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THE LAPLACIAN WITH COMPLEX MAGNETIC FIELDS

DAVID KREJČIŘÍK, THO NGUYEN DUC, AND NICOLAS RAYMOND

ABSTRACT. We study the two-dimensional magnetic Laplacian when the magnetic field is allowed to be complex-valued. Under the assumption that the imaginary part of the magnetic potential is relatively form-bounded with respect to the real part of the magnetic Laplacian, we introduce the operator as an m -sectorial operator. In two dimensions, sufficient conditions are established to guarantee that the resolvent is compact. In the case of non-critical complex magnetic fields, a WKB approach is used to construct semiclassical pseudomodes, which do not exist when the magnetic field is real-valued.

1. INTRODUCTION

Since the appearance of the highly influential paper [1] at the turn of the millennium, there has been a growing interest in Schrödinger operators with complex-valued electric potentials. This is explained not only by diverse physical motivations, but notably due to a new concept of quantum mechanics where observables can be represented by non-self-adjoint operators [26, 21]. Moreover, the mathematical studies lead to unprecedented spectral properties such as the existence of pseudomodes [9, 10, 20, 3] or the existence of eigenvalues accumulating at non-zero points of the essential spectrum [4, 8, 5].

It is about time that someone would address the case of Schrödinger operators where the magnetic potential is complex-valued now. This is the subject of the present paper. Our motivation is not merely our mathematical curiosity, but also strong physical motivations. Among these, let us mention superconductors [13, 14], quantum statistical physics [24, 2], speculations about a novel type of magnetic monopoles [25], stability of black holes in general relativity [16, 17] and the concept of quasi-self-adjointness again [19]. What is more, complexifying the magnetic field is mathematically much more challenging than its electric counterpart. That is probably why rigorous results do not exist in the literature and the objective of this paper is to fill in this gap.

1.1. The problem. We are concerned with the differential expression

$$(-ih\nabla - \mathbf{A})^2 = \sum_{j=1}^d (-ih\partial_{x_j} - A_j)^2, \quad (1.1)$$

where $d \geq 1$ is the dimension, $\mathbf{A} = (A_1, \dots, A_d) : \mathbb{R}^d \rightarrow \mathbb{C}^d$ is the complexified (magnetic) vector potential and h is a small positive (semiclassical) parameter. If the imaginary part of \mathbf{A} is non-zero, it is not even clear that (1.1) leads to a well defined operator in $L^2(\mathbb{R}^d)$. More specifically, our first concern is

- (I) to identify the right subspace of $L^2(\mathbb{R}^d)$ as a domain on which (1.1) is realised as a closed operator with non-empty resolvent set.

Our next curiosity is about the realisation of complexified “magnetic bottles”, *i.e.*,

- (II) to find conditions on \mathbf{A} which guarantee that the operator has compact resolvent.

Since it is the pseudospectrum which describes non-self-adjoint phenomena, our last task is

- (III) to construct pseudomodes in the semiclassical limit $h \rightarrow 0$.

Of course, other problems could be raised, but already these three tasks lead to considerable mathematical challenges.

1.2. Main results.

1.2.1. *The magnetic Laplacian as an m -sectorial operator.* In this paper, we deal with task (I) by assuming that the imaginary part of the vector potential is relatively form-bounded with respect to the real part of the magnetic Laplacian. More specifically, we always make the following hypothesis.

Assumption 1.1. *Let $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^d)$ and assume that there exist two constants $a \in (0, 1)$ and $b \geq 0$ such that*

$$\int_{\mathbb{R}^d} |(\text{Im } \mathbf{A})u|^2 dx \leq a \int_{\mathbb{R}^d} |(-ih\nabla - \text{Re } \mathbf{A})u|^2 dx + b \int_{\mathbb{R}^d} |u|^2 dx, \quad (1.2)$$

for all $u \in C_c^\infty(\mathbb{R}^d)$.

We define the sesquilinear form naturally associated with (1.1), *i.e.*,

$$Q_{h,\mathbf{A}}(u, v) := \int_{\mathbb{R}^d} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ih\nabla - \overline{\mathbf{A}})v} dx, \quad (1.3)$$

$$\text{Dom}(Q_{h,\mathbf{A}}) := \{u \in L^2(\mathbb{R}^d) : (-ih\nabla - \text{Re } \mathbf{A})u \in L^2(\mathbb{R}^d, \mathbb{C}^d), (\text{Im } \mathbf{A})u \in L^2(\mathbb{R}^d, \mathbb{C}^d)\}.$$

The objective of task (I) is fulfilled by the following theorem.

Theorem 1.2. *Under Assumption 1.1, $Q_{h,\mathbf{A}}$ is densely defined, closed and sectorial. Then, the operator defined by*

$$\begin{aligned} \text{Dom}(\mathcal{L}_{h,\mathbf{A}}) &:= \{u \in \text{Dom}(Q_{h,\mathbf{A}}) : \exists f \in L^2(\mathbb{R}^d), \forall v \in \text{Dom}(Q_{h,\mathbf{A}}), Q_{h,\mathbf{A}}(u, v) = \langle f, v \rangle\}, \\ \mathcal{L}_{h,\mathbf{A}} &:= f, \end{aligned}$$

is m -sectorial.

While the closed representative $\mathcal{L}_{h,\mathbf{A}}$ of (1.1) is easily introduced in all dimensions, it is well known that its spectral analysis becomes cumbersome in higher dimensions even in the self-adjoint case. Therefore, we modestly restrict to dimension $d = 2$ in the sequel. In this case, the magnetic field

$$\mathbf{B} := \text{curl } \mathbf{A} = \partial_{x_1} A_2 - \partial_{x_2} A_1$$

is a scalar function. Then pointwise sufficient conditions which guarantee Assumption 1.1 are contained in the following proposition.

Proposition 1.3. *Let $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C}^2)$ and $\mathbf{B} \in L^1_{\text{loc}}(\mathbb{R}^2)$. Assume one of the following conditions:*

(C1): *there exist $\varepsilon_1 \in (0, 1)$, $C_1 \in \mathbb{R}$ such that*

$$|\text{Im } \mathbf{A}(x)|^2 \leq \pm \varepsilon_1 h \text{Re } \mathbf{B}(x) + C_1, \quad \forall x \in \mathbb{R}^2;$$

(C2): *there exist $\varepsilon_2 \in (0, \frac{1}{2})$, $C_2 \in \mathbb{R}$ such that*

$$|\text{Im } \mathbf{A}(x)|^2 \leq \pm \varepsilon_2 h \text{Im } \mathbf{B}(x) + C_2, \quad \forall x \in \mathbb{R}^2.$$

Then Assumption 1.1 holds.

Of course, the conditions are automatically satisfied if $\text{Im } \mathbf{A} = 0$ and if $\text{Re } \mathbf{B}$ is bounded from below or from above. If $\text{Im } \mathbf{B} = 0$, note that one can choose a real vector potential \mathbf{A} , but that one can also choose a complex gauge (leading to non unitarily equivalent operators).

In practice, when \mathbf{A} and \mathbf{B} are continuous on \mathbb{R}^2 , it suffices to verify **(C1)** or **(C2)** for large values of $|x|$. Furthermore, if $\text{Im } \mathbf{A}$ is bounded and $\text{Re } \mathbf{B}(x)$ (respectively, $\text{Im } \mathbf{B}(x)$) does not change sign for large $|x|$, then **(C1)** (respectively, **(C2)**) holds.

Example 1 (Homogeneous magnetic field). Unfortunately, Assumption 1.1 excludes the important case of constant magnetic field $\mathbf{B} = c \in \mathbb{C}$ corresponding to

$$\mathbf{A}(x) := c \frac{1}{2} (-x_2, x_1) \quad \text{or} \quad \mathbf{A}(x) := c(0, x_1),$$

unless $\text{Im } c = 0$. In fact, implementing task (I) in this case seems particularly non-trivial.

Example 2 (Complexified Miller–Simon’s potential). Consider

$$\mathbf{A}(x) := \left(-\frac{cx_2}{(1+|x|)^\alpha}, \frac{cx_1}{(1+|x|)^\alpha} \right),$$

where $c := c_1 + ic_2 \in \mathbb{C}$ with $c_1, c_2 \in \mathbb{R}$ and $\alpha > 0$. When c is real, the magnetic Laplacian with this type of magnetic potential was considered in [23]. When $c_2 \neq 0$ and if we assume further that $\alpha \geq 1$ then it can be checked that both condition **(C1)** and **(C2)** are satisfied. Indeed, since $|\mathbf{A}|$ is bounded and that

$$\mathbf{B}(x) = c \left(\frac{2}{(1+|x|)^\alpha} - \frac{\alpha|x|}{(1+|x|)^{\alpha+1}} \right)$$

is also bounded on \mathbb{R}^2 , we can therefore find sufficient large constants C_1 and C_2 such that **(C1)** and **(C2)** hold. With more effort, we can see that $\text{Dom}(Q_{h,\mathbf{A}}) = H^1(\mathbb{R}^2)$ and $\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = H^2(\mathbb{R}^2)$. In particular, when $c_1 = 0$, we have an example of purely imaginary magnetic fields such that our magnetic Laplacian is well defined.

Example 3. (Purely imaginary exponential magnetic potentials) Now we give an example of an unbounded purely imaginary \mathbf{A} for which **(C1)** fails but **(C2)** still holds. Consider

$$\mathbf{A}(x) := ice^{|x|^2}(-x_2, x_1), \quad (1.4)$$

where $c \in (-h, h)$ and $h > 0$. It can be checked that

$$|\text{Im } \mathbf{A}(x)|^2 = c^2|x|^2e^{|x|^2}, \quad \text{Im } \mathbf{B}(x) = 2c(|x|^2 + 1)e^{|x|^2}.$$

Hence, we have

$$|\text{Im } \mathbf{A}(x)|^2 \leq \frac{c}{2} \text{Im } \mathbf{B}(x), \quad \forall x \in \mathbb{R}^2.$$

By choosing $C_2 = 0$ and $\varepsilon_2 \in \left(\frac{|c|}{2h}, \frac{1}{2}\right)$, **(C2)** is verified.

1.2.2. *Compactness of the resolvent.* Let us now consider task (II) about the compactness of the resolvent (which implies that the spectrum is purely discrete).

Theorem 1.4. *Let \mathbf{A} satisfy Assumption 1.1 and $\text{Re } \mathbf{A} \in L_{\text{loc}}^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Assume one of the following conditions:*

- (H1):** $\text{Re } \mathbf{B} \in C^0(\mathbb{R}^2)$ and $\lim_{|x| \rightarrow +\infty} |\text{Re } \mathbf{B}(x)| = +\infty$,
- (H2):** $\text{Im } \mathbf{B} \in C^0(\mathbb{R}^2)$ and $\lim_{|x| \rightarrow +\infty} |\text{Im } \mathbf{B}(x)| = +\infty$,
- (H3):** $\text{Im } \mathbf{A} \in C^0(\mathbb{R}^2, \mathbb{R}^2)$ and $\lim_{|x| \rightarrow +\infty} |\text{Im } \mathbf{A}(x)| = +\infty$.

Then $\mathcal{L}_{h,\mathbf{A}}$ has compact resolvent.

The first condition of the theorem extends the usual magnetic bottle realisation [15] to the non-self-adjoint setting. More interestingly, it follows that the magnetic Laplacian of Example 3 (with $c \neq 0$) has compact resolvent despite $\text{Re } \mathbf{B} = 0$.

