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SOME FKG INEQUALITIES FOR STOCHASTIC PROCESSES

ALEXANDRE LEGRAND

ABSTRACT. This paper is interested in proving correlation inequalities of the FKG-type for various stochastic processes in continuous time. The pivotal tool which yields these correlation inequalities is an approximation with (possibly conditioned) Markov chains and random walks. In particular, we prove FKG inequalities for Lévy processes, Bessel processes and several conditioned Brownian processes. As a side result, we also provide a necessary and sufficient condition for a random walk distribution in \mathbb{Z} to satisfy the well-known “FKG lattice condition”.

The *FKG inequality* is a correlation inequality named after Fortuine, Kasteleyn and Ginibre [9], who proved a sufficient condition for its validity in the Ising and other lattice models (see also [11]). It has seen many applications and generalizations in statistical mechanics and general probability theory, see e.g. [2, 6, 12, 18, 25, 27, 29] and references therein, or more recently [19, 20, 21, 34]; and it is a recurrent tool in proofs of stochastic domination results, as well as computations of lower bounds on first moments or event probabilities. It is also equivalent to the notion of “*association*” of random variables, which has been extensively studied, see e.g. [7]. In [1], Barbato proved that the FKG inequality holds for the Brownian motion, more precisely on the space $\mathcal{C}([0, T], \mathbb{R})$ of continuous functions from $[0, T]$ to \mathbb{R} endowed with the canonical Wiener measure; and this statement in a functional space has already seen several applications: the reader can refer to [5, 10, 26, 33] for some examples.

The present paper is interested in FKG inequalities on functional spaces, and aims to contribute to this literature in a few directions. First, we present a necessary and sufficient condition for a d -dimensional Lévy process to satisfy the FKG inequality: not only this extends Barbato’s result beyond the 1-dimensional Brownian case, but this also provides an alternate proof. Then, we show an FKG inequality for a class of conditional distributions of discrete-time Markov chain trajectories. Not only this allows us to prove that many other stochastic processes in continuous time satisfy the FKG inequality, such as Bessel processes and conditioned Brownian processes; this result also relates to several important issues around FKG inequalities, such as establishing necessary and sufficient conditions for the well-known *FKG lattice condition* to hold, or obtaining FKG inequalities for conditional distributions —a topic which has notably aroused interest in the domains of spin systems (see [23] and references therein) as well as percolation [3, 4].

1. MAIN RESULTS

1.1. The FKG inequality and Lévy processes. Throughout this paper, $(E, \mathcal{E}, \preceq)$ denotes a measurable set endowed with a measurable (partial) order, i.e. it satisfies $\{(u, v) \in E^2; u \preceq v\} \in \mathcal{E}^{\otimes 2}$. A function $f : E \rightarrow \mathbb{R}$ is said to be *increasing* (or non-decreasing) on E if, for all $u, v \in E$, $u \preceq v$ implies $f(u) \leq f(v)$. This paper will mostly be interested in the following cases:

(i) $E = \mathbb{R}^d$, $\mathcal{E} = \text{Bor}(\mathbb{R}^d)$ and \preceq is the *product partial order*: i.e. for $u, v \in E$, $u \preceq v$ if and only if $u_i \leq v_i$ for all $1 \leq i \leq d$,

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(ii) $E = \mathcal{C}([0, T], \mathbb{R})$, $\mathcal{E} = \mathcal{E}_{\mathcal{C}}$ is the σ -algebra induced by the uniform convergence topology, and for $u, v \in E$, $u \preceq v$ if and only if $u(t) \leq v(t)$ for all $t \in [0, T]$.

(iii) $E = \mathcal{D}([0, T], \mathbb{R}^d)$ is the set of right-continuous with left-hand limits (rcll) functions on $[0, T]$ taking values in \mathbb{R}^d , $\mathcal{E} = \mathcal{E}_{\mathcal{D}}$ is the σ -algebra induced by the Skorokhod metric, and for $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ in $\mathcal{D}([0, T], \mathbb{R}^d)$, $u \preceq v$ if and only if $u_i(t) \leq v_i(t)$ for all $t \in [0, T]$, $1 \leq i \leq d$.

Definition 1.1. Let \mathbf{P} a probability measure on $(E, \mathcal{E}, \preceq)$, and X a random variable with law \mathbf{P} : then X is *associated* (or *positively correlated*) if for all increasing functions $f, g \in L^2(\mathbf{P})$, one has

$$(1.1) \quad \mathbf{E}[f(X)g(X)] \geq \mathbf{E}[f(X)]\mathbf{E}[g(X)].$$

We shall abusively write that \mathbf{P} is associated whenever $X \sim \mathbf{P}$ is. When X is associated, we may also say that it “satisfies the FKG inequality”. Our first result is a necessary and sufficient condition for a d -dimensional Lévy process to satisfy the FKG inequality (in particular it encompasses [1, Theorem 4]).

Theorem 1.2. *Let $X = (X_t)_{t \in [0, T]}$ be a d -dimensional Lévy process, $d \geq 1$, started from $X_0 = 0$. Let ν denote its Lévy measure on \mathbb{R}^d , and $\Sigma = (\sigma_{i,j})_{1 \leq i, j \leq d}$ the covariance matrix of its Gaussian component. Then, X is associated in $\mathcal{D}([0, T], \mathbb{R}^d)$ if and only if the following two conditions hold:*

(i) $\sigma_{i,j} \geq 0$ for all $1 \leq i, j \leq d$,

(ii) $\text{Supp}(\nu) \subset (\mathbb{R}_+)^d \cup (\mathbb{R}_-)^d$.

In particular, if X is a 1-dimensional Lévy process then it is associated.

The matter of association for infinitely divisible vectors has already been studied before, see notably [28, 30, 32] and the notion of “infinite association” in [13]. However, to the author’s knowledge, the association of Lévy processes on the functional space $\mathcal{D}([0, T], \mathbb{R}^d)$ has not been considered previously. The proof of this theorem relies on the notion of infinite association, together with classical arguments on random walks and weak convergence of measures. Let us mention that, even though the proof displayed here is simpler than that of [1] (which relies on non-trivial measure theory and the introduction of a new order relation on $\mathcal{C}([0, T], \mathbb{R}^d)$) in the case of the Brownian motion, it seems that some of the arguments in [1] could be extended to continuous random processes with non-stationary increments.

