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Sana Louhichi

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# SHARP RATES OF CONVERGENCE IN THE HAUSDORFF METRIC FOR COMPACTLY SUPPORTED STATIONARY MARKOV CHAINS OR STATIONARY $\beta$ -MIXING SEQUENCES

SANA LOUHICHI

**ABSTRACT.** We study rates of convergence, in mean, for the Hausdorff metric between a finite set of stationary random variables and their common support, which is supposed to be a compact subset of  $\mathbb{R}^d$ . We propose two different approaches for this study. The first approach is based on the notion of minimal index. This notion is introduced in this paper. It is in the spirit of the extremal index which is very used in the extreme value theory. The second approach is based on a  $\beta$ -mixing condition together with a local type dependence assumption. More precisely, all our results concern stationary  $\beta$ -mixing sequences satisfying a tail condition, known as the  $(a, b)$ -standard assumption, together with a local type dependence condition or stationary sequences satisfying the  $(a, b)$ -standard assumption and having a positive minimal index. We prove that the optimal rates of the i.i.d. setting can be reached. We apply our results to stationary Markov chains on a ball, or to a class of Markov chains on a circle or on a torus. We study with simulations the particular examples of a Möbius Markov chain on the unit circle and of a Markov chain on the unit square wrapped on a torus.

**Keywords:** Hausdorff metric, stationary dependent random variables,  $\beta$ -mixing, Möbius Markov chain, Markov chain on a torus, geometrically ergodic Markov chain, compact support, rates of convergence, extremal index, local dependence condition, Bonferroni-type inequality.

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## 1. INTRODUCTION

The Hausdorff metric,  $d_H$ , is an useful measurement for determining how similar one set or shape is to another. More formally, given two compact sets  $A$  and  $B$  of  $\mathbb{R}^d$ ,

$$(1) \quad d_H(A, B) = \max \left( \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{x \in B} \inf_{y \in A} \|x - y\| \right),$$

where  $\|\cdot\|$  denotes the Euclidean distance. Classically, in practical applications, a Hausdorff metric that is very close to 0 would indicate a great similarity between the considered sets or shapes. Due to its utility in comparing shapes and sets, the Hausdorff metric finds applications in a variety of fields such as in computer vision and image processing ([25]), medical imaging ([28]), pattern recognition ([27]), robotics ([9]), machine learning ([21]), topological data analysis ([5, 10, 14, 20]) or in statistics and directional statistics ([7, 8, 14, 15]).

For this later field, circular data is a key concept. Circular data  $x_1, \dots, x_n$  are those for which the natural support is the unit circle or its toroidal extensions ([15, 26]). They may serve as models for wind directions, orientations of strata, and movement of animals, among others. Suppose that the data  $x_1, \dots, x_n$ , living in a same compact set, is a realisation of stationary random variables  $X_1, \dots, X_n$  compactly supported and not necessarily independent. Studying the Hausdorff metric from the observable random cloud  $\{X_1, \dots, X_n\}$  to their common support and how much it is close to 0 as  $n$  grows, shall then be helpful to deduce some information on this common support which is, in generally, unknown.

From now, let  $(X_i)_{i \in \mathbb{N}}$  be a stationary sequence of  $\mathbb{R}^d$ -valued random variables. Let  $\mu$  be the distribution of  $X_1$ , and thus of all the  $X_i$ 's. Suppose that  $\mu$  is supported on a compact set  $\mathbb{M}$  of  $\mathbb{R}^d$ , meaning that  $\mathbb{M}$  is the smallest closed set having probability 1. More formally,

$$(2) \quad \mathbb{M} = \bigcap_{C \subset \mathbb{R}^d, P(\overline{C})=1} \overline{C},$$

where  $\overline{C}$  means the closure of the set  $C$  in Euclidean space. Denote by  $\mathbb{X}_n$  the set  $\{X_1, \dots, X_n\}$  which is viewed as a subset of  $\mathbb{R}^d$ . We are interested by the evaluation of the Hausdorff metric,

$d_H$ , of  $\mathbb{X}_n$  to the support  $\mathbb{M}$ , more precisely to give a sharp upper bound for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$ , the expectation of  $d_H(\mathbb{X}_n, \mathbb{M})$ , by suitably controlling  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon)$  for positive  $\epsilon$ .

In topological data analysis, upper bounds for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  are useful since they lead, thanks to the stability theorem, to upper bounds for the bottleneck distance between suitable persistence diagrams. We refer the reader to the seminal paper [5], where, in particular, optimal upper bounds for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  are obtained for i.i.d. random sequence  $(X_i)_{i \in \mathbb{N}}$ .

One of our objectives is to extend the estimates of the i.i.d. case to the dependent one. Although this generalization is useful in order to model real phenomena, only few works have addressed these questions. To our knowledge the only papers that deal with the dependent framework are [6] (for the trajectories of a reflected Brownian motion) and more recently [1] where the authors give estimation results and optimal rates on the R-convex hull of stationary dependent random variables using a kernel density estimation approach and [14] for topological reconstruction of compact supports of various class of stationary dependent random variables. Even in the area of topological data analysis, only few works have been explored for dependent data. We refer for instance to the recent paper [18] who gave a concentration inequality for persistent Betti numbers or to the more recent paper [22] for the estimation of topological signatures (both in the dependent context).

Now, we come back to the main purpose of this paper which is to give sharp upper bounds for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  by suitably controlling  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon)$  for positive  $\epsilon$  and for stationary sequence  $(X_i)_{i \in \mathbb{N}}$ . In [14] we gave upper bounds for  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon)$  for different types of weak dependence of the stationary sequence  $(X_n)_{n \in \mathbb{N}}$ . Those upper bounds met the purpose of [14] which was to establish the asymptotic  $(\epsilon, \alpha)$ -density in  $\mathbb{M}$  of the stationary sequence (see Definition 1.1 in [14]). The proofs, there, used a clustering technique which consists of grouping the random variables into clusters and treating them as random variables in a larger space. While this approach accommodates multiple dependency types, it may not yield optimal speeds of convergence for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  because of the *curse of dimensionality* due to the Euclidean distance.

In this paper, our starting point to control  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  is Proposition 1.1 below, which is true for any stationary  $\mathbb{R}^d$ -valued sequence of random variables compactly supported (for its proof we refer the reader to Proposition 3.1 in [14], with  $k = n$  and  $r = 1$  there). The statement of this proposition and its proof are already done along the lines of [5, 7, 8, 10]. Its proof uses a nice geometrical result, proved in [20], relating the  $\epsilon$ -covering number of a compact set by closed balls of radius  $\epsilon$  to its  $\epsilon$ -packing number, i.e., to the maximal length of chains of points whose pairwise distances are bounded below by  $\epsilon$  (see Lemma 5.2 in [20]).

**Proposition 1.1.** *Let  $(X_n)_{n \geq 0}$  be a stationary  $\mathbb{R}^d$ -valued sequence of random variables compactly supported. Let  $\mathbb{M}$  be this common support. Then, for any  $\epsilon > 0$ ,*

$$\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) \leq \frac{\sup_{x \in \mathbb{M}} \mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2)}{1 - \sup_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| > \epsilon/4)}.$$

Recall that  $\mathbb{X}_n = \{X_1, \dots, X_n\}$ .

In view of Proposition 1.1, to bound  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon)$  we have, mainly, to control the two quantities  $\mathbb{P}(\|X_1 - x\| \leq \epsilon)$  and  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon)$ . To control the first term  $\mathbb{P}(\|X_1 - x\| \leq \epsilon)$ , we assume the  $(a, b)$ -standard assumption for  $\mu$  (the distribution of  $X_1$ ) as was done in the i.i.d. case. The  $(a, b)$ -standard assumption was used, in the i.i.d context, for set estimation problems under the Hausdorff metric ([7], [8]) and also for a statistical analysis of persistence diagrams ([5], [10]). This  $(a, b)$ -standard assumption gives a lower bound for

$\mathbb{P}(\|X_1 - x\| \leq \epsilon)$ , uniformly in  $x \in \mathbb{M}$ . This lower bound is a power of  $\epsilon$ . We summarise this notion in Definition 1.2 below.

**Definition 1.2.** *Let  $X$  be a compactly supported  $\mathbb{R}^d$ -random variable. Let  $\mathbb{M}$  be its support. This random variable  $X$  satisfies the  $(a, b)$ -standard assumption if there exist  $a > 0$ ,  $b > 0$  and  $\epsilon_0 > 0$  such that for any  $0 < \epsilon \leq \epsilon_0$ ,*

$$(3) \quad \inf_{x \in \mathbb{M}} \mathbb{P}(\|X - x\| \leq \epsilon) \geq a\epsilon^b.$$

In [14] we needed also, in order to establish the convergence in probability of  $d_H(\mathbb{X}_n, \mathbb{M})$ , a lower bound for  $\inf_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| \leq \epsilon)$  which is not necessarily a power of  $\epsilon$ . This specific form of the lower bound as a power of  $\epsilon$  allows to get precise rates of convergence with the Hausdorff metric as we shall see later.

Now to control the second term  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon)$  appearing in Proposition 1.1, we use two different approaches. The *first approach* is based on the following remark. For i.i.d. random variables satisfying the  $(a, b)$ -standard assumption, a uniform (on  $x \in \mathbb{M}$ ) upper bound for  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon)$  is easily obtained since, for i.i.d. random variables, it holds :

$$(4) \quad \mathbb{P}\left(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon\right) = (\mathbb{P}(\|X_1 - x\| > \epsilon))^n.$$

The situation becomes more complicated when the variables are no longer i.i.d. For this reason, we introduce the notion of minimal index  $\theta^*$  of a stationary sequence (see Definition 2.1 below): instead of having a power  $n$  in (4), we have a power  $n\theta^*$  for some  $\theta^* \in ]0, 1]$ , also an inequality instead of an equality. Finally to be more general, the upper bound that we propose for  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon)$  is up to some positive constant  $c$ ,  $(1 - \kappa_\epsilon)^{n\theta^*}$  where  $\kappa_\epsilon$  is in  $]0, 1[$ . We called in Definition 2.1,  $\theta^*$  the minimal index and  $\kappa_\epsilon$  a marginal lower bound of the sequence. Theorem 2.2 proves that the rate (of convergence to 0 of  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$ ) for i.i.d. sequences is reached for stationary sequences satisfying the  $(a, b)$ -standard assumption and having  $\theta^* \in ]0, 1]$ . In Subsection 2.1, we give some examples of calculations of  $\theta^*$ . The reader accustomed to the theory of extreme values will no doubt think of the extremal index. The extremal index introduced in [19], which is connected to the asymptotic distribution of the maximum, has a nice meaning. It is a measure of the extent of clustering in the extremes of a stationary process since it represents the reciprocal of the mean cluster size. Clearly in view of their definitions, these two indexes are not the same. But they may have an analogous meaning. In fact, Definition 2.1 (i.e. an inequality in (4) with  $p\theta^*$  instead of  $p$  in its right hand side) suggests that the minimum over  $p$  random variables is controlled by  $p\theta^*$  independent "clusters" with a same size. So each cluster has the size  $p/(p\theta^*)$ , that is  $1/\theta^*$ . Of course, this explanation remains intuitive at this stage. The evaluation of the minimal index for an  $m$ -dependent sequence agrees with this interpretation. We prove in Proposition 2.3 that the minimal index for stationary  $m$ -dependent sequence is  $\theta^* = 1/(m + 1)$ . In Proposition 2.4, we give sufficient conditions for a stationary sequence to have  $\theta^* = 1$  and to satisfy the requirements of Theorem 2.2 (see Corollary 2.5). We apply Proposition 2.4 to stationary Markov chains. The results are announced in Proposition 2.6, in Corollary 2.7 and in Corollary 2.8. Here the minimal index of this Markov chain is  $\theta^* = 1$ . The explicit values of  $a$  and  $b$  of the  $(a, b)$ -standard assumptions are also given. The proofs for Markov chains are based on some calculations in [14]. In Proposition 2.9, we give sufficient conditions for a stationary sequence to have  $0 < \theta^* < 1$  and to satisfy the requirements of Theorem 2.2 (see Corollary 2.10). We apply those results to stationary Markov chains (see Proposition 2.11 and Corollary 2.12).

The *second approach* to bound the second term  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2)$  appearing in Proposition 1.1, is based on a mixing assumption and on a local type dependence condition. We are interested in this paper by the  $\beta$ -mixing assumption. We prove in Proposition 3.1 below that, for  $\beta$ -mixing sequences, the control of  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2)$  needs a suitable control of  $\mathbb{P}^k(\min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4)$  for  $kp \leq n$ . The control of this later probability needs a local dependence condition in the spirit of the well known Leadbetter's anti-clustering condition  $D'(u_n)$  ([13], [19]). This local dependence condition allows to give a lower bound for

$$\frac{\mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon/4)}{p\mathbb{P}(\|X_1 - x\| \leq \epsilon/4)},$$

by using a Bonferroni-type inequality (see Lemma 5.1). The result is summarized in Proposition 3.2. Once we have bounded  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon)$ , we deduce a bound for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$ . This is the purpose of Theorem 3.4: for a stationary  $\beta$ -mixing sequence under a polynomial decay of the  $\beta$ -coefficients together with the  $(a, b)$ -standard assumption and a local type dependence condition, the optimal rate of the i.i.d. setting, proved in [5], can be reached for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  (see Theorem 3.4, Corollaries 3.5 and 3.6 below for precise statements).

In conclusion, this paper extends known optimal bounds for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  to stationary random sequences having a minimal index  $\theta^* \in ]0, 1]$  or to  $\beta$ -mixing random variables  $(X_i)_{i \in \mathbb{N}}$ , thus extending the framework of independence. This shall open the scope of applications. As such, we apply Theorem 2.2 and Theorem 3.4 to some stationary Markov chains. In Subsection 4.1 we discuss an example of a Markov chain on a closed ball for which Theorem 2.2 applies. Subsection 4.2 gives a class of Markov chains for which our two approaches works: Proposition 4.2 for the first approach and Proposition 4.3 for the second approach. In fact, Proposition 4.3 proves that the considered Markov chain is geometrically ergodic and then  $\beta$ -mixing (recall that geometrically ergodic Markov chains are  $\beta$ -mixing (see Theorem 3.7 in [3])). Two explicit examples are studied: a Markov chain on the circle and on a Torus. The first example was introduced by [15] to model the wind direction. Both Theorem 2.2 and Theorem 3.4 apply to this Möbius Markov chain on the circle. We illustrate the result, in Subsection 4.2, by some simulations. The second example studies with simulations the case of a stationary Markov chain on a Torus. This model satisfies the requirements of Theorem 2.2 (see Subsection 4.3, Proposition 4.4).