1.2.3. *Semiclassical pseudomodes.* Finally, let us present our implementation of task (III) about the construction of semiclassical pseudomodes. To this purpose, let us restrict to magnetic potentials $\mathbf{A} \in C^\infty(\mathbb{R}^2, \mathbb{C}^2)$ satisfying Assumption 1.1. In the plane \mathbb{R}^2 , we define the subset

$$\Gamma := \left\{ x \in \mathbb{R}^2 \left| \begin{array}{l} \operatorname{Im} \mathbf{A}(x) = 0, \quad \mathbf{B}(x) \neq 0, \quad \partial_{\bar{z}} \mathbf{B}(x) \neq 0, \\ Q_1(x) > 0, \quad Q_1(x)Q_3(x) - Q_2^2(x) > 0 \end{array} \right. \right\},$$

where

$$\begin{aligned} Q_1(x) &:= \frac{1}{4} \operatorname{Re} \left[\mathbf{B}(x) \left(1 + \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(x) \right) \right] + \frac{1}{2} \partial_{x_1} \operatorname{Im} A_1(x), \\ Q_2(x) &:= \frac{1}{4} \operatorname{Im} \left[\mathbf{B}(x) \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(x) \right] + \frac{1}{4} (\partial_{x_1} \operatorname{Im} A_2 + \partial_{x_2} \operatorname{Im} A_1(x)), \\ Q_3(x) &:= \frac{1}{4} \operatorname{Re} \left[\mathbf{B}(x) \left(1 - \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(x) \right) \right] + \frac{1}{2} \partial_{x_2} \operatorname{Im} A_2(x), \end{aligned} \quad (1.5)$$

and $\partial_z := \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ stand for the usual Wirtinger derivatives. Our main result is the following theorem.

Theorem 1.5. *Let $x^0 \in \Gamma$ and assume that \mathbf{B} is real-analytic at x^0 . Then there exist constants $C, h_0 > 0$ and a family of functions $(u_h)_{0 < h \leq h_0} \subset C_c^\infty(\mathbb{R}^2)$ such that, for all $h \in (0, h_0)$,*

$$\|(\mathcal{L}_{h, \mathbf{A}} - h\mathbf{B}(x^0)) u_h\| \leq \exp\left(-\frac{C}{h^{1/7}}\right) \|u_h\|. \quad (1.6)$$

Note that $\Gamma = \emptyset$ if $\operatorname{Im} \mathbf{A} = 0$, because $Q_1 Q_3 - Q_2^2 = 0$ in this case. It follows that the existence of the pseudomode $(u_h)_{0 < h \leq h_0}$ is possible only in the present non-self-adjoint setting. Below we provide examples of magnetic potentials for which $\Gamma \neq \emptyset$; these include polynomial and oscillating complex magnetic fields.

In order to prove Theorem 1.5, we use the ideas of the magnetic WKB strategy as developed in [6, 12] in the self-adjoint case to find approximations of eigenfunctions. In our non-self-adjoint development, it is remarkable that the magnetic potential \mathbf{A} does not need to be analytic for this result to hold. This technical outcome is due to the use of a formal gauge, which allows for the local transformation to an appropriate analytic potential at x^0 , see Section 3.1.

We stress that Theorem 1.5 is not covered by [10]. Indeed, the Weyl symbol of $\mathcal{L}_{h, \mathbf{A}}$ is

$$p(x, \xi) = |\xi - \operatorname{Re} \mathbf{A}|^2 - |\operatorname{Im} \mathbf{A}|^2 - 2i\langle \xi - \operatorname{Re} \mathbf{A}, \operatorname{Im} \mathbf{A} \rangle.$$

Note that $p(x, \xi) = 0$ is equivalent to $\xi - \operatorname{Re} \mathbf{A} = \pm(\operatorname{Im} \mathbf{A})^\perp$ and that

$$\begin{aligned} \{\operatorname{Re} p, \operatorname{Im} p\}(x, \xi) \\ = -4(\xi - \operatorname{Re} \mathbf{A}) \cdot \partial_x \langle \xi - \operatorname{Re} \mathbf{A}, \operatorname{Im} \mathbf{A} \rangle - 2\partial_x (|\xi - \operatorname{Re} \mathbf{A}|^2 - |\operatorname{Im} \mathbf{A}|^2) \cdot \operatorname{Im} \mathbf{A}, \end{aligned}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Given $x^0 \in \Gamma$, then for any point $(x^0, \xi^0) \in \mathbb{R}^4$ such that $p(x^0, \xi^0) = 0$, we have $\{\operatorname{Re} p, \operatorname{Im} p\}(x^0, \xi^0) = 0$ (since $\operatorname{Im} \mathbf{A}(x^0) = 0$), so [10, Thm. 1.2] does not apply.

1.3. Structure of the paper. In Section 2 we deal with tasks (I) and (II); in particular, we introduce the operator $\mathcal{L}_{h, \mathbf{A}}$ and prove Proposition 1.3 and Theorem 1.4. Task (III) is considered in Section 3; namely, we establish Theorem 1.5 and provide the specific examples which the theorem applies to.

2. SECTORIAL MAGNETIC LAPLACIANS

The main purpose of this section is to realise the differential expression (1.1) as an m-sectorial operator in $L^2(\mathbb{R}^d)$ via a sesquilinear form and study its spectral property compact resolvent.

Assume $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^d)$. Expanding $(-ih\nabla - \mathbf{A})^2$, the action of this operator should be

$$(-ih\nabla - \text{Re } \mathbf{A})^2 - (\text{Im } \mathbf{A})^2 - i(\text{Im } \mathbf{A} \cdot (-ih\nabla - \text{Re } \mathbf{A}) + (-ih\nabla - \text{Re } \mathbf{A}) \cdot \text{Im } \mathbf{A}).$$

Aiming at the variational definition, this suggests considering the form domain

$$\mathcal{V}_{h,\mathbf{A}} := \{u \in L^2(\mathbb{R}^d) : (-ih\nabla - \text{Re } \mathbf{A})u \in L^2(\mathbb{R}^d, \mathbb{C}^d), (\text{Im } \mathbf{A})u \in L^2(\mathbb{R}^d, \mathbb{C}^d)\} \quad (2.1)$$

equipped with the natural inner product

$$\langle u, v \rangle_{\mathcal{V}_{h,\mathbf{A}}} := \langle u, v \rangle + \langle (-ih\nabla - \text{Re } \mathbf{A})u, (-ih\nabla - \text{Re } \mathbf{A})v \rangle + \langle (\text{Im } \mathbf{A})u, (\text{Im } \mathbf{A})v \rangle, \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\mathbb{R}^d)$, linear in the first component.

Remark 2.1. Since $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^d)$, it follows that for all $u \in L^2(\mathbb{R}^d)$,

$$(\text{Re } \mathbf{A})u \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^d), \quad (\text{Im } \mathbf{A})u \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^d).$$

In particular, $(-ih\nabla - \text{Re } \mathbf{A})u$ and $(\text{Im } \mathbf{A})u$ can be understood in the sense of distributions in (2.1).

Just as in the self-adjoint case, we have the following properties of the form domain.

Proposition 2.2. *Let $h > 0$ and $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^d)$, the following holds.*

- (a) $(\mathcal{V}_{h,\mathbf{A}}, \langle \cdot, \cdot \rangle_{\mathcal{V}_{h,\mathbf{A}}})$ is a Hilbert space.
- (b) $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{V}_{h,\mathbf{A}}$.
- (c) Under the assumption 1.1, $\mathcal{V}_{h,\mathbf{A}} = \mathcal{V}_{h,\text{Re } \mathbf{A}}$, and two norms $\|\cdot\|_{\mathcal{V}_{h,\text{Re } \mathbf{A}}}$ and $\|\cdot\|_{\mathcal{V}_{h,\mathbf{A}}}$ are equivalent.

Proof. The proof of (a) is standard, and the proof of the density result in (b) follows the same steps as in [22, Theorem 7.22], so we omit the details here. Using (b), we extend the assumption (1.2) to hold on the space $\mathcal{V}_{h,\mathbf{A}}$. Consequently, statements in (c) are established. \square

After exploring several key properties of the space $\mathcal{V}_{h,\mathbf{A}}$, we are now ready to prove our first main theorem.

Proof of Theorem 1.2. We recall the sesquilinear form $Q_{h,\mathbf{A}} : \mathcal{V}_{h,\mathbf{A}} \times \mathcal{V}_{h,\mathbf{A}} \rightarrow \mathbb{C}$ defined in (1.3). It is evident that $Q_{h,\mathbf{A}}$ is densely defined. Writing $\mathbf{A} = \text{Re } \mathbf{A} + i\text{Im } \mathbf{A}$, we expand $Q_{h,\mathbf{A}}$ as follows

$$\begin{aligned} Q_{h,\mathbf{A}}(u, v) &= \int_{\mathbb{R}^d} (-ih\nabla - \text{Re } \mathbf{A} - i\text{Im } \mathbf{A})u \cdot \overline{(-ih\nabla - \text{Re } \mathbf{A} + i\text{Im } \mathbf{A})v} dx \\ &= \int_{\mathbb{R}^d} (|(-ih\nabla - \text{Re } \mathbf{A})u|^2 - |\text{Im } \mathbf{A}|^2|u|^2) dx - 2i\text{Re} \langle \text{Im } \mathbf{A} u, (-ih\nabla - \text{Re } \mathbf{A})u \rangle. \end{aligned}$$

Under the assumption (1.2) (which extends to $\mathcal{V}_{h,\mathbf{A}}$ by the density of $C_c^\infty(\mathbb{R}^d)$ in $\mathcal{V}_{h,\mathbf{A}}$), we obtain that for all $u \in \mathcal{V}_{h,\mathbf{A}}$,

$$\text{Re } Q_{h,\mathbf{A}}(u, u) \geq (1 - a) \int_{\mathbb{R}^d} |(-ih\nabla - \text{Re } \mathbf{A})u|^2 dx - b\|u\|^2, \quad (2.3)$$

where $a \in (0, 1)$ and $b \geq 0$ are constants from (1.2).

Next, we estimate the imaginary part of $Q_{h,\mathbf{A}}(u, u)$:

$$\begin{aligned} |\operatorname{Im} Q_{h,\mathbf{A}}(u, u)| &\leq 2\|\operatorname{Im} \mathbf{A} u\| \|(-ih\nabla - \operatorname{Re} \mathbf{A})u\| \\ &\leq \|\operatorname{Im} \mathbf{A} u\|^2 + \|(-ih\nabla - \mathbf{A})u\|^2 \\ &\leq (1+a)\|(-ih\nabla - \mathbf{A})u\|^2 + b\|u\|^2. \end{aligned}$$

Combining these inequalities, we deduce

$$|\operatorname{Im} Q_{h,\mathbf{A}}(u, u)| \leq \frac{1+a}{1-a} \operatorname{Re} Q_{h,\mathbf{A}}(u, u) + \frac{2b}{1-a} \|u\|^2.$$

This shows that the form $Q_{h,\mathbf{A}}$ is sectorial.