Let us also mention that, throughout this paper, we will only consider processes which are started from a deterministic initial value, e.g. $X_0 = 0 \in \mathbb{R}^d$ almost surely; but all our results can be extended to generic (random) initial values with the following proposition. For a general random variable X on some space (E, \mathcal{E}) , we denote its law with \mathbf{P}_X .

Proposition 1.3. *Let I a generic set and $E \subset (\mathbb{R}^d)^I$ be a partially ordered measurable space stable by translation, e.g. $(\mathbb{R}^d)^n$, $\mathcal{C}([0, T], \mathbb{R}^d)$ or $\mathcal{D}([0, T], \mathbb{R}^d)$, $d \geq 1$. Let Y be a random process in E , X_0 a random vector in \mathbb{R}^d , and denote their laws \mathbf{P}_Y and \mathbf{P}_{X_0} respectively. Assume that \mathbf{P}_{X_0} is associated, and that for almost every X_0 , $\mathbf{P}_Y(\cdot | X_0)$ is associated. Then, \mathbf{P}_{Y+X_0} is associated.*

Notice that Y and X_0 are not assumed to be independent.

1.2. Conditional association of Markov chains. The second result of this paper is that many Markov chain (or random walk) distributions, after conditioning on a quite generic event, are associated. In particular, we deduce from this the association of several stochastic processes in continuous time which can be approximated by well-chosen Markov chains. Let us introduce the following definitions.

Definition 1.4. We say that an event $A \in \mathcal{E}$ is *max/min-stable* if for $u, v \in A$, one has $u \vee v \in A$ and $u \wedge v \in A$, where $u \vee v$ (resp. $u \wedge v$) denotes the smallest common upper bound (resp. largest lower bound) of u and v in E .

In this paper we always assume that for all $u, v \in E$, $u \vee v$ and $u \wedge v$ are well-defined in E —which is the case if $E = \mathbb{R}^d$, $E = \mathcal{C}([0, T], \mathbb{R})$ or $E = \mathcal{D}([0, T], \mathbb{R}^d)$.

Definition 1.5. Let $\mathbf{p} := (p_{x,y})_{x,y \in \mathcal{X}}$ a probability transition kernel on some countable subset $\mathcal{X} \subset \mathbb{R}$ (i.e. $p_{x,y} \geq 0$ and for all $x \in \mathcal{X}$, $\sum_y p_{x,y} = 1$). For $\mathcal{X}_1 \subset \mathcal{X}$, we say that \mathbf{p} has \mathcal{X}_1 -*unfavorable crossings* (abbreviated \mathcal{X}_1 -u.c.) if for all $u_1, v_1 \in \mathcal{X}_1$, $u_2, v_2 \in \mathcal{X}$ one has,

$$(1.2) \quad p_{u_1, u_2} \times p_{v_1, v_2} \leq p_{u_1 \vee v_1, u_2 \vee v_2} \times p_{u_1 \wedge v_1, u_2 \wedge v_2}.$$

For \mathbf{P} a (purely atomic) probability measure on \mathbb{R} , we say that \mathbf{P} has \mathcal{X}_1 -u.c. if the transition kernel of the random walk with increment distribution \mathbf{P} has \mathcal{X}_1 -u.c..

For the sake of clarity, we now provide some examples of Markov chains on $\mathcal{X} := \mathbb{Z}$ which have unfavorable crossings. If \mathbf{P} is an atomic measure on some measurable space E , we write abusively $\mathbf{P}(u) := \mathbf{P}(\{u\})$ for $u \in E$.

Lemma 1.6. (i) Discrete Laplace random walk: let $\beta > 0$ and $\mathbf{P}(z) := C e^{-\beta|z|} \mathbf{1}_{z \in \mathbb{Z}}$, where $C = C(\beta)$ is a normalizing constant. Then \mathbf{P} has \mathbb{Z} -u.c. (i.e. the transition kernel of the random walk with increment distribution \mathbf{P} has \mathbb{Z} -u.c.).

(ii) Markov chain with unit steps: assume $p_{i,i+1} + p_{i,i-1} = 1$ for all $i \in \mathbb{Z}$. Then \mathbf{p} has both $(2\mathbb{Z})$ -u.c. and $(2\mathbb{Z} + 1)$ -u.c..

We also present some counter-examples.

Example 1.1. (i) Let $\alpha > 1$ and $\mathbf{P}(z) := C(1 + |z|)^{-\alpha} \mathbf{1}_{z \in \mathbb{Z}}$, where $C = C(\alpha)$ is a normalizing constant. Then \mathbf{P} does not have \mathbb{Z} -u.c..

(ii) Let $\gamma \in [0, 1]$, and $\mathbf{P} := \frac{1-\gamma}{2}(\delta_{-1} + \delta_{+1}) + \gamma\delta_0$. Then \mathbf{P} has \mathbb{Z} -u.c. if and only if $\gamma \geq 1/3$.

The proofs of Lemma 1.6 and Example 1.1 are quite straightforward and are left as an exercise to the reader (nonetheless let us mention that, with the exception of Lemma 1.6.(ii), all those statements are consequences of Proposition 1.11 below).

We may now state our central result. Recall that for any probability measure \mathbf{P} on (E, \mathcal{E}) and $A \in \mathcal{E}$ such that $\mathbf{P}(A) > 0$, the measure \mathbf{P} conditioned to A is defined with $\mathbf{P}(B|A) := \mathbf{P}(B \cap A)/\mathbf{P}(A)$, $B \in \mathcal{E}$. We have the following.

Theorem 1.7. Let $X^n = (X_k)_{k=1}^n$ be the first n steps of a Markov chain on some set $\mathcal{X} \subset \mathbb{R}$ started from some $x_0 \in \mathcal{X}$. Assume \mathcal{X} is countable, locally finite, and let \mathbf{p} denote the transition kernel of X^n . Let $A \subset \mathcal{X}^n$, and assume the following:

(H1) for all $k \in \{0, \dots, n-1\}$, \mathbf{p} has $\text{Supp}(\mathbf{P}_{X_k})$ -u.c.,

(H2) $\mathbf{P}_{X^n}(A) > 0$ and A is max/min-stable,

then, $\mathbf{P}_{X^n}(\cdot|A)$ is associated in \mathbb{R}^n .

