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# Small-time controllability on the group of diffeomorphisms for Schrödinger equations

Karine Beauchard\*, Eugenio Pozzoli†

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## Abstract

In this work, we establish a link between the small-time approximate controllability of bilinear Schrödinger PDEs (posed on a boundaryless Riemannian manifold  $M$ ) and the control in the group  $\text{Diff}_c^0(M)$  of the diffeomorphisms, isotopic to the identity and with compact support, of the underlying manifold  $M$ .

More precisely, under a density assumption on the Lie algebra generated by the control potential and the Laplacian, we show that compositions  $|J_P|^{1/2}(\psi_0 \circ P)$  of the initial wavefunction  $\psi_0 \in L^2(M, \mathbb{C})$  with any diffeomorphism  $P \in \text{Diff}_c^0(M)$  can be approximately reached, in arbitrarily small times, by controlled solutions of the Schrödinger equation (here,  $|J_P|$  denotes the determinant of the Jacobian of  $P$ ). We illustrate this property on two examples, posed respectively on the torus  $\mathbb{T}^d$  and on the euclidean space  $\mathbb{R}^d$ .

As a physical application, we obtain in particular the small-time approximate control of the quantum particle's averaged positions. This yields also new small-time approximate controllability properties between families of eigenstates on  $\mathbb{T}^d$ .

To prove the result, we first construct solutions of the Schrödinger equation that approximately evolve, arbitrarily fast, along any unitary transport flow on  $L^2(M, \mathbb{C})$ . In this way, we control the composition with any diffeomorphism that can be decomposed as a product of flows on  $M$ . We then combine this property with a result of Thurston on the simplicity of the group  $\text{Diff}_c^0(M)$  to conclude.

## 1 Introduction

### 1.1 The model

Let  $M$  be a boundaryless Riemannian manifold (for instance  $M = \mathbb{T}^d$  or  $\mathbb{R}^d$  with  $d \in \mathbb{N}^*$ ). We consider the initial value problem for Schrödinger equations of the form

$$\begin{cases} i\partial_t \psi(t, x) = \left( -\Delta + V(x) + \sum_{j=1}^m u_j(t) W_j(x) \right) \psi(t, x), & (t, x) \in (0, T) \times M, \\ \psi(0, \cdot) = \psi_0, \end{cases} \quad (1)$$

where  $\Delta$  is the Laplace-Beltrami operator of  $M$ , the functions  $V, W_1, \dots, W_m : M \rightarrow \mathbb{R}$  are real valued potentials, and the functions  $u_1, \dots, u_m : (0, T) \rightarrow \mathbb{R}$  are real valued controls.

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The time-independent part  $-\Delta + V$  is usually referred to as *the drift*. The time-dependent potential  $\sum_{j=1}^m u_j(t)W_j(x)$  is possibly unbounded on  $L^2(M, \mathbb{C})$ .

System (1) describes the dynamics of a quantum particle on the manifold  $M$ , with free (kinetic plus potential) energy  $-\Delta + V$ , in interaction with additional external fields with potentials  $W_j$  that can be switched on and off. It is used to model a variety of physical situations, such as atoms in optical cavities, and molecular dynamics.

When well defined, the solution of (1) at time  $t$ , associated with a time-dependent function  $u = (u_1, \dots, u_m)$ , and an initial state  $\psi_0$ , is denoted by  $\psi(t; u, \psi_0)$  and lives in the unitary sphere  $\mathcal{S}$  of  $L^2(M)$

$$\mathcal{S} := \{\psi \in L^2(M, \mathbb{C}) ; \|\psi\|_{L^2(M)} = 1\}. \quad (2)$$

Our goal is to investigate where the system can be steered in arbitrarily small times, when arbitrarily large controls are allowed. More precisely, we introduce the notion of small-time (approximately) reachable states.

**Definition 1** (Small time  $\mathcal{H}$ -approximate controllability). *Let  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$  be a normed space, subset of  $L^2(M, \mathbb{C})$ .*

*For  $\psi_0, \psi_1 \in \mathcal{H} \cap \mathcal{S}$ , we say that  $\psi_1$  is small-time  $\mathcal{H}$ -approximately (resp. exactly) reachable from  $\psi_0$  if for any  $\varepsilon > 0$ , there exist a time  $T \in [0, \varepsilon]$ , a global phase  $\theta \in [0, 2\pi)$  and a control  $u : [0, T] \rightarrow \mathbb{R}^m$  such that*

- *the Cauchy problem (1) has a unique solution  $\psi \in C^0([0, T], \mathcal{H})$ , and*
- *$\|\psi(T; u, \psi_0) - e^{i\theta}\psi_1\|_{\mathcal{H}} < \varepsilon$  (resp.  $\psi(T; u, \psi_0) = e^{i\theta}\psi_1$ ).*

*We denote with  $\text{Adh}_{\mathcal{H}}\text{Reach}_{\text{st}}(\psi_0)$  (resp.  $\text{Reach}_{\text{st}}(\psi_0)$ ) the set of small-time  $\mathcal{H}$ -approximately (resp. exactly) reachable states from  $\psi_0$ .*

Note that the wavefunction is defined up to global phases, hence the state  $e^{i\theta}\psi_1$  for some constant  $\theta \in \mathbb{R}$ , is physically the same as  $\psi_1$ .

Since the beginning of quantum control, time-optimality has represented a challenge both in physics and mathematics (see, e.g., the pioneering work [34]).

Our goal here is to provide insights on the sets  $\text{Adh}_{\mathcal{H}}\text{Reach}_{\text{st}}(\psi_0)$ ,  $\psi_0 \in \mathcal{S}$ , which describes the states that can be reached *instantaneously*, i.e. in time approximately zero. More generally, we wish to elucidate its relation with the Lie algebra generated by  $\Delta - V, W_1, \dots, W_m$ .

## 1.2 A motivating example and a question

We consider a  $d$ -dimensional quantum harmonic oscillator, with tunable frequency and position of the quadratic potential. This system is governed by (1), with

$$M = \mathbb{R}^d, \quad V = 0, \quad m = d + 1, \quad W_j(x) = x_j \text{ for } j = 1, \dots, d, \quad W_{d+1}(x) = |x|^2. \quad (3)$$

Notice that we do not assume  $u_{d+1}$  to be positive. Classical and elementary arguments (see Proposition 51 in the Appendix) prove that, for any  $\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C})$ ,

$$\{\sigma^{d/2}\psi_0(\sigma(x - q)); \sigma > 0, q \in \mathbb{R}^d\} \subset \text{Adh}_{L^2}\text{Reach}_{\text{st}}(\psi_0). \quad (4)$$

It is thus possible to reach in small time the compositions of the initial data by any diffeomorphism of  $\mathbb{R}^d$  of the form  $P_{\sigma, q}(x) = \sigma(x - q)$ . Physically, it corresponds to the small-time control of the spread  $\sigma$  and the position  $q$  of the state  $\psi_0$ . In short, we will say that it is possible to "control in small time the family of diffeomorphisms  $\{P_{\sigma, q}; \sigma > 0, q \in \mathbb{R}^d\}$ ". A natural question that we answer in this work is then the following.

(Q) Are there equations of the form (1) for which it is possible to control, in small times, the whole group of diffeomorphisms of the underlying manifold?

It is surely not possible for (3), as shown in [14] (see, in particular, Theorem 23 part 3). The intuition behind this obstruction is that the Lie algebra generated by  $i\Delta, i|x|^2, ix_1, \dots, ix_m$  is finite-dimensional, and spanned e.g. as

$$\text{Span}_{\mathbb{R}}\{i\Delta, i|x|^2, \langle x, \nabla \rangle, ix_1, \dots, ix_d, \partial_{x_1}, \dots, \partial_{x_d}, i\}.$$

### 1.3 Results

**Definition 2** ( $\text{Diff}_c^0(M)$ ). We denote with  $\text{Diff}_c^0(M)$  the group of diffeomorphisms, of a smooth manifold  $M$ , which are  $C^\infty$ , isotopic to the identity, and with compact support (i.e. for every  $P \in \text{Diff}_c^0(M)$  there exists  $K \subset M$  compact such that  $P = \text{Id}$  on  $M \setminus K$ ).

**Definition 3** (Unitary composition operator  $\mathcal{L}_P$ ). For  $P \in \text{Diff}_c^0(M)$ , we denote with  $\mathcal{L}_P$  the unitary composition operator associated with  $P$ :

$$\mathcal{L}_P : \psi \in L^2(M, \mathbb{C}) \mapsto |J_P|^{1/2}(\psi_0 \circ P) \in L^2(M, \mathbb{C}). \quad (5)$$

where  $J_P$  is the determinant of the Jacobian matrix of  $P$ . This operator preserves the  $L^2(M, \mathbb{C})$ -norm:  $\|\mathcal{L}_P \psi\|_{L^2} = \|\psi\|_{L^2}$ .

A more precise formulation of the question we study in this work is then the following.

(Q') Are there equations of the form (1) such that,

$$\forall \psi_0 \in L^2(M, \mathbb{C}), \quad \{\mathcal{L}_P \psi_0; P \in \text{Diff}_c^0(M)\} \subset \text{Adh}_{L^2} \text{Reach}_{\text{st}}(\psi_0)? \quad (6)$$

We shall furnish a positive answer to (Q') for two examples of equations of the form (1). We shall also highlight some physically relevant consequences of this property, such as the control among particular steady states.

**A system on  $M = \mathbb{T}^d$ .** The first example is the following Schrödinger equation

$$\begin{cases} i\partial_t \psi(t, x) = \left( -\Delta + V(x) + \sum_{j=1}^d (u_{2j-1}(t) \sin + u_{2j}(t) \cos) \langle b_j, x \rangle \right) \psi(t, x), & (t, x) \in (0, T) \times \mathbb{T}^d, \\ \psi(0, \cdot) = \psi_0, \end{cases} \quad (7)$$

where  $V \in L^\infty(\mathbb{T}^d, \mathbb{R})$ , and

$$b_1 = (1, 0, \dots, 0), \quad b_2 = (0, 1, \dots, 0), \quad b_{d-1} = (0, \dots, 1, 0), \quad b_d = (1, \dots, 1). \quad (8)$$

This system is inspired by the recent results obtained in [29]. In the one-dimensional case  $d = 1$ , this system describes the orientation in the plane of a rigid molecule, controlled by the dipolar interactions with two orthogonal electric fields of constant direction and time-variable intensity. It is used in physics and chemistry as a model for rotational molecular dynamics (see, e.g., [35] and reference therein). We obtain that compositions of the initial wavefunction with any diffeomorphism are small-time approximately reachable states.

**Theorem 4.** Let  $V \in L^\infty(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for every  $P \in \text{Diff}^0(\mathbb{T}^d)$ ,  $\psi_0 \in L^2(\mathbb{T}^d, \mathbb{C})$  and  $\varepsilon > 0$ , there exist  $T \in [0, \varepsilon]$ ,  $\theta \in \mathbb{R}$  and  $u : [0, T] \rightarrow \mathbb{R}^{2d}$  piecewise constant such that

$$\|\psi(T; u, \psi_0) - e^{i\theta} \mathcal{L}_P \psi_0\|_{L^2(\mathbb{T}^d)} < \varepsilon.$$

In other words, for every  $\psi_0 \in L^2(\mathbb{T}^d, \mathbb{C})$  we have

$$\{\mathcal{L}_P \psi_0; P \in \text{Diff}^0(\mathbb{T}^d)\} \subset \text{Adh}_{L^2(\mathbb{T}^d)} \text{Reach}_{st}(\psi_0).$$

For  $k \in \mathbb{Z}^d$  and  $J \subset \{1, \dots, d\}$ , the function  $f_{k,J} : \mathbb{T}^d \rightarrow \mathbb{R}$  defined by

$$f_{k,J}(x) := \prod_{j \in J} \sin(k_j x_j) \prod_{j \in J^c} \cos(k_j x_j). \quad (9)$$

is an eigenfunction of the Laplacian operator  $\Delta$  with (degenerate) eigenvalue  $-|k|^2$ .

As an application of Theorem 4, by choosing specific diffeomorphisms of the torus (i.e. rotations), we obtain the following result of small-time approximate controllability between particular eigenstates  $f_{k,J}$ . Notice that eigenstates are particularly relevant in physics as they represent quantum steady states.

**Corollary 5.** Let  $V \in L^\infty(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for any  $k \in \mathbb{Z}^d$ ,  $J_1, J_2 \subset \{1, \dots, d\}$ , and  $\varepsilon > 0$ , there exist  $T \in [0, \varepsilon]$ ,  $\theta \in \mathbb{R}$  and  $u : [0, T] \rightarrow \mathbb{R}^{2d}$  piecewise constant such that

$$\|\psi(T; u, f_{k,J_1}) - e^{i\theta} f_{k,J_2}\|_{L^2(\mathbb{T}^d)} < \varepsilon.$$

In other words, for every  $k \in \mathbb{Z}^d$  and  $J \subset \{1, \dots, d\}$ , we have

$$\{f_{k,J'}, J' \subset \{1, \dots, d\}\} \subset \text{Adh}_{L^2(\mathbb{T}^d)} \text{Reach}_{st}(f_{k,J}).$$

To prove Corollary 5, it suffices to apply Theorem 4 with initial state  $\psi_0 = f_{k,J_1}$  and diffeomorphism  $P$  defined as

$$\begin{cases} x_j \mapsto x_j + \pi/(2k_j) & \text{if } j \in (J_1 \setminus J_2) \cup (J_2 \setminus J_1) \\ x_j \mapsto x_j & \text{otherwise.} \end{cases}$$

Under appropriate regularity assumptions on  $V$ , the  $L^2$ -norm can be replaced by more regular norms in the conclusions of Theorem 4 and Corollary 5 (see Theorem 35 and Corollary 36).

**A system on  $M = \mathbb{R}^d$ .** The second example is the following Schrödinger equation

$$\begin{cases} i\partial_t \psi(t, x) = \left( -\Delta + V(x) + \sum_{j=1}^d u_j(t) x_j + u_{d+1}(t) e^{-|x|^2/2} \right) \psi(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\ \psi(0, \cdot) = \psi_0, \end{cases} \quad (10)$$

where

$$V \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}) \quad \text{and} \quad \exists a, b \geq 0, \forall x \in \mathbb{R}^d, V(x) \geq -a|x|^2 - b. \quad (11)$$

This system is inspired by the recent results obtained in [30]. It models the dipolar interaction of a quantum particle with controls coupling to its positions  $x_j$ , and an additional control concentrated around the origin as a Gaussian function. Analogously to Theorem 4, we obtain the following result.

**Theorem 6.** *Let  $V$  satisfying (11). System (10) satisfies the following property: for every  $P \in \text{Diff}_c^0(\mathbb{R}^d)$ ,  $\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C})$  and  $\varepsilon > 0$ , there exist  $T \in [0, \varepsilon]$ ,  $\theta \in \mathbb{R}$  and  $u : [0, T] \rightarrow \mathbb{R}^{d+1}$  piecewise constant such that*

$$\|\psi(T; u, \psi_0) - e^{i\theta} \mathcal{L}_P \psi_0\|_{L^2(\mathbb{R}^d)} < \varepsilon. \quad (12)$$

*In other words, for every  $\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C})$  we have*

$$\{\mathcal{L}_P \psi_0; P \in \text{Diff}_c^0(\mathbb{R}^d)\} \subset \text{Adh}_{L^2(\mathbb{R}^d)} \text{Reach}_{\text{st}}(\psi_0).$$

**Definition 7** (Averaged position/momentum). *For  $\psi \in H^1(\mathbb{R}^d, \mathbb{C})$  such that  $|x|\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ , we define its averaged position  $\langle x \rangle \psi \in \mathbb{R}^d$  and its average momentum  $\langle p \rangle \psi \in \mathbb{R}^d$  by*

$$\langle x \rangle \psi = \int_{\mathbb{R}^d} x |\psi(x)|^2 dx, \quad \langle p \rangle \psi = \int_{\mathbb{R}^d} i \nabla \psi(x) \overline{\psi(x)} dx.$$

As an application of (the proof of) Theorem 6, we obtain the small-time approximate simultaneous controllability of the quantum particle's averaged position and momentum.

**Corollary 8.** *Let  $\psi_0 \in H^1(\mathbb{R}^d, \mathbb{C})$  be such that  $|x|\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\bar{x}, \bar{p} \in \mathbb{R}^d$  and  $\varepsilon > 0$ . There exist  $T \in [0, \varepsilon]$  and  $u : [0, T] \rightarrow \mathbb{R}^{d+1}$  piecewise constant such that*

$$|\langle x \rangle \psi(T, u, \psi_0) - \bar{x}| < \varepsilon \quad \text{and} \quad |\langle p \rangle \psi(T, u, \psi_0) - \bar{p}| < \varepsilon.$$

The latter corollary is obtained in particular by using as specific diffeomorphisms the translations along the  $x_j$ -axis: such diffeomorphisms do not have compact support, but we can nevertheless reach them, as it will be clear from the proof of Theorem 6.

Under appropriate regularity assumptions on  $V$ , the  $L^2$ -norm can be replaced by more regular norms in the conclusions of Theorem 6, see Theorem 44.

## 1.4 Proof strategy

We will use the following notation for flows.

**Definition 9** (Flows  $\phi_f^s$ ). *For a smooth (time-independent) vector field  $f$  on  $M$ , then  $\phi_f^s$  denotes the flow associated with  $f$  at time  $s$ : for every  $x_0 \in M$ ,  $x(s) = \phi_f^s(x_0)$  is the solution (when well defined) of*

$$\begin{cases} \frac{dx}{ds}(s) = f(x(s)), \\ x(0) = x_0. \end{cases}$$

The main novelty of this paper, from the point of view of the technique, is in linking the problem of small-time global approximate controllability of Schrödinger PDEs (on a manifold  $M$ ) with the problem of global controllability in the group of diffeomorphisms of  $M$ .