The paper is organized as follows. In Section 2, the first approach is described. The second approach is described in Section 3. Explicit examples are discussed in Section 4. All the proofs are given in Section 5.

From now on and in all the paper, the notation  $a_n = O(b_n)$  (respectively  $a_n = o(b_n)$ ) means, as usual, that there exists a positive constant  $C$  such that for  $n$  large enough  $a_n \leq Cb_n$  (respectively  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ ). The notation  $[\cdot]$  means the integer part. The notation  $a \wedge b$  means  $\min(a, b)$  and finally  $cst$  denotes a positive constant that may be different from line to line.

## 2. AN APPROACH BASED ON THE MINIMAL INDEX $\theta^*$

We introduce the following definition, that allows to give uniformly on  $x \in \mathbb{M}$ , an upper bound for the main term of Proposition 1.1, which is  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon)$ .

**Definition 2.1.** *Let  $(X_i)_{i \in \mathbb{N}}$  be a stationary sequence of  $\mathbb{R}^d$ -valued random variables. Suppose that  $X_1$  is with compact support  $\mathbb{M}$ . We say that this sequence  $(X_i)_{i \geq 0}$  has a minimal index  $\theta^*$  with a marginal lower bound  $\kappa_\epsilon$ , if there exist a positive constant  $c$ ,  $\theta^* \in ]0, 1]$ ,  $\epsilon_0 > 0$ ,  $n_0 \in \mathbb{N}$*

such that for any  $n \geq n_0$  and any  $\epsilon \in ]0, \epsilon_0]$ ,

$$(5) \quad \sup_{x \in \mathbb{M}} \mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon \right) \leq c(1 - \kappa_\epsilon)^{n\theta^*},$$

for a constant  $\kappa_\epsilon \in ]0, 1[$ .

Clearly, compactly supported i.i.d. random variables have a minimal index  $\theta^* = 1$  with a marginal lower bound  $\kappa_\epsilon \in ]0, 1[$  if for  $\epsilon$  small enough  $\inf_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| \leq \epsilon) \geq \kappa_\epsilon$ .

We have now what we need, to announce our first result on the rates of  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  and using this notion  $\theta^*$ .

**Theorem 2.2.** *Let  $(X_n)_{n \geq 0}$  be a stationary sequence of  $\mathbb{R}^d$ -valued random variables. Suppose that  $X_1$  is supported on a compact set  $\mathbb{M}$  and satisfies the  $(a, b)$ -standard assumption. Suppose moreover that  $(X_n)_{n \geq 0}$  has a minimal index  $\theta^* \in ]0, 1[$  with a marginal lower bound  $\kappa_\epsilon = a\epsilon^b$ . Then,*

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) = O \left( \left( \frac{\ln n}{n} \right)^{1/b} \right).$$

For i.i.d random variables whose common distribution satisfies the  $(a, b)$ -standard assumption, the rate in Theorem 2.2 is optimal (see [5]).

**2.1. Examples of calculation of  $\theta^*$ .** The purpose of this section is to apply Theorem 2.2 to stationary  $m$ -dependent random sequences and to a class of some stationary random sequences including some stationary Markov chains. All compactly supported. We specify, for each example, the value of the minimal index  $\theta^*$  as is introduced in Definition 2.1.

**2.1.1. Stationary  $m$ -dependent random sequences.** Recall that the random sequence  $(X_i)_{i \in \mathbb{N}}$  is  $m$ -dependent for some  $m \geq 0$  if the two  $\sigma$ -fields  $\sigma(X_i, i \leq k)$  and  $\sigma(X_i, i \geq k + m + 1)$  are independent for every  $k$ . In particular, 0-dependent is the same as independent.

**Proposition 2.3.** *Let  $(X_i)_{i \in \mathbb{N}}$  be a sequence of stationary  $m$ -dependent random variables compactly supported. Suppose that  $X_1$  satisfies the  $(a, b)$ -standard assumption. Then  $(X_i)_{i \in \mathbb{N}}$  has a minimal index*

$$\theta^* = \frac{1}{m + 1},$$

with a marginal lower bound  $\kappa_\epsilon = a\epsilon^b$ . The conclusion of Theorem 2.2 holds for this  $m$ -dependent random sequence  $(X_i)_{i \in \mathbb{N}}$ .

**2.1.2. Stationary random sequences and Markov chains with  $\theta^* = 1$ .** In the sequel, we denote by  $\mathbb{P}_{X_0, \dots, X_{n-1}}$  or by  $\mathbb{P}(\cdot | X_0, \dots, X_{n-1})$ , the conditional distribution known  $X_0, \dots, X_{n-1}$ .

**Proposition 2.4.** *Let  $(X_n)_{n \geq 0}$  be a stationary  $\mathbb{R}^d$ -valued random sequence compactly supported. Suppose that, for a positive  $\epsilon \leq \epsilon_0$  there exists a positive constant  $\kappa_\epsilon \in ]0, 1[$  such that, for any  $x_0 \in \mathbb{M}$ , and any  $n \geq 1$*

$$(6) \quad \inf_{x \in \mathbb{M}} \mathbb{P}_{X_0, \dots, X_{n-1}}(\|X_n - x\| \leq \epsilon) \geq \kappa_\epsilon,$$

a.s. Then for any  $n \geq 1$ ,

$$\mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon \right) \leq (1 - \kappa_\epsilon)^n.$$

That is  $(X_i)_{i \in \mathbb{N}}$  has a minimal index  $\theta^* = 1$  with a marginal lower bound  $\kappa_\epsilon$ , as soon as the bound (6) is satisfied.

An immediate consequence of Proposition 2.4 is the following corollary.

**Corollary 2.5.** *Let  $(X_n)_{n \geq 0}$  be a stationary  $\mathbb{R}^d$ -valued random sequence compactly supported. Suppose that (6) is satisfied with  $\kappa_\epsilon = a\epsilon^b$ . Then the conclusion of Theorem 2.2 holds true.*

**Proof of Corollary 2.5.** We deduce from,

$$\mathbb{P}(\|X_1 - x\| \leq \epsilon) = \mathbb{P}(\|X_n - x\| \leq \epsilon) = \mathbb{P}(\mathbb{P}_{X_0, \dots, X_{n-1}}(\|X_n - x\| \leq \epsilon))$$

that if (6) is satisfied with  $\kappa_\epsilon = a\epsilon^b$  then the distribution of  $X_1$  satisfies the  $(a, b)$ -standard assumption. This fact together with the conclusion of Proposition 2.4 are enough to guarantee all the requirements of Theorem 2.2. The conclusion of Theorem 2.2 holds.  $\square$

*Application to stationary Markov chains.* We suppose in all this paragraph that  $(X_n)_{n \geq 0}$  is a Markov chain satisfying Assumption  $(\mathcal{A}_1)$  below.

$(\mathcal{A}_1)$  The Markov chain  $(X_n)_{n \geq 0}$  has an invariant measure  $\mu$  with compact support  $\mathbb{M}$  (and then the chain is stationary).

Under Assumption  $(\mathcal{A}_1)$ , Proposition 2.4 is reduced to the following proposition (we denote by  $\mathbb{P}_{x_0}$  (respectively by  $\mathbb{P}_\mu$ ) the conditional distribution known  $X_0 = x_0$  (respectively known that  $X_0$  is distributed as  $\mu$ )).

**Proposition 2.6.** *Let  $(X_n)_{n \geq 0}$  be a Markov chain satisfying Assumption  $(\mathcal{A}_1)$ . Suppose that, for a positive  $\epsilon \leq \epsilon_0$  there exists a positive constant  $\kappa_\epsilon \in ]0, 1[$  such that, for any  $x_0 \in \mathbb{M}$ ,*

$$(7) \quad \inf_{x \in \mathbb{M}} \mathbb{P}_{x_0}(\|X_1 - x\| \leq \epsilon) \geq \kappa_\epsilon.$$

Then for any  $n \geq 1$ ,

$$\mathbb{P}_\mu \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon \right) \leq (1 - \kappa_\epsilon)^n.$$

That is,  $(X_i)_{i \in \mathbb{N}}$  has a minimal index  $\theta^* = 1$  with a marginal lower bound  $\kappa_\epsilon$ , as soon as the bound (7) is satisfied.

We deduce from Proposition 2.6, the following corollary.

**Corollary 2.7.** *Let  $(X_n)_{n \geq 0}$  be a Markov chain satisfying Assumption  $(\mathcal{A}_1)$ . Suppose that (7) is satisfied with  $\kappa_\epsilon = a\epsilon^b$ . Then the conclusion of Theorem 2.2 holds true.*

The proof of Corollary 2.7 is exactly as that of Corollary 2.5 and is omitted, in fact it is based on  $\mathbb{P}_\mu(\|X_1 - x\| \leq \epsilon) = \int \mathbb{P}_{x_0}(\|X_1 - x\| \leq \epsilon) \mu(dx_0)$ . Our purpose now is to give sufficient conditions under which the lower bound (7) is satisfied. For this, we consider the following assumption introduced in [14].

$(\mathcal{A}_2)$  The transition probability kernel  $K$ , defined for  $x \in \mathbb{M}$ , by

$$K(x, \cdot) = \mathbb{P}(X_1 \in \cdot | X_0 = x)$$

is absolutely continuous with respect to some measure  $\nu$  on  $\mathbb{M}$ , i.e. there exists a positive measure  $\nu$  and a positive function  $k$  such that for any  $x \in \mathbb{M}$ ,  $K(x, dy) = k(x, y)\nu(dy)$ . Suppose that, for some  $b > 0$  and  $\epsilon_0 > 0$ ,

$$(8) \quad V_d := \inf_{x \in \mathbb{M}} \inf_{0 < \epsilon < \epsilon_0} \left( \frac{1}{\epsilon^b} \int_{B(x, \epsilon) \cap \mathbb{M}} \nu(dx_1) \right) > 0,$$

and that there exists a positive constant  $\kappa$  such that  $\inf_{x \in \mathbb{M}, y \in \mathbb{M}} k(x, y) \geq \kappa > 0$ .



**Corollary 2.8.** *Under Assumptions  $(\mathcal{A}_1)$  and  $(\mathcal{A}_2)$ , the bound (7) is satisfied with*

$$\kappa_\epsilon = \kappa V_d \epsilon^b,$$

*and the conclusion of Theorem 2.2 holds true.*

2.1.3. *Stationary random sequences and Markov chains with  $\theta^* < 1$ .* The purpose of this paragraph is to give sufficient conditions for a stationary random sequence to have  $\theta^* < 1$ .

**Proposition 2.9.** *Let  $(X_n)_{n \geq 0}$  be a stationary  $\mathbb{R}^d$ -valued random sequence compactly supported. Suppose that, for  $0 < \epsilon \leq \epsilon_0$ , there exists a positive integer  $m \geq 1$  and a positive constant  $\kappa_\epsilon \in ]0, \alpha[$  (for fixed  $\alpha \in ]0, 1[$ ) such that, for any  $x_0 \in \mathbb{M}$ , and any integer  $n \geq 1$ ,*

$$(9) \quad \inf_{x \in \mathbb{M}} \mathbb{P}_{X_0, \dots, X_{n-1}}(\|X_{n+m} - x\| \leq \epsilon) \geq \kappa_\epsilon,$$

*a.s. Then for any  $n \geq m + 1$ ,*

$$\mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon \right) \leq \frac{1}{1 - \alpha} (1 - \kappa_\epsilon)^{n/(m+1)}.$$

*That is  $(X_i)_{i \in \mathbb{N}}$  has a minimal index  $\theta^* = \frac{1}{m+1}$  with a marginal lower bound  $\kappa_\epsilon$ , as soon as the lower bound (9) is satisfied.*

We deduce the following corollary (its proof is omitted since it is the same as that of Corollary 2.5).

**Corollary 2.10.** *Let  $(X_n)_{n \geq 0}$  be a stationary  $\mathbb{R}^d$ -valued random sequence compactly supported. Suppose that (9) is satisfied with  $\kappa_\epsilon = a\epsilon^b$ . Then the conclusion of Theorem 2.2 holds true.*

*Application to stationary Markov chains.* Proposition 2.9 applied to stationary Markov chains gives the following.

**Proposition 2.11.** *Let  $(X_n)_{n \geq 0}$  be a Markov chain  $\mathbb{R}^d$ -valued, satisfying Assumption  $(\mathcal{A}_1)$ . Suppose that, there exists a positive integer  $m \geq 1$  and  $\alpha \in ]0, 1[$ , such that for any positive  $\epsilon \leq \epsilon_0$ , there exists a positive constant  $\kappa_\epsilon \in ]0, \alpha[$  such that, for any  $x_0 \in \mathbb{M}$ ,*

$$(10) \quad \inf_{x \in \mathbb{M}} \mathbb{P}_{x_0}(\|X_{m+1} - x\| \leq \epsilon) \geq \kappa_\epsilon.$$

*Then for any  $n \geq m + 1$ ,*

$$\mathbb{P}_\mu \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon \right) \leq \frac{1}{1 - \alpha} (1 - \kappa_\epsilon)^{n/(m+1)}.$$

*That is,  $(X_i)_{i \in \mathbb{N}}$  has a minimal index  $\theta^* = \frac{1}{m+1}$  with a marginal lower bound  $\kappa_\epsilon$ , as soon as the lower bound (10) is satisfied.*

We deduce the following corollary (its proof is omitted since it is as that of Corollary 2.5 and Corollary 2.7.)