Remark 1.2. One can clearly take $A := \mathcal{X}^n$ in the theorem, hence \mathbf{P}_{X^n} is associated as soon as \mathbf{p} satisfies (H1).

Example 1.3. (i) Let X be a random walk on \mathbb{Z} with increment distribution \mathbf{P} . If \mathbf{P} has \mathbb{Z} -u.c., then (H1) holds.

(ii) Let X be a Markov chain on \mathbb{Z} such that $p_{i,i+1} + p_{i,i-1} = 1$ for all $i \in \mathbb{Z}$: then (H1) holds (recall Lemma 1.6.(ii)).

Remark 1.4. One could be tempted to replace the assumption (H1) with “ \mathbf{p} has \mathcal{X} -u.c.”, however the latter is a strictly stronger assumption since it excludes some periodic Markov chains (e.g. the simple random walk: compare Lemma 1.6.(ii) with Example 1.1.(ii) for $\gamma = 0$).

We then deduce from Theorem 1.7 that several 1-dimensional stochastic processes are associated.

Corollary 1.8. *Let $E = \mathcal{C}([0, T], \mathbb{R})$ for some $T > 0$. The following Brownian processes $(B_t)_{t \in [0, T]}$, started from $B_0 = 0$, are associated in $\mathcal{C}([0, T], \mathbb{R})$:*

- (1) *the Brownian bridge, $B_T = x$, $x \in \mathbb{R}$,*
- (2) *the Brownian excursion, $B_T = 0$, $B_t \geq 0$ for all $t \in [0, T]$,*
- (3) *the Brownian meander, $B_t \geq 0$ for all $t \in [0, T]$,*
- (4) *the Brownian motion conditioned to remain in some (time-dependent) interval, $B_t \in [a(t), b(t)]$ for all $t \in [0, T]$, where $a, b \in \mathcal{D}([0, T], [-\infty, +\infty])$ satisfy $\mathbf{P}_\omega(\forall t \in [0, T], \omega_t \in [a(t), b(t)]) > 0$, for \mathbf{P}_ω the canonical Wiener measure on $\mathcal{C}([0, T], \mathbb{R})$.*

Corollary 1.9. *Let $E = \mathcal{C}([0, T], \mathbb{R}_+)$. The Bessel process $(X_t)_{t \in [0, T]}$ of index $\nu \in (-1, +\infty)$, that is with generator $\frac{1}{2} \frac{d^2}{dx^2} + \frac{2\nu+1}{2x} \frac{d}{dx}$, and started from some $x_0 \geq 0$, is associated.*

The case of Bessel processes with indices $\nu \leq -1$ is discussed in Remark 3.1 below. Let us clarify that these corollaries of Theorem 1.7 are not meant to be exhaustive, but illustrative; the reader may think of many similar applications other than Corollaries 1.8, 1.9.

1.3. The FKG lattice condition. Let us comment on Theorem 1.7. The matter of proving association (or other correlation inequalities) for conditional distributions has been investigated in various settings, see e.g. [3, 4, 23] or even [26, Lemma 2.3]. However, many of those results make strong assumptions on the conditioning set $A \subset E$. In Theorem 1.7, the assumption (H2) on the set A is minimal, whereas the crucial assumption (H1) only involves the transition kernel of the Markov chain *without conditioning*.

The assumption (H1) is closely related to the following correlation property, called the “FKG (lattice) condition”.

Definition 1.10. Let $(E, \mathcal{E}, \preceq, \mathbf{P})$ be a partially ordered probability space, and assume \mathbf{P} is a purely atomic measure.¹ We say that \mathbf{P} satisfies the *FKG (lattice) condition* if,

$$(1.3) \quad \mathbf{P}(u \vee v) \mathbf{P}(u \wedge v) \geq \mathbf{P}(u) \mathbf{P}(v), \quad \text{for } \mathbf{P}\text{-a.e. } u, v \in E.$$

The FKG condition is central to the proof of Theorem 1.7. Indeed, we prove in Lemma 3.1 below that, in the case of a Markov chain trajectory X^n in $\mathcal{X} \subset \mathbb{R}$, (H1) is a sufficient condition for \mathbf{P}_{X^n} to satisfy (1.3); and, in general, the FKG lattice condition implies the FKG inequality (see notably [9, 29], and Lemma 3.3 below). However, those two properties are not equivalent: counter-examples in various settings have been established in the literature, see e.g. [8, 23].

Determining necessary and sufficient conditions for a class of probability measures to satisfy the FKG condition is a difficult problem in general, see [15]. In the case of random walks, it is well known that any 1-dimensional random walk satisfies the FKG inequality (see Proposition 2.7 below); and in this paper we give a full characterization of 1-dimensional lattice random walks which satisfy the FKG condition.

¹The FKG condition can also be formulated for some non-atomic measures, but for the sake of conciseness we do not consider this case in the present paper.

Proposition 1.11. *Let $\mathbf{P} =: (p_k)_{k \in \mathbb{Z}}$ a (non-trivial) probability measure on \mathbb{Z} . Let $a := \gcd(w_1 - w_2; w_1, w_2 \in \text{Supp}(\mathbf{P}))$: in particular, there exists $b \in \mathbb{Z}$ such that $\text{Supp}(\mathbf{P}) \subset a\mathbb{Z} + b$. Let $X = (X_k)_{k \geq 1}$ a random walk with increment distribution \mathbf{P} started from some $x_0 \in \mathbb{Z}$. Then, the following three statements are equivalent.*

- (i) *for all $n \in \mathbb{N}$, $X^n := (X_k)_{k=1}^n$ satisfies the FKG lattice condition (1.3),*
- (ii) *for all $b' \in \mathbb{Z}$, \mathbf{P} has $(a\mathbb{Z} + b')$ -u.c.,*
- (iii) *the function $k \mapsto p_k$ is log-concave on $a\mathbb{Z} + b$.*

Notice that Lemma 1.6.(i) and Example 1.1, as well as Lemma 1.6.(ii) in the random walk case, are implied by this proposition. The fact that the statement (iii) implies the FKG lattice condition is quite standard, so the most interesting result is the converse implication. In particular this proposition complements Theorem 1.7 by showing that, in the case of random walks, the assumption (H1) and the FKG condition are equivalent; whereas in the case of a general Markov chains \mathbf{P}_{X^n} , we only prove that (H1) implies (1.3). It would be interesting to determine if the converse also holds for generic Markov chains, or to find a counter-example.