We now verify the closedness of the form. Consider the form

$$t(u, v) := \langle (-i\nabla - \operatorname{Re} \mathbf{A})u, (-i\nabla - \operatorname{Re} \mathbf{A})v \rangle,$$

which is sectorial and closed on $\mathcal{V}_{h,\operatorname{Re} \mathbf{A}}$. By applying [18, Thm. VI.1.33] and noting that the form $\langle \operatorname{Im} \mathbf{A} \cdot, \operatorname{Im} \mathbf{A} \cdot \rangle$ is t -bounded, we conclude that the form $\operatorname{Re} Q_{h,\mathbf{A}}$ is closed. By a remark in [18, Sec. VI.1.3], the sectorial form $Q_{h,\mathbf{A}}$ is also closed.

Consequently, $Q_{h,\mathbf{A}}$ gives rise to an m -sectorial operator $\mathcal{L}_{h,\mathbf{A}}$ via the standard representation theorem [18, Thm. VI.2.1]. \square

From now on, we restrict to dimension $d = 2$.

Lemma 2.3. *Let $\mathbf{A} \in L_{\operatorname{loc}}^2(\mathbb{R}^2, \mathbb{C}^2)$ and $\mathbf{B} \in L_{\operatorname{loc}}^1(\mathbb{R}^2)$. Then, for all $u \in C_c^\infty(\mathbb{R}^2)$, we have*

$$\begin{aligned} \left| \int_{\mathbb{R}^2} h \operatorname{Re} \mathbf{B} |u|^2 dx \right| &\leq \int_{\mathbb{R}^2} |(-ih\nabla - \operatorname{Re} \mathbf{A})u|^2 dx, \\ \left| \int_{\mathbb{R}^2} h \operatorname{Im} \mathbf{B} |u|^2 dx \right| &\leq \int_{\mathbb{R}^2} |(-ih\nabla - \operatorname{Re} \mathbf{A})u|^2 dx + \int_{\mathbb{R}^2} |(\operatorname{Im} \mathbf{A})u|^2 dx. \end{aligned}$$

Proof. Since $\mathbf{A} \in L_{\operatorname{loc}}^2(\mathbb{R}^2, \mathbb{C}^2)$ and $\mathbf{B} \in L_{\operatorname{loc}}^1(\mathbb{R}^2)$, all the integrals appearing in the statements of this lemma are well defined. Let $u \in C_c^\infty(\mathbb{R}^2)$. By observing that

$$[(-ih\partial_{x_1} - A_1), (-ih\partial_{x_2} - A_2)] = ih\mathbf{B},$$

we have

$$ih\mathbf{B}|u|^2 = \bar{u}(-ih\partial_{x_1} - A_1)(-ih\partial_{x_2} - A_2)u - \bar{u}(-ih\partial_{x_2} - A_2)(-ih\partial_{x_1} - A_1)u.$$

Integrating by parts, we get

$$\int_{\mathbb{R}^2} ih\mathbf{B}|u|^2 dx = \langle (-ih\partial_{x_2} - A_2)u, (-ih\partial_{x_1} - \bar{A}_1)u \rangle - \langle (-ih\partial_{x_1} - A_1)u, (-ih\partial_{x_2} - \bar{A}_2)u \rangle.$$

By writing out the real and imaginary parts of A_1 and A_2 , we find that

$$\begin{aligned} \int_{\mathbb{R}^2} ih\mathbf{B}|u|^2 dx &= 2\operatorname{Im} \langle (\operatorname{Im} A_2)u, (-ih\partial_{x_1} - \operatorname{Re} A_1)u \rangle - 2\operatorname{Im} \langle (\operatorname{Im} A_1)u, (-ih\partial_{x_2} - \operatorname{Re} A_2)u \rangle \\ &\quad + 2i\operatorname{Im} \langle (-ih\partial_{x_2} - \operatorname{Re} A_2)u, (-ih\partial_{x_1} - \operatorname{Re} A_1)u \rangle. \end{aligned}$$

In other words, we have

$$\int_{\mathbb{R}^2} h \operatorname{Re} \mathbf{B} |u|^2 dx = 2\operatorname{Im} \langle (-ih\partial_{x_2} - \operatorname{Re} A_2)u, (-ih\partial_{x_1} - \operatorname{Re} A_1)u \rangle,$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} h \operatorname{Im} \mathbf{B} |u|^2 dx &= 2\operatorname{Im} \langle (\operatorname{Im} A_1)u, (-ih\partial_{x_2} - \operatorname{Re} A_2)u \rangle \\ &\quad - 2\operatorname{Im} \langle (\operatorname{Im} A_2)u, (-ih\partial_{x_1} - \operatorname{Re} A_1)u \rangle. \end{aligned}$$

The conclusion follows from the Cauchy–Schwarz inequality. \square

Now we are in a position to establish Proposition 1.3.

Proof of Proposition 1.3. When $\text{Im } \mathbf{A}$ and $\text{Re } \mathbf{B}$ satisfy **(C1)**, then Assumption 1.1 is a direct consequence of the first inequality in Lemma 2.3.

Now, we assume that $\text{Im } \mathbf{A}$ and $\text{Im } \mathbf{B}$ satisfy **(C2)** with the plus sign (for instance). From the second inequality in Lemma 2.3,

$$\begin{aligned} \int_{\mathbb{R}^2} |(\text{Im } \mathbf{A})u|^2 dx &\leq \varepsilon_2 h \int_{\mathbb{R}^2} \text{Im } \mathbf{B} |u|^2 dx + C_2 \|u\|^2 \\ &\leq \varepsilon_2 \int_{\mathbb{R}^2} |(-ih\nabla - \text{Re } \mathbf{A})u|^2 dx + \varepsilon_2 \int_{\mathbb{R}^2} |(\text{Im } \mathbf{A})u|^2 dx + C_2 \|u\|^2. \end{aligned}$$

This implies that Assumption 1.1 holds with $a = \frac{\varepsilon_2}{1-\varepsilon_2} \in (0, 1)$ and $b = \frac{C_2}{1-\varepsilon_2}$. \square

Now we turn to task (II) about the compactness of the resolvent of $\mathcal{L}_{h,\mathbf{A}}$.

Proof of Theorem 1.4. Thanks to [7, Prop. 4.24], it is equivalent to proving that the injection

$$(\text{Dom}(\mathcal{L}_{h,\mathbf{A}}), \|\cdot\|_{\mathcal{L}_{h,\mathbf{A}}}) \hookrightarrow (L^2(\mathbb{R}^2), \|\cdot\|)$$

is compact, where $\|\cdot\|_{\mathcal{L}_{h,\mathbf{A}}} := \|\mathcal{L}_{h,\mathbf{A}} \cdot\| + \|\cdot\|$ is the graph norm. From (2.3) and Proposition 2.2(c), there exist $\gamma > 0$ and $\mu > 0$ such that for all $u \in \text{Dom}(\mathcal{L}_{\mathbf{A}})$, we have

$$\gamma \|u\|_{\mathcal{V}_{h,\mathbf{A}}}^2 \leq |Q_{h,\mathbf{A}}(u, u) + \mu \|u\|^2| \leq \|(\mathcal{L}_{h,\mathbf{A}} + \mu)u\| \|u\| \leq \left(\frac{1}{2} + \mu\right) \|u\|_{\mathcal{L}_{h,\mathbf{A}}}^2,$$

which shows that the injection

$$(\text{Dom}(\mathcal{L}_{h,\mathbf{A}}), \|\cdot\|_{\mathcal{L}_{h,\mathbf{A}}}) \hookrightarrow (\mathcal{V}_{h,\mathbf{A}}, \|\cdot\|_{\mathcal{V}_{h,\mathbf{A}}})$$

is continuous. Since the space of compact operators forms an ideal within the space of bounded operator, it remains to explain why $(\mathcal{V}_{h,\mathbf{A}}, \|\cdot\|_{\mathcal{V}_{h,\mathbf{A}}})$ is compactly embedded in $L^2(\mathbb{R}^2)$.

Let us consider $D := \{u \in \mathcal{V}_{h,\mathbf{A}} : \|u\|_{\mathcal{V}_{h,\mathbf{A}}} \leq 1\}$ and prove its precompactness in $L^2(\mathbb{R}^2)$ by means of the Kolmogorov–Riesz theorem (see [7, Thm. 4.14]). We only need to check the following:

- (i) For all $\varepsilon > 0$, there exists $\omega \subset\subset \mathbb{R}^2$ such that $\int_{\mathbb{R}^2 \setminus \omega} |u|^2 dx \leq \varepsilon^2$, for all $u \in D$.
- (ii) For all $\varepsilon > 0$ and for all $\omega \subset\subset \mathbb{R}^2$, there exists $\delta > 0$ such that

$$\int_{\omega} |u(x+s) - u(x)|^2 dx \leq \varepsilon^2$$

for all $s \in \mathbb{R}^2$ with $|s| \leq \delta$ and for all $u \in D$.

The equi-integrability condition (i) is satisfied as long as at least one of the assumptions **(H1)**, **(H2)**, or **(H3)** holds. This follows from Assumption 1.1, Lemma 2.3, Proposition 2.2(b), and the continuity of the functions considered in assumptions **(H1)**, **(H2)**, and **(H3)**. Below, we provide a detailed proof of (i) under the assumption **(H1)**. Since the proof is identical under the other assumptions, we will omit those cases.

Assume that $\text{Re } \mathbf{B}$ satisfies **(H1)**. Since $\text{Re } \mathbf{B}$ is continuous, by the multivariate intermediate value theorem [11, Thm. 1.9.5] and the connectedness of the punctured disk in \mathbb{R}^2 , the condition $\lim_{|x| \rightarrow +\infty} |\text{Re } \mathbf{B}(x)| = +\infty$ implies that

$$\lim_{|x| \rightarrow +\infty} \text{Re } \mathbf{B}(x) = +\infty \quad \text{or} \quad \lim_{|x| \rightarrow +\infty} -\text{Re } \mathbf{B}(x) = +\infty.$$

Without loss of generality, we assume the first possibility. By adding a constant $C > 0$ such that $\text{Re } \mathbf{B} + C \geq 0$ on \mathbb{R}^2 (to apply Fatou's lemma in the following step), and using the first inequality in Lemma 2.3, we obtain

$$(C+1) \|u\|_{\mathcal{V}_{h,\mathbf{A}}}^2 \geq \int_{\mathbb{R}^2} (\text{Re } \mathbf{B} + C) |u|^2 dx, \quad \forall u \in C_c^\infty(\mathbb{R}^2).$$

To extend this inequality to the space $\mathcal{V}_{h,\mathbf{A}}$, we utilise the density of $C_c^\infty(\mathbb{R}^2)$ in $\mathcal{V}_{h,\mathbf{A}}$. More precisely, let $u \in \mathcal{V}_{h,\mathbf{A}}$, there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^2)$ such that $u_n \xrightarrow{n \rightarrow +\infty} u$ in $\mathcal{V}_{h,\mathbf{A}}$. By considering a subsequence, still denoted by u_n , such that $u_n(x) \xrightarrow{n \rightarrow +\infty} u(x)$ for almost every $x \in \mathbb{R}^2$, Fatou's lemma yields that

$$\begin{aligned} (C+1)\|u\|_{\mathcal{V}_{h,\mathbf{A}}}^2 &= \lim_{n \rightarrow +\infty} (C+1)\|u_n\|_{\mathcal{V}_{h,\mathbf{A}}}^2 \geq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^2} (\operatorname{Re} \mathbf{B} + C)|u_n|^2 dx \\ &\geq \int_{\mathbb{R}^2} (\operatorname{Re} \mathbf{B} + C)|u|^2 dx, \quad \forall u \in \mathcal{V}_{\mathbf{A}}. \end{aligned}$$

