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Inverse resonance problem on a perturbed infinite hyperbolic cylinder

Valentin Arrigoni*

Abstract

We study an inverse resonance problem on a radially and compactly perturbed infinite hyperbolic cylinder. Using the symmetries of this kind of geometry, we are led to study a stationary Schrödinger equation on the line with a potential V which is the sum of a Pöschl-Teller potential and a perturbation which we consider integrable and compactly supported. We define the resonances as the poles of the reflection coefficients with a negative imaginary part. We prove that, under some assumptions on the support of the compact perturbation, we are able to solve the question of uniqueness in the inverse resonance problem. We also give asymptotics of the resonances and show that they are asymptotically localised on two logarithmic branches and according to the localisation of the support of q , sometimes also on parallel lines to the imaginary axis.

1 Introduction

An inverse problem consists in the determination of causes of a phenomenon based on empirical observations of its effects. In the inverse problem we are interested in, we consider a one-dimensional stationary Schrödinger equation,

$$-y'' + V(x)y = z^2y, \quad (x, z) \in \mathbb{R} \times \mathbb{C}, \quad (1)$$

with a potential $V \in L^1(\mathbb{R})$ being a compactly supported perturbation of a Pöschl-Teller potential :

$$V(x) = \frac{\lambda}{\cosh^2(x)} + q(x), \quad \lambda \in \mathbb{R}^*, \quad q \in L^1_{comp}(\mathbb{R}). \quad (2)$$

We also consider its two specific solutions with prescribed asymptotics denoted f^\pm :

$$f^\pm(x, z) \underset{x \rightarrow \pm\infty}{\sim} e^{\pm izx},$$

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called Jost solutions. From these solutions, we can define three Wronskians, $w(z) = [f^-(x, z), f^+(x, z)]$ and $s^\pm(z) = [f^+(x, \mp z), f^-(x, \pm z)]$. Under the general assumption $V \in L^1(\mathbb{R})$, the function w is always defined for $z \in \mathbb{C}^+$ and its zeros with a positive imaginary part correspond to *eigenvalues* for (1). Indeed, for such a z , f^+ and f^- are linearly dependent which ensure they are both in $L^2(\mathbb{R})$. For our specific potentials V given by (2), we shall show that the function w can be meromorphically continued to \mathbb{C}^- and its zeros with a negative imaginary part are then called *resonances*. Like the eigenvalues, for z a resonance, f^+ and f^- are still linearly dependent, but they are no longer in $L^2(\mathbb{R})$. One can understand them as "eigenvalues without eigenvectors".

We are interested here in an inverse resonance problem, that is we address the question whether a potential V can be uniquely determined from the spectral data consisting in the set of eigenvalues and resonances. The answer of such a question is typically no when working with Schrödinger operator on the line. Indeed, it is well known from the work of Korotyaev [12] who treated the case of compactly supported potentials that we have to add some *extra data* to the eigenvalues and resonances in order to conclude to uniqueness. In this paper, we shall prove a result similar to Korotyaev's one for the class of compactly supported perturbations of a Pöschl-Teller potential. More precisely, denote by

$$\mathcal{R}^\pm = \frac{s^\pm}{w}$$

the left (or right) reflection coefficient. The first result of this paper can be stated as follows: the potential V is essentially ¹ determined from the spectral data consisting in the poles (eigenvalues and resonances) and zeros of one of these coefficients.

From the general Gelfand-Levitan-Marchenko theory (see for instance [14]), for a potential $V \in L^1(\mathbb{R})$, the data consisting in one of the reflection coefficient, the eigenvalues and corresponding norming constants is sufficient to determine the potential V . When working with our specific potentials (2), only one of the reflection coefficient is needed to recover the potential. Indeed, since we work with a compactly supported perturbation of a Pöschl-Teller potential, we will show that the reflection coefficients can be meromorphically extended to the whole complex plane. As a consequence, the knowledge of the eigenvalues and norming constants follows from the knowledge of one of the reflection coefficient alone. This analyticity will allow us to use powerful tools from complex analysis such as Hadamard's factorisation theorem, Stirling's formula, Euler's reflection formula, . . . in order to determine one of the reflection coefficient from the eigenvalues and resonances (the poles of this coefficient) and its zeros.

Let us now recall the existing results in one-dimensional inverse resonance problems. Considering first the case of halfline Schrödinger operators, Korotyaev proved in [11] that we only need to know the eigenvalues and the resonances to solve the uniqueness of the inverse resonance problem for a compactly supported potential. He also solved the characterization problem in that paper.

¹We refer to Theorem 3 for the precise statement.

Marletta, Shterenberg and Weikard established in [15] stability results from the small resonances, always for a compactly supported potential in \mathbb{R}^+ . Considering non-compactly supported potentials on the half-line, Borthwick, Boussaïd and Daudé [3] studied the problem of uniqueness and the localisation of the resonances in the case of a compactly perturbed exponentially decreasing potential. As already mentioned, on the whole real line, more spectral data are needed in order to solve the inverse resonance problem. Korotyaev proved in [12] the uniqueness and characterisation problems for compactly supported potentials. Few years later, Bledsoe [1] proved stability estimates from the small resonances still in the case of compactly supported potentials. This work can be understood as a generalisation of the previous results to non-compactly supported potentials on the whole line, as we work in the class of Pöschl-Teller potential with a compactly supported perturbation.

At last, we want to point out that this equation comes from a geometrical problem. Let \mathfrak{C} be an infinite hyperbolic cylinder

$$\mathfrak{C} \simeq (\mathbb{R})_x \times (\mathbb{S}^{d-1})_\omega$$

where \mathbb{S}^{d-1} is the $(d-1)$ -dimensional sphere ($d \geq 2$), equipped with the metric $g := dx^2 + \cosh^2(x)d\omega^2$. Let Δ_g be the Laplace-Beltrami operator on this infinite hyperbolic cylinder. Then

$$\Delta_g = \partial_x^2 + (d-1)\tanh(x)\partial_x + \Delta_{\mathbb{S}^{d-1}}\cosh^{-2}(x), \quad (3)$$

where $\Delta_{\mathbb{S}^{d-1}}$ is the Laplace-Beltrami operator on the sphere \mathbb{S}^{d-1} . This operator is, through conjugation by $\cosh^{\frac{d-1}{2}}(x)$, equivalent to

$$\partial_x^2 + \frac{\Delta_{\mathbb{S}^{d-1}} + (d-1)(3-d)/4}{\cosh^2(x)} + \frac{d-1}{4}$$

We would like to understand how waves propagate on this infinite hyperbolic cylinder and their scattering properties, because this model can be considered as a preliminary toy model in the analysis of black holes spacetime and their gravitational waves. In particular, we are interested in the stationary scattering governed by the Helmholtz equation:

$$-\Delta_g f = \nu^2 f. \quad (4)$$

We denote by H_k the subspace of spherical harmonics of degree k . We recall that

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{k \geq 0} H_k,$$

and for every integer k , for every $Y_k \in H_k$,

$$-\Delta_{\mathbb{S}^{d-1}} Y_k = k(k+d-2)Y_k.$$

We refer to [7] for more information concerning spherical harmonics. We are looking for solutions $v \in L^2((\mathbb{R})_x \times (\mathbb{S}^{d-1})_\omega; dx \otimes d\omega)$ to Equation (4) such that

$$v(x, \omega) = \sum_{k=0}^{+\infty} v_k(x) Y_k(\omega).$$

Then, functions v_k satisfy the ordinary differential equation

$$-v_k''(x) + \frac{k(k+d-2) + (d-1)(3-d)/4}{\cosh^2(x)} v_k(x) = z^2 v_k(x),$$

where $z^2 = \nu^2 - \frac{d-1}{4}$. This is exactly Equation (1) without the compact perturbation q . Note that the equation (1) with a potential V being the sum of a Pöschl-Teller potential and a compactly supported perturbation corresponds to an infinite hyperbolic cylinder that is radially compactly perturbed.² We refer to [6] and [2] for further information on the geometry, spectral theory and inverse resonance problem of infinite hyperbolic cylinder.

A second problem addressed in this paper is the localisation of the resonances in the lower half-plane. For compactly supported potential, precise asymptotics were obtained by Zworski in [20]. Similar results have already been obtain in [3], for non compactly supported potential on the half-line only. We extend their work to our class of potentials (2) defined on the whole line under the additional hypothesis that the compactly supported perturbation q (or one of its derivatives) has some discontinuities. When the Pöschl-Teller potential is not perturbed, the resonances can be computed explicitly and lie on two vertical half-lines symmetric with respect to the negative imaginary axis. When we add a perturbation with a discontinuity, there is always creation of resonances that lie asymptotically on two logarithmic branches symmetric with respect to the imaginary negative axis. But, depending on the localisation of the support of q , we can still observe or not some resonances created by the initial Pöschl-Teller potential (see [3] for a similar result). This result confirms the numerical observations stated by Jaramillo, Panosso Macedo and Al Sheikh [10].

This work is organised as follows. In a first section, we will give some classical definitions of scattering theory, some results on hypergeometric functions and results in the case of the non-perturbed Pöschl-Teller potential. Then in a second section, we are going to state and prove the main theorems of this work, concerning the solution of an inverse resonance problem for a compactly perturbed Pöschl-Teller potential. Eventually, we study how resonances are perturbed by adding a compactly supported potential. This paper includes an appendix on hypergeometric functions, where we recall some standard results we use in the core of our analysis.

2 Pöschl-Teller potentials

We start with the following one-dimensional stationary Schrödinger equation

$$-y''(x) + \frac{\lambda}{\cosh^2(x)} y(x) = z^2 y(x), \quad (5)$$

²More precisely, if we consider the metric

$$g = dx^2 + a^2(x) d\omega^2, \quad a(x) = \cosh(x) + a_0(x), \quad a_0 \in C_0^2(\mathbb{R}),$$

on the hyperbolic cylinder \mathfrak{C} , then the stationary scattering would be exactly governed by equation (1).

where λ is a real fixed parameter, and z a complex spectral parameter. The class of Pöschl-Teller potentials: $\frac{\lambda}{\cosh^2(x)}$, was introduced for the first time in [16] in the study of quantum anharmonic oscillators. Its Jost solutions denoted f_0^\pm are solutions of (5) such that

$$f_0^\pm(x, z) \underset{x \rightarrow \pm\infty}{\sim} e^{\pm izx}. \quad (6)$$

They can explicitly be expressed in terms of hypergeometric functions: consider the change of variable $\zeta := \frac{1}{2}(1 - \tanh x) = (1 + e^{2x})^{-1}$ and the function $f(x) = \cosh^{iz}(x)y(\zeta)$, where f is a solution to (5). Then we obtain an hypergeometric differential equation

$$\zeta(1 - \zeta)y'' + (c - (a + b + 1)\zeta)y' - aby = 0, \quad (7)$$

with

$$\begin{aligned} a &:= \frac{1}{2} - iz + \sqrt{\frac{1}{4} - \lambda}, \\ b &:= \frac{1}{2} - iz - \sqrt{\frac{1}{4} - \lambda}, \\ c &:= 1 - iz. \end{aligned}$$

Here, the notation $\sqrt{\cdot}$ must be understood as

$$\sqrt{x} = \begin{cases} \sqrt{x} & \text{if } x \in [0, +\infty) \\ i\sqrt{-x} & \text{if } x \in (-\infty, 0]. \end{cases}$$

A solution to (7) is given by an hypergeometric function

$$\zeta \mapsto F(a, b, c; \zeta) := {}_2F_1(a, b, c; \zeta) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{\zeta^n}{n!},$$

where $(x)_n$ is the Pochhammer symbol, defined by $(x)_0 := 1$ and for every positive integer n

$$(x)_n := x(x+1)\dots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

Note that $F(a, b, c; \zeta)$ are power series with radius of convergence 1. Furthermore

$$\lim_{\zeta \rightarrow 0} F(a, b, c; \zeta) = 1. \quad (8)$$

In the following, every hypergeometric function is denoted F instead of ${}_2F_1$. Equation (7) is invariant under the change of variable $\zeta' = 1 - \zeta$, so $\zeta \mapsto F(a, b, c; 1 - \zeta)$ is also a solution to (7). These two solutions are linearly independent if a and b are not non positive integers. We refer to the monograph [17]

for further information on hypergeometric functions. Hence two independent solutions, for a and b not non positive integers, to Equation (7) are given by

$$x \mapsto F\left(a, b, c; \frac{1}{1+e^{2x}}\right) \text{ and } x \mapsto F\left(a, b, c; \frac{e^{2x}}{1+e^{2x}}\right).$$