As a final comment, let us mention that one can see the association of some conditional distributions as an intermediary property between the FKG condition and FKG inequality —see e.g. [23], and the notions of “downward FKG” and “downward conditional association” therein. Therefore, since we prove in Lemma 3.1 below that the assumption (H1) actually implies the FKG condition, it is not surprising that it yields conditional association for all set A satisfying (H2). On the other hand, when (H1) fails but \mathbf{P}_{X^n} still satisfies the FKG inequality, it would be interesting to determine if there exists (non-trivial) sets $A \subset \mathcal{X}^n$ such that $\mathbf{P}_{X^n}(\cdot|A)$ is still associated.

Organization of the paper. In Section 2 we prove that general random walks are associated, which yields Theorem 1.2 with a weak convergence argument combined with the notion of “infinite association”. We also prove Proposition 1.3. In Section 3 we prove Theorem 1.7, and we apply it to some well-chosen conditioned Markov chains in order to deduce Corollaries 1.8 and 1.9. Finally, in Section 4 we prove Proposition 1.11.

2. ASSOCIATION FOR LÉVY PROCESSES

In this section, we recall classical results on the FKG inequality, and we use them to prove Theorem 1.2. We also prove Proposition 1.3 at the end of the section.

2.1. Classical results on association. Let us recall the following.

Lemma 2.1. *Let $(E, \mathcal{E}, \preceq)$ a metric space endowed with its Borel σ -algebra and a measurable partial order. Let (\mathbf{P}_n) , $n \geq 1$ a sequence of probability measures on (E, \mathcal{E}) , which converges to some probability measure \mathbf{P}_∞ . Assume that \mathbf{P}_n is associated for all $n \geq 1$; then \mathbf{P}_∞ is associated.*

This is a classical result that follows from a standard integral approximation. We leave the proof of the following lemma as an exercise to the reader.

Lemma 2.2. *Let $(E, \mathcal{E}, \preceq, \mathbf{P})$ a metric space endowed with its Borel σ -algebra, a measurable partial order and a probability measure. The inequality (1.1) holds for all increasing functions $f, g \in L^2(\mathbf{P})$ if and only if it holds for all Lipschitz-continuous, non-negative, bounded and increasing functions.*

Proof of Lemma 2.1. Let f, g be bounded, increasing and continuous functions: then, under the assumptions of Lemma 2.1, f, g and \mathbf{P}_n satisfy the inequality (1.1) for all $n \geq 1$. Taking the limit $n \rightarrow +\infty$, so does \mathbf{P}_∞ . Applying Lemma 2.2, this finishes the proof. \square

We also recall the following properties.

Lemma 2.3. *Let $(E_1, \mathcal{E}_1, \preceq_1, \mathbf{P}_1)$ be a partially ordered probability space. Let $(E_2, \mathcal{E}_2, \preceq_2)$ be a partially ordered measurable space. Let $f : E_1 \rightarrow E_2$ a measurable, increasing function, and let \mathbf{P}_2 be the push-forward measure of \mathbf{P}_1 by f . If \mathbf{P}_1 is associated, then so is \mathbf{P}_2 .*

The proof of this lemma is straightforward, see e.g. [1, Proposition 2] for the details.

Lemma 2.4. *Let $(E, \mathcal{E}, \preceq)$ be a totally ordered measurable space, and let \mathbf{P} be any probability measure on E . Then \mathbf{P} is associated.*

Proof. When the space E is totally ordered, a direct computation yields for any increasing functions f, g and probability measure \mathbf{P} that $\mathbf{E}[(f(X) - f(X'))(g(X) - g(X'))] \geq 0$, where X' is an independent copy of $X \sim \mathbf{P}$ (we leave the details to the reader), and this directly implies (1.1). \square

Lemma 2.5. *Let $(E, \mathcal{E}, \preceq)$ be a product of partially ordered measurable spaces, $E = \prod_{i=1}^n E_i$, $\mathcal{E} = \otimes_{i=1}^n \mathcal{E}_i$, endowed with the product partial order. Let $\mathbf{P} = \otimes_{i=1}^n \mathbf{P}_i$ be a product probability measure on $(E, \mathcal{E}, \preceq)$. Then \mathbf{P} is associated if and only if for all $1 \leq i \leq n$, \mathbf{P}_i is associated.*

This result is proven in [1, Theorem 3]. Let us mention that in the case E_i totally ordered for all $1 \leq i \leq n$, it can also be deduced from [29, Theorem 3].

2.2. Association for random walks and Lévy processes. We now turn to Theorem 1.2. We prove the following.

Lemma 2.6. *Let $X = (X_t)_{t \in [0, T]}$ be a d -dimensional Lévy process, $d \geq 1$, started from $X_0 = 0$. It is associated in $\mathcal{D}([0, T], \mathbb{R}^d)$ if and only if, for all $t \in [0, T]$, X_t is associated in \mathbb{R}^d . In particular, if $d = 1$ then X is associated.*

When X_t is associated in \mathbb{R}^d for all $t \in [0, T]$, we say that the infinitely divisible vector X_T is “infinitely associated” (see [13], and also [32]). In particular, it is proven in [13, Proposition 3] that the infinite association of X_T is equivalent to the conditions (i) and (ii) from Theorem 1.2. We also refer to [28] in the case of Gaussian vectors (i.e. $\nu(\mathbb{R}^d) = 0$), and [30, 32] for pure jump processes (i.e. $\Sigma = 0$). Therefore, we only have to show Lemma 2.6, and the theorem follows.

Remark 2.1. Let us mention that, for an infinite divisible vector Y in \mathbb{R}^d , $d \geq 2$, with Lévy measure ν , [32, Theorem 2.1] proves that the assumption $\text{Supp}(\nu) \subset (\mathbb{R}_+)^d \cup (\mathbb{R}_-)^d$ is *not necessary* for Y to be associated. In particular, this implies that there exists a Lévy process $(X_t)_{t \in [0, T]}$ such that X_T is associated in \mathbb{R}^d , but X is not associated in $\mathcal{D}([0, T], \mathbb{R}^d)$.