**Corollary 2.12.** *Let  $(X_n)_{n \geq 0}$  be a Markov chain satisfying Assumption  $(\mathcal{A}_1)$ . Suppose that the lower bound (10) holds with  $\kappa_\epsilon = a\epsilon^b$ . Then the conclusion of Theorem 2.2 holds true.*

3. AN APPROACH FOR  $\beta$ -MIXING RANDOM SEQUENCES

The main purpose of this section is to present a second approach to bound the quantity  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2)$  (which shall give, thanks to Proposition 1.1 together with the  $(a, b)$ -standard assumption, an upper bound for  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon)$ ). We focus on stationary  $\beta$ -mixing random sequences (introduced by [24]). Recall that the stationary random sequence  $(X_n)_{n \in \mathbb{N}}$  is  $\beta$ -mixing if its coefficient  $\beta_n$  tends to 0 when  $n$  tends to infinity. These coefficients  $\beta_n$  can be defined by,

$$\beta_n = \sup_{l \geq 1} \mathbb{E} \left\{ \sup_{B \in \sigma(\sigma_i, i \geq l+n)} |\mathbb{P}(B | \sigma(X_1, \dots, X_l)) - \mathbb{P}(B)| \right\},$$

we refer the reader to [4] and to [29] for this expression of  $\beta_n$ .

Geometrically ergodic Markov chains are an example of  $\beta$ -mixing random sequences with geometrically decaying mixing coefficients  $(\beta_n)_{n \geq 1}$  (cf. for instance Theorem 3.7 in [3] and the references therein). Recall that a stationary Markov chain, with a stationary measure  $\mu$ , is geometrically ergodic, if there exists a positive constant  $c$  and a Borel positive function  $a$  such that the following bound holds for any  $\mu$ -a.e.  $x \in \mathbb{R}^d$ : for any  $n \in \mathbb{N}^*$  and any  $B$  Boolean of  $\mathbb{R}^d$ , it holds,

$$(11) \quad |\mathbb{P}^n(x, B) - \mu(B)| \leq a(x)e^{-cn}.$$

Proposition 3.1 below gives a bound for  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2)$  and for stationary  $\beta$ -mixing random sequences. For its proof, we use a specific tool based on a Berbee's coupling argument [2] (see also [23]), available for  $\beta$ -mixing random sequences.

**Proposition 3.1.** *Let  $(X_n)_{n \geq 0}$  be a stationary sequence of  $\beta$ -mixing and  $\mathbb{R}^d$ -valued random variables. Suppose that  $X_1$  is supported on a compact set  $\mathbb{M}$ . Let  $p$  be a positive integer less than  $n/2$  and  $k = \lfloor n/(2p) \rfloor$ . Then, for any positive  $\epsilon$ ,*

$$\sup_{x \in \mathbb{M}} \mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2 \right) \leq k\beta_p + \sup_{x \in \mathbb{M}} \mathbb{P}^k \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right).$$

We see from Proposition 3.1 that, the control of  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2)$  needs a control of  $\mathbb{P}^k(\min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4)$ . We control this later term by introducing a local type dependence condition analogous to the well known Leadbetter's condition  $D'(u_n)$ , described in our case by the local dependence coefficient  $(\Lambda(n, p, \epsilon_n))_{n, p}$  (see (12) below). Propositions 3.1 and 1.1 together with a local dependence study give the following bound for  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon)$  under a  $\beta$ -mixing condition.

**Proposition 3.2.** *Let  $(X_n)_{n \geq 0}$  be a stationary sequence of  $\beta$ -mixing and  $\mathbb{R}^d$ -valued random variables. Suppose that  $X_1$  is supported on a compact set  $\mathbb{M}$  and that its distribution satisfies the  $(a, b)$ -standard assumption. Let  $p$  be a positive integer less than  $n/2$  and  $k = \lfloor n/(2p) \rfloor$ . Define, for a non-random sequence  $(\epsilon_n)_{n \geq 0}$  tending to 0 as  $n$  tends to infinity,*

$$(12) \quad \Lambda(n, p, \epsilon_n) = n \sup_{x \in \mathbb{M}} \left( \sum_{r=2}^p \mathbb{P}(\|X_1 - x\| \leq \epsilon_n, \|X_r - x\| \leq \epsilon_n) \right).$$

Then, for any positive  $\epsilon$  small enough,

$$\begin{aligned} \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) &\leq 1 \wedge \left( \frac{k\beta_p}{a'\epsilon^b} \right) \\ &+ 1 \wedge \left( \frac{\exp(-kpa'\epsilon^b)}{a'\epsilon^b} \exp(\Lambda(n, p, \epsilon_n)) \mathbb{1}_{0 < \epsilon \leq \epsilon_n} \right) + 1 \wedge \left( \frac{\exp(-kpa'\epsilon_n^b)}{a'\epsilon_n^b} \exp(\Lambda(n, p, \epsilon_n)) \mathbb{1}_{\epsilon_n < \epsilon} \right). \end{aligned}$$

Recall that  $a' = \frac{a}{4^b}$  and that  $c \wedge d$  means  $\min(c, d)$ .

An immediate consequence of Proposition 3.2 is the following corollary.

**Corollary 3.3.** *Suppose that all the requirements of Proposition 3.2 are satisfied. Let  $p_n \rightarrow \infty$  and  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  be such that  $p_n \leq \frac{n}{4}$ ,  $\frac{\exp(-n \frac{a'}{4} \epsilon_n^b)}{\epsilon_n^b} = O((\frac{\ln n}{n})^{1/b})$  and that*

$$(13) \quad \limsup_{n \rightarrow \infty} \Lambda(n, p_n, \epsilon_n) < \infty.$$

Then, for  $n$  large enough,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \frac{n}{2p_n} \beta_{p_n} + \int_0^\infty \left( 1 \wedge \left( \frac{\frac{n}{2p_n} \beta_{p_n}}{a' \epsilon^b} \right) \right) d\epsilon + cst \left( \frac{\ln n}{n} \right)^{1/b}.$$

Condition (13) is a local type dependence condition or an anti-clustering dependence condition. Its meaning is analogous to the well known Leadbetter's condition  $D'(u_n)$  ([19], [13]). Condition (13) means that an observation  $X_1$  in the small ball  $B(x, \epsilon_n)$  cannot be followed by an observation  $X_r$  in this ball, within an interval of length  $p_n$ . The size  $p_n$  affects the rates for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$  are shown in the following theorem.

**Theorem 3.4.** *Let  $(X_n)_{n \geq 0}$  be a stationary sequence of  $\beta$ -mixing and  $\mathbb{R}^d$ -valued random variables. Suppose that  $X_1$  is supported on a compact set  $\mathbb{M}$  and that its distribution satisfies the  $(a, b)$ -standard assumption. Let  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  such that  $\frac{\exp(-n \frac{a'}{4} \epsilon_n^b)}{\epsilon_n^b} = O((\frac{\ln n}{n})^{1/b})$  and that, for some  $\alpha \in ]0, 1]$ , this sequence  $(\epsilon_n)$  satisfies also,*

$$\limsup_{n \rightarrow \infty} \Lambda(n, [n^\alpha/4], \epsilon_n) < \infty,$$

where  $(\Lambda(n, p, \epsilon_n))_{n, p}$  is as defined in (12). Suppose that  $\beta_n = O(n^{-\gamma})$  for some  $\gamma > 0$ . Let  $s = \min(1, \frac{1}{b})$ . The following rates hold.

- If  $\gamma \geq \max(1, 1/b)$ ,  $b \neq 1$  and  $\frac{s+1/b}{s(1+\gamma)} \leq \alpha \leq 1$  then,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) = O\left(\left(\frac{\ln n}{n}\right)^{1/b}\right).$$

- If  $0 < \gamma < \max(1, 1/b)$ ,  $b \neq 1$  and  $\frac{1}{1+\gamma} < \alpha \leq 1$  then,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) = O\left(n^{-\alpha s \gamma + (1-\alpha)s}\right).$$

- If  $\gamma > 0$ ,  $b = 1$  and  $\frac{1}{1+\gamma} < \alpha \leq 1$  then,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) = O\left(\frac{\ln n}{n^{\min(-1+\alpha+\alpha\gamma, 1)}}\right).$$

The following corollary proves that the optimal rates of the i.i.d. case can be reached, under suitable decays of the mixing coefficient  $\beta_n$  together with a suitable control of the local-type dependence condition.

**Corollary 3.5.** *Suppose that all the requirements of Theorem 3.4 are satisfied. If  $\gamma \geq \max(1, 1/b)$  and  $\frac{s+1/b}{s(1+\gamma)} \leq \alpha \leq 1$  then,*

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) = O\left(\left(\frac{\ln n}{n}\right)^{1/b}\right).$$

Finally the following corollary gives a sufficient condition for the local-type dependence condition of Theorem 3.4 to hold.

**Corollary 3.6.** *Let  $(X_n)_{n \geq 0}$  be a stationary sequence of  $\beta$ -mixing and  $\mathbb{R}^d$ -valued random variables. Suppose that  $X_1$  is supported on a compact set  $\mathbb{M}$  and that its distribution satisfies the  $(a, b)$ -standard assumption. Suppose that,*

$$(14) \quad \limsup_{u \rightarrow 0} \sup_{x \in \mathbb{M}} \sup_{r \geq 2} u^{-2b} \mathbb{P}(\|X_1 - x\| \leq u, \|X_r - x\| \leq u) < \infty.$$

*Then the requirements of Theorem 3.4 are satisfied for some arbitrary  $\alpha < 1$ . The conclusion of Theorem 3.4 holds (for this  $\alpha < 1$ ).*

#### 4. EXPLICIT EXAMPLES

The purpose of this section is to give some explicit examples satisfying the requirements of Theorem 2.2 and/or Theorem 3.4.

**4.1. Stationary Markov chains on a ball of  $\mathbb{R}^d$ .** We recall the following example already studied in [14]. Let  $(X_n)_{n \geq 0}$  be a Markov chain defined, for  $n \geq 0$ , by

$$(15) \quad X_{n+1} = A_{n+1}X_n + B_{n+1},$$

where  $A_{n+1}$  is a  $(d \times d)$ -matrix,  $X_n \in \mathbb{R}^d$ ,  $B_n \in \mathbb{R}^d$ ,  $(A_n, B_n)_{n \geq 1}$  is an i.i.d. random sequence independent of  $X_0$ . Recall that for a matrix  $M$ ,  $\|M\|$  is the operator norm defined by  $\|M\| = \sup_{x \in \mathbb{R}^d, \|x\|=1} \|Mx\|$ . It is well known that for any  $n \geq 1$ ,  $X_n$  is distributed as  $\sum_{k=1}^n A_1 \cdots A_{k-1} B_k + A_1 \cdots A_n X_0$ , see for instance [17]. It is also well-known that the following conditions (see [12, 16]),

$$(16) \quad \mathbb{E}(\ln^+ \|A_1\|) < \infty, \mathbb{E}(\ln^+ \|B_1\|) < \infty, \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|A_1 \cdots A_n\| < 0 \text{ a.s.},$$

ensure the existence of a stationary solution to (15), and that  $\|A_1 \cdots A_n\|$  approaches 0 exponentially fast. If in addition  $\mathbb{E}\|B_1\|^\beta < \infty$  for some  $\beta > 0$ , then the series  $R := \sum_{i=1}^\infty A_1 \cdots A_{i-1} B_i$  converges a.s. and the distribution of  $X_n$  converges to that of  $R$ , independently of  $X_0$ . The distribution of  $R$  is, then, that of the stationary measure of the chain. Corollary 5.2 in [14] gives conditions under which Assumptions  $(\mathcal{A}_1)$  and  $(\mathcal{A}_2)$  are satisfied and thus  $\theta^* = 1$  for this Markov chain. We summarise these conditions in the following corollary (that we announce without proof).

**Corollary 4.1.** *Suppose that in the model (15), conditions (16) are satisfied, and moreover  $\|B_1\| \leq c < \infty$ . If the density of  $A_1 x + B_1$ ,  $f_{A_1 x + B_1}$ , satisfies  $\inf_{x, y \in \mathbb{M}} f_{A_1 x + B_1}(y) \geq \kappa > 0$  for some positive  $\kappa$ , then Assumptions  $(\mathcal{A}_1)$  and  $(\mathcal{A}_2)$  are satisfied with  $b = d$ ,  $\nu$  being the Lebesgue measure on  $\mathbb{R}^d$  and thus  $\theta^* = 1$  for this Markov chain. The conclusion of Theorem 2.2 holds.*

**4.2. The Möbius Markov chain on the circle.** Our purpose is to study an explicit example of a Markov chain on the unit circle, known as a Möbius Markov chain, for which both approaches are applicable. We first check that this model satisfies all the requirements of Theorem 2.2. Next we prove that this Markov chain is geometrically ergodic. The Möbius Markov chain on the circle is introduced in [15] and is defined as follows.

- Let  $X_0$  be a random variable which takes values on the unit circle.

- Define, for  $n \geq 1$ ,

$$X_n = \frac{X_{n-1} + \beta}{\beta X_{n-1} + 1} \epsilon_n,$$

where  $\beta \in ]-1, 1[$  and  $(\epsilon_n)_{n \geq 1}$  is a sequence of i.i.d. random variables which are independent of  $X_0$  and distributed as the wrapped Cauchy distribution with a common density,  $f_\varphi$ , with respect to the arc length measure  $\nu$  on the unit circle  $\partial B(0, 1)$ , i.e.,  $\forall z \in \partial B(0, 1)$ ,

$$f_\varphi(z) = \frac{1}{2\pi} \frac{1 - \varphi^2}{|z - \varphi|^2},$$

$\varphi \in [0, 1[$  being fixed.

The following proposition proves that the introduced Möbius Markov chain satisfies the requirements of the first approach.

**Proposition 4.2.** *Let  $(X_n)_{n \geq 0}$  be the Möbius Markov chain on the unit circle as defined above. Then all the requirements of Theorem 2.2 are satisfied. More precisely, this Markov chain admits a unique invariant distribution, denoted by  $\mu$ . If  $X_0$  is distributed as  $\mu$  then the  $(a, b)$ -standard assumption is satisfied by  $\mu$  with*

$$a = \frac{1}{2\pi} \frac{1 - \varphi}{1 + \varphi} \left( \inf_{u \in \partial B(0, 1)} \inf_{0 < \epsilon < \epsilon_0} \frac{\nu(\partial B(0, 1) \cap B(u, \epsilon))}{\epsilon} \right) > 0, \quad b = 1,$$

$\nu$  is the arc length measure on the unit circle. This Markov chain has a minimal index  $\theta^* = 1$  with a marginal lower bound  $\kappa_\epsilon = a\epsilon$ . The conclusion of Theorem 2.2 holds, that is,

$$\mathbb{E}(d_H(\mathbb{X}_n, \partial B(0, 1))) = O\left(\frac{\ln n}{n}\right).$$

Proposition 4.3 below proves that the Möbius Markov chain, as introduced in this section, satisfies also the requirements of the second approach.

