_0 + \min(-t_0, 12r^2)$  which implies  $t \leq 0$ .

Third we have  $t > t_0 - 25r^2 + \min(-t_0, 12r^2)$  and the condition  $r^2 \leq \zeta_0^2 \leq \min((\gamma^2 + t_0)/13, \gamma^2/25)$  implies  $t > -\gamma^2$ . Indeed, if  $-t_0 \leq 12r^2$ , then  $t > -25r^2 \geq -\gamma^2$ . And if  $-t_0 \geq 12r^2$ , then  $t > t_0 - 13r^2 \geq -\gamma^2$ . Remark that this implies that  $|t - t_0| < 25r^2$ .

Fourth and last we have  $|x| < |x_0| + |t - t_0||v_0| \leq |x_0| + 25r^2\gamma$  and the condition  $r^2 \leq \zeta_0^2 \leq (\gamma^3 - |x_0|)_+/(25\gamma)$  implies  $|x| < \gamma^3$ . Hence the conclusion  $5Q_r(z_0) \subset Q_\gamma$  holds. □

With such a localization function and such cylinders at hand, we scale the functions  $g$  and  $h$ .

**Lemma 12** (Localization). *Given  $\gamma > 1$  and non-negative functions  $g \in L^q(Q_\gamma)$  and  $h \in L^\sigma(Q_\gamma)$  for some  $\sigma > q > 1$  such that there is  $b > 1$  and  $\theta \in (0, 1)$  so that for all  $R > 0$  and  $Q_{\gamma R}(z_0) \subset Q_\gamma$  the inequality (1.2) holds. Let  $\mathbf{g}$  and  $\mathbf{h}$  be defined on  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  by the following formulas,*

$$\mathbf{g}(z) := \frac{|Q_{\zeta(z)}|^{\frac{1}{q}}}{C_0} \frac{g(z)}{\|g\|_{L^q(Q_\gamma)}}, \quad \mathbf{h}(z) := \frac{|Q_{\zeta(z)}|^{\frac{1}{q}}}{C_0} \frac{h(z)}{\|g\|_{L^q(Q_\gamma)}}$$

with  $C_0 = 2^{\frac{4d+2}{q}}$ .

(i) The function  $\mathbf{g}$  satisfies, for all  $z_0 \in \mathcal{Q}_\gamma$  and  $r > (2/3)\zeta(z_0)$  such that  $\mathcal{Q}_r(z_0) \subset \mathcal{Q}_\gamma$ ,

$$(3.1) \quad \int_{\mathcal{Q}_r(z_0)} \mathbf{g}^q \leq 1.$$

(ii) The functions  $\mathbf{g}$  and  $\mathbf{h}$  satisfy, for all  $z_0 = (t_0, x_0, v_0) \in \mathcal{Q}_\gamma$  and  $r \in (0, \zeta(z_0))$ ,

$$(3.2) \quad \int_{\mathcal{Q}_r(z_0)} \mathbf{g}^q \leq \bar{b} \left\{ \left( \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{g} \right)^q + \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{h}^q \right\} + \bar{\theta} \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{g}^q$$

with  $\bar{b} = Cb$  and  $\bar{\theta} = C\theta$  with  $C = 3^{4d+2}$ .

*Proof.* To prove (i), we estimate the mean of  $\mathbf{g}^q$  on  $\mathcal{Q}_r(z_0)$  with  $r > (2/3)\zeta(z_0)$ ,

$$\begin{aligned} \int_{\mathcal{Q}_r(z_0)} \mathbf{g}^q &= \frac{1}{C_0^q} \frac{1}{\|g\|_{L^q(\mathcal{Q}_\gamma)}^q} \int_{\mathcal{Q}_r(z_0)} \frac{|\mathcal{Q}_{\zeta(z)}|}{|\mathcal{Q}_r(z_0)|} g(z)^q dz \\ &= \frac{1}{\|g\|_{L^q(\mathcal{Q}_\gamma)}^q} \int_{\mathcal{Q}_r(z_0)} \frac{1}{C_0^q} \left( \frac{\zeta(z)}{r} \right)^{4d+2} g(z)^q dz. \end{aligned}$$

in order to obtain (3.1), it is enough to check that we can choose  $C_0 > 0$  such that

$$\forall z \in \mathcal{Q}_r(z_0), \quad \frac{1}{C_0^q} \left( \frac{\zeta(z)}{r} \right)^{4d+2} \leq 1.$$

If we pick  $C_0 = 2^{\frac{4d+2}{q}}$ , the previous inequality holds true as soon as

$$(3.3) \quad \forall z \in \mathcal{Q}_r(z_0), \quad \zeta(z) \leq 2r.$$

It is a consequence of the fact that  $r > (2/3)\zeta(z_0)$ ,  $z \in \mathcal{Q}_r(z_0)$  and Lemma 10 since it implies that for all  $z \in \mathcal{Q}_r(z_0) \subset \mathcal{Q}_\gamma$ , we have  $\zeta(z) \leq \zeta(z_0) + r/2$ .

We now check (3.2). We first remark thanks to Lemma 10 that for  $z \in \mathcal{Q}_r(z_0)$  when  $r \leq \zeta(z_0)$ , we get  $\frac{\zeta(z_0)}{2} \leq \zeta(z) \leq \frac{3\zeta(z_0)}{2}$ . This implies

$$(3.4) \quad C_0^q \|g\|_{L^q(\mathcal{Q}_\gamma)}^q \int_{\mathcal{Q}_r(z_0)} \mathbf{g}^q \leq \left( \frac{3}{2} \right)^{4d+2} |\mathcal{Q}_{\zeta(z_0)}| \int_{\mathcal{Q}_r(z_0)} \mathbf{g}^q,$$

$$(3.5) \quad \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{g}^q \leq 2^{4d+2} \|g\|_{L^q(\mathcal{Q}_\gamma)}^q \frac{C_0^q}{|\mathcal{Q}_{\zeta(z_0)}|} \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{g}^q.$$

The same computation yields,

$$(3.6) \quad \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{h}^q \leq 2^{4d+2} \|g\|_{L^q(\mathcal{Q}_\gamma)}^q \frac{C_0^q}{|\mathcal{Q}_{\zeta(z_0)}|} \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{h}^q.$$

We next estimate the mean of  $g$ ,

$$(3.7) \quad \left( \int_{\mathcal{Q}_{\gamma r}(z_0)} g \right)^q \leq 2^{4d+2} \|g\|_{L^q(\mathcal{Q}_\gamma)}^q \frac{C_0^q}{|\mathcal{Q}_{\zeta(z_0)}|} \left( \int_{\mathcal{Q}_{\gamma r}(z_0)} \mathbf{g} \right)^q.$$

Combining (1.2) with (3.4), (3.5), (3.6) and (3.7) yields (3.2) with  $\bar{b} = Cb$  and  $\bar{\theta} = C\theta$  and  $C = 3^{4d+2}$ .  $\square$



### 3.2. Covering of the superlevel sets of $\mathbf{g}$ .

**Lemma 13** (Covering of superlevel sets). *Given  $\gamma > 1$  and a non-negative function  $g \in L^q(Q_\gamma)$  for some  $q > 1$  and  $\mathbf{g}$  be defined on  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  from  $g$  as in Lemma 12. Let  $s > 1$  and consider the union  $\mathcal{G}$  of all cylinders  $Q_i \subset Q_\gamma$  such that  $Q_i = Q_r(z_0)$  with  $z_0 \in Q_\gamma$  and  $r \leq \zeta(z_0)$  and such that,*

$$(3.8) \quad \int_{5Q_i} \mathbf{g}^q < s^q \leq \int_{Q_i} \mathbf{g}^q.$$

Then we have:  $\{\mathbf{g} > s\} \cap Q_\gamma \subset \mathcal{G}$ , up to a set of measure zero.