Lemma 2.6 is a consequence of the following claim. Let us mention that in the case $d = 1$ this result dates back to [31], but we present a full proof for the sake of completeness.

Proposition 2.7. *Let \mathbf{P} be any probability measure on \mathbb{R}^d , and let $X^n = (X_k)_{k=1}^n$ a random walk in \mathbb{R}^d with independent increments with law \mathbf{P} , started from $X_0 = 0$ a.s.. Then, \mathbf{P}_{X^n} is associated in $(\mathbb{R}^d)^n$ if and only if \mathbf{P} is associated in \mathbb{R}^d . In particular if $d = 1$, then \mathbf{P}_{X^n} is associated.*

Proof. Let $d \geq 1$. Since the application $(x_1, \dots, x_n) \mapsto x_1$ is increasing on $(\mathbb{R}^d)^n$ for the product order \preceq , it is clear by Lemma 2.3 that, if \mathbf{P}_{X^n} is associated, then so is $\mathbf{P}_{X_1} = \mathbf{P}$. Conversely, assume now that \mathbf{P} is associated and define,

$$(2.1) \quad \begin{aligned} & (\mathbb{R}^d)^n \mapsto (\mathbb{R}^d)^n \\ \varphi : & (z_i)_{1 \leq i \leq n} \rightarrow \left(\sum_{i=1}^k z_i \right)_{1 \leq k \leq n} . \end{aligned}$$

Notice that φ is increasing, i.e. for $u, v \in (\mathbb{R}^d)^n$, $u \preceq v$ implies $\varphi(u) \preceq \varphi(v)$. Let $U^n := \varphi^{-1}(X^n)$; then $\mathbf{P}_{U^n} = \mathbf{P}^{\otimes n}$ is a product measure. By Lemma 2.5, we deduce that \mathbf{P}_{U^n} is associated. Since \mathbf{P}_{X^n} is the push-forward of \mathbf{P}_{U^n} by φ , by Lemma 2.3 it is also associated.

Finally, when $d = 1$ we have by Lemma 2.4 that any probability measure \mathbf{P} on \mathbb{R} is associated, thus so is \mathbf{P}_{X^n} . \square

Proof of Lemma 2.6. Let $t \in [0, T]$: we clearly have that the application $(x_s)_{s \in [0, T]} \mapsto x_t$ from $\mathcal{D}([0, T], \mathbb{R}^d)$ to \mathbb{R}^d is increasing for the product order. So, by Lemma 2.3, if a random process $(X_t)_{t \in [0, T]}$ is associated then so is X_t for any $t \in [0, T]$.

Let us now prove the converse implication. Let $X = (X_t)_{t \in [0, T]}$ be a d -dimensional Lévy process. For all $n \in \mathbb{N}$ and $0 \leq k \leq Tn$, define, $Y_k^n := X_{k/n}$. Then for all $n \in \mathbb{N}$, Y^n is a random walk which satisfies $Y_n^n = X_1$. Writing $\widehat{X}_t^n = Y_{\lfloor tn \rfloor}^n$, $t \in [0, T]$, this defines a rcll interpolation of Y^n , and we deduce from the classical result [17, Theorem 23.14] that \widehat{X}^n converges weakly to X for the Skorokhod topology in $\mathcal{D}([0, T], \mathbb{R}^d)$. Since we assumed that $X_{1/n}$ is associated for all $n \in \mathbb{N}$, we deduce from Proposition 2.7 that the random walk Y^n is associated. Moreover \widehat{X}^n is the image of Y^n by some increasing function $\varphi_n : (\mathbb{R}^d)^{\lfloor Tn \rfloor} \rightarrow \mathcal{D}([0, T], \mathbb{R}^d)$, so for all $n \in \mathbb{N}$ it is associated by Lemma 2.3 (we leave the details to the reader). Recalling Lemma 2.1, this finishes the proof of the converse implication.

Finally, when $d = 1$ the result follows directly from Lemma 2.4. This completes the proof of Lemma 2.6 and Theorem 1.2. \square

Remark 2.2. In the case of the d -dimensional Brownian motion $(B_t)_{t \in [0, T]}$, the proof of Theorem 1.2 can be simplified substantially: indeed, it follows from Lemmata 2.4 and 2.5 that, for all $t \in [0, T]$, B_t is associated in \mathbb{R}^d ; and the proof of Lemma 2.6 follows from Donsker's theorem instead of [17, Theorem 23.14].

2.3. Proof of Proposition 1.3. This is straightforward: let $f, g : E \rightarrow \mathbb{R}$ be increasing functions (one can also assume that they are bounded, recall Lemma 2.2), then one has

$$\begin{aligned} \mathbf{E}[f(Y + X_0)g(Y + X_0)] &= \mathbf{E}[\mathbf{E}[f(Y + X_0)g(Y + X_0)|X_0]] \\ &\geq \mathbf{E}[\mathbf{E}[f(Y + X_0)|X_0]\mathbf{E}[g(Y + X_0)|X_0]] \\ (2.2) \quad &\geq \mathbf{E}[\mathbf{E}[f(Y + X_0)|X_0]]\mathbf{E}[\mathbf{E}[g(Y + X_0)|X_0]] = \mathbf{E}[f(Y + X_0)]\mathbf{E}[g(Y + X_0)], \end{aligned}$$

where the first inequality comes from the fact that, for almost every X_0 , the functions $y \mapsto f(y + X_0)$, $y \mapsto g(y + X_0)$ are increasing on E and $\mathbf{P}_Y(\cdot|X_0)$ is associated; and the second inequality comes from the observation that $x \mapsto \mathbf{E}[f(Y + X_0)|X_0 = x]$, $x \mapsto \mathbf{E}[g(Y + X_0)|X_0 = x]$ are increasing functions on \mathbb{R}^d , and that \mathbf{P}_{X_0} is associated. This finishes the proof of the proposition. \square

3. CONDITIONAL ASSOCIATION FOR MARKOV CHAINS

In this section we prove Theorem 1.7, and then Corollaries 1.8, 1.9. Recall that, for the sake of simplicity, we only consider Markov chains and processes started from a deterministic initial value a.s..