**Proposition 4.3.** *Let  $(X_n)_{n \geq 0}$  be the Möbius Markov chain on the unit circle as defined above. This Markov chain is  $\beta$ -mixing with  $\beta_n = O(e^{-cn})$  (for some  $c > 0$ ). It satisfies all the requirements of Theorem 3.4 and  $\mathbb{E}(d_H(\mathbb{X}_n, \partial B(0, 1))) = O\left(\frac{\ln n}{n}\right)$ .*

The purpose now is to simulate a Möbius Markov chain on the unit circle and to illustrate the rate of  $\mathbb{E}(d_H(\mathbb{X}_n, \partial B(0, 1)))$ . More precisely, we simulate,

- a random variable  $X_0$  uniformly distributed on the unit circle  $\partial B(0, 1)$ , that is  $X_0$  has the density,

$$f(z) = \frac{1}{2\pi}, \quad \forall z \in \partial B(0, 1).$$

- For  $n \geq 1$ ,

$$X_n = X_{n-1} \epsilon_n,$$

where  $(\epsilon_n)_{n \geq 1}$  is a sequence of i.i.d. random variables which are independent of  $X_0$  and distributed as the wrapped Cauchy distribution with a common density with respect to the arc length measure  $\nu$  on the unit circle  $\partial B(0, 1)$ ,

$$f_\varphi(z) = \frac{1}{2\pi} \frac{1 - \varphi^2}{|z - \varphi|^2}, \quad \varphi \in [0, 1[, \quad z \in \partial B(0, 1).$$

It is proved, in [15], that this Markov chain is stationary. Its stationary measure is the uniform law on the unit circle.

Simulations give the following numerical values and illustrations, which do not contradict the theoretical result of both Proposition 4.2 and Proposition 4.3.

	n	Hausdorff dist.
1	20	0.44273078
2	40	0.42544579
3	60	0.23801556
4	80	0.23807300
5	100	0.13752374
6	120	0.13511955
7	140	0.11108553
8	160	0.12247550
9	180	0.11113827
10	200	0.09572337
11	220	0.08870232
12	240	0.06918471
13	260	0.11012207
14	280	0.07965723
15	300	0.07060258

Table 1. Behavior of  $d_H(\mathbb{X}_n, \partial B(0, 1))$  with  $n$ . We used the function `distFct` from the library `(TDA)` of the software `R` as described in [11].

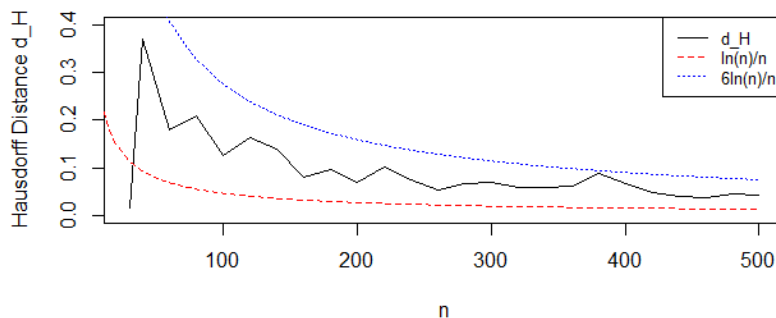


FIGURE 1.  $d_H(\mathbb{X}_n, \partial B(0, 1))$  and the rate  $(\ln n)/n$ .

**4.3. A Markov chain on a square wrapped on a torus.** Recall that, for  $x \in \mathbb{R}$ ,  $[x]$  denotes the integer part of  $x$  and  $x - [x]$  denotes its fractional part. Clearly,  $0 \leq x - [x] < 1$ . Define the Markov chain  $(\Phi_n)_{n \geq 0} = (\theta_n, \phi_n)_{n \geq 0}$  on the square  $[0, 1[ \times [0, 1[$  with opposite edges

identified, by

$$\begin{cases} \theta_{n+1} &= \theta_n + \epsilon_{n+1} - [\theta_n + \epsilon_{n+1}] \\ \phi_{n+1} &= \phi_n + \eta_{n+1} - [\phi_n + \eta_{n+1}], \end{cases}$$

where  $(\epsilon_i)_{i \geq 0}$  and  $(\eta_i)_{i \geq 0}$  are two independent sequences of i.i.d. random variables all uniformly distributed over  $[0, 1]$ . Suppose also that, for each  $n$ ,  $\theta_n$  (respectively  $\phi_n$ ) is independent of  $\epsilon_{n+1}$  (respectively  $\eta_{n+1}$ ). The following proposition proves that the requirements of Theorem 2.2 are satisfied.

**Proposition 4.4.** *The following properties hold.*

- $(\Phi_n)_{n \geq 0}$  is a stationary Markov chain with the uniform over  $[0, 1] \times [0, 1]$  stationary distribution.
- For any positive  $\epsilon$  small enough and any couples  $(x, y) \in [0, 1[ \times [0, 1[$  and  $(u, v) \in [0, 1[ \times [0, 1[$ ,

$$\mathbb{P} \left( \sqrt{|\theta_1 - x|^2 + |\phi_1 - y|^2} \leq \epsilon \mid \theta_0 = u, \phi_0 = v \right) \geq \frac{\epsilon^2}{2}$$

- $\Phi_1$  satisfies the  $(a, b)$ -standard assumption with  $a = \frac{1}{2}$  and  $b = 2$ .
- The Markov chain  $(\Phi_n)_{n \geq 0}$  has a minimal index  $\theta^* = 1$  with a marginal lower bound  $\kappa_\epsilon = \frac{\epsilon^2}{2}$ .

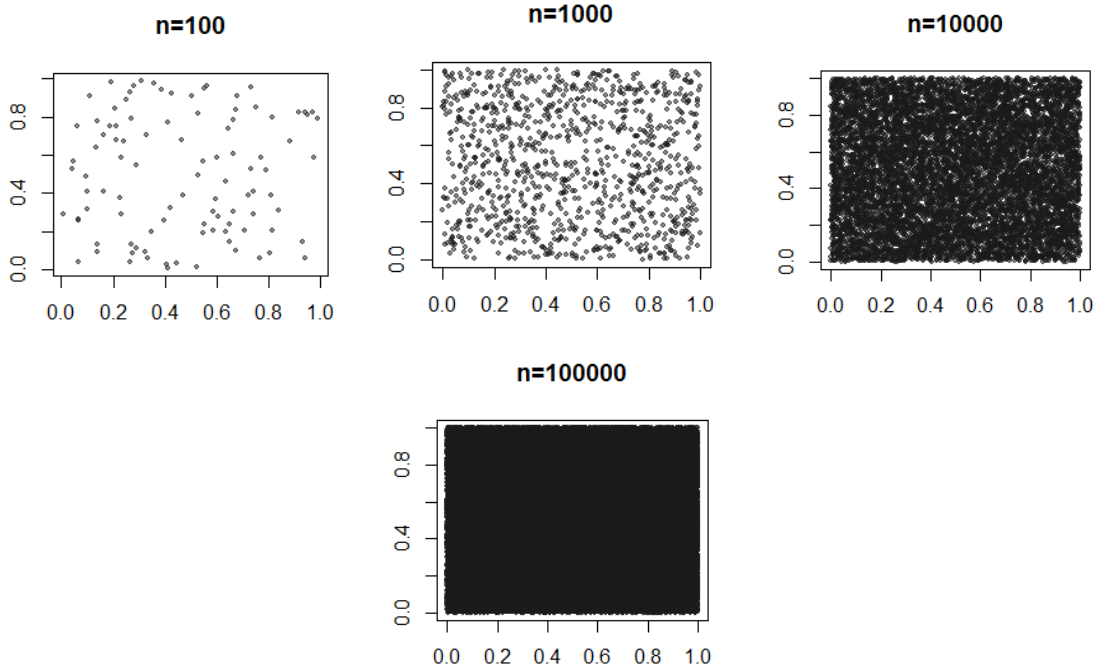


FIGURE 2. A Markov chain on  $[0, 1[ \times [0, 1[$  with opposite edges identified. Different realisations of the set  $\{\Phi_1, \dots, \Phi_n\}$  with different values of  $n$ .

The random torus on  $[0, 1[ \times [0, 1[$  can be represented parametrically in 3D-dimension using the following equations,

$$\begin{cases} X_n = (R + r \cos(2\pi\theta_n)) \cos(2\pi\phi_n) \\ Y_n = (R + r \cos(2\pi\theta_n)) \sin(2\pi\phi_n) \\ Z_n = r \sin(2\pi\theta_n) \end{cases}$$

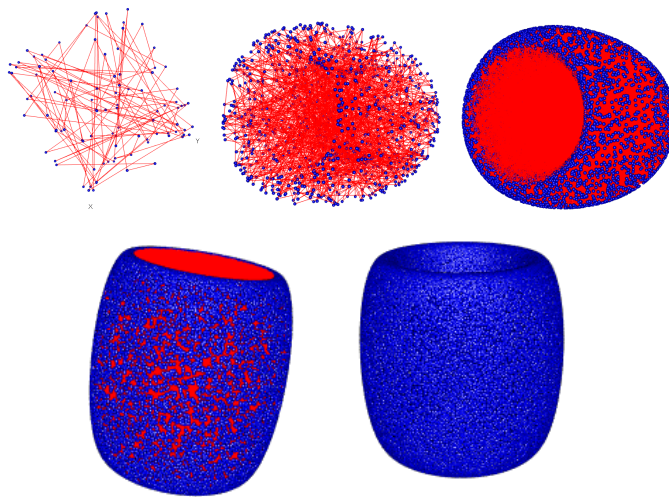


FIGURE 3. Different realisations of the set  $\mathbb{X}_n = \{(X_i, Y_i, Z_i)_{1 \leq i \leq n}\}$  for different values of  $n$ . From the left to the right  $n = 100, 1000, 10000, 50000, 100000$ . The realisations of  $(X_i, Y_i, Z_i)_{1 \leq i \leq n}$  are in blue. The paths of the Markov chain are in red. (Here  $R = 0.9$  and  $r = 0.3$ ).

	n	Hausdorff dist.
1	30	0.01520626
2	40	0.37171930
3	60	0.17933311
4	80	0.20862514
5	100	0.12695697
6	120	0.16236784
7	140	0.13807599
8	160	0.08060261
9	180	0.09467337
10	200	0.06865655
11	220	0.10108801
12	240	0.07552198
13	260	0.05321497
14	280	0.06503609
15	300	0.06932031

Table 2. Behavior (with  $n$ ) of the Hausdorff distance between a set of the realisations of  $\mathbb{X}_n = \{(X_i, Y_i, Z_i)_{1 \leq i \leq n}\}$  and the torus with  $R = 0.9$  and  $r = 0.3$ .



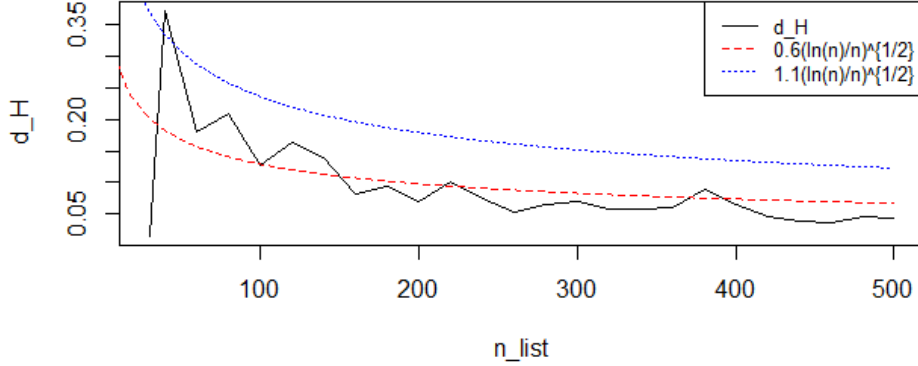


FIGURE 4. The rate  $((\ln n)/n)^{1/2}$  and the behavior of the Hausdorff distance between the set of realisations of  $\mathbb{X}_n = \{(X_i, Y_i, Z_i)_{1 \leq i \leq n}\}$  and the torus with  $n$ , (here  $R = 0.9$  and  $r = 0.3$ ). Proposition 4.4 together with Theorem 2.2 ensure that the rate of convergence is  $(\ln n/n)^{1/2}$  who agrees with this figure.