*Proof.* We notice that both sets  $\{\mathbf{g} > s\} \cap Q_\gamma$  and  $\mathcal{G}$  are contained in  $Q_\gamma$ . We prove that  $\mathbf{g}(z) \leq s$  for almost all  $z = (t, x, v) \in Q_\gamma \setminus \mathcal{G}$ .

We pick  $z_r$  in such a way that  $z \in Q_r(z_r)$  and  $5Q_r(z_r) = Q_{5r}(z_{5r})$ . It is sufficient to consider  $z_r = z_0(\min(-t, r^2/2), 0, 0)$ . Moreover  $z_r \in Q_\gamma$  and  $Q_r(z_r) \subset Q_\gamma$  for  $r > 0$  small enough.

Thanks to the Lebesgue differentiation result, Lemma 5, it is sufficient to prove that for  $r > 0$  small enough,

$$(3.9) \quad \int_{Q_r(z_r)} \mathbf{g}^q < s^q.$$

We prove (3.9) by contradiction. Lemma 10 yields that  $\zeta(z_r) \leq \zeta(z) + r/2$ , which implies that when  $r > \zeta(z)$ , we have  $r > (2/3)\zeta(z_r)$ . In particular, (3.1) holds which implies that (3.9) is true for  $r > \zeta(z)$ . The continuity of the function  $r \mapsto \int_{Q_r(z_r)} \mathbf{g}^q$  on  $(0, +\infty)$  implies that if (3.9) is not true for all  $r > 0$ , there is  $\bar{r} = \max\{r > 0 : \int_{Q_r(z_r)} \mathbf{g}^q = s^q\} \leq \zeta(z) \leq \zeta(z_{\bar{r}}) + \bar{r}/2$  so that  $\bar{r} \leq 2\zeta(z_{\bar{r}})$ . Thanks to Lemma 11,  $5Q_{\bar{r}}(z_{\bar{r}}) = Q_{5\bar{r}}(z_{5\bar{r}}) \subset Q_\gamma$  which leads to

$$\int_{5Q_{\bar{r}}(z_{\bar{r}})} \mathbf{g}^q < s^q = \int_{Q_{\bar{r}}(z_{\bar{r}})} \mathbf{g}^q.$$

We deduce that the cylinder  $Q_{\bar{r}}(z_{\bar{r}})$  is included in  $\mathcal{G}$ . This implies that  $z \in \mathcal{G}$ , which contradicts  $z \in Q_\gamma \setminus \mathcal{G}$ .  $\square$

### 3.3. Reverse Hölder inequality on superlevel sets.

**Lemma 14** (Reverse Hölder on superlevel sets). *Given  $\gamma > 1$  and non-negative functions  $g \in L^q(Q_\gamma)$  and  $h \in L^\sigma(Q_\gamma)$ , for some  $\sigma > q > 1$  such that there is  $b > 1$  and  $\theta \in (0, 1)$  so that for all  $R > 0$  and all  $z_0 \in Q_\gamma$  such that  $Q_{\gamma R}(z_0) \subset Q_\gamma$  the inequality (1.2) holds. Let  $\mathbf{g}$  and  $\mathbf{h}$  be defined on  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  as in Lemma 12. We consider the superlevel sets of  $\mathbf{g}$  and  $\mathbf{h}$ :*

$$\forall t \geq 1, \quad \mathbf{g}(t) := \{z : \mathbf{g}(z) > t\} \quad \text{and} \quad \mathbf{h}(t) := \{z : \mathbf{h}(z) > t\}.$$

If  $\bar{\theta} \leq \bar{\theta}_0 := (2.4^q 5^{8d+4})^{-1}$ , then we have

$$\forall t \geq 1, \quad \int_{\mathbf{g}(t)} \mathbf{g}(z)^q dz \leq at^{q-1} \int_{\mathbf{g}(t)} \mathbf{g}(z) dz + a \int_{\mathbf{h}(t)} \mathbf{h}(z)^q dz,$$

where  $a = 2\bar{b}5^{8d+4}4^q$ , with  $\bar{b}$  appearing in (3.2).

*Proof.* In order to prove this inequality, we introduce a parameter  $s$  larger than  $t$  such that  $s = \alpha t$  with  $\alpha > 1$  to be chosen later. We split the integral we want to estimate as follows

$$\int_{\mathbf{g}(t)} \mathbf{g}^q = \int_{\mathbf{g}(s)} \mathbf{g}^q + \int_{\mathbf{g}(t) \setminus \mathbf{g}(s)} \mathbf{g}^q.$$

We estimate the second integral as follows:

$$(3.10) \quad \int_{\mathbf{g}(t) \setminus \mathbf{g}(s)} \mathbf{g}^q \leq s^{q-1} \int_{\mathbf{g}(t)} \mathbf{g} \leq (\alpha t)^{q-1} \int_{\mathbf{g}(t)} \mathbf{g}.$$

As for the first integral, we use the covering of  $\mathfrak{g}(s)$  obtained in Lemma 13:  $\mathfrak{g}(s) \subset \mathcal{G} = \cup_i Q_i$ . Thanks to Lemma 4 (Vitali's covering argument), we can extract a countable family  $I$  of disjoint cylinders  $(Q_i)_{i \in I}$  such that  $\mathcal{G} \subset \cup_{i \in I} 5Q_i$ . Since  $\mathfrak{g}(s) \subset \cup_{i \in I} 5Q_i$  with  $Q_i$  such that  $\int_{5Q_i} \mathfrak{g}^q < s^q \leq \int_{Q_i} \mathfrak{g}$ , we have

$$(3.11) \quad \int_{\mathfrak{g}(s)} \mathfrak{g}^q \leq \sum_{i \in I} \int_{5Q_i} \mathfrak{g}^q \leq 5^{4d+2} s^q \sum_{i \in I} |Q_i| \leq 5^{4d+2} s^q |\mathcal{G}|.$$

