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Convergence rate for random walk approximations of mean field BSDEs

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Abstract

We study the rate of convergence w.r.t. a Wasserstein type distance for random walk approximation of mean field BSDEs. This article continues [\[BGGL21\]](#), where the rate of convergence of a Donsker-type theorem for standard BSDEs is studied.

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Keywords: Mean field backward stochastic differential equations, convergence rate, scaled random walk, Wasserstein distance

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1 Introduction

In this article we fix $0 \leq t_o < T$ and assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_s)_{s \in [t_o, T]})$, satisfying the *usual conditions*, carrying a one-dimensional standard Brownian motion $B = (B_s)_{s \in [t_o, T]}$ starting identically at zero, such that \mathcal{F}_{t_o} and B are independent and $\mathcal{F}_s = \mathcal{F}_{t_o} \vee \sigma(B_r - B_{t_o} : r \in [t_o, s])$, and assume $\mathcal{F} = \mathcal{F}_T$. We are concerned with the discretization of solutions to mean field BSDEs of the form

$$\begin{aligned}
Y_s^{t,x} &= g\left(B_T^{t,x}\right) + \int_s^T f\left(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, [Y_r^{t_o, \xi}], [Z_r^{t_o, \xi}]\right) dr \\
&\quad - \int_s^T Z_r^{t,x} dB_r, \qquad t_o \leq t \leq s \leq T, \quad (1.1)
\end{aligned}$$

for

$$(t_o, \xi) \in [0, T] \times L^{2\varepsilon}(\Omega, \mathcal{F}_{t_o}, \mathbb{P}) \quad \text{under } \textit{Assumption 2.2}.$$

So, the process $B_r^{t,x} := x + B_r - B_t$, $t \leq r \leq T$ is a Brownian motion starting at time t in $x \in \mathbb{R}$. The notation $[\eta]$ will stand for the law of the random variable η . As generators we consider maps

$$f : \Omega \times [t_o, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}.$$

The set $\mathcal{P}_2(\mathbb{R})$ is the set of probability measures with a finite second-order moment, endowed with the Wasserstein distance i.e.

$$\mathcal{W}_2(\mu, \mu') := \inf_{\pi} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - x'|^2 d\pi(x, x') \right)^{1/2}$$

for $(\mu, \mu') \in \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$, the infimum being taken over the probability distributions π on $\mathbb{R} \times \mathbb{R}$ whose marginals on \mathbb{R} are respectively μ and μ' . Notice that if X and X' are random variables of order 2 with values in \mathbb{R} and defined on the same probability space, then by definition we have

$$\mathcal{W}_2([X], [X']) \leq \left[\mathbb{E}|X - X'|^2 \right]^{1/2}. \quad (1.2)$$

Moreover, $(Y^{t_o, \xi}, Z^{t_o, \xi})$ solves (1.1) with (t, x) replaced by (t_o, ξ) . The assumptions on g and f will be specified below.

Mean field BSDEs (also called McKean-Vlasov BSDEs) have been introduced in [BDLP07] and [BLP07] in a more particular framework: in [BDLP07], the authors study the mean field problem in a Markovian setting and prove the existence and the uniqueness of the solution when the terminal condition is of type $\eta = \mathbb{E}[g(x, X_T)]_{|x=X_T}$ where X is a driving adapted stochastic process, and the generator is defined by $\mathbb{E}[f(s, \lambda, \Lambda_s)]_{|\lambda=\Lambda_s}$ where $\Lambda_s = (X_s, Y_s, Z_s)$. In [BLP07], the authors extend the result of existence and uniqueness to a more general framework and link the mean field BSDE to non local partial differential equation.

Mean field forward BSDEs (mean field FBSDEs in the following) arise naturally in the probabilistic analysis of mean field games or mean field control problems. Mean field games are introduced by [LL06a] and [LL06b] to model games with interactions between many similar players. In this theory, each player's dynamics and cost take into account the empirical distribution of all agents. For example, in [AD19], the authors model pedestrian crowd by using a mean field game approach and solve the problem with a mean field FBSDE. Optimal control problems of mean field type can be interpreted as large population limits of cooperative games, where the players collaborate to optimize a joint objective [Lac17]. For example, in [AC21], the authors solve an optimal portfolio with mean field risk minimization problem. It is therefore a relevant challenge to solve such equations numerically.

Numerical methods for mean field FBSDEs are of natural interest already for some time (see [Ala15], [CT15], [CCD19]). Usually, forward McKean-Vlasov SDEs are solved by using particle algorithms (see for example [AKH02], [TV03], [Bos15]) in which the McKean-Vlasov term is approximated by the empirical measure of a large number of interacting particles with independent noise. Adapting such algorithms to the backward problem is not obvious as the high dimension of the involved Brownian motion (given by the number of particles) induces, a priori, a high dimension backward problem with negative consequences for the numerical implementation. The above mentioned papers on numerical methods for mean field FBSDEs do not use particle systems. In [CT15], the authors present a cubature method for decoupled mean field FBSDE. In [CCD19], the authors consider the case of strongly coupled mean field FBSDE. They propose a scheme whose principle is to implement recursively Picard iterations on small time intervals. The scheme is based on a variation of the method of continuation. In [GMW22], the authors propose several algorithms to solve mean field FBSDEs based on neural networks to estimate the solution or its gradient through minimization problems. We also refer to [CCSJ21], [FZ20] and [HHL23] for other papers based on deep learning methods. In [ABGL22] the authors solve non Markovian mean field BSDEs by using Wiener chaos expansion and particle system approximation. In [ZM23], the authors propose first order and Crank-Nicholson numerical schemes to solve certain kinds of mean field BSDEs. They provide L^p error estimates for the proposed schemes.

In this work we propose to approximate the solution $(Y^{t,x}, Z^{t,x})$ to (1.1) using a scaled random walk. For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ we set $h := \frac{T}{n}$ and we consider for $t_k := kh$, $k = 0, 1, \dots, n$, the scaled random walk

$$B_{t_k}^n := \sqrt{h} \sum_{m=1}^k \zeta_m,$$

where $(\zeta_k)_{k \geq 1}$ is an i.i.d. sequence of Rademacher random variables. One of the first studies on the approximation of standard BSDEs (i.e. f may depend on Y and Z , but not on their laws) by a random walk scheme is due to Briand, Delyon and Mémin in [BDM01]. They

proposed an approximation based on Donsker's theorem. They showed that the solution (Y, Z) to a standard BSDE can be approximated by the solution (Y^n, Z^n) to the BSDE

$$Y_t^n = G(B^n) + \int_{]t, T]} f(B_{s-}^n, Y_{s-}^n, Z_s^n) d\langle B^n \rangle_s + \int_{]t, T]} Z_s^n dB_s^n, \quad 0 \leq t \leq T.$$

where

$$B_t^n := B_{\underline{t}}^n \quad \text{for } t \in [0, T], \quad \text{where } \underline{t} := \max \{t_k \mid k \in \{0, \dots, n\} \text{ and } t \geq t_k\}$$

and the quadratic variation of $(B_t^n)_{t \in [0, T]}$ is deterministic and it holds

$$\langle B^n \rangle_s = h \sum_{k=1}^n \delta_{t_k}((0, s]).$$

They proved, in full generality (meaning that $G(B)$ is only required to be a square integrable random variable) that (Y^n, Z^n) converges weakly to (Y, Z) . However, the question of the rate of convergence was left open. For general path-dependent terminal conditions $G(B)$ this question is open, the Markovian case $G(B) = g(B_T)$ has been studied in [BGGL21]: Assuming **Assumption 2.3 without the singularity** $(T - t \vee t')^{-\frac{1}{2}}$ it was shown in [BGGL21, Theorem 12] that

$$\mathcal{W}_p(Y_s^{t,x}, Y_s^{n,t,x}) \leq c_{(1.3)} (1 + |x|)^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}}, \quad (1.3)$$

$$\mathcal{W}_p(Z_s^{t,x}, Z_s^{n,t,x}) \leq c_{(1.4)} \frac{(1 + |x|)^\varepsilon}{\sqrt{T-s}} n^{-\alpha \wedge \frac{\varepsilon}{2}} \quad (1.4)$$

for any $p \in [1, \infty[$ and constants $c_{(1.3)}, c_{(1.4)} > 0$ depending at most on $(T, \varepsilon, \alpha, f, g, p)$. This result exploits the PDE structure behind the Markovian case and a result of Emmanuel Rio [Ri09] which directly implies that there exists a constant $c_p > 0$ such that $\mathcal{W}_p(B_1^n, \mathcal{G}) \leq c_p n^{-1/2}$ for $n \geq 1$ and $p \geq 1$, when \mathcal{G} is a standard Gaussian random variable. The rate in (1.3) and (1.4) improve [GLL20a] and [GLL20b], where the L_2 distance was studied which resulted in an order of $n^{-1/4}$ for smooth coefficients.