## 5. PROOFS

5.1. **Proof of Theorem 2.2.** We have from the assumptions of Theorem 2.2 and the definition of  $\theta^*$  (letting  $a' = a/4^b$ ), for any  $0 < \epsilon \leq 4\epsilon_0 =: \epsilon'_0$ ,

$$\inf_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| \leq \epsilon/4) \geq a' \epsilon^b$$

$$\sup_{x \in \mathbb{M}} \mathbb{P}\left(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/4\right) \leq c(1 - a' \epsilon^b)^{n\theta^*} \leq c \exp(-a' \theta^* n \epsilon^b),$$

(the last bound is obtained since for any  $x \in [0, 1]$ ,  $1 - x \leq e^{-x}$ ). The conclusion of Proposition 1.1 together with the two last bounds, give for any  $0 < \epsilon \leq \epsilon'_0$ ,

$$(17) \quad \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) \leq 1 \wedge \left(c \frac{\exp(-a' \theta^* n \epsilon^b)}{a' \epsilon^b}\right).$$

We have, a.s., since  $\mathbb{X}_n$  is a subset of  $\mathbb{M}$ ,  $d_H(\mathbb{X}_n, \mathbb{M}) \leq \text{diam}(\mathbb{M})$ . Hence  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) \geq \epsilon) = 0$  for any  $\epsilon \geq C$  where  $C$  is a positive constant satisfying  $C > \max(\text{diam}(\mathbb{M}), \epsilon'_0)$ . We have (noting by  $a'' = a'/c$  and by  $cst$  a positive constant that does not depend on  $n$ ),

$$(18) \quad \begin{aligned} \mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) &= \int_0^\infty \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) d\epsilon = \int_0^C \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) d\epsilon \\ &\leq \int_0^C \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) d\epsilon + \int_0^{\epsilon'_0} \left(1 \wedge \frac{\exp(-a' \theta^* n \epsilon^b)}{a'' \epsilon^b}\right) d\epsilon \\ &\leq C \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon'_0) + \int_0^\infty \left(1 \wedge \frac{\exp(-a' \theta^* n \epsilon^b)}{a'' \epsilon^b}\right) d\epsilon \\ &\leq cst \exp(-a' \theta^* n \epsilon_0^b) + \int_0^\infty \left(1 \wedge \frac{\exp(-a' \theta^* n \epsilon^b)}{a'' \epsilon^b}\right) d\epsilon, \end{aligned}$$

the last bound is obtained thanks to (17). We have also, using the same calculations as for the i.i.d. case (see for instance [5], Section B.2),

$$(19) \quad \int_0^\infty \left( 1 \wedge \frac{\exp(-a'\theta^*n\epsilon^b)}{a''\epsilon^b} \right) d\epsilon \leq cst \left( \frac{\ln n}{n} \right)^{1/b}.$$

Clearly,

$$(20) \quad \exp(-a'\theta^*n\epsilon_0^b) = o\left(\left(\frac{\ln n}{n}\right)^{1/b}\right).$$

The proof of Theorem 2.2 is complete by combining the bounds (18), (19) and (20).  $\square$

**5.2. Proof of Proposition 2.3.** The random sequence  $(X_i)_{i \in \mathbb{N}}$  is stationary and  $m$ -dependent, so that the random variables:

$$X_1, X_{m+2}, X_{2m+3}, \dots, X_{1+k(m+1)}, \quad k \in \mathbb{N}$$

are i.i.d. Hence, for any  $\epsilon \in ]0, \epsilon_0]$ ,  $x \in \mathbb{M}$  and  $n \geq m + 1$ ,

$$\begin{aligned} \mathbb{P}\left(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon\right) &\leq \mathbb{P}\left(\min_{0 \leq k \leq [n/(m+1)]-1} \|X_{1+k(m+1)} - x\| > \epsilon\right) \\ &\leq \left(\mathbb{P}(\|X_1 - x\| > \epsilon)\right)^{[n/(m+1)]} \leq \left(\mathbb{P}(\|X_1 - x\| > \epsilon)\right)^{n/(m+1)-1} \end{aligned}$$

The  $(a, b)$ -standard assumption satisfied by the distribution of  $X_1$  gives, for any  $0 < \epsilon \leq \epsilon_0$ ,

$$\mathbb{P}(\|X_1 - x\| > \epsilon) = 1 - \mathbb{P}(\|X_1 - x\| \leq \epsilon) \leq 1 - a\epsilon^b.$$

Consequently, for any  $0 < \epsilon \leq \epsilon_0$ ,

$$\mathbb{P}\left(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon\right) \leq (1 - a\epsilon^b)^{n/(m+1)-1} \leq \frac{(1 - a\epsilon^b)^{n/(m+1)}}{1 - a\epsilon_0^b}.$$

The requirement of Definition 2.1 is then satisfied with,

$$\theta^* = \frac{1}{m+1}, \quad \kappa_\epsilon = a\epsilon^b, \quad c = \frac{1}{1 - a\epsilon_0^b}.$$

The first part of Proposition 2.3 is then proved. All the requirements of Theorem 2.2 are satisfied. The conclusion of Theorem 2.2 holds.  $\square$

**5.3. Proof of Propositions 2.4 and 2.6.** The proof of this proposition is analogous to that of Lemma 7.2 of [14]. We have, letting  $k$  be a positive integer,

$$\mathbb{P}\left(\min_{1 \leq i \leq k} \|X_i - x\| > \epsilon\right) = \mathbb{E}\left(\prod_{i=1}^{k-1} \mathbb{1}_{\{X_i \notin B(x, \epsilon)\}} \mathbb{E}_{\mathcal{F}_{k-1}}(\mathbb{1}_{\{X_k \notin B(x, \epsilon)\}})\right),$$

where  $\mathcal{F}_{k-1} = \sigma(X_0, \dots, X_{k-1})$  and  $B(x, \epsilon) = \{y, \|x - y\| \leq \epsilon\}$ . We deduce from (6) that,

$$\begin{aligned} \mathbb{P}\left(\min_{1 \leq i \leq k} \|X_i - x\| > \epsilon\right) &\leq (1 - \kappa_\epsilon) \mathbb{E}\left(\prod_{i=1}^{k-1} \mathbb{1}_{\{X_i \notin B(x, \epsilon)\}}\right) \\ &\leq (1 - \kappa_\epsilon) \mathbb{P}\left(\min_{1 \leq i \leq k-1} \|X_i - x\| > \epsilon\right). \end{aligned}$$

The proof of Proposition 2.4 is complete by induction on  $k$ . Recall that, for  $k = 1$ ,

$$(21) \quad \mathbb{P}(\|X_1 - x\| > \epsilon) = 1 - \mathbb{P}(\mathbb{P}(\|X_1 - x\| \leq \epsilon | X_0, \dots, X_{n-1})) \leq 1 - \kappa_\epsilon,$$

by (6). Now in the case of stationary Markov chains, we have,

$$\mathbb{E}_{\mathcal{F}_{k-1}}(\mathbb{1}_{\{X_k \notin B(x, \epsilon)\}}) = \mathbb{E}_{X_{k-1}}(\mathbb{1}_{\{X_k \notin B(x, \epsilon)\}}) = \mathbb{E}_{X_0}(\mathbb{1}_{\{X_1 \notin B(x, \epsilon)\}}),$$

and by (7),

$$\mathbb{E}_{X_0}(\mathbb{1}_{\{X_1 \notin B(x, \epsilon)\}}) \leq 1 - \kappa_\epsilon,$$

almost surely. So that, Proposition (2.4) applies. More precisely, we obtain, for any  $k \geq 1$ ,

$$\mathbb{P}_\mu \left( \min_{1 \leq i \leq k} \|X_i - x\| > \epsilon \right) \leq (1 - \kappa_\epsilon)^k.$$

The proof of Proposition 2.6 is complete.

**5.4. Proof of Corollary 2.8.** We have, for any  $x_0 \in \mathbb{M}$ , using  $(\mathcal{A}_2)$ ,

$$\mathbb{P}_{x_0}(\|X_1 - x\| \leq \epsilon) = \int \mathbb{1}_{\{y \in B(x, \epsilon)\}} K(x_0, dy) = \int \mathbb{1}_{\{y \in B(x, \epsilon) \cap \mathbb{M}\}} k(x_0, y) \nu(dy).$$

So that, using again  $(\mathcal{A}_2)$ ,

$$\mathbb{P}_{x_0}(\|X_1 - x\| \leq \epsilon) \geq \kappa \int \mathbb{1}_{\{y \in B(x, \epsilon) \cap \mathbb{M}\}} \nu(dy) \geq \kappa \epsilon^b V_d > 0.$$

The bound (7) is then satisfied with  $\kappa_\epsilon = \kappa \epsilon^b V_d$ . The rest of the proof of Corollary 2.8 follows from Corollary 2.7.

**5.5. Proofs of Propositions 2.9 and 2.11.** Let  $m$  and  $k$  be two positive integers for which  $(m+1)k \leq n$ . Then,

$$\begin{aligned} & \mathbb{P} \left( \min_{1 \leq i \leq k} \|X_{i(m+1)} - x\| > \epsilon \right) \\ &= \mathbb{E} \left( \prod_{i=1}^{k-1} \mathbb{1}_{\{X_{i(m+1)} \notin B(x, \epsilon)\}} \mathbb{E} \left( \mathbb{1}_{\{X_{k(m+1)} \notin B(x, \epsilon)\}} \mid (X_0, \dots, X_{(k-1)(m+1)}) \right) \right) \\ &\leq (1 - \kappa_\epsilon) \mathbb{E} \left( \prod_{i=1}^{k-1} \mathbb{1}_{\{X_{i(m+1)} \notin B(x, \epsilon)\}} \right), \end{aligned}$$

the last bound is obtained thanks to (9). So that, we deduce using an induction on  $k$  (the case  $k = 1$  follows from (9), as in (21)),

$$\mathbb{P} \left( \min_{1 \leq i \leq k} \|X_{i(m+1)} - x\| > \epsilon \right) \leq (1 - \kappa_\epsilon)^k,$$

and for  $k = \lfloor n/(m+1) \rfloor$  (recall that  $\kappa_\epsilon \in ]0, \alpha[$ ), we obtain,

$$\begin{aligned} & \mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon \right) \leq \mathbb{P} \left( \min_{1 \leq i \leq k} \|X_{i(m+1)} - x\| > \epsilon \right) \leq (1 - \kappa_\epsilon)^{\lfloor n/(m+1) \rfloor} \\ &\leq (1 - \kappa_\epsilon)^{n/(m+1)-1} \leq \frac{1}{1 - \alpha} (1 - \kappa_\epsilon)^{n/(m+1)}. \end{aligned}$$

The proof of Proposition 2.9 is complete. Let us now prove Proposition 2.11. We get thanks to the stationary assumption of the Markov chain,

$$\mathbb{E} \left( \mathbb{1}_{\{X_{k(m+1)} \notin B(x, \epsilon)\}} \mid (X_0, \dots, X_{(k-1)m}) \right) = \mathbb{E}_{X_{(k-1)(m+1)}} \left( \mathbb{1}_{\{X_{k(m+1)} \notin B(x, \epsilon)\}} \right) = \mathbb{E}_{X_0}(\mathbb{1}_{\{X_{m+1} \notin B(x, \epsilon)\}}).$$

This last fact proves Proposition 2.11 from Proposition 2.9.

**5.6. Proof of Proposition 3.1.** Define, for a positive integer  $p < n/2$ ,  $k = \lfloor \frac{n}{2p} \rfloor$ , recall that  $\lfloor \cdot \rfloor$  denotes the integer part. Define also, for  $1 \leq i \leq k$ , the sets of indices,

$$I_{i,2p} = \{2p(i-1) + 1, \dots, ip + (i-1)p\}.$$

We need for the proof of this proposition the following Berbee's coupling (we refer, for instance, to [23] page 116 for a clear formulation): there exists a random sequence of i.i.d. blocks  $\{\xi_j, j \in I_{i,2p}\}$  (onto a richer probability space) such that, for any  $1 \leq i \leq k$ , the following three properties hold,

- ( $\mathcal{P}_1$ )  $\{\xi_j, j \in I_{i,2p}\}$  and  $\{X_j, j \in I_{i,2p}\}$  are identically distributed
- ( $\mathcal{P}_2$ )  $\mathbb{P}(\{\xi_j, j \in I_{i,2p}\} \neq \{X_j, j \in I_{i,2p}\}) \leq \beta_p$
- ( $\mathcal{P}_3$ )  $\{\xi_j, j \in I_{i,2p}\}$  is independent of  $(\{X_j, j \in I_{l,2p}\})_{1 \leq l \leq i-1}$ , for  $i \geq 2$ .

Our purpose is to control  $\mathbb{P}(\min_{1 \leq i \leq n} \|X_i - x\| > \epsilon)$ . For this, we use Berbee's coupling on the blocs of variables having the set of indices  $(I_{l,2p})_l$ , as defined above. More precisely, define for  $x \in \mathbb{M}$ ,  $X_l(x)$  and  $\xi_l(x)$ , respectively, by

$$(22) \quad X_l(x) \in \operatorname{argmin}_{X_i, i \in I_{l,2p}} \|X_i - x\|, \quad \xi_l(x) \in \operatorname{argmin}_{\xi_i, i \in I_{l,2p}} \|\xi_i - x\|,$$

so that,

$$\|X_l(x) - x\| = \min_{i \in I_{l,2p}} \|X_i - x\|,$$

and

$$\|\xi_l(x) - x\| = \min_{i \in I_{l,2p}} \|\xi_i - x\|.$$

Let  $x \in \mathbb{M}$  be fixed. Clearly,

$$(23) \quad \min_{1 \leq i \leq n} \|X_i - x\| \leq \min_{1 \leq l \leq k} \min_{i \in I_{l,2p}} \|X_i - x\|$$

and,

$$\min_{i \in I_{l,2p}} \|X_i - x\| = \|X_l(x) - x\| \leq \|X_l(x) - \xi_l(x)\| + \|\xi_l(x) - x\|.$$

Hence,

$$\begin{aligned} \min_{1 \leq l \leq k} \min_{i \in I_{l,2p}} \|X_i - x\| &\leq \min_{1 \leq l \leq k} (\|X_l(x) - \xi_l(x)\| + \|\xi_l(x) - x\|) \\ &\leq \max_{1 \leq l \leq k} \|X_l(x) - \xi_l(x)\| + \min_{1 \leq l \leq k} \|\xi_l(x) - x\|. \end{aligned}$$

Consequently, by (23),

$$\min_{1 \leq i \leq n} \|X_i - x\| \leq \max_{1 \leq l \leq k} \|X_l(x) - \xi_l(x)\| + \min_{1 \leq l \leq k} \|\xi_l(x) - x\|,$$

and

$$(24) \quad \begin{aligned} &\mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2 \right) \\ &\leq \mathbb{P} \left( \max_{1 \leq l \leq k} \|X_l(x) - \xi_l(x)\| + \min_{1 \leq l \leq k} \|\xi_l(x) - x\| > \epsilon/2 \right). \end{aligned}$$

Now (recall that  $x \in \mathbb{M}$ ),

$$\begin{aligned}
& \mathbb{P} \left( \max_{1 \leq l \leq k} \|X_l(x) - \xi_l(x)\| + \min_{1 \leq l \leq k} \|\xi_l(x) - x\| > \epsilon/2 \right) \\
& \leq \mathbb{P} \left( \max_{1 \leq l \leq k} \|X_l(x) - \xi_l(x)\| > \epsilon/4 \right) + \mathbb{P} \left( \min_{1 \leq l \leq k} \|\xi_l(x) - x\| > \epsilon/4 \right) \\
& \leq \sup_{x \in \mathbb{M}} \mathbb{P} \left( \max_{1 \leq l \leq k} \|X_l(x) - \xi_l(x)\| > \epsilon/4 \right) + \sup_{x \in \mathbb{M}} \mathbb{P} \left( \min_{1 \leq l \leq k} \|\xi_l(x) - x\| > \epsilon/4 \right) \\
(25) \quad & \leq k \sup_{x \in \mathbb{M}} \max_{1 \leq l \leq k} \mathbb{P} (\|X_l(x) - \xi_l(x)\| > \epsilon/4) + \sup_{x \in \mathbb{M}} \mathbb{P} \left( \min_{1 \leq l \leq k} \|\xi_l(x) - x\| > \epsilon/4 \right).
\end{aligned}$$

We get, using (24) and (25),

$$\begin{aligned}
& \mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2 \right) \leq k \sup_{x \in \mathbb{M}} \max_{1 \leq l \leq k} \mathbb{P} (\|X_l(x) - \xi_l(x)\| > \epsilon/4) \\
& + \sup_{x \in \mathbb{M}} \mathbb{P} \left( \min_{1 \leq l \leq k} \|\xi_l(x) - x\| > \epsilon/4 \right) \\
(26) \quad & =: I(\epsilon/4) + II(\epsilon/4).
\end{aligned}$$

Let us control the two terms  $I(\epsilon/4)$  and  $II(\epsilon/4)$  on the right hand side of the last bound.