Given $\varepsilon > 0$, using this estimate and the unboundedness of $\operatorname{Re} \mathbf{B}$ at infinity, there exists a constant $R > 0$ such that

$$\int_{|x| > R} |u|^2 dx < \varepsilon, \quad \forall u \in D.$$

Let us now consider (ii). Let $\varepsilon > 0$ and $\omega \subset \subset \mathbb{R}^2$, we consider a function $\chi \in C_c^\infty(\mathbb{R}^2)$ and $\chi = 1$ in some neighbourhood of $\bar{\omega}$. For all $u \in D$,

$$-ih\nabla(\chi u) = \operatorname{Re} \mathbf{A}\chi u + (-ih\nabla\chi)u + \chi(-ih\nabla - \operatorname{Re} \mathbf{A})u \in L^2(\mathbb{R}^2),$$

where we used the fact that $\operatorname{Re} \mathbf{A} \in L_{\text{loc}}^\infty(\mathbb{R}^2, \mathbb{R}^2)$. There exists $C > 0$ such that, for all $u \in D$,

$$\|\chi u\|_{H^1(\mathbb{R}^2)} \leq C.$$

Then, we notice that, for $|s|$ small enough such that $\chi(x+s) = 1$ on ω , we have, for all $u \in D$,

$$\begin{aligned} \int_{\omega} |u(x+s) - u(x)|^2 dx &= \int_{\omega} |(\chi u)(x+s) - (\chi u)(x)|^2 dx \\ &\leq \int_{\mathbb{R}^2} |(\chi u)(x+s) - (\chi u)(x)|^2 dx \leq C^2|s|^2, \end{aligned}$$

where we used [7, Prop. 2.94]. □

3. WKB CONSTRUCTION OF PSEUDOMODES

This section is concerned with task (III) about the construction of semiclassical pseudomodes. In particular, we establish Theorem 1.5. Throughout this section, we assume that $\mathbf{B} \in C^\infty(\mathbb{R}^2)$.

Definition 3.1. We say that \mathbf{B} is real-analytic at x^0 , when, in the neighbourhood of x^0 , \mathbf{B} is the sum of a converging series:

$$\mathbf{B}(x_1, x_2) = \sum_{m,n \geq 0} b_{mn}(x_1 - x_1^0)^m (x_2 - x_2^0)^n, \quad b_{mn} \in \mathbb{C}.$$

The main technical result of this section is the following theorem.

Theorem 3.2. *Let $x^0 \in \Gamma$ and assume that \mathbf{B} is real-analytic at x^0 . Then, there exist*

- i) a neighbourhood \mathcal{U} of x^0 in \mathbb{R}^2 ;
- ii) a real-analytic function P on \mathcal{U} satisfying

$$\operatorname{Re} P(x) = Q(x_1 - x_1^0, x_2 - x_2^0) + \mathcal{O}(|x - x^0|^3), \quad (3.1)$$

where $Q(u, v) := Q_1(x^0)u^2 - 2Q_2(x^0)uv + Q_3(x^0)v^2$ is a positive definite quadratic form on \mathbb{R}^2 with Q_1, Q_2, Q_3 defined as in (1.5);

- iii) a sequence of real-analytic functions $(a_j)_{j \in \mathbb{N}}$ on \mathcal{U} with $a_0(x^0) = 1$ and $a_j(x^0) = 0$ for $j \geq 1$;

such that, for all $N \in \mathbb{N}$,

$$e^{P/h} (\mathcal{L}_{h,\mathbf{A}} - h\mathbf{B}(x^0)) \left(e^{-P/h} \sum_{j=0}^N h^j a_j \right) = \mathcal{O}(h^{N+2}) \quad (3.2)$$

locally uniformly on \mathcal{U} .

After a translation, we can assume that $x^0 = 0$.

Notation 3.3 (Complexification of a real-analytic function). Assume that a is real-analytic near the point $0 \in \mathbb{R}^2$. We denote by \tilde{a} the function defined near $0 \in \mathbb{C}^2$ by

$$\tilde{a}(z, w) := a \left(\frac{z+w}{2}, \frac{z-w}{2i} \right).$$

Note that

$$\tilde{a}(z, \bar{z}) = a(\operatorname{Re} z, \operatorname{Im} z), \quad \partial_z \tilde{a} = \widetilde{\partial_z a}, \quad \partial_w \tilde{a} = \widetilde{\partial_{\bar{z}} a}. \quad (3.3)$$

3.1. A choice of the magnetic potential.

Lemma 3.4. *There exists a real-analytic and complex-valued function φ in a neighbourhood Ω of 0 such that*

$$\Delta\varphi = \mathbf{B}, \quad \varphi(x_1, x_2) = \frac{\mathbf{B}(0)}{4} (x_1^2 + x_2^2) + \mathcal{O}(|x|^3).$$

Proof. By considering the complexification of \mathbf{B} in the neighbourhood of 0

$$\tilde{\mathbf{B}}(z, w) = \sum_{(\alpha, \beta) \in \mathbb{N}^2} a_{\alpha, \beta} z^\alpha w^\beta,$$

with $a_{0,0} = \mathbf{B}(0)$, we introduce the power series

$$\tilde{\varphi}(z, w) = \frac{1}{4} \sum_{(\alpha, \beta) \in \mathbb{N}^2} \frac{a_{\alpha, \beta}}{(\alpha+1)(\beta+1)} z^{\alpha+1} w^{\beta+1}.$$

Then we get

$$4\partial_z \partial_w \tilde{\varphi}(z, w) = \tilde{\mathbf{B}}(z, w).$$

The function $z \mapsto \tilde{\varphi}(z, \bar{z})$ satisfies the required properties. \square

Let φ be the function given by Lemma 3.4, and define

$$\mathcal{M} := (-\partial_{x_2} \varphi, \partial_{x_1} \varphi).$$

Then \mathcal{M} satisfies

$$\frac{\partial \mathcal{M}_2}{\partial x_1} - \frac{\partial \mathcal{M}_1}{\partial x_2} = \mathbf{B} \quad \text{on } \Omega.$$

By the Poincaré lemma, there exists a function $\theta \in C^\infty(\Omega, \mathbb{C})$ such that

$$\mathcal{M} = \mathbf{A} + \nabla \theta \quad \text{on } \Omega. \quad (3.4)$$

From this, we obtain

$$\mathcal{L}_{h,\mathcal{M}} = e^{i\theta/h} \mathcal{L}_{h,\mathbf{A}} e^{-i\theta/h} \quad \text{on } \Omega. \quad (3.5)$$

3.2. WKB analysis. In this section, we construct a pseudomode for the operator $\mathcal{L}_{h,\mathcal{M}}$. More precisely, for $N \in \mathbb{N}$, we look for a pseudomode in the form

$$u_h(x) = e^{-S(x)/h} \sum_{j=0}^N a_j(x) h^j,$$

attached to a quasi-eigenvalue $\lambda(h) = h\mu$. Here, S and a_j are real-analytic functions defined in the neighbourhood of $0 \in \mathbb{R}^2$, and $\mu \in \mathbb{C}$.

Let us consider the formal conjugated operator acting locally:

$$\begin{aligned} \mathcal{L}_{h,\mathcal{M}}^S &:= e^{S/h} \mathcal{L}_{h,\mathcal{M}} e^{-S/h} \\ &= (-ih\partial_1 - \mathcal{M}_1 + i\partial_{x_1}S)^2 + (-ih\partial_2 - \mathcal{M}_2 + i\partial_{x_2}S)^2 \\ &= E_0 + hE_1 + h^2E_2, \end{aligned}$$

where the differential expressions E_0 , E_1 and E_2 are given by

$$\begin{aligned} E_0 &:= (-\mathcal{M}_1 + i\partial_{x_1}S)^2 + (-\mathcal{M}_2 + i\partial_{x_2}S)^2, \\ E_1 &:= \Delta S + 2(\nabla S + i\mathcal{M}) \cdot \nabla, \\ E_2 &:= -\Delta. \end{aligned}$$

Then, we have

$$e^{S/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda(h)) u_h(x) = [E_0 + h(E_1 - \mu) + h^2E_2] \sum_{j=0}^N a_j(x) h^j = \sum_{j=0}^{N+2} \phi_j(x) h^j,$$

where the functions ϕ_j are explicitly given by

$$\begin{aligned} h^0 &: & E_0 a_0 &=: \phi_0, \\ h^1 &: & E_0 a_1 + (E_1 - \mu) a_0 &=: \phi_1, \\ h^2 &: & E_0 a_2 + (E_1 - \mu) a_1 + E_2 a_0 &=: \phi_2, \\ &\vdots & & \\ h^N &: & E_0 a_N + (E_1 - \mu) a_{N-1} + E_2 a_{N-2} &=: \phi_N, \\ h^{N+1} &: & (E_1 - \mu) a_N + E_2 a_{N-1} &=: \phi_{N+1}, \end{aligned} \tag{3.6}$$

and the last function ϕ_{N+2} is

$$h^{N+2} : E_2 a_N =: \phi_{N+2}.$$

3.2.1. The eikonal equation. Let us find S such that $E_0 = 0$, *i.e.*,

$$(-\mathcal{M}_1 + i\partial_{x_1}S)^2 + (-\mathcal{M}_2 + i\partial_{x_2}S)^2 = 0.$$

It is equivalent to the equation

$$(i\partial_{x_1}S - \partial_{x_2}S - \mathcal{M}_1 - i\mathcal{M}_2)(i\partial_{x_1}S + \partial_{x_2}S - \mathcal{M}_1 + i\mathcal{M}_2) = 0.$$

Let us choose S such that

$$i\partial_{x_1}S - \partial_{x_2}S - \mathcal{M}_1 - i\mathcal{M}_2 = 0.$$

Then we have

$$2\partial_{\bar{z}}S = \mathcal{M}_2 - i\mathcal{M}_1 = 2\partial_{\bar{z}}\varphi.$$

In particular, $S - \varphi$ is holomorphic. By using Notation 3.3, this suggests taking

$$\tilde{S}(z, w) = \tilde{\varphi}(z, w) + f(z),$$

where $f(z)$ is a holomorphic function (in a neighbourhood of $0 \in \mathbb{R}^2$) to be determined later in the neighbourhood of $0 \in \mathbb{C}$. Note that $\Delta S = \mathbf{B}$.

3.2.2. *Towards the transport equations.* Now, we consider the operator E_1 .