Equation (8) implies that these two hypergeometric functions satisfy

$$F\left(a, b, c; \frac{1}{1+e^{2x}}\right) \xrightarrow{x \rightarrow +\infty} 1 \text{ and } F\left(a, b, c; \frac{e^{2x}}{1+e^{2x}}\right) \xrightarrow{x \rightarrow -\infty} 1.$$

Hence Jost solutions are given by:

$$f_0^+(x, z) = (2 \cosh(x))^{iz} F\left(a, b, c; \frac{1}{1+e^{2x}}\right), \quad (9)$$

$$f_0^-(x, z) = (2 \cosh(x))^{iz} F\left(a, b, c; \frac{e^{2x}}{1+e^{2x}}\right). \quad (10)$$

Since the Pöschl-Teller potential is even, we have $f_0^+(-x, z) = f_0^-(x, z)$. Furthermore, we can also write Jost solutions (see Appendix, Proposition 9)

$$f_0^+(x, z) = e^{izx} F\left(c-a, c-b, c; \frac{1}{1+e^{2x}}\right),$$

$$f_0^-(x, z) = e^{-izx} F\left(c-a, c-b, c; \frac{e^{2x}}{1+e^{2x}}\right).$$

Let $[f, g] := fg' - f'g$ denote the Wronskian of functions f and g . Equation (5) has no first-order terms, then the Wronskian of two solutions is constant. We can so define two specific Wronskians, which are independent of the variable x , named Jost functions w_0 and s_0^\pm , by

$$w_0(z) := [f_0^-(\cdot, z), f_0^+(\cdot, z)]$$

and

$$s_0^\pm(z) := [f_0^+(\cdot, \mp z), f_0^-(\cdot, \pm z)].$$

Relations between hypergeometric functions (details are given in the Appendix, see Proposition 11) give explicit expressions for w_0 and s_0^\pm , for $z \in \mathbb{C} \setminus i\mathbb{Z}^*$:

$$w_0(z) = 2iz \frac{\Gamma(c-1)\Gamma(c)}{\Gamma(a)\Gamma(b)} = \frac{-2\Gamma(c)^2}{\Gamma(a)\Gamma(b)} \quad (11)$$

and

$$s_0^\pm(z) = 2iz \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = 2 \frac{\Gamma(c)\Gamma(2-c)}{\Gamma(c-a)\Gamma(c-b)}.$$

There are three types of zeros of w_0 . These interpretations still hold for any potential V . We will give a short explanation of their differences. Zeros with positive imaginary part are called eigenvalues. Indeed, if $w_0(z) = 0$, functions

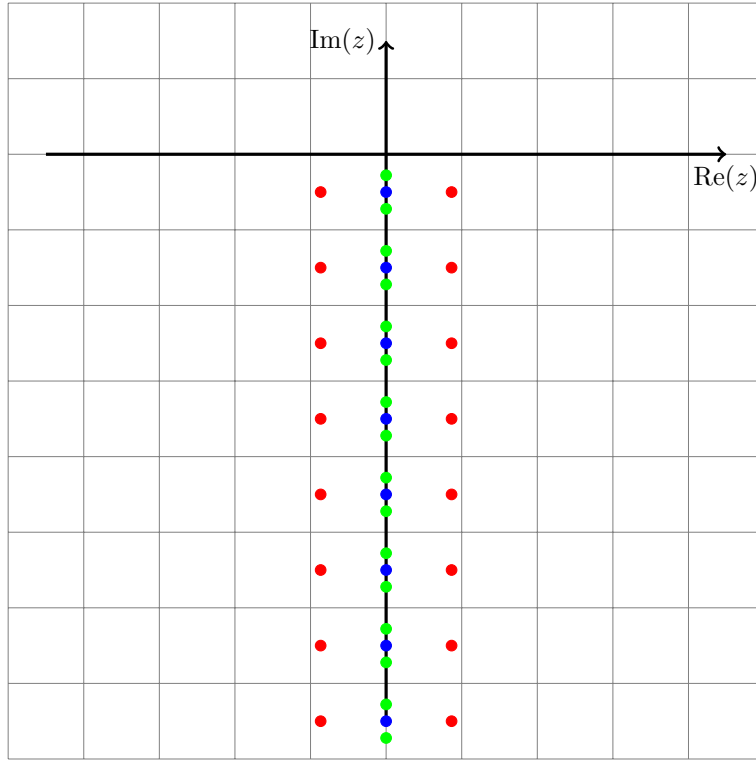


Figure 1: Resonances in the lower half-plane:
 $\lambda = 1/5$ in green, $\lambda = 1/4$ in blue and $\lambda = 1$ in red.

f^+ and f^- are linearly dependent, and using Equation (6), $f^+ \in L^2(\mathbb{R})$ (and so is f^-). Zeros with a negative imaginary part are quite different. They are called resonances, and they can be understood as "eigenvalues without eigenvectors", due to the fact that for such z , $f^+ \notin L^2(\mathbb{R})$ (and so is $f^- \notin L^2(\mathbb{R})$). We refer to [8, 19] for detailed presentations of resonances. Their localisation is possible, due to the explicit formula (11). For $\lambda \leq 1/4$, resonances appear for $z = -i \left(n + \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda} \right)$, for any $n \geq 0$. While $\lambda > 1/4$, they are localised at $z = \pm \sqrt{\lambda - \frac{1}{4}} - i \left(n + \frac{1}{2} \right)$, for any $n \geq 0$. Finally, when $w_0(0) = 0$, we say there is a half-bound state, or a zero resonance. We observe that, when λ increases, the pair of resonances for the same integer n gets closer along the imaginary axis, then for $\lambda = 1/4$ they are equal and finally split apart when λ keeps increasing, See Figure 1.

A straightforward calculation shows that $[f_0^\pm(\cdot, z), f_0^\pm(\cdot, -z)] = \mp 2iz$, so for every $z \neq 0$, we have two fundamental systems of solutions for Equation (5): $(f_0^\pm(\cdot, z), f_0^\pm(\cdot, -z))$. Every solution to Equation (5) can be written in these

basis. So

$$f_0^\pm(\cdot, z) = \frac{w_0(z)}{2iz} f_0^\mp(\cdot, -z) + \frac{s_0^\mp(z)}{2iz} f_0^\mp(\cdot, z), \quad z \in \mathbb{C} \setminus i\mathbb{Z}.$$

Let $\mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0$ and \mathcal{D}_0 be the functions, defined for $z \in \mathbb{C} \setminus i\mathbb{Z}$, by

$$\mathcal{A}_0(z) := \frac{w_0(z)}{2iz} \quad \text{and} \quad \mathcal{B}_0(z) := \frac{s_0^-(z)}{2iz}$$

and

$$\mathcal{D}_0(z) := \frac{w_0(z)}{2iz} \quad \text{and} \quad \mathcal{C}_0(z) := \frac{s_0^+(z)}{2iz}.$$

The transmission coefficient \mathcal{T}_0 is defined by $\mathcal{T}_0(z) := 2iz/w_0(z)$ and the right and left reflection coefficients \mathcal{R}_0^\pm are defined by $\mathcal{R}_0^\pm(z) := s_0^\pm(z)/w_0(z)$. They play an important role in the resolution of our inverse problem. See [1, 14]. They are the coefficients of the scattering matrix:

$$\mathcal{S}_0(z) := \begin{pmatrix} \mathcal{T}_0(z) & \mathcal{R}_0^-(z) \\ \mathcal{R}_0^+(z) & \mathcal{T}_0(z) \end{pmatrix}.$$

This matrix corresponds to the measured quantity in scattering experiments.

Note that the hypergeometric function $F(a, b, c; \zeta)$ are actually not define at complex number $c \in \{0, -1, -2, \dots\}$ (hence for $z \in \{-i, -2i, -3i, \dots\}$). These poles are created by the term $\Gamma(c)$ due to the presence of the Pochhammer coefficient $(c)_n$ at the denominator. To extend them to the whole complex plane, one can consider the normalised function

$$\mathbb{F}(a, b, c; \zeta) := \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n \zeta^n}{\Gamma(c+n) n!}.$$

We refer to the monographs of Lebedev [13] or Temme [17] for further information on hypergeometric functions (see also the Appendix). Jost functions (9) and (10) are modify accordingly with this change:

$$\begin{aligned} \mathbf{f}_0^+(x, z) &:= (2 \cosh(x))^{iz} \mathbb{F}\left(a, b, c; \frac{1}{1 + e^{2x}}\right) \\ \mathbf{f}_0^-(x, z) &:= (2 \cosh(x))^{iz} \mathbb{F}\left(a, b, c; \frac{e^{2x}}{1 + e^{2x}}\right), \end{aligned}$$

and so are the Jost functions w_0 and s_0^\pm , now defined on the whole complex plane by $W_0(z) := [\mathbf{f}_0^-(x, z), \mathbf{f}_0^+(x, z)]$ and $S_0^\pm(z) := [\mathbf{f}_0^+(x, \mp z), \mathbf{f}_0^-(x, \pm z)]$.

3 Compact perturbations of a Pöschl-Teller potential

We now consider a perturbation q of the Pöschl-Teller potential. We assume that this function q is integrable and compactly supported, with its support in

$[\alpha, \beta]$. With this new potential, Equation (1) becomes

$$-y''(x) + \underbrace{\left(\frac{\lambda}{\cosh^2(x)} + q(x) \right)}_{:=V(x)} y(x) = z^2 y(x). \quad (12)$$

In terms of operator theory, we are interested in the study of the operator

$$H := -\frac{d^2}{dx^2} + \frac{\lambda}{\cosh^2(x)} + q(x).$$

It is a perturbation of the operator

$$H_0 := -\frac{d^2}{dx^2} + \frac{\lambda}{\cosh^2(x)}.$$

Let $D(H)$ (respectively $D(H_0)$) be the domain of the operator H (respectively H_0) in $L^2(\mathbb{R})$:

$$D(H) = \{f \in L^2(\mathbb{R}) \mid f, f' \in AC(\mathbb{R}), Hf \in L^2(\mathbb{R})\},$$

The solutions of $Hf = z^2 f$ are linked with those of $H_0 f = z^2 f$, using transformation operators. See [14]. In the end, if we assume that V is integrable at $+\infty$, we can find an integral kernel K^+ such that the operator X defined by

$$(Xf)(x) = f(x) + \int_x^{+\infty} K^+(x, t) f(t) dt,$$

maps $D(H_0)$ into $D(H)$ and satisfies the equality:

$$XH_0 = HX.$$

This kernel K^+ is defined as a fixed point of an integral equation

$$\begin{aligned} K^+(x, t) &= \frac{1}{2} \int_{\frac{1}{2}(x+t)}^{\beta} q(\tau) d\tau \\ &+ \int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) K^+(\tau - u, \tau + u) du d\tau, \end{aligned}$$

for $x \leq t$. If we assume that V is integrable at $-\infty$, we can find a kernel K^- such that the operator X is defined by

$$(Xf)(x) = f(x) + \int_{-\infty}^x K^-(x, t) f(t) dt,$$

and the kernel K^- is a fixed point of the integral equation

$$K^-(x, t) = \frac{1}{2} \int_{\alpha}^{\frac{1}{2}(x+t)} q(\tau) d\tau + \int_{\alpha}^{\frac{1}{2}(x+t)} \int_{\frac{t-x}{2}}^0 \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) K^-(\tau - u, \tau + u) du d\tau,$$

for $t \leq x$. Note that we can consider K^+ as a continuous function with the assumption

$$K^+(x, t) = 0, \quad t \geq 2\beta - x,$$

and similarly for K^- with the assumption

$$K^-(x, t) = 0, \quad t \leq 2\alpha - x.$$

We are interested on those kernels to be continuous. Let E_β (respectively E_α) the set of continuous functions on $\{(x, t) \in \mathbb{R}^2 \mid x \leq t\}$ (respectively $\{(x, t) \in \mathbb{R}^2 \mid t \leq x\}$) and with a support in $\{(x, t) \in \mathbb{R}^2 \mid x \leq t \leq x - 2\beta\}$ (respectively $\{(x, t) \in \mathbb{R}^2 \mid x - 2\alpha \leq t \leq x\}$). We endow these two sets with the supremum norm.