In this paper we aim at extending the result obtained in [BGGL21] to the case of mean field BSDEs of type (1.1). Our random walk approximation of (1.1) can be written for $t \geq t_o$ as

$$\begin{aligned} Y_s^{n,t,x} &= g(B_T^{n,t,x}) + \int_{]s, T]} f(r \wedge t_{n-1}, B_{r-}^{n,t,x}, Y_{r-}^{n,t,x}, Z_r^{n,t,x}, [Y_r^{n,t_o,\xi}], [Z_r^{n,t_o,\xi}]) d\langle B^n \rangle_r \\ &\quad - \int_{]s, T]} Z_r^{n,t,x} dB_r^n, \quad s \in [t, T]. \end{aligned} \quad (1.5)$$

where we put, for $x \in \mathbb{R}$ and $0 \leq t \leq s \leq T$,

$$B_s^{n,t,x} := x + B_s^n - B_t^n.$$

(We use $r \wedge t_{n-1}$ for f because it is defined on $[t_o, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ only.) We aim to estimate the error between $(Y^{t,x}, Z^{t,x})$ and $(Y^{n,t,x}, Z^{n,t,x})$ under a suitable distance. We refer to **Theorem 1.1** for a precise statement. To study this error we introduce

$$\mathbf{f}_n(r, x, y, z) := f(r \wedge t_{n-1}, x, y, z, [Y_r^{t_o,\xi}], [Z_{r \wedge t_{n-1}}^{t_o,\xi}]) \quad (1.6)$$

and freezing the distributions $[Y_s^{t_0, \xi}]$ and $[Z_{s \wedge t_{n-1}}^{t_0, \xi}]$ we consider the BSDEs driven by a Brownian motion (for $s \in [t, T]$)

$$Y_{f_n, s}^{t, x} = g\left(B_T^{t, x}\right) + \int_s^T f_n\left(r, B_r^{t, x}, Y_{f_n, r}^{t, x}, Z_{f_n, r}^{t, x}\right) dr - \int_s^T Z_{f_n, r}^{t, x} dB_r, \quad (1.7)$$

and by a random walk (for $s \in [t, T]$)

$$Y_{f_n, s}^{n, t, x} = g(B_T^{n, t, x}) + \int_{]s, T]} f_n(r, B_{r^-}^{n, t, x}, Y_{f_n, r^-}^{n, t, x}, Z_{f_n, r}^{n, t, x}) d\langle B^n \rangle_r - \int_{]s, T]} Z_{f_n, r}^{n, t, x} dB_r^n. \quad (1.8)$$

Then we split our problem into three steps by introducing $(Y_{f_n, s}^{t, x}, Z_{f_n, s}^{t, x})$ and $(Y_{f_n, s}^{n, t, x}, Z_{f_n, s}^{n, t, x})$ into the study of the error:

$$\begin{aligned} \text{error}((Y^{t, x}, Z^{t, x}), (Y^{n, t, x}, Z^{n, t, x})) &\leq \text{error}((Y^{t, x}, Z^{t, x}), (Y_{f_n}^{t, x}, Z_{f_n}^{t, x})) \\ &\quad + \text{error}((Y_{f_n}^{t, x}, Z_{f_n}^{t, x}), (Y_{f_n}^{n, t, x}, Z_{f_n}^{n, t, x})) \\ &\quad + \text{error}((Y_{f_n}^{n, t, x}, Z_{f_n}^{n, t, x}), (Y^{n, t, x}, Z^{n, t, x})). \end{aligned} \quad (1.9)$$

Besides that we are in the mean field setting and have to handle the approximation of the mean field terms in parallel to the classical terms, there is another problem with the above splitting: Although we can interpret (1.7) and (1.8) as the classical setting without mean field terms, in order to estimate the corresponding error $((Y_{f_n}^{t, x}, Z_{f_n}^{t, x}), (Y_{f_n}^{n, t, x}, Z_{f_n}^{n, t, x}))$ we cannot use the estimate (1.4) because the singularity in time (if squared) is not integrable. We will therefore prove a new version of [BGGL21, Theorem 12] (especially a new version of (1.4)) in Lemma 7.6 where the convergence rate and the singularity are connected.

Now we formulate the main result, the convergence rate for the distributions on the path space in Wasserstein sense. Let $\Gamma(B, B^n)$ denote the set of all the processes $(\mathcal{B}, \mathcal{B}^n)$ with \mathcal{B} continuous and \mathcal{B}^n being càdlàg that have the same finite dimensional distributions as B and B^n , respectively.

Theorem 1.1. *Suppose that Assumption 2.2 holds. Then for any $p \geq 2$ and $\beta > \frac{1}{2}$ there is a constant $c = c(T, \varepsilon, \alpha, \beta, |g|_\varepsilon, |f|_{\alpha; t}^{\text{loc}}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, f_0, \xi) > 0$ such that for all $n \geq n_0$,*

$$\begin{aligned} \inf \left\{ \left(\mathbb{E} \sup_{s \in [t, T]} |\mathcal{Y}_s^{t, x} - \mathcal{Y}_s^{n, t, x}|^p + \mathbb{E} \left(\int_t^T |\mathcal{Z}_s^{t, x} - \mathcal{Z}_s^{n, t, x}|^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right\} \\ \leq c p^{\frac{1}{2} \vee (\beta + \varepsilon)} (1 + |x|)^\varepsilon \frac{|\log(n+1)|^{\varepsilon + \beta}}{n^{\alpha \wedge \frac{\varepsilon}{2}}}, \end{aligned}$$

where the processes $(\mathcal{Y}^{t, x}, \mathcal{Z}^{t, x})$ and $(\mathcal{Y}^{n, t, x}, \mathcal{Z}^{n, t, x})$ solve (1.1) and (1.5) with driving processes \mathcal{B} and \mathcal{B}^n , respectively, and the infimum is taken over all $(\mathcal{B}, \mathcal{B}^n) \in \Gamma(B, B^n)$.

The proof and a local version (Theorem 8.1) of Theorem 1.1 will be given in Section 8.

The article is organized as follows: Section 2 introduces the notation and assumptions. In Section 3 we recall facts about function spaces describing an exponential tail behaviour. Section 4 states results on the random walk approximation of the Brownian motion. Regularity properties of u and ∇u (where u represents the solution of the PDE associated to the standard BSDE) are proven in Section 5. After recalling the discrete scheme in Section 6, we establish in Section 7 the convergence rates of the three error terms introduced in (1.9). Section 8 contains the proof of Theorem 1.1, but also the more general version Theorem 8.1 which can be applied to any coupling (B, B^n) of the Brownian motion with the random walk.