By using the expression of \mathcal{M} and the choice of \widetilde{S} , we have

$$\begin{aligned} (\nabla S + i\mathcal{M}) \cdot \nabla &= (\partial_{x_1} S + i\mathcal{M}_1) \partial_{x_1} + (\partial_{x_2} S + i\mathcal{M}_2) \partial_{x_2} \\ &= (\partial_{x_1} \varphi - i\partial_{x_2} \varphi + f'(z)) \partial_{x_1} + (\partial_{x_2} \varphi + i\partial_{x_1} \varphi + if'(z)) \partial_{x_2} \\ &= 2(2\partial_z \varphi + f'(z)) \partial_{\bar{z}}. \end{aligned}$$

Thus, for any real-analytic function a near the point x^0 , we have

$$\widetilde{E_1} a = \left[4(2\partial_z \widetilde{\varphi} + f'(z)) \partial_w + \widetilde{\mathbf{B}} \right] \widetilde{a}.$$

Note also that

$$\widetilde{E_2} a = -4\partial_z \partial_w \widetilde{a}.$$

From (3.6), we are led to the the system of the transport equations

$$\begin{aligned} h^1 : & \quad \left[4(2\partial_z \widetilde{\varphi} + f'(z)) \partial_w + \widetilde{\mathbf{B}} - \mu \right] \widetilde{a}_0 = 0, \\ h^2 : & \quad \left[4(2\partial_z \widetilde{\varphi} + f'(z)) \partial_w + \widetilde{\mathbf{B}} - \mu \right] \widetilde{a}_1 = 4\partial_z \partial_w \widetilde{a}_0, \\ & \quad \vdots \\ h^{N+1} : & \quad \left[4(2\partial_z \widetilde{\varphi} + f'(z)) \partial_w + \widetilde{\mathbf{B}} - \mu \right] \widetilde{a}_N = 4\partial_z \partial_w \widetilde{a}_{N-1}. \end{aligned}$$

3.2.3. *Choosing f and determining S .* Since $0 \in \Gamma$, we have $\partial_{\bar{z}} \mathbf{B}(0) \neq 0$ and thus

$$\partial_w \widetilde{\mathbf{B}}(0) = \widetilde{\partial_z \mathbf{B}}(0) = \partial_z \mathbf{B}(0) \neq 0.$$

In virtue of the holomorphic implicit function theorem, there exists a unique holomorphic function w , in the neighborhood of $0 \in \mathbb{C}$, such that

$$w(0) = 0, \quad \widetilde{\mathbf{B}}(z, w(z)) = \mathbf{B}(0). \quad (3.7)$$

In particular,

$$w'(0) = -\frac{\partial_z \widetilde{\mathbf{B}}}{\partial_w \widetilde{\mathbf{B}}}(0) = -\frac{\partial_z \mathbf{B}}{\partial_z \mathbf{B}}(0). \quad (3.8)$$

Remark 3.5. Note that the function w solves the same effective equation as in [6], but that we are *not* in the case of a magnetic well.

In order to solve the above transport equations, we will choose a function $f(z)$ such that

$$2\partial_z \widetilde{\varphi}(z, w(z)) + f'(z) = 0.$$

Lemma 3.6. *Let φ be given in Lemma 3.4, w be the holomorphic function given in (3.7) and θ given in (3.4). Letting*

$$f(z) := -\int_{[0,z]} 2\partial_z \widetilde{\varphi}(\zeta, w(\zeta)) d\zeta + \text{Im } \theta(0), \quad (3.9)$$

we have

$$f(0) = \text{Im } \theta(0), \quad f'(0) = 0, \quad f''(0) = \frac{\mathbf{B}(0)}{2} \frac{\partial_z \mathbf{B}}{\partial_z \mathbf{B}}(0).$$

In particular,

$$\text{Re } S(x) = \text{Im } \theta(0) + \widetilde{Q}_1 x_1^2 - 2\widetilde{Q}_2 x_1 x_2 + \widetilde{Q}_3 x_2^2 + \mathcal{O}(|x|^3),$$

where

$$\begin{aligned}\tilde{Q}_1 &= \frac{1}{4} \operatorname{Re} \left[\mathbf{B}(0) \left(1 + \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(0) \right) \right], & \tilde{Q}_2 &= \frac{1}{4} \operatorname{Im} \left[\mathbf{B}(0) \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(0) \right], \\ \tilde{Q}_3 &= \frac{1}{4} \operatorname{Re} \left[\mathbf{B}(0) \left(1 - \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(0) \right) \right].\end{aligned}$$

Proof. A straightforward computation gives

$$f''(0) = -2\partial_{zz}^2 \tilde{\varphi}(0) - 2w'(0)\partial_{zw}^2 \tilde{\varphi}(0).$$

Due to Lemma 3.4, we have

$$\partial_{zz}^2 \tilde{\varphi}(0) = 0, \quad \partial_{zw}^2 \tilde{\varphi}(0) = \frac{\mathbf{B}(0)}{4},$$

and we get the value of $f''(0)$ by using (3.8).

Since $S(x_1, x_2) = \varphi(x_1, x_2) + f(x_1 + ix_2)$, we can write

$$S(x_1, x_2) = \operatorname{Im} \theta(0) + \frac{\mathbf{B}(0)}{4} \left[x_1^2 + x_2^2 + \frac{\partial_z \mathbf{B}(0)}{\partial_{\bar{z}} \mathbf{B}(0)} (x_1 + ix_2)^2 \right] + \mathcal{O}(|x|^3),$$

so that

$$\begin{aligned}\operatorname{Re} S(x) &= \operatorname{Im} \theta(0) + \frac{1}{4} \operatorname{Re} \left[\mathbf{B}(0) \left(1 + \frac{\partial_z \mathbf{B}(0)}{\partial_{\bar{z}} \mathbf{B}(0)} \right) \right] x_1^2 + \frac{1}{4} \operatorname{Re} \left[\mathbf{B}(0) \left(1 - \frac{\partial_z \mathbf{B}(0)}{\partial_{\bar{z}} \mathbf{B}(0)} \right) \right] x_2^2 \\ &\quad - \frac{1}{2} \operatorname{Im} \left[\mathbf{B}(0) \frac{\partial_z \mathbf{B}(0)}{\partial_{\bar{z}} \mathbf{B}(0)} \right] x_1 x_2 + \mathcal{O}(|x|^3) \\ &= \operatorname{Im} \theta(0) + \tilde{Q}_1 x_1^2 - 2\tilde{Q}_2 x_1 x_2 + \tilde{Q}_3 x_2^2 + \mathcal{O}(|x|^3).\end{aligned}$$

□

3.2.4. *Solving the first transport equation.* With the choice (3.9), we can write

$$8[\partial_z \tilde{\varphi}(z, w) - \partial_z \tilde{\varphi}(z, w(z))] \partial_w \tilde{a}_0(z, w) + \left(\tilde{\mathbf{B}}(z, w) - \mu \right) \tilde{a}_0(z, w) = 0. \quad (3.10)$$

Taking $w = w(z)$ and using (3.7), we see that (3.10) has a holomorphic solution \tilde{a}_0 such that $\tilde{a}_0(0) \neq 0$ if and only if

$$\mu = \mathbf{B}(0).$$

Let us explain this. The Taylor formula gives

$$\begin{aligned}\partial_z \tilde{\varphi}(z, w) - \partial_z \tilde{\varphi}(z, w(z)) &= (w - w(z))V(z, w), \\ \tilde{\mathbf{B}}(z, w) - \tilde{\mathbf{B}}(z, w(z)) &= (w - w(z))F(z, w),\end{aligned}$$

where

$$V(z, w) = \int_0^1 \tilde{\mathbf{B}}(z, w(z) + t(w - w(z))) dt, \quad F(z, w) = \int_0^1 \partial_w \tilde{\mathbf{B}}(z, w(z) + t(w - w(z))) dt.$$

Notice that $V(0) = \mathbf{B}(0) \neq 0$, so that near $0 \in \mathbb{C}^2$, we have $V(z, w) \neq 0$. By using $\mu = \mathbf{B}(0)$, the first transport equation becomes

$$\partial_w \tilde{a}_0(z, w) + \frac{1}{8} \frac{F(z, w)}{V(z, w)} \tilde{a}_0(z, w) = 0.$$

Therefore, we have

$$\tilde{a}_0(z, w) = \mathcal{A}_0(z) J(z, w), \quad J(z, w) := \exp \left(- \int_{[w(z), w]} \frac{1}{8} \frac{F(z, u)}{V(z, u)} du \right),$$

where $\mathcal{A}_0(z)$ is a holomorphic function to be determined such that $\mathcal{A}_0(0) = 1$.

3.2.5. *Solving the second transport equation.* Let us now consider the second transport equation

$$(w - w(z)) [8V(z, w)\partial_w + F(z, w)] \tilde{a}_1(z, w) = 4\partial_z \partial_w \tilde{a}_0(z, w). \quad (3.11)$$

Letting $w = w(z)$, we necessarily get that

$$4\partial_z \partial_w \tilde{a}_0(z, w(z)) = 0,$$

which means that

$$\partial_w J(z, w(z)) \mathcal{A}'_0(z) + \partial_z \partial_w J(z, w(z)) \mathcal{A}_0(z) = 0.$$

This allows us to determine \mathcal{A}_0 of the previous step. From the definition of J , we have

$$\partial_w J(z, w(z)) = -\frac{1}{8} \frac{F(z, w(z))}{V(z, w(z))},$$

and thus

$$\partial_w J(0, w(0)) = -\frac{1}{8} \frac{\partial_w \tilde{\mathbf{B}}}{\tilde{\mathbf{B}}}(0) = -\frac{1}{8} \frac{\partial_{\bar{z}} \mathbf{B}}{\mathbf{B}}(0) \neq 0.$$

This leads to the choice

$$\mathcal{A}_0(z) = \exp \left(- \int_{[0, z]} \frac{\partial_z \partial_w J}{\partial_w J}(u, w(u)) du \right),$$

which is not allowed in the case of a magnetic well in the self-adjoint case. In particular,

$$\tilde{a}_0(z, w) = \exp \left(- \int_{[0, z]} \frac{\partial_z \partial_w J}{\partial_w J}(u, w(u)) du \right) \exp \left(- \int_{[w(z), w]} \frac{1}{8} \frac{F(z, u)}{V(z, u)} du \right).$$

With this choice, (3.11) becomes

$$\partial_w \tilde{a}_1(z, w) + \frac{1}{8} \frac{F(z, w)}{V(z, w)} \tilde{a}_1(z, w) = \frac{1}{2} \frac{T_0(z, w)}{V(z, w)},$$

with

$$T_0(z, w) = \int_0^1 \partial_w^2 \partial_z \tilde{a}_0(z, w(z) + t(w - w(z))) dt.$$

We have

$$\tilde{a}_1(z, w) = J(z, w) \int_{[w(z), w]} \frac{T_0(z, u)}{2J(z, u)V(z, u)} du + \mathcal{A}_1(z) J(z, w),$$

where $\mathcal{A}_1(z)$ is a holomorphic function to be determined with $\mathcal{A}_1(0) = 0$.