**Control of  $I(\epsilon/4)$ .** We deduce from (22) that the random variable  $X_l(x)$  (respectively  $\xi_l(x)$ ) belongs to the sigma-fields generated by  $\{X_j, j \in I_{l,2p}\}$  (respectively by  $\{\xi_j, j \in I_{l,2p}\}$ ). Hence, by (22),

$$(\|X_l(x) - \xi_l(x)\| > \epsilon/4) \implies (\{\xi_j, j \in I_{l,2p}\} \neq \{X_j, j \in I_{l,2p}\}),$$

so that, (by the construction of the random sequence  $(\xi_j)_j$ , more precisely by Property  $(\mathcal{P}_2)$ ),

$$\mathbb{P} (\|X_l(x) - \xi_l(x)\| > \epsilon/4) \leq \mathbb{P} (\{\xi_j, j \in I_{l,2p}\} \neq \{X_j, j \in I_{l,2p}\}) \leq \beta_p,$$

and,

$$(27) \quad I(\epsilon/4) \leq k\beta_p$$

**Control of  $II(\epsilon/4)$ .** By construction, the random variables  $(\xi_l(x))_{1 \leq l \leq k}$  are i.i.d., each of them is distributed as  $X_1(x)$ , since  $\xi_1(x)$  and  $X_1(x)$  are identically distributed. Hence,

$$\begin{aligned}
& \sup_{x \in \mathbb{M}} \mathbb{P} \left( \min_{1 \leq l \leq k} \|\xi_l(x) - x\| > \epsilon/4 \right) \leq \sup_{x \in \mathbb{M}} \mathbb{P}^k (\|\xi_1(x) - x\| > \epsilon/4) \\
& \leq \sup_{x \in \mathbb{M}} \mathbb{P}^k (\|X_1(x) - x\| > \epsilon/4) = \sup_{x \in \mathbb{M}} \mathbb{P}^k \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right),
\end{aligned}$$

the last equality is obtained by using the definition of  $X_1(x)$  (in 22). Hence,

$$(28) \quad II(\epsilon/4) \leq \sup_{x \in \mathbb{M}} \mathbb{P}^k \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right).$$

We deduce, collecting (27), (28) together with (26),

$$\mathbb{P} \left( \min_{1 \leq i \leq n} \|X_i - x\| > \epsilon/2 \right) \leq k\beta_p + \sup_{x \in \mathbb{M}} \mathbb{P}^k \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right).$$

The last bound completes the proof of Proposition 3.1.  $\square$

**5.7. Proof of Proposition 3.2.** We combine Proposition 3.1 together with Proposition 1.1. We obtain, noting that  $(a+b)\wedge 1 \leq (a\wedge 1)+(b\wedge 1)$  for  $a, b > 0$  and that  $1 - \sup_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| > \epsilon/4) \geq \inf_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| \leq \epsilon/4)$ ,

$$\begin{aligned} \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) &\leq 1 \wedge \left( k\beta_p \left( \inf_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| \leq \epsilon/4) \right)^{-1} \right) \\ &+ 1 \wedge \left( \left( \inf_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| \leq \epsilon/4) \right)^{-1} \sup_{x \in \mathbb{M}} \mathbb{P}^k \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right) \right). \end{aligned}$$

Recall that, for reals  $a$  and  $b$ ,  $a \wedge b = \min(a, b)$ . The last bound together with the  $(a, b)$ -standard assumption give,

$$(29) \quad \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) \leq 1 \wedge 4^b \frac{k\beta_p}{a\epsilon^b} + 1 \wedge \left( 4^b \frac{1}{a\epsilon^b} \sup_{x \in \mathbb{M}} \mathbb{P}^k \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right) \right).$$

In order to control  $\mathbb{P}^k(\min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4)$ , we need the following lemma.

**Lemma 5.1.** *Let  $(\epsilon_n)_{n \geq 0}$  be a non-random positive fixed sequence. Let  $(\Lambda(n, p, \epsilon_n))_{n, p}$  be as defined in (12). The following statements hold.*

- If  $0 < \epsilon \leq \epsilon_n$  then,

$$\left( \mathbb{P} \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right) \right)^k \leq \exp(-kp \mathbb{P}(\|X_1 - x\| \leq \epsilon/4)) \exp(\Lambda(n, p, \epsilon_n)).$$

- If  $\epsilon > \epsilon_n$  then,

$$\left( \mathbb{P} \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4 \right) \right)^k \leq \exp(-kp \mathbb{P}(\|X_1 - x\| \leq \epsilon_n/4)) \exp(\Lambda(n, p, \epsilon_n)).$$

**Proof of Lemma 5.1.** We have, using the trivial bound  $\ln(1-x) \leq -x$  for  $x \in ]0, 1[$ ,

$$\left( \mathbb{P} \left( \min_{1 \leq i \leq p} \|X_i - x\| > \epsilon \right) \right)^k \leq \exp(-k \mathbb{P} \left( \min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon \right)).$$

Recall the following Bonferroni-type inequality, for any events  $(A_i)_{1 \leq i \leq p}$ ,

$$\sum_{i=1}^p \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq p} \mathbb{P}(A_i \cap A_j) \leq \mathbb{P} \left( \bigcup_{i=1}^p A_i \right) \leq \sum_{i=1}^p \mathbb{P}(A_i).$$

Let  $A_i$  be the event  $(\|X_i - x\| \leq \epsilon)$  for positive  $\epsilon$ . So that,  $\bigcup_{i=1}^p A_i \subset (\min_{1 \leq i \leq p} (\|X_i - x\| \leq \epsilon))$  and for any positive  $\epsilon$ ,

$$\begin{aligned} &\mathbb{P} \left( \min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon \right) \\ &\geq p \mathbb{P}(\|X_1 - x\| \leq \epsilon) - \sum_{1 \leq i < j \leq p} \mathbb{P}(\|X_i - x\| \leq \epsilon, \|X_j - x\| \leq \epsilon) \\ (30) \quad &\geq p \mathbb{P}(\|X_1 - x\| \leq \epsilon) - p \sum_{r=2}^p \mathbb{P}(\|X_1 - x\| \leq \epsilon, \|X_r - x\| \leq \epsilon). \end{aligned}$$

- If  $0 < \epsilon \leq \epsilon_n$  then,

$$\mathbb{P}(\|X_1 - x\| \leq \epsilon, \|X_r - x\| \leq \epsilon) \leq \mathbb{P}(\|X_1 - x\| \leq \epsilon_n, \|X_r - x\| \leq \epsilon_n)$$

and by (30),

$$\begin{aligned} & \mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon) \\ & \geq p\mathbb{P}(\|X_1 - x\| \leq \epsilon) - p \sum_{r=2}^p \mathbb{P}(\|X_1 - x\| \leq \epsilon_n, \|X_r - x\| \leq \epsilon_n). \end{aligned}$$

Consequently (since  $kp \leq n$ ),

$$\exp(-k\mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon)) \leq \exp(-kp\mathbb{P}(\|X_1 - x\| \leq \epsilon)) \exp(\Lambda(n, p, \epsilon_n)).$$

- If  $\epsilon > \epsilon_n$  then,

$$(31) \quad \mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon_n) \leq \mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon)$$

and

$$\exp(-k\mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon)) \leq \exp(-k\mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon_n)).$$

We have, by (30),

$$(32) \quad \begin{aligned} & \mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon_n) \\ & \geq p\mathbb{P}(\|X_1 - x\| \leq \epsilon_n) - p \sum_{r=2}^p \mathbb{P}(\|X_1 - x\| \leq \epsilon_n, \|X_r - x\| \leq \epsilon_n). \end{aligned}$$

Finally, we obtain, combining (31) and (32),

$$\exp(-k\mathbb{P}(\min_{1 \leq i \leq p} \|X_i - x\| \leq \epsilon)) \leq \exp(-kp\mathbb{P}(\|X_1 - x\| \leq \epsilon_n)) \exp(\Lambda(n, p, \epsilon_n)).$$

The proof of Lemma 5.1 is complete.  $\square$

Lemma 5.1 and the  $(a, b)$ -standard assumption give,

$$\begin{aligned} & \frac{\sup_{x \in \mathbb{M}} \mathbb{P}^k(\min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4)}{\inf_{x \in \mathbb{M}} \mathbb{P}(\|X_1 - x\| \leq \epsilon/4)} \\ & \leq 4^b \frac{1}{a\epsilon^b} \sup_{x \in \mathbb{M}} \mathbb{P}^k\left(\min_{1 \leq i \leq p} \|X_i - x\| > \epsilon/4\right) \\ & \leq 4^b \frac{1}{a\epsilon^b} \exp(-kpa/4^b\epsilon^b) \exp(\Lambda(n, p, \epsilon_n)) \mathbb{1}_{0 < \epsilon \leq \epsilon_n} + 4^b \frac{1}{a\epsilon_n^b} \exp(-kpa/4^b\epsilon_n^b) \exp(\Lambda(n, p, \epsilon_n)) \mathbb{1}_{\epsilon_n < \epsilon} \end{aligned}$$

This last bound together with (29) prove that for any  $\epsilon \leq \epsilon'_0 = 4\epsilon_0$ , (since for positive reals  $b, c, 1 \wedge (b+c) \leq (1 \wedge b) + (1 \wedge c)$ ),

$$(33) \quad \begin{aligned} & \mathbb{P}(d_H(X_n, \mathbb{M}) > \epsilon) \\ & \leq 1 \wedge \frac{k\beta_p}{a'\epsilon^b} + \left(4^b \frac{1}{a\epsilon^b} \exp(-kpa/4^b\epsilon^b) \exp(\Lambda(n, p, \epsilon_n)) \mathbb{1}_{0 < \epsilon \leq \epsilon_n}\right) \wedge 1 \\ & + \left(4^b \frac{1}{a\epsilon_n^b} \exp(-kpa/4^b\epsilon_n^b) \exp(\Lambda(n, p, \epsilon_n)) \mathbb{1}_{\epsilon_n < \epsilon}\right) \wedge 1. \end{aligned}$$

The proof of Proposition 3.2 is thus complete.

**5.8. Proof of Corollary 3.3.** We have, a.s., since  $\mathbb{X}_n$  is a subset of  $\mathbb{M}$ ,  $d_H(\mathbb{X}_n, \mathbb{M}) \leq \text{diam}(\mathbb{M})$ . Hence  $\mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) \geq \epsilon) = 0$  for any  $\epsilon \geq C$  where  $C$  is a positive constant satisfying  $C > \max(\text{diam}(\mathbb{M}), \epsilon'_0)$ . We have (noting by  $a'' = a'/c$  and by  $cst$  a positive constant that does not depend on  $n$ ) and using (33),

$$\begin{aligned} \mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) &= \int_0^\infty \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) d\epsilon = \int_0^C \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) d\epsilon \\ &\leq \int_{\epsilon'_0}^C \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) d\epsilon + \int_0^{\epsilon'_0} \left(1 \wedge \frac{k\beta_p}{a'\epsilon^b}\right) d\epsilon \\ (34) \quad &+ cst \exp(\Lambda(n, p, \epsilon_n)) \int_0^{\epsilon'_0} \left(1 \wedge \frac{\exp(-a'pk\epsilon^b)}{a'\epsilon^b}\right) d\epsilon + \frac{cst}{\epsilon_n^b} \exp(-kpa'\epsilon_n^b) \exp(\Lambda(n, p, \epsilon_n)). \end{aligned}$$

We have from (33) and for  $n$  large enough such that  $\epsilon_n < \epsilon'_0$ ,

$$\begin{aligned} &\int_{\epsilon'_0}^C \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon) d\epsilon \leq C \mathbb{P}(d_H(\mathbb{X}_n, \mathbb{M}) > \epsilon'_0) \\ &\leq 1 \wedge \frac{k\beta_p}{a'\epsilon_0^b} + \frac{1}{a'\epsilon_n^b} \exp(-kpa'\epsilon_n^b) \exp(\Lambda(n, p, \epsilon_n)) \\ (35) \quad &\leq cst k\beta_p + \frac{1}{a'\epsilon_n^b} \exp(-kpa'\epsilon_n^b) \exp(\Lambda(n, p, \epsilon_n)). \end{aligned}$$

Hence, by (34), (35) and (19), we get for  $n$  large enough,

$$\begin{aligned} \mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) &\leq cst k\beta_p + \\ &\frac{2}{a'\epsilon_n^b} \exp(-kpa'\epsilon_n^b) \exp(\Lambda(n, p, \epsilon_n)) + cst \exp(\Lambda(n, p, \epsilon_n)) \left(\frac{\ln(pk)}{pk}\right)^{1/b} + \int_0^{\epsilon'_0} \left(1 \wedge \frac{k\beta_p}{a'\epsilon^b}\right) d\epsilon. \end{aligned}$$

Suppose now that  $p \leq \frac{n}{4}$ , then

$$\frac{n}{2} \geq pk \geq p\left(\frac{n}{2p} - 1\right) \geq \frac{n}{4}.$$

So that, for  $n$  large enough,

$$\begin{aligned} \mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) &\leq cst k\beta_p + \frac{cst}{a'\epsilon_n^b} \exp(-n\frac{a'}{4}\epsilon_n^b) \exp(\Lambda(n, p, \epsilon_n)) \\ (36) \quad &+ cst \exp(\Lambda(n, p, \epsilon_n)) \left(\frac{\ln n}{n}\right)^{1/b} + \int_0^{\epsilon'_0} \left(1 \wedge \frac{k\beta_p}{a'\epsilon^b}\right) d\epsilon. \end{aligned}$$

Let  $p_n \rightarrow \infty$  and  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  be such that  $p_n \leq n/4$ ,  $\frac{\exp(-n\frac{a'}{4}\epsilon_n^b)}{\epsilon_n^b} = O\left(\left(\frac{\ln n}{n}\right)^{1/b}\right)$  and that

$$\limsup_{n \rightarrow \infty} \Lambda(n, p_n, \epsilon_n) < \infty.$$

Then by (36), we obtain for  $n$  large enough,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \frac{n}{2p_n} \beta_{p_n} + \int_0^\infty \left(1 \wedge \left(\frac{\frac{n}{2p_n} \beta_{p_n}}{a'\epsilon^b}\right)\right) d\epsilon + cst \left(\frac{\ln n}{n}\right)^{1/b}.$$

The proof of Corollary 3.3 is then complete.