2 Notation and assumptions

If E is a metric space and $1 \leq p < \infty$, then $\mathcal{P}_p(E)$ will denote the space of probability measures on $(E, \mathcal{B}(E))$ with finite p th moment.

Definition 2.1. For $p \geq 1$ and $\mu, \mu' \in \mathcal{P}_p(\mathbb{R})$ we introduce the p -Wasserstein distance $\mathcal{W}_p(\mu, \mu')$ by setting

$$\mathcal{W}_p(\mu, \mu') := \inf_{\gamma \in \Gamma(\mu, \mu')} \left(\int_{\mathbb{R}^2} |x - y|^p \gamma(dx, dy) \right)^{\frac{1}{p}},$$

where $\Gamma(\mu, \mu')$ is the set of all probability measures on $\mathcal{P}_p(\mathbb{R}^2)$ with marginals μ and μ' .

Assumption 2.2. *There exist $0 < \varepsilon \leq 1$ and $0 < \alpha \leq 1$ such that it holds:*

(i) *The function $g : \mathbb{R} \rightarrow \mathbb{R}$ is ε -Hölder continuous: for all $(x, x') \in \mathbb{R}^2$ one has*

$$|g(x) - g(x')| \leq |g|_\varepsilon |x - x'|^\varepsilon, \quad \text{where} \quad |g|_\varepsilon := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\varepsilon}.$$

(ii) *The function $f : [t_o, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ is locally α -Hölder continuous in time, ε -Hölder continuous in space and Lipschitz continuous with respect to (y, z, μ, ν) : for all (t, x, y, z, μ, ν) and $(t', x', y', z', \mu', \nu')$ in $[t_o, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$ one has*

$$\begin{aligned} & |f(t, x, y, z, \mu, \nu) - f(t', x', y', z', \mu', \nu')| \\ & \leq |f|_{\alpha; t}^{\text{loc}} \frac{|t - t'|^\alpha}{(T - t \vee t')^{\frac{1}{2}}} + |f|_{\varepsilon; x} |x - x'|^\varepsilon \\ & \quad + L_f \left(|y - y'| + |z - z'| + \mathcal{W}_2(\mu, \mu') + \mathcal{W}_2(\nu, \nu') \right). \end{aligned} \quad (2.1)$$

(iii) *One has $\int_{t_o}^T |f(t, 0, 0, 0, \delta_0, \delta_0)|^2 dt < \infty$.*

Under Assumption 2.2 there exists a unique solution $(Y^{t_o, \xi}, Z^{t_o, \xi})$ to

$$\begin{aligned} Y_s^{t_o, \xi} &= g\left(B_T^{t_o, \xi}\right) + \int_s^T f\left(r, B_r^{t_o, \xi}, Y_r^{t_o, \xi}, Z_r^{t_o, \xi}, [Y_r^{t_o, \xi}], [Z_r^{t_o, \xi}]\right) dr - \int_s^T Z_r^{t_o, \xi} dB_r, \\ & \quad t_o \leq s \leq T, \end{aligned} \quad (2.2)$$

see [Li18, Theorem A.1]. In Section 5.1 we consider generators with frozen measure components, that is, generators of the type $f : [t_o, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. For clarity, we rephrase Assumption 2.2 for these generators:

Assumption 2.3. *There exist $0 < \varepsilon \leq 1$ and $0 < \alpha \leq 1$ such that [Assumption 2.2\(i\)](#) holds and the generator $f : [t_o, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies*

$$|f(t, x, y, z) - f(t', x', y', z')| \leq |f|_{\alpha \wedge \frac{\varepsilon}{2}; t}^{loc} \frac{|t - t'|^{\alpha \wedge \frac{\varepsilon}{2}}}{(T - t \vee t')^{\frac{1}{2}}} + |f|_{\varepsilon; x} |x - x'|^\varepsilon + L_f (|y - y'| + |z - z'|) \quad (2.3)$$

for all $(t, x, y, z), (t', x', y', z') \in [t_o, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Remark 2.4. Although global Hölder continuity in time for the generator was assumed in [\[BGGL21\]](#), the results can still be proved under local Hölder continuity given in (2.3). We will discuss the necessary changes in [Section 7.2](#).

We collect the notation we use in the sequel:

$$\begin{aligned} T > 0 &: \text{ terminal time} \\ \varepsilon \in (0, 1] &: \text{ Hölder continuity in the space variable } x \\ \beta \in \{0\} \cup \left(\frac{1}{2}, \infty\right) &: \text{ exponent for logarithmic weight} \\ |g|_\varepsilon &:= \sup_{x \neq y} \left[\frac{|g(x) - g(y)|}{|x - y|^\varepsilon} \right] \\ |f|_{\alpha; t}^{loc} &:= \sup_{t \neq t', x, y, z, \mu, \nu} \left[\frac{|f(t, x, y, z, \mu, \nu) - f(t', x, y, z, \mu, \nu)|}{|t - t'|^\alpha} (T - t \vee t')^{\frac{1}{2}} \right] \\ |f|_{\varepsilon; x} &:= \sup_{x \neq x', t, y, z, \mu, \nu} \left[\frac{|f(t, x, y, z, \mu, \nu) - f(t, x', y, z, \mu, \nu)|}{|x - x'|^\varepsilon} \right] \\ L_f \text{ and } L_f &: \text{ Lipschitz constants for the generators } f \text{ and } f \\ c_{f_0}^{(2)} &:= \|f(\cdot, 0, 0, 0, \delta_0, \delta_0)\|_{L^2([t_o, T])} \\ c_{f_0} &:= \|f(\cdot, 0, 0, 0)\|_{L^1([t_o, T])} \\ f_0 &:= f(0, 0, 0, 0, \delta_0, \delta_0) \\ n_0 &= n_0(T, L_f, t_o) \end{aligned}$$

Remark 2.5. We will use $n \geq n_0(T, L_f, t_o)$ in the sequel. In particular, this ensures $n_0 > TL_f$, so that we have a unique solution to (1.5) and (1.8) (see [\[BDM01\]](#)). Our assumption on n_0 also guarantees the requirements for [Lemma 7.5](#) (that generalizes [\[BGGL21, Lemma 14\]](#)) and that $t_{n-1} > t_o$.

3 The function spaces M_φ

We assume an increasing bijection $\psi : [1, \infty) \rightarrow [1, \infty)$ and define for a random variable $f : \Omega \rightarrow \mathbb{R}$ the quantity

$$|f|_{M_\varphi^0} := \inf \left\{ c > 0 : \mathbb{P}(|f| > \lambda) \leq e^{1-\psi(\lambda/c)} \text{ for } \lambda \geq c \right\} \quad \text{for } \varphi(t) := \frac{1}{\psi^{-1}(1 + \log(1/t))}.$$

As particular functions we consider

$$\psi_\gamma(x) := x^\gamma \quad \text{and} \quad \varphi_\gamma(t) := \frac{1}{\sqrt[\gamma]{1 + \log(1/t)}} \quad \text{for } \gamma \in [1, \infty).$$

We will use the following two statements:

Lemma 3.1 ([Gei97, Lemma 5.2]). *If there is an $a > 1$ such that*

$$\inf_{\lambda \geq 1} \frac{\psi(a\lambda)}{\psi(\lambda)} > 1,$$

then one has

$$\left| \sup_{1 \leq i \leq N} \frac{|f_i|}{\psi^{-1}(1 + \log i)} \right|_{M_\varphi^0} \leq c_{(3.1)} \sup_{1 \leq i \leq N} |f_i|_{M_\varphi^0} \quad (3.1)$$

for all random variables $f_1, \dots, f_N : \Omega \rightarrow \mathbb{R}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $c_{(3.1)} = c_{(3.1)}(\varphi) > 0$.