Roughly speaking, the strategy consists first in showing that, given any smooth and compactly supported real valued function  $\varphi = \varphi(x)$  on  $M$ , there exists a control function  $u = u(t)$  such that the solution  $\psi$  to the Schrödinger equation (7) or (10) is approximately solving a unitary (i.e.,  $L^2$ -norm-preserving) transport equation along the gradient vector field  $f = \nabla \varphi$ , of the form (for a rigorous statement, we refer to Propositions 23 and 31)

$$\begin{cases} \partial_t \psi(t, x) = \langle f(x), \nabla_x \psi(t, x) \rangle + \frac{1}{2} \text{div} f(x) \psi(t, x), \\ \psi(0, \cdot) = \psi_0. \end{cases}$$

The dynamics can then be reinitialized from a new initial state of the form  $\mathcal{L}_P \psi_0$ , where  $P = \phi_f^1$ , and then controlled again along another arbitrary gradient vector field, and so on.

The arbitrariness of the potential function  $\varphi$  is a consequence of the local phase control proved in the recent works [29,30]: i.e., the small-time approximate reachability from  $\psi_0$  of any state of the form  $e^{i\varphi}\psi_0, \varphi \in L^2(M, \mathbb{C})$  (we recalled it in Propositions 22 and 30). This fact is itself a consequence of a property of density (in an appropriate functional space) of a particular subspace of  $\text{Lie}\{\Delta - V, W_1, \dots, W_m\}$ , shared by both systems (7) and (10) (for the rigorous definition of this property, we refer to the proofs of Propositions 22, 30).

Once we have proved the reachability of gradient vector fields, we also show the reachability of their Lie brackets (Propositions 24, 32); then, the Lie algebra generated by gradient vector fields turns out to be dense in the algebra of all smooth compactly supported vector fields (as we prove in Propositions 26, 34). This means that, for any smooth vector field  $f$  on  $M$  with compact support (not necessarily a gradient), we can approximately reach in small times the state  $\mathcal{L}_P\psi_0$  for  $P = \phi_f^1$ , and analogously for products of them. Hence, given any diffeomorphism  $P$  that can be written as a finite product of flows of vector fields, we can approximate in small time the state  $\mathcal{L}_P\psi_0$ .

Thanks to a result of Thurston on the simplicity of the group  $\text{Diff}_c^0(M)$  [46], every diffeomorphism isotopic to the identity and with compact support can be decomposed as a finite product of flows of vector fields, hence Theorems 4 and 6 follow.

## 1.5 Literature review

The mathematical control theory of bilinear Schrödinger PDEs has undergone a vast development in the last two decades. Such theoretical problems find in fact their origins in applications of quantum control, originally in physics and chemistry (e.g. absorption spectroscopy), and more lately in informatics and engineering (e.g. quantum computation). We report here on the mathematical advances on this subject.

### 1.5.1 Obstruction to exact controllability

Since the seminal work [7] of Ball, Marsden and Slemrod on the bilinear control of PDEs, it is known that if the drift generates a group of bounded operators on an Hilbert space, where the control operators are bounded, then exact controllability is never achieved. This obstruction is due to the fact that the reachable set has empty interior (see also [19,26,27] for recent developments). This obstruction holds e.g. for the system (7), in any  $H^s(\mathbb{T}^d, \mathbb{C})$ ,  $s \in \mathbb{N}$  with controls in  $L_{loc}^1(\mathbb{R}, \mathbb{R}^m)$ . Different notions of controllability have then been investigated, such as exact controllability in more regular spaces, or approximate controllability. In this paper we analyse the latter one.

### 1.5.2 Exact controllability in more regular spaces

The case of a 1D Schrödinger equation with Dirichlet boundary conditions was the first one where positive results were obtained: local exact controllability was first proved in [8, 9] with Nash-Moser techniques, and then in [13] with a classical inverse mapping theorem. Other results of global (resp. local) exact controllability in regular spaces have followed, for different models, in [40,42] (resp. [18]).

### 1.5.3 Approximate controllability

Global approximate controllability of bilinear Schrödinger equations were firstly obtained in [24,32,41]. If the drift  $-\Delta + V$  has compact resolvent, (1) is known to be globally approx-

imately controllable in large times, generically w.r.t. the potentials  $V, W_0, \dots, W_m$  [36], for a manifold  $M$  of arbitrary dimension.

E.g., in the one-dimensional case, global approximate controllability in large times of (7) with  $V = 0$  was established in particular in [16]. Concerning (10), global approximate controllability (even in large times) is an open problem (in fact, its Hamiltonian has not a compact resolvent). When the Hamiltonian presents continuous parts in the spectrum, some results of large-time approximate controllability between bound states are also available [10, 23, 38].

#### 1.5.4 Small-time approximate controllability

The first positive mathematical results of small-time control of Schrödinger equations are very recent [14, 17, 25, 28–30].

Obstructions were previously known when the potentials are all (sub)quadratic [11, 12], due to the conservation of Gaussian states for small times (we refer also to [15] for different semi-classical obstructions). Also, an example of a small-time globally approximately controllable conservative PDE was obtained in [20], where the Laplacian is replaced with  $|\Delta|^\alpha$ ,  $\alpha > 5/2$ , in (7) for  $d = 1, V = 0$ .

A renewed interest in this subject has arrived after the work [29], where small-time approximate control of local phases was proved for (a state-nonlinear version of) system (7). Similar conclusions are obtained in [30] for a system similar to (a state-nonlinear version of) (10). These results were inspired by saturation and low mode forcing strategies previously used for controlling Navier-Stokes and Euler systems in the pioneering articles [3, 5]. We also mention the recent contributions on the small-time control, with similar techniques, of other bilinear PDEs [31, 43].

Recently, using finite-dimensional geometric techniques, we showed the existence of small-time globally approximately controllable equations of the form (1) [14]. In particular, we showed that it is possible in the presence of a quadratic control potential. In the present work we develop a new infinite-dimensional geometric approach, and we apply it to get results in the presence of more general (i.e., not quadratic) control potentials. The capability of globally approximately controlling in small times systems (7) and (10) remains an open question.

#### 1.5.5 Approximate controllability in the group of diffeomorphisms

Related infinite-dimensional geometric methods have been also reintroduced recently to study the problem of approximate controllability in the group of diffeomorphisms [1, 4, 6], as a natural extension of control problems for ensembles of particles (see also [2]).

With respect to this recent literature, the contribution of our work is in showing the relation between the two problems of approximately controlling bilinear Schrödinger PDEs on  $L^2(M, \mathbb{C})$  and diffeomorphisms on  $M$ .

### 1.6 Structure of the paper

In Section 2 we introduce some control notions and recall some tools from functional analysis. In Section 3, we show Theorem 4 and Corollary 5. In Section 4, we prove Theorem 6. In Section 5, we adapt Theorem 4 and Corollary 5 to higher regularities. In Section 6, we adapt Theorem 6 to higher regularities and prove Corollary 8.



## 2 Preliminaries

In this section, we introduce the set of small-time reachable operators, prove its semi-group structure and its closure. We also recall some classical tools of functional analysis that we shall need in the rest of the paper.

### 2.1 Well posedness for piecewise constant controls

For the scope of this work it is sufficient to consider, as control space, the space  $PWC$  of piecewise constant functions. We recall the operation of concatenation: for every  $T_1, T_2 > 0$ ,  $u_1 \in PWC(0, T_1)$  and  $u_2 \in PWC(0, T_2)$  then  $u_1 \# u_2 \in PWC(0, T_1 + T_2)$ , where, for every  $t \in (0, T_1 + T_2)$ ,

$$u_1 \# u_2(t) = \begin{cases} u_1(t) & \text{if } t \in (0, T_1), \\ u_2(t - T_1) & \text{if } t \in (T_1, T_1 + T_2). \end{cases}$$

**Proposition 10.** [45, Corollary page 199] *Let  $V$  satisfying (11). Then  $-\Delta + V$  is essentially self-adjoint on  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ .*

**Definition 11.** *Given two densely defined linear operators  $A$  and  $B$  with domains  $D(A)$  and  $D(B)$  on an Hilbert space  $\mathcal{H}$ ,  $B$  is said to be  $A$ -bounded if  $D(A) \subset D(B)$  and there exist  $a, b \geq 0$  such that for all  $\psi \in D(A)$*

$$\|B\psi\| < a\|A\psi\| + b\|\psi\|.$$

The infimum of such  $a$  is called the relative bound of  $B$ .

**Proposition 12.** (Kato-Rellich Theorem) [45, Theorem X.12] *If  $A$  is self-adjoint and  $B$  is symmetric and  $A$ -bounded with relative bound  $a < 1$ , then  $A + B$  is self-adjoint on  $D(A)$  and essentially self-adjoint on any core of  $A$ .*

In light of Propositions 10 and 12, we can define the solutions of the two systems (7), (10), associated with piecewise constant controls, by composition of time-independent unitary propagators associated with self-adjoint operators (see, e.e., [44, Definition p.256 & Theorem VIII.7]). For instance, for system (10), given a subdivision  $0 = t_0 < \dots < t_N = T$ , a piecewise constant control  $u : [0, T] \rightarrow \mathbb{R}^{d+1}$  defined as  $u(t) = (u_1^j, \dots, u_{d+1}^j) \in \mathbb{R}^{d+1}$  when  $t \in [t_{j-1}, t_j]$ , and an initial condition  $\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C})$ , the solution  $\psi \in C^0([0, T], L^2(\mathbb{R}^d, \mathbb{C}))$  of (10) is defined by

$$\begin{aligned} & \psi(t; u, \psi_0) \\ &= e^{i(t-t_{j-1})(\Delta - V - \sum_{k=1}^d u_k^j x_k - u_{d+1}^j e^{-|x|^2/2})} e^{i\tau_{j-1}(\Delta - V - \sum_{k=1}^d u_k^{j-1} x_k - u_{d+1}^{j-1} e^{-|x|^2/2})} \\ & \dots e^{i\tau_1(\Delta - V - \sum_{k=1}^d u_k^1 x_k - u_{d+1}^1 e^{-|x|^2/2})} \psi_0. \end{aligned}$$

where  $\tau_l = (t_l - t_{l-1})$  for  $l = 1, \dots, N$ .

### 2.2 Small-time approximately reachable operators

We introduce the notion of small-time approximately reachable operators.

**Definition 13** (Small-time  $\mathcal{H}$ -approximately reachable operator,  $\mathcal{H}$ -STAR). *Let  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$  be a normed  $\mathbb{C}$ -vector space, subset of  $L^2(M, \mathbb{C})$ .*

- *For  $T > 0$ , an operator  $L \in \mathcal{L}(\mathcal{H}) \cap \text{Isom}(L^2(M, \mathbb{C}))$  is  $\mathcal{H}$ -approximately (resp.  $\mathcal{H}$ -exactly) reachable in time  $T$  if, for every  $\psi_0 \in \mathcal{H} \cap \mathcal{S}$  and  $\varepsilon > 0$ , there exists  $\theta \in \mathbb{R}$  and  $u \in PWC(0, T)$  such that  $\|\psi(T; u, \psi_0) - e^{i\theta} L\psi_0\|_{\mathcal{H}} < \varepsilon$  (resp.  $\psi(T; u, \psi_0) = e^{i\theta} L\psi_0$ ).*

- Given  $T \geq 0$ , the operator  $L$  is  $\mathcal{H}$ -**approximately (resp.  $\mathcal{H}$ -exactly) reachable in time  $T^+$**  if, for every  $\psi_0 \in \mathcal{H} \cap \mathcal{S}$  and  $\varepsilon > 0$ , there exists  $T_1 \in [T, T + \varepsilon]$ ,  $\theta \in \mathbb{R}$  and  $u \in PWC(0, T_1)$  such that  $\|\psi(T_1; u, \psi_0) - e^{i\theta} L\psi_0\|_{\mathcal{H}} < \varepsilon$  (resp.  $\psi(T_1; u, \psi_0) = e^{i\theta} L\psi_0$ ).
- The operator  $L$  is **small-time  $\mathcal{H}$ -approximately (resp.  $\mathcal{H}$ -exactly) reachable** if it is  $\mathcal{H}$ -approximately (resp.  $\mathcal{H}$ -exactly) reachable in time  $0^+$ . Then, we use the abbreviation: “the operator  $L$  is  $\mathcal{H}$ -STAR”.

As an example, our main results (Theorem 4 with  $M = \mathbb{T}^d$  and Theorem 6 with  $M = \mathbb{R}^d$ ) state that for every  $P \in \text{Diff}_c^0(M)$ , the unitary composition operator  $\mathcal{L}_P$  (see (5)) is  $L^2$ -STAR. The following Lemma will be largely used in this work. It states that the set of  $\mathcal{H}$ -STAR operators is a subsemigroup of  $\mathcal{L}(\mathcal{H}) \cap \text{Isom}(L^2(M, \mathbb{C}))$  closed for the topology of the pointwise convergence. It already appeared in [14] (we prove it also here for the sake of completeness).

- Lemma 14.** 1. If the operator  $L_1$  (resp.  $L_2$ ) is  $\mathcal{H}$ -approximately (resp.  $\mathcal{H}$ -exactly) reachable in time  $T_1$  (resp.  $T_2$ ), then  $L_2 L_1$  is  $\mathcal{H}$ -approximately (resp.  $\mathcal{H}$ -exactly) reachable in time  $T_1 + T_2$ .
2. Let  $T \geq 0$  and  $(L_n)_{n \in \mathbb{N}}$  be a sequence of operators that are  $\mathcal{H}$ -approximately reachable in time  $T^+$  and  $L \in \mathcal{L}(\mathcal{H}) \cap \text{Isom}(L^2(M, \mathbb{C}))$  such that, for every  $\psi \in \mathcal{H}$ ,  $\|(L_n - L)\psi\|_{\mathcal{H}} \rightarrow 0$ . Then the operator  $L$  is  $\mathcal{H}$ -approximately reachable in time  $T^+$ .

*Proof.* 1. Let  $\psi_0 \in \mathcal{H} \cap \mathcal{S}$  and  $\varepsilon > 0$ . There exists  $u_2 \in PWC(0, T_2)$  such that

$$\|\psi(T_2; u_2, L_1 \psi_0) - L_2 L_1 \psi_0\|_{\mathcal{H}} < \frac{\varepsilon}{2}. \quad (13)$$

There exists  $u_1 \in PWC(0, T_1)$  such that

$$\|\psi(T_1; u_1, \psi_0) - L_1 \psi_0\|_{\mathcal{H}} < \frac{\varepsilon}{2\|\psi(T_2; u_2, \cdot)\|_{\mathcal{L}_c(\mathcal{H})}}. \quad (14)$$

Then  $u := u_1 \sharp u_2 \in PWC(0, T_1 + T_2)$ . Moreover, by using the triangular inequality, (13) and (14), we obtain

$$\begin{aligned} & \|\psi(T_1 + T_2; u, \psi_0) - L_2 L_1 \psi_0\|_{\mathcal{H}} \\ & \leq \|\psi(T_2; u_2, \psi(T_1; u_1, \psi_0)) - \psi(T_2; u_2, L_1 \psi_0)\|_{\mathcal{H}} + \|\psi(T_2; u_2, L_1 \psi_0) - L_2 L_1 \psi_0\|_{\mathcal{H}} \\ & \leq \|\psi(T_2; u_2, \cdot)\|_{\mathcal{L}(\mathcal{H})} \|\psi(T_1; u_1, \psi_0) - L_1 \psi_0\|_{\mathcal{H}} + \varepsilon/2 < \varepsilon. \end{aligned}$$

2. Let  $\psi_0 \in \mathcal{H} \cap \mathcal{S}$  and  $\varepsilon > 0$ . There exists  $n \in \mathbb{N}$  such that  $\|(L_n - L)\psi_0\|_{\mathcal{H}} < \varepsilon/2$ . There exists  $T_1 \in [T, T + \varepsilon]$ , and  $u \in \mathcal{U}(0, T_1)$  such that  $\|\psi(T_1; u, \psi_0) - L_n \psi_0\|_{\mathcal{H}} < \varepsilon/2$ . Then, by triangular inequality and homogeneity

$$\|\psi(T_1; u, \psi_0) - L\psi_0\|_{\mathcal{H}} \leq \|\psi(T_1; u, \psi_0) - L_n \psi_0\|_{\mathcal{H}} + \|(L_n - L)\psi_0\|_{\mathcal{H}} < \varepsilon.$$

□

## 2.3 Some useful tools

The proof of the following proposition can be found in [14].

**Proposition 15.** Let  $A, B$  be self-adjoint operators on the Hilbert space  $\mathcal{H}$ , and suppose that  $e^{iB}$  is an isomorphism of  $D$ , where  $D$  is a core for  $A$ . Then, for any  $t \in \mathbb{R}$ ,

$$e^{-iB} e^{itA} e^{iB} = \exp(e^{-iB} itA e^{iB}).$$

**Proposition 16.** [44, Theorem VIII.21 & Theorem VIII.25(a)] Let  $(A_n)_{n \in \mathbb{N}}$ ,  $A$  be self-adjoint operators on an Hilbert space  $\mathcal{H}$ , with a common core  $D$ . If  $\|(A_n - A)\psi\|_{\mathcal{H}} \xrightarrow{n \rightarrow \infty} 0$  for any  $\psi \in D$ , then  $\|(e^{iA_n} - e^{iA})\psi\|_{\mathcal{H}} \xrightarrow{n \rightarrow \infty} 0$  for any  $\psi \in \mathcal{H}$ .

