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EQUIDISTRIBUTION SPEED OF PERIODIC POINTS FOR COMPLEX POLYNOMIALS

TIEN-CUONG DINH AND LUCAS KAUFMANN

ABSTRACT. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial map of degree $d \geq 2$. We show that the periodic points of f of period n equidistribute towards the equilibrium measure of f exponentially fast. This quantifies a theorem of Lyubich.

1. INTRODUCTION

Let $f(x)$ be a complex polynomial of degree $d \geq 2$ in the variable $x \in \mathbb{C}$ and $f : \mathbb{C} \rightarrow \mathbb{C}$ the associated holomorphic map of topological degree d . The dynamics of such maps, that is, the study of the behavior of the iterates $f^n := f \circ \dots \circ f$ as $n \rightarrow \infty$ has been the subject of intense study for over a hundred years, see [BM01, Mil06] for an overview.

It follows from the seminal works of Brolin [Bro65], Freire-Lopes-Mañé [FLMn83] and Lyubich [Lyu82, Lyu83] that f admits a canonical invariant measure μ , called the *equilibrium measure of f* . This is a probability measure on \mathbb{C} characterized by various dynamical properties. In particular, the support of μ is the Julia set of f and it is the unique invariant probability measure of maximal entropy. We highlight two important properties satisfied by μ , namely two equidistribution theorems.

The first equidistribution theorem concerns iterated pre-images of non-exceptional points, see [Bro65, FLMn83, Lyu83]. More precisely, there exists an exceptional set $\mathcal{E}_0 \subset \mathbb{C}$ containing at most one point such that

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{1}{d^n} \sum_{x \in f^{-n}(a)} \delta_x = \mu \quad \text{for every } a \in \mathbb{C} \setminus \mathcal{E}_0,$$

where the convergence is in the weak sense. Moreover, \mathcal{E}_0 is empty unless f is conjugated to the map $x \mapsto x^d$. The above convergence can be quantified and is exponentially fast in the following sense. If $a \in \mathbb{C} \setminus \mathcal{E}_0$, then there exist constants $1 < \lambda < d$ and $C > 0$ such that for every test function φ on \mathbb{C} of class \mathcal{C}^2 we have

$$(1.2) \quad \left| \left\langle \frac{1}{d^n} \sum_{x \in f^{-n}(a)} \delta_x, \varphi \right\rangle - \int_{\mathbb{C}} \varphi d\mu \right| \leq C \|\varphi\|_{\mathcal{C}^2} \frac{1}{\lambda^n}.$$

Moreover, the constant C is proportional to $1 + \log^+(1/\text{dist}(a, \mathcal{E}))$, where $\mathcal{E} = \mathcal{E}_0 \cup \{\infty\}$ is the exceptional set of f viewed as an endomorphism of the Riemann sphere \mathbb{P}^1 , that is, the maximal f^{-1} -invariant finite subset of \mathbb{P}^1 . By replacing \mathcal{E} by a larger finite set, namely the orbit of periodic critical points of f , the exponent λ can be taken to be arbitrarily close to d . See [DS10b] and Theorem 2.6 below for precise statements. By the interpolation theory of Banach spaces, an analogous result holds for test functions of class \mathcal{C}^α with $0 < \alpha \leq 2$, where λ is replaced by $\lambda^{\alpha/2}$.

A second equidistribution theorem satisfied by μ concerns the periodic points of f , see [Lyu83]. For each $n \geq 1$, let $P_n = \{x \in \mathbb{C} : f^n(x) = x\}$ be the set of periodic points of

period n of f . Then,

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{1}{d^n} \sum_{a \in P_n} \delta_a = \mu,$$

where again the convergence is in the weak sense. In this case, however, a quantitative convergence has only been obtained in the arithmetic setting, that is, for polynomials with algebraic coefficients, see [FRL06, Oku16]. In this work, we prove that the convergence is exponentially fast for any polynomial.

Theorem 1.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree $d \geq 2$ and let μ be its equilibrium measure. Let $P_n = \{x \in \mathbb{C} : f^n(x) = x\}$. Then, there exists a constant $0 < \xi < 1$ such that, for any function φ of class \mathcal{C}^α , $0 < \alpha \leq 1$ on \mathbb{C} we have*

$$(1.4) \quad \left| \left\langle \frac{1}{d^n} \sum_{a \in P_n} \delta_a, \varphi \right\rangle - \int_{\mathbb{C}} \varphi d\mu \right| \leq C_\alpha \|\varphi\|_{\mathcal{C}^\alpha} \xi^{\alpha n},$$

for some constant $C_\alpha > 0$ independent of n and φ .

One might expect that the optimal rate in (1.4) should be $\xi = \underline{d}^{-1/2}$ where $\underline{d} < d$ is arbitrarily close to d , as this is the best known speed for the equidistribution of pre-images for \mathcal{C}^1 observables, as discussed above. Also, the rate $n^{1/2}d^{-n/2}$ is achieved in the arithmetic case for Lipschitz observables, as shown by Favre–Rivera-Letelier and complemented by Okuyama. See [FRL06, Oku16]. However, obtaining the optimal rate for general maps f seems challenging. Our methods seem to be adapted to prove the convergence of the solutions of equations of the type $f^n(x) = g(x)$ for a given polynomial $g(x)$ towards μ in the spirit of [Lyu83, FRL06, Oku16]. They are general and can be extended to cover the case of rational maps of \mathbb{P}^1 . In this case however, some extra technical results must be added. We chose to present here only the polynomial case in order to have a more accessible presentation.

Structure of the proof. We now outline the main strategy of our proof. Let Γ_n be the graph of f^n viewed in $\mathbb{C} \times \mathbb{C}$ or in the compactification $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\Delta \subset \mathbb{C} \times \mathbb{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal. Let $[\Gamma_n]$ and $[\Delta]$ be the corresponding integration currents. Then, the measure $d^{-n} \sum_{a \in P_n} \delta_a$ is naturally identified with the measure $d^{-n}[\Gamma_n] \wedge [\Delta]$ on $\mathbb{C} \times \mathbb{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ and (1.3) is equivalent to the weak convergence

$$d^{-n}[\Gamma_n] \wedge [\Delta] \longrightarrow \pi_1^* \mu \wedge [\Delta],$$

where $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the projection on the first coordinate. Our goal is to show that the above convergence is exponentially fast.

The exponentially fast equidistribution of pre-images (1.1) can be used to show that

$$(1.5) \quad d^{-n}[\Gamma_n] \longrightarrow \pi_1^* \mu,$$

exponentially fast as currents, see Proposition 2.8 below. Therefore, in order to prove (1.4) from (1.5), we need to show that

$$(1.6) \quad \lim_{n \rightarrow \infty} (d^{-n}[\Gamma_n] \wedge [\Delta]) = \left(\lim_{n \rightarrow \infty} d^{-n}[\Gamma_n] \right) \wedge [\Delta]$$

with control on the convergence rate.

In general, one cannot exchange the order of limits and wedge products of currents as above. A simple counter-example is given by considering the intersection of the currents

$n^{-1}[\{y = x^n\}]$ with $[y = 0]$ in \mathbb{C}^2 . Here, we need to show this property in our case, moreover with a control on the speed, which is even more challenging.

Our strategy to overcome the above difficulty is the following:

Step 1 – Perturbed diagonals: We first show that (1.6) holds with an exponential rate if we replace Δ by small perturbations. Namely, we show that there is a sequence of curves $\Delta_n \subset \mathbb{C} \times \mathbb{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ converging to Δ satisfying

$$(1.7) \quad \lim_{n \rightarrow \infty} (d^{-n}[\Gamma_n] \wedge [\Delta_n]) = \left(\lim_{n \rightarrow \infty} d^{-n}[\Gamma_n] \right) \wedge [\Delta]$$

exponentially fast. Moreover, one can choose Δ_n to be $e^{-\lambda^n}$ -close to Δ for some $\lambda > 1$, see Proposition 3.2. This is an application of Moser-Trudinger exponential estimate in the Sobolev space $W^{1,2}$.

Step 2 – Counting transverse branches: In order to be able to replace $[\Delta_n]$ by $[\Delta]$ in (1.7), one needs to show that many branches of the graph Γ_n cross both Δ and Δ_n in a sufficiently transverse way. This amounts to controlling the number of points of Γ_n that are too close to being tangent to Δ . To handle this, we lift Γ_n and Δ to the space of first jets. More precisely, if (x, y) denote the coordinates on $\mathbb{C} \times \mathbb{C}$, we consider the space $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ with coordinates (x, y, t) and the curves

$$\widehat{\Gamma}_n := \{(x, y, t) : y = f^n(x), t = (f^n)'(x)\} \quad \text{and} \quad \widehat{\Delta} := \{(x, y, t) : y = x, t = 1\}.$$

We show, using methods from pluripotential theory, that there is a sequence of neighborhoods \mathcal{U}_n shrinking to $\widehat{\Delta}$ at a suitable rate such that the mass of $d^{-n}[\widehat{\Gamma}_n]$ over \mathcal{U}_n is exponentially small. See Proposition 4.4. In order to do that, we compare the slices at $t = 1$ and $t = 0$. The advantage of the slice $t = 0$ is that $\widehat{\Gamma}_n \cap \{t = 0\}$ corresponds to the critical points of f^n , which by the chain rule can be treated using the equidistribution of inverse images, see Lemma 4.3.

Step 3 – From Δ_n to Δ : From Step 2 we infer that most of the branches of Γ_n are transverse to Δ , and by removing a small number of them we can show that sufficiently many branches cross both Δ_n and Δ in nearby points. Together with (1.7), this yields our main theorem.

This article is organized as follows. In Section 2 we recall some fundamental material needed in the proof of our main theorem. In Section 3 we achieve Step 1 above, namely, we prove that the desired exponentially fast convergence holds by considering small perturbations of the diagonal. In Section 4 we study the lifts $\widehat{\Gamma}_n$ as above and show that their mass near $\widehat{\Delta}$ is small, yielding Step 2. In Section 5 we use these results to construct many branches of Γ_n sufficiently transversal to Δ . Finally, in Section 6 we combine the previous results and provide the proof of Theorem 1.1.

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2. PRELIMINARY RESULTS

We denote by \mathbb{P}^1 the complex projective line. It is naturally identified with the one point compactification $\mathbb{C} \cup \{\infty\}$ of \mathbb{C} where $x \in \mathbb{C}$ is identified with $[x : 1]$ and ∞ is identified with $[1 : 0]$. Here, the brackets denote homogeneous coordinates. For a polynomial map $f : \mathbb{C} \rightarrow \mathbb{C}$ we shall also denote by $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ its natural extension to \mathbb{P}^1 by setting $f(\infty) = \infty$.

In this article we denote by ω_{FS} the Fubini-Study form, that is, the unique unitary invariant $(1,1)$ -form on \mathbb{P}^1 normalized by $\int_{\mathbb{P}^1} \omega_{\text{FS}} = 1$. The $(1,1)$ -form ω_{FS} induces a Kähler metric on \mathbb{P}^1 . We'll denote by dist the associated distance function.

The use of positive closed currents is crucial throughout this work. We refer to [Dem] and the appendix of [DS10a] for the necessary background.

2.1. Test functions and norms. For the proof of our main theorem we need to work with different spaces of test functions and their corresponding norms. Even if our maps act on an one-dimensional space, we'll later work on manifolds of higher dimensions, so we introduce our function spaces in a more general setting. We recall here a few basic definitions and results. We refer the appendix of [DS10a] for an overview.

Let (X, ω) be a compact Kähler manifold of dimension k . A quasi-p.s.h. function on X is locally the difference of a plurisubharmonic (p.s.h.) function and a smooth one. A function on X is called d.s.h. if it is equal outside a pluripolar set to the difference of two quasi-p.s.h. functions. We denote by $\text{DSH}(X)$ the space of d.s.h. functions equipped with the norm

$$\|\varphi\|_{\text{DSH}} := |\langle \omega^k, \varphi \rangle| + \min \|S^\pm\|,$$

where the above minimum is taken over all positive closed $(1,1)$ -currents on X such that $\text{dd}^c \varphi = S^+ - S^-$. Here, $\text{d}^c := \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $\text{dd}^c = \frac{i}{\pi} \partial \bar{\partial}$. If μ is the equilibrium measure of a rational map on \mathbb{P}^1 (see Section 2.3 below) the above norm is equivalent to the norm

$$\|u\|_\mu := |\langle \mu, \varphi \rangle| + \min \|S^\pm\|,$$

see [DS10a, Proposition A.43].

Let $W^{1,2}(X)$ be the Sobolev space of L^2 functions whose gradient is also in L^2 . The complex Sobolev space $W^*(X)$ is the subspace of $W^{1,2}(X)$ formed by functions ψ such that $i\partial\psi \wedge \bar{\partial}\psi \leq R$ for some positive closed $(1,1)$ -current R on X . This space is equipped with the norm

$$\|\psi\|_* := |\langle \omega^k, \psi \rangle| + \min \|R\|^{1/2},$$

where the minimum is taken over all positive closed $(1,1)$ -currents R as above, see [Vig07]. When X is of dimension $k = 1$ the spaces $W^*(X)$ and $W^{1,2}(X)$ coincide. In this case, the norm has a simpler expression, namely

$$\|\psi\|_* := \|\psi\|_{L^1(\omega)} + \|\partial\psi\|_{L^2}, \quad \text{where} \quad \|\partial\psi\|_{L^2} := \left(\int_X i\partial\psi \wedge \bar{\partial}\psi \right)^{1/2}.$$

Theorem 2.1 (Skoda/Moser-Trudinger). *Let (X, ω) be a compact Kähler manifold of dimension k and $d\text{vol} = \omega^k/k!$ be the associated volume form.*

(1) *Let \mathcal{F} be a bounded subset of $\text{DSH}(X)$. There are constants $\alpha, C > 0$ such that*

$$\int_X e^{\alpha|\varphi|} d\text{vol} \leq C \quad \text{for all} \quad \varphi \in \mathcal{F}.$$

(2) Let \mathcal{G} be a bounded subset of $W^*(X)$. There are constants $\alpha, C > 0$ such that

$$\int_X e^{\alpha|\psi|^2} d\text{vol} \leq C \quad \text{for all } \psi \in \mathcal{G}.$$

As a corollary, one obtains the following estimates. Here and in the remainder of this article, we'll say that a function $\varphi : X \rightarrow \mathbb{R}$ is (M, α) -Hölder continuous for some $M > 0$ and $0 < \alpha \leq 1$ if $|\varphi(x) - \varphi(x')| \leq M \text{dist}(x, x')^\alpha$ for all $x, x' \in X$, where dist denotes the distance on X induced by the Kähler metric ω .