Lemma 3.2 ([GG24, Section 18.3]). *For $\gamma \in [1, \infty)$ there is a constant $c_{(3.2)} = c_{(3.2)}(\gamma) \geq 1$ such that*

$$\frac{1}{c_{(3.2)}} |f|_{M_{\varphi_\gamma}^0} \leq \sup_{p \in [1, \infty)} \frac{\|f\|_{L^p}}{\sqrt[p]{p}} \leq c_{(3.2)} |f|_{M_{\varphi_\gamma}^0}. \quad (3.2)$$

We will use the Hoeffding inequality [Hoe63] (see also [GG24, Section 7.12.4]) that says in the language of this section that

$$\sup_{n \in \mathbb{N}} \left\| \frac{\zeta_1 + \dots + \zeta_n}{\sqrt{n}} \right\|_{M_{\varphi_2}^0} < \infty, \quad (3.3)$$

where $(\zeta_k)_{k \geq 1}$ is an i.i.d. sequence of Rademacher random variables.

4 Random walk approximation of the Brownian motion

One direction of the following result is based on a result of J. Komlós, P. Major and G. Tusnády [KMT75, Theorem 1] which states (for the special case of Rademacher random variables we consider here) that if $(g_i)_{i=1}^\infty$ are i.i.d. standard Gaussian random variables, then there is a sequence of Borel functions $f_n : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, such that

$$(\zeta_n)_{n \in \mathbb{N}} := (f_n(g_1, g_2, \dots, g_{2n}))_{n \in \mathbb{N}}$$

are independent Rademacher variables and it holds for $S_k := \sum_{j=1}^k \zeta_j$ and $T_k := \sum_{j=1}^k g_j$, that

$$\mathbb{P} \left(\sup_{1 \leq k \leq n} |S_k - T_k| > C_{(4.1)} \log n + x \right) \leq K_{(4.1)} e^{-\lambda_{(4.1)} x} \quad \text{for } x > 0 \text{ and } n \geq 1, \quad (4.1)$$

where $C_{(4.1)}, K_{(4.1)}, \lambda_{(4.1)} > 0$ are constants. We also need the following relations:

Lemma 4.1 ([GG24, Theorem 18.2.11, Example 18.2.14], [Gei97, Lemma 5.2]).

(i) For $p \in [1, \infty)$ there exists a constant $c_{(4.2),p} > 0$ such that

$$\left\| \sup_{1 \leq k \leq n} |g_k| \right\|_{L^p} \geq \frac{1}{c_{(4.2),p}} \sqrt{\log(n+1)} \quad (4.2)$$

for all $n \in \mathbb{N}$ and i.i.d. standard Gaussian random variables g_1, \dots, g_n .

(ii) Assume $\alpha \in (0, \infty)$ to be fixed and i.i.d. random variables h_1, \dots, h_n such that $\|h_1\|_{L^q} \leq \sqrt[q]{q}$ for $q \in [1, \infty)$. Then for $p \in [1, \infty)$ one has

$$\left\| \sup_{1 \leq k \leq n} |h_k| \right\|_{L^p} \leq c_{(4.3)} \sqrt[p]{p} \sqrt[p]{\log(n+1)} \quad (4.3)$$

with $c_{(4.3)} = c_{(4.3)}(\alpha) > 0$.

The following theorem gives the path-wise convergence rate for the random walk approximation.

Theorem 4.2. There is a constant $c_{(4.4)} = c_{(4.4)}(T) > 0$ such that for all $p \in [1, \infty)$ and $n \in \mathbb{N}$ it holds that

$$\frac{1}{2} \sqrt{\frac{T}{2n}} \frac{1}{c_{(4.2),p}} \sqrt{\log(n+1)} \leq \inf_{(\mathcal{B}, \mathcal{B}^n) \in \Gamma(B, B^n)} \left\{ \left(\mathbb{E} \sup_{t \in [0, T]} |B_t - \mathcal{B}_t^n|^p \right)^{\frac{1}{p}} \right\} < c_{(4.4)} p \frac{\log(n+1)}{n^{\frac{1}{2}}}. \quad (4.4)$$

Proof. For the lower bound we observe that

$$\begin{aligned} & \left(\mathbb{E} \sup_{t \in [0, T]} |B_t - B_t^n|^p \right)^{1/p} \\ & \geq \left\| \sup_{m=1, \dots, n-1} \left| \sqrt{\frac{T}{n}} \sum_{k=1}^m \zeta_k - B_{\frac{2mT}{2n}} \right| \vee \left| \sqrt{\frac{T}{n}} \sum_{k=1}^m \zeta_k - B_{\frac{(2m+1)T}{2n}} \right| \right\|_{L^p} \\ & \geq \sqrt{\frac{T}{n}} \left\| \sup_{m=1, \dots, n-1} \left| \sum_{k=1}^m \zeta_k - \sum_{k=1}^{2m} \frac{1}{\sqrt{2}} g_k \right| \vee \left| \sum_{k=1}^m \zeta_k - \sum_{k=1}^{2m+1} \frac{1}{\sqrt{2}} g_k \right| \right\|_{L^p} \\ & \geq \frac{1}{2} \sqrt{\frac{T}{2n}} \left\| \sup_{k=1, \dots, n-1} |g_k| \right\|_{L^p} \\ & \geq \frac{1}{2} \sqrt{\frac{T}{2n}} \frac{1}{c_{(4.2),p}} \sqrt{\log(n+1)}, \end{aligned}$$

where we used that $\inf_{a \in \mathbb{R}} (|a| \vee |a - b|) \geq \frac{1}{2}|b|$ for any $b \in \mathbb{R}$, while the last inequality follows from [Lemma 4.1](#).

Upper bound: We may assume $n \geq 2$. We put $(\Omega', \mathcal{F}', \mathbb{P}') := (\Omega, \mathcal{F}, \mathbb{P}) \otimes (\Omega, \mathcal{F}, \mathbb{P})$. Then $(\Omega', \mathcal{F}', \mathbb{P}')$ carries a Brownian motion $(B_t)_{t \in [0, 2T]}$. We choose $n \in \mathbb{N}$ and put $t_k = k \frac{T}{n}$, $k = 0, 1, \dots, 2n$. This provides us with the $2n$ i.i.d. Gaussian random variables

$$(B_{t_k} - B_{t_{k-1}})_{k=1}^{2n}$$

from which the ζ_1, \dots, ζ_n can be constructed following [KMT75, Theorem 1]. Then for $(B_t^n)_{t \in [0, T]}$ as defined above with $B_{t_k}^n = \sqrt{h} \sum_{m=1}^k \zeta_m$ we have

$$\begin{aligned} \sup_{t \in [0, T]} |B_t - B_t^n|^p &= \left(\max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} |B_t - B_{t_{k-1}}^n|^p \right) \vee |B_{t_n} - B_{t_n}^n|^p \\ &\leq 2^{p-1} \left(\max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} (|B_t - B_{t_{k-1}}|^p + |B_{t_{k-1}} - B_{t_{k-1}}^n|^p) \right) \vee |B_{t_n} - B_{t_n}^n|^p \\ &\leq 2^{p-1} \left(\max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} |B_t - B_{t_{k-1}}|^p + \max_{1 \leq k \leq n} |B_{t_k} - B_{t_k}^n|^p \right). \end{aligned}$$

FIRST TERM: Here we prove that

$$\left(\mathbb{E} \max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} |B_t - B_{t_{k-1}}|^p \right)^{1/p} \leq c_{(4.5)} \sqrt{\frac{\log(n+1)}{n}}. \quad (4.5)$$

To verify this, we define

$$h_k := \sup_{t \in [t_{k-1}, t_k]} |B_t - B_{t_{k-1}}|.$$

The sequence is an i.i.d. sequence of random variables. By definition,

$$\left(\mathbb{E} \max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} |B_t - B_{t_{k-1}}|^p \right)^{1/p} = \left\| \max_{1 \leq k \leq n} |h_k| \right\|_{L^p}.$$