**Proposition 17.** (Trotter-Kato product formula) [44, Theorem VIII.31] Let  $A, B$  be self-adjoint operators on the Hilbert space  $\mathcal{H}$  such that  $A + B$  is essentially self-adjoint on  $D(A) \cap D(B)$ . Then, for every  $\psi_0 \in \mathcal{H}$ ,

$$\left\| \left( e^{i\frac{A}{n}} e^{i\frac{B}{n}} \right)^n \psi_0 - e^{i(A+B)} \psi_0 \right\|_{\mathcal{H}} \xrightarrow{n \rightarrow +\infty} 0.$$

### 3 Transport equations on $\mathbb{T}^d$

In this section, we consider the system (7) on the torus  $\mathbb{T}^d$  and we prove Theorem 4 and Corollary 5. The proof strategy is presented in Subsection 3.1. It relies on the simplicity of the group  $\text{Diff}_c^0(M)$  proved by Thurston [46], and the small-time approximate reachability of transport operators along vector fields.

To prove this last property, the first step is the small-time approximate control of the phase, developed in Subsection 3.2. It allows to obtain the small-time approximate reachability of transport operators along smooth **gradient** vector fields in Subsection 3.3. Finally, in Subsection 3.4, we obtain the same property for any smooth vector field.

#### 3.1 Proof strategy: from transport operators to diffeomorphisms

In this section we outline the strategy for proving Theorem 4, and give a proof of Corollary 5.

**Definition 18** (Unitary transport operator associated with a vector field). For  $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ , the unitary transport operator associated with  $f$  is defined by

$$D(\mathcal{T}_f) := \{\varphi \in L^2(\mathbb{T}^d, \mathbb{C}); \langle f, \nabla \varphi \rangle \in L^2(\mathbb{T}^d, \mathbb{C})\}, \quad \mathcal{T}_f \varphi = \langle f, \nabla \varphi \rangle + \frac{1}{2} \text{div}(f) \varphi. \quad (15)$$

In the first expression,  $\nabla \varphi$  denotes the distributional derivative in  $\mathcal{D}'(\mathbb{T}^d)$ .  $\mathcal{T}_f$  is skew adjoint, thus the group  $(e^{t\mathcal{T}_f})_{t \in \mathbb{R}}$  is well defined on  $L^2(\mathbb{T}^d, \mathbb{C})$ .

**Lemma 19.** Let  $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$  a vector field and  $P := \phi_f^1$  be the associated flow at time 1 (see Definition 9). Then the unitary composition operator associated with  $P$  (see (5)) satisfies  $\mathcal{L}_P = e^{\mathcal{T}_f}$ . Moreover for every  $t \in \mathbb{R}$  and  $\varphi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,

$$(e^{t\mathcal{T}_f} \varphi)(x) = \varphi(\phi_f^t(x)) e^{\frac{1}{2} \int_0^t \text{div} f(\phi_f^s(x)) ds}. \quad (16)$$

*Proof.* Let  $\varphi \in L^2(\mathbb{T}^d, \mathbb{C})$ . Then  $e^{t\mathcal{T}_f} \varphi = \psi(t, \cdot)$  where  $\psi$  is the solution of the transport equation

$$\begin{cases} \partial_t \psi(t, x) = \langle f(x), \nabla_x \psi(t, x) \rangle + \frac{1}{2} (\text{div} f)(x) \psi(t, x), & (t, x) \in \mathbb{R} \times \mathbb{T}^d, \\ \psi(0, \cdot) = \varphi. \end{cases}$$

The characteristic method proves that  $\psi(t, x)$  is given by the right hand side of (16).

By definition of  $P$ , for every  $x \in \mathbb{T}^d$ ,  $dP(x) = R(1)$  where the resolvent  $R \in C^0(\mathbb{R}, \mathcal{M}_d(\mathbb{R}))$  solves the linear equation

$$\begin{cases} \frac{dR}{dt}(t) = Df(\phi_f^t(x)) R(t) \\ R(0) = I_d. \end{cases}$$

By Liouville formula

$$J_P(x) = \det(dP(x)) = \det(R(1)) = e^{\int_0^1 \operatorname{tr} Df(\phi_f^s(x)) ds} = e^{\int_0^1 \operatorname{div} f(\phi_f^s(x)) ds}$$

therefore, for every  $\varphi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,

$$(\mathcal{L}_P \varphi)(x) = |J_P(x)|^{1/2} \varphi \circ P(x) = e^{\frac{1}{2} \int_0^1 \operatorname{div} f(\phi_f^s(x)) ds} \varphi(\phi_f^1(x)) = (e^{\mathcal{T}_f} \varphi)(x).$$

□

To get Theorem 4 and Corollary 5, we use the following key property: any diffeomorphism of  $\mathbb{T}^d$  isotopic to the identity, can be decomposed into a finite product of flows. This is a straightforward consequence of a deep result by Thurston [46], about the simplicity of the group  $\operatorname{Diff}_c^0(M)$  (see Definition 2); we recall this property in the following statement.

**Proposition 20.** *Let  $M$  be a boundaryless connected smooth manifold and  $P \in \operatorname{Diff}_c^0(M)$ . Then, there exist  $n \in \mathbb{N}^*$  and smooth vector fields  $f_1, \dots, f_n$  with compact support on  $M$  such that  $P = \phi_{f_n}^1 \circ \dots \circ \phi_{f_1}^1$ . As a consequence,  $\mathcal{L}_P = \mathcal{L}_{\phi_{f_n}^1} \circ \dots \circ \mathcal{L}_{\phi_{f_1}^1}$ .*

*Proof.* By [46] (see also [33, 37]), the group  $\operatorname{Diff}_c^0(M)$  is simple. Thus, in order to get Proposition 20, it suffices to prove that  $F_c(M)$  is a normal subgroup of  $\operatorname{Diff}_c^0(M)$ , where

$$F_c(M) := \{\phi_{f_n}^1 \circ \dots \circ \phi_{f_1}^1; n \in \mathbb{N}, f_j \text{ smooth compactly supported vector field on } M\}.$$

Let  $P \in \operatorname{Diff}_c^0(M)$  and  $X \in F_c(M)$ : there exist  $n \in \mathbb{N}$  and  $f_1, \dots, f_n$  smooth compactly supported vector fields on  $M$  such that  $X = \phi_{f_n}^1 \circ \dots \circ \phi_{f_1}^1$ . Consider the smooth compactly supported vector field defined as the pushforward of  $f_j$  by  $P$ :

$$g_j(x) := (P \star f_j)(x) = DP(P^{-1}(x)) f_j(P^{-1}(x)), \quad x \in M.$$

Then

$$\phi_{g_j}^1(x) = P(\phi_{f_j}^1(P^{-1}(x))), \quad x \in M.$$

Hence,  $PXP^{-1} = \phi_{g_n}^1 \circ \dots \circ \phi_{g_1}^1 \in F_c(M)$ . □

Taking into account Proposition 20 and Lemma 14, in order to get Theorem 4, it suffices to prove the following statement.

**Proposition 21.** *Let  $V \in L^\infty(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for every  $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ , the unitary transport operator  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR.*

Therefore, the goal of the following subsections is to prove Proposition 21.

## 3.2 Pre-requisite: small-time control of the phase

To prove Proposition 21, we use the small-time approximate control of the phase.

**Proposition 22.** *Let  $V \in L^\infty(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for every  $\varphi \in L^2(\mathbb{T}^d, \mathbb{R})$ , the operator  $e^{i\varphi}$  is  $L^2$ -STAR.*

For the sake of completeness, we propose below a proof of Proposition 22, which is an adaptation to our functional framework of the one of [29, Theorem A]. It relies on a saturating argument introduced in [3] for studying the additive controllability of Navier-Stokes systems, and readapted to the bilinear control of Schrödinger equations in [29].

*Proof. Step 1:* We prove that, for every  $\varphi \in \mathcal{H}_0 := \text{span}_{\mathbb{R}}\{\sin\langle b_j, x \rangle, \cos\langle b_j, x \rangle; j \in \{1, \dots, d\}\}$ , the operator  $e^{i\varphi}$  is  $L^2$ -STAR. Let  $\alpha \in \mathbb{R}^{2d}$  and  $\varphi : x \in \mathbb{T}^d \mapsto \sum_{j=1}^d (\alpha_{2j-1} \sin + \alpha_{2j} \cos)\langle b_j, x \rangle$ . For any  $\tau > 0$ , the operator  $L_\tau := e^{i\tau(\Delta-V)+i\varphi}$  is  $L^2$ -exactly reachable in time  $\tau$ , because associated with the constant controls  $u_j(t) = -\alpha_j/\tau$ . For  $\tau > 0$ , the operator  $A_\tau := \tau(\Delta - V) + \varphi$  is self-adjoint on  $D(A_\tau) := H^2(\mathbb{T}^d, \mathbb{C})$ , because  $V, \varphi \in L^\infty(\mathbb{T}^d, \mathbb{R})$ . The multiplicative operator  $A_0 := \varphi$  is self-adjoint on  $L^2(\mathbb{T}^d, \mathbb{C})$ .  $H^2(\mathbb{T}^d, \mathbb{C})$  is a common core of  $A_\tau$  and  $A_0$ . For every  $\psi \in H^2(\mathbb{T}^d, \mathbb{C})$ ,  $\|(A_\tau - A_0)\psi\|_{L^2} = \tau\|(\Delta - V)\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Thus, by Proposition 16, for every  $\psi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,  $\|(L_\tau - e^{i\varphi})\psi\|_{L^2} = \|(e^{iA_\tau} - e^{iA_0})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that  $e^{i\varphi}$  is  $L^2$ -STAR.

*Step 2:* We prove that, if  $\varphi \in C^2(\mathbb{T}^d, \mathbb{R})$  and  $e^{i\lambda\varphi}$  is  $L^2$ -STAR for every  $\lambda \in \mathbb{R}$  then  $e^{-i|\nabla\varphi|^2}$  is  $L^2$ -STAR. Let  $\tau > 0$ . By Lemma 14, the operator

$$\tilde{L}_\tau := e^{i\frac{\varphi}{\sqrt{\tau}}} e^{i\tau(\Delta-V)} e^{-i\frac{\varphi}{\sqrt{\tau}}}$$

is  $L^2$ -approximately reachable in time  $\tau^+$ .

The operator  $\tau(\Delta - V)$  is self-adjoint on  $H^2(\mathbb{T}^d, \mathbb{C})$ . The multiplicative operator  $\varphi/\sqrt{\tau}$  is self-adjoint on  $L^2(\mathbb{T}^d, \mathbb{C})$ .  $e^{i\varphi/\sqrt{\tau}}$  is an isomorphism of  $H^2(\mathbb{T}^d, \mathbb{C})$  because  $\varphi \in C^2(\mathbb{T}^d, \mathbb{R})$ . Thus, by Lemma 15 and standard computations

$$\tilde{L}_\tau = \exp\left(i\tau e^{i\frac{\varphi}{\sqrt{\tau}}}(\Delta - V)e^{-i\frac{\varphi}{\sqrt{\tau}}}\right) = \exp\left(i\tau(\Delta - V) - \sqrt{\tau}(2\langle \nabla\varphi, \nabla \rangle + \Delta\varphi) - i|\nabla\varphi|^2\right).$$

The operator

$$\tilde{A}_\tau := \tau e^{i\frac{\varphi}{\sqrt{\tau}}}(\Delta - V)e^{-i\frac{\varphi}{\sqrt{\tau}}} = \tau(\Delta - V) + i\sqrt{\tau}(2\langle \nabla\varphi, \nabla \rangle + \Delta\varphi) - |\nabla\varphi|^2$$

is self-adjoint on  $H^2(\mathbb{T}^d, \mathbb{C})$ . The multiplicative operator  $\tilde{A}_0 := -|\nabla\varphi|^2$  is self-adjoint on  $L^2(\mathbb{T}^d, \mathbb{C})$  because  $\nabla\varphi \in L^\infty(\mathbb{T}^d, \mathbb{R})$ .  $H^2(\mathbb{T}^d, \mathbb{C})$  is a common core of  $\tilde{A}_\tau$  and  $\tilde{A}_0$ . For every  $\psi \in H^2(\mathbb{T}^d, \mathbb{C})$ ,

$$\|(\tilde{A}_\tau - \tilde{A}_0)\psi\|_{L^2} = \|\tau(\Delta - V)\psi + i\sqrt{\tau}(2\langle \nabla\varphi, \nabla \rangle + \Delta\varphi)\psi\|_{L^2} \xrightarrow{\tau \rightarrow 0} 0.$$

Thus, by Proposition 16, for every  $\psi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,

$$\|(\tilde{L}_\tau - e^{-i|\nabla\varphi|^2})\psi\|_{L^2} = \|(e^{i\tilde{A}_\tau} - e^{i\tilde{A}_0})\psi\|_{L^2} \xrightarrow{\tau \rightarrow 0} 0.$$

By Lemma 14, this proves that  $e^{-i|\nabla\varphi|^2}$  is  $L^2$ -STAR.

*Step 3: Iteration.* We define by induction an increasing sequence of sets  $(\mathcal{H}_j)_{j \in \mathbb{N}}$ :  $\mathcal{H}_0$  is defined in Step 1 and, for every  $j \in \mathbb{N}^*$ ,  $\mathcal{H}_j$  is the largest vector space whose elements can be written as

$$\varphi_0 - \sum_{k=1}^N |\nabla\varphi_k|^2; N \in \mathbb{N}, \varphi_0, \dots, \varphi_N \in \mathcal{H}_{j-1}.$$

Let  $\mathcal{H}_\infty := \cup_{j \in \mathbb{N}} \mathcal{H}_j$ . Thanks to Lemma 14, Steps 1 and 2, for every  $\varphi \in \mathcal{H}_\infty$ , the operator  $e^{i\varphi}$  is  $L^2$ -STAR. Moreover, the proof of [29, Proposition 2.6] shows that  $\mathcal{H}_\infty$  contains any trigonometric polynomial, in particular,  $\mathcal{H}_\infty$  is dense in  $L^2(\mathbb{T}^d, \mathbb{R})$ .

*Step 4: Conclusion.* Let  $\varphi \in L^2(\mathbb{T}^d, \mathbb{R})$ . There exists  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{H}_\infty$  such that  $\|\varphi_n - \varphi\|_{L^2} \rightarrow 0$  as  $n \rightarrow \infty$ . Up to an extraction, one may assume that  $\varphi_n \rightarrow \varphi$  almost everywhere on  $\mathbb{T}^d$ , as  $n \rightarrow \infty$ . The dominated convergence theorem proves that, for every  $\psi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,  $\|(e^{i\varphi_n} - e^{i\varphi})\psi\|_{L^2} \rightarrow 0$  as  $n \rightarrow \infty$ . Finally, Step 3 and Lemma 14 prove that the operator  $e^{i\varphi}$  is  $L^2$ -STAR.  $\square$

### 3.3 Small-time reachability of transport operators along gradient vector fields

In this section, we prove Proposition 21 for gradient vector fields: this is the content of the next result.

**Proposition 23.** *Let  $V \in L^\infty(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: if  $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})$  and  $f = \nabla\varphi$  then the unitary transport operator  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR.*

*Proof.* Let  $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})$  and  $f := 2\nabla\varphi$ . Let  $n \in \mathbb{N}^*$  and  $\tau > 0$ .

*Step 1:* We prove that the operator  $L_{\tau,n}$  is  $L^2$ -approximately reachable in time  $\tau^+$ , where

$$L_{\tau,n} := \left( e^{\frac{i|\nabla\varphi|^2}{\tau n}} e^{i\frac{\varphi}{\tau}} e^{i\frac{\tau}{n}(\Delta - V)} e^{-i\frac{\varphi}{\tau}} \right)^n.$$

By Proposition 22, the operators  $e^{\frac{i|\nabla\varphi|^2}{\tau n}}$ , and  $e^{\pm i\frac{\varphi}{\tau}}$ , are  $L^2$ -STAR because  $\varphi, |\nabla\varphi|^2 \in L^2(\mathbb{T}^d, \mathbb{R})$ . Moreover the operator  $e^{i\frac{\tau}{n}(\Delta - V)}$  is exactly reachable in time  $\tau/n$  because associated with the constant control  $u = 0$ . Thus, Lemma 14 proves that  $L_{\tau,n}$  is  $L^2$ -approximately reachable in time  $\tau^+$ .

*Step 2:* We prove that  $L_\tau$  is  $L^2$ -approximately reachable in time  $\tau^+$ , where

$$L_\tau := \exp(i\tau(\Delta - V) + 2\langle \nabla\varphi, \nabla \rangle + \Delta\varphi) = \exp(i\tau(\Delta - V) + \mathcal{T}_f).$$

The operator  $\frac{\tau}{n}(\Delta - V)$  is self-adjoint on  $H^2(\mathbb{T}^d, \mathbb{C})$ . The multiplicative operator  $\frac{\varphi}{\tau}$  is self-adjoint on  $L^2(\mathbb{T}^d, \mathbb{C})$  because  $\varphi \in L^\infty(\mathbb{T}^d, \mathbb{C})$ . The operator  $e^{i\frac{\varphi}{\tau}}$  is an isomorphism of  $H^2(\mathbb{T}^d, \mathbb{C})$  thus, by Lemma 15

$$L_{\tau,n} = \left( e^{\frac{i|\nabla\varphi|^2}{\tau n}} \exp\left(i\frac{\tau}{n} e^{i\frac{\varphi}{\tau}} (\Delta - V) e^{-i\frac{\varphi}{\tau}}\right) \right)^n.$$

The operator  $\tau e^{i\frac{\varphi}{\tau}} (\Delta - V) e^{-i\frac{\varphi}{\tau}}$  is self-adjoint on  $H^2(\mathbb{T}^d, \mathbb{C})$ . The multiplicative operator  $\frac{|\nabla\varphi|^2}{\tau n}$  is self-adjoint on  $L^2(\mathbb{T}^d, \mathbb{C})$  because  $\nabla\varphi \in L^\infty(\mathbb{T}^d, \mathbb{C})$ . Their sum is self-adjoint on  $H^2(\mathbb{T}^d, \mathbb{C})$  by Proposition 12. Thus, by Proposition 17,

$$\forall \psi \in L^2(\mathbb{R}^d, \mathbb{C}), \quad \|(L_{\tau,n} - \tilde{L}_\tau)\psi\|_{L^2} \xrightarrow{n \rightarrow \infty} 0, \quad (17)$$

where

$$\tilde{L}_\tau := \exp\left(\frac{i|\nabla\varphi|^2}{\tau} + i\tau e^{i\frac{\varphi}{\tau}} (\Delta - V) e^{-i\frac{\varphi}{\tau}}\right).$$

Standard computations prove that

$$i\tau e^{i\frac{\varphi}{\tau}} (\Delta - V) e^{-i\frac{\varphi}{\tau}} = i\tau(\Delta - V) + 2\langle \nabla\varphi, \nabla \rangle + \Delta\varphi - \frac{i|\nabla\varphi|^2}{\tau}$$

thus  $\tilde{L}_\tau = L_\tau$ . Then Step 1, (17) and Lemma 14 prove that  $L_\tau$  is  $L^2$ -approximately reachable in time  $\tau^+$ .