Let Φ_β be the function defined on E_β by:

$$\Phi_\beta(L)(x, t) := \frac{1}{2} \int_{\frac{1}{2}(x+t)}^{\beta} q(\tau) d\tau + \int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) L(\tau - u, \tau + u) du d\tau.$$

Similarly, we can define a function Φ_α on E_α . Following the proof of [3, Lemma 3], we obtain the existence of an integer n such that Φ_β^n is a contraction mapping, and similarly for Φ_α . According to Banach's fixed point theorem, Φ_β and Φ_α have each a unique fixed point. We summarise in the following proposition the main properties of these two kernels:

Proposition 1. *The kernels K^\pm are real-valued, and their supports are included in $\{(x, t) \in \mathbb{R}^2 \mid x \leq t \leq 2\beta - x\}$ for K^+ and $\{(x, t) \in \mathbb{R}^2 \mid 2\alpha - x \leq t \leq x\}$ for K^- . K^\pm also satisfy:*

$$K^+(x, 2\beta - x) = 0 \text{ and } K^+(x, x) = \frac{1}{2} \int_x^{\beta} q(s) ds,$$

and

$$K^-(x, 2\alpha - x) = 0 \text{ and } K^-(x, x) = \frac{1}{2} \int_{\alpha}^x q(s) ds. \quad (13)$$

Furthermore these functions are continuous and they satisfy the estimates

$$|K^\pm(x, t)| \leq \frac{1}{2} \|q\|_1 \exp \left(\|V\|_{1,1} + \left\| \frac{\lambda}{\cosh^2(\cdot)} \right\|_{1,1} \right), \quad (14)$$

where $\|f\|_{1,1} := \int_{\mathbb{R}} |t| |f(t)| dt$.

We have also information on the derivatives of K^\pm :

Proposition 2. *Let q be a real-valued, integrable and compactly supported function. Let $k \geq 0$ be an integer. The derivatives*

$$\partial_{\frac{t+x}{2}} \partial_{\frac{t-x}{2}} K^\pm, \quad \partial_{\frac{t-x}{2}} \partial_{\frac{t+x}{2}} K^\pm$$

exist and are equal to

$$\left(\frac{\lambda}{\cosh^2(t)} - V(x) \right) K^\pm(x, t)$$

and we also have

$$\partial_x^2 K^\pm(x, t) - \partial_t^2 K^\pm(x, t) = \left(V(x) - \frac{\lambda}{\cosh^2(t)} \right) K^\pm(x, t)$$

in a distributional sense in $\{(x, t) \in \mathbb{R}^2 \mid x < t\}$ for K^+ ($\{(x, t) \in \mathbb{R}^2 \mid t < x\}$ for K^-) when q is at most continuous, and in the strong sense on $\{(x, t) \in \mathbb{R}^2 \mid x \leq t\}$ for K^+ ($\{(x, t) \in \mathbb{R}^2 \mid t \leq x\}$ for K^-) when q is at least C^1 . Furthermore, if $q \in W^{k,1}$, then $K^\pm \in W^{k+1,1}$. Finally, if the $(p-1)^{\text{th}}$ derivative of q has a discontinuity at $x = \beta$ or the $(r-1)^{\text{th}}$ derivative of q has a discontinuity at $x = \alpha$, we have

$$\partial_x^p K^+(x^-, (2\beta - x)^-) = -\frac{1}{4} q^{(p-1)}(\beta^-) \neq 0 \quad (15)$$

and

$$\partial_x^r K^-(x^+, (2\alpha - x)^+) = -\frac{1}{4} q^{(r-1)}(\alpha^+) \neq 0, \quad (16)$$

where $f^{(l)}$ indicate the l^{th} derivative of the function f .

Proof. A similar proof of this proposition is given in [3, Lemma 4] when $q \in C^k$. We give the proof when q is only integrable, and for K^+ . For $L \in E_\beta$, and because of the continuity of translation in $L^1(\mathbb{R})$, the quantity

$$\int_0^{\frac{t-x}{2}} \left(V\left(\frac{t+x}{2} - u\right) - \frac{\lambda}{\cosh^2\left(\frac{t+x}{2} + u\right)} \right) L\left(\frac{t+x}{2} - u, \frac{t+x}{2} + u\right) du$$

is absolutely continuous in $\frac{t-x}{2}$ and continuous in $\frac{t+x}{2}$, and the quantity

$$\int_{\frac{t+x}{2}}^\beta \left(V\left(\tau - \frac{t-x}{2}\right) - \frac{\lambda}{\cosh^2\left(\tau + \frac{t-x}{2}\right)} \right) L\left(\tau - \frac{t-x}{2}, \tau + \frac{t-x}{2}\right) d\tau$$

is absolutely continuous in $\frac{t+x}{2}$, and continuous in $\frac{t-x}{2}$. Then we have, according to the fundamental theorem of calculus,

$$\partial_{\frac{t+x}{2}} \left[\int_{\frac{t+x}{2}}^\beta \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) L(\tau - u, \tau + u) du d\tau \right]$$

$$= - \int_0^{\frac{t-x}{2}} \left(V \left(\frac{t+x}{2} - u \right) - \frac{\lambda}{\cosh^2 \left(\frac{t+x}{2} + u \right)} \right) L \left(\frac{t+x}{2} - u, \frac{t+x}{2} + u \right) du. \quad (17)$$

Using Fubini-Tonelli's theorem, we also have

$$\begin{aligned} & \partial_{\frac{t-x}{2}} \left[\int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) L(\tau - u, \tau + u) du d\tau \right] \\ &= \int_{\frac{t+x}{2}}^{\beta} \left(V \left(\tau - \frac{t-x}{2} \right) - \frac{\lambda}{\cosh^2 \left(\tau + \frac{t-x}{2} \right)} \right) L \left(\tau - \frac{t-x}{2}, \tau + \frac{t-x}{2} \right) d\tau. \end{aligned} \quad (18)$$

In the sense of distribution, and on the set $\{(x, t) \in \mathbb{R}^2 \mid x < t\}$, we have

$$\begin{aligned} & \partial_{\frac{t+x}{2}} \partial_{\frac{t-x}{2}} \left[\int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) L(\tau - u, \tau + u) du d\tau \right] \\ &= \partial_{\frac{t-x}{2}} \partial_{\frac{t+x}{2}} \left[\int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) L(\tau - u, \tau + u) du d\tau \right] \\ &= \left(V(x) - \frac{\lambda}{\cosh^2(t)} \right) L(x, t). \end{aligned}$$

It directly follows that K^+ satisfies the wave equation

$$\partial_x^2 K^+(x, t) - \partial_t^2 K^+(x, t) = \left(V(x) - \frac{\lambda}{\cosh^2(t)} \right) K^+(x, t),$$

always in a distributional sense. For the last point, we prove that $K^+(x, t) - \frac{1}{2} \int_{\frac{t+x}{2}}^{\beta} q(s) ds$ is $W^{k,1}$. Using the definition of K^+ , and the fact that $q \in W^{k-2,1}$

$$K^+(x, t) = \sum_{m=1}^{+\infty} K_m(x, t)$$

where $K_1(x, t) := \frac{1}{2} \int_{\frac{t+x}{2}}^{\beta} q(s) ds$ and

$$K_{m+1}(x, t) := \int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) K_m^+(\tau - u, \tau + u) du d\tau.$$

For $l \in \mathbb{N}$, the application

$$L \mapsto \int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau - u) - \frac{\lambda}{\cosh^2(\tau + u)} \right) L(\tau - u, \tau + u) du d\tau$$

maps $W^{l,1}$ to $W^{\min(l+1,k-1),1}$. By induction, it follows that $K_m \in W^{k-1,1}$ for any m , and so $K^+ \in W^{k-1,1}$. To conclude, we remark that if $q \in W^{k-1,1}$, then the formulas (17) and (18) show that for $K^+ \in W^{k-1,1}$,

$$\partial_{\frac{t+x}{2}} \left[\int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau-u) - \frac{\lambda}{\cosh^2(\tau+u)} \right) K^+(\tau-u, \tau+u) du d\tau \right]$$

and

$$\partial_{\frac{t-x}{2}} \left[\int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau-u) - \frac{\lambda}{\cosh^2(\tau+u)} \right) K^+(\tau-u, \tau+u) du d\tau \right]$$

are $W^{k-1,1}$ in $\frac{t+x}{2}$ and $\frac{t-x}{2}$ respectively. Moreover, for $q \in W^{k-2,1}$, the quantity $\left(V(x) - \frac{\lambda}{\cosh^2(t)} \right) K^+(x, t)$ is $W^{k-2,1}$, and so the quantity

$$\int_{\frac{t+x}{2}}^{\beta} \int_0^{\frac{t-x}{2}} \left(V(\tau-u) - \frac{\lambda}{\cosh^2(\tau+u)} \right) K^+(\tau-u, \tau+u) du d\tau$$

is $W^{k,1}$ on $\{(x, t) \mid x \leq t \leq x - 2\beta\}$. From that, using $K^+(x, t) = 0$ for $t \leq 2\beta - x$, we straightly deduce

$$\partial_x^p K^+(x^-, (2\beta - x)^-) = \frac{-1}{4} \partial_x^{p-1} q(\beta^-). \quad \square$$

We are now going to summarise the main properties of the kernels in the following property. Using this transformation operators theory, we can finally define

$$f^+(x, z) = f_0^+(x, z) + \int_x^{2\beta-x} K^+(x, t) f_0^+(t, z) dt \mathbf{1}_{]-\infty, \beta[}(x)$$

and

$$f^-(x, z) = f_0^-(x, z) + \int_{2\alpha-x}^x K^-(x, t) f_0^-(t, z) dt \mathbf{1}_{]_{\alpha}, +\infty[}(x).$$

These two functions f^{\pm} still satisfy (12) and (6), so they are Jost solutions associated with Equation (12). As we did in the previous part, we can define Jost functions w and s^{\pm} as the two following Wronskians

$$w(z) := [f^-(\cdot, z), f^+(\cdot, z)]$$

and

$$s^{\pm}(z) := [f^+(\cdot, \mp z), f^-(\cdot, \pm z)].$$

We define for $z \in \mathbb{C} \setminus i\mathbb{Z}$ the coefficients $\mathcal{A}(z) := w(z)/2iz$, $\mathcal{B}(z) := s^-(z)/2iz$, $\mathcal{C}(z) := s^+(z)/2iz$ and $\mathcal{D}(z) := w(z)/2iz$, the transmission coefficient $\mathcal{T}(z) := 2iz/w(z)$, the right and left reflection coefficients $\mathcal{R}^{\pm}(z) := s^{\pm}(z)/w(z)$ and finally the scattering matrix

$$\mathcal{S}(z) := \begin{pmatrix} \mathcal{T}(z) & \mathcal{R}^-(z) \\ \mathcal{R}^+(z) & \mathcal{T}(z) \end{pmatrix}.$$