Corollary 2.2. *Let (X, ω) be a compact Kähler manifold.*

(1) *Let \mathcal{F} be a bounded subset of $\text{DSH}(X)$. There is a constant $A > 0$ (independent of M and γ) such that if $\varphi \in \mathcal{F}$ is (M, γ) -Hölder continuous for some constants $M \geq 1$ and $0 < \gamma \leq 1$, then*

$$\|\varphi\|_\infty \leq A\gamma^{-1}(1 + \log M).$$

(2) *Let \mathcal{G} be a bounded subset of $W^*(X)$. There is a constant $A > 0$ (independent of M and γ) such that if $\psi \in \mathcal{G}$ is (M, γ) -Hölder continuous for some constants $M \geq 1$ and $0 < \gamma \leq 1$, then*

$$\|\psi\|_\infty \leq A\gamma^{-1/2}(1 + \log M)^{1/2}.$$

Proof. See [DS10b, Prop 2.1] for a proof of (1) in the case $\gamma = 1$. The case $0 < \gamma \leq 1$ can be proved similarly. For a proof of (2) we refer to [DKW20, Prop 4.3]. We note that version in [DKW20] states a weaker version analogous to (1), but the proof there actually gives the stronger form stated in (2). \square

2.2. Potentials on \mathbb{C} and \mathbb{P}^1 . Let ν be a Borel measure on \mathbb{C} viewed as a positive $(1, 1)$ -current on \mathbb{C} . A subharmonic function u on \mathbb{C} such that $\text{dd}^c u = \nu$ is called a *potential* of ν . By Poincaré's Lemma every measure admits a global potential on \mathbb{C} (for instance, the logarithmic potential below). Furthermore, any two potentials differ by a harmonic function. In particular, the regularity of the potential depends only on ν and not on the choice of u . We assume from now on that ν and all measures appearing below have compact support in \mathbb{C} .

The *logarithmic potential* of ν is the subharmonic function defined by

$$u(x) = \int \log|x - z| d\nu(z) = N * \nu(x),$$

where $N(z) = \log|z|$ is the Newton potential on \mathbb{C} . From the fact that $\text{dd}^c N(z) = \delta_0$, it follows that $\text{dd}^c u = \nu$.

Lemma 2.3. *Let $0 < s \leq \frac{1}{10}$ and denote by $\mathcal{L}_s := (\pi s^2)^{-1} \mathbf{1}_{|t-1| \leq s} \text{id}t \wedge \bar{d}t$ the normalized Lebesgue measure on the disc $\mathbb{D}(1; s)$ viewed as a positive $(1, 1)$ -form in the variable t . There exists a d.s.h. function χ_s such that*

$$\mathcal{L}_s(t) - \delta_0(t) = \text{dd}^c \chi_s(t)$$

with the following properties:

(1) *There exist constants $c_1, c_2 > 0$ independent of s such that*

$$-c_1 |\log s| \leq \chi_s(t) \leq c_2 (2 - \log|t|) \quad \text{for } |t| \leq 3.$$

(2) *There exists a constant $C > 0$ independent of s such that $|\chi_s(t)| \leq C/|t|$ for $|t| \geq 2$.*

Proof. Let u_s (resp. v_0) be logarithmic potential of \mathcal{L}_s (resp. δ_0). By definition

$$u_s(t) = \int \log |t - z| d\mathcal{L}_s(z) \quad \text{and} \quad v_0(t) = \log |t|.$$

Set $\chi_s := u_s - v_0$. Clearly $\text{dd}^c \chi_s(t) = \mathcal{L}_s(t) - \delta_0(t)$. We now prove the desired estimates for χ_s .

For the upper bound in (1), we observe that $-v_0 \leq 2 - \log |t|$, so we only need to bound u_s from above. If z is in the support of \mathcal{L}_s we have $|z| \leq 1 + s \leq 2$ so $|t - z| \leq 5$ when $|t| \leq 3$. It follows that $\log |t - z|$ is uniformly bounded from above for $|t| \leq 3$ and z in the support of \mathcal{L}_s . Hence u_s is uniformly bounded from above over $\{|t| \leq 3\}$. Since $2 - \log |t| > 0$ there, it follows that $u_s(t) \leq c'_2(2 - \log |t|)$ for some constant $c_2 > 0$. The upper bound in (1) follows.

For the lower bound in (1), we first note that for $|t| \leq 3$ we have $-v_0(t) \geq -\log 3 \geq -|\log s|$ because $0 < s \leq \frac{1}{10}$. Therefore, we only need to bound u_s from below. Consider the new variable $\tau = t - 1$, so that \mathcal{L}_s corresponds to the normalized Lebesgue measure on the disc $\{|\tau| \leq s\}$ whose logarithmic potential is

$$\tilde{u}_s(\tau) = \frac{1}{\pi s^2} \int_{|\tau| \leq s} \log |\tau - z| idz \wedge d\bar{z}.$$

The above function is a radial subharmonic function in τ . In particular \tilde{u}_s can be written as a non-decreasing convex function of $\log |\tau|$, see [Dem, Corollary 5.14]. Therefore, in order to obtain a lower bound for \tilde{u}_s (and consequently for $u_s(t) = \tilde{u}_s(t + 1)$), it is enough to bound $\tilde{u}_s(0)$ from below. By a direct computation in polar coordinates, using integration by parts and recalling that $0 < s < \frac{1}{10}$, we have

$$\tilde{u}_s(0) = \frac{2}{s^2} \int_0^s r \log r dr = -|\log s| - \frac{1}{2} \geq -c_1 |\log s|$$

for a suitable constant $c_1 > 0$. This finishes the proof of (1).

Let us now prove (2). By definition, we have $\chi_s(t) = \int (\log |t - z| - \log |t|) d\mathcal{L}_s(z)$. By radial symmetry, it is enough to prove the estimate for t real and $t > 3$. In this case, we have $|\log |t - z| - \log |t|| = |\log |1 - \frac{z}{t}||$. From the mean value theorem, the fact that $t > 3$ and $\frac{1}{2} \leq |z| \leq 2$ on the support of \mathcal{L}_s , it follows that $|\log |1 - \frac{z}{t}|| \leq M |\frac{z}{t}|$ for all such t and z , for a suitable constant $M > 0$. Therefore

$$|\chi_s(t)| \leq \frac{M}{|t|} \int |z| d\mathcal{L}_s(z) \leq \frac{2M}{|t|},$$

yielding (2). This finishes the proof of the lemma. \square

Lemma 2.4. *Let $\mathcal{L} := idz \wedge d\bar{z}$ be the Lebesgue measure on \mathbb{C} . Let ϱ be a measurable function such that $0 \leq \varrho \leq 1$ and $\text{supp} \varrho \subset \mathbb{D}(0; 1)$. Set $\nu = \varrho \cdot \mathcal{L}$ and let u be the logarithmic potential of ν . Then, u is Lipschitz continuous. Moreover, there exists a constant $C > 0$ independent of ϱ such that $\|u\|_{\text{Lip}} \leq C$.*

Before giving the proof of the above lemma, we need to introduce some notation. Let $\tilde{u}(x) := \lim_{r \rightarrow 0} \int_{\mathbb{D}(x, r)} u(x) dx$, where the symbol $\int_{\mathbb{D}(x, r)}$ denotes the average over $\mathbb{D}(x, r)$. Then $\tilde{u} = u$ almost everywhere and the points where equality holds are called Lebesgue points of u . If u is subhamornic, then all points are Lebesgue points, that is, $\tilde{u} = u$ everywhere.

For a measurable function h defined on an open subset U of \mathbb{C} , consider its Hardy-Littlewood maximal function

$$\mathcal{M}_h^r(a) := \sup_{0 < s < r} \int_{\mathbb{D}(a,s)} |h(x)| dx, \quad r > 0, a \in U_r,$$

where $U_r = \{z \in U; d(z, U^c) > r\}$. We need the following classical result. We state only a particular case which is enough for our purposes, see [BHa93] for a more general statement.

Theorem 2.5. *Let U be an open subset of \mathbb{C} and $u \in W_{\text{loc}}^{1,1}(U)$. If $x \in U_r$ is such that $\mathcal{M}_{|\nabla u|}^r(x) < +\infty$ then x is a Lebesgue point of u . Furthermore, for every $x, y \in U_r$ such that $|x - y| \leq \frac{r}{3}$ and $\mathcal{M}_{|\nabla u|}^r(x) < \infty, \mathcal{M}_{|\nabla u|}^r(y) < \infty$ we have*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C |x - y| (\mathcal{M}_{|\nabla u|}^r(x) + \mathcal{M}_{|\nabla u|}^r(y)),$$

where $C > 0$ is a universal constant.

Proof of Lemma 2.4. Let u be the logarithmic potential of ν . By definition, $u(x) = N * \nu(x)$, where $N(z) = \log |z|$. Since $\frac{\partial N}{\partial z}(z) = -\frac{1}{2} \frac{\bar{z}}{|z|^2}$, we have $|\frac{\partial N}{\partial z}(z)| \leq \frac{1}{2|z|}$ and similarly for $\frac{\partial N}{\partial \bar{z}}$. As $\frac{\partial u}{\partial z} = \frac{\partial N}{\partial z} * \nu$ and $\nu = \varrho \cdot \mathcal{L} \leq 1_{|x| \leq 1} \mathcal{L}$, it follows that $\frac{\partial u}{\partial z} \in L_{\text{loc}}^1$ and similarly for $\frac{\partial u}{\partial \bar{z}}$. Hence $u \in W_{\text{loc}}^{1,1}(U)$. Moreover,

$$\left| \frac{\partial u}{\partial z}(x) \right| = \left| \frac{\partial N}{\partial z} * \nu(x) \right| \leq \frac{1}{2} \int \frac{1}{|x - z|} \varrho(z) d\mathcal{L}(z) \leq \frac{1}{2} \int_{|z| \leq 1} \frac{1}{|x - z|} d\mathcal{L}(z) = \pi$$

and similarly for $\frac{\partial u}{\partial \bar{z}}$. We conclude that ∇u is bounded by a constant independent of ϱ . Hence the maximal function $\mathcal{M}_{|\nabla u|}^r$ is bounded by a constant independent of ϱ . The lemma now follows from Theorem 2.5 and the fact that $\tilde{u} = u$ when u is subharmonic. \square

2.3. Basic equidistribution results. We now recall some basic equidistribution results for holomorphic endomorphisms of \mathbb{P}^1 . Most of them are also valid, after suitable modifications, for endomorphisms of higher dimensional projective spaces. See [DS10a] for an overview.

Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a holomorphic endomorphism of degree $d \geq 2$. The *equilibrium measure* of f is defined by

$$\mu := \lim_{n \rightarrow \infty} \frac{1}{d^n} (f^n)^* \omega_{\text{FS}}.$$

Then, μ probability measure on \mathbb{P}^1 whose support is the Julia set of f .

If $f : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial map, then μ has compact support in \mathbb{C} and

$$\mu = dd^c G, \quad \text{where } G(x) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |f^n(x)|.$$

The following result, although not stated in this exact form, is contained in [DS10b]. Recall that the exceptional set \mathcal{E} of f is the largest finite subset of \mathbb{P}^1 such that $f^{-1}(\mathcal{E}) = \mathcal{E}$. For polynomials, \mathcal{E} always contains the point at infinity. Moreover, $\mathcal{E} = \{\infty\}$ unless the polynomial is conjugated to $f(x) = x^d$, in which case $\mathcal{E} = \{0, \infty\}$.

We'll use from now on the notation $\log^* t = 1 + |\log t|$ for $t \in \mathbb{R}$.

Theorem 2.6 (Equidistribution of pre-images). *Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a holomorphic endomorphism of degree $d \geq 2$ and let μ be its equilibrium measure. For $a \in \mathbb{P}^1$, let $\mu_a^n := d^{-n} (f^n)^* \delta_a$. Let φ be a quasi-p.s.h. function on \mathbb{P}^1 such that $\|dd^c \varphi\|_* \leq 1$. Assume furthermore that φ is (M, α) -Hölder for some $M > 0$ and $0 < \alpha \leq 1$.*

(1) Let \mathcal{E} be the exceptional set of f . Then, there exist constants $C_\alpha > 0$ and $1 < \delta < d$ independent of a, M, α, n and φ such that

$$(2.1) \quad |\langle \mu_a^n - \mu, \varphi \rangle| \leq C_\alpha \log^* \text{dist}(a, \mathcal{E}) \frac{\log^* M}{\delta^n}.$$

(2) Let \mathcal{E}' be the orbit of the periodic critical points of f . Then, for every $1 < \underline{d} < d$ there exists a constant $C_{\underline{d}, \alpha} > 0$ independent of a, M, α, n and φ as above such that

$$(2.2) \quad |\langle \mu_a^n - \mu, \varphi \rangle| \leq C_{\underline{d}, \alpha} \log^* \text{dist}(a, \mathcal{E}') \frac{\log^* M}{\underline{d}^n}.$$

It is well known that μ is Hölder regular in the following sense: there exist constants $0 < \gamma \leq 2$ and $c > 0$ such that $\mu(\mathbb{D}(a, r)) \leq cr^\gamma$ for all $a \in \mathbb{P}^1$ and all $r \geq 0$. Hereon, the largest such γ will be referred to as *the Hölder exponent of μ* .