**5.9. Proof of Theorem 3.4.** The task now is to calculate the integral  $\int_0^{\epsilon'_0} \left(1 \wedge \frac{k\beta_p}{a'\epsilon^b}\right) d\epsilon$  and to deduce a bound for  $\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M}))$ . We do this by discussing on the values of  $b$ . We suppose that  $k\beta_p < 1$ ,

(1) If  $0 < b < 1$  then  $\int_0^{\epsilon'_0} \frac{1}{\epsilon^b} d\epsilon < \infty$  and

$$\int_0^{\epsilon'_0} \left(1 \wedge \frac{k\beta_p}{a'\epsilon^b}\right) d\epsilon \leq cst \, k\beta_p = cst \, (k\beta_p)^{\min(1, 1/b)}.$$

(2) If  $b > 1$  and  $\left(\frac{k\beta_p}{a'}\right)^{1/b} < \epsilon'_0$  then,

$$\int_0^{\epsilon'_0} \left(1 \wedge \frac{k\beta_p}{a'\epsilon^b}\right) d\epsilon \leq \left(\frac{k\beta_p}{a'}\right)^{1/b} + \frac{k\beta_p}{a'} \int_{\left(\frac{k\beta_p}{a'}\right)^{1/b}}^{\infty} \frac{1}{\epsilon^b} d\epsilon \leq cst \, (k\beta_p)^{1/b} = cst \, (k\beta_p)^{\min(1, 1/b)}.$$

(3) If  $b = 1$ ,  $k\beta_p \leq C_{k,p} < 1$  and  $\left(\frac{C_{k,p}}{a'}\right) < \epsilon'_0$ , then

$$\int_0^{\epsilon'_0} \left(1 \wedge \frac{k\beta_p}{a'\epsilon}\right) d\epsilon \leq \int_0^{\epsilon'_0} \left(1 \wedge \frac{C_{k,p}}{a'\epsilon}\right) d\epsilon \leq cst(C_{k,p} - C_{k,p} \ln(C_{k,p})).$$

So that, we obtain by Corollary 3.3 since  $k\beta_p \leq (k\beta_p)^{\min(1, 1/b)}$  (recall that  $k = k_n, p = p_n$  and that  $k_n\beta_{p_n} < C_{k_n, p_n} < 1$ ),

(1) if  $b \neq 1$  then,

$$(37) \quad \mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \left( (k\beta_p)^{\min(1, 1/b)} + \left(\frac{\ln n}{n}\right)^{1/b} \right).$$

(2) If  $b = 1$  then,

$$(38) \quad \mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \left( C_{k,p} - C_{k,p} \ln(C_{k,p}) + \frac{\ln n}{n} \right).$$

The task now is to choose, in the last bound, suitable values of  $p = p_n$ . Let  $s = \min(1/b, 1)$  and  $p = \lceil n^\alpha/4 \rceil$  for some  $\alpha \in ]0, 1]$  and  $k = \lfloor \frac{n}{2p} \rfloor$ . Recall that  $\beta_p = O(p^{-\gamma})$ . We have,

$$(39) \quad (k\beta_p)^s \leq cst \, n^{(1-\alpha)s - (\alpha\gamma)s}.$$

So that, we obtain when  $b \neq 1$  (thanks to (37)),

$$(40) \quad \mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \left( n^{(1-\alpha)s - (\alpha\gamma)s} + \left(\frac{\ln n}{n}\right)^{1/b} \right).$$

**A. Suppose that  $b \neq 1$  and  $\beta_n = O(n^{-\gamma})$  for some  $\gamma \geq \max(1, 1/b)$ .** Choose  $\alpha$  such that,

$$\frac{1 + 1/b}{1 + \gamma} \leq \frac{s + 1/b}{s(1 + \gamma)} \leq \alpha \leq 1,$$

(such a value of  $\alpha$  exists since  $s \leq 1$ ,  $s + 1/b \leq s(1 + \gamma)$  and  $\gamma \geq \max(1, 1/b)$ ). Hence, by (40),

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \left(\frac{\ln n}{n}\right)^{1/b} \left(1 + \frac{n^{(1-\alpha)s - \alpha\gamma s + 1/b}}{(\ln n)^{1/b}}\right).$$

Consequently, since  $(1 - \alpha)s - \alpha\gamma s + 1/b \leq 0$ ,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) = O\left(\left(\frac{\ln n}{n}\right)^{1/b}\right).$$

**B. Suppose that  $b \neq 1$  and  $\beta_n = O(n^{-\gamma})$  for some  $0 < \gamma < \max(1, 1/b)$ .** Each  $\alpha \in ]\frac{1}{1+\gamma}, 1]$  satisfies

$$(1 - \alpha)s - \alpha\gamma s < 0, \quad 1/b + (1 - \alpha)s - \alpha\gamma s > 0,$$

since  $s\gamma < 1/b$ . So that, for any  $\alpha \in ](1 + \gamma)^{-1}, 1]$ ,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq n^{(1-\alpha)s - \alpha\gamma s} \left(1 + \frac{(\ln n)^{1/b}}{n^{(1-\alpha)s - \alpha\gamma s + 1/b}}\right),$$

and that,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) = O(n^{(1-\alpha)s - \alpha\gamma s}).$$

**C. Suppose that  $b = 1$ .** In this case,  $s = 1$  and by (39),  $C_{k,p} = n^{1-\alpha-\alpha\gamma}$  (recall that  $C_{k,p}$  is an upper bound for  $k\beta_p$  less than one). So that, for any  $\alpha \in ](1 + \gamma)^{-1}, 1]$ ,  $1 - \alpha - \alpha\gamma < 0$  and Inequality (38) gives,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \left(n^{1-\alpha-\alpha\gamma} \ln(n) + \frac{\ln n}{n}\right)$$

so that,

$$\mathbb{E}(d_H(\mathbb{X}_n, \mathbb{M})) \leq cst \frac{\ln(n)}{n^{\min(-1+\alpha+\alpha\gamma, 1)}}.$$

The proof of Theorem 3.4 is complete.  $\square$

**5.10. Proof of Corollary 3.5.** According to Theorem 3.4, the rate  $O\left(\left(\frac{\ln n}{n}\right)^{1/b}\right)$  is reached as soon as,

- $\gamma \geq \max(1, 1/b)$ ,  $b \neq 1$  and  $\frac{s+1/b}{s(1+\gamma)} \leq \alpha \leq 1$ , or
- $\gamma > 0$ ,  $b = 1$ ,  $\frac{1}{1+\gamma} < \alpha \leq 1$  and  $-1 + \alpha + \alpha\gamma \geq 1$ .

The second case implies that  $\frac{2}{1+\gamma} \leq \alpha \leq 1$  so that  $\gamma \geq 1$ . This situation is included in the first case since, here,  $s = 1$  and  $b = 1$ . The proof of Corollary 3.5 is complete.

**5.11. Proof of Corollary 3.6.** Let  $\epsilon_n^b = \frac{1}{\sqrt{n^{1+\alpha}}}$ , for fixed  $\alpha \in ]0, 1[$ . Then,

$$\frac{1}{a\epsilon_n^b} \exp(-na\epsilon_n^b) \leq \sqrt{n^{1+\alpha}} \exp(-cst\sqrt{n^{1-\alpha}}) = O\left(\left(\frac{\ln n}{n}\right)^{1/b}\right).$$

We have thanks to (14),  $\Lambda(n, [n^\alpha/4], \epsilon_n) \leq cst n^{1+\alpha} \epsilon_n^{2b} \leq cst$ , by this choice of  $\epsilon_n^b$ . All the requirements of Theorem 3.4 are satisfied for this  $\alpha \in ]0, 1[$ . The proof of Corollary 3.6 is complete.

**5.12. Proof of Proposition 4.2.** Our main reference for this proof is [15]. We summarize Kato's results in the following proposition (but only for  $\beta \in ]-1, 1[$ , the case of our interest.) We note by,

$$\begin{aligned} A &= \sqrt{(1-\varphi)^2 + 4\beta^2\varphi} \\ \lambda_1 &= \frac{1}{2}(1 + \varphi - A) \\ \lambda_2 &= \frac{1}{2}(1 + \varphi + A) \end{aligned}$$

**Proposition 5.2.** *Consider the Möbius Markov chain (as introduced in Proposition 4.2). The following properties hold.*

- (1) *For any  $x$  in the unit circle  $\partial B(0, 1)$ , the conditional distribution of  $X_n$  given  $X_0 = x$  is, for each  $n \geq 1$ , the wrapped Cauchy on the unit circle  $C^*(\phi_n(x))$  i.e. having a density  $\pi_n$  (with respect to the arc length on the circle),*

$$\pi_n(z) = \frac{1}{2\pi} \frac{1 - |\phi_n(x)|^2}{|z - \phi_n(x)|^2},$$

where, if  $\beta \neq 0$  and  $\varphi > 0$  then

$$\phi_n(x) = \frac{\lambda_1^n(1 - \varphi + A)x + \lambda_2^n(\varphi - 1 + A)x + 2(\lambda_2^n - \lambda_1^n)\beta\varphi}{2(\lambda_2^n - \lambda_1^n)\beta x + \lambda_2^n(1 - \varphi + A) + \lambda_1^n(\varphi - 1 + A)},$$

if  $\beta = 0$  then  $\phi_n(x) = \varphi^n x$  and finally if  $\varphi = 0$  then  $\phi_n(x) = 0$  for each  $n \geq 1$ .

- (2) *The conditional distribution of  $X_n$  given  $X_0 = x$  converges in law, as  $n$  tends to infinity, to the wrapped Cauchy on the unit circle  $C^*(\phi_\infty)$  with,*

$$\phi_\infty = \frac{\varphi - 1 + A}{2\beta},$$

if  $\beta \neq 0$  and  $\phi_\infty = 0$  otherwise.

- (3) *The wrapped Cauchy on the unit circle  $C^*(\phi_\infty)$  is the unique invariant measure of this Möbius Markov chain (denoted by  $\mu$ ). Recall that  $C^*(\phi_\infty)$  has a density on the unit circle defined, for  $z \in \partial B(0, 1)$ , by*

$$\pi(z) = \frac{1}{2\pi} \frac{1 - |\phi_\infty|^2}{|z - \phi_\infty|^2}.$$

This Markov chain has, then, a unique invariant measure  $\mu$  on the unit circle. So that Assumption  $(\mathcal{A}_1)$  is satisfied with  $\mathbb{M} = \partial B(0, 1)$ . The task now is to check Assumption  $(\mathcal{A}_2)$ . We have, for  $x \in \partial B(0, 1)$ ,

$$(41) \quad K(x, dz) = \mathbb{P}(X_1 \in \nu(dz) | X_0 = x) = k(x, z)\nu(dz),$$

where  $\nu$  is the arc length measure on the unit circle and for  $x, z \in \partial B(0, 1)$ ,

$$(42) \quad k(x, z) = \frac{1}{2\pi} \frac{1 - |\phi_1(x)|^2}{|z - \phi_1(x)|^2},$$

with

$$\phi_1(x) = \frac{\varphi x + \beta\varphi}{\beta x + 1}, \quad |\phi_1(x)| = \varphi.$$

It is proved in [14] (see the proof of Proposition 5.3 there) that,

$$(43) \quad k(x, z) \geq \frac{1}{2\pi} \frac{1 - \varphi^2}{(1 + \varphi)^2} = \frac{1}{2\pi} \frac{1 - \varphi}{1 + \varphi} =: \kappa > 0,$$

and, for positive  $\epsilon_0$  sufficiently small,

$$(44) \quad V_d := \inf_{u \in \partial B(0,1)} \inf_{0 < \epsilon < \epsilon_0} \left( \epsilon^{-1} \int_{\partial B(0,1) \cap B(u,\epsilon)} \nu(dx_1) \right) > 0.$$

Assumption  $(\mathcal{A}_2)$  is satisfied with  $b = 1$ , thanks to (41), (43) and to (44). The requirements of Theorem 2.2 are satisfied thanks to Corollary 2.8. In particular  $\kappa_\epsilon = \kappa V_d \epsilon$ . From this, we deduce that  $a = \kappa V_d$ . The value of  $\theta^* = 1$  follows from Proposition 2.6. The conclusion of Theorem 2.2 holds. The proof of Proposition 4.2 is complete.