Again, because of the use of the hypergeometric functions, f^\pm may have poles at $z \in i\mathbb{Z}$. To extend them to the whole complex plane, one have to consider the normalised Jost solutions

$$\mathbf{f}^+(x, z) := \mathbf{f}_0^+(x, z) + \int_x^{2\beta-x} K^+(x, t) \mathbf{f}_0^+(t, z) dt \mathbf{1}_{(-\infty, \beta)}(x),$$

and

$$\mathbf{f}^-(x, z) := \mathbf{f}_0^-(x, z) + \int_{2\alpha-x}^x K^-(x, t) \mathbf{f}_0^-(t, z) dt \mathbf{1}_{(\alpha, +\infty)}(x).$$

Modified respectively, the normalised Jost functions W and S^\pm are defined similarly as W_0 and S_0^\pm before:

$$W(z) := [\mathbf{f}^-(x, z), \mathbf{f}^+(x, z)] \text{ and } S^\pm(z) := [\mathbf{f}^+(x, \mp z), \mathbf{f}^-(x, \pm z)].$$

When $z \in \mathbb{C} \setminus (-i\mathbb{N})$ for W and $z \in \mathbb{C} \setminus i\mathbb{Z}^*$ for S^\pm , we have

$$W(z) = \frac{w(z)}{\Gamma(1-iz)^2} \text{ and } S^\pm(z) = \frac{s^\pm(z)}{\Gamma(1-iz)\Gamma(1+iz)}.$$

With this normalisation, we have $[\mathbf{f}^\pm(x, z), \mathbf{f}^\pm(x, -z)] = \frac{\mp 2iz}{\Gamma(1-iz)\Gamma(1+iz)}$. So, the expressions of the normalised Jost solutions in the basis $(\mathbf{f}^\pm(\cdot, -z), \mathbf{f}^\pm(\cdot, z))$ are now:

$$\mathbf{f}^\pm(\cdot, z) = \frac{W(z)}{\Gamma(1-iz)\Gamma(1+iz)} \mathbf{f}^\mp(\cdot, -z) + \frac{S^\mp(z)}{\Gamma(1-iz)\Gamma(1+iz)} \mathbf{f}^\mp(\cdot, z), \quad z \in \mathbb{C} \setminus i\mathbb{Z}.$$

Coefficients A , B , C and D are defined as before, *i.e.*

$$A(z) := \frac{W(z)}{\Gamma(1-iz)\Gamma(1+iz)} \text{ and } B(z) := \frac{S^-(z)}{\Gamma(1-iz)\Gamma(1+iz)},$$

$$D(z) := \frac{W(z)}{\Gamma(1-iz)\Gamma(1+iz)} \text{ and } C(z) := \frac{S^+(z)}{\Gamma(1-iz)\Gamma(1+iz)}.$$

Finally, we can define, always for $z \in \mathbb{C} \setminus i\mathbb{Z}$, the normalised transmission coefficient $T(z) := \frac{1}{A(z)} = \frac{2iz}{W(z)\Gamma(1-iz)\Gamma(1+iz)}$, the two normalised reflection coefficients $R^-(z) := \frac{B(z)}{A(z)} = \frac{S^-(z)}{W(z)}$ and $R^+(z) := \frac{C(z)}{A(z)} = \frac{S^+(z)}{W(z)}$, and the normalised scattering matrix $S(z)$. We remark here that $T(z)\Gamma(1-iz)^2 = \mathcal{T}(z)$ and

$$R^\pm(z) \frac{\Gamma(1+iz)}{\Gamma(1-iz)} = \mathcal{R}^\pm(z). \quad (19)$$

From now, we assume that q is real. First, we start with a lemma on normalised Jost functions:

Lemma 1. *Then, the functions W and S^\pm are entire and their growth order is at most one. Moreover, they satisfy:*

1. $\overline{W(z)} = W(-\bar{z}), \overline{S^\pm(z)} = S^\pm(-\bar{z});$
2. $S^-(z) = S^+(-z);$
3. For $z \in \mathbb{R}^*: |A(z)|^2 = 1 + |B(z)|^2;$
4. $W(z)W(-z) - \frac{4z^2}{\Gamma(1-iz)^2\Gamma(1+iz)^2} = S^\pm(z)S^\pm(-z);$
5. $W(0) = -S^\pm(0).$

This lemma is similar to [1, Lemma 2.1], except for the third point. So we give a short proof. For a such z , a straightforward calculation shows:

$$\begin{aligned}
& -\frac{2iz}{\Gamma(1-iz)\Gamma(1+iz)} \\
&= [\mathbf{f}^+(x, z), \mathbf{f}^+(x, -z)] \\
&= [A(z)\mathbf{f}^-(x, -z) + B(z)\mathbf{f}^-(x, z), A(-z)\mathbf{f}^-(x, z) + B(-z)\mathbf{f}^-(x, -z)] \\
&= A(z)A(-z)[\mathbf{f}^-(x, -z), \mathbf{f}^-(x, z)] + B(z)B(-z)[\mathbf{f}^-(x, z), \mathbf{f}^-(x, -z)] \\
&= -\frac{2iz}{\Gamma(1-iz)\Gamma(1+iz)} (|A(z)|^2 - |B(z)|^2),
\end{aligned}$$

and $A(-z) = \overline{A(z)}, B(-z) = \overline{B(z)}$ are consequences of the first two items. The first, the fourth and the fifth points of Lemma 1 show that 0 is at most a simple zero of W . Furthermore, the second and the third points justify that \mathcal{S} is a unitary matrix. Since these functions are entire of growth order at most one, Hadamard's factorisation theorem gives the next expressions. Consider $\{W_n \in \mathbb{C} \mid 0 < |W_1| \leq |W_2| \leq \dots\}$ the set of non-vanishing zeros of W listed according to multiplicity. This set is composed of two types of zeros: the zeros of w denoted w_n which are in the scattering data of the problem, and the $-ik$, where $k \in \mathbb{N}$ and ik is an eigenvalue, which are also known. Then, by Hadamard's factorisation theorem we have

$$W(z) = z^m e^{a_0 + a_1 z} \prod_{n=1}^{+\infty} \left(1 - \frac{z}{W_n}\right) e^{z/W_n}, \quad (20)$$

where $a_0, a_1 \in \mathbb{C}$ and $m \in \{0, 1\}$. We are going to give another expression for w , and so for W . Using the definition of w , we can prove that, for $x \geq \beta$ and

$z \in \mathbb{C} \setminus i\mathbb{Z}^*$,

$$\begin{aligned}
w(z) &= 2iz \frac{\Gamma(c-1)\Gamma(c)}{\Gamma(a)\Gamma(b)} - F(c-a, c-b, c; 1/(1+e^{2x})) \\
&\quad \times \int_{2\alpha-x}^x e^{-iz(t-x)} \partial_x K^-(x, t) F(c-a, c-b, c; e^{2t}/(1+e^{2t})) dt \\
&\quad - K^-(x, x) F(c-a, c-b, c; 1/(1+e^{2x})) F(c-a, c-b, c; e^{2x}/(1+e^{2x})) \\
&\quad + \int_{2\alpha-x}^x K^-(x, t) e^{-iz(t-x)} F(c-a, c-b, c; e^{2t}/(1+e^{2t})) dt \\
&\quad \left[iz F(c-a, c-b, c; 1/(1+e^{2x})) \right. \\
&\quad \left. + \frac{-2e^{2x}(c-a)(c-b)}{(1+e^{2x})^2 c} F(c-a+1, c-b+1, c+1; 1/(1+e^{2x})) \right].
\end{aligned}$$

The quantity $\frac{\Gamma(c-1)\Gamma(c)}{\Gamma(a)\Gamma(b)}$ tends to 1 as z tends to infinity in the upper half-plane, using Stirling's formula:

$$\Gamma(z) \sim (2\pi z)^{\frac{1}{2}} e^{-z} z^z, \quad |\operatorname{Im}(z)| \leq \pi - \varepsilon, \quad \varepsilon > 0, \quad z \rightarrow \infty.$$

We also have:

$$\begin{aligned}
&\int_{2\alpha-x}^x K^-(x, t) e^{-iz(t-x)} F(c-a, c-b, c; e^{2t}/(1+e^{2t})) dt \\
&= \int_{2(\alpha-x)}^0 K^-(x, u+x) e^{-izu} F(c-a, c-b, c; e^{2(u+x)}/(1+e^{2(u+x)})) du.
\end{aligned}$$

Let $\delta > 0$ such that $|\zeta| \leq 1 - \delta$. Then we have for $s \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned}
\left| \frac{|\zeta|^{s+1} \frac{(|c-a|)_{s+1} (|b-a|)_{s+1}}{(|c|)_{s+1} (s+1)!}}{|\zeta|^s \frac{(|c-a|)_s (|b-a|)_s}{(|c|)_s s!}} \right| &= |\zeta| \frac{(|c-a|+s)(|c-b|+s)}{|c+s|(s+1)} \\
&\leq |\zeta| \frac{(|c-a|+s)(|c-b|+s)}{(s+m)^2} \frac{(s+m)^2}{|c+s|(s+1)} \\
&\leq |\zeta| \frac{(s+m)^2}{(s+1)^2} \\
&\leq |\zeta|(1+r) \\
&\leq (1-\delta)(1+r),
\end{aligned}$$

where $m := \max(|c-a|, |c-b|) + 1$, $r > 0$ such that $(1-\delta)(1+r) < 1$ and a choice of s_0 such that for $s \geq s_0$, we have $\frac{(s+m)^2}{(s+1)^2} \leq 1+r$. Then by cutting the

sum of the hypergeometric function in two, we obtain:

$$\begin{aligned} & \left| \int_{2(\alpha-x)}^0 K^-(x, u+x) e^{-izu} F(c-a, c-b, c; e^{2(u+x)}/(1+e^{2(u+x)})) du \right| \\ & \leq \int_{2(\alpha-x)}^0 |K^-(x, u+x)| \sum_{s=0}^{s_0-1} \frac{(|c-a|)_s (|b-a|)_s}{|(c)_s s!} \left(\frac{e^{2(u+x)}}{1+e^{2(u+x)}} \right)^s du \\ & + \int_{2(\alpha-x)}^0 |K^-(x, u+x)| \sum_{s=s_0}^{+\infty} \frac{(|c-a|)_s (|b-a|)_s}{|(c)_s s!} \left(\frac{e^{2(u+x)}}{1+e^{2(u+x)}} \right)^s du, \end{aligned}$$

where we use $|e^{-izu}| \leq 1$ for $u \in (2(\alpha-x), 0)$ and $\text{Im}(z) \geq 0$. The quantity $\sum_{s=s_0}^{+\infty} \frac{(|c-a|)_s (|b-a|)_s}{|(c)_s s!} \left(\frac{e^{2(u+x)}}{1+e^{2(u+x)}} \right)^s$ is bounded by

$$\begin{aligned} & \frac{(|c-a|)_{s_0} (|b-a|)_{s_0}}{|(c)_{s_0} s_0!} \left(\frac{e^{2(u+x)}}{1+e^{2(u+x)}} \right)^{s_0} \sum_{s=s_0}^{+\infty} ((1-\delta)(1+r))^{s-s_0} \\ & \leq \frac{(|c-a|)_{s_0} (|b-a|)_{s_0}}{|(c)_{s_0} s_0!} \left(\frac{e^{2(u+x)}}{1+e^{2(u+x)}} \right)^{s_0} \frac{1}{1-(1-\delta)(1+r)}. \end{aligned}$$

Then, using Estimates (14), the ratios of these two integrals to $2iz$ tend to 0 when z tends to ∞ . So $w(z) \sim 2iz$ when $z \rightarrow \infty$ in the closed upper half-plane. From this result, we can deduce a universal asymptotic for W in this domain, *i.e.* an asymptotic which doesn't depend on the potential:

$$W(z) \sim \frac{2iz}{\Gamma(1-iz)^2}, \quad z \rightarrow \infty, \quad \text{Im}(z) \geq 0. \quad (21)$$

This asymptotic is going to be useful to uniquely determine the function W , and so the denominator of both reflection coefficients.