3.1. Proof of Theorem 1.7. The proof of our main result is achieved in three short steps, which are contained in the following lemmata. Recall Definitions 1.5 and 1.10, as well as the assumptions (H1) and (H2) from Theorem 1.7.

Lemma 3.1. *Let $n \geq 1$, and let $X^n = (X_1, \dots, X_n)$ be the first n steps of a Markov chain on some countable subset $\mathcal{X} \subset \mathbb{R}$, started from $X_0 = x_0 \in \mathcal{X}$ and with transition kernel \mathbf{p} . If \mathbf{p} satisfies (H1), then \mathbf{P}_{X^n} satisfies the FKG condition.*

Proof. For $(u_j)_{1 \leq j \leq n}$, write $u^n := (u_1, \dots, u_n)$. Since $\mathbf{p} = (p_{x,y})_{x,y \in \mathcal{X}}$ satisfies (H1), this implies that for \mathbf{P}_{X^n} -a.e. $u^n, v^n \in \mathcal{X}^n$, one has

$$(3.1) \quad \begin{aligned} \mathbf{P}_{X^n}(u^n \vee v^n) \mathbf{P}_{X^n}(u^n \wedge v^n) &= \prod_{j=0}^{n-1} p_{u_j \vee v_j, u_{j+1} \vee v_{j+1}} \times \prod_{j=0}^{n-1} p_{u_j \wedge v_j, u_{j+1} \wedge v_{j+1}} \\ &\geq \prod_{j=0}^{n-1} p_{u_j, u_{j+1}} \times \prod_{j=0}^{n-1} p_{v_j, v_{j+1}} = \mathbf{P}_{X^n}(u^n) \mathbf{P}_{X^n}(v^n), \end{aligned}$$

where $u_0 = v_0 := x_0$, which finishes the proof. \square

Lemma 3.2. *Let $(E, \mathcal{E}, \preceq, \mathbf{P})$ be a partially ordered probability space, and assume that \mathbf{P} is purely atomic and satisfies the FKG condition. Let $A \in \mathcal{E}$ which satisfies (H2). Then, $\mathbf{P}(\cdot|A)$ satisfies the FKG condition.*

Proof. This is straightforward. For \mathbf{P} -a.e. $u, v \in A$, (H2) implies that $u \wedge v, u \vee v \in A$; in particular,

$$\mathbf{P}(u \vee v|A) \mathbf{P}(u \wedge v|A) = \frac{1}{\mathbf{P}(A)^2} \mathbf{P}(u \vee v) \mathbf{P}(u \wedge v) \geq \frac{1}{\mathbf{P}(A)^2} \mathbf{P}(u) \mathbf{P}(v) = \mathbf{P}(u|A) \mathbf{P}(v|A),$$

which finishes the proof. \square

Lemma 3.3. *Let \mathbf{P} a probability measure on a countable, locally finite subset of $(\mathbb{R}^n, \text{Bor}(\mathbb{R}^n), \preceq)$ which satisfies the FKG condition. Then it is associated.*

Proof. This is essentially the content of [29]. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ a non-negative, bounded and increasing function, and define the probability measure \mathbf{Q} on \mathbb{R}^n with

$$\frac{d\mathbf{Q}}{d\mathbf{P}}(u) = \frac{g(u)}{\mathbf{E}g}, \quad \text{for } \mathbf{P}\text{-a.e. } u \in \mathbb{R}^n,$$

where $\mathbf{E} = \mathbf{E}_{\mathbf{P}}$ denotes the expectation with respect to \mathbf{P} (we assume that $\mathbf{E}g > 0$, otherwise $g \equiv 0$ \mathbf{P} -a.e. and (1.1) holds for all f). Since \mathbf{P} satisfies (1.3) and g is increasing, one clearly has for \mathbf{P} -a.e. $u, v \in \mathbb{R}^n$,

$$\mathbf{Q}(u \vee v) \mathbf{P}(u \wedge v) = \frac{g(u \vee v)}{\mathbf{E}g} \mathbf{P}(u \vee v) \mathbf{P}(u \wedge v) \geq \frac{g(u)}{\mathbf{E}g} \mathbf{P}(u) \mathbf{P}(v) = \mathbf{Q}(u) \mathbf{P}(v).$$

Therefore, it follows² from [29, Theorem 3] that for $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a bounded increasing function, one has $\mathbf{E}_{\mathbf{Q}}f \geq \mathbf{E}_{\mathbf{P}}f$, which is exactly (1.1). Recalling Lemma 2.2, the inequality (1.1) also holds for all increasing $f, g \in L^2(\mathbf{P})$, finishing the proof. \square

With these lemmata at hand, the proof of Theorem 1.7 is immediate.

Proof of Theorem 1.7. Under the assumptions (H1, H2), Lemmata 3.1, 3.2 imply that the measure $\mathbf{P}_{X^n}(\cdot|A)$ satisfies the FKG condition. By Lemma 3.3, it is associated, completing the proof. \square

²Indeed, since $\text{Supp}(\mathbf{P})$ is locally finite, there exists a σ -finite counting measure μ on some set $A \subset \mathbb{R}$ such that $\text{Supp}(\mathbf{P}) \subset A^n$. Letting $\omega := \mu^{\otimes n}$ in [29], this yields the result (details are left to the reader).

3.2. Proof of the corollaries. Let $T > 0$, $X^n = (X_1, \dots, X_n)$ a random vector in \mathbb{R}^n , and $X_0 := x_0 \in \mathbb{R}$. One can define a rcll (resp. continuous) interpolation of X in $\mathcal{D}([0, T], \mathbb{R})$ (resp. $\mathcal{C}([0, T], \mathbb{R})$) by letting, for $t \in [0, T]$,

$$(3.2) \quad \begin{aligned} \widehat{X}_t^n &:= \sum_{j=0}^n X_j \mathbf{1}_{[Tj/n, T(j+1)/n)}(t), \\ \text{resp. } \widetilde{X}_t^n &:= \sum_{j=0}^n [X_j + (\frac{nt}{T} - j)(X_{j+1} - X_j)] \mathbf{1}_{[Tj/n, T(j+1)/n)}(t). \end{aligned}$$

Both are increasing functions of X^n ; hence by Lemma 2.3, if the vector X^n is associated, then the processes \widehat{X}^n and \widetilde{X}^n are associated as well. Moreover by Lemma 2.1, if the sequence of processes \widehat{X}^n (or \widetilde{X}^n) converges weakly, then the limiting process is associated as well. Therefore, each corollary of Theorem 1.7 will be proven by determining a sequence of Markov chains X^n with transition kernel \mathbf{p}^n , and sets A_n in $\mathcal{E}_{\mathcal{D}}$ (or $\mathcal{E}_{\mathcal{C}}$) such that,

- for all $n \geq 1$, \mathbf{p}^n satisfies (H1),
- for all $n \geq 1$, A_n satisfies (H2) under $\mathbf{P}_{\widehat{X}^n}$ (or $\mathbf{P}_{\widetilde{X}^n}$),
- the sequence of processes \widehat{X}^n (or \widetilde{X}^n) conditioned on A_n , $n \geq 1$, converges weakly to the desired process.