**5.13. Proof of Proposition 4.3.** We have now to prove the geometric ergodicity property. This allows to deduce that this Markov chain is  $\beta$ -mixing with geometrically decaying mixing coefficients  $(\beta_n)_{n \geq 1}$  (cf. for instance Theorem 3.7 in [3] and the references therein). We will prove this property, only, in the case when  $\beta \neq 0$  and  $\varphi > 0$  using Proposition 5.2 above (which is already proved in [15]). The others cases are much easier. Clearly, for a measurable subset  $A$  of the unit circle,  $\partial B(0, 1)$ ,

$$\mathbb{P}^n(x, A) - \mu(A) = \int_A \frac{1}{2\pi} \left( \frac{1 - |\phi_n(x)|^2}{|z - \phi_n(x)|^2} - \frac{1 - |\phi_\infty|^2}{|z - \phi_\infty|^2} \right) \nu(dz).$$

So that,

$$(45) \quad |\mathbb{P}^n(x, A) - \mu(A)| \leq \int_A \frac{1}{2\pi} \left| \frac{1 - |\phi_n(x)|^2}{|z - \phi_n(x)|^2} - \frac{1 - |\phi_\infty|^2}{|z - \phi_\infty|^2} \right| \nu(dz).$$

Now we have, (using the fact that  $|a_1 a_2 - b_1 b_2| \leq 2|a_1 - b_1| + 2|a_2 - b_2|$  for  $|a_1|, |a_2|, |b_1|, |b_2|$  all less than 2), for any  $z$  belonging to the unit circle,

$$(46) \quad \begin{aligned} & \left| \frac{1 - |\phi_n(x)|^2}{|z - \phi_n(x)|^2} - \frac{1 - |\phi_\infty|^2}{|z - \phi_\infty|^2} \right| \\ & \leq \left| \frac{(1 - |\phi_n(x)|^2)|z - \phi_\infty|^2 - |z - \phi_n(x)|^2(1 - |\phi_\infty|^2)}{|z - \phi_n(x)|^2 |z - \phi_\infty|^2} \right| \\ & \leq 2 \left| \frac{(1 - |\phi_n(x)|^2) - (1 - |\phi_\infty|^2)}{|z - \phi_n(x)|^2 |z - \phi_\infty|^2} \right| + 2 \left| \frac{|z - \phi_\infty|^2 - |z - \phi_n(x)|^2}{|z - \phi_n(x)|^2 |z - \phi_\infty|^2} \right| \\ & \leq 8 \frac{|\phi_n(x) - \phi_\infty|}{|z - \phi_n(x)|^2 |z - \phi_\infty|^2}. \end{aligned}$$

Now, for any  $x \in \partial B(0, 1)$ ,

$$\begin{aligned} & \frac{(\varphi - 1 + A)x + 2\beta\varphi}{2\beta x + 1 - \varphi + A} = \frac{\frac{(\varphi - 1 + A)}{2\beta}x + \varphi}{x + \frac{1 - \varphi + A}{2\beta}} \\ & = \frac{(\varphi - 1 + A)}{2\beta} \frac{x + \varphi \frac{2\beta}{(\varphi - 1 + A)}}{x + \frac{1 - \varphi + A}{2\beta}} = \phi_\infty, \end{aligned}$$

since, thanks to the expression of  $A$ ,  $\varphi \frac{2\beta}{(\varphi-1+A)} = \frac{1-\varphi+A}{2\beta}$ . So that, for  $\tilde{b} = \frac{\lambda_1}{\lambda_2} \in ]0, 1[$ ,

$$\begin{aligned} & \phi_n(x) - \phi_\infty \\ &= \frac{\tilde{b}^n(1-\varphi+A)x + (\varphi-1+A)x + 2\beta\varphi(1-\tilde{b}^n)}{2\beta x(1-\tilde{b}^n) + (1-\varphi+A) + \tilde{b}^n(\varphi-1+A)} - \frac{(\varphi-1+A)x + 2\beta\varphi}{2\beta x + 1 - \varphi + A} \\ &=: \frac{\alpha(x)\tilde{b}^n + \beta(x)}{\gamma(x)\tilde{b}^n + \mu(x)} - \frac{\beta(x)}{\mu(x)} \\ &= \frac{(\alpha(x) - \gamma(x)\frac{\beta(x)}{\mu(x)})\tilde{b}^n}{\gamma(x)\tilde{b}^n + \mu(x)} = \frac{(\alpha(x) - \gamma(x)\phi_\infty)\tilde{b}^n}{\gamma(x)\tilde{b}^n + \mu(x)}. \end{aligned}$$

We obtain (recall that  $x$  is in the unit circle and can be seen as a complex number with  $|x| = 1$ ),

$$|\phi_n(x) - \phi_\infty| \leq \frac{3A + 2\beta + \varphi - 1}{|2\beta - \varphi + 1 - A|\tilde{b}^n + |2\beta + \varphi - 1 - A|} \tilde{b}^n$$

We deduce from the last equality, that there exists a positive constant  $c$  (independent of  $x$ ) such that for any  $n \in \mathbb{N} \setminus \{0\}$

$$|\phi_n(x) - \phi_\infty| \leq c\tilde{b}^n.$$

The last bound together with (45) and (46) give,

$$|\mathbb{P}^n(x, A) - \mu(A)| \leq 8c\tilde{b}^n \int_A \frac{1}{2\pi} \frac{1}{|z - \phi_n(x)|^2 |z - \phi_\infty|^2} \nu(dz).$$

Recall that, for any  $z \in \partial B(0, 1)$ ,  $|z - \phi_n(x)| \geq |1 - |\phi_n(x)||$ . Hence, (recall that  $\phi_\infty \neq 1$  and that  $\phi_n(x) \neq 1$ ),

$$\begin{aligned} |\mathbb{P}^n(x, A) - \mu(A)| &\leq \frac{8c\tilde{b}^n}{(1 - |\phi_n(x)|)^2} \int_A \frac{1}{2\pi} \frac{1}{|z - \phi_\infty|^4} \nu(dz) \\ &\leq \frac{8c}{(1 - |\phi_n(x)|)^2 |1 - \phi_\infty|^4} \tilde{b}^n. \end{aligned}$$

Since  $\tilde{b} \in ]0, 1[$  and  $\sup_{n \in \mathbb{N}} |\phi_n(x)| < \infty$ , the bound (11) is satisfied. The purpose now is to check (14) (with  $b = 1$ ). We have,

$$\begin{aligned} & u^{-2} \mathbb{P}_\mu(\|X_1 - x\| \leq u, \|X_{r+1} - x\| \leq u) \\ &= u^{-2} \mathbb{P}_\mu(\|X_1 - x\| \leq u, \mathbb{P}_{X_1, X_r}(\|X_{r+1} - x\| \leq u)) \\ &= u^{-2} \mathbb{P}_\mu(\|X_1 - x\| \leq u, \mathbb{P}_{X_r}(\|X_{r+1} - x\| \leq u)) \\ (47) \quad &= u^{-2} \mathbb{P}_\mu(\|X_1 - x\| \leq u, \mathbb{P}_{X_0}(\|X_1 - x\| \leq u)) \end{aligned}$$

Recall first (arguing as in the proof of (5.5) in [14]) that,  $\nu(\partial B(0, 1) \cap B(x, u)) \leq cst u$ , for any  $x \in \partial B(0, 1)$ . We also have (recall that the conditional density of  $X_1$  known  $X_0 = x_0$  is given

in 42),

$$\begin{aligned}
\mathbb{P}_{x_0}(\|X_1 - x\| \leq u) &= \int_{\partial B(0,1) \cap B(x,u)} \frac{1}{2\pi} \frac{1 - |\phi_1(x)|^2}{|z - \phi_1(x)|^2} \nu(dz) \\
&\leq \int_{\partial B(0,1) \cap B(x,u)} \frac{1}{2\pi} \frac{1 - |\phi_1(x)|^2}{(|z| - |\phi_1(x)|)^2} \nu(dz) \\
&\leq \int_{\partial B(0,1) \cap B(x,u)} \frac{1}{2\pi} \frac{1 - \varphi^2}{(1 - \varphi)^2} \nu(dz) \\
(48) \quad &\leq \frac{1}{2\pi} \frac{1 + \varphi}{1 - \varphi} \nu(\partial B(0,1) \cap B(x,u)) \leq cst u,
\end{aligned}$$

and,

$$(49) \quad \mathbb{P}_\mu(\|X_1 - x\| \leq u) = \int_{\partial B(0,1)} \mathbb{P}_{x_0}(\|X_1 - x\| \leq u) \mu(dx_0) \leq cst u.$$

Condition (14) with  $b = 1$  follows from (47), (48) and (49). We complete the proof of Proposition 4.3 by applying Corollary 3.6 together with Corollary 3.5 (recall that, here  $b = 1$  and  $\gamma$  is any real greater than one since  $\beta_n = O(e^{-cn})$ , for some positive  $c$ ).

**5.14. Proof of Proposition 4.4.** Recall that if  $\epsilon$  is a random variable uniformly distributed over  $[0, 1]$  then for any  $u \in [0, 1]$ , the random variable  $u + \epsilon - [u + \epsilon]$  is also uniformly distributed over  $[0, 1]$ . From this, we deduce that the stationary Markov chain  $(\Phi_n)_{n \geq 0}$  has the uniform over the unit square stationary distribution. Let us now prove the second statement of Proposition 4.4. We have for any  $u, v, x, y \in [0, 1[$  (recall that the two random variables  $\epsilon_1, \eta_1$  are independent and that the two couples  $(\epsilon_1, \eta_1)$  and  $(\theta_0, \phi_0)$  are independent),

$$\begin{aligned}
&\mathbb{P} \left( \sqrt{|\theta_1 - x|^2 + |\phi_1 - y|^2} \leq \epsilon \mid \theta_0 = u, \phi_0 = v \right) \\
&= \mathbb{P} (|u + \epsilon_1 - [u + \epsilon_1] - x|^2 + |v + \eta_1 - [v + \eta_1] - y|^2 \leq \epsilon^2) \\
&\geq \mathbb{P} (|u + \epsilon_1 - [u + \epsilon_1] - x|^2 \leq \epsilon^2/2, |v + \eta_1 - [v + \eta_1] - y|^2 \leq \epsilon^2/2) \\
(50) \quad &\geq \mathbb{P} (|u + \epsilon_1 - [u + \epsilon_1] - x|^2 \leq \epsilon^2/2) \mathbb{P} (|v + \eta_1 - [v + \eta_1] - y|^2 \leq \epsilon^2/2)
\end{aligned}$$

Let us control (letting  $\epsilon' = \epsilon/\sqrt{2}$ ),  $\mathbb{P} (|u + \epsilon_1 - [u + \epsilon_1] - x| \leq \epsilon')$ . Clearly,

$$\begin{aligned}
I &:= \mathbb{P} (|u + \epsilon_1 - [u + \epsilon_1] - x| \leq \epsilon') \\
&= \mathbb{P} (-\epsilon' + x \leq u + \epsilon_1 - [u + \epsilon_1] \leq \epsilon' + x).
\end{aligned}$$

- If  $-\epsilon' + x \leq 0 \leq 1 \leq \epsilon' + x$  then  $I = 1$ , so that  $I \geq \epsilon'$ .
- If  $0 \leq -\epsilon' + x \leq \epsilon' + x \leq 1$  then (recall that  $u + \epsilon_1 - [u + \epsilon_1]$  is uniformly distributed over  $[0, 1]$ ),  $I = 2\epsilon'$ .
- If  $-\epsilon' + x \leq 0 \leq \epsilon' + x \leq 1$  then,

$$\begin{aligned}
I &= \mathbb{P} (-\epsilon' + x \leq u + \epsilon_1 \leq \epsilon' + x, u + \epsilon_1 < 1) \\
&\quad + \mathbb{P} (-\epsilon' + x \leq u + \epsilon_1 - 1 \leq \epsilon' + x, u + \epsilon_1 \geq 1) \\
&= \mathbb{P} (0 \leq \epsilon_1 \leq \epsilon' + x - u) \\
&\quad + \mathbb{P} (u - \epsilon' - x \leq 1 - \epsilon_1 \leq u + \epsilon' - x, u \geq 1 - \epsilon_1).
\end{aligned}$$

– If  $0 < \epsilon' + x - u$  (we already have  $\epsilon' + x - u < 1$ ,  $x \in [0, 1[$ ) then,

$$\begin{aligned} I &= \mathbb{P}(0 \leq \epsilon_1 \leq \epsilon' + x - u) + \mathbb{P}(0 \leq 1 - \epsilon_1 \leq u) \\ &= \epsilon' + x - u + u \geq \epsilon'. \end{aligned}$$

– If  $\epsilon' + x - u < 0$  then,

$$\begin{aligned} I &= \mathbb{P}(u - \epsilon' - x \leq 1 - \epsilon_1 \leq u + \epsilon' - x, u \geq 1 - \epsilon_1) \\ &= \mathbb{P}(u - \epsilon' - x \leq 1 - \epsilon_1 \leq u) = \epsilon' + x \geq \epsilon'. \end{aligned}$$

• If  $0 \leq -\epsilon' + x \leq 1 \leq \epsilon' + x$  then,

$$\begin{aligned} I &= \mathbb{P}(-\epsilon' + x \leq u + \epsilon_1 \leq \epsilon' + x, u + \epsilon_1 < 1) \\ &\quad + \mathbb{P}(-\epsilon' + x \leq u + \epsilon_1 - 1 \leq \epsilon' + x, u + \epsilon_1 \geq 1) \\ &= \mathbb{P}(-\epsilon' + x - u \leq \epsilon_1 \leq 1 - u) \\ &\quad + \mathbb{P}(u - \epsilon' - x \leq 1 - \epsilon_1 \leq u + \epsilon' - x, u + \epsilon_1 \geq 1). \end{aligned}$$

– If  $0 \leq -\epsilon' + x - u$  then (recall that  $1 - x \geq 0$ ),

$$\begin{aligned} I &= \mathbb{P}(-\epsilon' + x - u \leq \epsilon_1 \leq 1 - u) = 1 - u - (-\epsilon' + x - u) \\ &= 1 - x + \epsilon' \geq \epsilon'. \end{aligned}$$

– If  $-\epsilon' + x - u \leq 0$  then (recall that  $u - \epsilon' - x \leq 1 - \epsilon' - x \leq 0$ ),

$$I = \mathbb{P}(0 \leq \epsilon_1 \leq 1 - u) + \mathbb{P}(0 \leq 1 - \epsilon_1 \leq u) = 1 - u + u = 1 \geq \epsilon'.$$

So that we always have  $I \geq \epsilon'$  and

$$\mathbb{P}(|u + \epsilon_1 - [u + \epsilon_1] - x|^2 \leq \epsilon^2/2) \mathbb{P}(|v + \eta_1 - [v + \eta_1] - y|^2 \leq \epsilon^2/2) \geq \epsilon^2/2.$$

This last bound together with (50) prove the second item of Proposition 4.4. This immediately implies that the distribution of  $(\theta_1, \phi_1)$  satisfies the  $(a, b)$ -standard assumption with  $a = 1/2$  and  $b = 2$ . The bound (7) of Proposition 2.6 is satisfied and proves the last item of Proposition 4.4. The proof of Proposition 4.4 is now complete.

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SANA LOUHICHI: UNIV. GRENOBLE ALPES, CNRS, GRENOBLE INP\*, LJK 38000 GRENOBLE, FRANCE. \*INSTITUT OF ENGINEERING UNIV. GRENOBLE ALPES, 700 AVENUE CENTRALE, 38401 SAINT-MARTIN-D’HÈRES, FRANCE.

*Email address:* [sana.louhichi@univ-grenoble-alpes.fr](mailto:sana.louhichi@univ-grenoble-alpes.fr)