Proposition 3. *The Jost function W associated with a certain potential V defined as $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$ is fully determined by its zeros, *i.e.* the poles of reflection coefficients.*

Proof. Let V and \tilde{V} be two potentials such that, for every real numbers x , we have $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$ and $\tilde{V}(x) = \frac{\lambda}{\cosh^2(x)} + \tilde{q}(x)$. We also assume that they share the same scattering data, so for every positive integer n , $W_n = \tilde{W}_n$. We can use the Hadamard's factorisation theorem to obtain an expression for \tilde{W} , the Jost function associated with the potential \tilde{V} :

$$\tilde{W}(z) = z^{\tilde{m}} e^{\tilde{a}_0 + \tilde{a}_1 z} \prod_{n=1}^{+\infty} \left(1 - \frac{z}{\tilde{W}_n} \right) e^{z/\tilde{W}_n}, \quad (22)$$

where $\tilde{a}_0, \tilde{a}_1 \in \mathbb{C}$ and $\tilde{m} \in \{0, 1\}$. Then, because of (20) and (22),

$$\begin{aligned} \frac{W(z)}{\tilde{W}(z)} &= \frac{z^m e^{a_0 + a_1 z} \prod_{n=1}^{+\infty} \left(1 - \frac{z}{W_n} \right) e^{z/W_n}}{z^{\tilde{m}} e^{\tilde{a}_0 + \tilde{a}_1 z} \prod_{n=1}^{+\infty} \left(1 - \frac{z}{\tilde{W}_n} \right) e^{z/\tilde{W}_n}} \\ &= z^{m-\tilde{m}} e^{a_0 - \tilde{a}_0} e^{(a_1 - \tilde{a}_1)z}. \end{aligned}$$

Furthermore, (21) entails that $\frac{W(z)}{\tilde{W}(z)} \rightarrow 1$ as $z \rightarrow \infty$ with $\text{Im}(z) \geq 0$. Necessarily $m = \tilde{m}$, $a_0 = \tilde{a}_0$, $a_1 = \tilde{a}_1$ and so $W = \tilde{W}$. \square

As we can see, no further assumptions on the potential than q been integrable, real and compactly supported are required to conclude on the uniqueness of the Jost function W . We now consider S^\pm for which we have more assumptions on q . As we did with W , because they are also entire functions of growth order at most one, we can use the Hadamard's factorisation theorem and so write

$$S^\pm(z) = (\mp 1)^l z^l e^{b_0 \mp b_1 z} \prod_{n=1}^{+\infty} \left(1 \pm \frac{z}{S_n} \right) e^{\mp z/S_n}, \quad (23)$$

where $b_0, b_1 \in \mathbb{C}$, $l \geq m$. The S_n are the non-vanishing zeros of S^- such that $0 < |S_1| \leq |S_2| \leq \dots$, listed according to multiplicity. The situation is the same as before with W : some S_n are zeros of s^- which are denoted s_n , and the other ones of the form $\pm ik$, where ik is an eigenvalue and $k \in \mathbb{N}$. Now, using Lemma 1, we have

$$W(z)W(-z) - \frac{4z^2}{\Gamma(1-iz)^2\Gamma(1+iz)^2} = S^\pm(z)S^\pm(-z).$$

Evaluating this identity at $z = 0$, we obtain that the right-hand side has a zero of order $2l$. The left-hand side is perfectly known, so l is uniquely determined. To obtain the quantities b_0 and b_1 , we have, as we previously did for the Jost function w , to give another expression of s^+ . The link between S^+ and S^- (or s^+ and s^-) presented in the second item of Lemma 1 allows to work with only one of them. Let $\tilde{c} := c(-z) = 1 + iz$, and similarly for \tilde{a} and \tilde{b} . If $x \geq \beta$, then

$$\begin{aligned} s^+(z) &= 2 \frac{\Gamma(1-iz)\Gamma(1+iz)}{\Gamma(c-a)\Gamma(c-b)} \\ &+ \left[\int_{2\alpha-x}^x e^{-iz(t+x)} (\partial_x + \partial_t) K^-(x, t) F\left(c-a, c-b, c; \frac{e^{2t}}{1+e^{2t}}\right) dt \right. \\ &+ \left. \int_{2\alpha-x}^x e^{-iz(t+x)} K^-(x, t) \partial_t F\left(c-a, c-b, c; \frac{e^{2t}}{1+e^{2t}}\right) dt \right] \\ &\times F(\tilde{c} - \tilde{a}, \tilde{c} - \tilde{b}, \tilde{c}; \frac{1}{1+e^{2x}}) \end{aligned} \quad (24)$$

$$\begin{aligned} &+ \frac{2e^{2x}}{(1+e^{2x})^2} \frac{(\tilde{c} - \tilde{a})(\tilde{c} - \tilde{b})}{\tilde{c}} \times F(\tilde{c} - \tilde{a} + 1, \tilde{c} - \tilde{b} + 1, \tilde{c} + 1; \frac{1}{1+e^{2x}}) \\ &\times \int_{2\alpha-x}^x e^{-iz(t+x)} K^-(x, t) F\left(c-a+1, c-b+1, c+1; \frac{e^{2t}}{1+e^{2t}}\right) dt. \end{aligned} \quad (25)$$

These different writings of s^+ allow us to find asymptotics for it.

Lemma 2. Let V be a potential such that $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$ for all real numbers x . Assume that $\text{supp } q \subset [\alpha, \beta] \subset \mathbb{R}_-$. Let $z_n = i(\frac{4n+1}{2})$, for every integer n . Then,

$$s^+(z_n) \underset{n \rightarrow \infty}{\sim} 2 \frac{\Gamma(1 - iz_n)\Gamma(1 + iz_n)}{\Gamma(c-a)\Gamma(c-b)}.$$

The same lemma exists for a perturbation with its support on \mathbb{R}_+ .

Lemma 3. Let V be a potential such that $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$ for all real numbers x . Assume that $\text{supp } q \subset [\alpha, \beta] \subset \mathbb{R}_+$. Let $z_n = -i(\frac{4n+1}{2})$, for every integer n . Then,

$$s^-(z_n) \underset{n \rightarrow \infty}{\sim} 2 \frac{\Gamma(1 - iz_n)\Gamma(1 + iz_n)}{\Gamma(c-a)\Gamma(c-b)}.$$

Proof. We are going to do the proof for $\text{supp } q \subset \mathbb{R}_-$, and so give an asymptotic for s^+ . To conclude in the case $\text{supp } q \subset \mathbb{R}_+$, we can use the symmetries of the problem under the change of variable $\tilde{x} = -x$. A function q with its support in \mathbb{R}_+ has now its support in \mathbb{R}_- with this change of variable. Using (25), and consider $x = \beta$. Then $x \leq 0$, and we also have

$$\Gamma(1 - iz_n)\Gamma(1 + iz_n) = \Gamma\left(1 + \frac{4n+1}{2}\right)\Gamma\left(1 - \frac{4n+1}{2}\right) = \frac{(4n+1)\pi}{2},$$

Finally, arguments used in the proof of (21) implies that the integral terms tends to zero. \square

This lemma means that, under assumptions on the location of the support,

$$S^\pm(z_n) \underset{n \rightarrow \infty}{\sim} \frac{2}{\Gamma(c-a)\Gamma(c-b)}.$$

Identically to what we did for W , using Lemma 2 and (23), we can show that S^\pm are completely determined by their zeros and the assumption on the location of the support of q . We can now state an inverse-problem result:

Theorem 1. A real-valued, integrable and compactly supported perturbation q of a Pöschl-Teller potential is uniquely determined by the zeros and poles of one of its reflection coefficients in the following cases:

- $\text{supp } q \subset \mathbb{R}_+$;
- $\text{supp } q \subset \mathbb{R}_-$.

Proof. We previously showed that under these two assumptions, Jost functions are fully determined. So are both reflection coefficients R^\pm . \square

In the case of a potential V such that q have a support that contains zero, we are going to use other assumptions: we assume now that there is no half-bound state, i.e. $W(0) \neq 0$ or there is at least one eigenvalue which is not in $i\mathbb{N}$. Using $W(0) = -S^\pm(0)$, the knowledge of both W and the link between norming constants and residues of reflection coefficients at $z = ik_0$,

$$\int_{-\infty}^{+\infty} |\mathbf{f}^\pm(x, ik_0)|^2 dx = i \frac{S^\pm(ik_0)\dot{W}(ik_0)}{\frac{4k_0^2}{\Gamma(1+k_0)^2\Gamma(1-k_0)^2}} > 0,$$

where \dot{W} denotes the derivative of W with respect to the complex variable z , e^{b_0} is completely determined in these two cases, see [1, Theorem 2.2]. According to the work of Bledsoe [1, Proposition 2.3 and Lemma 2.4], we have the two following lemmas:

Lemma 4. *The complex number b_1 is an imaginary number.*

Lemma 5. *Let V and \tilde{V} be two potentials defined as before, i.e. $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$ and $\tilde{V}(x) = \frac{\lambda}{\cosh^2(x)} + \tilde{q}(x)$. Then $\tilde{V}(x) = V(x - \alpha)$ for every real number x and for α a real number if and only if $W = \tilde{W}$ and $S^\pm(z) = e^{\mp 2i\alpha z} \tilde{S}^\pm(z)$.*

There is a misprint in the statement of Lemma 5 in [1]. It is written $S^\pm(z) = e^{\pm 2i\alpha z} \tilde{S}^\pm(z)$. Using a proof by contradiction, we are going to prove that b_1 is also fully determined by the two previous assumptions. Assume that $b_1 \neq \tilde{b}_1$. Because of Lemma 4, they are both imaginary numbers, so one can write them $b_1 = i\beta_1$ and $\tilde{b}_1 = i\tilde{\beta}_1$ with $\beta_1, \tilde{\beta}_1 \in \mathbb{R}$. Let $\alpha := \beta_1 - \tilde{\beta}_1$ be a non-zero real number. Then we have

$$S^\pm(z) = e^{\mp i\alpha z} \tilde{S}^\pm(z). \quad (26)$$

Using Lemma 5, Equation (26) and $W = \tilde{W}$, we have for every real number x

$$\tilde{V}(x) = V\left(x - \frac{\alpha}{2}\right). \quad (27)$$

This result holds for every real x , so one can choose x large enough, such that $x \notin \text{supp } q$ and $x \notin \text{supp } \tilde{q}$. For such a real number x , Equality (27) becomes

$$\frac{\lambda}{\cosh^2(x)} = \frac{\lambda}{\cosh^2\left(x - \frac{\alpha}{2}\right)}.$$

This is absurd, so we have a contradiction with the assumption that α is a non-zero real number. This means that b_1 is fully determined under the assumptions "no half-bound state" or "existence of an eigenvalue which is not in $i\mathbb{N}$ ". Actually, what we have shown here is that if $b_0 = \tilde{b}_0$, then necessarily $b_1 = \tilde{b}_1$. So we have another inverse problem theorem:

Theorem 2. *A real-valued, integrable and compactly supported perturbation q of a Pöschl-Teller potential is uniquely determined by the zeros and poles of one of its reflection coefficients in the following cases:*

- *There is no half-bound state;*
- *There is at least one eigenvalue which is not in $i\mathbb{N}$.*

Finally, in the case of a function q such that zero is in its support and which has a half-bound state and eigenvalues in $i\mathbb{N}$, in order to be able to conclude, we can add a quantity to our scattering data: this result is stated in the work of Korotyaev [12]. As we said, we are able to determine, with Lemma 1 the quantity e^{2b_0} . Let $p := \text{sign}(i^l e^{b_0})$, p is well-defined because $i^l e^{b_0}$ is a real number. Suppose that p is given, *i.e* p is in the scattering data set with the poles and zeros of reflection coefficients. Then we will be able to uniquely determine b_0 . Following Theorem 2, using Lemma 5, we then obtain b_1 . We now state the last theorem of uniqueness of this work:

Theorem 3. *A real-valued, integrable and compactly supported perturbation q of a Pöschl-Teller potential is uniquely determined by the zeros and poles of one of its reflection coefficients and the quantity $p = \text{sign}(i^l e^{b_0})$.*

One can see these three theorems as adjustments of results stated in the works of Bledsoe [1] and Korotyaev [12]. They have shown similar theorems of uniqueness considering a compact supported perturbation of the zero potential. The main difference with their works is that, because of the presence of a Pöschl-Teller potential, we have to deal with more different cases. Due to the translation invariance of the support, Theorem 1 doesn't exist for them. These three theorems show that the normalised reflection coefficients of a potential V such that $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$ are completely determined under different assumptions. Then, because of the well-known relation (19) between T , R^\pm and \mathcal{T} , \mathcal{R}^\pm , these latter will also be fully determined. Finally, the Gelfand-Levitan-Marchenko theory will prove that the potential is uniquely determined. See [14, Part 6.5]

4 Localisation of the resonances at infinity for a compactly supported perturbation

We can determine the resonances when $q = 0$ (see Figure 1). We would like to know how these "initial" resonances behave when we perturb a Pöschl-Teller potential with a real-valued, integrable and compactly supported potential q . We are going to detail the reasoning only in the case of a potential q which satisfies the condition $0 \in (\alpha, \beta)$. If this assumption is not satisfied, we observe, in addition to logarithmic branches that some initial resonances are still present asymptotically.