By Doob's maximal inequality we get for $q \in [2, \infty)$ that

$$\|h_1\|_{L^q} \leq \frac{q}{q-1} \|B_{t_1}\|_{L^q} = \frac{q}{q-1} \sqrt{\frac{T}{n}} \|B_1\|_{L^q} \leq \frac{q}{q-1} \sqrt{\frac{T}{n}} \kappa \sqrt{q} \leq [2\sqrt{T}\kappa] \sqrt{\frac{q}{n}},$$

where $\kappa > 0$ is an absolute constant. In other words,

$$\|\sqrt{n}h_1\|_{L^q} \leq [2\sqrt{T}\kappa] \sqrt{q}.$$

Using (4.3) implies

$$\left\| \max_{1 \leq k \leq n} \sqrt{n}|h_k| \right\|_{L^p} \leq [c_{(4.3)} 2\sqrt{T}\kappa] \sqrt{p} \sqrt{\log(n+1)}.$$

SECOND TERM: By the inequality (4.1), for $x \geq 1$ and $n \geq 2$, we have for $S_k := \sum_{j=1}^k \zeta_j$ and $T_k := \sum_{j=1}^k \frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{h}}$

$$\begin{aligned} \mathbb{P} \left(\sup_{1 \leq k \leq n} |S_k - T_k| > (C_{(4.1)} + 1)x \log(n+1) \right) &\leq \mathbb{P} \left(\sup_{1 \leq k \leq n} |S_k - T_k| > C_{(4.1)} \log n + x \right) \\ &\leq K_{(4.1)} e^{-\lambda_{(4.1)} x}. \end{aligned}$$

This is the same as

$$\mathbb{P} \left(\sup_{1 \leq k \leq n} \left[\sqrt{n} |B_{t_k} - B_{t_k}^n| \right] > (C_{(4.1)} + 1)x \log(n+1) \right) \leq K_{(4.1)} e^{-\lambda_{(4.1)} x}.$$

And this implies that

$$\left\| \sup_{1 \leq k \leq n} \left[\sqrt{n} |B_{t_k} - B_{t_k}^n| \right] \right\|_{L^p} \leq c_{(4.6)} p \log(n+1) \quad (4.6)$$

with $c_{(4.6)} = c_{(4.6)}(C_{(4.1)}, K_{(4.1)}, \lambda_{(4.1)}) > 0$. \square

5 Regularity properties of u and ∇u

In this section we generalize some regularity results obtained in [BGGL21]. If $f : [t_o, T[\times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a generator satisfying Assumption 2.3, then we define

$$u(t, x) := Y_t^{t, x}.$$

We will use that, a.s.,

$$Y_s^{t, x} = u(s, B_s^{t, x}) \quad \text{and} \quad Z_s^{t, x} = \nabla u(s, B_s^{t, x}) \quad (5.1)$$

on $[0, T[$. For this, we need regularity properties of $u(t, x)$ and $\nabla u(t, x)$.

5.1 Regularity results for the case with frozen mean field terms

In this section we generalize some regularity results obtained in [BGGL21]. Investigating the proof of [BGGL21, Lemma 6] about regularity properties of $u(t, x)$, one can see that the global Hölder continuity of f in time is not used to show regularity properties for u . Therefore, the lemma holds true under Assumption 2.3 as well.

Lemma 5.1 ([BGGL21, Lemma 6]). *Under Assumption 2.3 there exists a constant $c_{(5.2)} = c_{(5.2)}(T, \varepsilon, |g|_\varepsilon, |f|_{\varepsilon; x}, Lf, c_{f_0}) > 0$ such that for $u(t, x) := Y_t^{t, x}$ and for all $(t, x) \in [t_o, T] \times \mathbb{R}$,*

$$|u(t, x)| \leq c_{(5.2)} (1 + |x|)^\varepsilon, \quad |u(t, \cdot)|_{\varepsilon; x} \leq c_{(5.2)}, \quad |u(\cdot, x)|_{\varepsilon/2; t} \leq c_{(5.2)} (1 + |x|)^\varepsilon. \quad (5.2)$$

We introduce for $\beta > \frac{1}{2}$ the function $\Phi_\beta :]0, \infty[\rightarrow]0, \infty[$ given by $\Phi_\beta(r) := \left(\log \left(\frac{1}{r \wedge \hat{r}} \right) \right)^\beta$, with $\hat{r} = e^{-2\beta}$. If $\beta = 0$ we let $\Phi_0 := 1$. We derive some properties of the function Φ_β for later use.

Lemma 5.2. *Let $\beta > 1/2$ or $\beta = 0$ and $\hat{r} = e^{-2\beta}$. Then:*

- (i) Φ_β is non-increasing and $\Phi_\beta(r) = \left(\log \left(\frac{1}{r \wedge \hat{r}} \right) \right)^\beta \geq 1$ for all $r \in]0, \infty[$.
- (ii) The map $]0, \infty[\ni r \mapsto r^a \Phi_\beta(r^2)$ is non-decreasing for all $a \geq 1$.
- (iii) $\Psi_{2\beta}(r) := r \left(\log \left(\frac{1}{r \wedge \hat{r}} \right) \right)^{2\beta} = r \Phi_\beta(r)^2$ is non-decreasing and concave, and, if $\beta > \frac{1}{2}$, bounded by $\Psi_{2\beta}(\hat{r}) = e^{-2\beta} (2\beta)^{2\beta}$.

Proof. If $\beta = 0$, then all the properties (i)–(iii) are clearly true, so we can focus on the case $\beta > \frac{1}{2}$.

(i) If $\beta > 1/2$, then it follows that $\left(\log\left(\frac{1}{r}\right)\right)^\beta \geq 1$ for all $r \in]0, e^{-2\beta}]$, and obviously Φ_β is non-increasing.

(ii) We see that $r \mapsto r^a \Phi_\beta(r^2)$ is non-decreasing for $a \geq 1$ since for $r < \hat{r}$ we have

$$\frac{d}{dr} (r^a \Phi_\beta(r^2)) = r^{a-1} \left(\log\left(\frac{1}{r^2}\right)\right)^{\beta-1} \left(a \log\left(\frac{1}{r^2}\right) - 2\beta\right),$$

which is positive for small r . In fact, $\left(a \log\left(\frac{1}{r^2}\right) - 2\beta\right) = 0 \iff r = e^{-\frac{\beta}{a}}$ implies that the expression on the RHS is positive for all $r \in]0, e^{-\frac{\beta}{a}}[$, but $\hat{r} = e^{-2\beta}$.

(iii) For $r < \hat{r}$ we have

$$\Psi'_{2\beta}(r) = \left(\log\left(\frac{1}{r}\right)\right)^{2\beta-1} \left(\log\left(\frac{1}{r}\right) - 2\beta\right).$$

Hence the RHS is non-negative for all $r \in]0, e^{-2\beta}]$, and one easily checks that $\Psi''_{2\beta} \leq 0$ for $r \in]0, 1[$. \square

Remark 5.3. By the help of Φ_β we achieve and will use square integrability in time since for $\beta > 1/2$ it holds

$$\int_0^T \frac{1}{s|\Phi_\beta(s^\varepsilon)|^2} ds < \infty.$$

We will show regularity properties of ∇u by the help of the following lemma.