The covering is such that each  $Q_i = Q_{r_i}(z_i)$  with  $r_i \in (0, \zeta(z_i))$ , allowing us to apply (3.2):

$$s^q \leq \int_{Q_i} \mathfrak{g}^q \leq \bar{b} \left\{ \left( \int_{Q_i^{(\gamma)}} \mathfrak{g} \right)^q + \int_{Q_i^{(\gamma)}} \mathfrak{h}^q \right\} + \bar{\theta} \int_{Q_i^{(\gamma)}} \mathfrak{g}^q$$

where  $Q_i^{(\gamma)}$  denotes  $Q_{\gamma r_i}(z_i)$ , with  $Q_i = Q_{r_i}(z_i)$ . Using  $(A + B)^{\frac{1}{q}} \leq A^{\frac{1}{q}} + B^{\frac{1}{q}}$  for all  $A, B > 0$ , we get

$$\frac{\alpha t}{\bar{b}^{1/q}} |Q_i^{(\gamma)}| < \int_{Q_i^{(\gamma)}} \mathfrak{g} + |Q_i^{(\gamma)}|^{1-\frac{1}{q}} \left( \int_{Q_i^{(\gamma)}} \mathfrak{h}^q \right)^{1/q} + |Q_i^{(\gamma)}|^{1-\frac{1}{q}} \left( \frac{\bar{\theta}}{\bar{b}} \int_{Q_i^{(\gamma)}} \mathfrak{g}^q \right)^{1/q}.$$

We now estimate the three terms on the right hand side of the previous inequality. First observe that

$$\int_{Q_i^{(\gamma)}} \mathfrak{g}(z) dz = \int_{Q_i^{(\gamma)} \cap \mathfrak{g}(t)} \mathfrak{g}(z) dz + \int_{Q_i^{(\gamma)} \cap \mathfrak{g}(t)^c} \mathfrak{g}(z) dz \leq \int_{Q_i^{(\gamma)} \cap \mathfrak{g}(t)} \mathfrak{g}(z) dz + t |Q_i^{(\gamma)}|,$$

As for the second term, we proceed in a similar way:

$$\begin{aligned} |Q_i^{(\gamma)}|^{1-\frac{1}{q}} \left( \int_{Q_i^{(\gamma)}} \mathfrak{h}^q dz \right)^{1/q} &\leq |Q_i^{(\gamma)}|^{1-\frac{1}{q}} \left( \int_{Q_i^{(\gamma)} \cap \mathfrak{h}(t)} \mathfrak{h}^q dz + t^q |Q_i^{(\gamma)}| \right)^{1/q} \\ &\leq t |Q_i^{(\gamma)}| \left( \frac{1}{|Q_i^{(\gamma)}|} \int_{Q_i^{(\gamma)} \cap \mathfrak{h}(t)} \left( \frac{\mathfrak{h}}{t} \right)^q dz + 1 \right)^{1/q} \\ &\leq t |Q_i^{(\gamma)}| + t^{1-q} \int_{Q_i^{(\gamma)} \cap \mathfrak{h}(t)} \mathfrak{h}^q dz \end{aligned}$$

where we have used  $(A + 1)^{1/q} \leq 1 + A$ , for any  $A \geq 0$  and  $q \geq 1$ . We continue similarly for the third term:

$$|Q_i^{(\gamma)}|^{1-\frac{1}{q}} \left( \frac{\bar{\theta}}{\bar{b}} \int_{Q_i^{(\gamma)}} \mathfrak{g}^q dz \right)^{1/q} \leq t |Q_i^{(\gamma)}| + \frac{\bar{\theta}}{\bar{b}} t^{1-q} \int_{Q_i^{(\gamma)} \cap \mathfrak{g}(t)} \mathfrak{g}^q dz.$$

Choosing  $\alpha = 4b^{1/q} > 1$ , the four last inequalities imply

$$|Q_i^{(\gamma)}| < \frac{1}{t} \left( \int_{Q_i^{(\gamma)} \cap \mathfrak{g}(t)} \mathfrak{g} dz + t^{1-q} \int_{Q_i^{(\gamma)} \cap \mathfrak{h}(t)} \mathfrak{h}^q dz + \frac{\bar{\theta}}{\bar{b}} t^{1-q} \int_{Q_i^{(\gamma)} \cap \mathfrak{g}(t)} \mathfrak{g}^q dz \right).$$

Since  $\gamma > 1$ ,  $Q_i \subset Q_i^{(\gamma)}$  and  $\mathcal{G}$  is covered by the family of cylinders  $Q_i^{(\gamma)}$  with  $Q_i$  as in Lemma 13. Thanks to the Vitali covering result Lemma 4, there is a set  $J \subset \mathbb{N}$  of indices such that  $(Q_j^{(\gamma)})_{j \in J}$  are disjoint and

$\mathcal{G} \subset \cup_{j \in J} 5Q_j^{(\gamma)}$ . Keeping in mind that  $s = \alpha t$ , we deduce from (3.11) since the cylinders  $Q_j^{(\gamma)}$  are disjoint,

$$\begin{aligned}
\int_{G(s)} \mathbf{g}^q &\leq 5^{4d+2} s^q |\mathcal{G}| \\
&\leq (\alpha t)^q 5^{8d+4} \sum_{j \in J} |Q_j^{(\gamma)}| \\
&\leq (\alpha t)^q 5^{8d+4} \sum_{j \in J} \frac{1}{t} \left( \int_{Q_j^{(\gamma)} \cap \mathfrak{g}(t)} \mathbf{g} \, dz + t^{1-q} \int_{Q_j^{(\gamma)} \cap \mathfrak{h}(t)} \mathbf{h}^q \, dz + \frac{\bar{\theta}}{b} t^{1-q} \int_{Q_j^{(\gamma)} \cap \mathfrak{g}(t)} \mathbf{g}^q \, dz \right) \\
(3.12) \quad &\leq (\alpha t)^q 5^{8d+4} \frac{1}{t} \left( \int_{\mathfrak{g}(t)} \mathbf{g} \, dz + t^{1-q} \int_{\mathfrak{h}(t)} \mathbf{h}^q \, dz + \frac{\bar{\theta}}{b} t^{1-q} \int_{\mathfrak{g}(t)} \mathbf{g}^q \, dz \right).
\end{aligned}$$

Combining (3.10) and (3.12) we deduce for  $\bar{\theta} \leq \bar{\theta}_0 := \frac{1}{2.4q5^{8d+4}}$ ,

$$\int_{\mathfrak{g}(t)} \mathbf{g}(z)^q \, dz \leq at^{q-1} \int_{\mathfrak{g}(t)} \mathbf{g}(z) \, dz + a \int_{\mathfrak{h}(t)} \mathbf{h}(z)^q \, dz$$

for any  $a \geq 2 \max\{\alpha^{q-1}, 5^{8d+4}\alpha^q\}$ . Since  $\alpha \geq 1$  and  $\alpha^q = 4^q b$ , this condition reduces to  $a \geq 2\bar{b}5^{8d+4}4^q$ .  $\square$