3.2.6. *Induction.* By induction, the solution of the $j+1$ -th transport equation can be written as

$$\tilde{a}_{j+1}(z, w) = J(z, w) \int_{[w(z), w]} \frac{T_j(z, u)}{2J(z, u)V(z, u)} du + \mathcal{A}_{j+1}(z) J(z, w),$$

with

$$T_j(z, w) = \int_0^1 \partial_w^2 \partial_z \tilde{a}_j(z, w(z) + t(w - w(z))) dt$$

and \mathcal{A}_{j+1} is determined by the constraint (coming from the $j+2$ -th equation):

$$4\partial_z \partial_w \tilde{a}_{j+1}(z, w(z)) = 0. \quad (3.12)$$

We have

$$\partial_w \tilde{a}_{j+1}(z, w) = \partial_w J(z, w) \int_{[w(z), w]} \frac{T_j(z, u)}{2J(z, u)V(z, u)} du + \frac{T_j(z, w)}{2V(z, w)} + \mathcal{A}_{j+1}(z) \partial_w J(z, w),$$

so that

$$\begin{aligned} & \partial_z \partial_w \tilde{a}_{j+1}(z, w(z)) \\ &= \left[-\partial_w J w'(z) \frac{T_j}{2V} + \partial_z J \frac{T_j}{2V} + \partial_z \left(\frac{T_j}{2V} \right) + \mathcal{A}'_{j+1} \partial_w J + \mathcal{A}_{j+1} \partial_{zw}^2 J \right] (z, w(z)). \end{aligned}$$

Since $J(z, w(z)) = 1$, we have $\partial_z J(z, w(z)) = -w'(z) \partial_w J(z, w(z))$. Therefore, \mathcal{A}_{j+1} satisfies the equation

$$\partial_w J(z, w(z)) \mathcal{A}'_{j+1}(z) + \partial_{zw}^2 J(z, w(z)) \mathcal{A}_{j+1}(z) = G_j(z),$$

where

$$G_j(z) := - \left[\partial_z J \frac{T_j}{V} + \partial_z \left(\frac{T_j}{2V} \right) \right] (z, w(z)).$$

Thus,

$$\mathcal{A}_{j+1}(z) = \mathcal{A}_0(z) \int_{[0, z]} \frac{G_j(u)}{\mathcal{A}_0(u)} du.$$

Therefore, we have, for all $j \geq 0$,

$$\tilde{a}_{j+1}(z, w) = J(z, w) \int_{[w(z), w]} \frac{T_j(z, u)}{2J(z, u)V(z, u)} du + \tilde{a}_0(z, w) \int_{[0, z]} \frac{G_j(u)}{\mathcal{A}_0(u)} du, \quad (3.13)$$

where

$$T_j(z, w) = \int_0^1 \partial_w^2 \partial_z \tilde{a}_j(z, w(z) + t(w - w(z))) dt,$$

and

$$G_j(u) = \left[\left(\frac{\partial_z J}{\mathbf{B}(0)} + \frac{1}{2} \partial_z \left(\frac{1}{V} \right) \right) \partial_w^2 \partial_z \tilde{a}_j + \frac{1}{2\mathbf{B}(0)} \partial_w^2 \partial_z^2 \tilde{a}_j + \frac{w'(u)}{4\mathbf{B}(0)} \partial_w^3 \partial_z \tilde{a}_j \right] (u, w(u)).$$

3.3. Proof of Theorem 3.2. We have constructed holomorphic functions $\tilde{S}(z, w)$ and $(\tilde{a}_j(z, w))_{j \in \mathbb{N}_0}$ which are defined in a neighbourhood of $0 \in \mathbb{C}^2$. Now we take $w = \bar{z}$ and recall Notation 3.3. For all $N \geq 0$, we have

$$e^{S/h} (\mathcal{L}_{h, \mathcal{M}} - h\mu) \left(e^{-S/h} \sum_{j=0}^N h^j a_j \right) = (-\Delta a_N) h^{N+2}.$$

From (3.5), we have

$$e^{P/h} (\mathcal{L}_{h, \mathbf{A}} - h\mu) \left(e^{-P/h} \sum_{j=0}^N h^j a_j \right) = (-\Delta a_N) h^{N+2}, \quad P := S + i\theta. \quad (3.14)$$

From the definitions of θ and \mathcal{M} , we have $\nabla \theta(0) = \mathcal{M}(0) - \mathbf{A}(0) = -\mathbf{A}(0)$ and thus $\nabla \operatorname{Im} \theta(0) = -\operatorname{Im} \mathbf{A}(0) = 0$. Notice that

$$\begin{aligned} \operatorname{Im} \partial_1^2 \theta(0) &= \operatorname{Im} (-\partial_{12}^2 \varphi(0) - \partial_1 A_1(0)) = -\partial_1 \operatorname{Im} A_1(0), \\ \operatorname{Im} \partial_2^2 \theta(0) &= \operatorname{Im} (\partial_{12}^2 \varphi(0) - \partial_2 A_2(0)) = -\partial_2 \operatorname{Im} A_2(0), \\ \operatorname{Im} \partial_{12}^2 \theta(0) &= \operatorname{Im} (\partial_{11}^2 \varphi(0) - \partial_1 A_2(0)) \\ &= \frac{\operatorname{Im} \mathbf{B}(0)}{2} - \partial_1 \operatorname{Im} A_2(0) \\ &= -\frac{1}{2} (\partial_1 \operatorname{Im} A_2(0) + \partial_2 \operatorname{Im} A_1(0)). \end{aligned}$$

From Lemma 3.6 and using Taylor expansion for $\operatorname{Im} \theta$, we deduce that

$$\operatorname{Re} P(x) = Q_1(0)x_1^2 - 2Q_2(0)x_1x_2 + Q_3(0)x_2^2 + \mathcal{O}(|x|^3).$$

This concludes the proof of Theorem 3.2.

3.4. Proof of Theorem 1.5.

3.4.1. Preliminaries.

Notation 3.7. We will use the following notation for a polydisc

$$P(0; R_1, R_2) := \{(z, w) \in \mathbb{C}^2 : |z| < R_1 \text{ and } |w| < R_2\}.$$

For $K \subset \mathbb{C}^2$, we write

$$\|\tilde{a}(z, w)\|_K := \sup_{(z, w) \in K} |\tilde{a}(z, w)|.$$

When $K = P(0; R_1, R_2)$, we simply write $\|\cdot\|_K = \|\cdot\|_{R_1, R_2}$. For $m, n \in \mathbb{N}$, we denote $\llbracket m, n \rrbracket := [m, n] \cap \mathbb{Z}$.

Lemma 3.8. *For a holomorphic function $\tilde{a}(z, w)$ defined in a neighbourhood of $P(0; R_1, R_2)$, we have, for all $(z, w) \in P(0; R_1, R_2)$,*

$$|\partial_w^2 \partial_z \tilde{a}(z, w)| \leq \frac{2R_1 R_2 \|\tilde{a}\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - |w|)^3}$$

and

$$|\partial_w^2 \partial_z^2 \tilde{a}(z, w)| \leq \frac{4R_1 R_2 \|\tilde{a}\|_{R_1, R_2}}{(R_1 - |z|)^3 (R_2 - |w|)^3}, \quad |\partial_w^3 \partial_z \tilde{a}(z, w)| \leq \frac{6R_1 R_2 \|\tilde{a}\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - |w|)^4}.$$

Proof. For example, the Cauchy formula gives

$$\begin{aligned} |\partial_w^2 \partial_z \tilde{a}(z, w)| &= \left| \frac{2}{(2\pi i)^2} \int_{|z|=R_1} \int_{|w|=R_2} \frac{\tilde{a}(\xi_1, \xi_2)}{(\xi_1 - z)^2 (\xi_2 - w)^3} d\xi_1 d\xi_2 \right| \\ &\leq \frac{2R_1 R_2 \|\tilde{a}\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - |w|)^3}. \end{aligned}$$

□

Lemma 3.9. *Let $(a_j)_{j \in \mathbb{N}}$ be the real-analytic sequence given in Theorem 3.2, then there exist constants $m > 0$, $\delta_0 > 0$ such that, for all $j \geq 0$ and for all $x \in D(x^0, \delta_0)$,*

$$|a_j(x)| \leq m^{j+1} j^{7j}, \quad |\nabla a_j(x)| \leq m^{j+1} j^{7j}, \quad |\Delta a_j(x)| \leq m^{j+1} j^{7j}. \quad (3.15)$$

Proof. Let us prove that there exists $m > 0$ such that, for all $j \geq 0$ and all (z, w) in a neighbourhood of $0 \in \mathbb{C}^2$,

$$|\tilde{a}_j(z, w)| \leq m^{j+1} j^{7j}. \quad (3.16)$$

The key to get the growth control is the recursion relation (3.13): we see that \tilde{a}_{j+1} is related to derivatives $\partial_w^2 \partial_z \tilde{a}_j$, $\partial_w^2 \partial_z^2 \tilde{a}_j$, and $\partial_w^3 \partial_z \tilde{a}_j$.

Consider $C, R_1, R_2 > 0$ such that, for all $z \in D(0, R_1)$,

$$|w(z)| \leq C|z| < R_2. \quad (3.17)$$

For all $(z, w) \in P(0; R_1, R_2)$ and for all $t \in [0, 1]$, we have

$$|w(z) + t(w - w(z))| < \max(C|z|, |w|) < R_2. \quad (3.18)$$

Then, Lemma 3.8 yields that, for all $(z, w) \in P(0; R_1, R_2)$,

$$\begin{aligned} |T_j(z, w)| &\leq \int_0^1 |\partial_w^2 \partial_z \tilde{a}_j(z, w(z) + t(w - w(z)))| dt \\ &\leq \int_0^1 \frac{2R_1 R_2 \|\tilde{a}_j\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - |w(z) + t(w - w(z))|)^3} dt \\ &\leq \frac{2R_1 R_2 \|\tilde{a}_j\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - \max\{C|z|, |w|\})^3}. \end{aligned}$$

Then, the first term in (3.13) can be controlled, for all $(z, w) \in P(0; R_1, R_2)$,

$$\begin{aligned}
& \left| J(z, w) \int_{[w(z), w]} \frac{T_j(z, u)}{2J(z, u)V(z, u)} du \right| \\
&= \left| J(z, w) \int_0^1 \frac{T_j(z, w(z) + t(w - w(z)))}{2(JV)(z, w(z) + t(w - w(z)))} (w - w(z)) dt \right| \\
&\leq \|J\|_{R_1, R_2} \left\| \frac{1}{JV} \right\|_{R_1, R_2} \frac{R_1 R_2 \|\tilde{a}_j\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - \max\{C|z|, |w|\})^3} |w - w(z)| \\
&\leq \|J\|_{R_1, R_2} \left\| \frac{1}{JV} \right\|_{R_1, R_2} \frac{2R_1 R_2^2 \|\tilde{a}_j\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - \max\{C|z|, |w|\})^3}.
\end{aligned}$$

For the second term in (3.13), we have, for all $(z, w) \in P(0; R_1, R_2)$,

$$\begin{aligned}
& \left| \tilde{a}_0(z, w) \int_{[0, z]} \frac{G_j(u)}{\mathcal{A}_0(u)} du \right| \\
&\leq |z| \|\tilde{a}_0\|_{R_1, R_2} \left\| \frac{1}{\mathcal{A}_0} \right\|_{R_1} \\
&\quad \times \int_0^1 \left| \left[\left(\frac{\partial_z J}{\mathbf{B}(0)} + \frac{1}{2} \partial_z \left(\frac{1}{V} \right) \right) \partial_w^2 \partial_z \tilde{a}_j + \frac{1}{2\mathbf{B}(0)} \partial_w^2 \partial_z^2 \tilde{a}_j + \frac{w'(tz)}{4\mathbf{B}(0)} \partial_w^3 \partial_z \tilde{a}_j \right] (tz, w(tz)) \right| dt \\
&\leq |z| \|\tilde{a}_0\|_{R_1, R_2} \left\| \frac{1}{\mathcal{A}_0} \right\|_{R_1} \left[\left(\frac{\|\partial_z J\|_{R_1, R_2}}{|\mathbf{B}(0)|} + \frac{1}{2} \left\| \partial_z \left(\frac{1}{V} \right) \right\|_{R_1, R_2} \right) \int_0^1 |\partial_w^2 \partial_z \tilde{a}_j(tz, w(tz))| dt \right. \\
&\quad \left. + \frac{1}{2|\mathbf{B}(0)|} \int_0^1 |\partial_w^2 \partial_z^2 \tilde{a}_j(tz, w(tz))| dt + \frac{\|w'\|_{R_1}}{4|\mathbf{B}(0)|} \int_0^1 |\partial_w^3 \partial_z \tilde{a}_j(tz, w(tz))| dt \right].
\end{aligned}$$