Lemma 5.4. *For any continuous function $F : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $H(u, s) := \frac{B_s - B_u}{s - u}$ for $0 \leq u < s \leq T$ we have the following: if $\varepsilon > 0$, $c_{(5.3)} > 0$ and $0 \leq r \leq t < s \leq T$, the relation*

$$|F(s, x) - F(s, x')| \leq c_{(5.3)} \frac{|x - x'|^\varepsilon \Phi_\beta(|x - x'|^{2\varepsilon})}{(T - s)^\gamma \Phi_\beta(|T - s|^\varepsilon)} = c_{(5.3)} \frac{\Psi_{2\beta}(|x - x'|^{2\varepsilon})^{\frac{1}{2}}}{(T - s)^\gamma \Phi_\beta(|T - s|^\varepsilon)}, \quad (5.3)$$

where $\gamma \geq 0$ and $\beta = 0$ or $\beta > \frac{1}{2}$ and $\bar{\beta} = 0$ or $\bar{\beta} > \frac{1}{2}$, implies

$$\begin{aligned} & |\mathbb{E}(F(s, B_s^{r,x})H(r, s) - F(s, B_s^{t,x})H(t, s))| \\ & \leq c_{(5.3)} \frac{\sqrt{b_\varepsilon} \Phi_\beta(|t - r|^\varepsilon b_\varepsilon) \Phi_\beta(|s - t|^\varepsilon b_\varepsilon)}{(T - s)^\gamma \Phi_\beta((T - s)^\varepsilon)} \frac{(t - r)^{\frac{\varepsilon}{2}} \Phi_{\bar{\beta}}((t - r)^\varepsilon)}{(s - t)^{\frac{1}{2}} \Phi_{\bar{\beta}}((s - t)^\varepsilon)}. \end{aligned} \quad (5.4)$$

In particular,

(i) for $\gamma = \beta = 0$ we have that $|F(s, x) - F(s, x')| \leq c_{(5.3)} |x - x'|^\varepsilon$ which implies for all $\bar{\beta} > \frac{1}{2}$

$$|\mathbb{E}(F(s, B_s^{r,x})H(r, s) - F(s, B_s^{t,x})H(t, s))| \leq c_{(5.3)} \frac{\Psi_{2\bar{\beta}}(|t - r|^\varepsilon)^{\frac{1}{2}}}{(s - t)^{\frac{1}{2}} \Phi_{\bar{\beta}}(|s - t|^\varepsilon)} \quad (5.5)$$

(ii) for $\gamma = \frac{1}{2}$, $\beta > \frac{1}{2}$ we have that $|\mathbf{F}(s, x) - \mathbf{F}(s, x')| \leq c_{(5.3)} \frac{|x-x'|^\varepsilon \Phi_\beta(|x-x'|^{2\varepsilon})}{(T-s)^{\frac{1}{2}} \Phi_\beta(|T-s|^\varepsilon)}$ which implies for $\bar{\beta} = 0$ that

$$|\mathbb{E}(\mathbf{F}(s, B_s^{r,x})H(r, s) - \mathbf{F}(s, B_s^{t,x})H(t, s))| \leq c_{(5.3)} \frac{\Phi_\beta(|s-t|^\varepsilon b_\varepsilon)}{(s-t)^{\frac{1}{2}}} \frac{\Psi_{2\beta}(|t-r|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta(|T-s|^\varepsilon)}, \quad (5.6)$$

where $b_\varepsilon := \mathbb{E}|B_1|^{2\varepsilon} \leq 1$.

Proof. We have

$$\begin{aligned} & |\mathbb{E}[\mathbf{F}(s, B_s^{r,x})H(r, s)] - \mathbb{E}[\mathbf{F}(s, B_s^{t,x})H(t, s)]| \\ &= |\mathbb{E}[(\mathbf{F}(s, B_s^{r,x}) - \mathbf{F}(s, B_s^{t,x}))H(r, s)] + \mathbb{E}[(\mathbf{F}(s, B_s^{t,x}) - \mathbf{F}(t, x))(H(r, s) - H(t, s))]| \\ &\leq \mathbb{E}[|\mathbf{F}(s, B_s^{r,x}) - \mathbf{F}(s, B_s^{t,x})| |H(r, s)|] + \mathbb{E}[|\mathbf{F}(s, B_s^{t,x}) - \mathbf{F}(t, x)| |H(r, s) - H(t, s)|]. \end{aligned} \quad (5.7)$$

For the first term on the RHS we get by applying the Cauchy-Schwartz inequality that

$$\begin{aligned} & \mathbb{E}\left[|\mathbf{F}(s, B_s^{r,x}) - \mathbf{F}(s, B_s^{t,x})| \frac{|B_s - B_r|}{s-r}\right] \\ &\leq \frac{c_{(5.3)}}{(T-s)^\gamma \Phi_\beta((T-s)^\varepsilon)} \mathbb{E}\left(\Psi_{2\beta}(|B_t - B_r|^{2\varepsilon})^{\frac{1}{2}} \frac{|B_s - B_r|}{s-r}\right) \\ &\leq \frac{c_{(5.3)}}{(T-s)^\gamma \Phi_\beta((T-s)^\varepsilon)} (\mathbb{E}\Psi_{2\beta}(|B_t - B_r|^{2\varepsilon}))^{\frac{1}{2}} \frac{1}{(s-r)^{\frac{1}{2}}}. \end{aligned} \quad (5.8)$$

Since by Lemma 5.2 the function $\Psi_{2\beta}$ is concave, Jensen's inequality permits to move the expectation inside, and recalling that $b_\varepsilon = \mathbb{E}|B_1|^{2\varepsilon}$ we have

$$(\mathbb{E}\Psi_{2\beta}(|B_t - B_r|^{2\varepsilon}))^{\frac{1}{2}} \leq \Psi_{2\beta}(\mathbb{E}|B_t - B_r|^{2\varepsilon})^{\frac{1}{2}} = \sqrt{b_\varepsilon} |t-r|^{\varepsilon/2} \Phi_\beta(|t-r|^\varepsilon b_\varepsilon). \quad (5.9)$$

By Lemma 5.2(i) the function $\Phi_{\bar{\beta}}$ is non-increasing, implying that $\frac{\Phi_{\bar{\beta}}((t-r)^\varepsilon)}{\Phi_{\bar{\beta}}((s-r)^\varepsilon)} \geq 1$. Furthermore, since $(s-r)^{\frac{1}{2}} \Phi_{\bar{\beta}}((s-r)^\varepsilon) \geq (s-t)^{\frac{1}{2}} \Phi_{\bar{\beta}}((s-t)^\varepsilon)$ (see Lemma 5.2(ii)), we get from (5.9) that

$$\begin{aligned} \frac{(\mathbb{E}\Psi_{2\beta}(|B_t - B_r|^{2\varepsilon}))^{\frac{1}{2}}}{(s-r)^{\frac{1}{2}}} &\leq \Phi_\beta(|t-r|^\varepsilon b_\varepsilon) \sqrt{b_\varepsilon} \frac{(t-r)^{\frac{\varepsilon}{2}} \Phi_{\bar{\beta}}((t-r)^\varepsilon)}{(s-r)^{\frac{1}{2}} \Phi_{\bar{\beta}}((s-r)^\varepsilon)} \\ &\leq \Phi_\beta(|t-r|^\varepsilon b_\varepsilon) \sqrt{b_\varepsilon} \frac{(t-r)^{\frac{\varepsilon}{2}} \Phi_{\bar{\beta}}((t-r)^\varepsilon)}{(s-t)^{\frac{1}{2}} \Phi_{\bar{\beta}}((s-t)^\varepsilon)}. \end{aligned} \quad (5.10)$$