### 3.4. Reverse Hölder inequality on level set functions.

**Lemma 15** (Reducing to the monodimensional case). *Given  $\gamma > 1$  and non-negative functions  $g \in L^q(Q_\gamma)$  and  $h \in L^\sigma(Q_\gamma)$  for some  $\sigma > q > 1$  such that there is  $b > 1$  and  $\theta \in (0, 1)$  so that for all  $R > 0$  and all  $z_0 \in Q_\gamma$  such that  $Q_{\gamma R}(z_0) \subset Q_\gamma$  the inequality (1.2) holds. Let  $\mathbf{g}$  and  $\mathbf{h}$  be defined on  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  as in Lemma 12 and the superlevel sets  $\mathfrak{g}$  and  $\mathfrak{h}$  of  $\mathbf{g}$  and  $\mathbf{h}$  as defined in Lemma 14. The functions*

$$\mathcal{G}(t) := \int_{\mathfrak{g}(t)} \mathbf{g}(z) \, dz \quad \text{and} \quad \mathcal{H}(t) := \int_{\mathfrak{h}(t)} \mathbf{h}(z) \, dz$$

satisfy for all  $r > 1$ ,

$$(3.13) \quad \forall t \geq 1, \quad - \int_t^{+\infty} s^{r-1} d\mathcal{G}(s) = \int_{\mathfrak{g}(t)} \mathbf{g}(z)^r \, dz,$$

$$(3.14) \quad \forall t \geq 1, \quad - \int_t^{+\infty} s^{r-1} d\mathcal{H}(s) = \int_{\mathfrak{h}(t)} \mathbf{h}(z)^r \, dz$$

and

$$(3.15) \quad \forall t \geq 1, \quad - \int_t^{+\infty} s^{q-1} d\mathcal{G}(s) \leq at^{q-1} \mathcal{G}(t) - a \int_t^{+\infty} s^{q-1} d\mathcal{H}(s).$$

*Proof.* The layer cake representation yields

$$\begin{aligned}
\mathcal{G}(t) &= \int_{\mathbb{R}^{2d+1}} \mathbf{g}(z) \chi_{\{\mathbf{g}(z) > t\}} \, dz \\
&= \int_{\mathbb{R}^{2d+1}} \int_0^{+\infty} \chi_{\{\mathbf{g}(z) \chi_{\{\mathbf{g}(z) > t\}} > s\}} \, ds \, dz \\
&= \int_{\mathbb{R}^{2d+1}} \int_0^t \chi_{\{\mathbf{g}(z) > s\}} \chi_{\{\mathbf{g}(z) > t\}} \, ds \, dz + \int_{\mathbb{R}^{2d+1}} \int_t^{+\infty} \chi_{\{\mathbf{g}(z) > s\}} \chi_{\{\mathbf{g}(z) > t\}} \, ds \, dz \\
&= \int_{\mathbb{R}^{2d+1}} \int_0^t \chi_{\{\mathbf{g}(z) > t\}} \, ds \, dz + \int_{\mathbb{R}^{2d+1}} \int_t^{+\infty} \chi_{\{\mathbf{g}(z) > s\}} \, ds \, dz \\
&= t|\mathfrak{g}(t)| + \int_t^{+\infty} |\mathfrak{g}(s)| \, ds
\end{aligned}$$

where we have used the Fubini theorem in the last line. This means:

$$(3.16) \quad d\mathcal{G}(s) = sd|\mathbf{g}(s)|.$$

Moreover, we deduce that for any  $r \geq 1$ :

$$\begin{aligned} u(t) &:= \int_{\mathbf{g}(t)} \mathbf{g}(z)^r dz = \int_{\mathbb{R}^{2d+1}} \mathbf{g}(z)^r \chi_{\{\mathbf{g}(z) > t\}} dz \\ &= \int_{\mathbb{R}^{2d+1}} \int_0^{+\infty} \chi_{\{\mathbf{g}(z)^r \chi_{\{\mathbf{g}(z) > t\}} > s\}} ds dz \\ &= \int_{\mathbb{R}^{2d+1}} \int_0^{t^r} \chi_{\{\mathbf{g}(z)^r > s\}} \chi_{\{\mathbf{g}(z) > t\}} ds \\ &+ \int_{\mathbb{R}^{2d+1}} \int_{t^r}^{+\infty} \chi_{\{\mathbf{g}(z)^r > s\}} \chi_{\{\mathbf{g}(z) > t\}} ds dz \\ &= \int_{\mathbb{R}^{2d+1}} \int_0^{t^r} \chi_{\{\mathbf{g}(z) > t\}} ds + \int_{\mathbb{R}^{2d+1}} \int_{t^r}^{+\infty} \chi_{\{\mathbf{g}(z)^r > s\}} ds \\ &= t^r |\mathbf{g}(t)| + \int_{t^r}^{+\infty} |\mathbf{g}(s^{1/r})| ds \end{aligned}$$

where we have used the Fubini theorem again. So we have

$$(3.17) \quad du(s) = s^r d|\mathbf{g}(s)|.$$

Combining (3.16) and (3.17), we deduce  $du(s) = s^{r-1} d\mathcal{G}(s)$ , which implies (3.13). Identity (3.14) is obtained by replacing  $\mathbf{g}(t)$  and  $\mathbf{g}(z)$  with  $\mathbf{h}(t)$  and  $\mathbf{h}(z)$ . Finally (3.15) follows from Lemma 14.  $\square$

### 3.5. Proof of the improved integrability in dimension one.

**Lemma 16** (The dimension one case). *The functions  $\mathcal{G}$  and  $\mathcal{H}$  defined in Lemma 15 satisfy,*

$$\begin{aligned} & - \int_1^{+\infty} t^{p-1} d\mathcal{G}(t) \\ & \leq \left( \frac{(p-1) - (p-q)}{(p-1) - a(p-q)} \right) \left( - \int_1^{+\infty} t^{q-1} d\mathcal{G}(t) \right) + \frac{2a(p-1)}{(p-1) - a(p-q)} \left( - \int_1^{+\infty} t^{p-1} d\mathcal{H}(t) \right) \end{aligned}$$

for  $p \in [q, p^*)$  with  $p^* = \min\left(\sigma, q + \frac{q-1}{a-1}\right)$ . We recall that  $a = 2(75)^{4d+2} 4^q b > 1$ .

*Proof.* Consider for  $r \geq 1$ ,

$$I(r) := - \int_1^{+\infty} t^{r-1} d\mathcal{G}(t) \quad \text{and} \quad J(r) := - \int_1^{+\infty} t^{r-1} d\mathcal{H}(t).$$

Pick  $p \in [q, p^*)$  and consider for  $L > 1$ ,

$$I_L(p) = - \int_1^L t^{p-q} t^{q-1} d\mathcal{G}(t).$$

We perform an integration by parts,

$$\begin{aligned}
I_L(p) &= - \int_1^L t^{p-q} t^{q-1} d\mathcal{G}(t) \\
&= \left[ t^{p-q} \left( \int_t^{+\infty} s^{q-1} d\mathcal{G}(s) \right) \right]_1^L + (p-q) \int_1^L t^{p-q-1} \left( - \int_t^{+\infty} s^{q-1} d\mathcal{G}(s) \right) dt \\
&= L^{p-q} \int_L^{+\infty} s^{q-1} d\mathcal{G}(s) + I(q) + (p-q) \int_1^L t^{p-q-1} \left( - \int_t^{+\infty} s^{q-1} d\mathcal{G}(s) \right) dt.
\end{aligned}$$

We now use (3.15) and get,

$$\begin{aligned}
(3.18) \quad I_L(p) &\leq L^{p-q} \int_L^{+\infty} s^{q-1} d\mathcal{G}(s) + I(q) + a(p-q) \int_1^L t^{p-2} \mathcal{G}(t) dt \\
&\quad + a(p-q) \int_1^L t^{p-q-1} \left( - \int_t^{+\infty} s^{q-1} d\mathcal{H}(s) \right) dt.
\end{aligned}$$