We suppose that the $(p - 1)^{th}$ derivative of q has a discontinuity at $x = \beta$ and the $(r - 1)^{th}$ derivative of q has a discontinuity at $x = \alpha$, where $p, q \geq 1$. A similar work for a compactly perturbed exponentially decreasing potential on

the half-line has already been done in [3]. We give a new expression for the Jost function w , evaluating this Wronskian at $x = 0$:

$$\begin{aligned}
w(z) &= [f^-(0, z), f^+(0, z)] \\
&= f^-(0, z)(f^+)'(0, z) - (f^-)'(0, z)f^+(0, z) \\
&= f^-(0, z)(f^+)'(0, z) \left(1 - \frac{(f^-)'(0, z)f^+(0, z)}{f^-(0, z)(f^+)'(0, z)} \right) \\
&= f^-(0, z)(f^+)'(0, z) \left(1 - \frac{m_-(z)}{m_+(z)} \right) \tag{28}
\end{aligned}$$

with $m_+(z) := \frac{(f^+)'(0, z)}{f^+(0, z)}$ and $m_-(z) := \frac{(f^-)'(0, z)}{f^-(0, z)}$. We mention here that the two quantities m_+ and m_- are called Weyl-Titchmarsh functions. Resonances are either zeros of $f^\pm(0, z)$ (or their derivatives) or zeros of the third factor in (28). We are going to study both of these two factors in different domains of the lower half-plane.

Definition 1. Let S_1 and S_2 be two domains of the lower half-plane defined by

$$S_1 := \{z \in \mathbb{C}^- \mid -|\operatorname{Re}(z)| \geq \delta \operatorname{Im}(z)\},$$

and

$$S_2 := \{z \in \mathbb{C}^- \mid -|\operatorname{Re}(z)| < \delta \operatorname{Im}(z)\},$$

for $\delta > 0$ and $\mathbb{C}^- := \{z \in \mathbb{C} \mid \operatorname{Im}(z) < 0\}$. We also define a set \mathcal{B} by

$$\mathcal{B} := \bigcup_{n=0}^{+\infty} B(-i(n+1), \eta),$$

for $\eta > 0$, and where $B(x, r)$ stands for the open ball of center x and radius r . Finally, let Z be the set of zeros of $f^-(0, z)(f^+)'(0, z)$:

$$Z := \{z \in \mathbb{C} \mid f^-(0, z)(f^+)'(0, z) = 0\}.$$

One can understand Z as the domain where the quantity m_-/m_+ is not defined. See Figure 2 for a representation of these domains. We start by the determination of asymptotics of the third factor in the expression (28) of w .

Lemma 6. We have:

$$f^+(0, z) \sim (-1)^p \frac{\partial_t^p K^+(0, 2\beta^-)}{(iz)^{p+1}} e^{2iz\beta}, \quad \operatorname{Im}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus \mathcal{B}.$$

Let $C > 0$, we have

$$f^+(0, z) \rightarrow 1, \quad \operatorname{Re}(z) \rightarrow \infty, \quad |\operatorname{Im}(z)| \leq C, \quad z \in \mathbb{C}^- \setminus \mathcal{B}.$$

Proof. Let us start with the first statement. The notations o and \sim holds for z tending to ∞ and $z \in \mathbb{C} \setminus \mathcal{B}$. Since

$$f^+(0, z) = f_0^+(0, z) + \int_0^{2\beta} K^+(0, t) f_0^+(t, z) dt,$$

we will show that $f_0^+(0, z) = o\left(\int_0^{2\beta} K^+(0, t) f_0^+(t, z) dt\right)$. We have, after $(p+1)$ integrations by parts,

$$\begin{aligned} & \int_0^{2\beta} K^+(0, t) f_0^+(t, z) dt = \\ & o(1) + (-1)^p \frac{\partial_t^p K^+(0, 2\beta)}{(iz)^{p+1}} F\left(c-a, c-b, c; \frac{1}{1+e^{4\beta}}\right) e^{2iz\beta} [1 + o(1)]. \quad (*) \end{aligned}$$

Equation (15) tells us that $\partial_t^p K^+(0, 2\beta) = -\frac{1}{4}q^{(p-1)}(\beta^-) \neq 0$, since we assume that the $(p-1)^{th}$ derivative of q has a discontinuity at $x = \beta$. The second $o(1)$ corresponds to the remaining integral term after all the integrations by parts. We are now going to use Lemma 11 to find the asymptotic of f_0^+ in this domain. We know that:

$$f_0^+(0, z) = 2^{iz} \frac{\sqrt{\pi} \Gamma\left(\frac{a}{2} + \frac{b}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{a}{2} + \frac{1}{2}\right) \Gamma\left(\frac{b}{2} + \frac{1}{2}\right)}.$$

We use now Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z},$$

to modify the expression of gamma function, so we will be able to use Stirling formula. To use Euler's reflection formula, we have to consider the domain without some discs centred around zeros of $\frac{1}{\Gamma(\frac{a}{2} + \frac{1}{2})}$ and $\frac{1}{\Gamma(\frac{b}{2} + \frac{1}{2})}$ of radius $\varepsilon > 0$.

So, for $z \in \mathbb{C} \setminus \mathbb{Z}$, we have

$$\begin{aligned} & \frac{\sqrt{\pi} \Gamma\left(\frac{a}{2} + \frac{b}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{a}{2} + \frac{1}{2}\right) \Gamma\left(\frac{b}{2} + \frac{1}{2}\right)} = \frac{\sqrt{\pi} \Gamma(c)}{\Gamma\left(\frac{a}{2} + \frac{1}{2}\right) \Gamma\left(\frac{b}{2} + \frac{1}{2}\right)} \\ & = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \frac{a}{2}\right) \Gamma\left(\frac{1}{2} - \frac{b}{2}\right) \sin\left(\pi\left(\frac{a}{2} + \frac{1}{2}\right)\right) \sin\left(\pi\left(\frac{b}{2} + \frac{1}{2}\right)\right) \pi}{\Gamma(1-c) \pi^2 \sin(\pi c)} \\ & = \frac{1}{\sqrt{\pi}} \frac{\sin\left(\pi\left(\frac{a}{2} + \frac{1}{2}\right)\right) \sin\left(\pi\left(\frac{b}{2} + \frac{1}{2}\right)\right) \Gamma\left(\frac{1}{2} - \frac{a}{2}\right) \Gamma\left(\frac{1}{2} - \frac{b}{2}\right)}{\sin(\pi c) \Gamma(1-c)}. \end{aligned}$$

Then, we apply Stirling formula

$$\Gamma(z) \sim (2\pi z)^{\frac{1}{2}} e^{-z} z^z, \quad |\text{Arg}(z)| \leq \pi - \varepsilon, \quad \varepsilon > 0,$$

on the last term:

$$\begin{aligned} & \frac{\Gamma\left(\frac{a}{2} + \frac{b}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{a}{2}\right) \Gamma\left(\frac{1}{2} - \frac{b}{2}\right)} \sim \frac{\sqrt{2\pi\left(\frac{1}{2} - \frac{a}{2}\right)} \sqrt{2\pi\left(\frac{1}{2} - \frac{b}{2}\right)}}{\sqrt{2\pi(1-c)}} \\ & \quad \times \frac{e^{-\left(\frac{1}{2} - \frac{a}{2}\right)} e^{-\left(\frac{1}{2} - \frac{b}{2}\right)}}{e^{-(1-c)}} \times \frac{\left(\frac{1}{2} - \frac{a}{2}\right)^{\frac{1}{2} - \frac{a}{2}} \left(\frac{1}{2} - \frac{b}{2}\right)^{\frac{1}{2} - \frac{b}{2}}}{(1-c)^{1-c}}. \end{aligned}$$

The first two terms are $O(\sqrt{iz})$. Consider the last one, we have:

$$\frac{\left(\frac{1}{2} - \frac{a}{2}\right)^{\frac{1}{2} - \frac{a}{2}} \left(\frac{1}{2} - \frac{b}{2}\right)^{\frac{1}{2} - \frac{b}{2}}}{(1-c)^{1-c}} = \frac{\left(\frac{1}{2} - \frac{a}{2}\right)^{\frac{1}{2}} \left(\frac{1}{2} - \frac{b}{2}\right)^{\frac{1}{2}}}{(1-c)} \times \frac{\left(\frac{1}{2} - \frac{a}{2}\right)^{-\frac{a}{2}} \left(\frac{1}{2} - \frac{b}{2}\right)^{-\frac{b}{2}}}{(1-c)^{-c}}.$$

The first term is also $O(\sqrt{iz})$. Then

$$\begin{aligned} & \frac{\left(\frac{1}{2} - \frac{a}{2}\right)^{-\frac{a}{2}} \left(\frac{1}{2} - \frac{b}{2}\right)^{-\frac{b}{2}}}{(1-c)^{-c}} \\ &= \frac{e^{(1-iz)\log(iz)}}{e^{\left(\frac{1}{4} - \frac{iz}{2} + \sqrt{\frac{1}{4} - \lambda}\right) \log\left(\frac{1}{4} + \frac{iz}{2} - \sqrt{\frac{1}{4} - \lambda}\right)} e^{\left(\frac{1}{4} - \frac{iz}{2} - \sqrt{\frac{1}{4} - \lambda}\right) \log\left(\frac{1}{4} + \frac{iz}{2} + \sqrt{\frac{1}{4} - \lambda}\right)}} \\ &= \frac{e^{\log(iz)}}{e^{\left(\frac{1}{4} + \sqrt{\frac{1}{4} - \lambda}\right) \log\left(\frac{1}{4} + \frac{iz}{2} - \sqrt{\frac{1}{4} - \lambda}\right)} e^{\left(\frac{1}{4} - \sqrt{\frac{1}{4} - \lambda}\right) \log\left(\frac{1}{4} + \frac{iz}{2} + \sqrt{\frac{1}{4} - \lambda}\right)}} \\ &\times \frac{e^{-iz\log(iz)}}{e^{-\frac{iz}{2} \log\left(\frac{1}{4} + \frac{iz}{2} - \sqrt{\frac{1}{4} - \lambda}\right)} e^{-\frac{iz}{2} \log\left(\frac{1}{4} + \frac{iz}{2} - \sqrt{\frac{1}{4} + \lambda}\right)}}. \end{aligned}$$

The first term of this product is still a $O(\sqrt{iz})$. For the second one:

$$\frac{e^{-iz\log(iz)}}{e^{-\frac{iz}{2} \log\left(\frac{1}{4} + \frac{iz}{2} - \sqrt{\frac{1}{4} - \lambda}\right)} e^{-\frac{iz}{2} \log\left(\frac{1}{4} + \frac{iz}{2} - \sqrt{\frac{1}{4} + \lambda}\right)}} \sim K e^{iz\log\left(\frac{1}{2}\right)} \rightarrow 0, z \rightarrow \infty.$$