In a similar way we estimate the second term of (5.7), that is,

$$\begin{aligned} & \mathbb{E}[|\mathbf{F}(s, B_s^{t,x}) - \mathbf{F}(s, x)| |H(r, s) - H(t, s)|] \\ &\leq \frac{c_{(5.3)}}{(T-s)^\gamma \Phi_\beta((T-s)^\varepsilon)} (\mathbb{E}\Psi_{2\beta}(|B_s - B_t|^{2\varepsilon}))^{\frac{1}{2}} \frac{(t-r)^{\frac{1}{2}}}{(s-r)^{\frac{1}{2}} (s-t)^{\frac{1}{2}}}, \end{aligned} \quad (5.11)$$

where we used that $\mathbb{E}|H(r, s) - H(t, s)|^2 \leq \frac{t-r}{(s-r)(s-t)}$ (see the proof of [BGGL21, Proposition 10]). From $\frac{t-r}{s-r} \leq 1$ it follows

$$\frac{(t-r)^{\frac{1}{2}}}{(s-t)^{\frac{1}{2}} (s-r)^{\frac{1}{2}}} \leq \frac{(t-r)^{\frac{\varepsilon}{2}}}{(s-t)^{\frac{1}{2}} (s-r)^{\frac{\varepsilon}{2}}}$$

by which we get thanks to (5.9) and Lemma 5.2(ii) that

$$\begin{aligned}
(\mathbb{E}\Psi_{2\beta}(|B_s - B_t|^{2\varepsilon}))^{\frac{1}{2}} \frac{(t-r)^{\frac{1}{2}}}{(s-t)^{\frac{1}{2}}(s-r)^{\frac{1}{2}}} &\leq \Phi_\beta(|s-t|^\varepsilon b_\varepsilon) \frac{\sqrt{b_\varepsilon}(t-r)^{\frac{\varepsilon}{2}}}{(s-t)^{\frac{1-\varepsilon}{2}}(s-r)^{\frac{\varepsilon}{2}}} \\
&\leq \Phi_\beta(|s-t|^\varepsilon b_\varepsilon) \frac{\sqrt{b_\varepsilon}(t-r)^{\frac{\varepsilon}{2}}}{(s-t)^{\frac{1-\varepsilon}{2}}(s-r)^{\frac{\varepsilon}{2}}} \frac{\Phi_{\bar{\beta}}((t-r)^\varepsilon)}{\Phi_{\bar{\beta}}((s-r)^\varepsilon)} \\
&= \Phi_\beta(|s-t|^\varepsilon b_\varepsilon) \frac{\sqrt{b_\varepsilon}(t-r)^{\frac{\varepsilon}{2}} \Phi_{\bar{\beta}}((t-r)^\varepsilon)}{(s-t)^{\frac{1}{2}} \Phi_{\bar{\beta}}((s-r)^\varepsilon)}. \quad (5.12)
\end{aligned}$$

The assertion (5.4) follows by inserting (5.10) and (5.12) into (5.8) and (5.11), respectively. Here we recall that by Lemma 5.2(i) we have $\Phi_\beta \geq 1$ so that

$$\Phi_\beta(|t-r|^\varepsilon b_\varepsilon) + \Phi_\beta(|s-t|^\varepsilon b_\varepsilon) \leq 2\Phi_\beta(|t-r|^\varepsilon b_\varepsilon) \Phi_\beta(|s-t|^\varepsilon b_\varepsilon). \quad \square$$

The next proposition is a modification of [BGGL21, Lemma 7 and Proposition 10]. More precisely, it generalizes the inequality of Lemma 7(b)(i) giving the possibility to weaken the singularity in time by slightly reducing the regularity in space, while in relation to Proposition 10 the Hölder continuity in time is reduced to achieve a weaker singularity in time. The items (i) and (ii) in the following proposition are proven in [BGGL21, Lemma 7].

Proposition 5.5. *Let Assumption 2.3 hold and put $F(s, x) := f(s, x, u(s, x), \nabla u(s, x))$.*

(i) *The function u belongs to $\mathcal{C}^{0,1}([t_0, T] \times \mathbb{R})$ and, for all $(t, x) \in [t_0, T] \times \mathbb{R}$, we have,*

$$Z_s^{t,x} = \nabla u(s, B_s^{t,x}) \quad \text{for a.e. } (s, \omega) \in [t, T] \times \Omega, \quad (5.13)$$

as well as

$$\nabla u(t, x) = \mathbb{E} \left(g(B_T^{t,x}) \frac{B_T - B_t}{T - t} \right) + \mathbb{E} \left(\int_t^T F(s, B_s^{t,x}) \frac{B_s - B_t}{s - t} ds \right). \quad (5.14)$$

Consequently, for $\mathbb{E}_r := \mathbb{E}[\cdot | \mathcal{F}_r]$,

$$\nabla u(r, B_r^{t,x}) = \mathbb{E}_r \left(g(B_T^{t,x}) \frac{B_T - B_r}{T - r} \right) + \mathbb{E}_r \left(\int_r^T F(s, B_s^{t,x}) \frac{B_s - B_r}{s - r} ds \right) \quad \text{a.s. for } r \in [t, T]. \quad (5.15)$$

(ii) *There exists a constant $c_{(5.16)} = c_{(5.16)}(T, \varepsilon, |g|_\varepsilon, |f|_{\varepsilon;x}, L_f, c_{f_0}) > 0$ such that*

$$|\nabla u(t, x)| \leq \frac{c_{(5.16)}}{(T-t)^{(1-\varepsilon)/2}} \quad \text{for all } (t, x) \in [t_0, T] \times \mathbb{R}. \quad (5.16)$$

(iii) *There exists a constant $c_{(5.17)} = c_{(5.17)}(T, \varepsilon, \beta, |g|_\varepsilon, |f|_{\varepsilon;x}, L_f, c_{f_0}) > 0$ such that, for all $t \in [t_0, T]$, $x, y \in \mathbb{R}$ and for all $\beta > \frac{1}{2}$ or $\beta = 0$,*

$$|\nabla u(t, x) - \nabla u(t, y)| \leq c_{(5.17)} \frac{\Psi_{2\beta}(|x-y|^{2\varepsilon})^{\frac{1}{2}}}{(T-t)^{\frac{1}{2}} \Phi_\beta((T-t)^\varepsilon)}. \quad (5.17)$$

(iv) There exists a constant $c_{(5.18)} = c_{(5.18)}(T, \varepsilon, \beta, |g|_\varepsilon, |f|_{\varepsilon;x}, L_f, c_{f_0}) > 0$ such that for $\beta > \frac{1}{2}$ or $\beta = 0$ and for all $x \in \mathbb{R}$,

$$|\nabla u(t, x) - \nabla u(r, x)| \leq c_{(5.18)} \frac{\Psi_{2\beta}(|t-r|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T-t)^{\frac{1}{2}} \Phi_\beta(|T-t|^\varepsilon)} \quad \text{for all } t_0 \leq r < t < T, \quad (5.18)$$

where $b_\varepsilon := \mathbb{E}|B_1|^{2\varepsilon}$.

Proof. (iii) The representation (5.15) yields to

$$\begin{aligned} \|\nabla u(r, B_r^{t,x}) - \nabla u(r, B_r^{t,y})\|_{L^2} &\leq \left\| \mathbb{E}_r \left(\left(g(B_T^{t,x}) - g(B_T^{t,y}) \right) \frac{B_T - B_r}{T - r} \right) \right\|_{L^2} \\ &\quad + \int_r^T \left\| \mathbb{E}_r \left(\left(F(s, B_s^{t,x}) - F(s, B_s^{t,y}) \right) \frac{B_s - B_r}{s - r} \right) \right\|_{L^2} ds. \end{aligned} \quad (5.19)$$

Let us start with the g difference. Throughout the paper we will repeatedly use that one can add terms like $g(B_r^{t,x}) \frac{B_T - B_r}{T - r}$ to the calculation since their conditional expectation w.r.t. \mathcal{F}_r is zero. Note that

$$\begin{aligned} &\mathbb{E}_r \left(\left(g(B_T^{t,x}) - g(B_T^{t,y}) \right) \frac{B_T - B_r}{T - r} \right) \\ &= \mathbb{E}_r \left(\left(g(B_T^{t,x}) - g(B_T^{t,y}) - g(B_r^{t,x}) + g(B_r^{t,y}) \right) \frac{B_T - B_r}{T - r} \right) \\ &\leq \mathbb{E}_r \left(\underbrace{|g(B_T^{t,x}) - g(B_T^{t,y}) - g(B_r^{t,x}) + g(B_r^{t,y})|}_{:=A} \frac{|B_T - B_r|}{T - r} \right). \end{aligned}$$