Corollary 2.7. *Let f and μ_a^n be as in Theorem 2.6. Fix $0 < r < 1$. There exist constants $c > 0$ and $0 < \beta < 1$ that depend on r , such that for any $a, b \in \mathbb{P}^1$, we have*

$$\mu_a^n(\mathbb{D}(b; r^n)) \leq c \log^* \text{dist}(a, \mathcal{E}) \beta^n.$$

Proof. After a change of coordinate, one can assume that $b = 0$. Let ψ_n be the function given by

$$\psi_n(x) = \frac{\max(\log |x|, \frac{1}{2} \log r^n) - \max(\log |x|, \log r^n)}{-\frac{1}{2} \log r^n}, \quad x \in \mathbb{C}.$$

Then, ψ_n is a d.s.h. function with $\|\text{dd}^c \psi_n\|_* \lesssim 1/|\log r^n|$, $\mathbf{1}_{\mathbb{D}(0; r^n)} \leq \psi_n \leq \mathbf{1}_{\mathbb{D}(0; r^{n/2})}$ and $\|\psi_n\|_{\text{Lip}} \lesssim 1/r^n$.

From Theorem 2.6 for $\alpha = 1$, we obtain

$$\mu_a^n(\mathbb{D}(0; r^n)) \leq \langle \mu_a^n, \psi_n \rangle \lesssim \langle \mu, \psi_n \rangle + \log^* \text{dist}(a, \mathcal{E}) \log^* r^n \cdot \delta^{-n}$$

for some $1 < \delta < d$. Let $\gamma > 0$ be the Hölder exponent of μ . Then $\mu(\mathbb{D}(0; r)) \leq cr^\gamma$, so $\langle \mu, \psi_n \rangle \leq \mu(\mathbb{D}(0; r^{n/2})) \leq cr^{\gamma n/2}$.

Choose $1 < \delta_1 < \delta$. The above computations show that $\mu_a^n(\mathbb{D}(0; r^n))$ is bounded by a constant times

$$r^{\gamma n/2} + \log^* \text{dist}(a, \mathcal{E}) (1 + n |\log r|) \delta^{-n} \lesssim \log^* \text{dist}(a, \mathcal{E}) (r^{\gamma n/2} + \delta_1^{-n}),$$

Taking $\beta := \max(r^{\gamma/2}, \delta_1^{-1})$ gives the desired result. \square

The graph point of view. An alternative way of looking at the equidistribution theorems above is to use the graphs of iterates of f . We'll see in Proposition 2.8 below that these graphs, suitably normalized, converge to a canonical current.

We begin by setting some notation that will be kept throughout the text. Denote by (x, y) the coordinates in $\mathbb{C} \times \mathbb{C}$. Let

$$\Gamma_n := \{y = f^n(x)\} = \{(x, f^n(x)) : x \in \mathbb{C}\}$$

be the graph of f^n , viewed as subset of $\mathbb{C} \times \mathbb{C}$. We still denote by Γ_n its compactification inside $\mathbb{P}^1 \times \mathbb{P}^1$. Let $[\Gamma_n]$ be the integration current along Γ_n and

$$\llbracket \Gamma_n \rrbracket := \frac{1}{d^n} [\Gamma_n]$$

be the associated normalized current of unit mass.

The following result can be seen as a version of Theorem 2.6 in terms of the above currents.

Proposition 2.8. *Let K be a compact subset of $\mathbb{C} \times \mathbb{C}$. Fix $1 < \underline{d} < d$. There exists a constant $C_{\underline{d},K} > 0$ such that for every test form ϕ on $\mathbb{C} \times \mathbb{C}$ of class \mathcal{C}^1 with support contained in K one has*

$$|\langle [\Gamma_n] - \pi_1^* \mu, \phi \rangle| \leq C_{\underline{d},K} \|\phi\|_{\mathcal{C}^1} \underline{d}^{-n/2}.$$

Proof. Fix a polydisc $\mathbb{D}(0; A)^2$ containing K . We divide the proof into four cases.

1st case: $\phi = \varphi(x, y) dx \wedge d\bar{x}$, where $\varphi(x, y)$ is a function supported by K . Using that the $\pi_1|_{\Gamma_n}$ is one-to-one, we get

$$\left| \int_{\Gamma_n} \varphi(x, y) dx \wedge d\bar{x} \right| \leq \|\varphi\|_{\infty} \int_{\Gamma_n \cap \mathbb{D}(0; A)^2} dx \wedge d\bar{x} = \|\varphi\|_{\infty} \pi A^2,$$

so $|\langle [\Gamma_n], \phi \rangle| \leq d^{-n} \|\varphi\|_{\infty} \pi A^2$. Since $\langle \pi_1^* \mu, \phi \rangle = 0$ for such ϕ by a degree reason and $\|\varphi\|_{\infty} \lesssim \|\phi\|_{\mathcal{C}^1}$ the desired estimate follows.

2nd case: $\phi = \varphi(x, y) dy \wedge d\bar{y}$, where $\varphi(x, y)$ is a function supported by K . We will prove the estimate using interpolation between \mathcal{C}^0 and \mathcal{C}^2 functions. For this, let us first assume that φ is of class \mathcal{C}^2 . For simplicity, one can assume that $\|\varphi\|_{\mathcal{C}^2} = 1$. Notice first that

$$\langle \pi_1^* \mu, \phi \rangle = \int_{|y| \leq A} \left(\int_{\mathbb{P}^1} \varphi(x, y) d\mu(x) \right) dy \wedge d\bar{y}.$$

Since both $\|dd^c \varphi\|_*$ and $\|\varphi\|_{\text{Lip}}$ are bounded by a constant, Theorem 2.6-(2) for $\alpha = 1$ gives that, for any $y \in \mathbb{P}^1$, one has that

$$\int_{\mathbb{P}^1} \varphi(x, y) d\mu(x) = d^{-n} \sum_{f^n(x)=y} \varphi(x, y) + O(\underline{d}^{-n} \log^* \text{dist}(y, \mathcal{E}')),$$

where the implicit constants in the big O notation are independent of y, n and φ . Also, $\int_{\Gamma_n} \phi = \int_{\Gamma_n} \varphi(x, y) dy \wedge d\bar{y} = \int_{|y| \leq A} \sum_{f^n(x)=y} \varphi(x, y) dy \wedge d\bar{y}$ by Fubini's theorem. Hence, using that $\log^* \text{dist}(\cdot, \mathcal{E}')$ is integrable, we get

$$\begin{aligned} \langle \pi_1^* \mu, \phi \rangle &= \int_{|y| \leq A} \left[d^{-n} \sum_{f^n(x)=y} \varphi(x, y) + O(\underline{d}^{-n} \log^* \text{dist}(y, \mathcal{E}')) \right] dy \wedge d\bar{y} \\ &= \frac{1}{d^n} \int_{|y| \leq A} \sum_{f^n(x)=y} \varphi(x, y) dy \wedge d\bar{y} + O(\underline{d}^{-n}) \\ &= \frac{1}{d^n} \int_{\Gamma_n} \phi + O(\underline{d}^{-n}) \\ &= \langle [\Gamma_n], \phi \rangle + O(\underline{d}^{-n}), \end{aligned}$$

where again the implicit constants are independent of y, n and φ .

The above computation proves that $|\langle [\Gamma_n] - \pi_1^* \mu, \phi \rangle| \leq C'_{\underline{d},K} \|\phi\|_{\mathcal{C}^2} \underline{d}^{-n}$ when φ is of class \mathcal{C}^2 for some constant $C'_{\underline{d},K} > 0$. It is clear that $|\langle [\Gamma_n] - \pi_1^* \mu, \phi \rangle| \leq C''_{\underline{d},K} \|\phi\|_{\mathcal{C}^0}$ for some constant $C''_{\underline{d},K} > 0$ when φ is of class \mathcal{C}^0 . A standard interpolation argument (see [Tri78, §2.7.2]) yields $|\langle [\Gamma_n] - \pi_1^* \mu, \phi \rangle| \leq C_{\underline{d},K} \|\phi\|_{\mathcal{C}^1} \underline{d}^{-n/2}$ for some constant $C_{\underline{d},K} > 0$.

3rd case: $\phi = \varphi(x, y)dx \wedge d\bar{y}$, where $\varphi(x, y)$ is a function supported by K . Using that $dy|_{\Gamma_n} = (f^n)'(x)dx|_{\Gamma_n}$, that $|y| = |f^n(x)| \leq A$ over K and Cauchy-Schwarz inequality we have that

$$\begin{aligned} \left| \int_{\Gamma_n} \varphi(x, y)dx \wedge d\bar{y} \right| &= \left| \int_{\Gamma_n \cap \mathbb{D}(0; A)^2} \varphi(x, f^n(x)) \overline{(f^n)'(x)} dx \wedge d\bar{x} \right| \\ &\leq \left(\int_{\Gamma_n \cap \mathbb{D}(0; A)^2} |\varphi(x, f^n(x))|^2 dx \wedge d\bar{x} \right)^{1/2} \left(\int_{\Gamma_n \cap \mathbb{D}(0; A)^2} |(f^n)'(x)|^2 dx \wedge d\bar{x} \right)^{1/2} \\ &\leq \|\varphi\|_{\infty} \pi^{1/2} A \cdot c_A d^{m/2}. \end{aligned}$$

In the last step we have used that $\int_{\Gamma_n \cap \mathbb{D}(0; A)^2} |(f^n)'(x)|^2 dx \wedge d\bar{x} = \int_{\Gamma_n \cap \mathbb{D}(0; A)^2} idy \wedge d\bar{y}$ and the last quantity is bounded by a constant times d^n , which is the total mass of Γ_n in $\mathbb{P}^1 \times \mathbb{P}^1$. See also Lemma 4.2 below.

We conclude that $|\langle [\Gamma_n], \phi \rangle| \leq c'_A d^{-n/2} \|\varphi\|_{\infty}$. Since $\langle \pi_1^* \mu, \phi \rangle = 0$ for such ϕ and $\|\varphi\|_{\infty} \lesssim \|\phi\|_{\mathcal{C}^1}$ the desired estimate follows.

4th case: $\phi = \varphi(x, y)d\bar{x} \wedge dy$, where $\varphi(x, y)$ is a function supported by K . The proof is analogous to the one in the 3rd case

Since every test form ϕ on $\mathbb{C} \times \mathbb{C}$ of class \mathcal{C}^1 with support contained in K is a finite sum of forms of one of the above types, the proposition follows. \square

3. SMALL PERTURBATIONS OF Δ

In this section we show that the expected convergence $[\Gamma_n] \wedge [\Delta] \rightarrow \pi_1^* \mu \wedge [\Delta]$ holds with exponential speed if we replace $[\Delta]$ in the left hand side by small perturbations Δ_n converging to Δ .

Lemma 3.1. *After conjugating f by an automorphism of \mathbb{C} , we can assume that there is a constant $c > 0$ such that $|f^n(0)| \geq c$ and $|f^n(x)| \geq 4^{d^n}$ for all $n \geq 1$ and $|x| \geq 1/2$.*

Proof. We first replace f by $f_1 = \tau^{-1} \circ f \circ \tau$, where $\tau : x \mapsto x + x_0$ is a translation sending the origin to a point x_0 in the basin of infinity of f . Then $f_1^n(0) = \tau^{-1} \circ f^n(x_0)$ converges to ∞ as $n \rightarrow \infty$. In particular $|f_1^n(0)| \geq M$ for every n and a constant $M > 0$. Now, let $f_2 = H_R^{-1} \circ f_1 \circ H_R$, where $H_R(x) = Rx$, $R > 0$. We can choose R large enough so that $|x| \geq 1$ is contained in the basin of infinity of f_2 . In particular $|f_2(x)| \geq (8|x|)^d$ when $|x| \geq 1/2$ and R is large enough, and in this case $|f_2^n(x)| \geq 4^{d^n}$. This gives the last assertion of the lemma. For the first assertion, we note that $|f_2^n(0)| = R^{-1}|f_1^n(0)| \geq R^{-1}M$ for all $n \geq 1$. The lemma follows. \square

Using Lemma 3.1 we can assume that $(0, 0) \notin \Gamma_n$ for every $n \geq 1$. Consider the radial projection on $\mathbb{C} \times \mathbb{C}$ centered at the origin, namely

$$\Pi : (\mathbb{C} \times \mathbb{C}) \setminus \{(0, 0)\} \rightarrow \mathbb{P}^1, \quad \Pi(x, y) = [x : y].$$

Note that $\Pi|_{\Gamma_n}$ is well-defined and holomorphic.

For $t \in \mathbb{C}$, let

$$(3.1) \quad \Delta_t := \Pi^{-1}([1 : t]) = \{tx - y = 0\}.$$

Observe that $\Delta_1 = \Delta$. To ease the notation we shall identify t with $[1 : t]$. We'll consider values of t close to 1.

The Fubini-Study form ω_{FS} on \mathbb{P}^1 induces a probability measure on \mathbb{P}^1 , that we denote by m_{FS} .