Furthermore, we have

$$\begin{aligned} \frac{\sin\left(\pi\left(\frac{a}{2} + \frac{1}{2}\right)\right) \sin\left(\pi\left(\frac{b}{2} + \frac{1}{2}\right)\right)}{\sin(\pi c)} &= \frac{i \cos\left(\pi \times \sqrt{\frac{1}{4} - \lambda}\right)}{\tanh(z)} \\ &\rightarrow i \cos\left(\pi \times \sqrt{\frac{1}{4} - \lambda}\right), z \rightarrow \infty. \end{aligned}$$

So $f_0^+(0, z)$ has at most a polynomial growth at infinity in this domain,

$$f_0^+(0, z) = O(z^m), m \in \mathbb{N} \quad (**)$$

and so, because of the exponential growth of (*) and the polynomial growth of (**) we have

$$\begin{aligned} f^+(0, z) &\sim \int_0^{2\beta} K^+(0, t) f_0^+(t, z) dt \\ &\sim (-1)^p \frac{\partial_t^p K^+(0, 2\beta^-)}{(iz)^{p+1}} e^{2iz\beta}. \end{aligned}$$

We now consider the second statement. The integral term tends to 0 as $\text{Re}(z)$ tends to ∞ using (*) and the fact that the imaginary part is bounded. Furthermore, using Proposition (8), we have that $f_0^+(0, z) \rightarrow 1$ when $\text{Re}(z)$ tends to ∞ . This gives the result. \square

We obtain similar results for f^- :

Lemma 7. *We have:*

$$f^-(0, z) \sim (-1)^r \frac{\partial_t^r K^-(0, 2\alpha^+)}{(-iz)^{r+1}} e^{-2iz\alpha}, \quad \text{Im}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus \mathcal{B}.$$

Let $C > 0$, we have:

$$f^-(0, z) \rightarrow 1, \quad \text{Re}(z) \rightarrow \infty, \quad |\text{Im}(z)| \leq C, \quad z \in \mathbb{C}^- \setminus \mathcal{B}.$$

Using these two lemmas, we can determine asymptotics for $\frac{m_-}{m_+}$:

Lemma 8. *We assume that q is compactly supported, with a support satisfying $0 \in (\alpha, \beta)$. We have:*

$$\frac{m_-(z)}{m_+(z)} \rightarrow -1, \quad |z| \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus (\mathcal{B} \cup Z).$$

Proof. The notations o and \sim holds for z tending to ∞ . We first assume that $\text{Im}(z) \rightarrow \infty$. Using result of Lemma 6, we can prove that

$$f^+(0, z) \sim (-1)^p \frac{\partial_t^p K^+(0, 2\beta^-)}{(iz)^{p+1}} e^{2iz\beta}.$$

Similarly to what we did in Lemma 6, we have:

$$(f^+)'(0, z) \sim (-1)^{p-1} \frac{\partial_t^{p-1} \partial_x K^+(0, 2\beta^-)}{(iz)^p} e^{2iz\beta}.$$

So the function m_+ is equivalent to

$$\begin{aligned} m_+(z) &\sim \frac{(-1)^{p-1} \frac{\partial_t^{p-1} \partial_x K^+(0, 2\beta^-)}{(iz)^p} e^{2iz\beta}}{(-1)^p \frac{\partial_t^p K^+(0, 2\beta^-)}{(iz)^{p+1}} e^{2iz\beta}} \\ &\sim - \frac{\partial_t^{p-1} \partial_x K^+(0, 2\beta^-) \times iz}{\partial_t^p K^+(0, 2\beta^-)} \\ &\sim -iz, \quad \text{Im}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus (\mathcal{B} \cup Z). \end{aligned}$$

because $K^+(x, 2\beta-x) = 0$ for every x real and $\partial_t^{p-1} \partial_x K^+(0, 2\beta^-) = \partial_t^p K^+(0, 2\beta^-)$ (see the proof of (15)). Using the same sketch of proof, and Lemma 7, we have

the result for m_- :

$$\begin{aligned}
m_-(z) &\sim \frac{(-1)^{r-1} \frac{\partial_t^{r-1} \partial_x K^+(0, 2\beta^-)}{(-iz)^r} e^{-2iz\alpha}}{(-1)^r \frac{\partial_t^r K^+(0, 2\beta^-)}{(-iz)^{r+1}} e^{-2iz\alpha}} \\
&\sim \frac{\partial_t^{r-1} \partial_x K^-(0, 2\alpha^+) \times iz}{\partial_t^r K^-(0, 2\alpha^+)} \\
&\sim iz, \quad \text{Im}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus (\mathcal{B} \cup Z).
\end{aligned}$$

Then, combining these two equivalents:

$$\frac{m_-(z)}{m_+(z)} \rightarrow -1, \quad \text{Im}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus (\mathcal{B} \cup Z),$$

and so the result is proved. Now, if $\text{Re}(z) \rightarrow \infty$, we have

$$f^+(0, z) \sim 1$$

Similarly to what we did in Lemma 6, we have:

$$(f^+)'(0, z) \sim iz.$$

So the quantity m_+ is equivalent to

$$m_+(z) \sim iz, \quad \text{Re}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus (\mathcal{B} \cup Z).$$

Similarly,

$$m_-(z) \sim -iz, \quad \text{Re}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus (\mathcal{B} \cup Z),$$

and so

$$\frac{m_-(z)}{m_+(z)} \rightarrow -1, \quad \text{Re}(z) \rightarrow \infty, \quad z \in \mathbb{C}^- \setminus (\mathcal{B} \cup Z),$$

and so the result is proved when $|z| \rightarrow \infty$. \square

In the case of an even perturbation q , we immediately have $\frac{m_-(z)}{m_+(z)} = -1$, and so $w(z) = 2f^+(0, z)(f^+)'(0, z)$.

Proposition 4. *Let V be a potential defined by $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$, for all $x \in \mathbb{R}$ and q is real-valued, integrable and compactly supported such that the $(p-1)^{\text{th}}$ derivative of q has a discontinuity at $x = \beta$, the $(r-1)^{\text{th}}$ derivative of q has a discontinuity at $x = \alpha$, with $p, r \geq 1$ and $0 \in (\alpha, \beta)$. For $z \in S_1 \setminus (\mathcal{B} \cup Z)$, asymptotically, the function w has no zero in this domain.*

Proof. We consider this domain instead of S_1 to be away from the poles of hypergeometric functions and to have a well-defined quantity $\frac{m_-(z)}{m_+(z)}$. The term $1 - \frac{m_-(z)}{m_+(z)}$ tends to 2, so it will not vanish for sufficiently large values of z . Let now consider z_0 a resonance of w_0 , outside a sufficiently large disc centred at 0. Let z be a complex number on the boundary of $(S_1 \setminus (\mathcal{B} \cup Z)) \cap \overline{D(z_0, \varepsilon)}$ with

$\varepsilon > 0$, denoted $\partial \left((S_1 \setminus (\mathcal{B} \cup Z)) \cap \overline{D(z_0, \varepsilon)} \right)$. We will now recall two results: the term $\int_{2\alpha}^0 K^-(0, t) f_0^-(t, z) dt$ has an exponential growth with (*), and the term $f^-(0, z)$ has a polynomial growth with (**) (these two results (*) and (**) are stated for f^+ , but they are still true for f^- up to obvious modifications). Then, for z large enough (and so for z_0 a resonance large enough) we have the following estimates:

$$\begin{aligned} \left| f^-(0, z) - \int_{2\alpha}^0 K^-(0, t) f_0^-(t, z) dt \right| &= |f_0^-(0, z)| \\ &< \left| \int_{2\alpha}^0 K^-(0, t) f_0^-(t, z) dt \right|. \end{aligned}$$

We can apply Rouché's theorem on the two meromorphic functions $f^-(0, z)$ and $\int_{2\alpha}^0 K^-(0, t) f_0^-(t, z) dt$, seen as function of the complex parameter z . As the dominant function has no zero asymptotically on $\overline{D(z_0, \varepsilon)} \cap (S_1 \setminus (\mathcal{B} \cup Z))$ for z_0 a zero of w_0 , so $f^-(0, z)$ has no zero asymptotically on this set. The same reasoning for $(f^+)'(0, z)$ shows that this function has no zero in this domain neither. Using (28), we conclude that w has no zero asymptotically on this set. \square

We are now interested on the localisation of resonances in the domain S_2 :

Proposition 5. *Let V be a potential defined by $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$, for all $x \in \mathbb{R}$ and q is real-valued, integrable and compactly supported such that the $(p-1)^{th}$ derivative of q has a discontinuity at $x = \beta$, the $(r-1)^{th}$ derivative of q has a discontinuity at $x = \alpha$, $p, r \geq 1$ and $0 \in (\alpha, \beta)$. In the domain $S_2 \setminus Z$, the resonances β_j for $j \in \mathbb{N}$ satisfy the asymptotics:*

$$\begin{aligned} \beta_{\pm j} &\sim \mp \frac{\pi}{2(\beta - \alpha)} \left(2|j| + \frac{p+r-2}{2} \pm (\text{sign}(A) + 1) \right) \\ &\quad - \frac{i(p+r)}{2(\beta - \alpha)} \log \left(\frac{|j|\pi}{\beta - \alpha} \right) + \frac{i}{2(\beta - \alpha)} \log |A|(p+r-2)!, \end{aligned}$$

for $A = \frac{(-1)^p \partial_x^{r-1} \partial_x K^-(0, 2\alpha^+) \times \partial_x^p K^+(0, 2\beta^-)}{(p+r-2)!}$ a real constant. The notation \sim holds for j tending to ∞ .

Proof. From Proposition 8, since for any $z \in S_2 \setminus Z$, $\zeta = z$ satisfies assumption 1, or 2, or 3, or 4 of Proposition 8, we can use this asymptotic (see Appendix). So, after integrations by parts (as we did in Proposition 4) we have, for $z \in \mathbb{C} \setminus Z$,

$$\begin{aligned} w(z) &= 2izF \left(c-a, c-b, c; \frac{1}{2} \right)^2 F \left(c-a, c-b, c; \frac{1}{1+e^{4\beta}} \right) \\ &\quad \times F \left(c-a, c-b, c; \frac{e^{4\alpha}}{1+e^{4\alpha}} \right) \left[1 + \underset{|z| \rightarrow +\infty}{o(1)} - \frac{C e^{2iz(\beta-\alpha)}}{(iz)^{p+r}} \left(1 + \underset{|z| \rightarrow +\infty}{o(1)} \right) \right] \\ &\quad \times \left(1 - \frac{m_-(z)}{m_+(z)} \right), \end{aligned}$$

where

$$C = (-1)^p \partial_t^{r-1} \partial_x K^-(0, 2\alpha^+) \times \partial_x^p K^+(0, 2\beta^-).$$