*Step 3:* We prove that  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR. The operator  $A_\tau := \tau(\Delta - V) - i\mathcal{T}_f$  is self-adjoint on  $H^2(\mathbb{T}^d, \mathbb{C})$ . The operator  $A_0 := -i\mathcal{T}_f$  is self-adjoint.  $H^2(\mathbb{T}^d, \mathbb{C})$  is a common core of  $A_\tau$  and  $A_0$ . For every  $\psi \in H^2(\mathbb{T}^d, \mathbb{C})$ ,  $\|(A_\tau - A_0)\psi\|_{L^2} = \tau\|(\Delta - V)\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Thus, by Proposition 16, for every  $\psi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,  $\|(L_\tau - e^{\mathcal{T}_f})\psi\|_{L^2} = \|(e^{iA_\tau} - e^{iA_0})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . This convergence, Step 2 and Lemma 14 prove that  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR.  $\square$

### 3.4 Small-time reachability of transport operators along any vector fields

We introduce the set

$$\mathfrak{L} := \{f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d); \forall t \in \mathbb{R}, e^{t\mathcal{T}_f} \text{ is } L^2\text{-STAR}\}, \quad (18)$$

the set  $\mathfrak{G}$  of gradient vector fields and the Lie algebra  $\mathfrak{L}_0$  generated by  $\mathfrak{G}$  in  $C^\infty(\mathbb{T}^d, \mathbb{C})$

$$\mathfrak{G} := \{\nabla\varphi; \varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})\}, \quad \mathfrak{L}_0 := \text{Lie}(\mathfrak{G}).$$

By Proposition 23,  $\mathfrak{G} \subset \mathfrak{L}$ . The goal of this section is to prove the following result.

**Proposition 24.**  *$\mathfrak{L}$  is a Lie algebra, thus  $\mathfrak{L}_0 \subset \mathfrak{L}$ .*

*Proof.* Clearly, if  $f \in \mathfrak{L}$  and  $\lambda \in \mathbb{R}$  then  $\lambda f \in \mathfrak{L}$ . Thus, it suffices to prove that  $\mathfrak{L}$  is stable by summation and Lie bracket.

*Step 1:  $\mathfrak{L}$  is stable by summation.* It suffices to prove that  $(f, g \in \mathfrak{L} \implies e^{\mathcal{T}_{f+g}}$  is  $L^2$ -STAR) because, for every  $t \in \mathbb{R}$ ,  $tf, tg \in \mathfrak{L}$  and  $t\mathcal{T}_{f+g} = \mathcal{T}_{tf+tg}$ . Let  $f, g \in \mathfrak{L}$ . By Lemma 14, for every  $n \in \mathbb{N}^*$ , the operator  $(e^{\frac{1}{n}\mathcal{T}_f} e^{\frac{1}{n}\mathcal{T}_g})^n$  is  $L^2$ -STAR. The operators  $\mathcal{T}_f$  and  $\mathcal{T}_g$  are skew adjoint on  $L^2(\mathbb{T}^d, \mathbb{C})$  and the operator  $\mathcal{T}_f + \mathcal{T}_g = \mathcal{T}_{f+g}$  is essentially skew adjoint on  $D(\mathcal{T}_f) \cap D(\mathcal{T}_g)$ . Thus, by Proposition 17, for every  $\varphi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,  $\|(e^{\frac{1}{n}\mathcal{T}_f} e^{\frac{1}{n}\mathcal{T}_g})^n \varphi - e^{\mathcal{T}_{f+g}} \varphi\|_{L^2} \rightarrow 0$  as  $n \rightarrow \infty$ . By Lemma 14, the operator  $e^{\mathcal{T}_{f+g}}$  is  $L^2$ -STAR.

*Step 2:  $\mathfrak{L}$  is stable by Lie bracket, i.e.  $(f, g \in \mathfrak{L} \implies [f, g] := (Dg)f - (Df)g \in \mathfrak{L})$ .* It suffices to prove that  $(f, g \in \mathfrak{L} \implies e^{\mathcal{T}_{[f,g]}}$  is  $L^2$ -STAR), because, for every  $t \in \mathbb{R}$ ,  $tf \in \mathfrak{L}$  and  $t\mathcal{T}_{[f,g]} = \mathcal{T}_{[tf,g]}$ . For  $t \in \mathbb{R}^*$  and  $n \in \mathbb{N}^*$ , the operator

$$L_{t,n} := \left( e^{\frac{-1}{tn}\mathcal{T}_f} e^{-t\mathcal{T}_g} e^{\frac{1}{tn}\mathcal{T}_f} e^{t\mathcal{T}_g} \right)^n$$

is  $L^2$ -STAR thanks to Lemma 14. The transport operators  $\mathcal{T}_f, \mathcal{T}_g$  are skew-adjoint. By (16),  $e^{t\mathcal{T}_g}$  is an isomorphism of  $C^\infty(\mathbb{T}^d, \mathbb{C})$ . Thus by applying Lemma 15 we get

$$L_{t,n} = \left( e^{\frac{-1}{tn}\mathcal{T}_f} \exp \left( e^{-t\mathcal{T}_g} \frac{1}{tn} \mathcal{T}_f e^{t\mathcal{T}_g} \right) \right)^n.$$

By Proposition 17,

$$\forall \varphi \in L^2(\mathbb{R}^d, \mathbb{C}), \quad \|(L_{t,n} - L_t)\varphi\|_{L^2} \xrightarrow{n \rightarrow \infty} 0,$$

where

$$L_t = \exp \left( \frac{-1}{t} \mathcal{T}_f + e^{-t\mathcal{T}_g} \frac{1}{t} \mathcal{T}_f e^{t\mathcal{T}_g} \right).$$

By Lemma 14, this proves that  $L_t$  is  $L^2$ -STAR. By combining Lemma 25 below, with Proposition 16, we obtain, for every  $\varphi \in L^2(\mathbb{T}^d, \mathbb{C})$ ,  $\|(L_t - e^{\mathcal{T}_{[f,g]}})\varphi\|_{L^2} \rightarrow 0$  as  $t \rightarrow 0$ . Finally, Lemma 14 proves that the operator  $e^{\mathcal{T}_{[f,g]}}$  is  $L^2$ -STAR.  $\square$

**Lemma 25.** *Let  $f, g \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$  vector fields and  $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{C})$ . Then*

$$\|(e^{-t\mathcal{T}_g} \mathcal{T}_f e^{t\mathcal{T}_g} - \mathcal{T}_f - t\mathcal{T}_{[f,g]}) \varphi\|_{L^2} = o_{t \rightarrow 0}(t).$$

*Proof. Step 1: We prove*

$$(e^{-t\mathcal{T}_g} \mathcal{T}_f e^{t\mathcal{T}_g} \varphi)(x) = \langle (\phi_g^t \star f)(x), \nabla \varphi(x) \rangle + \frac{1}{2} (\operatorname{div} f(\phi_g^{-t}(x)) + tG(t, x)) \varphi(x)$$

where  $(\phi_g^t \star f)$  is the push forward of the vector field  $f$  by the diffeomorphism  $\phi_g^t$ , i.e.

$$(\phi_g^t \star f)(x) := D\phi_g^t(\phi_g^{-t}(x))f(\phi_g^{-t}(x)) = (D\phi_g^{-t}(x))^{-1}f(\phi_g^{-t}(x))$$

and

$$G(t, x) := \int_{-1}^0 D(\operatorname{div} g)(\phi_g^{t\theta}(x)) D\phi_g^{t\theta}(\phi_g^{-t}(x)) (\phi_g^t \star f)(x) d\theta.$$

Using (16) and (35), we obtain

$$\begin{aligned} (e^{-t\mathcal{T}_g} \mathcal{T}_f e^{t\mathcal{T}_g} \varphi)(x) &= (\mathcal{T}_f e^{t\mathcal{T}_g} \varphi)(\phi_g^{-t}(x)) e^{\frac{1}{2} \int_0^{-t} \operatorname{div} g(\phi_g^s(x)) ds} \\ &= \left( \langle f, \nabla(e^{t\mathcal{T}_g} \varphi) \rangle + \frac{1}{2} \operatorname{div}(f)(e^{t\mathcal{T}_g} \varphi) \right) (\phi_g^{-t}(x)) e^{-\frac{1}{2} \int_{-t}^0 \operatorname{div} g(\phi_g^s(x)) ds}. \end{aligned} \quad (19)$$

We deduce from the expression (16) that

$$(e^{t\mathcal{T}_g} \varphi)(\phi_g^{-t}(x)) = \varphi(x) e^{\frac{1}{2} \int_{-t}^0 \operatorname{div} g(\phi_g^s(x)) ds}. \quad (20)$$

We deduce from (16) and the chain rule that

$$\langle f(x), \nabla(e^{t\mathcal{T}_g} \varphi)(x) \rangle = D(e^{t\mathcal{T}_g} \varphi)(x) f(x) \quad (21)$$

$$= \left( D\varphi(\phi_g^t(x)) D\phi_g^t(x) f(x) + \frac{1}{2} \int_0^t D(\operatorname{div} g)(\phi_g^s(x)) D\phi_g^s(x) f(x) ds \varphi(\phi_g^t(x)) \right) e^{\frac{1}{2} \int_0^t \operatorname{div} g(\phi_g^s(x)) ds}. \quad (22)$$

We deduce from the resolvent relation  $D\phi_g^s(y)(D\phi_g^t(y))^{-1} = D\phi_g^{s-t}(y)$  that

$$\begin{aligned} \langle f, \nabla(e^{t\mathcal{T}_g} \varphi) \rangle(\phi_g^{-t}(x)) &= \left( D\varphi(x) (\phi_g^t \star f)(x) \right. \\ &\quad \left. + \frac{1}{2} \int_0^t D(\operatorname{div} g)(\phi_g^{s-t}(x)) D\phi_g^{s-t}(\phi_g^{-t}(x)) (\phi_g^t \star f)(x) ds \varphi(x) \right) e^{\frac{1}{2} \int_{-t}^0 \operatorname{div} g(\phi_g^s(x)) ds}. \end{aligned} \quad (23)$$

We conclude Step 1 by gathering (19), (20) and (23), in which we use the change of variable  $s - t = t\theta$ .

*Step 2: Conclusion.* We deduce from Step 1 and the relation

$$\mathcal{T}_{[f,g]} = \langle [f, g], \nabla \rangle + \frac{1}{2} (D(\operatorname{div} g)f - D(\operatorname{div} f)g)$$

that

$$\begin{aligned} (e^{-t\mathcal{T}_g} \mathcal{T}_f e^{t\mathcal{T}_g} - \mathcal{T}_f - t\mathcal{T}_{[f,g]}) \varphi &= \langle \phi_g^t \star f - f - t[f, g], \nabla \varphi \rangle + \frac{t}{2} (G(t, \cdot) - G(0, \cdot)) \varphi \\ &\quad + \frac{1}{2} (\operatorname{div} f \circ \phi_g^{-t} - \operatorname{div} f + t D(\operatorname{div} f)g) \varphi. \end{aligned} \quad (24)$$

Thus

$$\begin{aligned} \|(e^{-t\mathcal{T}_g} \mathcal{T}_f e^{t\mathcal{T}_g} - \mathcal{T}_f - t\mathcal{T}_{[f,g]}) \varphi\|_{L^2} &\leq \|\phi_g^t \star f - f - t[f, g]\|_{L^\infty} \|\nabla \varphi\|_{L^2} \\ &\quad + \frac{t}{2} \|G(t, \cdot) - G(0, \cdot)\|_{L^\infty} \|\varphi\|_{L^2} \\ &\quad + \frac{1}{2} \|\operatorname{div} f \circ \phi_g^{-t} - \operatorname{div} f + t D(\operatorname{div} f)g\|_{L^\infty} \|\varphi\|_{L^2}. \end{aligned} \quad (25)$$



The expression  $F(t, x) := (\phi_g^t \star f)(x)$  defines  $F \in C^\infty(\mathbb{R} \times \mathbb{T}^d, \mathbb{R}^d)$  thus, by Taylor formula, for every  $(t, x) \in \mathbb{R} \times \mathbb{T}^d$ ,

$$\left| F(t, x) - F(0, x) - t \frac{\partial F}{\partial t}(0, x) \right| \leq \frac{|t|^2}{2} \sup \left\{ \left| \frac{\partial^2 F}{\partial \tau^2}(\tau, y) \right|; (\tau, y) \in [0, t] \times \mathbb{T}^d \right\}.$$

Moreover, the chain rule and the resolvent relation

$$\frac{\partial}{\partial t} D\phi_g^t(x) = Dg(\phi_g^t(x)) D\phi_g^t(x)$$

prove that

$$\frac{\partial F}{\partial t}(t, x) = (D\phi_g^{-t}(x))^{-1} Dg(\phi_g^{-t}(x)) f(\phi_g^{-t}(x)) - (D\phi_g^{-t}(x))^{-1} Df(\phi_g^{-t}(x)) g(\phi_g^{-t}(x))$$

thus

$$\frac{\partial F}{\partial t}(0, x) = [f, g](x).$$

The continuous function  $(t, x) \mapsto \partial_t^2 F(t, x)$  is bounded on the compact set  $[0, 1] \times \mathbb{T}^d$ , thus

$$\|\phi_g^t \star f - f - t[f, g]\|_{L^\infty} = o_{t \rightarrow 0}(t). \quad (26)$$

The same argument applied to the expression  $F(t, x) := \operatorname{div} f(\phi_g^{-t}(x))$  gives

$$\|\operatorname{div} f \circ \phi_g^{-t} - \operatorname{div} f + tD(\operatorname{div} f)g\|_{L^\infty} = o_{t \rightarrow 0}(t). \quad (27)$$

The same argument applied to  $G$  proves

$$\|G(t, \cdot) - G(0)\|_{L^\infty} = o_{t \rightarrow 0}(1). \quad (28)$$

The estimates (25), (26), (27), (28) give the conclusion.  $\square$

**Proposition 26.**  $\mathfrak{L}_0$  contains any linear combination of vector fields of the form

$$f(x) = \prod_{i \in I} \cos(\ell_i x_i) \prod_{i \in I^c} \sin(\ell_i x_i) e_k \quad (29)$$

where  $I$  is a subset of  $\{1, \dots, d\}$ ,  $k \in \{1, \dots, d\}$ ,  $(\ell_1, \dots, \ell_d) \in \mathbb{N}^d$ , and  $e_k = \partial_{x_k}$ . As a consequence, Proposition 21 holds.

The proof is inspired by [6, Lemma 6.5]: the main difference and difficulty here, is that we only have access to *gradient* vector fields.

*Proof. Step 1:*  $\mathfrak{L}_0$  contains any vector field of the form  $f(x) = \cos(\ell x_j) e_j$  (resp.  $\sin(\ell x_j) e_j$ ) for  $j \in \{1, \dots, d\}$  and  $\ell \in \mathbb{N}^*$ . Indeed  $f = \nabla \varphi$  where  $\varphi(x) = \sin(\ell x_j) / \ell$  (resp.  $\cos(\ell x_j) / \ell$ ).

*Step 2:*  $\mathfrak{L}_0$  contains  $f(x) = e_j$  for any  $j \in \{1, \dots, d\}$ . By Step 1,  $\mathfrak{L}_0$  contains

$$[\cos(x_j) e_j, \sin(x_j) e_j] = \cos(x_j) \frac{\partial}{\partial x_j} (\sin(x_j) e_j) - \sin(x_j) \frac{\partial}{\partial x_j} (\cos(x_j) e_j) \quad (30)$$

$$= (\cos^2(x_j) + \sin^2(x_j)) e_j = e_j. \quad (31)$$

*Step 3:*  $\mathfrak{L}_0$  contains  $f(x) = \cos(x_j) e_k$  (resp.  $\sin(x_j) e_k$ ) for any  $j \neq k \in \{1, \dots, d\}$ . Indeed,  $\mathfrak{L}_0$  contains

$$-\frac{1}{2} [\nabla \sin(x_j) \sin(x_k), \nabla \cos(x_k)] + \frac{1}{2} [\nabla \sin(x_j) \cos(x_k), \nabla \sin(x_k)] = \sin(x_j) e_k$$

$$\text{(resp. } \frac{1}{2}[\nabla \cos(x_j) \cos(x_k), \nabla \sin(x_k)] - \frac{1}{2}[\nabla \cos(x_j) \sin(x_k), \nabla \cos(x_k)] = \cos(x_j)e_k).$$

These elementary calculations can be carried out as in Step 2.