Proposition 3.2. *Let $\varphi = \varphi(x)$ be a function on \mathbb{C} of class \mathcal{C}^1 and set $\tilde{\varphi}(x, y) = \pi_1^* \varphi(x, y) = \varphi(x)$ on $\mathbb{C} \times \mathbb{C}$. Let $1 < \underline{d} < d$. For $n \geq 1$ and $\lambda > 1$ let*

$$(3.2) \quad m_n := \frac{\mathbf{1}_{\mathbb{D}(1; e^{-\lambda^n})} m_{\text{FS}}}{m_{\text{FS}}(\mathbb{D}(1; e^{-\lambda^n}))}$$

be the normalized Fubini-Study measure on the disk $\mathbb{D}(1; e^{-\lambda^n})$. Then, there exist constants $\lambda > 1$, $0 < \beta < 1$ and a subset $\mathcal{B}_n \subset \mathbb{D}(1; e^{-\lambda^n})$ with $m_n(\mathcal{B}_n) \geq 1 - \beta^n$ such that, for all $t_n \in \mathcal{B}_n$ one has

$$|\langle [\Gamma_n] \wedge [\Delta_{t_n}] - \pi_1^* \mu \wedge [\Delta], \tilde{\varphi} \rangle| \leq C \|\varphi\|_{\mathcal{C}^1} \underline{d}^{-n/2}.$$

We now begin the proof of the above proposition. From now on, we fix $\varphi = \varphi(x)$ a compactly supported function on \mathbb{C} . We'll often identify φ with $\tilde{\varphi} = \pi_1^* \varphi$. Consider the $(1, 1)$ -current

$$\Psi_n := \varphi [\Gamma_n] = d^{-n} \varphi [\Gamma_n],$$

which has compact support on $\mathbb{C} \times \mathbb{C}$ and let

$$\psi_n = \Pi_* \Psi_n = d^{-n} \Pi_*(\varphi [\Gamma_n]).$$

Observe that Π is well-defined on the support of Ψ_n and ψ_n is a continuous function on \mathbb{P}^1 with compact support in \mathbb{C} .

Lemma 3.3. *Let φ and ψ_n be as above. Let ν be a probability measure on \mathbb{P}^1 . Then*

- (1) $\langle [\Gamma_n] \wedge \Pi^* \nu, \varphi \rangle = \langle \nu, \psi_n \rangle$. In particular $\psi_n(t) = \langle [\Gamma_n] \wedge [\Delta_t], \varphi \rangle$ for $t \in \mathbb{C}$ and $\psi_n(1) = \langle [\Gamma_n] \wedge [\Delta], \varphi \rangle$.
- (2) $\langle \Pi^* \nu \wedge \pi_1^* \mu, \varphi \rangle = \int \varphi d\mu$.

Proof. Observe that, since Γ_n intersects the fibers of Π on isolated points, the intersection of currents $[\Gamma_n] \wedge \pi^* \nu$ is always a well defined positive measure on $\mathbb{C} \times \mathbb{C}$. By direct computation

$$\langle [\Gamma_n] \wedge \Pi^* \nu, \varphi \rangle = \langle \Pi^* \nu, \varphi [\Gamma_n] \rangle = \langle \nu, \Pi_*(\varphi [\Gamma_n]) \rangle = \langle \nu, \psi_n \rangle,$$

giving the first identity in (1). The last two identities in (1) follow from the the fact that for fixed $t \in \mathbb{C}$ we have $[\Delta_t] = \Pi^*(\delta_t)$ and $\Delta_1 = \Delta$.

We now prove (2). As in (1), we can readily see that the wedge product $\Pi^* \nu \wedge \pi_1^* \mu$ is well defined. Consider first the case $\nu = \delta_1$. Then $\langle \Pi^* \delta_1 \wedge \pi_1^* \mu, \varphi \rangle = \langle [\Delta] \wedge \pi_1^* \mu, \varphi \rangle = \int \varphi d\mu$. In general, we can write $\nu = \delta_1 + dd^c u$ for some d.s.h. function u on \mathbb{P}^1 . Then,

$$\langle \Pi^* \nu \wedge \pi_1^* \mu, \varphi \rangle = \int \varphi d\mu + \langle dd^c(u \circ \Pi) \wedge \pi_1^* \mu, \varphi \rangle = \int \varphi d\mu + \langle \pi_1^* \mu \wedge dd^c \varphi, u \circ \Pi \rangle.$$

Observe that $\pi_1^* \mu \wedge dd^c \varphi$ vanishes, since it is a $(2, 2)$ -current in the variable x . This gives (2) in general. \square

Recall that $t = [1 : t]$, $t \in \mathbb{C}$ denotes an affine coordinate in \mathbb{P}^1 .

Lemma 3.4. *Let $\nu(t)$ be a smooth probability measure on \mathbb{P}^1 with compact support on \mathbb{C} . Then $|\int \psi_n(t) d\nu(t) - \int \varphi d\mu| \leq C \|\varphi\|_{\mathcal{C}^1} \underline{d}^{-n/2}$ for some constant $C > 0$ independent of φ and n .*

Proof. Using Lemma 3.3 and applying Proposition 2.8 we have that

$$\left| \int \psi_n(t) d\nu(t) - \int \varphi d\mu \right| = |\langle d^{-n}[\Gamma_n] - \pi_1^* \mu, \varphi \Pi^* \nu \rangle| \lesssim \|\varphi\|_{\mathcal{E}^1} \underline{d}^{-n/2},$$

where in the last step we have used that the $(1, 1)$ -form $\phi := \varphi \Pi^* \nu$ is smooth, compactly supported in $\mathbb{C} \times \mathbb{C}$ and $\|\phi\|_{\mathcal{E}^1} \lesssim \|\varphi\|_{\mathcal{E}^1}$. \square

Recall from the Section 2 that the space $W^*(\mathbb{P}^1) = W^{1,2}(\mathbb{P}^1)$ is formed by the measurable functions whose $\|\cdot\|_*$ norm is finite.

Lemma 3.5. *Let φ and ψ_n be as above. Then $\|\partial\psi_n\|_{L^2} \leq d^{-n/2} \|\partial\varphi\|_{L^2}$.*

Proof. Let $\widehat{\psi}_n := d^n \psi_n$. We'll follow the proof of [DKW20, Proposition 2.1]. Denote by π_n the restriction of Π to Γ_n . Then $\widehat{\psi}_n = (\pi_n)_* \varphi$. The map π_n defines a branched covering of degree d^n . As in the proof of [DKW20, Proposition 2.1], using Cauchy-Schwarz inequality we can show that $i\partial\widehat{\psi}_n \wedge \bar{\partial}\widehat{\psi}_n \leq d^n i\partial\varphi \wedge \bar{\partial}\varphi$ pointwise outside the set of critical values of π_n . Since this set is finite, it doesn't contribute to the integral of the above $(1, 1)$ -forms. It follows that

$$\|\partial\widehat{\psi}_n\|_{L^2}^2 = \int i\partial\widehat{\psi}_n \wedge \bar{\partial}\widehat{\psi}_n \leq d^n \int i\partial\varphi \wedge \bar{\partial}\varphi = d^n \|\partial\varphi\|_{L^2}^2.$$

Since $\psi_n = d^{-n} \widehat{\psi}_n$ we get $\|\partial\psi_n\|_{L^2}^2 = d^{-2n} \|\partial\widehat{\psi}_n\|_{L^2}^2 \leq d^{-n} \|\partial\varphi\|_{L^2}^2$ and the result follows. \square

Recall that m_n is the normalized Fubini-Study measure on the disc $\mathbb{D}(1, e^{-\lambda^n})$ as in the statement of Proposition 3.2, where $\lambda > 1$ will be fixed below.

Lemma 3.6. *Let φ , ψ_n and ν be as above. Let $1 < \underline{d} < d$ and set $\lambda_0 := d/\underline{d}$. Then, there exist constants $1 < \lambda < \lambda_0$, $0 < \beta < 1$, independent of φ , and a subset $\mathcal{B}_n \subset \mathbb{D}(1, e^{-\lambda^n})$ such that $m_n(\mathcal{B}_n) \geq 1 - \beta^n$ and for all $t_n \in \mathcal{B}_n$ with n large enough we have*

$$\left| \int \psi_n(t) d\nu(t) - \psi_n(t_n) \right| \leq \|\varphi\|_{\mathcal{E}^1} \underline{d}^{-n/2}.$$

Proof. To ease notation, set $\mathbf{m}(\psi_n) := \int \psi_n(t) d\nu(t)$. We only need to consider n large enough. Since ψ_n depends linearly on φ we can assume that $\|\varphi\|_{\mathcal{E}^1} = 1$. In particular $\|\partial\varphi\|_{L^2}$ is bounded. Then $\|\partial\psi_n\|_{L^2} \lesssim d^{-n/2}$ by Lemma 3.5. By Poincaré–Sobolev's inequality, we have that $\|\mathbf{m}(\psi_n) - \psi_n\|_{L^1} \lesssim \|\partial\psi_n\|_{L^2}$. In particular, the functions $\widetilde{\psi}_n := d^{n/2}(\psi_n - \mathbf{m}(\psi_n))$, $n \geq 1$ belong to a bounded family in $W^*(\mathbb{P}^1)$. By Theorem 2.1-(2) there exist constants $c, \alpha > 0$, independent of φ , such that

$$m_{\text{FS}}(|\widetilde{\psi}_n| \geq M) \leq ce^{-\alpha M^2},$$

for all $M > 0$. Choose $M = (d/\underline{d})^{n/2} = \lambda_0^{n/2}$. Then,

$$m_{\text{FS}}\{|\psi_n - \mathbf{m}(\psi_n)| \geq \underline{d}^{-n/2}\} = m_{\text{FS}}(|\widetilde{\psi}_n| \geq \lambda_0^n) \leq ce^{-\alpha \lambda_0^n}.$$

Let $s_n = e^{-\lambda^n}$, where $\lambda > 1$ will be fixed later and let $m_n(t)$ be the normalized Lebesgue measure on the disc $\mathbb{D}(1, e^{-\lambda^n})$ as in the statement. By definition, for a Borel set $E \subset \mathbb{P}^1$ we have $m_n(E) = m_{\text{FS}}(E \cap \mathbb{D}(1, e^{-\lambda^n})) / m_{\text{FS}}(\mathbb{D}(1, e^{-\lambda^n})) \lesssim e^{2\lambda^n} m_{\text{FS}}(E \cap \mathbb{D}(1, e^{-\lambda^n})) \leq e^{2\lambda^n} m_{\text{FS}}(E)$. Hence, denoting by $\mathcal{B}'_n := \{|\psi_n - \mathbf{m}(\psi_n)| > \underline{d}^{-n/2}\}$ and $\mathcal{B}_n := (\mathcal{B}'_n)^c$, we have by the last paragraph that $m_n(\mathcal{B}'_n) \leq c'e^{2\lambda^n} e^{-\alpha \lambda_0^n}$ for some constant $c' > 0$. By choosing $1 < \lambda < \lambda_0$ sufficiently close to 1 we have $c'e^{2\lambda^n} e^{-\alpha \lambda_0^n} \leq \beta^n$ for some $0 < \beta < 1$ and large n . Hence $m_n(\mathcal{B}'_n) \leq \beta^n$ and $m_n(\mathcal{B}_n) \geq 1 - \beta^n$.

Recall that we have assumed that $\|\varphi\|_{\mathcal{C}^1} = 1$. We have thus proved that for φ of class \mathcal{C}^1 and $t_n \in \mathcal{B}_n$ as above we have $|\psi_n(t_n) - \mathbf{m}(\psi_n)| \leq \|\varphi\|_{\mathcal{C}^1} \underline{d}^{-n/2}$ and $m_n(\mathcal{B}_n) \geq 1 - \beta^n$. This is the desired result. \square

Proof of Proposition 3.2. Fix $1 < \underline{d} < d$ and let \mathcal{B}_n be as in Lemma 3.6. From Lemmas 3.3, 3.4, 3.6 and the fact that $\langle \pi_1^* \mu \wedge [\Delta], \tilde{\varphi} \rangle = \int \varphi d\mu$ we obtain that, for $t_n \in \mathcal{B}_n$ we have

$$|\langle [\Gamma_n] \wedge [\Delta_{t_n}] - \pi_1^* \mu \wedge [\Delta], \tilde{\varphi} \rangle| = \left| \psi_n(t_n) - \int \varphi d\mu \right| \leq (1 + C) \|\varphi\|_{\mathcal{C}^1} \underline{d}^{-n/2},$$

which is the desired result. \square

4. LIFT TO THE TANGENT BUNDLE

Let $X := \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ with coordinates (x, y, t) and compactify it by $\overline{X} := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We denote by $\mathbf{p}(x, y, t) = (x, y)$ and $\mathbf{q}(x, y, t) = t$ the two natural projections and set

$$z = x - y.$$

The graphs $\Gamma_n \subset \mathbb{C} \times \mathbb{C}$ admit natural lifts to X by setting

$$\widehat{\Gamma}_n = \{(x, f^n(x), (f^n)'(x)) : x \in \mathbb{C}\} \subset X.$$

We will denote by $[\widehat{\Gamma}_n]$ the corresponding integration current and

$$[[\widehat{\Gamma}_n]] := \frac{1}{d^n} [\widehat{\Gamma}_n]$$

its normalized version. These are positive closed currents of bidegree $(2, 2)$ viewed in X or \overline{X} .

For later use, we state some intermediary estimates. The first one implies that, in a sense, the currents $[[\widehat{\Gamma}_n]]$ converge to vertical currents with respect to the projection $\mathbf{p}(x, y, t) = (x, y)$, and they become vertical exponentially fast.