First, we perform an integration by parts in the third term of the right hand side of (3.18):

$$\begin{aligned}
a(p-q) \int_1^L t^{p-2} \mathcal{G}(t) dt &= a(p-q) \int_1^L \left( \frac{t^{p-1} - 1}{p-1} \right)' \mathcal{G}(t) dt \\
&= a(p-q) \left\{ \left[ \left( \frac{t^{p-1} - 1}{p-1} \right) \mathcal{G}(t) \right]_1^L - \frac{1}{p-1} \int_1^L (t^{p-1} - 1) d\mathcal{G}(t) \right\} \\
&= a(p-q) \left\{ \frac{L^{p-1} - 1}{p-1} \mathcal{G}(L) + \frac{I_L(p)}{p-1} + \frac{1}{p-1} \int_1^L d\mathcal{G}(t) \right\} \\
&= a(p-q) \left\{ \frac{L^{p-1}}{p-1} \mathcal{G}(L) + \frac{I_L(p)}{p-1} - \frac{1}{p-1} \mathcal{G}(1) \right\}.
\end{aligned}$$

Second, inequality (3.15) at  $t = 1$  yields

$$-\mathcal{G}(1) \leq \frac{1}{a} \int_1^{+\infty} s^{q-1} d\mathcal{G}(s) - \int_1^{+\infty} s^{q-1} d\mathcal{H}(s) = -\frac{1}{a} I(q) + J(q).$$

We thus proved the following estimate,

$$(3.19) \quad a(p-q) \int_1^L t^{p-2} \mathcal{G}(t) dt \leq a(p-q) \left\{ \frac{L^{p-1}}{p-1} \mathcal{G}(L) + \frac{I_L(p)}{p-1} - \frac{1}{a(p-1)} I(q) + \frac{1}{p-1} J(q) \right\}.$$

The fourth term of the right hand side of (3.18) is bounded by

$$\begin{aligned}
(3.20) \quad -a(p-q) \int_1^L t^{p-q-1} \int_t^{+\infty} s^{q-1} d\mathcal{H}(s) dt &\leq -a(p-q) \int_1^{+\infty} \int_1^{+\infty} t^{p-q-1} s^{q-1} 1_{s \geq t} d\mathcal{H}(s) dt \\
&= -a(p-q) \int_1^{+\infty} \left\{ \int_1^s t^{p-q-1} dt \right\} s^{q-1} d\mathcal{H}(s) \\
&= -a \int_1^{+\infty} s^{p-1} d\mathcal{H}(s) \\
&= aJ(p).
\end{aligned}$$

Combining (3.18), (3.19) and (3.20) finally yields,

$$I_L(p) \leq I(q) + a(p-q) \left\{ \frac{I_L(p)}{p-1} - \frac{1}{a(p-1)} I(q) + \frac{1}{p-1} J(q) \right\} + aJ(p) \\ + L^{p-q} \left[ \int_L^{+\infty} s^{q-1} d\mathcal{G}(s) + \frac{a(p-q)}{p-1} L^{q-1} \mathcal{G}(L) \right].$$

Observe that for  $p \in [q, p^*]$  with  $p^* = \min\left(\sigma, \frac{aq-1}{a-1}\right)$ , we have  $\frac{a(p-q)}{p-1} \leq 1$  and

$$\left[ \int_L^{+\infty} s^{q-1} d\mathcal{G}(s) + \frac{a(p-q)}{p-1} L^{q-1} \mathcal{G}(L) \right] \leq \int_L^{+\infty} s^{q-1} d\mathcal{G}(s) + L^{q-1} \mathcal{G}(L) \leq 0.$$

We thus proved that for all  $L > 1$ ,

$$I_L(p) \leq \left( \frac{(p-1) - (p-q)}{(p-1) - a(p-q)} \right) I(q) + \frac{a(p-q)}{(p-1) - a(p-q)} J(q) + \frac{a(p-1)}{(p-1) - a(p-q)} J(p)$$

Remark that  $J(q) \leq J(p)$  since  $q \leq p$ . Letting  $L \rightarrow +\infty$  allows us to conclude the proof.  $\square$

Recalling the definition of  $\mathcal{G}$  and  $\mathcal{H}$ , see Lemma 15, and formulas (3.13) and (3.14), Lemma 16 can be reformulated as follows.

**Lemma 17** (Going back to  $\mathbf{g}$  and  $\mathbf{h}$ ). *For  $p \in [q, p^*]$ ,*

$$\left( \int_{\mathfrak{g}(1)} \mathbf{g}^p dx \right)^{1/p} \leq C_{p,q,a} \left( \left( \int_{\mathfrak{g}(1)} \mathbf{g}^q dz \right)^{1/q} + \left( \int_{\mathfrak{h}(1)} \mathbf{h}^p dz \right)^{1/p} \right)$$

with  $C_{p,q,b} = \frac{2a(p-1)}{(p-1)-a(p-q)}$  with  $a = 2(75)^{4d+2} 4^q b$ .

### 3.6. Proof of Theorem 1.

*Proof of Theorem 1.* The localization function  $\zeta$  is bounded from below on  $\mathcal{Q}_1$  by  $c_\gamma > 0$  with

$$(3.21) \quad c_\gamma = \frac{1}{2} \min \left[ \frac{\gamma-1}{5}, \left( \frac{\gamma^2-1}{13} \right)^{\frac{1}{2}}, \left( \frac{\gamma^3-1}{25\gamma} \right)^{\frac{1}{2}} \right].$$

With such a lower bound at hand, we now write,

$$\left( \int_{\mathcal{Q}_1} \mathbf{g}^p \right)^{\frac{1}{p}} = C_0 \|g\|_{L^q(\mathcal{Q}_\gamma)} \left( \int_{\mathcal{Q}_1} |\mathcal{Q}_{\zeta(z)}|^{-p/q} \mathbf{g}^p \right)^{\frac{1}{p}} \\ \leq C_1 C_0 \|g\|_{L^q(\mathcal{Q}_\gamma)} \left( \int_{\mathcal{Q}_\gamma} \mathbf{g}^p \right)^{\frac{1}{p}} \\ \leq C_1 C_0 \|g\|_{L^q(\mathcal{Q}_\gamma)} \left( \int_{\mathfrak{g}(1)} \mathbf{g}^p \right)^{\frac{1}{p}} + C_1 |\mathcal{Q}_\gamma|^{\frac{1}{p}} C_0 \|g\|_{L^q(\mathcal{Q}_\gamma)}$$

with  $C_1 = |\mathcal{Q}_1|^{-1/q} c_\gamma^{-\frac{4d+2}{q}}$ . From Lemma 17, we have

$$C_0 \|g\|_{L^q(\mathcal{Q}_\gamma)} \left( \int_{\mathfrak{g}(1)} \mathbf{g}^p \right)^{\frac{1}{p}} \leq C_{p,q,b} \left( |\mathcal{Q}_1| \|\zeta\|_\infty^{\frac{4d+2}{q}} \|g\|_{L^q(\mathcal{Q}_\gamma)} + |\mathcal{Q}_1| \|\zeta\|_\infty^{\frac{4d+2}{q}} \|h\|_{L^p(\mathcal{Q}_\gamma)} \right).$$