*Step 4:* We prove by induction on  $\ell \in \mathbb{N}$  that  $\mathfrak{L}_0$  contains  $f(x) = \cos(\ell x_j)e_k$  (resp.  $\sin(\ell x_j)e_k$ ) for every  $j \neq k \in \{1, \dots, d\}$  and  $\ell \in \mathbb{N}^*$ . The initialization for  $\ell = 1$  is given by Step 3. We assume the property proved up to  $\ell$ . By Step 1 and the induction assumption,  $\mathfrak{L}_0$  contains

$$\begin{aligned} \frac{1}{\ell}[\sin(x_j)e_j, \cos(\ell x_j)e_k] + \frac{1}{\ell}[\cos(x_j)e_j, \sin(\ell x_j)e_k] &= \cos((\ell + 1)x_j)e_k, \\ \frac{1}{\ell}[\sin(x_j)e_j, \sin(\ell x_j)e_k] - \frac{1}{\ell}[\cos(x_j)e_j, \cos(\ell x_j)e_k] &= \sin((\ell + 1)x_j)e_k. \end{aligned}$$

*Step 5:* We prove the first statement of Proposition 26 by induction on the degree of  $f$ , defined by  $d(f) := \#\{i \in \{1, \dots, d\}; \ell_i \neq 0\}$ . The initialization for  $d(f) = 1$  is given by Step 4. Let  $s \in \mathbb{N}^*$ . We assume the property proved for any monomial of degree  $\leq s$ . Let  $f$  be a monomial with degree  $(s + 1)$ . Then  $f(x) = g(x) \cos(\ell_\alpha x_\alpha)e_k$  or  $g(x) \sin(\ell_\alpha x_\alpha)e_k$  where  $g$  is a monomial with degree  $s$  independent of  $x_\alpha$ .

*First case:*  $g(x)$  does not depend on  $x_k$ . By the induction assumption and Step 4,  $\mathfrak{L}_0$  contains

$$\begin{aligned} \frac{1}{\ell_\alpha}[g(x)e_\alpha, \sin(\ell_\alpha x_\alpha)e_k] &= g(x) \cos(\ell_\alpha x_\alpha)e_k, \\ -\frac{1}{\ell_\alpha}[g(x)e_\alpha, \cos(\ell_\alpha x_\alpha)e_k] &= g(x) \sin(\ell_\alpha x_\alpha)e_k. \end{aligned}$$

*Second case:*  $g(x)$  depends on  $x_k$ . Then  $\alpha \neq k$ . Thus there exists a monomial  $g_1(x)$  of degree  $s$  such that  $g(x) = \partial_{x_k} g_1(x)$ . By induction assumption and Step 4,  $\mathfrak{L}_0$  contains

$$\begin{aligned} [\cos(\ell_\alpha x_\alpha)e_k, g_1(x)e_k] &= \cos(\ell_\alpha)g(x)e_k, \\ [\sin(\ell_\alpha x_\alpha)e_k, g_1(x)e_k] &= \sin(\ell_\alpha)g(x)e_k. \end{aligned}$$

*Step 6:* We prove the last statement of Proposition 26. Let  $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$  a vector field. By applying Fejer theorem to  $f$  and  $\nabla f$ , we obtain a sequence  $(f_k)_{k \in \mathbb{N}}$  of trigonometric polynomials vector fields such that  $\|f_k - f\|_{W^{1,\infty}} \rightarrow 0$  as  $k \rightarrow \infty$ . By the first statement of Proposition 26 and Proposition 24, for every  $k \in \mathbb{N}$ , the operator  $e^{\mathcal{T}f_k}$  is  $L^2$ -STAR. By Lemma 14, to prove that  $e^{\mathcal{T}f}$  is  $L^2$ -STAR, it suffices to prove that

$$\forall \psi \in L^2(\mathbb{T}^d, \mathbb{C}), \quad \|(e^{\mathcal{T}f_k} - e^{\mathcal{T}f})\psi\|_{L^2} \xrightarrow{\epsilon \rightarrow 0} 0. \quad (32)$$

It suffices to prove (32) for every  $\psi$  in a dense subset of  $L^2(\mathbb{T}^d, \mathbb{C})$ . So we consider  $\psi \in L^\infty(\mathbb{T}^d, \mathbb{C})$ . Gronwall Lemma proves that

$$\forall t \in \mathbb{R}, \quad \|\phi_f^t - \phi_{f_k}^t\|_{L^\infty} \leq t\|f - f_k\|_{L^\infty} e^{t\|f\|_{C^1}}. \quad (33)$$

Thus, for every  $x \in \mathbb{T}^d$ ,

$$\int_0^1 |\operatorname{div} f(\phi_f^s(x)) - \operatorname{div} f_k(\phi_{f_k}^s(x))| ds \leq \|f\|_{C^2}\|f - f_k\|_{L^\infty} e^{\|f\|_{C^1}} + \|f - f_k\|_{W^{1,\infty}} \xrightarrow[k \rightarrow \infty]{} 0. \quad (34)$$

For almost every  $x \in \mathbb{T}^d$ , we have the convergence  $e^{\mathcal{T}f_k}\psi(x) \rightarrow e^{\mathcal{T}f}\psi(x)$  as  $k \rightarrow \infty$  and the domination, for  $k$  large enough  $|e^{\mathcal{T}f_k}\psi(x)| \leq \|\psi\|_{L^\infty} e^{\|f\|_{C^1} + 1}$ . The dominated convergence theorem proves that  $\|(e^{\mathcal{T}f_k} - e^{\mathcal{T}f})\psi\|_{L^2} \rightarrow 0$  as  $\epsilon \rightarrow 0$ .  $\square$

## 4 Transport equations on $\mathbb{R}^d$

In this section, we consider the system (10) on the whole space  $\mathbb{R}^d$  and we prove Theorem 6.

### 4.1 Proof strategy for Theorem 6

**Definition 27** (Unitary transport operator associated with a vector field). For  $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ , the unitary transport operator associated with  $f$  is defined by

$$\mathcal{T}_f \varphi = \langle f, \nabla \varphi \rangle + \frac{1}{2} \operatorname{div}(f) \varphi, \quad D(\mathcal{T}_f) := \{\varphi \in L^2(\mathbb{R}^d, \mathbb{C}); \mathcal{T}_f \varphi \in L^2(\mathbb{R}^d, \mathbb{C})\}. \quad (35)$$

Note that  $\mathcal{T}_f \varphi$  is well defined in  $\mathcal{D}'(\mathbb{R}^d)$ , which gives a sense to this definition.

**Lemma 28.** Let  $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$  be a globally Lipschitz vector field and  $P := \phi_f^1$  be the associated flow at time 1 (see Definition 9). Then the unitary composition operator associated with  $P$  (see (5)) satisfies  $\mathcal{L}_P = e^{\mathcal{T}_f}$ . Moreover for every  $t \in \mathbb{R}$  and  $\varphi \in L^2(\mathbb{R}^d, \mathbb{C})$ , (16) holds.

Lemma 28 can be proved as Lemma 19. Since  $f$  is globally Lipschitz, the flow  $\phi_f^t$  is well defined on  $\mathbb{R}^d$  for every  $t \in \mathbb{R}$  and  $\operatorname{div} f$  is bounded on  $\mathbb{R}^d$ .

The proof strategy for Theorem 6 is the same as for Theorem 4 in the previous section: taking into account Thurston's result (i.e. Proposition 20 with  $M = \mathbb{R}^d$ ) and Lemma 14, it suffices to prove the following result.

**Proposition 29.** Let  $V$  satisfying (11). System (10) satisfies the following property: for every  $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ , the operator  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR.

To prove Proposition 29, the first step is the small-time approximate control of the phase, developed in Subsection 4.2. It allows to obtain the small-time approximate reachability of transport operators along **gradient** vector fields in Subsection 4.3. Finally, in Subsection 4.4, we obtain the same property for any vector field.

### 4.2 Pre-requisite: small-time control of the phase

**Proposition 30.** Let  $V$  satisfying (11). System (10) satisfies the following properties:

- for every  $j \in \{1, \dots, d\}$  and  $u \in \mathbb{R}$ , the operator  $e^{u\partial_{x_j}}$  is  $L^2$ -STAR,
- for every  $\varphi \in L^2(\mathbb{R}^d, \mathbb{R})$ , the operator  $e^{i\varphi}$  is  $L^2$ -STAR

The proof of Proposition 30 is an adaptation to our framework of the one given in [30, Theorem 1] for a similar system.

*Proof. Step 1:* We prove that, for every  $\varphi \in \operatorname{span}\{x_1, \dots, x_d, e^{-|x|^2/2}\}$ , the operator  $e^{i\varphi}$  is  $L^2$ -STAR. Let  $\alpha \in \mathbb{R}^{d+1}$  and  $\varphi : x \in \mathbb{R}^d \mapsto \alpha_1 x_1 + \dots + \alpha_d x_d + \alpha_{d+1} e^{-|x|^2/2}$ . For any  $\tau > 0$ , the operator  $e^{i\tau(\Delta - V) + i\varphi}$  is  $L^2$ -exactly reachable in time  $\tau$  because associated with the constant controls  $u_j = -\alpha_j/\tau$ . The operator  $\tau(\Delta - V) + \varphi$  is essentially self-adjoint on  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  thus its closure  $A_\tau$  is self-adjoint. The multiplicative operator  $\varphi$  is self-adjoint on  $\{\psi \in L^2(\mathbb{R}^d, \mathbb{C}); \varphi\psi \in L^2(\mathbb{R}^d, \mathbb{C})\}$ .  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  is a common core of  $A_\tau$  and  $\varphi$ . For every  $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ ,  $\|(A_\tau - \varphi)\psi\|_{L^2} = \tau\|(\Delta - V)\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Thus, by Proposition 16, for every  $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\|(e^{i\tau(\Delta - V) + i\varphi} - e^{i\varphi})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Therefore, by Lemma 14, the operator  $e^{i\varphi}$  is  $L^2$ -STAR.

Step 2: We prove that, for every  $j \in \{1, \dots, d\}$  and  $u \in \mathbb{R}$ , the operator  $e^{u\partial_{x_j}}$  is  $L^2$ -STAR. Let  $j \in \{1, \dots, d\}$ ,  $u \in \mathbb{R}^*$ . By Step 1 and Lemma 14, for every  $\tau > 0$ , the operator

$$L_\tau := e^{\frac{iu x_j}{2\tau}} e^{i\tau(\Delta - V)} e^{-\frac{iu x_j}{2\tau}}$$

is  $L^2$ -approximately reachable in time  $\tau^+$ . The operator  $\tau(\Delta - V)$  is essentially self-adjoint on  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  by Proposition 10 thus its closure  $A$  is self-adjoint. The operator  $B := ux_j/(2\tau)$  is self-adjoint on  $D(B) := \{\psi \in L^2(\mathbb{R}^d, \mathbb{C}); x_j\psi \in L^2(\mathbb{R}^d, \mathbb{C})\}$ .  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  is a core of  $A$ . The operator  $e^{iB}$  is an isomorphism of  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ . Thus, by Proposition 15 and standard computations

$$L_\tau = \exp\left(i\tau e^{\frac{iu x_j}{2\tau}}(\Delta - V)e^{-\frac{iu x_j}{2\tau}}\right) = \exp\left(i\tau(\Delta - V) + u\partial_{x_j} - i\frac{u^2}{2\tau}\right). \quad (36)$$

By Definition 13,

$$L'_\tau := e^{i\frac{u^2}{2\tau}} L_\tau = \exp\left(i\tau(\Delta - V) + u\partial_{x_j}\right)$$

is also  $L^2$ -approximately reachable in time  $\tau^+$ . The operator

$$\tau e^{\frac{iu x_j}{2\tau}}(\Delta - V)e^{-\frac{iu x_j}{2\tau}} + \frac{u^2}{2\tau} = \tau(\Delta - V) - iu\partial_{x_j}$$

is essentially self-adjoint on  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  by Proposition 10, thus its closure  $A_\tau$  is self-adjoint. The operator  $A_0 := -iu\partial_{x_j}$  is self-adjoint on  $D(A_0) := \{\psi \in L^2(\mathbb{R}^d, \mathbb{C}); \partial_{x_j}\psi \in L^2(\mathbb{R}^d, \mathbb{C})\}$ .  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  is a common core of  $A_\tau$  and  $A_0$ . For every  $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ ,  $\|(A_\tau - A_0)\psi\|_{L^2} = \tau\|(\Delta - V)\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Thus, by Proposition 16, for every  $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\|(L'_\tau - e^{u\partial_{x_j}})\psi\|_{L^2} = \|(e^{iA_\tau} - e^{iA_0})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that the operator  $e^{u\partial_{x_j}}$  is  $L^2$ -STAR.

Step 3: We prove that, if  $\varphi \in C^1(\mathbb{R}^d, \mathbb{R})$  and  $e^{i\lambda\varphi}$  is  $L^2$ -STAR for every  $\lambda \in \mathbb{R}$  then the operator  $e^{-i\partial_{x_j}\varphi}$  is  $L^2$ -STAR. Let  $\tau > 0$ . The assumption on  $\varphi$ , Step 2 and Lemma 14 prove that the operator

$$\tilde{L}_\tau := e^{i\frac{\varphi}{\tau}} e^{\tau\partial_{x_j}} e^{-i\frac{\varphi}{\tau}}$$

is  $L^2$ -STAR. The characteristic method proves that, for every  $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ ,

$$\tilde{L}_\tau\psi : x \in \mathbb{R}^d \mapsto \psi(x + \tau e_j) e^{-i\frac{\varphi(x + \tau e_j) - \varphi(x)}{\tau}} \in \mathbb{C}.$$

The continuity of the translation on  $L^2(\mathbb{R}^d, \mathbb{C})$  and the dominated convergence theorem prove that, for every  $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\|(\tilde{L}_\tau - e^{-i\partial_{x_j}\varphi})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Finally, by Lemma 14, the operator  $e^{-i\partial_{x_j}\varphi}$  is  $L^2$ -STAR.

Step 4: Iteration. We define by induction an increasing sequence of sets  $(\mathcal{H}_j)_{j \in \mathbb{N}}$  by  $\mathcal{H}_0 = \text{span}\{e^{-|x|^2/2}\}$  and, for every  $j \in \mathbb{N}^*$ ,

$$\mathcal{H}_j := \text{span}_{\mathbb{R}} \left\{ \varphi_0 - \sum_{k=1}^d \partial_{x_k} \varphi_k; \varphi_0, \dots, \varphi_d \in \mathcal{H}_{j-1} \right\}$$

and  $\mathcal{H}_\infty := \cup_{j \in \mathbb{N}} \mathcal{H}_j$ . Thanks to Lemma 14, Steps 1 and 3, for every  $\varphi \in \mathcal{H}_\infty$ , the operator  $e^{i\varphi}$  is  $L^2$ -STAR. Moreover, by the proof of [30, Lemma 5.2],  $\mathcal{H}_\infty$  is dense in  $L^2(\mathbb{R}^d, \mathbb{C})$  because it contains the linear combinations of Hermite functions.

Step 5: Conclusion. Let  $\varphi \in L^2(\mathbb{R}^d, \mathbb{R})$ . There exists  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{H}_\infty$  such that  $\|\varphi_n - \varphi\|_{L^2} \rightarrow 0$  as  $n \rightarrow \infty$ . Up to an extraction, one may assume that  $\varphi_n \rightarrow \varphi$  almost everywhere on  $\mathbb{R}^d$ , as  $n \rightarrow \infty$ . The dominated convergence theorem proves that, for every  $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\|(e^{i\varphi_n} - e^{i\varphi})\psi\|_{L^2} \rightarrow 0$  as  $n \rightarrow \infty$ . Finally, Step 4 and Lemma 14 prove that the operator  $e^{i\varphi}$  is  $L^2$ -STAR. □

### 4.3 Small-time reachability of transport operators along gradient vector fields

**Proposition 31.** *Let  $V$  satisfying (11). System (10) satisfies the following property: if  $\varphi \in C^\infty \cap L^2(\mathbb{R}^d, \mathbb{R})$  and  $f := \nabla\varphi \in W^{1,\infty} \cap L^4(\mathbb{R}^d, \mathbb{R}^d)$  then the unitary transport operator  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR.*

*Proof.* Let  $\varphi \in C^\infty \cap L^2(\mathbb{R}^d, \mathbb{R})$  be such that  $\nabla\varphi \in W^{1,\infty} \cap L^4(\mathbb{R}^d, \mathbb{R}^d)$  and  $f := 2\nabla\varphi$ . Let  $n \in \mathbb{N}^*$  and  $\tau > 0$ . Proposition 30 and Lemma 14 prove that the operator

$$L_{\tau,n} := \left( e^{i\frac{|\nabla\varphi|^2}{n\tau}} e^{i\frac{\varphi}{\tau}} e^{i\frac{\tau}{n}(\Delta-V)} e^{-i\frac{\varphi}{\tau}} \right)^n$$

is  $L^2$ -approximately reachable in time  $\tau^+$ , because  $\varphi, |\nabla\varphi|^2 \in L^2(\mathbb{R}^d, \mathbb{R})$ .

*Step 1: We prove that*

$$L_{\tau,n} = \left( e^{i\frac{|\nabla\varphi|^2}{n\tau}} \exp\left(i\frac{\tau}{n} e^{i\frac{\varphi}{\tau}} (\Delta - V) e^{-i\frac{\varphi}{\tau}}\right) \right)^n. \quad (37)$$

The operator  $(\Delta - V)$  is essentially self-adjoint on  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  by Proposition 10, because  $V$  satisfies (11), thus its closure  $A$  is self-adjoint. The operator  $B := \varphi/\tau$  is self-adjoint on  $D(B) := \{\psi \in L^2(\mathbb{R}^d, \mathbb{C}); \varphi\psi \in L^2(\mathbb{R}^d, \mathbb{C})\}$ . The operator  $e^{iB}$  is an isomorphism of  $C_c^\infty(\mathbb{R}^d, \mathbb{R})$  because  $\varphi \in C^\infty(\mathbb{R}^d)$ . Thus, Lemma 15 proves (37).