Proof of Corollary 1.8. Let us prove the corollary simultaneously in the four cases, which we denote respectively with the subscripts $j \in \{1, 2, 3, 4\}$. Let Y^n be the first n steps of a random walk with discrete Laplace increment for some $\beta > 0$.³ We may take $\beta > 0$ such that the Laplace increments have unit variance, and we let $X^n := n^{-1/2}Y^n$. Then for all $n \geq 1$, it follows from Lemma 1.6.(i) that the transition kernel \mathbf{p}^n of the Markov chain X^n satisfies (H1).

We then define the sets $A_{j,n} = A_j$, $n \geq 1$ differently for each case $j \in \{1, 2, 3, 4\}$. Let,

$$(3.3) \quad A_j := \begin{cases} \{x \in \mathcal{D}([0, T], \mathbb{R}) \mid x_T = 0\} & \text{for } j = 1, \\ \{x \in \mathcal{D}([0, T], \mathbb{R}) \mid x_T = 0; \forall t, x_t \geq 0\} & \text{for } j = 2, \\ \{x \in \mathcal{D}([0, T], \mathbb{R}) \mid \forall t, x_t \geq 0\} & \text{for } j = 3, \\ \{x \in \mathcal{D}([0, T], \mathbb{R}) \mid \forall t, x_t \in [a(t), b(t)]\} & \text{for } j = 4, \end{cases}$$

Then, for n large enough, one clearly has in the four cases that $\mathbf{P}_{\widehat{X}^n}(A_j) > 0$ and A_j is max/min-stable, so (H2) holds.

By Theorem 1.7, we deduce in each case $j \in \{1, 2, 3, 4\}$ that $\mathbf{P}_{\widehat{X}^n}(\cdot | A_j)$ (similarly $\mathbf{P}_{\widetilde{X}^n}(\cdot | A_j)$) is associated for all $n \in \mathbb{N}$. Therefore, it only remains to confirm in each case that the conditioned random walk converges weakly to the desired limit. Those results are classical: for the Brownian bridge $j = 1$, this is the content of [24]. For the Brownian excursion $j = 2$, this can be found in [16]. For the Brownian meander $j = 3$, we refer to [14]. Finally in the case $j = 4$, by assumption the event A_4 has positive probability for the Wiener measure, therefore the weak convergence can be deduced directly from Donsker's theorem (details are left to the reader). This concludes the proof of the corollary. \square

³Of course, one could also consider a simple random walk (recall Lemma 1.6.(ii)), provided that they restrict themselves to $n \in 2\mathbb{N}$ in the cases $j \in \{1, 2\}$ below.

Proof of Corollary 1.9. For $\nu \in \mathbb{R}$, we may define a Markov chain $(Y_k)_{k \geq 0}$ on \mathbb{N} with transition probabilities that satisfy,

$$p_{i,i+1} = 1 - p_{i,i-1} := \frac{1}{2} \left(1 + \frac{2\nu + 1}{2i} + o(i^{-1}) \right), \quad \text{if } i \geq 1,$$

$$p_{0,1} = 1,$$

and $p_{i,j} = 0$ otherwise. Since Y only makes unitary steps, by Lemma 1.6.(ii) its transition kernel satisfies (H1). For $n \geq 1$, letting $X^n := n^{-1/2}(Y_1, \dots, Y_n)$ and \tilde{X}^n its rcl interpolation in $\mathcal{D}([0, T], \mathbb{R})$, one deduces from Theorem 1.7, Remark 1.2 and Lemma 2.3 that $\mathbf{P}_{\tilde{X}^n}$ is associated (this is a carbon copy of previous arguments, details are left to the reader). Moreover when $\nu > -1$, the convergence of the process \tilde{X}^n to the Bessel process of index ν is proven in [22, Theorem 5.1]. By Lemma 2.1, this concludes the proof of the corollary. \square

Remark 3.1. Corollary 1.9 is also expected to hold for indices $\nu \leq -1$ (i.e. when the barrier at 0 is absorbing), however this requires a convergence result to Bessel processes different from [22]. Since this is not the object of the present paper, we leave this consideration to later work.

4. CHARACTERIZATION OF THE FKG CONDITION FOR LATTICE RANDOM WALKS

In this section we prove Proposition 1.11. Recall Definitions 1.5 and 1.10.

Proof of Proposition 1.11. First, recall the assumption (H1) from Theorem 1.7, and notice that it is implied by (ii); so we deduce from Lemma 3.1 that (ii) \Rightarrow (i).

We now prove (i) \Rightarrow (ii). Recall that $a := \gcd(w_1 - w_2; w_1, w_2 \in \text{Supp}(\mathbf{P}))$, and the notation $u^k := (u_1, \dots, u_k)$. If (ii) fails, then there exists some large $k_0 \in \mathbb{N}$ and $u_{k_0} \leq v_{k_0}$ in $\text{Supp}(X_{k_0})$, $u_{k_0+1} \geq v_{k_0+1}$ in \mathbb{Z} , such that,

$$\mathbf{P}(u_{k_0+1} - u_{k_0})\mathbf{P}(v_{k_0+1} - v_{k_0}) > \mathbf{P}(u_{k_0+1} - v_{k_0})\mathbf{P}(v_{k_0+1} - u_{k_0}) \geq 0.$$