Lemma 4.1. *There is a constant $c > 0$ such that for every $A > 0$, $M \geq 1$ and $n \geq 1$*

$$\langle [[\widehat{\Gamma}_n]], id_x \wedge d\bar{x} \rangle_{\{|x| \leq A\}} \leq cA^2 d^{-n}$$

and

$$\langle [[\widehat{\Gamma}_n]], id_y \wedge d\bar{y} \rangle_{\{|x| \leq A, |t| \leq M\}} \leq cA^2 M^2 d^{-n}.$$

Proof. Since $\widehat{\Gamma}_n$ is a graph over $x \in \mathbb{D}(0, A)$, the first integral is equal to d^{-n} times the area of $\mathbb{D}(0, A)$. This gives the first inequality.

For $(x, y, t) \in \widehat{\Gamma}_n$, we have $dy = tdx$. Hence, the LHS of the second inequality is bounded by d^{-n} times

$$\int_{|x| \leq A} M^2 id_x \wedge d\bar{x} \lesssim A^2 M^2.$$

\square

Lemma 4.2. *For every $A > 0$ there is a constant $c_A > 0$ such that*

$$\int_{|z| \leq A} |(f^n)'(x)|^2 (id_x \wedge d\bar{x}) \leq c_A d^n.$$

Proof. Observe that for $(x, y, t) \in \widehat{\Gamma}_n$ if $|z| = |x - y| \leq A$ then both $|x|$ and $|y|$ are bounded by a constant $M > 0$, independent of n . This follows for instance from the fact that $|f(x)| \geq 2|x|$ outside some fixed disc in \mathbb{C} centered at the origin, so $|y| = |f^n(x)| \geq 2^n|x|$ there. Therefore, the integral in the statement is bounded by the area of Γ_n inside the bidisc $\mathbb{D}(0, M)^2$ in $\mathbb{C} \times \mathbb{C}$. The lemma follows. \square

Note that the diagonal $\Delta \subset \mathbb{C} \times \mathbb{C}$ lifts to

$$\widehat{\Delta} = \{x = y, t = 1\} \subset X.$$

One step of the proof of Theorem 1.1 will consist in bounding the number of branches of Γ_n that are nearly tangent to Δ . This will amount to controlling the behaviour of $\widehat{\Gamma}_n$ near $\widehat{\Delta}$. We'll obtain estimates on the mass of $[\widehat{\Gamma}_n]$ over neighbourhoods of $\widehat{\Delta}$ the form $\{|z| < r_n, |t-1| < s_n\}$ for suitable sequences $(r_n)_{n \geq 1}$ and $(s_n)_{n \geq 1}$ tending to zero as $n \rightarrow \infty$. In order to achieve this goal, we first estimate the mass at the slice $t = 0$. This slice is easier to handle due to the chain rule for $(f^n)'(x)$.

4.1. The slice $t = 0$. The lemma below bounds the mass of the slice $t = 0$, which correspond to the critical points of f^n . Recall from the statement of Theorem 2.6 that \mathcal{E} denotes the exceptional set of f viewed as a map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. This set is always contained in the critical set of f .

Lemma 4.3. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial map viewed as an endomorphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ as above. Let $0 < r < 1$ and $\epsilon_0 > 0$ be such that the ϵ_0 -neighborhood of \mathcal{E} contains no other critical point of f . Set*

$$\Omega_n = \Omega_n(r, \epsilon_0) := \{|z| < 8r^n, \text{dist}(x, \mathcal{E}) \geq \epsilon_0\} \subset X.$$

Then, for every r and ϵ_0 , there exist constants $c_1 > 0$ and $0 < \beta < 1$ (depending on r and ϵ_0) such that the mass of the measures $[\widehat{\Gamma}_n] \wedge [t = 0]$ over Ω_n is bounded by $c_1 \beta^n$.

Proof. It suffices to consider n large enough. Let \mathcal{C} be the critical set of f and \mathcal{C}_n that of f^n . By definition $[\widehat{\Gamma}_n] \wedge [t = 0]$ is the measure ϑ_n on $X \cap \{t = 0\} \simeq \mathbb{C} \times \mathbb{C}$ given by

$$\vartheta_n = \frac{1}{d^n} \sum_{x \in \mathcal{C}_n} \delta_{(x, f^n(x))}.$$

By the chain rule, $x \in \mathcal{C}_n$ if and only if $(f^n)'(x) = f'(f^{n-1}(x))f'(f^{n-2}(x)) \cdots f'(f(x))f'(x)$ vanishes, so $\mathcal{C}_n = \mathcal{C} \cup f^{-1}(\mathcal{C}) \cup \dots \cup f^{-(n-2)}(\mathcal{C}) \cup f^{-(n-1)}(\mathcal{C})$ and $f^n(\mathcal{C}_n) = f^n(\mathcal{C}) \cup f^{n-1}(\mathcal{C}) \cup \dots \cup f^2(\mathcal{C}) \cup f(\mathcal{C})$. Therefore, we can write

$$\vartheta_n = \frac{1}{d^n} \sum_{c \in \mathcal{C}} \sum_{p=1}^n \sum_{a \in f^{-n+p}(c)} \delta_{(a, f^p(c))},$$

where we count the points with multiplicity. Alternatively, one could rewrite the above identities for \mathcal{C}_n and $f^n(\mathcal{C}_n)$ as $[\mathcal{C}_n] = \sum_{p=0}^{n-1} (f^p)^*[\mathcal{C}]$, where the bracket denotes the corresponding integration currents and equivalently for $[f^n(\mathcal{C}_n)]$.

Claim: If $(a, f^p(c)) \in \Omega_n$ for some $c \in \mathcal{C}$, then $\text{dist}(c, \mathcal{E}) \geq \epsilon_0$.

Proof of claim: By the definition of Ω_n , we have $|a - f^p(c)| < 8r^n$ and $\text{dist}(a, \mathcal{E}) \geq \epsilon_0$. Assume by contradiction that $\text{dist}(c, \mathcal{E}) < \epsilon_0$. Then, $c \in \mathcal{E}$ by the assumption on ϵ_0 . By the invariance of \mathcal{E} , we have $f^p(c) \in \mathcal{E}$. Hence, $\text{dist}(a, \mathcal{E}) < 8r^n$, which is smaller than ϵ_0 for n large, a contradiction. The claim follows. \square

Fix $p = 1, \dots, n$ and $c \in \mathcal{C}$, and set $b := f^p(c)$. Notice that $(a, b) \in \Omega_n$ implies that $a \in \mathbb{D}(b, 8r^n)$. Hence, the mass of $\sum_{a \in f^{-n+p}(c)} \delta_{(a, f^p(c))}$ over Ω_n is bounded by the mass of $(f^{n-p})^* \delta_c = d^{n-p} \mu_c^p$ over $\mathbb{D}(b, r^n)$. We use here the notation of Theorem 2.6. From Corollary 2.7 and the above claim, one has $\mu_c^{n-p}(\mathbb{D}(b, 8r^n)) \leq \mu_c^{n-p}(\mathbb{D}(b, 8r^{n-p})) \lesssim \beta^{n-p}$ for some $0 < \beta < 1$ (depending on r). By increasing β if necessary, we can assume that $\beta > d^{-1}$.

Therefore

$$\vartheta_n(\Omega_n) \lesssim \frac{1}{d^n} (d-1) \sum_{p=1}^n d^{n-p} \beta^{n-p} = (d-1) \beta^n \sum_{p=1}^n d^{-p} \beta^{-p} \lesssim \beta^n$$

and the lemma follows. \square

Near the slice $t = 1$. In this section we estimate the mass of $[\widehat{\Gamma}_n]$ near the slice $t = 1$. This will be used later to construct good branches of Γ_n across Δ .

Proposition 4.4. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial map as above viewed as an endomorphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. For $n \geq 1$ and $\lambda > 1$ let $m_n(t)$ be the induced normalized Fubini-Study measure on $\mathbb{D}(1, e^{-\lambda^n})$ defined in (3.2), viewed also as a positive $(1, 1)$ -form in t . There exist $0 < r < 1$, $\lambda > 1$ and constants $c_2 > 0$ and $0 < \beta < 1$, depending on r and λ , such that the mass of the measure $[\widehat{\Gamma}_n] \wedge m_n(t)$ over $\{|z| \leq 4r^n\}$ is bounded by $c_2 \beta^n$.*

Remark 4.5. *When $f(x) = x^d$, the periodic points of order n satisfy either $|(f^n)'(x)| = d^n$ or $(f^n)'(x) = 0$. It follows that the curves $\widehat{\Gamma}_n$ do not intersect $\text{supp } m_n(t) \cap \{|z| \leq 4r^n\}$ and Proposition 4.4 is trivially true in this case. Therefore, in the proof we will be able to assume that f is not conjugated to the map $x \mapsto x^d$. We can avoid singling out this exceptional map by working outside a neighborhood of the exceptional set of f . The proof below still works, but requires the introduction of an additional cut-off function, making the estimates more technical. We have chosen to avoid it in order to simplify the proof.*

Observe that, from Lemma 3.1, after an affine change of coordinates in \mathbb{C} , we can assume that $(x, y, t) \in \widehat{\Gamma}_n$ and $|x| \geq 1/2$, then $|y| \geq 4^{d^n}$ and $|z| \geq 3^{d^n}$. In particular, we have that

$$(4.1) \quad \widehat{\Gamma}_n \cap \{|z| \leq 3\} \subset \{|x| \leq 1\}.$$

We can now prove Proposition 4.4.

Proof of Proposition 4.4. It is enough to consider n large. By the above discussion, we can assume that (4.1) holds. Let $0 < r < 1$ and $\lambda > 1$ be constants whose values will be fixed in the end of the proof and set $s_n := e^{-\lambda^n}$.

For each $n \geq 1$, let ρ_n be a cut-off function such that $\mathbf{1}_{\{|z| < 4r^n\}} \leq \rho_n \leq \mathbf{1}_{\{|z| < 8r^n\}}$ and $\|\rho_n\|_{\mathcal{G}^2} \lesssim r^{-2n}$. As in the statement of the proposition, we let $m_n(t)$ be the normalized Fubini-Study measure on $\mathbb{D}(1, s_n)$, viewed also as a positive $(1, 1)$ -form in t . Then, it is enough to show that

$$\langle [\widehat{\Gamma}_n] \wedge m_n(t), \rho_n(z) \rangle \leq c_2 \beta^n.$$

for some constant $c_2 > 0$.

Denote by $\mathcal{L}_n := (\pi s_n^2)^{-1} \mathbf{1}_{|t-1| \leq s_n} idt \wedge d\bar{t}$ the normalized Lebesgue measure on the disc $\mathbb{D}(1; s_n)$ viewed as a positive $(1, 1)$ -form in the variable t . Observe that $m_n(t) \leq C \mathcal{L}_n(t)$ for

some constant $C > 0$ independent of n . Hence, it is enough to show that

$$(4.2) \quad \langle \llbracket \widehat{\Gamma}_n \rrbracket \wedge \mathcal{L}_n(t), \rho_n(z) \rangle \leq c'_2 \beta^n.$$

for some constant $c'_2 > 0$.

By Lemma 2.3, we can write

$$\mathcal{L}_n(t) - \delta_0(t) = \text{dd}^c \chi_n(t)$$

for a d.s.h. function χ_n on \mathbb{C} with $-|\log s_n| \lesssim \chi_n(t) \lesssim 2 - \log |t|$ on $\mathbb{D}(0; 3)$ and $|\chi_n(t)| \lesssim 1/|t|$ for $|t| \geq 2$.

We have seen that $(x, y, t) \in \widehat{\Gamma}_n$, if $|z| = |x - y|$ is bounded by a constant $A > 0$, both $|x|$ and $|y|$ are bounded by a constant $M > 0$, independent of n . See the proof of Lemma 4.2. Moreover, if (x, y, t) and $|z| \leq M$, then $|t| \leq M'_n$ for some $M'_n > 0$. Since $\rho_n(z)$ has compact support, the integrals below all take place over some compact subset K_n of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$, so we can apply Stokes' formula. Then, the left-hand side of (4.2) equals

$$\begin{aligned} \langle \llbracket \widehat{\Gamma}_n \rrbracket, \rho_n(z) \mathcal{L}_n(t) \rangle &= \langle \llbracket \widehat{\Gamma}_n \rrbracket, \rho_n(z) \delta_0(t) \rangle + \langle \llbracket \widehat{\Gamma}_n \rrbracket, \rho_n(z) \text{dd}^c \chi_n(t) \rangle \\ &= \langle \llbracket \widehat{\Gamma}_n \rrbracket, \rho_n(z) \delta_0(t) \rangle + \langle \text{dd}^c(\rho_n(z)) \wedge \llbracket \widehat{\Gamma}_n \rrbracket, \chi_n(t) \rangle \\ &= \langle \llbracket \widehat{\Gamma}_n \rrbracket, \rho_n(z) \delta_0(t) \rangle + \langle \llbracket \widehat{\Gamma}_n \rrbracket, \chi_n(t) \text{dd}^c(\rho_n(z)) \rangle. \end{aligned}$$

By (4.1), we have that $\text{supp} \rho_n(z) \cap \widehat{\Gamma}_n \subset \Omega_n$, where Ω_n is the set from Lemma 4.3. By that lemma, we get that

$$\langle \llbracket \widehat{\Gamma}_n \rrbracket, \rho_n(z) \delta_0(t) \rangle \leq c_1 \beta^n.$$

for some constants $0 < \beta < 1$ and $c_1 > 0$.