*Step 2: We prove that  $L_\tau$  is  $L^2$ -reachable in time  $\tau^+$ , where*

$$L_\tau := \exp(i\tau(\Delta - V) + \mathcal{T}_f).$$

The operator  $e^{i\frac{\varphi}{\tau}}(\Delta - V)e^{-i\frac{\varphi}{\tau}}$  is essentially self-adjoint on  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  by Proposition 10, thus its closure  $A_1$  is self-adjoint. The multiplicative operator  $B_1 := |\nabla\varphi|^2/\tau$  is self-adjoint on  $L^2(\mathbb{R}^d, \mathbb{C})$  because  $|\nabla\varphi|^2 \in L^\infty(\mathbb{R}^d, \mathbb{R})$ .  $A_1 + B_1$  is self-adjoint on  $D(A_1)$  because  $B_1$  is bounded. Thus, by Proposition 17, for every  $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\|(L_{\tau,n} - L'_\tau)\psi\|_{L^2} \xrightarrow{n \rightarrow \infty} 0$  where

$$L'_\tau := \exp\left(i\left\{\frac{|\nabla\varphi|^2}{\tau} + \tau e^{i\frac{\varphi}{\tau}}(\Delta - V)e^{-i\frac{\varphi}{\tau}}\right\}\right)$$

and standard computations prove that

$$\tau e^{i\frac{\varphi}{\tau}}(\Delta - V)e^{-i\frac{\varphi}{\tau}} + \frac{|\nabla\varphi|^2}{\tau} = \tau(\Delta - V) - 2i\langle \nabla\varphi, \nabla \rangle - i\Delta\varphi = \tau(\Delta - V) - i\mathcal{T}_f \quad (38)$$

thus  $L'_\tau = L_\tau$ . By Lemma 14, the operator  $L_\tau$  is  $L^2$ -approximately reachable in time  $\tau^+$ .

*Step 3: We prove that the operator  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR.* The operator (38) defined on  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  has a self-adjoint closure  $A_\tau$  by Propositions 10 and 12. The operator  $A_0 := -i\mathcal{T}_f$  is self-adjoint because  $f \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ .  $C_c^\infty(\mathbb{R}^d, \mathbb{C})$  is a common core of  $A_\tau$  and  $A_0$ . For every  $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ ,  $\|(A_\tau - A_0)\psi\|_{L^2} = \tau\|(\Delta - V)\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Thus, by Proposition 16, for every  $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\|(L_\tau - e^{\mathcal{T}_f})\psi\|_{L^2} = \|(e^{iA_\tau} - e^{iA_0})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Finally, by Lemma 14, the operator  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR.  $\square$

### 4.4 Small-time reachability of transport operators along any vector field

Let  $(e_1, \dots, e_d)$  be the canonical basis of  $\mathbb{R}^d$ . We introduce the sets

$$\mathcal{L} := \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) \text{ globally Lipschitz ; } \forall t \in \mathbb{R}, e^{t\mathcal{T}_f} \text{ is } L^2\text{-STAR}\},$$

$$\mathfrak{G} := \{e_1, \dots, e_d\} \cup \{\nabla\varphi; \varphi \in C^\infty \cap L^2(\mathbb{R}^d, \mathbb{R}), \nabla\varphi \in W^{1,\infty} \cap L^4(\mathbb{R}^d, \mathbb{R}^d)\}$$

and the Lie subalgebra of  $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$  generated by  $\mathfrak{G}$ ,

$$\mathfrak{L}_0 := \text{Lie}(\mathfrak{G}).$$

By the first conclusion of Proposition 30 and Proposition 31,  $\mathfrak{G} \subset \mathfrak{L}$ .

**Proposition 32.**  $\mathfrak{L}$  is a Lie algebra thus  $\mathfrak{L}_0 \subset \mathfrak{L}$ .

The proof is the same as for Proposition 24. Only Lemma 25 requires the following adaptation.

**Lemma 33.** Let  $f, g \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$  globally Lipschitz and  $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ . Then

$$\|(e^{-t\mathcal{T}_g} \mathcal{T}_f e^{t\mathcal{T}_g} - \mathcal{T}_f - t\mathcal{T}_{[f,g]}) \varphi\|_{L^2} = o_{t \rightarrow 0}(t).$$

The proof of Lemma 33 is the same as for Lemma 25, with  $L^\infty(K)$ -norms in the estimate (25), where  $K \subset \mathbb{R}^d$  is the compact set defined by  $K = \text{Supp}(\varphi)$ .

**Proposition 34.**  $\mathfrak{L}_0$  contains any linear combination of vector fields of the form

$$\partial_{x_1}^{n_1} \dots \partial_{x_d}^{n_d} e^{-|x|^2/2} e_j \tag{39}$$

where  $n = (n_1, \dots, n_d) \in \mathbb{N}^d$  and  $j \in \{1, \dots, d\}$ . Thus Proposition 29 holds.

*Proof.* Step 1: We prove the first statement. Let  $n \in \mathbb{N}^d$  and  $j \in \{1, \dots, d\}$ . By definition of  $\mathfrak{L}_0$ , this Lie algebra contains the following vector fields

$$\begin{aligned} h_j &:= -4[\nabla x_j e^{-|x|^2/4}, \nabla e^{-|x|^2/4}] = (2 + |x|^2) e^{-|x|^2/2} e_j, \\ k_j &:= -8[\nabla \frac{x_j^2}{2} e^{-|x|^2/4}, \nabla x_j e^{-|x|^2/4}] = (8 - 2x_j^2 + x_j^2 |x|^2) e^{-|x|^2/2} e_j, \\ \text{ad}_{e_j}^2 h_j &= (-2x_j^2 - |x|^2 + x_j^2 |x|^2) e^{-|x|^2/2} e_j, \\ \ell_j &:= k_j - \text{ad}_{e_j}^2 h_j = (8 + |x|^2) e^{-|x|^2/2} e_j, \\ m_j &:= (\ell_j - h_j)/6 = e^{-|x|^2/2} e_j, \\ \text{ad}_{e_1}^{n_1} \dots \text{ad}_{e_d}^{n_d} m_j &= \partial_{x_1}^{n_1} \dots \partial_{x_d}^{n_d} e^{-|x|^2/2} e_j. \end{aligned}$$

Step 2: Density of  $\mathfrak{L}_0$ . By Step 1,

$$\text{Span}\{\psi_n(x) e_j; n \in \mathbb{N}^d, j \in \{1, \dots, d\}\} \subset \mathfrak{L}_0,$$

where the  $\psi_n$  are the Hermite functions on  $\mathbb{R}^d$ . Thus  $\mathfrak{L}_0$  is dense in  $L^2(\mathbb{R}^d)$ , but it is also, for any  $s \geq 0$ , dense in  $L^2((1 + |x|^2)^s dx)$  and in  $H^s(\mathbb{R}^d)$  (by Plancherel, because  $\widehat{\psi}_n = (i)^{|n|} \psi_n$ ).

Step 3: We prove Proposition 29. Let  $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ . By Step 2, there exists a sequence  $(f_k)_{k \in \mathbb{N}} \subset \mathfrak{L}_0$  such that  $\|f - f_k\|_{W^{1,\infty}} \rightarrow 0$  as  $k \rightarrow \infty$ . By Proposition 32, for every  $k \in \mathbb{N}$ , the operator  $e^{\mathcal{T}_{f_k}}$  is  $L^2$ -STAR. By Lemma 14, to prove that  $e^{\mathcal{T}_f}$  is  $L^2$ -STAR, it suffices to prove that

$$\forall \psi \in L^2(\mathbb{R}^d, \mathbb{C}), \quad \|(e^{\mathcal{T}_{f_k}} - e^{\mathcal{T}_f})\psi\|_{L^2} \xrightarrow[k \rightarrow \infty]{} 0. \tag{40}$$

It suffices to prove (40) for any  $\psi$  in a dense subset of  $L^2$  because the operators  $(e^{\mathcal{T}_{f_k}} - e^{\mathcal{T}_f})$  are bounded uniformly with respect to  $k$  (by 2). So we consider  $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ . By (33) and (34), for almost every  $x \in \mathbb{R}^d$ , we have the convergence  $e^{\mathcal{T}_{f_k}} \psi(x) \rightarrow e^{\mathcal{T}_f} \psi(x)$  as  $k \rightarrow \infty$  and the domination  $|e^{\mathcal{T}_{f_k}} \psi(x)| \leq \|\psi\|_\infty e^{\|f\|_\infty + 1} 1_K(x)$  where  $K$  is a compact subset of  $\mathbb{R}^d$  that contains  $\cup_{k \in \mathbb{N}} \text{Supp}(\psi)$ . The dominated convergence theorem gives the conclusion.  $\square$

## 5 Composition by diffeomorphism on $\mathbb{T}^d$ at higher regularity

In this section, we consider system (7) with state-space  $\mathcal{H} = H^{2s}(\mathbb{T}^d, \mathbb{C})$  where  $s \in \mathbb{N}^*$ . The goal of this section is to prove the following regular versions of Theorem 4 and Corollary 5.

**Theorem 35.** *Let  $s \in \mathbb{N}^*$  and  $V \in W^{2(s+1), \infty}(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for every  $\psi_0 \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ ,*

$$\{\mathcal{L}_P \psi_0; P \in \text{Diff}^0(\mathbb{T}^d)\} \subset \text{Adh}_{H^{2s}(\mathbb{T}^d)} \text{Reach}_{st}(\psi_0).$$

**Corollary 36.** *Let  $s \in \mathbb{N}^*$  and  $V \in W^{2(s+1), \infty}(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for any  $k \in \mathbb{Z}^d$ ,  $J_1, J_2 \subset \{1, \dots, d\}$ , and  $\varepsilon > 0$ , there exist  $T \in [0, \varepsilon]$ ,  $\theta \in \mathbb{R}$  and  $u : [0, T] \rightarrow \mathbb{R}^{2d}$  piecewise constant such that*

$$\|\psi(T; u, f_{k, J_1}) - e^{i\theta} f_{k, J_2}\|_{H^{2s}(\mathbb{T}^d)} < \varepsilon,$$

where the functions  $f_{k, J_i}$ ,  $i = 1, 2$ , are defined as in (9).

The proof of Theorem 35 follows the same strategy as the proof of Theorem 4: each time we obtain an  $L^2(\mathbb{T}^d, \mathbb{C})$ -convergence result, we prove an additional  $H^{2(s+1)}(\mathbb{T}^d, \mathbb{C})$ -bound, so that an interpolation argument provides the  $H^{2s}(\mathbb{T}^d, \mathbb{C})$ -convergence. Corollary 36 follows from Theorem 35 exactly as Corollary 5 does from 4 (see end of Sec. 3.1). In Section 5.1 we estimate the growth of the Sobolev norms of the solution. In Section 5.2, we prove a regular version of the small-time control of the phase. In Section 6.3, we prove a regular version of the small-time control of transport operators, and consequently Theorem 35 and Corollary 36.

### 5.1 Growth of Sobolev norms

Let us start by recalling the following well-posedness result.

**Proposition 37.** *Let  $s \in \mathbb{N}^*$ ,  $V \in W^{2s, \infty}(\mathbb{T}^d, \mathbb{R})$ ,  $T > 0$ ,  $u = (u_1, \dots, u_{2d}) : (0, T) \rightarrow \mathbb{R}^{2d}$  piecewise constant and  $\psi_0 \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ . The solution  $\psi$  of the Cauchy problem (7) belongs to  $C^0([0, T], H^{2s}(\mathbb{T}^d, \mathbb{C}))$ . Moreover, there exists  $C = C(s, V, \|u\|_{L^1}) > 0$  such that, for every  $t \in [0, T]$ ,  $\|\psi(t; u, \cdot)\|_{\mathcal{L}(H^{2s})} \leq C$ .*

*Proof.* When  $\|u\|_{L^1}$  is small enough, this can be proved by applying a fixed point argument in  $C^0([0, T], H^{2s}(\mathbb{T}^d, \mathbb{C}))$  on the Duhamel formula

$$\psi(t) = e^{i\Delta} \psi_0 - i \int_0^t e^{i\Delta(t-s)} \left( V + \sum_{j=1}^d (u_{2j-1}(s) \sin + u_{2j}(s) \cos) \langle b_j, x \rangle \right) \psi(s) ds$$

because the multiplicative operators  $V$ ,  $\sin \langle b_j, x \rangle$ ,  $\cos \langle b_j, x \rangle$  are bounded on  $H^{2s}(\mathbb{T}^d, \mathbb{C})$ . When  $\|u\|_{L^1}$  is not small enough, we iterate the previous result. The estimate is a consequence of Gronwall Lemma.  $\square$

We shall need the following regular version of the Trotter-Kato formula, whose proof can be found in [14].

**Proposition 38.** Let  $A, B, \mathcal{H}$  be as in Proposition 17 and  $X$  be a dense vector subspace of  $\mathcal{H}$  equipped with a norm  $\|\cdot\|_X$ . We assume there exists  $C > 0$  such that, for every  $t \in [0, 1]$ ,  $e^{itA}, e^{itB}, e^{i(A+B)}$  are uniformly bounded operators on  $X$  and

$$\|e^{tA}\|_{\mathcal{L}(X)}, \|e^{tB}\|_{\mathcal{L}(X)} \leq e^{Ct}. \quad (41)$$

Then, for every (strict) interpolation space  $Y$  between  $\mathcal{H}$  and  $X$ , and for every  $\psi_0 \in Y$ ,

$$\left\| \left( e^{i\frac{B}{n}} e^{i\frac{A}{n}} \right)^n \psi_0 - e^{i(A+B)} \psi_0 \right\|_Y \xrightarrow{n \rightarrow +\infty} 0$$

We conclude this section with an estimate on the growth of Sobolev norms.

**Proposition 39.** Let  $s \in \mathbb{N}^*$ ,  $V, g \in C^{2s, \infty}(\mathbb{T}^d, \mathbb{R})$  be such that  $\nabla V, \nabla g \in W^{2s-1, \infty}(\mathbb{T}^d, \mathbb{R}^d)$ . Let  $f \in W^{2s+1, \infty}(\mathbb{T}^d, \mathbb{R}^d)$  be a vector field, and  $a, b, c \in \mathbb{R}$ . For  $\tau \in (0, 1]$ , consider the self-adjoint operator

$$D(A_\tau) = H^2(\mathbb{T}^d, \mathbb{C}), \quad A_\tau = \tau^a(\Delta - V) + i\tau^b \mathcal{T}_f + \tau^c g.$$

There exists  $C = C(s, \nabla V, f, \nabla g) > 0$  such that for every  $(\tau, t) \in (0, 1] \times [0, 1]$ ,  $\|e^{itA_\tau}\|_{\mathcal{L}(H^{2s})} \leq e^{C(\tau^a + \tau^b + \tau^c)t}$ .

*Proof.* For smooth data,  $v(t) := e^{itA_\tau} \psi_0$  satisfies

$$i\partial_t v = -\tau^a(\Delta - V)v - i\tau^b \mathcal{T}_f v - \tau^c g v,$$

$Im \langle \Delta v, v \rangle_{H^{2s}} = Im \langle V \partial_x^\alpha v, \partial_x^\alpha v \rangle = Im \langle g \partial_x^\alpha v, \partial_x^\alpha v \rangle = 0$ , thus for any multi-index  $\alpha \in \mathbb{N}^d$  with  $|\alpha| \leq 2s$  we have

$$\begin{aligned} \partial_t \|\partial_x^\alpha v\|_{L^2}^2 &\leq \tau^a |\langle V v, \partial_x^{2\alpha} v \rangle_{L^2}| + \tau^b |\langle \mathcal{T}_f v, \partial_x^{2\alpha} v \rangle_{L^2}| + \tau^c |\langle g v, \partial_x^{2\alpha} v \rangle_{L^2}| \\ &\leq C(\tau^a + \tau^b + \tau^c) \|v\|_{H^{2s}}^2, \end{aligned}$$

where  $C = C(\|\nabla V\|_{W^{2s-1, \infty}}, \|f\|_{W^{2s+1, \infty}}, \|\nabla g\|_{W^{2s-1, \infty}})$ . Hence, we deduce that

$$\|v(t)\|_{H^{2s}} \leq e^{C(\tau^a + \tau^b + \tau^c)t} \|\psi_0\|_{H^{2s}}. \quad (42)$$

□

## 5.2 Small-time $H^{2s}$ -control of the phase on $\mathbb{T}^d$

**Proposition 40.** Let  $V \in W^{2(s+1), \infty}(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for every  $\varphi \in C^{2s}(\mathbb{T}^d, \mathbb{R})$ , the operator  $e^{i\varphi}$  is  $H^{2s}$ -STAR.

*Proof.* We follow the proof of Proposition 22, with the same notations.

*Step 1:* We prove that, for every  $\varphi \in \mathcal{H}_0 = \text{span}_{\mathbb{R}} \{\sin \langle e_j, x \rangle, \cos \langle e_j, x \rangle; j \in \{1, \dots, d\}\}$ , the operator  $e^{i\varphi}$  is  $H^{2s}$ -STAR. For any  $\tau > 0$ , the operator  $L_\tau = e^{i\tau(\Delta - V) + i\varphi}$  is  $H^{2s}$ -exactly reachable in time  $\tau$ . Let  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ . In the proof of Proposition 22, we proved that  $\|(L_\tau - e^{i\varphi})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Since  $V, \varphi \in W^{2(s+1), \infty}(\mathbb{T}^d, \mathbb{R})$ , Proposition 39 provides  $C > 0$  such that, for every  $\tau \in [0, 1]$ ,  $\|L_\tau\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Thus, by interpolation  $\|(L_\tau - e^{i\varphi})\psi\|_{H^{2s}} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that  $e^{i\varphi}$  is  $H^{2s}$ -STAR.