From Lemma 3.8, we have, for all $(z, w) \in P(0; R_1, R_2)$,

$$\begin{aligned}
& \int_0^1 |\partial_w^2 \partial_z \tilde{a}_j(tz, w(tz))| |z| dt \leq \frac{2R_1^2 R_2 \|\tilde{a}_j\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - C|z|)^2}, \\
& \int_0^1 |\partial_w^2 \partial_z^2 \tilde{a}_j(tz, w(tz))| |z| dt \leq \frac{4R_1^2 R_2 \|\tilde{a}_j\|_{R_1, R_2}}{(R_1 - |z|)^3 (R_2 - C|z|)^3},
\end{aligned}$$

and

$$\int_0^1 |\partial_w^3 \partial_z \tilde{a}_j(tz, w(tz))| |z| dt \leq \frac{6R_1^2 R_2 \|\tilde{a}_j\|_{R_1, R_2}}{(R_1 - |z|)^2 (R_2 - C|z|)^4}.$$

Let us consider $R_1^0, R_2^0, C_1, C > 0$ such that, for all $R_1 \leq 2R_1^0, R_2 \leq 2R_2^0$,

$$\|J\|_{R_1, R_2} + \left\| \frac{1}{JV} \right\|_{R_1, R_2} + \|\tilde{a}_0\|_{R_1, R_2} + \left\| \frac{1}{\mathcal{A}_0} \right\|_{R_1} + \|\partial_z J\|_{R_1, R_2} + \left\| \partial_z \left(\frac{1}{V} \right) \right\|_{R_1, R_2} + \|w'\|_{R_1} \leq C_1,$$

and, for all $z \in D(0, 2R_1^0)$,

$$|w(z)| \leq C|z| < 2R_2^0. \quad (3.19)$$

From (3.13), there exists a constant $C_2 > 0$ (depending only on R_1^0, R_2^0) such that, for all $\ell \geq 0$ and for all $(z, w) \in P(0; R_1, R_2)$,

$$|\tilde{a}_{\ell+1}(z, w)| \leq C_2 \frac{\|\tilde{a}_\ell\|_{R_1, R_2}}{(R_1 - |z|)^3 (R_2 - \max\{C|z|, |w|\})^4}. \quad (3.20)$$

We fix $j \in \mathbb{N}$ and let, for all $k \in \llbracket 0, j-1 \rrbracket$,

$$r_{1,k} := \left(2 - \frac{k}{j} \right) R_1^0, \quad r_{2,k} := \left(2 - \frac{k}{j} \right) R_2^0.$$

Thanks to (3.19), we see that (3.17) holds for $R_1 = r_{1,k}$ and $R_2 = r_{2,k}$ with a constant C defined in (3.19). Thus, the estimate (3.20) also holds for all $k \in \llbracket 0, j-1 \rrbracket$, *i.e.*, for all $(z, w) \in P(0; r_{1,k}, r_{2,k})$,

$$|\tilde{a}_{k+1}(z, w)| \leq C_2 \frac{\|\tilde{a}_k\|_{r_{1,k}, r_{2,k}}}{(r_{1,k} - |z|)^3 (r_{2,k} - \max\{C|z|, w\})^4}. \quad (3.21)$$

This holds in particular for all $(z, w) \in P(0; r_{1,k+1}, r_{2,k+1})$. From the second inequality in (3.19), $CR_1^0 < R_2^0$, and thus, for $|z| < r_{1,k+1}$, we have

$$C|z| < Cr_{1,k+1} = C \left(2 - \frac{k+1}{j}\right) R_1^0 < \left(2 - \frac{k+1}{j}\right) R_2^0 = r_{2,k+1}.$$

This shows that, for all $(z, w) \in P(0; r_{1,k+1}, r_{2,k+1})$,

$$\frac{1}{(r_{1,k} - |z|)^3 (r_{2,k} - \max\{C|z|, w\})^4} \leq \frac{1}{(r_{1,k} - r_{1,k+1})^3 (r_{2,k} - r_{2,k+1})^4} = \frac{j^7}{(R_1^0)^3 (R_2^0)^4}.$$

From (3.21) (with $\ell = k$), there exists $C_3 > 0$ such that, for all $k \in \llbracket 0, j-1 \rrbracket$,

$$\|\tilde{a}_{k+1}\|_{r_{1,k+1}, r_{2,k+1}} \leq C_3 j^7 \|\tilde{a}_k\|_{r_{1,k}, r_{2,k}}.$$

Multiplying these estimates, we get

$$\|\tilde{a}_j\|_{R_1^0, R_2^0} \leq (C_3 j^7)^j \|\tilde{a}_0\|_{2R_1^0, 2R_2^0}. \quad (3.22)$$

Then, the estimate (3.16) follows with $m = \max\{C_3, \|\tilde{a}_0\|_{2R_1^0, 2R_2^0}\}$.

Finally, the estimates on the derivatives of \tilde{a}_j are elementary consequences of (3.22). Indeed,

$$\begin{aligned} \widetilde{\partial_{x_1} a_j}(z, w) &= (\partial_z \tilde{a}_j + \partial_w \tilde{a}_j)(z, w), & \widetilde{\partial_{x_2} a_j}(z, w) &= i(\partial_z \tilde{a}_j - \partial_w \tilde{a}_j)(z, w), \\ \widetilde{\Delta a_j}(z, w) &= 4\partial_z \partial_w \tilde{a}_j(z, w), \end{aligned}$$

so that Cauchy estimates give, for all $(z, w) \in P(0; R_1^0, R_2^0)$,

$$\begin{aligned} \left| \widetilde{\partial_{x_1} a_j}(z, w) \right| &\leq \frac{R_1^0 R_2^0 (R_1^0 + R_2^0) \|\tilde{a}_j\|_{R_1^0, R_2^0}}{(R_1^0 - |z|)^2 (R_2^0 - |w|)^2}, & \left| \widetilde{\partial_{x_2} a_j}(z, w) \right| &\leq \frac{R_1^0 R_2^0 (R_1^0 + R_2^0) \|\tilde{a}_j\|_{R_1^0, R_2^0}}{(R_1^0 - |z|)^2 (R_2^0 - |w|)^2}, \\ \left| \widetilde{\Delta a_j}(z, w) \right| &\leq \frac{4R_1^0 R_2^0 \|\tilde{a}_j\|_{R_1^0, R_2^0}}{(R_1^0 - |z|)^2 (R_2^0 - |w|)^2}. \end{aligned}$$

Then, by using these estimates (3.22) for all $(z, w) \in P\left(0; \frac{R_1^0}{2}, \frac{R_2^0}{2}\right)$, this concludes the proof. \square

3.4.2. Proof of Theorem 1.5. Let us recall (3.1). Since Q is positive, there exist $\delta > 0$ and $M_1, M_2 > 0$ such that

$$M_1 |x|^2 \leq \operatorname{Re} P(x) \leq M_2 |x|^2, \quad \text{for all } x \in D(0, \delta). \quad (3.23)$$

By considering δ sufficiently small, we can assume that (3.15) holds.

Let $\chi \in C_c^\infty(\mathbb{R}^2)$ be a smooth cut-off function which is equal to 1 near 0 and has support in a compact subset of $D(0, \delta)$. We define our pseudomode as

$$u_h(x) := \chi(x) e^{-P(x)/h} \left[a_0(x) + \sum_{j=1}^{N(h)} h^j a_j(x) \right], \quad N(h) := \lfloor (emh)^{-1/7} \rfloor.$$

Here, m is a constant appearing in (3.15) and $\lfloor \cdot \rfloor$ denotes the floor function.

By construction in Theorem 3.2, for all $j \geq 1$, $a_j(0) = 0$. Thus, for all $x \in D(0, \delta)$,

$$|a_j(x)| = \left| \int_0^1 \frac{d}{dt} a_j(tx) dt \right| = \left| \int_0^1 x \cdot \nabla a_j(tx) dt \right| \leq m^{j+1} j^{7j} |x|.$$

Then, by using (3.15) and the definition of $N(h)$, we have, for some $C_1 > 0$,

$$\sum_{j=1}^{N(h)} h^j |a_j(x)| \leq \sum_{j=1}^{N(h)} h^j m^{j+1} j^{7j} |x| \leq \sum_{j=1}^{N(h)} m e^{-j} |x| \leq C_1 |x|. \quad (3.24)$$

Since $a_0(0) = 1$, there exists $C_2 > 0$ (independent of h) such that, in a neighbourhood of 0,

$$\left| a_0(x) + \sum_{j=1}^{N(h)} h^j a_j(x) \right| \geq |a_0(x)| - C_1 |x| \geq C_2.$$

By using (3.23), we get

$$\int_{\mathbb{R}^2} |u_h(x)|^2 dx \gtrsim h^{\frac{1}{2}}.$$

Let us now prove that for each $\varepsilon \in (0, 1)$, $e^{\varepsilon P/h} (\mathcal{L}_{h,\mathbf{A}} - h\mu) u_h(x)$ is exponentially small. We write

$$\begin{aligned} & e^{\varepsilon P/h} (\mathcal{L}_{h,\mathbf{A}} - h\mu) u_h(x) \\ &= \underbrace{e^{\varepsilon P/h} [\mathcal{L}_{h,\mathbf{A}}, \chi]}_{I_h} \left(e^{-P/h} \sum_{j=0}^N h^j a_j \right) + \underbrace{\chi e^{\varepsilon P/h} (\mathcal{L}_{h,\mathbf{A}} - h\lambda)}_{J_h} \left(e^{-P/h} \sum_{j=0}^N h^j a_j \right), \end{aligned}$$

where

$$[\mathcal{L}_{h,\mathbf{A}}, \chi] = -2h^2 \nabla \chi \cdot \nabla - h^2 \Delta \chi + 2h i \mathbf{A} \cdot \nabla \chi.$$

The term I_h is a function supported on $\text{supp}(\nabla \chi)$:

$$I_h = e^{-\frac{(1-\varepsilon)P}{h}} \left[-2h^2 \nabla \chi \cdot \sum_{j=0}^{N(h)} h^j \nabla a_j + (-h^2 \Delta \chi + 2h \nabla P \cdot \nabla \chi + 2h i \mathbf{A} \cdot \nabla \chi) \left(\sum_{j=0}^{N(h)} h^j a_j \right) \right].$$

Thanks to (3.24), we notice that $\sum_{j=0}^{N(h)} h^j a_j(x)$ is bounded uniformly in $x \in D(0, \delta)$ and in

h . Similarly, $\sum_{j=0}^{N(h)} h^j \nabla a_j(x)$ is also uniformly bounded.