By definition of $\text{Supp}(X_{k_0})$ there exists two paths $u^{k_0} = (u_1, \dots, u_{k_0})$ and $v^{k_0} = (v_1, \dots, v_{k_0})$ such that $\mathbf{P}_{X^{k_0}}(u^{k_0}) > 0$, $\mathbf{P}_{X^{k_0}}(v^{k_0}) > 0$. For $j \geq k_0$, let us define by induction,

$$u_{j+1} := \begin{cases} u_j + (u_{k_0+1} - u_{k_0}) & \text{if } j - k_0 \text{ is even,} \\ u_j + (v_{k_0+1} - v_{k_0}) & \text{if } j - k_0 \text{ is odd,} \end{cases}$$

$$\text{and } v_{j+1} := \begin{cases} v_j + (v_{k_0+1} - v_{k_0}) & \text{if } j - k_0 \text{ is even,} \\ v_j + (u_{k_0+1} - u_{k_0}) & \text{if } j - k_0 \text{ is odd.} \end{cases}$$

Then, one has $u_j \leq v_j$ if $j - k_0$ is even, and $v_j \leq u_j$ if $j - k_0$ is odd. Therefore, one observes for $m \geq k_0$ that,

$$\frac{\mathbf{P}_{X^m}(u^m \wedge v^m)\mathbf{P}_{X^m}(u^m \vee v^m)}{\mathbf{P}_{X^m}(u^m)\mathbf{P}_{X^m}(v^m)} \leq C \left(\frac{\mathbf{P}(u_{k_0+1} - v_{k_0})\mathbf{P}(v_{k_0+1} - u_{k_0})}{\mathbf{P}(u_{k_0+1} - u_{k_0})\mathbf{P}(v_{k_0+1} - v_{k_0})} \right)^m,$$

for some $C > 0$. Since the r.h.s. goes to 0 as $m \rightarrow +\infty$, there exists a large m and u^m, v^m such that (1.3) fails, which conclude the proof of (i) \Rightarrow (ii).

Let us now show that (iii) \Rightarrow (ii). Let $b' \in \mathbb{Z}$, $u_1, v_1 \in a\mathbb{Z} + b'$ and $u_2, v_2 \in \mathbb{Z}$. We may assume without loss of generality that $u_1 > v_1$, and that $u_2 < v_2$ (otherwise (1.2) clearly holds). Let us define,

$$(4.1) \quad w_1 = u_2 - u_1, \quad w_2 = u_2 - v_1, \quad w_3 = v_2 - u_1, \quad w_4 = v_2 - v_1.$$

In particular, one has $w_i \in (a\mathbb{Z} + b)$ for $i \in \{1, \dots, 4\}$, $w_1 < w_i < w_4$ for $i \in \{2, 3\}$ and $w_1 + w_4 = w_2 + w_3$. Moreover (1.2) can be rewritten $p_{w_1}p_{w_4} \leq p_{w_2}p_{w_3}$. If $p_{w_1} = 0$ or $p_{w_4} = 0$ then (1.2) clearly holds, otherwise one has

$$(4.2) \quad p_{w_2}p_{w_3} = p_{w_1}p_{w_4} \frac{p_{w_2}p_{w_3}}{p_{w_1}p_{w_4}} \geq p_{w_1}p_{w_4} \left(\frac{p_{w_2}}{p_{w_1}} \right)^{1 - \frac{w_4 - w_3}{w_2 - w_1}} = p_{w_1}p_{w_4},$$

where we used that $k \mapsto p_k$ is log-concave on $a\mathbb{Z} + b$.

It remains to prove (ii) \Rightarrow (iii). Let $w_1 \in a\mathbb{Z} + b$, and let us assume w.l.o.g. that $p_{w_1} > 0$. We first prove by induction that for all $i \geq 1$,

$$(4.3) \quad \left(\frac{p_{w_1+a}}{p_{w_1}} \right)^{(a)^{-1}} \geq \left(\frac{p_{w_1+ai}}{p_{w_1}} \right)^{(ai)^{-1}}.$$

This clearly holds for $i = 1$. We now assume that (4.3) holds for some $i \in \mathbb{N}$, and we define $w_2 := w_1 + a$, $w_3 := w_1 + ai$ and $w_4 := w_1 + a(i + 1)$. With this notation, we claim that one can define some points $u_1, u_2, v_1, v_2 \in \mathbb{Z}$ such that the identities (4.1) hold, and such that (ii) yields $p_{w_1}p_{w_4} \leq p_{w_2}p_{w_3}$ (we leave the details to the reader). Therefore,

$$\frac{p_{w_2}}{p_{w_1}} \geq \frac{p_{w_4}}{p_{w_3}} = \frac{p_{w_4} p_{w_1}}{p_{w_1} p_{w_3}} \geq \frac{p_{w_4}}{p_{w_1}} \left(\frac{p_{w_1}}{p_{w_2}} \right)^i,$$

where the second inequality follows from the induction hypothesis. Rearranging the terms, this shows that (4.3) holds for the index $(i + 1)$, finishing the proof of (4.3) by induction. With (4.3) at hand, we deduce that for all $w_1, w_2, w_3 \in a\mathbb{Z} + b$, $w_1 < w_2 < w_3$, one has

$$(4.4) \quad \frac{\log p_{w_2} - \log p_{w_1}}{w_2 - w_1} \geq \frac{\log p_{w_3} - \log p_{w_1}}{w_3 - w_1}.$$

Indeed, if (4.4) fails, then there exists some $i \in \{w_1, \dots, w_2 - 1\}$ such that,

$$\log p_{i+1} - \log p_i < \frac{\log p_{w_3} - \log p_{w_1}}{w_3 - w_1},$$

and this directly contradicts (4.3).

Similarly, we claim that, for all $w_1, w_2, w_3 \in a\mathbb{Z} + b$, $w_1 < w_2 < w_3$,

$$(4.5) \quad \left(\frac{p_{w_3}}{p_{w_1}} \right)^{(w_3-w_1)^{-1}} \geq \left(\frac{p_{w_3}}{p_{w_2}} \right)^{(w_3-w_2)^{-1}},$$

where the case $w_2 = w_3 - a$ is proven by (descending) induction on $w_1 \in a\mathbb{Z} + b$, and the general case follow from a proof by contradiction (this is a carbon copy of the proofs of (4.3,4.4), so we do not reproduce them). This finishes the proof that $k \mapsto p_k$ is log-concave. \square

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