*Step 2:* We prove that, if  $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})$  and  $e^{i\lambda\varphi}$  is  $H^{2s}$ -STAR for every  $\lambda \in \mathbb{R}$  then  $e^{-i|\nabla\varphi|^2}$  is  $H^{2s}$ -STAR. For any  $\tau > 0$ , the operator  $\tilde{L}_\tau = \exp(i\tau(\Delta - V) - \sqrt{\tau}(2\langle \nabla\varphi, \nabla \rangle + \Delta\varphi) - i|\nabla\varphi|^2)$  is  $H^{2s}$ -approximately reachable in time  $\tau^+$ . Let  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ . In the proof of Proposition 22, we proved that  $\|(\tilde{L}_\tau - e^{-i|\nabla\varphi|^2})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Since  $V \in W^{2s+2, \infty}(\mathbb{T}^d, \mathbb{R})$



and  $\varphi \in W^{2s+4,\infty}(\mathbb{T}^d, \mathbb{R})$ , Proposition 39 provides  $C > 0$  such that, for every  $\tau \in [0, 1]$ ,  $\|\tilde{L}_\tau\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Thus, by interpolation  $\|(\tilde{L}_\tau - e^{-i|\nabla\varphi|^2})\psi\|_{H^{2s}} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that  $e^{-i|\nabla\varphi|^2}$  is  $H^{2s}$ -STAR.

*Step 3: Iteration.* For every  $\varphi \in \mathcal{H}_\infty$ ,  $e^{i\varphi}$  is  $H^{2s}$ -STAR. Moreover, the set  $\mathcal{H}_\infty$  is dense in  $C^{2s}(\mathbb{T}^d, \mathbb{R})$ .

*Step 4: Conclusion.* Let  $\varphi \in C^{2s}(\mathbb{T}^d, \mathbb{R})$ . There exists  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{H}_\infty$  such that  $\|\varphi_n - \varphi\|_{C^{2s}} \rightarrow 0$  as  $n \rightarrow \infty$ . Then, for every  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ ,  $\|(e^{i\varphi_n} - e^{i\varphi})\psi\|_{H^{2s}} \rightarrow 0$  as  $n \rightarrow \infty$ . Finally, Step 3 and Lemma 14 prove that the operator  $e^{i\varphi}$  is  $H^{2s}$ -STAR.  $\square$

### 5.3 Small time $H^{2s}$ -reachability of transport operators on $\mathbb{T}^d$

Theorem 35 is a consequence of Lemma 14 and the following regular version of Proposition 23.

**Proposition 41.** *Let  $V \in W^{2(s+1),\infty}(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: if  $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})$  and  $f := \nabla\varphi$  then the operator  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.*

*Proof.* We follow the proof of Proposition 23, with the same notations.

*Step 1: The operator  $L_{\tau,n}$  is  $H^{2s}$ -approximately reachable in time  $\tau^+$ .* This is a consequence of Proposition 40.

*Step 2: We prove that  $L_\tau$  is  $H^{2s}$ -approximately reachable in time  $\tau^+$ .* In the application of Proposition 38 (instead of Proposition 17), the bounds (41) are given by Proposition 39.

*Step 3: We prove that  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.* Let  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ . In the proof of Proposition 23, we have proved that  $\|(L_\tau - e^{\mathcal{T}_f})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Since  $\varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})$ , Proposition 39 provides  $C > 0$  such that, for every  $\tau \in [0, 1]$ ,  $\|L_\tau\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Thus by interpolation,  $\|(L_\tau - e^{\mathcal{T}_f})\psi\|_{H^{2s}} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that the operator  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.  $\square$

**Proposition 42.** *Let  $V \in W^{2(s+1),\infty}(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for every vector field  $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ , the operator  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.*

To prove Proposition 42, we introduce the set

$$\mathfrak{L}^s := \{f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d); \forall t \in \mathbb{R}, e^{t\mathcal{T}_f} \text{ is } H^{2s}\text{-STAR}\}, \quad (43)$$

By Proposition 41,  $\mathfrak{O} \subset \mathfrak{L}^s$ .

**Proposition 43.**  $\mathfrak{L}^s$  is a Lie algebra, thus  $\mathfrak{L}_0 \subset \mathfrak{L}^s$ .

*Proof.* We follow the proof of Proposition 24.

*Step 1: We prove that  $\mathfrak{L}^s$  is stable by summation, i.e.  $(f, g \in \mathfrak{L}^s \implies e^{\mathcal{T}_{f+g}}$  is  $H^{2s}$ -STAR).* When using Trotter-Kato formula, we apply Proposition 38 instead of Proposition 17. Since  $f, g \in W^{2(s+1),\infty}(\mathbb{T}^d, \mathbb{R}^d)$ , Proposition 39 provides  $C > 0$  such that, for every  $t \in [0, 1]$ ,  $\|e^{t\mathcal{T}_f}\|_{\mathcal{L}(H^{2s})}, \|e^{t\mathcal{T}_g}\|_{\mathcal{L}(H^{2s})} \leq e^{Ct}$ , which corresponds to the assumption (41).

*Step 2: We prove that  $\mathfrak{L}^s$  is stable by Lie bracket i.e.  $(f, g \in \mathfrak{L}^s \implies e^{\mathcal{T}_{[f,g]}}$  is  $H^{2s}$ -STAR).* Let  $t \in \mathbb{R}^*$ . For every  $n \in \mathbb{N}^*$ , the operator  $L_{t,n}$  is  $H^{2s}$ -STAR. When using Trotter-Kato formula, instead of Proposition 17, we apply Proposition 38 with  $A \leftarrow \frac{1}{t}\mathcal{T}_f$  and  $B \leftarrow \frac{1}{t}e^{-t\mathcal{T}_g}\mathcal{T}_g e^{t\mathcal{T}_g}$ . An explicit expression for  $B$  is given in the proof of Lemma 25 (Step 1). An energy argument (as in the proof of Proposition 39) proves the existence of  $C > 0$  such that, for every  $\theta \in [0, 1]$ ,

$\|e^{\theta B}\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Here, the equation has time-depending coefficients, but this dependence is smooth, which allows the energy argument. Thus, for every  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ ,  $\|(L_{t,n} - L_t)\psi\|_{H^{2s}} \rightarrow 0$  as  $n \rightarrow \infty$  and Lemma 14 proves that  $L_t$  is  $H^{2s}$ -STAR.

The explicit expression given in the Step 1 of the proof of Lemma 25 proves that (writing  $L_t = e^{A_t}$ )

$$A_t = \langle F_t, \nabla \rangle + a_t \quad \text{where} \quad F_t := \frac{1}{t}(\phi_g^t \star f - f), \quad a_t = \frac{1}{2t}(\operatorname{div} f \circ \phi_g^{-t} - \operatorname{div} f) + \frac{1}{2}G(t, \cdot).$$

Since the vector fields  $f, g$  are smooth, there exists  $C > 0$  such that, for every  $t \in [0, 1]$ ,  $\|F_t\|_{W^{2s+3, \infty}}, \|a_t\|_{W^{2s+2, \infty}} \leq C$ . Thus, by Proposition 39, there exists  $C > 0$  such that, for every  $t \in [0, 1]$ ,  $\|L_t\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ .

For  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ , we have proved in the proof of Proposition 24 that  $\|(L_t - e^{\mathcal{T}_{[f, g]}})\psi\|_{L^2} \rightarrow 0$  as  $t \rightarrow 0$ , thus an interpolation argument proves that  $\|(L_t - e^{\mathcal{T}_{[f, g]}})\psi\|_{H^{2s}} \rightarrow 0$  as  $t \rightarrow 0$ . Finally Lemma 14 proves that  $e^{\mathcal{T}_{[f, g]}}$  is  $H^{2s}$ -STAR.  $\square$

*Proof of Proposition 42:* Let  $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$  a vector field. There exists a sequence  $(f_\epsilon)_{\epsilon > 0}$  of trigonometric polynomials such that  $\|f_\epsilon - f\|_{C^{2s+1}} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Propositions 26 and 43 prove that, for every  $\epsilon > 0$ ,  $e^{\mathcal{T}_{f_\epsilon}}$  is  $H^{2s}$ -STAR. For every  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ ,  $\|(e^{\mathcal{T}_{f_\epsilon}} - e^{\mathcal{T}_f})\psi\|_{H^{2s}} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Thus, by Lemma 14, the operator  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.  $\square$

*Proof of Theorem 35.* It suffices to combine Propositions 20, 42 and Lemma 14.  $\square$

## 6 Composition by diffeomorphism on $\mathbb{R}^d$ at higher regularity

The goal of this section is to prove Corollary 8 and the following higher regular versions of Theorem 6.

**Theorem 44.** *Let  $s \in \mathbb{N}$  and  $V \in W^{2(s+1), \infty}(\mathbb{R}^d, \mathbb{R})$ . For every  $\psi_0 \in H^{2s}(\mathbb{R}^d, \mathbb{C})$ , system (7) satisfies*

$$\{\mathcal{L}_P \psi_0; P \in \operatorname{Diff}_c^0(\mathbb{R}^d)\} \subset \operatorname{Adh}_{H^{2s}} \operatorname{Reach}_{st}(\psi_0).$$

### 6.1 Well-posedness

We point out one difference w.r.t. the previous section on  $\mathbb{T}^d$ , concerning the proof of the well-posedness of system (10) in  $H^{2s}(\mathbb{R}^d, \mathbb{C})$ . This is due to the fact that, contrarily to system (7), system (10) has unbounded control operators  $x_j$ , hence the proof of Proposition 37 does not work. Anyways, this difficulty can be handled, since  $\nabla x_j$  is bounded (see, e.g., [22]). We recall this fact in the following proposition.

**Proposition 45.** *Let  $s \in \mathbb{N}$ ,  $V \in W^{2s, \infty}(\mathbb{R}^d, \mathbb{R})$ ,  $T > 0$ ,  $u = (u_0, \dots, u_d) : (0, T) \rightarrow \mathbb{R}^{d+1}$  piecewise constant and  $\psi_0 \in H^{2s}(\mathbb{R}^d, \mathbb{C})$ . The solution  $\psi$  of the Cauchy problem (10) belongs to  $C^0([0, T], H^{2s}(\mathbb{R}^d, \mathbb{C}))$ . Moreover, there exists  $C = C(s, V, u) > 0$  such that, for every  $t \in [0, T]$ ,  $\|\psi(t; u, \cdot)\|_{\mathcal{L}(H^{2s})} \leq C$ .*

*Proof.* It suffices to prove the result when  $u = (u_0, u_1, \dots, u_{d+1}) \in \mathbb{R}^{d+1}$  is constant. Let  $v := (u_1, \dots, u_d) \in \mathbb{R}^d$ . For every  $T > 0$ , there exists  $C = C(T, v)$  such that for every  $t \in [0, T]$ ,  $\|e^{it(\Delta + \langle v, x \rangle)}\|_{\mathcal{L}(H^{2s})} \leq C$ ; indeed,

$$\|e^{it(\Delta + \langle v, x \rangle)}\psi_0\|_{H^s}^2 = \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{\psi}_0(\xi - vt)|^2 d\xi = \int_{\mathbb{R}^d} |\xi + vt|^{2s} |\widehat{\psi}_0(\xi)|^2 d\xi. \quad (44)$$

Thus, a fixed point argument in  $C^0([0, T], H^{2s}(\mathbb{R}^d, \mathbb{C}))$  inspired by the Duhamel formula

$$\psi(t) = e^{it(\Delta + \langle v, x \rangle)} \psi_0 + \int_0^t e^{i(t-s)(\Delta + \langle v, x \rangle)} (V + u_{d+1} e^{-|x|^2/2}) \psi(s) ds$$

proves the well-posedness in  $H^{2s}(\mathbb{R}^d, \mathbb{C})$ .  $\square$

## 6.2 Small-time $H^{2s}$ -control of the phase on $\mathbb{R}^d$

**Proposition 46.** *Let  $V \in W^{2(s+1), \infty}(\mathbb{R}^d, \mathbb{R})$ . System (10) satisfies the following properties:*

- for every  $j \in \{1, \dots, d\}$  and  $u \in \mathbb{R}$ , the operator  $e^{u\partial_{x_j}}$  is  $H^{2s}$ -STAR,
- for every  $\varphi \in H^\sigma(\mathbb{R}^d, \mathbb{R})$ ,  $\sigma > 2s + d/2$ , the operator  $e^{i\varphi}$  is  $H^{2s}$ -STAR

*Proof.* We follow the proof of Proposition 30, with the same notations.

*Step 1:* We prove that, for every  $\varphi \in \text{span}\{x_1, \dots, x_d, e^{-|x|^2/2}\}$ , the operator  $e^{i\varphi}$  is  $H^{2s}$ -STAR. Let  $\alpha \in \mathbb{R}^{d+1}$  and  $\varphi : x \in \mathbb{R}^d \mapsto \alpha_1 x_1 + \dots + \alpha_d x_d + \alpha_{d+1} e^{-|x|^2/2}$ . For any  $\tau > 0$ , the operator  $L_\tau = e^{i\tau(\Delta - V) + i\varphi}$  is  $H^{2s}$ -exactly reachable in time  $\tau$ . Let  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ . In the proof of Proposition 30, we proved that  $\|(L_\tau - e^{i\varphi})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Since  $\nabla V, \nabla \varphi \in W^{2(s+1), \infty}(\mathbb{R}^d, \mathbb{R})$ , Proposition 39 (where the only change is that  $\mathbb{T}^d$  is replaced with  $\mathbb{R}^d$ ) provides  $C > 0$  such that, for every  $\tau \in [0, 1]$ ,  $\|L_\tau\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Thus, by interpolation  $\|(L_\tau - e^{i\varphi})\psi\|_{H^{2s}} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that  $e^{i\varphi}$  is  $H^{2s}$ -STAR.

*Step 2:* We prove that, for every  $j \in \{1, \dots, d\}$  and  $u \in \mathbb{R}$ , the operator  $e^{u\partial_{x_j}}$  is  $H^{2s}$ -STAR. Let  $j \in \{1, \dots, d\}$ ,  $u \in \mathbb{R}^*$ . By Step 1 and Lemma 14, for every  $\tau > 0$ ,  $L'_\tau$  is  $H^{2s}$ -approximately reachable in time  $\tau^+$ . Let  $\psi \in H^{2s}(\mathbb{R}^d, \mathbb{C})$ . In the proof of Proposition 30 we proved that  $\|(L'_\tau - e^{u\partial_{x_j}})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Since  $V \in W^{2(s+1), \infty}(\mathbb{R}^d, \mathbb{R})$ , Proposition 39 (where the only change is that  $\mathbb{T}^d$  is replaced with  $\mathbb{R}^d$ ) provides  $C > 0$  such that, for every  $\tau \in [0, 1]$ ,  $\|L'_\tau\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Thus, by interpolation  $\|(L'_\tau - e^{u\partial_{x_j}})\psi\|_{H^{2s}} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that  $e^{u\partial_{x_j}}$  is  $H^{2s}$ -STAR.

*Step 3:* We prove that, if  $\varphi \in W^{2s+3, \infty}(\mathbb{R}^d, \mathbb{R})$  and  $e^{i\lambda\varphi}$  is  $H^{2s}$ -STAR for every  $\lambda \in \mathbb{R}$  then  $e^{-i\partial_{x_j}\varphi}$  is  $H^{2s}$ -STAR. For any  $\tau > 0$ , the operator  $\tilde{L}_\tau = \exp(\tau\partial_{x_j} - i\partial_{x_j}\varphi)$  is small-time  $H^{2s}$ -approximately reachable. Let  $\psi \in H^{2s}(\mathbb{R}^d, \mathbb{C})$ . In the proof of Proposition 30, we proved that  $\|(\tilde{L}_\tau - e^{-i\partial_{x_j}\varphi})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Since  $V, \partial_{x_j}\varphi \in W^{2s+2, \infty}(\mathbb{R}^d, \mathbb{R})$ , Proposition 39 (where the only change is that  $\mathbb{T}^d$  is replaced with  $\mathbb{R}^d$ ) provides  $C > 0$  such that, for every  $\tau \in [0, 1]$ ,  $\|\tilde{L}_\tau\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Thus, by interpolation  $\|(\tilde{L}_\tau - e^{-i\partial_{x_j}\varphi})\psi\|_{H^{2s}} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that the operator  $e^{-i\partial_{x_j}\varphi}$  is  $H^{2s}$ -STAR.

*Step 4: Iteration.* For every  $\varphi \in \mathcal{H}_\infty$ ,  $e^{i\varphi}$  is  $H^{2s}$ -STAR. Moreover, the set  $\mathcal{H}_\infty$  is dense in  $H^{2s}(\mathbb{T}^d, \mathbb{R})$  for any  $s > 0$ .

*Step 5: Conclusion.* Let  $\varphi \in H^\sigma(\mathbb{T}^d, \mathbb{R})$  with  $\sigma > 2s + d/2$ . There exists  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{H}_\infty$  such that  $\|\varphi_n - \varphi\|_{H^\sigma} \rightarrow 0$  as  $n \rightarrow \infty$  (and, in particular,  $\|\varphi_n^{(k)} - \varphi^{(k)}\|_{L^\infty} \rightarrow 0$ , as  $n \rightarrow \infty$ , for any  $k \leq 2s$ , being  $H^{\sigma-k}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$  for  $\sigma - k > d/2$ ). Then, for every  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ ,  $\|(e^{i\varphi_n} - e^{i\varphi})\psi\|_{H^{2s}} \rightarrow 0$  as  $n \rightarrow \infty$ . Finally, Step 4 and Lemma 14 prove that the operator  $e^{i\varphi}$  is  $H^{2s}$ -STAR.  $\square$

### 6.3 Small time $H^{2s}$ -reachability of transport operators on $\mathbb{R}^d$

Theorem 44 is a consequence of Lemma 14 and the following regular version of Proposition

**Proposition 47.** *Let  $V \in W^{2(s+1),\infty}(\mathbb{R}^d, \mathbb{R})$ . System (7) satisfies the following property: if  $\varphi, |f|^2 \in H^\sigma(\mathbb{R}^d, \mathbb{R})$ ,  $\sigma > 2s + d/2$ , where  $f := \nabla\varphi$ , then the operator  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.*

*Proof.* We follow the proof of Proposition 31, with the same notations.