Then, there exist $C, \tilde{C} > 0$ such that, for all $h \in (0, h_0)$ and for all $x \in D(0, \delta)$,

$$|I_h(x)| \leq C e^{-(1-\varepsilon)M_1|x|^2/h} \mathbf{1}_{\text{supp } \nabla \chi}(x) \leq C e^{-(1-\varepsilon)\tilde{C}/h}.$$

For J_h , we can notice that, by construction and by using Lemma 3.9,

$$|J_h(x)| = |\chi(x) e^{-(1-\varepsilon)P(x)/h} h^{N(h)+2} \Delta a_{N(h)}| \lesssim (hmN(h)^7)^{N(h)} \lesssim e^{-N(h)} \lesssim e^{-C/h^{1/7}},$$

where, in the last estimate, we used the fact that $N(h) > \frac{1}{(emh)^{1/7}} - 1$.

This concludes the proof of Theorem 1.5.

3.5. Examples. Of course, Theorem 1.5 is only interesting if one can ensure that Γ is not empty. Let us discuss conditions on \mathbf{A} and \mathbf{B} to ensure that $0 \in \Gamma$.

First, since $\partial_{\bar{z}} \mathbf{B}(0) \neq 0$, we observe that 0 cannot be a critical point of \mathbf{B} at 0. Then, also since $\text{Im } \mathbf{A}(0) = 0$, we write

$$\begin{aligned} \mathbf{B}(x_1, x_2) &= a + bx_1 + cx_2 + \mathcal{O}(|x|^2), \\ \text{Im } A_1(x_1, x_2) &= dx_1 + ex_2 + \mathcal{O}(|x|^2), \\ \text{Im } A_2(x_1, x_2) &= fx_1 + gx_2 + \mathcal{O}(|x|^2), \end{aligned}$$

in a neighbourhood of 0, where $a, b, c \in \mathbb{C}$ and $d, e, f, g \in \mathbb{R}$. Below, we will denote a_1, b_1, c_1 (respectively, a_2, b_2, c_2) as the real parts (respectively, imaginary parts) of a, b, c . Let us find conditions on these coefficients so that $0 \in \Gamma$. We have

$$\begin{aligned} \mathbf{B}(0) &= a_1 + ia_2, & \partial_z \mathbf{B}(0) &= \frac{1}{2} [b_1 + c_2 + i(b_2 - c_1)], & \partial_{\bar{z}} \mathbf{B}(0) &= \frac{1}{2} [b_1 - c_2 + i(b_2 + c_1)], \\ \partial_{x_1} \operatorname{Im} A_1(0) &= d, & \partial_{x_2} \operatorname{Im} A_1(0) &= e, & \partial_{x_1} \operatorname{Im} A_2(0) &= f, & \partial_{x_2} \operatorname{Im} A_2(0) &= g. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(0) &= \frac{b_1^2 + b_2^2 - c_1^2 - c_2^2 - i2(b_1 c_1 + b_2 c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2}, \\ \operatorname{Re} \left[B(0) \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(0) \right] &= \frac{a_1 (b_1^2 + b_2^2 - c_1^2 - c_2^2) + 2a_2 (b_1 c_1 + b_2 c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2}, \\ \operatorname{Im} \left[B(0) \frac{\partial_z \mathbf{B}}{\partial_{\bar{z}} \mathbf{B}}(0) \right] &= \frac{a_2 (b_1^2 + b_2^2 - c_1^2 - c_2^2) - 2a_1 (b_1 c_1 + b_2 c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2}, \end{aligned}$$

and thus,

$$\begin{aligned} Q_1(0) &= \frac{1}{2} \left[\frac{a_1 (b_1^2 + b_2^2 - b_1 c_2 + b_2 c_1) + a_2 (b_1 c_1 + b_2 c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2} + d \right], \\ Q_2(0) &= \frac{1}{2} \left[\frac{a_2 (b_1^2 + b_2^2 - b_1 c_2 + b_2 c_1) - a_1 (b_1 c_1 + b_2 c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2} + e \right], \\ Q_3(0) &= \frac{1}{2} \left[\frac{a_1 (c_1^2 + c_2^2 - b_1 c_2 + b_2 c_1) - a_2 (b_1 c_1 + b_2 c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2} + g \right]. \end{aligned}$$

Here, in $Q_2(0)$, we have used $f - e = a_2$ deduced from $\partial_{x_1} \operatorname{Im} A_2(0) - \partial_{x_2} \operatorname{Im} A_1(0) = \operatorname{Im} \mathbf{B}(0)$. Then, $0 \in \Gamma$ if and only if

$$a_1 \neq 0 \text{ or } a_2 \neq 0, \quad (3.25)$$

$$b_1 \neq c_2 \text{ or } b_2 \neq -c_1, \quad (3.26)$$

$$Q_1(0) > 0, \quad (3.27)$$

$$Q_1(0)Q_3(0) - Q_2^2(0) > 0. \quad (3.28)$$

Example 4 (Polynomial complex magnetic fields). In this example, we want to provide a class of polynomial \mathbf{B} such that $\mathcal{L}_{h, \mathbf{A}}$ is well defined and $0 \in \Gamma$. We consider

$$\mathbf{B}(x_1, x_2) = a + bx_1 + cx_2 + R(x_1, x_2),$$

where

- $a, b, c \in \mathbb{C}$ such that $a_1 > 0$, $a_2 = 0$ and $b_2 c_1 - b_1 c_2 > 0$,
- $R : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a polynomial such that

$$R(0) = \partial_{x_1} R(0) = \partial_{x_2} R(0) = 0 \quad \text{and} \quad |x|^4 = o(R(x)) \text{ as } |x| \rightarrow +\infty.$$

Let us choose the following magnetic potential

$$A_1(x_1, x_2) = 0, \quad A_2(x_1, x_2) = \int_0^{x_1} \mathbf{B}(s, x_2) ds.$$

Since R is real-valued, we have

$$\operatorname{Re} \mathbf{B}(x_1, x_2) = a_1 + b_1 x_1 + c_1 x_2 + R(x_1, x_2), \quad \operatorname{Im} A_2(x_1, x_2) = \frac{b_2}{2} x_1^2 + c_2 x_1 x_2.$$

From the asymptotic behaviour of R , we deduce that

$$|\operatorname{Im} \mathbf{A}(x)|^2 = o(\operatorname{Re} \mathbf{B}(x)), \quad \text{as } |x| \rightarrow +\infty.$$

Then, **(C1)** is satisfied (and thus Assumption 1.1 holds). This allows us to define $\mathcal{L}_{h,\mathbf{A}}$ by Theorem 1.2. Furthermore, since $|\operatorname{Re} \mathbf{B}(x)| \rightarrow +\infty$ as $|x| \rightarrow +\infty$, the operator has discrete eigenvalues by Theorem 1.4.

Since $a_1 > 0$ and $b_2c_1 - b_1c_2 > 0$, (3.25) and (3.26) hold. Note that $d = e = g = 0$ and $a_2 = 0$, we have

$$\begin{aligned} Q_1(0) &= \frac{1}{2} \frac{a_1 (b_1^2 + b_2^2 + b_2c_1 - b_1c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2}, \\ Q_2(0) &= -\frac{1}{2} \frac{a_1 (b_1c_1 + b_2c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2}, \\ Q_3(0) &= \frac{1}{2} \frac{a_1 (c_1^2 + c_2^2 + b_2c_1 - b_1c_2)}{(b_1 - c_2)^2 + (b_2 + c_1)^2}. \end{aligned}$$

Obviously, $Q_1(0) > 0$. Let us compute

$$Q_1(0)Q_3(0) - Q_2(0)^2 = \frac{a_1^2 [(b_1^2 + b_2^2 + b_2c_1 - b_1c_2)(c_1^2 + c_2^2 + b_2c_1 - b_1c_2) - (b_1c_1 + b_2c_2)^2]}{4 [(b_1 - c_2)^2 + (b_2 + c_1)^2]^2}.$$

Notice that

$$\begin{aligned} & (b_1^2 + b_2^2 + b_2c_1 - b_1c_2)(c_1^2 + c_2^2 + b_2c_1 - b_1c_2) - (b_1c_1 + b_2c_2)^2 \\ &= (b_2c_1 - b_1c_2) [(b_1 - c_2)^2 + (b_2 + c_1)^2]. \end{aligned}$$

This yields that

$$Q_1(0)Q_3(0) - Q_2(0)^2 = \frac{a_1^2 (b_2c_1 - b_1c_2)}{4(b_1 - c_2)^2 + (b_2 + c_1)^2} > 0.$$

Thus, we have $0 \in \Gamma$. By Theorem 1.5, there exist a pseudomode $u(\cdot, h)$ such that

$$\|(\mathcal{L}_{h,\mathbf{A}} - ha_1)u_h(x)\| \leq \exp\left(-\frac{C}{h^{1/7}}\right) \|u_h(x)\|, \quad \text{as } h \rightarrow 0. \quad (3.29)$$

Example 5 (Bounded oscillating magnetic fields). Consider the magnetic potential

$$A_1(x_1, x_2) = -\sin(x_1)x_2 + i \cos(x_2), \quad A_2(x_1, x_2) = i \cos(x_2).$$

We have

$$\mathbf{B}(x_1, x_2) = \sin(x_1) + i \sin(x_2).$$

Since $\operatorname{Im} \mathbf{A}$ is bounded, both conditions **(C1)** and **(C2)** hold, therefore, the magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}}$ is well-defined by Theorem 1.2. Let us find the set Γ in this explicit example. Take $x^0 = (x_1, x_2) \in \Gamma$, we have

$$\operatorname{Im} \mathbf{A}(x^0) = 0 \iff \cos(x_2) = 0.$$

Then, it implies that $\mathbf{B}(x^0) \neq 0$ and

$$\partial_{\bar{z}} \mathbf{B}(x^0) = \partial_z \mathbf{B}(x^0) = \frac{1}{2} \cos(x_1).$$

Therefore, it is obvious that

$$\partial_{\bar{z}} \mathbf{B}(x^0) \neq 0 \iff \cos(x_1) \neq 0.$$

By straightforward calculation, we obtain

$$Q_1(x^0) = \frac{1}{2} \sin(x_1), \quad Q_2(x^0) = 0, \quad Q_3(x^0) = -\frac{1}{2} \sin(x_2).$$

Therefore, it yields that

$$\begin{aligned} Q_1(x^0) > 0 &\iff \sin(x_1) > 0, \\ Q_1(x^0)Q_3(x^0) - Q_2^2(x^0) > 0 &\iff \sin(x_2) = -1. \end{aligned}$$

The above analysis leads to

$$\begin{aligned} \Gamma &= \{(x_1, x_2) \in \mathbb{R}^2 : \sin(x_1) > 0, \cos(x_1) \neq 0, \cos(x_2) = 0, \sin(x_2) = -1\} \\ &= \left\{ (0, \pi) \setminus \left\{ \frac{\pi}{2} \right\} + 2k\pi : k \in \mathbb{Z} \right\} \times \left\{ -\frac{\pi}{2} + 2\pi n : n \in \mathbb{Z} \right\}. \end{aligned}$$

Then, Theorem 1.5 states that: for each $x_1^0 \in (0, \pi) \setminus \{\frac{\pi}{2}\} + 2k\pi$ for some $k \in \mathbb{Z}$, we can construct a pseudomode $u(\cdot, h)$ such that

$$\|(\mathcal{L}_{h,\mathbf{A}} - h(\sin(x_1^0) - i)) u_h(x)\| \leq \exp\left(-\frac{C}{h^{1/7}}\right) \|u_h(x)\|, \quad \text{as } h \rightarrow 0. \quad (3.30)$$

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