Therefore, in order to conclude the proof, we need to show that

$$\langle \llbracket \widehat{\Gamma}_n \rrbracket \wedge \theta_n, \chi_n(t) \rangle \leq c_2 \beta^n,$$

for some constant $c_2 > 0$, where

$$\theta_n(z) := \text{dd}^c \rho_n(z).$$

Observe that θ_n is a smooth $(1, 1)$ -form supported by $\{|z| < 8r^n\}$ and $\|\theta_n\|_\infty \lesssim r^{-2n}$. In particular, the mass of θ_n is bounded uniformly in n .

We'll split the last integral into two parts. Let $0 \leq \eta(t) \leq 1$ be a cut-off function equal to 1 on $\mathbb{D}(0, 2)$ and 0 outside $\mathbb{D}(0, 3)$. Then,

$$\langle \llbracket \widehat{\Gamma}_n \rrbracket \wedge \theta_n(z), \chi_n(t) \rangle = \langle \llbracket \widehat{\Gamma}_n \rrbracket \wedge \theta_n(z), \eta(t) \chi_n(t) \rangle + \langle \llbracket \widehat{\Gamma}_n \rrbracket \wedge \theta_n(z), (1 - \eta(t)) \chi_n(t) \rangle.$$

The above two terms the left-hand side are handled separately in Lemmas 4.6 and 4.10 below. Combined, they give (4.2) and conclude the proof of the proposition. \square

Lemma 4.6. *In the above notations, one can choose constants $0 < r < 1$ and $\lambda > 1$ such that*

$$|\langle \llbracket \widehat{\Gamma}_n \rrbracket \wedge \theta_n(z), \eta(t) \chi_n(t) \rangle| \leq c_3 \beta^n$$

for some constants $c_3 > 0$ and $0 < \beta < 1$.

We first prove an auxiliary result. Let ν_n be the measure on \mathbb{C} defined by

$$(4.3) \quad \nu_n := \kappa_n \cdot \mathbf{1}_{\{|x| \leq 1\} \cap \{|(f^n)'(x)| \leq 3\}} \text{id}x \wedge \text{d}\bar{x},$$

where $\kappa_n > 0$ is chosen so that ν_n is a probability measure. Notice that κ_n is bounded from below by $(\int_{|x| \leq 1} \text{id}x \wedge \text{d}\bar{x})^{-1} = 1/\pi$. The role of ν_n will be clear later on.

Lemma 4.7. *There exists a constant $M > 0$ independent of n such that ν_n has a Lipschitz continuous potential on \mathbb{C} whose Lipschitz constant is $\leq M\kappa_n$.*

Proof. Set $\tilde{\nu}_n := \kappa_n^{-1}\nu_n = \mathbf{1}_{\{|x| \leq 1\} \cap \{|(f^n)'(x)| \leq 3\}} \cdot \mathcal{L}$, where \mathcal{L} denotes the Lebesgue measure. By Lemma 2.4, the measure $\tilde{\nu}_n$ has a Lipschitz potential with an uniformly bounded Lipschitz constant. As $\nu_n = \kappa_n \tilde{\nu}_n$, the lemma follows. \square

We'll need later the following lemma, which we'll apply to ν_n above.

Lemma 4.8. *Let ν be a probability measure with compact support on \mathbb{C} . Assume that ν has a Lipschitz continuous potential on \mathbb{C} whose Lipschitz constant is $\leq \kappa$ for some $\kappa \geq 1$. There exist constants $C > 0$ and $0 < \delta < 1$ independent of ν such that, for every $n \geq 1$ and every $p \geq 1$ we have*

$$\langle (f^p)_*(\nu), -\log |f'| \rangle \leq C\delta^{-p}(p + \log \kappa).$$

Moreover $\delta \geq (d-1)^{-1}$ unless f is conjugated to the map $x \mapsto x^d$.

Proof. Let u be the μ -potential of ν , that is, the unique d.s.h. function satisfying

$$\nu = \mu + dd^c u \quad \text{and} \quad \langle \mu, u \rangle = 0.$$

From the Lipschitz regularity of ν and the fact that μ has Hölder local potentials, we see that the function u is $(M\kappa, \alpha)$ -Hölder for some constants $M \geq 1$ and $0 < \alpha < 1$.

From the invariance of μ we get $(f^p)_*(\nu) = \mu + dd^c u_p$ for $u_{n,p} := (f^p)_*u$. Notice that $\|dd^c u_p\| \leq 1$ and $\langle \mu, u_p \rangle = 0$ for every $p \geq 0$.

By Poincaré-Lelong formula we have $dd^c \log |f'| = \sum_{c \in \mathcal{C}} \delta_c - (d-1)\delta_\infty$. Since $(f^p)_*\nu = \mu + dd^c u_p$, $(f^p)_*\mu = \mu$ and $u_p = (f^p)_*u$ we have

$$\begin{aligned} \langle (f^p)_*(\nu), -\log |f'| \rangle &= -\langle \mu, \log |f'| \rangle - \langle u_p, dd^c \log |f'| \rangle \\ &= -\langle \mu, \log |f'| \rangle - \sum_{c \in \mathcal{C}} u_p(c) + (d-1)u_p(\infty). \end{aligned}$$

Since ∞ is mapped to itself by f with multiplicity d , it follows that $u_p(\infty) = (f^p)_*u(\infty) = d^p u(\infty)$. We thus have

$$(4.4) \quad \langle (f^p)_*(\nu), -\log |f'| \rangle = -\langle \mu, \log |f'| \rangle - \sum_{c \in \mathcal{C}} u_p(c) + (d-1)d^p u(\infty).$$

Claim 1: $u(\infty) \leq 0$.

Proof of claim: Let $G : \mathbb{C} \rightarrow \mathbb{R}^{\geq 0}$ be the Green's function of f and use the same notation for its extension to \mathbb{P}^1 . Then $dd^c G = \mu - \delta_\infty$. Therefore,

$$\begin{aligned} u(\infty) &= \langle \delta_\infty, u \rangle = \langle \delta_\infty - \mu, u \rangle = \langle -dd^c G, u \rangle = \langle G, -dd^c u \rangle \\ &= \langle G, \mu - \nu \rangle = \langle G, -\nu \rangle \leq 0, \end{aligned}$$

where we have used that $\langle \mu, u \rangle = 0$, from the definition of u , and $\langle \mu, G \rangle = 0$ because the Green's function vanishes on the support of μ . This proves the claim. \square

From the above claim and (4.4) we get that

$$(4.5) \quad \langle (f^p)_*(\nu), -\log |f'| \rangle \leq -\langle \mu, \log |f'| \rangle - \sum_{c \in \mathcal{C}} u_p(c).$$

The term $\langle \mu, \log |f'| \rangle$ is finite, hence bounded by a constant independent of p . We now bound the remaining term.

Claim 2: Let $K := \overline{\mathbb{D}(0; R)} \subset \mathbb{C} \subset \mathbb{P}^1$ be a large disc containing \mathcal{C} and such that $\mathbb{P}^1 \setminus K$ is contained in the basin of ∞ of f . Let $\kappa \geq 1$ be as in the statement of Lemma 4.8. There are constants $M > 0$, $A \geq 1$ and $0 < \alpha, \delta < 1$ independent of p such that u_p is $(M\kappa A^p, \alpha\delta^p)$ -Hölder over K . Moreover $\delta \geq (d-1)^{-1}$ unless f is conjugated to the map $x \mapsto x^d$.

Remark 4.9. We can avoid singling out the exceptional map $x \mapsto x^d$ by working over $\text{dist}(x, \mathcal{E}') \geq \epsilon_0$, where \mathcal{E}' is the orbit of the periodic critical points of f . In this case we obtain an exponent δ which arbitrarily close to 1 after iteration. The weaker form of the above claim is enough for our purposes and the proof becomes less technical.

Proof of claim: Observe that since $\mathbb{P}^1 \setminus K$ is contained in the basin of ∞ and R is large, we have that $f^{-1}(K) \subset K$. We have seen above that u is $(M\kappa, \alpha)$ -Hölder for some constants $M > 0$ and $0 < \alpha < 1$. Let $l \geq 1$ be the maximal local multiplicity of f over K and set $\delta := 1/l$. Hence, f is given locally around any point of K by $\zeta \mapsto \zeta^m$ for some $m \in \{1, 2, \dots, l\}$. Notice that $l \leq d-1$ unless f is conjugated to the map $x \mapsto x^d$. It follows that for $x, x' \in K$ we can write $f^{-1}(x) = \{y_1, \dots, y_d\}$ and $f^{-1}(x') = \{y'_1, \dots, y'_d\}$ with $\text{dist}(y_j, y'_j) \leq B \text{dist}(x, x')^\delta$ for some constant $B \geq 1$. Furthermore, $y_j, y'_j \in K$ for all $j = 1, \dots, d$. See [DS10b, Proposition 4.1] for details.

For $p = 1$ we have $u_1(x) = f_* u(x) = \sum_{j=1}^d u(y_j)$ and similarly for x' , hence $|u_1(x) - u_1(x')| \leq dM\kappa_n B \text{dist}(x, x')^{\alpha\delta}$. By induction, it follows that

$$|u_p(x) - u_p(x')| \leq M\kappa d^p B^p \text{dist}(x, x')^{\alpha\delta^p}.$$

This proves the claim with $A = Bd$. \square

We now resume the proof of Lemma 4.8. Recall that we need to bound from above the right-hand side of (4.5). Notice that the u_p belong to $\mathcal{F} := \{v \in \text{DSH}(\mathbb{P}^1) : \|v\|_\mu \leq 1\}$, which is a bounded family inside $\text{DSH}(\mathbb{P}^1)$, see Section 2. By the claim above u_p is $(M\kappa A^p, \alpha\delta^p)$ -Hölder. Hence, by Corollary 2.2-(1)

$$\|u_p\|_\infty \lesssim \delta^{-p}(p + \log \kappa).$$

Therefore, the right-hand side of (4.5) is bounded by a constant times $\delta^{-p}(p + \log \kappa)$. This proves the lemma. \square

We can now prove Lemma 4.6.

Proof of Lemma 4.6. Let $\theta_n = \theta_n^+ - \theta_n^-$, where θ_n^\pm are respectively the positive and negative parts of θ_n . Denoting by $|\theta_n| = \theta_n^+ + \theta_n^-$, we have

$$|\langle [\widehat{\Gamma}_n] \wedge \theta_n(z), \eta(t)\chi_n(t) \rangle| \leq \langle [\widehat{\Gamma}_n] \wedge |\theta_n(z)|, |\eta(t)\chi_n(t)| \rangle.$$

Recall that $s_n = e^{-\lambda n}$ and $-|\log s_n| \lesssim \chi_n(t) \lesssim 2 - \log |t|$. As $|\log |t|| \lesssim 2 - \log |t| \lesssim |\log s_n| - \log |t|$ on $\mathbb{D}(0; 3)$, we have that

$$|\eta(t)\chi_n(t)| \lesssim \eta(t)|\log s_n| - \eta(t)\log |t|.$$

Notice that the right hand-side above is positive for large n .

Setting

$$\mathcal{J}_n^{(1)} := \langle [\widehat{\Gamma}_n] \wedge |\theta_n(z)|, \eta(t)|\log s_n| \rangle \quad \text{and} \quad \mathcal{J}_n^{(2)} := \langle [\widehat{\Gamma}_n] \wedge |\theta_n(z)|, -\eta(t)\log |t| \rangle,$$

the desired estimate boils down to showing that

$$(4.6) \quad \mathcal{J}_n^{(1)} + \mathcal{J}_n^{(2)} \leq c_2'' \beta^n$$

for some constants $c_2'' > 0$ and $0 < \beta < 1$.

We'll use that, over $\widehat{\Gamma}_n$, $|z| \leq 3$ implies $|x| \leq 1$ (see (4.1)). Observe that

$$(4.7) \quad dz|_{\widehat{\Gamma}_n} = dx - dy|_{\widehat{\Gamma}_n} = (1 - (f^n)'(x))dx|_{\widehat{\Gamma}_n} = (1 - t)dx|_{\widehat{\Gamma}_n}$$

and $|1 - t| \leq 4$ over the support of $\eta(t)$.

Using that $\|\theta_n\|_\infty \lesssim r^{-2n}$, recalling that $\theta_n(z)$ is supported by $\{|z| \leq 8r^n\}$ and using (4.1), we obtain that

$$(4.8) \quad |\theta_n(z)| \lesssim \frac{1}{r^{2n}} |1 - t|^2 \mathbf{1}_{|x| \leq 1} dx \wedge d\bar{x} \quad \text{over } \widehat{\Gamma}_n.$$

Since $|t| \leq 3$ over the support of $\eta(t)$, we get that

$$(4.9) \quad \mathcal{J}_n^{(1)} \lesssim \frac{|\log s_n|}{r^{2n}} \left| \langle [\widehat{\Gamma}_n], \mathbf{1}_{|x| \leq 1} dx \wedge d\bar{x} \rangle \right| \lesssim \frac{|\log s_n|}{r^{2n} d^n},$$

where we have used the first inequality in Lemma 4.1.

We now bound $\mathcal{J}_n^{(2)}$. Using (4.8), we have

$$\mathcal{J}_n^{(2)} \lesssim \frac{1}{d^n r^{2n}} \int_{\widehat{\Gamma}_n \cap \{|x| \leq 1\} \cap \{|t| \leq 3\}} -\log |(f^n)'(x)| dx \wedge d\bar{x}$$

Let ν_n be the measure defined in (4.3). Since the projection $(x, y, t) \mapsto x$ is a biholomorphism over $\widehat{\Gamma}_n$, the last integral above equals

$$\begin{aligned} \frac{1}{\kappa_n d^n r^{2n}} \langle \nu_n, -\log |(f^n)'| \rangle &= \frac{1}{\kappa_n d^n r^{2n}} \sum_{p=0}^{n-1} \langle \nu_n, -\log |f'| \circ f^p \rangle \\ &= \frac{1}{\kappa_n d^n r^{2n}} \sum_{p=0}^{n-1} \langle (f^p)_*(\nu_n), -\log |f'| \rangle, \end{aligned}$$

where in the first equality we have used the chain rule for $(f^n)'$ and in the second one the change of variables formula.