*Step 1:* The operator  $L_{\tau,n}$  is  $H^{2s}$ -approximately reachable in time  $\tau^+$ . This is a consequence of the second statement of Proposition 46.

*Step 2:* We prove that  $L_\tau$  is  $H^{2s}$ -approximately reachable in time  $\tau^+$ . In the application of Proposition 38 (instead of Proposition 17), the bounds (41) are given by Proposition 39 (where the only change is that  $\mathbb{T}^d$  is replaced with  $\mathbb{R}^d$ ). Notice that the hypothesis of Proposition 39 are satisfied since  $H^\sigma(\mathbb{R}^d) \subset W^{2s+1,\infty}(\mathbb{R}^d)$  for  $\sigma > 2s + d/2$ .

*Step 3:* We prove that  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR. Let  $\psi \in H^{2s}(\mathbb{T}^d, \mathbb{C})$ . In the proof of Proposition 31, we have proved that  $\|(L_\tau - e^{\mathcal{T}_f})\psi\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$ . Since  $\varphi, f \in W^{2s+1,\infty}(\mathbb{R}^d)$ , Proposition 39 (where the only change is that  $\mathbb{T}^d$  is replaced with  $\mathbb{R}^d$ ) provides  $C > 0$  such that, for every  $\tau \in [0, 1]$ ,  $\|L_\tau\|_{\mathcal{L}(H^{2(s+1)})} \leq C$ . Thus by interpolation,  $\|(L_\tau - e^{\mathcal{T}_f})\psi\|_{H^{2s}} \rightarrow 0$  as  $\tau \rightarrow 0$ . By Lemma 14, this proves that the operator  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.  $\square$

**Proposition 48.** *Let  $V \in W^{2(s+1),\infty}(\mathbb{T}^d, \mathbb{R})$ . System (7) satisfies the following property: for every vector field  $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ , the operator  $e^{\mathcal{T}_f}$  is  $H^{2s}$ -STAR.*

To prove Proposition 48, we introduce the sets

$$\mathfrak{L}^s := \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}^d); \nabla f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d), \forall t \in \mathbb{R}, e^{t\mathcal{T}_f} \text{ is } H^{2s}\text{-STAR}\}, \quad (45)$$

$$\mathfrak{G} := \{e_1, \dots, e_d\} \cup \{\nabla\varphi; \varphi \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})\}$$

where  $\mathcal{S}(\mathbb{R}^d, \mathbb{R})$  denotes the Schwartz class, and the Lie subalgebra of  $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$  generated by  $\mathfrak{G}$ ,

$$\mathfrak{L}_0 := \text{Lie}(\mathfrak{G}).$$

By Proposition 47,  $\mathfrak{G} \subset \mathfrak{L}^s$ .

**Proposition 49.**  $\mathfrak{L}^s$  is a Lie algebra, thus  $\mathfrak{L}_0 \subset \mathfrak{L}^s$ .

The proofs of Propositions 49 and 48 are completely analogous to the proofs of Proposition 43 and 42, so we omit the details.

*Proof of Theorem 44.* It suffices to combine Propositions 20, 48 and Lemma 14.  $\square$

### 6.4 Proof of Corollary 8

We introduce the normed space  $H_h^1(\mathbb{R}^d)$  defined by

$$H_h^1(\mathbb{R}^d) = \{f \in H^1(\mathbb{R}^d, \mathbb{C}); |x|f \in L^2(\mathbb{R}^d, \mathbb{C})\}, \quad \|f\|_{H_h^1} = (\|\nabla f\|_{L^2}^2 + \|xf\|_{L^2}^2)^{1/2}.$$

Let  $\psi_0 \in H_h^1(\mathbb{R}^d)$ ,  $\bar{p}, \bar{q} \in \mathbb{R}^d$ . We define  $p', q' \in \mathbb{R}^d$  by  $q' = \bar{q} - \langle x \rangle \psi_0$  and  $p' = \bar{p} - \langle p \rangle \psi_0$ . The function  $\psi_1 \in H_h^1$  defined by

$$\psi_1(x) = \psi_0(x - q')e^{-i\langle p', x \rangle}$$

satisfies, for every  $\theta \in \mathbb{R}$   $\langle x \rangle \psi_1 e^{i\theta} = \bar{q}$  and  $\langle p \rangle \psi_1 e^{i\theta} = \bar{p}$ . Then, for every  $T > 0$ ,  $\theta \in \mathbb{R}$  and  $u : (0, T) \rightarrow \mathbb{R}$  piecewise constant

$$\begin{aligned} |\langle x \rangle \psi(T; u, \psi_0) - \bar{q}| &\leq \|\psi(T; u, \psi_0) - \psi_1 e^{i\theta}\|_{L^2} (\|x\psi(T; u, \psi_0)\|_{L^2} + \|x\psi_1\|_{L^2}), \\ |\langle p \rangle \psi(T; u, \psi_0) - \bar{p}| &\leq \|\psi(T; u, \psi_0) - \psi_1 e^{i\theta}\|_{L^2} (\|\nabla \psi(T; u, \psi_0)\|_{L^2} + \|\nabla \psi_1\|_{L^2}). \end{aligned}$$

To conclude, it suffices to prove the existence of  $M > 0$  such that, for every  $\epsilon > 0$ , there exists  $T \in [0, \epsilon]$ ,  $\theta \in \mathbb{R}$  and  $u : [0, T] \rightarrow \mathbb{R}^{d+1}$  piecewise constant such that

$$\|\psi(T; u, \psi_0) - \psi_1 e^{i\theta}\|_{L^2} < \epsilon \quad \text{and} \quad \|\psi(T; u, \psi_0)\|_{H_h^1} \leq M.$$

Given  $\epsilon > 0$  we fix  $\tau \in [0, \epsilon/4)$  such that

$$\|e^{\frac{i\langle q', x \rangle}{2\tau}} e^{i\tau(\Delta-V)} e^{-\frac{i\langle q', x \rangle}{2\tau}} - e^{\frac{i|q|^2}{2\tau}} \psi_0(x - q')\|_{H^1} < \frac{\epsilon}{4\|e^{-i\langle p', x \rangle}\|_{\mathcal{L}(H^1)}}. \quad (46)$$

Notice that this is possible because  $\|e^{\frac{i\langle q', x \rangle}{2\tau}} e^{i\tau(\Delta-V)} e^{-\frac{i\langle q', x \rangle}{2\tau}} - e^{\frac{i|q|^2}{2\tau}} \psi_0(x - q')\|_{L^2} \rightarrow 0$  as  $\tau \rightarrow 0$  (cf. first statement of Proposition 30) and  $\|e^{\frac{i\langle q', x \rangle}{2\tau}} e^{i\tau(\Delta-V)} e^{-\frac{i\langle q', x \rangle}{2\tau}}\|_{\mathcal{L}(H^2)}$  is bounded uniformly w.r.t.  $\tau \in [0, 1]$  (which can be seen by combining (36) and Proposition 39), hence (46) follows from an interpolation argument.

Fix then  $\sigma \in [0, \tau)$  such that

$$\|e^{i\sigma(\Delta-V - \frac{\langle q', x \rangle}{2\sigma\tau})} \psi_0 - e^{-\frac{i\langle q', x \rangle}{2\tau}} \psi_0\|_{H^1} < \frac{\epsilon}{4\|e^{-i\langle p', x \rangle} e^{\frac{i\langle q', x \rangle}{2\tau}} e^{i\tau(\Delta-V)}\|_{\mathcal{L}(H^1)}},$$

and  $\delta \in [0, \tau)$  such that

$$\|e^{i\delta(\Delta-V + \frac{\langle q', x \rangle}{2\tau\delta})} e^{i\tau(\Delta-V)} e^{i\sigma(\Delta-V - \frac{\langle q', x \rangle}{2\tau\sigma})} \psi_0 - e^{\frac{i\langle q', x \rangle}{2\tau}} e^{i\tau(\Delta-V)} e^{i\sigma(\Delta-V - \frac{\langle q', x \rangle}{2\tau\sigma})} \psi_0\|_{H^1} < \frac{\epsilon}{4\|e^{-i\langle p', x \rangle}\|_{\mathcal{L}(H^1)}}$$

(these estimates are possible thanks to  $L^2$ -convergence and  $\mathcal{L}(H^2)$ -uniform boundedness, as recalled for (46)).

Fix finally  $\gamma \in [0, \tau)$  such that

$$\begin{aligned} &\|e^{i\gamma(\Delta-V - \frac{\langle p', x \rangle}{\gamma})} e^{i\delta(\Delta-V + \frac{\langle q', x \rangle}{2\tau\delta})} e^{i\tau(\Delta-V)} e^{i\sigma(\Delta-V - \frac{\langle q', x \rangle}{2\tau\sigma})} \psi_0 \\ &- e^{-i\langle p', x \rangle} e^{i\delta(\Delta-V + \frac{\langle q', x \rangle}{2\tau\delta})} e^{i\tau(\Delta-V)} e^{i\sigma(\Delta-V - \frac{\langle q', x \rangle}{2\tau\sigma})} \psi_0\|_{H^1} < \epsilon/4. \end{aligned}$$

By defining the control

$$u_{\tau, \delta, \sigma, \gamma} := \left(\frac{-p'}{\gamma}\right)|_{[0, \gamma]} \# \left(\frac{q'}{2\tau\delta}\right)|_{[0, \delta]} \# 0|_{[0, \tau]} \# \left(-\frac{q'}{2\tau\sigma}\right)|_{[0, \sigma]},$$

on the time interval  $[0, T]$ , with  $T := \sigma + \tau + \delta + \gamma < \epsilon$ , (notice that  $\psi(T; u_{\tau, \delta, \sigma, \gamma}, \psi_0) = e^{i\gamma(\Delta-V - \frac{\langle p', x \rangle}{\gamma})} e^{i\delta(\Delta-V + \frac{\langle q', x \rangle}{2\tau\delta})} e^{i\tau(\Delta-V)} e^{i\sigma(\Delta-V - \frac{\langle q', x \rangle}{2\tau\sigma})} \psi_0$ ) we obtain that

$$\|\psi(T; u_{\tau, \delta, \sigma, \gamma}, \psi_0) - e^{i\theta} \psi_1\|_{H^1} < \epsilon,$$

where  $\theta = |q|^2/(2\tau)$ . We are left to show the uniform boundedness of  $\|x\psi(T; u_{\tau, \delta, \sigma, \gamma}, \psi_0)\|_{L^2}$  w.r.t.  $\gamma, \sigma, \delta < \tau < \epsilon/4 < 1$ . We shall need the following elementary estimates.

**Lemma 50.** *There exists  $C = C(\|\nabla V\|_{L^\infty})$  such that for any  $\psi_0 \in H_h^1(\mathbb{R}^d)$ ,  $u \in \mathbb{R}^d$ ,  $t \geq 0$ ,*

$$\|e^{it(\Delta - V + \langle u, x \rangle)} \psi_0\|_{H^1} \leq \|\psi_0\|_{H^1} + t(C + |u|)\|\psi_0\|_{L^2}, \quad (47)$$

$$\|x_j e^{it(\Delta - V + \langle u, x \rangle)} \psi_0\|_{L^2} \leq \|x\psi_0\|_{L^2} + t\|\psi_0\|_{H^1} + \frac{C + |u|}{2}t^2\|\psi_0\|_{L^2}. \quad (48)$$

*Proof.* Denote  $A_u = \Delta - V + \langle x, u \rangle$ . Define  $w(t) := \partial_{x_j} e^{itA_u} \psi_0$ , we have

$$i\partial_t w = A_u w - [A_u, \partial_{x_j}] e^{itA_u} \psi_0 = A_u w + (\partial_{x_j} V + u) e^{itA_u} \psi_0.$$

We obtain (47) using Duhamel's formula

$$w(t) = e^{itA_u} \partial_{x_j} \psi_0 - i \int_0^t e^{i(t-s)A_u} (\partial_{x_j} V + u) e^{isA_u} \psi_0 ds,$$

and the unitarity of  $e^{itA_u}$  in  $L^2(\mathbb{R}^d)$ . Define then  $v(t) := x_j e^{itA_u} \psi_0$ , we have

$$i\partial_t v = A_u v - [A_u, x_j] e^{itA_u} \psi_0 = A_u v - \partial_{x_j} e^{itA_u} \psi_0.$$

We finally obtain (48) using Duhamel's formula, the unitarity of  $e^{itA_u}$  in  $L^2(\mathbb{R}^d)$ , and (47).  $\square$

Applying repeatedly Lemma 50 we finally obtain the existence of  $M > 0$  such that

$$\|x_j \psi(T; u_{\tau, \delta, \sigma, \gamma}, \psi_0)\|_{L^2} = \|x_j e^{i\gamma(\Delta - V - \frac{\langle p', x \rangle}{\gamma})} e^{i\delta(\Delta - V + \frac{\langle q', x \rangle}{2\tau\delta})} e^{i\tau(\Delta - V)} e^{i\sigma(\Delta - V - \frac{\langle q', x \rangle}{2\tau\sigma})} \psi_0\|_{L^2} \leq M$$

for all  $0 < \gamma, \sigma, \delta < \tau < \epsilon/4 < 1$ , which concludes the proof of Corollary 8.

## Appendix: toy model

**Proposition 51.** *Let  $\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C})$ ,  $\bar{q} \in \mathbb{R}^d$ ,  $\bar{\sigma}, T, \epsilon > 0$  and  $\psi_1 \in L^2(\mathbb{R}^d, \mathbb{C})$  be defined by  $\psi_1(x) := \bar{\sigma}^{d/2} \psi_0(\bar{\sigma}(x - \bar{q}))$ . There exist  $u \in C^\infty([0, T], \mathbb{R}^d)$ ,  $u_0 \in C^\infty([0, T], \mathbb{R})$  and  $\bar{\theta} \in \mathbb{R}$  such that the solution of*

$$\begin{cases} i\partial_t \psi(t, x) = (-\Delta - u_0(t)|x|^2 - \langle u(t), x \rangle) \psi(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\ \psi(0, \cdot) = \psi_0, \end{cases} \quad (49)$$

satisfies  $\|\psi(T) - \psi_1 e^{i\bar{\theta}}\|_{L^2} < \epsilon$ .

The proof relies on the explicit representation formula (57) inspired by [21, Section 4]. This strategy is also used in [39].

*Proof.* First, there exists  $\delta > 0$  such that,

$$\forall \zeta \in [0, \delta], \quad \|e^{i\zeta(\Delta - |x|^2)} \psi_0 - \psi_0\|_{L^2} < \epsilon. \quad (50)$$

Let  $A \in C^\infty([0, T], (0, \infty))$  be such that

$$A(0) = 1, \quad A(T) = \bar{\sigma}^2, \quad A'(0) = A'(T) = 0, \quad \int_0^T A(t) dt < \delta. \quad (51)$$

We define  $a, b, \zeta, u_0 \in C^\infty([0, T], \mathbb{R})$  by

$$a(t) = -\frac{1}{8} \frac{A'(t)}{A(t)}, \quad b(t) = \frac{1}{\sqrt{A(t)}}, \quad \zeta(t) = \int_0^t A(s) ds, \quad u_0(t) = -4a(t)^2 + \frac{1}{b(t)^4} - a'(t). \quad (52)$$

By (51) and (52) the function  $(a, b, \zeta)$  solves the following ODE on  $[0, T]$

$$\begin{cases} a'(t) = -4a(t)^2 + \frac{1}{b(t)^4} - u_0(t), & (a, b, \zeta)(0) = (0, 1, 0), \\ b'(t) = 4a(t)b(t), & (a, b)(T) = (0, \frac{1}{\bar{\sigma}}), \\ \zeta'(t) = \frac{1}{b(t)^2}, & \zeta(T) \in [0, \delta]. \end{cases} \quad (53)$$

Let  $p \in C_c^\infty((0, T), \mathbb{R}^d)$  be such that

$$\int_0^T p(t) dt = \bar{q}. \quad (54)$$

We define  $q, u \in C^\infty([0, T], \mathbb{R}^d)$  by

$$q(t) = \int_0^t p(s) ds, \quad u(t) = -2u_0(t)q(t) - \frac{1}{2}p'(t). \quad (55)$$

By (54) and (55), the function  $(q, p)$  solves the following ODE on  $[0, T]$

$$\begin{cases} q'(t) = p(t), & (q, p)(0) = 0, \\ p'(t) = -4u_0(t)q(t) - 2u(t), & (q, p)(T) = (\bar{q}, 0). \end{cases} \quad (56)$$

Basic computations and the ODEs (53), (56) prove that the solution of (49) satisfies

$$\psi(t, x) = \phi\left(\zeta(t), \frac{x - q(t)}{b(t)}\right) \frac{1}{b(t)^{d/2}} e^{i(a(t)|x - q(t)|^2 + \frac{1}{2}\langle p(t), x \rangle + \theta(t))} \quad (57)$$

where  $\phi(s, \cdot) = e^{is(\Delta - |x|^2)}\psi_0$  and  $\theta \in C^\infty([0, T], \mathbb{R})$  is defined by

$$\theta(t) = \int_0^t \left( u_0(s)|q(s)|^2 - \frac{|p(s)|^2}{4} \right) ds.$$

In particular

$$\psi(T, x) = \phi(\zeta(T), \bar{\sigma}(x - \bar{q})) \bar{\sigma}^{d/2} e^{i\theta(T)}.$$

Thus, using (53) and (50), we obtain

$$\left\| \psi(T) - \psi_0(\bar{\sigma}(\cdot - \bar{q})) e^{i\theta(T)} \right\|_{L^2} = \left\| \left( e^{i\zeta(T)(\Delta - |x|^2)} - I \right) \psi_0 \right\|_{L^2} < \epsilon.$$

□

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