Recall now that  $\zeta$  is bounded by  $\gamma/5$  and conclude by combining the two last inequalities with

$$C_G = C_1 C_0 |\mathcal{Q}_\gamma|^{\frac{1}{p}} + C_1 C_{p,q,b} |\mathcal{Q}_1| (\gamma/5)^{\frac{4d+2}{q}}.$$

We recall that  $C_0 = 2^{\frac{4d+2}{q}}$  (see Lemma 12) and  $C_1 = |\mathcal{Q}_1|^{-1/q} c_\gamma^{-\frac{4d+2}{q}}$  with  $c_\gamma$  given by (3.21).  $\square$

#### 4. APPLICATION TO KINETIC EQUATIONS

**4.1. A zeroth order Poincaré-Wirtinger inequality.** The proof of the reverse Hölder inequality for the gradient of solutions to the kinetic Fokker-Planck equation makes use of a *local hypoelliptic* Poincaré-Wirtinger inequality. The proof of this inequality relies on the following zeroth order Poincaré-Wirtinger inequality.

**Lemma 18** (Zeroth order Poincaré-Wirtinger inequality). *Given a function  $g = g(t, x) \in L^q(\mathcal{Q})$  with  $q \in (1, +\infty)$  in some convex open set  $\mathcal{Q} \subset \mathbb{R}^{1+d}$ , then*

$$\left\| g - \int_{\mathcal{Q}} g \right\|_{L^q(\mathcal{Q})} \leq C_{PW}^0 \|\nabla_{t,x} g\|_{W^{-1,q}(\mathcal{Q})}$$

for some constant  $C_{PW}^0 > 0$  depending on  $\mathcal{Q}$ .

In order to establish such an inequality, we first solve the divergence equation for some  $L^p$  function  $G$  with zero mean on  $\mathcal{Q}$ . The solution is not unique, but it is crucial to our argument, and it is a remarkable fact, that it is possible construct a solution that vanishes at the boundary of  $\mathcal{Q}$ . Such constructions are detailed in [Gal11] and go back to [Bog79]. The following statement is extracted from [Gal11, Lemma III.3.1, p. 162].

**Proposition 19** (The divergence equation). *Let  $p \in (1, +\infty)$  and let  $\mathcal{Q}$  be a cylinder of the form  $(-r^2, 0] \times B_{r^3} \subset \mathbb{R} \times \mathbb{R}^d$ . Then there is  $C_{\text{div}} > 0$  so that for any  $G \in L^p(\mathcal{Q})$  with  $\int_{\mathcal{Q}} G(t, x) dt dx = 0$ , there exists a vector-valued solution  $\vec{h} \in W_0^{1,p}(\mathcal{Q}, \mathbb{R}^{1+d})$  (the standard Sobolev space with Dirichlet conditions) to*

$$\nabla_{t,x} \cdot \vec{h} = G \text{ on } \mathcal{Q}$$

which satisfies

$$\|h\|_{W_0^{1,p}(\mathcal{Q})} \leq C_{\text{div}} \|G\|_{L^p(\mathcal{Q})}.$$

With such a proposition at hand, we can prove the previous lemma.

*Proof of Lemma 18.* For any real number  $a \in \mathbb{R}$ , we write  $[a]^{q-1} := |a|^{q-2}a$  if  $a \neq 0$  and  $[0]^{q-1} = 0$ . In particular,  $|a|^q = [a]^{q-1}a$ . We then consider the vector field  $\vec{h}$  given by Proposition 19 for

$$G := \left[ g - \int_{\mathcal{Q}} g \right]^{q-1} - \int_{\mathcal{Q}} \left[ g - \int_{\mathcal{Q}} g \right]^{q-1}.$$

Let us check that  $G \in L^p(\mathcal{Q})$  where  $p$  satisfies  $\frac{1}{p} + \frac{1}{q} = 1$ . We write  $G$  as  $G_0 - \int_{\mathcal{Q}} G_0$  with  $G_0 = \left[ g - \int_{\mathcal{Q}} g \right]^{q-1}$ . Then

$$\|G_0\|_{L^p(\mathcal{Q})}^p \leq \int_{\mathcal{Q}} \left| g - \int_{\mathcal{Q}} g \right|^{p(q-1)} = \int_{\mathcal{Q}} \left| g - \int_{\mathcal{Q}} g \right|^q.$$

This means that

$$\|G_0\|_{L^p(\mathcal{Q})} \leq \left\| g - \int_{\mathcal{Q}} g \right\|_{L^q(\mathcal{Q})}^{q-1}.$$

Moreover, Jensen's inequality yields,

$$\left\| \int_{\mathcal{Q}} G_0 \right\|_{L^p(\mathcal{Q})} \leq \left\| g - \int_{\mathcal{Q}} g \right\|_{L^q(\mathcal{Q})}^{q-1}$$

so that

$$\|G\|_{L^p(\mathcal{Q})} \leq 2 \left\| g - \int_{\mathcal{Q}} g \right\|_{L^q(\mathcal{Q})}^{q-1}.$$

We now compute

$$\begin{aligned}
\int_Q \left| g(t, x) - \int_Q g \right|^q dt dx &= \int_Q G \left( g - \int_Q g \right) \\
&= \int_Q \left( g(t, x) - \int_Q g \right) \nabla_{t,x} \cdot h(t, x) dt dx \\
&= - \int_Q \nabla_{t,x} g(t, x) \cdot h(t, x) dt dx \\
&\leq \| \nabla_{t,x} g(t, x) \|_{W^{-1,q}(Q)} \| h \|_{W_0^{1,p}(Q)} \\
&\leq C_{\text{div}} \| \nabla_{t,x} g(t, x) \|_{W^{-1,q}(Q)} \| G \|_{L^p(Q)}.
\end{aligned}$$

The estimate of the  $L^p$  norm of  $G$  concludes the proof with  $C_{PW}^0 = 2C_{\text{div}}$ .  $\square$

**4.2. Poincaré inequality in  $L^q$ .** We prove here a *local hypoelliptic* Poincaré-Wirtinger inequality for solutions to (1.1), following ideas of [AAMN21].

**Theorem 20** (Local hypoelliptic Poincaré-Wirtinger inequality). *Let  $Q = Q_r$  be a cylinder of  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  for some  $r > 0$ . For any function  $f \in L^2(Q)$  and any  $1 < q \leq 2$ , we have,*

$$(4.1) \quad \| f - \langle\langle f \rangle\rangle_Q \|_{L^q(Q)} \leq C_{hpw} \left( \| (\partial_t + v \cdot \nabla_x) f \|_{L^q(Q; W^{-1,q}(B))} + \| \nabla_v f \|_{L^q(Q)} \right)$$

where  $Q = (-r^2, 0] \times B_{r^3}$  and  $B = B_r$  and the average is defined as

$$\langle\langle f \rangle\rangle_Q := \int_Q f(t, x, v) dt dx dv.$$

The constant  $C_{hpw} > 0$  only depends on  $Q$ .