Our main theorem is clearly true for the map $x \mapsto x^d$, so we can assume that $\delta^{-1} \leq d - 1$ in Lemma 4.8. From Lemma 4.7 and Lemma 4.8 applied to ν_n we get that

$$(4.10) \quad \mathcal{J}_n^{(2)} \lesssim \frac{1}{\kappa_n d^n r^{2n}} \sum_{p=0}^{n-1} \delta^{-p} (p + \log^+ \kappa_n) \lesssim \frac{1}{d^n r^{2n}} n \delta^{-n} \leq n \frac{(d-1)^n}{d^n} \frac{1}{r^{2n}}.$$

Recall that $\kappa_n \geq 1/\pi$, so $\log^+ \kappa_n / \kappa_n$ is uniformly bounded.

Combining (4.9), (4.10) and using that $s_n = e^{-\lambda^n}$ we get that

$$\mathcal{J}_n^{(1)} + \mathcal{J}_n^{(2)} \lesssim \frac{\lambda^n}{r^{2n} d^n} + n \frac{(d-1)^n}{d^n} \frac{1}{r^{2n}}.$$

Choosing both $0 < r < 1$ and $\lambda > 1$ sufficiently close to 1 yields (4.6) for suitable constants $c_3 > 0$ and $0 < \beta < 1$. The lemma follows. \square

Lemma 4.10. *In the above notations, one can choose constants $0 < r < 1$ and $\lambda > 1$ such that*

$$\left| \langle [\widehat{\Gamma}_n] \wedge \theta_n(z), (1 - \eta(t)) \chi_n(t) \rangle \right| \leq c_4 \beta^n$$

for some constants $c_4 > 0$ and $0 < \beta < 1$.

Proof. Recall that $|\chi_n(t)| \lesssim 1/|t|$ for $|t| \geq 2$ and $\|\theta_n(z)\|_\infty = O(r^{-2n})$. Using (4.7), we get that

$$\begin{aligned} |\theta_n(z), (1 - \eta(t))\chi_n(t)|_{\widehat{\Gamma}_n} &\lesssim \frac{1}{r^{2n}} \frac{|1-t|^2}{|t|} \mathbf{1}_{|t| \geq 2, |z| \leq 3} \text{id}x \wedge d\bar{x}|_{\widehat{\Gamma}_n} \\ &\lesssim \frac{1}{r^{2n}} (1 + |(f^n)'(x)|) \mathbf{1}_{|t| \geq 2, |z| \leq 3} \text{id}x \wedge d\bar{x}|_{\widehat{\Gamma}_n}. \end{aligned}$$

Using Cauchy-Schwarz, we get that $|\langle [\widehat{\Gamma}_n] \wedge \theta_n(z), (1 - \eta(t))\chi_n(t) \rangle|$, is bounded by a constant times

$$\frac{1}{r^{2n} d^n} \int_{|z| < 3} (1 + |(f^n)'(x)|) \text{id}x \wedge d\bar{x} \lesssim \frac{1}{r^{2n} d^n} \left[\int_{|z| < 3} \text{id}x \wedge d\bar{x} \right]^{1/2} \left[\int_{|z| < 3} (1 + |(f^n)'(x)|^2) \text{id}x \wedge d\bar{x} \right]^{1/2}.$$

Using Lemmas 4.1 and 4.2, we see that the last quantity is $\lesssim r^{-2n} d^{-n} d^{n/2}$. The result follows by taking $0 < r < 1$ sufficiently close to 1 and setting $\beta = r^{-2} d^{-1/2}$. \square

5. TRANSVERSE BRANCHES ACROSS Δ

In this section we show that we can produce many branches of Γ_n crossing Δ in a sufficiently transversal way. Together with the equidistribution given by Proposition 3.2, this will yield Theorem 1.1.

As above, let $\pi(x, y) = x - y = z$. For $t \in \mathbb{C}^*$ let

$$\pi_t : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}, \quad \pi_t(x, y) = tx - y.$$

Note that $\pi_t|_{\Gamma_n}$ is a branched covering of degree d^n . Indeed, the fibers over $c \in \mathbb{C}$ correspond to the solutions of the polynomial equation $f^n(x) = tx - c$. Observe that $\Delta_t = \{tx - y = 0\} = \pi_t^{-1}(0)$, see (3.1).

For fixed $t \in \mathbb{C}$ consider the coordinate system

$$(z_t, w_t) := (tx - y, tx + y)$$

on $\mathbb{C} \times \mathbb{C}$. We'll work with values of t close to $t = 1$. When $t = 1$ we recover the coordinate $z = x - y$ as above. In these new coordinates, the projection π_t is given by $(z_t, w_t) \mapsto z_t$. Moreover,

$$(5.1) \quad \Delta_t = \{z_t = 0\} \quad \text{and} \quad \Delta = \{(1+t)z_t = (t-1)w_t\}.$$

Observe that when $|x| \leq 1$ as in (4.1) (see also Lemma 3.1) and if $|t-1| \leq s_n$ for $s_n = e^{-\lambda^n}$ as in Proposition 4.4 we have for n large, that

$$(5.2) \quad |z_t| \leq 2r^n \implies |z| = |z_t - (t-1)x| \leq |z_t| + s_n \leq 2r^n + e^{-\lambda^n} \leq 3r^n.$$

The lemma below says that, for many values of t near $t = 1$, the projection π_t admits many inverse branches above $\{|z_t| < r^n\}$. Recall from (3.2) that $m_n(t)$ is the normalized Fubini-Study measure on the disc $\mathbb{D}(1, e^{-\lambda^n})$.

Lemma 5.1. *Let $0 < r < 1$ and $s_n = e^{-\lambda^n}$ be as in Proposition 4.4. There exist $0 < \alpha_0, \beta_0 < 1$ and a subset $\mathcal{A}_n \subset \mathbb{D}(1, s_n)$ with $m_n(\mathcal{A}_n) \geq 1 - \alpha_0^n$ such that for every $t_n \in \mathcal{A}_n$ the restriction of π_{t_n} to $\Gamma_n \cap \pi_{t_n}^{-1}(\mathbb{D}(0; 2r^n)) = \Gamma_n \cap \{|z_{t_n}| < 2r^n\}$ admits at least $d^n(1 - \beta_0^n)$ inverse branches.*

Proof. Denote by $\tilde{\pi}_t$ the restriction of π_t to $\Gamma_n \cap \pi_t^{-1}(\mathbb{D}(0; 2r^n)) = \Gamma_n \cap \{|z_t| < 2r^n\}$. By Riemann-Hurwitz theorem [Din05, Lemme 3.4] the number of inverse branches of $\tilde{\pi}_t$ is at least $\deg \tilde{\pi}_t(1 - 2\tau_n) = d^n(1 - 2\tau_n)$, where $d^n\tau_n$ is the number of tangencies between $\Gamma_n \cap \{|z_t| < 2r^n\}$ and the fibers of π_t , i.e., the lines $tx - y = \text{constant}$.

For each $n \geq 1$ and $t \in \mathbb{C}$, denote by $\nu_n^t := \llbracket \widehat{\Gamma}_n \rrbracket \wedge \mathbf{q}^* \delta_t$ the slice of the current $\llbracket \widehat{\Gamma}_n \rrbracket$ at t with respect to the projection $\mathbf{q}(x, y, t) = t$. Equivalently, one may write $\nu_n^t = \frac{1}{d^n} \sum_{\{x: (f^n)'(x)=t\}} \delta_{(x, (f^n)(x), (f^n)'(x))}$. Notice that the total mass of ν_n^t is $1 - d^{-n}$ and $\tau_n = \|\mathbf{1}_{\{|z_t| \leq r^n\}} \nu_n^t\|$.

By Proposition 4.4 if we view $m_n(t)$ as a positive measure and also as a positive $(1, 1)$ -form in t , we have that the mass of the measure

$$\tilde{\nu}_n = \int \nu_n^t dm_n(t) = \llbracket \widehat{\Gamma}_n \rrbracket \wedge m_n(t)$$

over the set $\{|z| \leq 3r^n\}$ is bounded by $c_2\beta^n$ for $c_2 > 0$ and $0 < \beta < 1$ (depending on r and s , but not on n). In view of (5.2), for every $t \in \mathbb{D}(1, e^{-\lambda^n})$, the mass of $\tilde{\nu}_n$ over $\{|z_t| \leq 2r^n\}$ is also bounded by $c_2\beta^n$. In particular, for every $\beta < \beta_0 < 1$ we have that $m_n\{t : \|\mathbf{1}_{\{|z_t| \leq r^n\}} \nu_n^t\| \geq \beta_0^n\} \lesssim (\beta/\beta_0)^n$. The lemma follows by setting $\mathcal{A}_n := \{t \in \mathbb{D}(1, s_n) : \|\mathbf{1}_{\{|z_t| \leq r^n\}} \nu_n^t\| < \beta_0^n\} = \{t : \tau_n < \beta_0^n\}$, and choosing $\beta/\beta_0 < \alpha_0 < 1$. \square

In the next result, we will select only the branches given by Lemma 5.1 that are sufficiently transversal to Δ_t in a suitable sense.

Lemma 5.2. *Let $0 < r < 1$ and $\mathcal{A}_n \subset \mathbb{D}(1, s_n)$ be as in Lemma 5.1. Then, there exists $0 < \beta_1 < 1$ and $\kappa > 1$ such that for n large enough we have that, for every $t_n \in \mathcal{A}_n$, the restriction of π_{t_n} to $\Gamma_n \cap \pi_{t_n}^{-1}(\mathbb{D}(0; 2r^n)) = \Gamma_n \cap \{|z_{t_n}| < 2r^n\}$ admits at least $d^n(1 - \beta_1^n)$ inverse branches $\tilde{g} : \{|z_{t_n}| < 2r^n\} \rightarrow \Gamma_n \cap \{|z_{t_n}| < 2r^n\}$ such that in the coordinates (z_{t_n}, w_{t_n}) we have $\tilde{g}(z_{t_n}) = (z_t, g(z_{t_n}))$ with $|g'| \leq \kappa^n$ over $\{|z_{t_n}| < r^n\}$.*

Proof. As in the proof of Lemma 5.1, denote by $\tilde{\pi}_{t_n}$ the restriction of π_{t_n} to $\Gamma_n \cap \pi_{t_n}^{-1}(\mathbb{D}(0; 2r^n)) = \Gamma_n \cap \{|z_{t_n}| < 2r^n\}$. By that lemma, for $t_n \in \mathcal{A}_n$ there are at least $N_n := d^n(1 - \beta_0^n)$ inverse branches of $\tilde{\pi}_{t_n}$. Denote by $\tilde{g}_j : \{|z_{t_n}| < 2r^n\} \rightarrow \Gamma_n \cap \{|z_{t_n}| < 2r^n\}$, $j = 1, \dots, N_n$ such branches. We can write $\tilde{g}_j(z_{t_n}) = (z_{t_n}, g_j(z_{t_n}))$ for some holomorphic functions g_j on $\{|z_{t_n}| < r^n\}$. Recall that $\Gamma_n \cap \{|z_{t_n}| < 2r^n\} \subset \Gamma_n \cap \{|z| < 3r^n\}$.

Let $\omega = idx \wedge d\bar{x} + idy \wedge d\bar{y}$ be the standard Kähler form on $\mathbb{C} \times \mathbb{C}$ and denote by $\mathbf{A}(\Gamma)$ the area of a complex curve Γ inside $\mathbb{C} \times \mathbb{C}$ with respect to ω . Using Lemma 4.2, we see that

$$\mathbf{A}(\Gamma_n \cap \{|z| < 3r^n\}) = \int_{\Gamma_n \cap \{|z| < 3r^n\}} \omega = \int_{\{|z| < 3r^n\}} (1 + |(f^n)'(x)|^2) idx \wedge d\bar{x} \leq cd^n$$

for some constant $c > 0$, independent of r and n . Note that $\Gamma_n \cap \{|z| < 1\} \subset \{|x| < M\}$ for large n for some constant M . See also Lemma ??.

Denoting by $\Gamma(\tilde{g}_j)$ the graph of the branch \tilde{g}_j , we have that

$$\sum_{j=1}^{N_n} \mathbf{A}(\Gamma(\tilde{g}_j)) \leq \mathbf{A}(\Gamma_n \cap \{|z| < 3r^n\}) \leq cd^n.$$

For $0 < \delta < \delta' < 1$ denote by ℓ_n the number of branches satisfying $\mathbf{A}(\Gamma(\tilde{g}_j)) > \delta^{-n}$ and set $k_n = N_n - \ell_n$. The above estimates show that $\ell_n \leq cd^n\delta^n \leq d^n(\delta')^n$. Taking

$\max(\beta_0, \delta') < \beta_1 < 1$, we conclude that there are $k_n \geq d^n(1 - \beta_0^n - (\delta')^n) \geq d^n(1 - \beta_1^n)$ branches whose graph satisfies $\mathbf{A}(\tilde{\Gamma}(g_j)) \leq \delta^{-n}$. We call such \tilde{g}_j the *good branches*.