Then, using Hardy [9] and Cartwright [4, 5] method presented in [3, 20] the zeros of the fourth factor in the previous expression of w

$$1 + \underset{|z| \rightarrow +\infty}{o(1)} - \frac{C e^{4iz\beta}}{(iz)^{p+r}} \left(1 + \underset{|z| \rightarrow +\infty}{o(1)} \right)$$

are localised in the lower half-plane with the following asymptotics:

$$\begin{aligned} \beta_{\pm j} = & \mp \frac{\pi}{2(\beta - \alpha)} \left(2|j| + \frac{p+r}{2} - 1 \pm (\text{sign}(A) + 1) \right) \\ & - \frac{i(p+r)}{2(\beta - \alpha)} \log \left(\frac{|j|\pi}{\beta - \alpha} \right) + \frac{i}{2(\beta - \alpha)} \log |A| (p+r-2)! + o(1) \end{aligned}$$

where $A = \frac{C}{(p+r-2)!}$. We are now going to use once again Rouché's theorem. Indeed, we know the localisation of zeros of $f^-(0, z)(f^+)'(0, z)$. But we also know that for such points $\beta_{\pm j}$, the function w cannot be written $w(z) = f^-(0, z)(f^+)'(0, z) \left(1 - \frac{m_-(z)}{m_+(z)} \right)$. So we are going to prove that, on arbitrarily small discs around these $\beta_{\pm j}$, the functions w and $f^-(0, z)(f^+)'(0, z)$ have the same zeros: the $\beta_{\pm j}$. Let r be a positive real number and $K := \overline{B(\beta_{\pm j}, r)}$ for j a large enough integer. Then, for $z \in \partial K$:

$$\begin{aligned} & |w(z) - 2f^-(0, z)(f^+)'(0, z)| \\ & = |f^-(0, z)(f^+)'(0, z)| \left| 1 - \frac{m_-(z)}{m_+(z)} - 2 \right| < |f^-(0, z)(f^+)'(0, z)2| \end{aligned}$$

asymptotically, because 2 is defined as the limit of the term $1 - \frac{m_-(z)}{m_+(z)}$ when $|z| \rightarrow \infty$ and $z \in \mathbb{C}^- \setminus (\mathcal{B} \cap \mathcal{Z})$. When j tends to $+\infty$, this condition is satisfied for $z \in \partial K$. So, because of Rouché's theorem, functions w and $2f^-(0, z)(f^+)'(0, z)$ have their zeros in the compact K , for every $r > 0$. Then, asymptotically, resonances are the $\beta_{\pm j}$. Furthermore, because of the first item of Lemma 1, we have $\beta_{-j} = -\beta_j$. \square

Finally, assume that $0 \notin (\alpha, \beta)$. Then we suppose that $\text{supp } q \subset \mathbb{R}_+$. We have that $f^-(0, z) = f_0^-(0, z)$, because the integral term vanishes. So

$$m_-(z) \sim -iz \tan\left(\frac{\pi}{2}a\right) \tan\left(\frac{\pi}{2}b\right), \quad z \rightarrow \infty, \quad \text{Im}(z) < 0.$$

Then, we have to adapt the results of Propositions 4 and 5 to this new asymptotic. We reformulate here Proposition 4:

Proposition 6. *Let V be a potential defined by $V(x) = \frac{\lambda}{\cosh^2(x)} + q(x)$, for all $x \in \mathbb{R}$ and q is real-valued, integrable and compactly supported such that the $(p-1)^{\text{th}}$ derivative of q has a discontinuity at $x = \beta$, the $(r-1)^{\text{th}}$ derivative*

of q has a discontinuity at $x = \alpha$, with $p, r \geq 1$ and $\text{supp } q \subset \mathbb{R}_+$. For $z \in S_1 \setminus (\mathcal{B} \cup \mathcal{Z})$, the resonances α_j for $j \in \mathbb{N}$ satisfy the asymptotics:

$$\alpha_{\pm j} \sim -i \left(2|j| - 1 \pm \sqrt{\frac{1}{4} - \lambda - \frac{1}{2}} \right).$$

The notation \sim holds for j tending to ∞ .

Proof. The proof of this Proposition is quite similar to the one of Proposition 5. Since $\alpha \geq 0$, $f^-(0, z) = f_0^-(0, z)$. The zeros of $f^-(0, z)$ are exactly the $\alpha_{\pm j}$, the quantity $(f^+)'(0, z)$ has still an exponential growth, so asymptotically the zeros of $f^-(0, z)(f^+)'(0, z)$ are the $\alpha_{\pm j}$. To conclude that these zeros are also the zeros of w , as we did in Proposition 5, we have to apply Rouché's theorem on small discs around the $\alpha_{\pm j}$: Let r be a positive real number and $K := \overline{B(\alpha_{\pm j}, r)}$ for j a large enough integer. Then, for $z \in \partial K$:

$$\begin{aligned} & |w(z) - 2f^-(0, z)(f^+)'(0, z)| \\ &= |f^-(0, z)(f^+)'(0, z)| \left| 1 - \frac{m_-(z)}{m_+(z)} - 2 \right| < |2f^-(0, z)(f^+)'(0, z)| \end{aligned}$$

asymptotically. So, using Rouché's theorem, holomorphic functions w and $2f^-(0, z)(f^+)'(0, z)$ have the same zeros in the compact K , for every $r > 0$. We can conclude that w and $f^-(0, z)(f^+)'(0, z)$ have the same zeros in S_1 . \square

In S_2 , we observe the same behaviour for the resonances as in Proposition (5): they are still localised on logarithmic branches.

In the case $\text{supp } q \subset \mathbb{R}_-$, we can use the symmetries of the problem under the change of variable $\tilde{x} = -x$. A potential q with its support in \mathbb{R}_- has now its support in \mathbb{R}_+ , where we know how to conclude.

We observe that the resonances initially created by a Pöschl-Teller potential and localised on parallel lines with respect to the imaginary axis are asymptotically on two logarithmic branches when we consider a compactly supported perturbation of this potential. See Figure 2 for a representation of these results.

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A Appendix: a few results on hypergeometric functions

We recall here some results on hypergeometric functions. These results are presented in [13, 17] for example. The first proposition gives the derivative of an hypergeometric function in both cases, normalised or not.

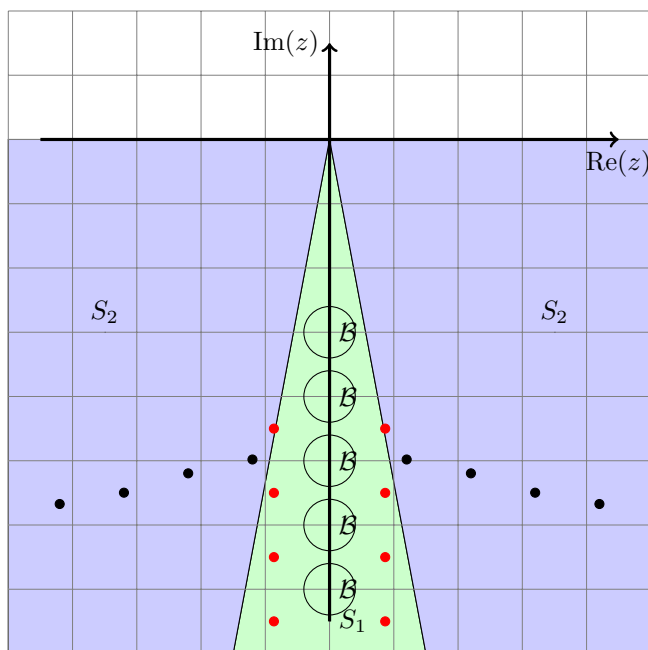


Figure 2: Location of resonances:
in the Pöschl-Teller case in red, and with a perturbation in black
(asymptotically) such that $0 \in (\alpha, \beta)$ and $\lambda = 1$.

Definition 2. Let a, b, c and ζ be complex numbers, with $|\zeta| < 1$. Then, an hypergeometric function is defined by

$$\zeta \mapsto F(a, b, c; \zeta) := {}_2F_1(a, b, c; \zeta) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{\zeta^n}{n!},$$

where $(x)_n$ is the Pochhammer symbol, defined by $(x)_0 := 1$ and for every positive integer n

$$(x)_n := x(x+1) \dots (x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

This is a power series with radius of convergence 1. We also know some results on its asymptotics:

Proposition 7. Let a, b, c and ζ be complex numbers, with $|\zeta| < 1$. Then:

$$F(a, b, c; \zeta) \rightarrow 1, \quad \zeta \rightarrow 0.$$

Proposition 8. Let a, b, c and ζ be complex numbers, with $|\zeta| < 1$. We assume that a, b and ζ are fixed complex numbers. Then, there are four different cases:

1. a and/or $b \in \{0, -1, -2, \dots\}$;
2. For ζ such that $\operatorname{Re}(\zeta) < 1/2$, $\delta > 0$ and c such that $|c+n| \geq \delta > 0$, for every $n \in \{0, -1, -2, \dots\}$;
3. For ζ such that $\operatorname{Re}(\zeta) = 1/2$, and $|\arg(c)| \leq \pi - \delta$;
4. For ζ such that $\operatorname{Re}(\zeta) > 1/2$, and $\alpha_- - 1/2\pi + \delta \leq \arg(c) \leq \alpha_+ + 1/2\pi - \delta$ where

$$\alpha_{\pm} := \arctan\left(\frac{\arg(\zeta) - \arg(1-\zeta) \mp \pi}{\ln|1-\zeta^{-1}|}\right);$$

we have the following formula, for $m \in \mathbb{N}$,

$$F(a, b, c; \zeta) = \sum_{n=0}^{m-1} \frac{(a)_n (b)_n}{(c)_n} \frac{\zeta^n}{n!} + O(c^{-m}).$$

This result have been proved by Wagner in [18].

Proposition 9. Let a, b and ζ be complex numbers. Let c be a complex number such that $c \in \mathbb{C} \setminus \{0, -1, \dots\}$. Then

$$\frac{d}{d\zeta} F(a, b, c; \zeta) = \frac{ab}{c} F(a+1, b+1, c+1; \zeta)$$

and

$$\frac{d}{d\zeta} \mathbb{F}(a, b, c; \zeta) = ab \mathbb{F}(a+1, b+1, c+1; \zeta).$$

The following proposition is useful to consider the limit as z tends to infinity of $F(a, b, c; \zeta)$, as we have shown earlier that, to be able to consider such a limit, the parameters a and b must be constant.

Proposition 10. *Let a, b, c and ζ be complex numbers such that $|\text{Arg}(1-\zeta)| < \pi$. Then we have*

$$F(a, b, c; \zeta) = (1 - \zeta)^{c-a-b} F(c-a, c-a, c; \zeta).$$

The next two lemmas give values of two different Wronskians. They are useful, combine with the next proposition, to calculate Wronskians such that w_0 and s_0^\pm .

Lemma 9. *Let a, b, c and ζ be complex numbers. Let f_1 and f_2 be two functions defined by*

$$f_1(\zeta) := F(a, b, c; \zeta)$$

and

$$f_2(\zeta) := \zeta^c F(a-c+1, b-c+1, 2-c; \zeta).$$

Then

$$[f_1(\zeta), f_2(\zeta)] = (1-c)\zeta^{-c}(1-\zeta)^{c-a-b-1}.$$

Lemma 10. *Let a, b, c and ζ be complex numbers. Let f_3 and f_4 be two functions defined by*

$$f_3(\zeta) := F(a, b, a+b+c-1; 1-\zeta)$$

and

$$f_4(\zeta) := (1-\zeta)^{c-a-b} F(c-a, c-b, c-a-b+1; 1-\zeta).$$

Then

$$[f_3(\zeta), f_4(\zeta)] = (a+b-c)\zeta^{-c}(1-\zeta)^{c-a-b-1}.$$

This following proposition makes explicit the link between hypergeometric functions at $1-\zeta$ and hypergeometric functions at ζ , and vice versa. These formulae are called "Kummer's formulae", and there exist 20 of them actually. We list here only the two that are used in our work.

Proposition 11 (Kummer's formulae). *Let a, b, c and ζ be complex numbers. We have, for $z \in \mathbb{C} \setminus i\mathbb{Z}^*$:*

- $f_3(\zeta) = \frac{\Gamma(1-c)\Gamma(c)}{\Gamma(a-c+1)\Gamma(b-c+1)} f_1(\zeta) + \frac{\Gamma(c-1)\Gamma(c)}{\Gamma(a)\Gamma(b)} f_2(\zeta);$
- $f_1(\zeta) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} f_3(\zeta) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} f_4(\zeta).$

We may remark that, when the coefficients a , b and c satisfy the equality: $\frac{a}{2} + \frac{b}{2} + \frac{1}{2} = c$, then we have the following formula for an hypergeometric function at $\zeta = \frac{1}{2}$, useful when we wanted to know the behaviour of hypergeometric function in the domain S_1 :

Lemma 11. *Let a and b be complex numbers. Then we have:*

$$F\left(a, b, \frac{a}{2} + \frac{b}{2} + \frac{1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{a}{2} + \frac{b}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{a}{2} + \frac{1}{2}\right)\Gamma\left(\frac{b}{2} + \frac{1}{2}\right)}.$$

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