*Proof.* The proof is a variation along the ideas in [AAMN21, Theorem 7.2-(1)], where we replace the Gaussian Poincaré inequality with the Poincaré-Wirtinger inequality in  $v$ :

$$(4.2) \quad \int_B |g - \langle g \rangle_B|^q dv \leq C_{PW} \int_B |\nabla_v g|^q dv \quad \text{with} \quad \langle g \rangle_B := \frac{1}{|B|} \int_B g(v) dv.$$

We use the partial  $v$ -average  $\langle f \rangle_B(t, x)$  as an intermediate term in the inequality:

$$\int_Q |f - \langle\langle f \rangle\rangle_Q|^q dt dx dv \leq 2 \int_Q |f - \langle f \rangle_B|^q dt dx dv + 2 \int_Q |\langle f \rangle_B - \langle\langle f \rangle\rangle_Q|^q dt dx dv.$$

The first term in the right hand side is estimated with (4.2) applied at each  $(t, x) \in Q$ :

$$\int_Q |f - \langle f \rangle_B|^q dt dx dv \leq C_{PW} \int_Q |\nabla_v f|^q dt dx dv.$$

To estimate the second term we first remove the  $v$ -integration and apply Lemma 18:

$$(4.3) \quad \begin{aligned} \int_{Q \times B} |\langle f \rangle_B - \langle\langle f \rangle\rangle_{Q \times B}|^q dt dx dv &= |B| \int_Q |\langle f \rangle_B - \langle\langle f \rangle\rangle_{Q \times B}|^q dt dx \\ &\leq |B| C_{PW}^0 \| \nabla_{t,x} \langle f \rangle_B \|_{W^{-1,q}(Q)}^q. \end{aligned}$$

We are left with proving the following hypoelliptic estimate

$$(4.4) \quad \| \nabla_{t,x} \langle f \rangle_B \|_{W^{-1,q}(Q)}^2 \leq C_{hypo} \left( \| \nabla_v f \|_{L^q(Q \times B)}^2 + \| (\partial_t + v \cdot \nabla_x) f \|_{L^q(Q; W^{-1,q}(B))}^2 \right)$$



for some constant  $C_{hypo} > 0$  depending on  $Q$ . Let us denote  $v_0 := 1$ , and  $\partial_0 := \partial_t$ ,  $\partial_i := \partial_{x_i}$  for  $i = 1, \dots, d$ . Consider  $\varphi(t, x) \in W_0^{1,p}(Q)$  and  $\chi_i \in C_c^\infty(B)$ ,  $i = 0, 1, \dots, d$  such that

$$\forall i, j = 0, 1, \dots, d, \quad \int_B \chi_i(v) v_j \, dv = \delta_{ij}.$$

We then integrate any derivative  $\partial_i \langle f \rangle_B$  for  $i = 0, 1, \dots, d$ , against  $\varphi$  and compute:

$$\begin{aligned} & \int_Q \partial_i \langle f \rangle_B(t, x) \varphi(t, x) \, dt \, dx \\ &= \int_{Q \times B} \partial_i \langle f \rangle_B(t, x) \varphi(t, x) v_i \chi_i(v) \, dt \, dx \, dv \\ &= \int_{Q \times B} (\partial_t + v \cdot \nabla_x) \langle f \rangle_B(t, x) \varphi(t, x) \chi_i(v) \, dt \, dx \, dv \\ &= \int_{Q \times B} (\partial_t + v \cdot \nabla_x) (\langle f \rangle_B(t, x) - f(t, x, v)) \varphi(t, x) \chi_i(v) \, dt \, dx \, dv \\ &\quad + \int_{Q \times B} (\partial_t + v \cdot \nabla_x) f(t, x, v) \varphi(t, x) \chi_i(v) \, dt \, dx \, dv =: I_1^i + I_2^i. \end{aligned}$$

The first term is controlled from the Poincaré-Wirtinger inequality in  $v$  again:

$$|I_1^i| \leq \|f - \langle f \rangle_B\|_{L^q(Q \times B)} \left\| (\partial_t + v \cdot \nabla_x) (\varphi \chi_i) \right\|_{L^p(Q \times B)} \lesssim \|\nabla_v f\|_{L^q(Q \times B)}.$$

The second term is controlled by

$$|I_2^i| \leq \left\| (\partial_t + v \cdot \nabla_x) f \right\|_{L^q(Q; W^{-1,q}(B))} \|\varphi\|_{L^p(Q)} \|\chi_i\|_{W_0^{1,p}(B)}.$$

This establishes the hypoelliptic estimate (4.4) and concludes the proof of (4.1).  $\square$

**Corollary 21** (Local  $L^q$  estimate). *For any function  $f \in L^2(Q)$  solution to (1.1) on  $Q = Q \times B$  we have for any  $1 < q \leq 2$ ,*

$$(4.5) \quad \|f - \langle \langle f \rangle \rangle_Q\|_{L^q(Q)} \leq C'_{hpw} \left( \|\nabla_v f\|_{L^q(Q)} + \|S\|_{L^2(Q)} \right).$$

The constant  $C'_{hpw} > 0$  only depends on  $Q$  and the ellipticity constant  $\Lambda$ .

*Proof.* Let us now use the equation (1.1) to prove that

$$\left\| (\partial_t + v \cdot \nabla_x) f \right\|_{L^q(Q; W^{-1,q}(B))} \lesssim \|\nabla_v f\|_{L^q(Q \times B)} + \|S\|_{L^q(Q)}.$$

Consider  $\varphi \in L^p(Q; W_0^{1,p}(B))$  and compute

$$\begin{aligned} \left| \int_Q (\partial_t + v \cdot \nabla_x) f \varphi \, dz \right| &\leq \left| \int_Q (\mathcal{A} \nabla_v f) \cdot \nabla_v \varphi \, dt \, dx \, dv \right| + \left| \int_Q (B \cdot \nabla_v f) \varphi \, dt \, dx \, dv \right| + \left| \int_Q S \varphi \, dt \, dx \, dv \right| \\ &\leq \Lambda \left( \|\nabla_v f\|_{L^q(Q)} + \|S\|_{L^q(Q)} \right) \|\varphi\|_{L^p(Q; W_0^{1,p}(B))} \end{aligned}$$

which concludes the proof.  $\square$

**Corollary 22** (Gain of integrability of the solution controlled by the gradient). *Let  $f$  be a weak solution to (1.1) in  $Q$  (as per Definition 7). Then for all cylinders  $Q_r(z_0)$  such that  $Q_{2r}(z_0) \subset Q$ ,  $f$  satisfies*

$$\left\| f - \langle \langle f \rangle \rangle_{Q_r(z_0)} \right\|_{L^p(Q_r(z_0))} \leq C_r \left( \|\nabla_v f\|_{L^2(Q_{2r}(z_0))} + \|S\|_{L^2(Q_{2r}(z_0))} \right),$$

where  $p = \frac{6(2d+1)}{6d+1} > 2$  and  $C_r > 0$  only depends on the dimension  $r, d, \lambda$  and  $\Lambda$ .