Let $\omega_t = idz_t \wedge d\bar{z}_t + idw_t \wedge d\bar{w}_t$, where $(z_t, w_t) = (tx - y, tx + y)$ as above. Observe that, since $|t|$ is bounded away from zero, we have that $c_1\omega \leq \omega_t \leq c_2\omega$ for some constants $0 < c_1 \leq c_2$. It follows that, for the good branches as above, we have $\int_{\Gamma(\tilde{g}_j)} \omega_t \leq c_2 \mathbf{A}(\Gamma(\tilde{g}_j)) \leq c_2 \delta^{-n}$. On the other hand, using that $\tilde{g}_j(z_t) = (z_t, g_j(z_t))$, we have

$$\int_{\Gamma(\tilde{g}_j)} \omega_{t_n} = \int_{\{|z_{t_n}| < 2r^n\}} (1 + |g'_j(z_{t_n})|^2) idz_t \wedge d\bar{z}_t \geq \|g'_j\|_{L^2(\{|z_{t_n}| < 2r^n\})}^2.$$

We conclude that $\|g'_j\|_{L^2(\{|z_{t_n}| < 2r^n\})} \leq c_2^{1/2} \delta^{-n/2}$. By Cauchy's estimate, it follows that $\|g'_j\|_{L^\infty(\{|z_{t_n}| < r^n\})} \lesssim \delta^{-n/2} r^{-n}$. The result follows by taking $\kappa > 1/\delta^{1/2} r^{-1}$. \square

6. PROOF OF THEOREM 1.1

In this section we'll give the proof Theorem 1.1. The conclusion clearly holds to the map $\Xi : x \mapsto x^d$ as it can be checked directly. Therefore, we'll assume from now on that f not conjugated to Ξ in order to apply Proposition 4.4 and its consequences. Note however that by Remark 4.5 this is not strictly necessary.

It is enough to prove the theorem for φ of class \mathcal{C}^1 , since the general case follows by standard interpolation arguments, see [Tri78, §2.7.2]. We begin by observing that if φ is a test function on \mathbb{C} and $\tilde{\varphi} = \pi^* \varphi$, then

$$\left\langle \frac{1}{d^n} \sum_{a \in P_n} \delta_a, \varphi \right\rangle = \langle [\Gamma_n] \wedge [\Delta], \tilde{\varphi} \rangle \quad \text{and} \quad \langle \pi_1^* \mu \wedge [\Delta], \tilde{\varphi} \rangle = \int \varphi d\mu.$$

Therefore, showing (1.4) for $\alpha = 1$ is equivalent to showing that

$$(6.1) \quad \left| \langle [\Gamma_n] \wedge [\Delta] - \pi_1^* \mu \wedge [\Delta], \tilde{\varphi} \rangle \right| \leq C \|\varphi\|_{\mathcal{C}^1} \xi^n$$

for some $0 < \xi < 1$. As in the previous sections, we shall identify φ and $\tilde{\varphi}$.

Recall from Sections 5 and 3 that we have projections $\pi_t : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$, $t \in \mathbb{C}$ and $\Pi : (\mathbb{C} \times \mathbb{C}) \setminus \{(0, 0)\} \rightarrow \mathbb{P}^1$ given by $\pi_t(x, y) = tx - y$ and $\Pi(x, y) = [x : y]$ respectively. As before, we identify $t \in \mathbb{C}$ with the point $[1 : t] \in \mathbb{P}^1$. Recall also the notation

$$\Delta_t = \Pi^{-1}(t) = \{tx - y = 0\} = \pi_t^{-1}(0).$$

By Lemma 5.2 and Proposition 3.2, after renaming the constants δ_j, β appearing in the statements, we can find real numbers $0 < r < 1$ and $s_n = e^{-\lambda n}$ with $\lambda > 1$, constants $0 < \alpha, \beta < 1$, $\kappa > 1$, and subsets \mathcal{A}_n and \mathcal{B}_n of $\mathbb{D}(1; s_n)$ with the following properties

- (P1) $m_n(\mathcal{A}_n) \geq 1 - \alpha^n$ and $m_n(\mathcal{B}_n) \geq 1 - \alpha^n$, where m_n denotes the normalized Fubini-Study measure on $\mathbb{D}(1; s_n)$.
- (P2) For every $t_n \in \mathcal{A}_n$ the restriction of π_{t_n} to $\Gamma_n \cap \{|z_{t_n}| < r^n\} \subset \Gamma_n \cap \{|z| < 2r^n\}$ admits at least $d^n(1 - \beta^n)$ inverse branches of the form $z_{t_n} \mapsto (z_{t_n}, g(z_{t_n}))$ in the coordinates $(z_{t_n}, w_{t_n}) = (t_n x - y, t_n x + y)$ with $\|g'\|_\infty \leq \kappa^n$. We call such maps *good transverse branches*.
- (P3) For every $t_n \in \mathcal{B}_n$ we have that $\left| \langle [\Gamma_n] \wedge [\Delta_{t_n}] - \pi_1^* \mu \wedge [\Delta], \tilde{\varphi} \rangle \right| \leq C \|\varphi\|_{\mathcal{C}^1} d^{-n/2}$, where $1 < d < d$ can be taken arbitrarily close to d and $C > 0$ is a constant independent of n and φ .

The proof of Theorem 1.1 will follow from these properties.

Proof of Theorem 1.1. Fix $\varphi = \varphi(x)$ a compactly supported function on \mathbb{C} of class \mathcal{C}^1 . It is enough to prove (6.1) for n large enough. When n is large, (P1) implies that both $m_n(\mathcal{A}_n)$ and $m_n(\mathcal{B}_n)$ are larger than $1/2$. Since m_n is a probability measure, we have $\mathcal{A}_n \cap \mathcal{B}_n \neq \emptyset$. Therefore, we can choose for every large n a point $t_n \in \mathcal{A}_n \cap \mathcal{B}_n$ satisfying (P2) and (P3) above simultaneously. From here until the end of the proof we fix a large n and such a point t_n . We also fix coordinates (z_{t_n}, w_{t_n}) as in (P2) above.

From (P3), the estimate (6.1) will follow if we show that

$$|\langle [\Gamma_n] \wedge [\Delta_{t_n}] - [\Gamma_n] \wedge [\Delta], \tilde{\varphi} \rangle| \leq C \|\varphi\|_{\mathcal{C}^1} \xi^n.$$

for some constants $C > 0$ and $1/d^{1/2} \leq \xi < 1$. By setting

$$\mu_n := [\Gamma_n] \wedge [\Delta] \quad \text{and} \quad \nu_n := [\Gamma_n] \wedge [\Delta_{t_n}]$$

this is equivalent to showing that

$$(6.2) \quad |\langle \nu_n - \mu_n, \tilde{\varphi} \rangle| \leq C \|\varphi\|_{\mathcal{C}^1} \xi^n.$$

Observe that μ_n and ν_n are probability measures on $\mathbb{C} \times \mathbb{C}$. Let p_ℓ , $\ell = 1, \dots, d^n$ be the points in the support of ν_n , i.e. the points in the intersection $\Gamma_n \cap \Delta_{t_n}$, counted with multiplicity. We label them as follows: we denote by p_ℓ^G the ones through which passes a good transverse branch for $\pi_{t_n}|_{\Gamma_n \cap \{|z_{t_n}| \leq r^n\}}$ and by p_ℓ^B the remaining ones. Notice that the points p_ℓ^G have multiplicity one and by (P2) there are at least $d^n - \lfloor d^n \beta^n \rfloor$ many of them.

After applying Lemma 3.1, we see that the supports of ν_n and μ_n all belong to a fixed compact set of $\mathbb{C} \times \mathbb{C}$ and are bounded away from the origin. Notice that the point with coordinates $(z_{t_n}, w_{t_n}) = (0, g(0))$ is one the p_ℓ^G above. In particular, we see that for any good branch g we have $|g(0)| \leq M$ for some constant $M > 0$ independent of g .

Let q_ℓ , $\ell = 1, \dots, d^n$ be the points in the support of μ_n , i.e. the points in the intersection $\Gamma_n \cap \Delta$, counted with multiplicity. By (5.1), a sufficient condition for a point $q = (z_{t_n}, w_{t_n})$ to belong to $\Gamma_n \cap \Delta$ is that its coordinates yield a solution of the system

$$\begin{cases} w_{t_n} = g(z_{t_n}) & (q \in \Gamma_n) \\ (t_n - 1)w_{t_n} = (1 + t_n)z_{t_n} & (q \in \Delta), \end{cases}$$

where g is one the good inverse branches of $\pi_{t_n}|_{\Gamma_n \cap \{|z_{t_n}| \leq r^n\}}$. Since $t_n \neq 1$, it is enough to solve the equation

$$h(z_{t_n}) := g(z_{t_n}) - \frac{1 + t_n}{t_n - 1} z_{t_n} = 0$$

for z_{t_n} in order to obtain a solution of the above system.

Claim: For every good branch g , the associated equation $h(z_{t_n}) = 0$ admits a single solution on the disc $\{|z_{t_n}| \leq 6Me^{-\lambda^n}\}$ when n is large enough.

Proof of claim: Fix a good branch g . Recall that the point with coordinates $(0, g(0))$ is one the p_ℓ^G introduced above. We'll apply Rouché's Theorem on $\{|z_{t_n}| \leq Ae^{-\lambda^n}\}$, where $A > 0$ will be determined later. Since, $|g(0)| \leq M$ and $|g'| \leq \kappa^n$, we have the following inequality on $\{|z_{t_n}| = Ae^{-\lambda^n}\}$ for n large:

$$|g(z_{t_n})| \leq |g(z_{t_n}) - g(0)| + |g(0)| \leq A\kappa^n e^{-\lambda^n} + M \leq 2M.$$

Now $|t_n - 1| \leq e^{-\lambda^n}$ implies $\frac{1+t_n}{t_n-1} \geq \frac{1}{2}e^{\lambda^n}$, so $|\frac{1+t_n}{t_n-1}z_{t_n}| \geq \frac{A}{2}$ over $\{|z_{t_n}| = Ae^{-\lambda^n}\}$. By choosing $A = 6M$, we get that

$$|g(z_{t_n})| \leq \left| \frac{1+t_n}{t_n-1} z_{t_n} \right| \quad \text{on} \quad \{|z_{t_n}| = 6Me^{-\lambda^n}\}.$$

By Rouché's Theorem, the functions $\frac{1+t_n}{t_n-1}z_{t_n}$ and $h(z_{t_n}) = g(z_{t_n}) - \frac{1+t_n}{t_n-1}z_{t_n}$ have the same number of zeroes on the disc $\{|z_{t_n}| \leq 6Me^{-\lambda^n}\}$, namely a single one. This proves the claim. \square

We have thus shown that:

For each good transverse branch g , there is a single a single point $q = (z_{t_n}, g(z_{t_n}))$ in the graph of g with $|z_{t_n}| \leq Ae^{-\lambda^n}$ and $q \in \Gamma_n \cap \Delta$.

If we denote by $\mathcal{G}_n = \{p_\ell^G\}$ the set of good points, the above statement can be rephrased by saying that we have an injective map $\mathcal{G}_n \rightarrow \Gamma_n \cap \Delta = \text{supp}\mu_n$ sending $p_\ell^G = (0, g(0))$ to the corresponding point q as above, which is one of the points q_k of the support of μ_n . After re-indexing, we can assume that p_ℓ^G is sent to q_ℓ . We denote by q'_k the remaining points of $\text{supp}\mu_n$.

Using the above claim and recalling that $|g'| \leq \kappa^n$, we have

$$\text{dist}(p_\ell^G, q_\ell)^2 \lesssim |z_{t_n}|^2 + |g(0) - g(z_{t_n})|^2 \leq A^2(1 + \kappa^{2n})e^{-2\lambda^n},$$

so there exists a constant $C_1 > 0$ independent of n and ℓ such that $\text{dist}(p_\ell^G, q_\ell) \leq C_1\kappa^n e^{-\lambda^n}$.

We can thus write

$$\begin{aligned} \nu_n - \mu_n &= d^{-n} \left(\sum_{\ell=1}^{d^n - \lfloor d^n \beta^n \rfloor} \delta_{p_\ell^G} + \sum_{\ell=1}^{\lfloor d^n \beta^n \rfloor} \delta_{p_\ell^B} - \sum_{\ell=1}^{d^n - \lfloor d^n \beta^n \rfloor} \delta_{q_\ell} + \sum_{\ell=1}^{\lfloor d^n \beta^n \rfloor} \delta_{q'_\ell} \right) \\ &= d^{-n} \sum_{\ell=1}^{d^n - \lfloor d^n \beta^n \rfloor} (\delta_{p_\ell^G} - \delta_{q_\ell}) + d^{-n} \sum_{\ell=1}^{\lfloor d^n \beta^n \rfloor} (\delta_{p_\ell^B} - \delta_{q'_\ell}) \\ &=: \Upsilon_n^G + \Upsilon_n^B, \end{aligned}$$

where $\Upsilon_n^{G/B}$ are signed measures on $\mathbb{C} \times \mathbb{C}$.

Since $\text{dist}(p_\ell^G, q_\ell) \leq C_1\kappa^n e^{-\lambda^n}$, there exists a constant $C_2 > 0$ such that

$$|\langle \Upsilon_n^G, \tilde{\varphi} \rangle| \leq C_2\kappa^n e^{-\lambda^n} \|\varphi\|_{\mathcal{G}^1}.$$

As $\|\Upsilon_n^B\| = 2\beta^n$, we conclude that

$$|\langle \nu_n - \mu_n, \tilde{\varphi} \rangle| \leq C_2\kappa^n e^{-\lambda^n} \|\varphi\|_{\mathcal{G}^1} + 2\beta^n \|\varphi\|_\infty \leq C \|\varphi\|_{\mathcal{G}^1} \beta^n.$$

for some constant $C > 0$. By choosing $\xi = \max(\beta, 1/d^{1/2})$ we obtain (6.2). This completes the proof of the theorem. \square

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