*Proof.* The result follows from the combination of Proposition 9 and Corollary 21 with  $q = 2$  applied to  $f - \langle\langle f \rangle\rangle_{\mathcal{Q}_r(z_0)}$ , which is a weak solution to (1.1).  $\square$

#### 4.3. Proof of the gain of integrability of the velocity gradient.

*Proof of Theorem 2.* We first prove that the velocity gradient of any solution  $\bar{f}$  to (1.1) satisfies (1.2) for the cylinders  $\mathcal{Q}_1$  and  $\mathcal{Q}_\gamma$  and for  $q = 2$ . We use Proposition 8, an interpolation inequality of  $L^2$  between  $L^p$  and  $L^{p'}$  where  $p > 2$  is defined in Corollary 22 and  $p' := p/(p-1) \in (1, 2)$  and a Young inequality:

$$\begin{aligned} \|\nabla_v \bar{f}\|_{L^2(\mathcal{Q}_1)}^2 &\leq C \|\bar{f} - \langle\langle \bar{f} \rangle\rangle_{\mathcal{Q}_{(\gamma+1)/2}}\|_{L^2(\mathcal{Q}_{(\gamma+1)/2})}^2 + C \|\bar{S}\|_{L^2(\mathcal{Q}_\gamma)}^2 \\ &\leq \|\bar{f} - \langle\langle \bar{f} \rangle\rangle_{\mathcal{Q}_{(\gamma+1)/2}}\|_{L^p(\mathcal{Q}_{(\gamma+1)/2})} \|\bar{f} - \langle\langle \bar{f} \rangle\rangle_{\mathcal{Q}_{(\gamma+1)/2}}\|_{L^{p'}(\mathcal{Q}_{(\gamma+1)/2})} + C \|\bar{S}\|_{L^2(\mathcal{Q}_\gamma)}^2 \\ &\leq \varepsilon \|\bar{f} - \langle\langle \bar{f} \rangle\rangle_{\mathcal{Q}_{(\gamma+1)/2}}\|_{L^p(\mathcal{Q}_{(\gamma+1)/2})}^2 + C \|\bar{f} - \langle\langle \bar{f} \rangle\rangle_{\mathcal{Q}_{(\gamma+1)/2}}\|_{L^{p'}(\mathcal{Q}_{(\gamma+1)/2})}^2 + C \|\bar{S}\|_{L^2(\mathcal{Q}_\gamma)}^2 \end{aligned}$$

we now use Corollary 22 on the first term and Corollary 21 with  $q = p'$  on the second term:

$$\begin{aligned} &\leq \varepsilon C \|\nabla_v \bar{f}\|_{L^2(\mathcal{Q}_\gamma)}^2 + C \|\nabla_v \bar{f}\|_{L^{p'}(\mathcal{Q}_{(\gamma+1)/2})}^2 + C \|\bar{S}\|_{L^2(\mathcal{Q}_\gamma)}^2 \\ &\leq \varepsilon C \|\nabla_v \bar{f}\|_{L^2(\mathcal{Q}_\gamma)}^2 + C' \|\nabla_v \bar{f}\|_{L^2(\mathcal{Q}_{(\gamma+1)/2})}^{2\delta} \|\nabla_v \bar{f}\|_{L^1(\mathcal{Q}_{(\gamma+1)/2})}^{2(1-\delta)} + C \|\bar{S}\|_{L^2(\mathcal{Q}_\gamma)}^2 \\ &\leq 2\varepsilon C \|\nabla_v \bar{f}\|_{L^2(\mathcal{Q}_\gamma)}^2 + C'' \|\nabla_v \bar{f}\|_{L^1(\mathcal{Q}_{(\gamma+1)/2})}^2 + C \|\bar{S}\|_{L^2(\mathcal{Q}_\gamma)}^2 \end{aligned}$$

where we have used an interpolation inequality of  $L^{p'}$  between  $L^1$  and  $L^2$  with  $\delta \in (0, 1)$  such that  $\frac{1}{p'} = \frac{\delta}{2} + \frac{1-\delta}{1}$  and a Young inequality in the last line. From the previous inequality with  $\theta = 2\varepsilon C$  with  $\varepsilon$  small enough so that  $\theta < \theta_0$  follows

$$(4.6) \quad \int_{\mathcal{Q}_1} |\nabla_v \bar{f}|^2 \leq \theta \int_{\mathcal{Q}_\gamma} |\nabla_v \bar{f}|^2 + C(\theta) \left( \int_{\mathcal{Q}_\gamma} |\nabla_v \bar{f}| \right)^2 + C(\theta) \int_{\mathcal{Q}_\gamma} \bar{S}^2.$$

Now let us prove that the velocity gradient of any weak solution  $f$  to (1.1) satisfies (1.2) for any cylinder such that  $\mathcal{Q}_{\gamma r}(z_0) \subset \mathcal{Q}_{\gamma R}(z)$  with  $z_0 = (t_0, x_0, v_0) \in \mathcal{Q}$ . Consider such a solution  $f$  and a cylinder  $\mathcal{Q}_r(z_0)$  such that  $\mathcal{Q}_{\gamma r}(z_0) \subset \mathcal{Q}_{\gamma R}(z)$ . We apply (4.6) to the solution  $\bar{f}(t, x, v) = f(r^2 t + t_0, r^3 x + x_0 + r^2 t v_0, r v + v_0)$  of the equation (1.1) with different coefficients  $\bar{B} = rB$ ,  $\bar{S} = r^2 S$  satisfying the same conditions, so that

$$f(t, x, v) = \bar{f}\left(\frac{t-t_0}{r^2}, \frac{x-x_0-(t-t_0)v_0}{r^3}, \frac{v-v_0}{r}\right).$$

This yields

$$\begin{aligned} \int_{\mathcal{Q}_r(z_0)} |\nabla_v f|^2 &= \frac{1}{r^2} \int_{\mathcal{Q}_1} |\nabla_v \bar{f}|^2, \\ \int_{\mathcal{Q}_{\gamma r}(z_0)} |\nabla_v f| &= \frac{1}{r} \int_{\mathcal{Q}_\gamma} |\nabla_v \bar{f}|, \\ \int_{\mathcal{Q}_{\gamma r}(z_0)} |\nabla_v f|^2 &= \frac{1}{r^2} \int_{\mathcal{Q}_\gamma} |\nabla_v \bar{f}|^2, \\ \int_{\mathcal{Q}_{\gamma r}(z_0)} S^2 &= \frac{1}{r^4} \int_{\mathcal{Q}_\gamma} \bar{S}^2. \end{aligned}$$

Combining those inequalities with (4.6) and the fact that  $r \leq 1$  gives

$$\int_{Q_r(z_0)} |\nabla_v f|^2 \leq \theta \int_{Q_{\gamma r}(z_0)} |\nabla_v f|^2 + C(\theta) \left[ \left( \int_{Q_{\gamma r}(z_0)} |\nabla_v f| \right)^2 + \int_{Q_{\gamma r}(z_0)} S^2 \right],$$

which is (1.2) with  $g = |\nabla_v f|$  and  $q = 2$  and  $b = C(\theta)$  and  $h = S$ . We now apply Theorem 1 and get the desired gain of integrability for  $\nabla_v f$ .  $\square$

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