Since g is ε -Hölder continuous we have the estimates

$$A \leq 2|g|_\varepsilon |x - y|^\varepsilon \quad \text{and} \quad A \leq 2|g|_\varepsilon |B_T - B_r|^\varepsilon,$$

and therefore

$$A \leq 2|g|_\varepsilon \min\{|x - y|^\varepsilon, |B_T - B_r|^\varepsilon\}.$$

Setting $s := T - r \in (0, T]$ we are interested in the estimate of

$$\begin{aligned} M &:= \mathbb{E} \left(\min\{|x - y|, |B_s|\}^\varepsilon \frac{|B_s|}{s} \right) \\ &= \frac{1}{s} \int_{|B_s| \leq |x-y|} |B_s|^{1+\varepsilon} d\mathbb{P} + \frac{|x-y|^\varepsilon}{s} \int_{|B_s| > |x-y|} |B_s| d\mathbb{P} \\ &=: T_1 + T_2. \end{aligned}$$

For T_1 we use that $\frac{B_s}{\sqrt{s}}$ and B_1 have the same distribution and get

$$T_1 = \frac{1}{s} \int_{|z| \leq \frac{|x-y|}{\sqrt{s}}} |z|^{1+\varepsilon} s^{\frac{1+\varepsilon}{2}} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$$

$$\leq \begin{cases} \sqrt{\frac{2}{\pi}} s^{\frac{\varepsilon-1}{2}} \left(\frac{|x-y|}{\sqrt{s}}\right)^{2+\varepsilon} & \text{if } \frac{|x-y|}{\sqrt{s}} \leq 1 \\ s^{\frac{\varepsilon-1}{2}} \mathbb{E}|B_1|^{1+\varepsilon} & \text{if } \frac{|x-y|}{\sqrt{s}} > 1 \end{cases}$$

$$= \frac{|x-y|^\varepsilon}{\sqrt{s}} \left(\sqrt{\frac{2}{\pi}} \left(\frac{|x-y|}{\sqrt{s}}\right)^2 \mathbf{1}_{\{|x-y| \leq \sqrt{s}\}} + \left(\frac{\sqrt{s}}{|x-y|}\right)^\varepsilon \mathbb{E}|B_1|^{1+\varepsilon} \mathbf{1}_{\{|x-y| > \sqrt{s}\}} \right).$$

Similarly, we have for T_2 that

$$T_2 = \frac{|x-y|^\varepsilon}{s} \int_{|z| > \frac{|x-y|}{\sqrt{s}}} |z| s^{\frac{1}{2}} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$$

$$\leq \frac{|x-y|^\varepsilon}{\sqrt{s}} \left(\mathbb{E}|B_1| \mathbf{1}_{\{|x-y| \leq \sqrt{s}\}} + \sqrt{\frac{2}{\pi}} e^{-|x-y|^2/2s} \mathbf{1}_{\{|x-y| > \sqrt{s}\}} \right).$$

Hence there exists a constant $c_\varepsilon > 0$ such that

$$M \leq c_\varepsilon \frac{|x-y|^\varepsilon}{\sqrt{s}} \mathbf{1}_{\{|x-y| \leq \sqrt{s}\}} + c_\varepsilon s^{\frac{\varepsilon-1}{2}} \mathbf{1}_{\{|x-y| > \sqrt{s}\}},$$

where we used that $e^{-\frac{z^2}{2}} < z^{-\varepsilon}$ for $z := \frac{|x-y|}{\sqrt{s}} > 1$.

Since Ψ_β is increasing on $(0, \infty)$, we get using $\Phi_\beta(r) = \left(\log\left(\frac{1}{r\wedge\bar{r}}\right)\right)^\beta$ that

$$c_\varepsilon s^{\frac{\varepsilon-1}{2}} \mathbf{1}_{\{|x-y| > \sqrt{s}\}} = c_\varepsilon \frac{1}{s^{\frac{1}{2}} \Phi_\beta(s^\varepsilon)} s^{\frac{\varepsilon}{2}} \Phi_\beta(s^\varepsilon) \mathbf{1}_{\{|x-y| > \sqrt{s}\}} \leq c_\varepsilon \frac{1}{s^{\frac{1}{2}} \Phi_\beta(s^\varepsilon)} |x-y|^\varepsilon \Phi_\beta(|x-y|^{2\varepsilon}).$$

For the case $|x-y| \leq \sqrt{s}$ we use that $\Phi_\beta(s^\varepsilon) \leq \Phi_\beta(|x-y|^{2\varepsilon})$ and get as well

$$c_\varepsilon \frac{|x-y|^\varepsilon}{\sqrt{s}} \mathbf{1}_{\{|x-y| \leq \sqrt{s}\}} \leq c_\varepsilon \frac{1}{s^{\frac{1}{2}} \Phi_\beta(s^\varepsilon)} |x-y|^\varepsilon \Phi_\beta(|x-y|^{2\varepsilon}).$$

Consequently,

$$\left| \mathbb{E}_r \left(\left(g(B_T^{t,x}) - g(B_T^{t,y}) \right) \frac{B_T - B_r}{T - r} \right) \right| \leq c_\varepsilon \frac{|x-y|^\varepsilon \Phi_\beta(|x-y|^{2\varepsilon})}{(T-r)^{\frac{1}{2}} \Phi_\beta((T-r)^\varepsilon)} = c_\varepsilon \frac{\Psi_{2\beta}(|x-y|^{2\varepsilon})^{\frac{1}{2}}}{(T-r)^{\frac{1}{2}} \Phi_\beta((T-r)^\varepsilon)}.$$

We continue with the estimate of the second term on the RHS of (5.19). By the conditional Cauchy-Schwarz inequality we obtain

$$\left\| \mathbb{E}_r \left((F(s, B_s^{t,x}) - F(s, B_s^{t,y})) \frac{B_s - B_r}{s - r} \right) \right\|_{L^2} \leq \|F(s, B_s^{t,x}) - F(s, B_s^{t,y})\|_{L^2} \frac{1}{\sqrt{s-r}}.$$

Using (2.1) for f , we have

$$\begin{aligned} & |F(s, B_s^{t,x}) - F(s, B_s^{t,y})| \\ & \leq |f_{|\varepsilon;x}| |x-y|^\varepsilon + L_f |u(s, B_s^{t,x}) - u(s, B_s^{t,y})| + L_f |\nabla u(s, B_s^{t,x}) - \nabla u(s, B_s^{t,y})|, \end{aligned} \tag{5.20}$$

and the Hölder continuity of u stated in [Lemma 5.1](#) yields

$$|F(s, B_s^{t,x}) - F(s, B_s^{t,y})| \leq (|f|_{\varepsilon;x} + c_{(5.2)} L_f) |x - y|^\varepsilon + L_f |\nabla u(s, B_s^{t,x}) - \nabla u(s, B_s^{t,y})|. \quad (5.21)$$

By combining the above estimates we conclude from [\(5.19\)](#) that

$$\begin{aligned} \|\nabla u(r, B_r^{t,x}) - \nabla u(r, B_r^{t,y})\|_{L^2} &\leq \frac{c_\varepsilon \Psi_{2\beta}(|x - y|^{2\varepsilon})^{\frac{1}{2}}}{(T - r)^{\frac{1}{2}} \Phi_\beta((T - r)^\varepsilon)} + 2(|f|_{\varepsilon;x} + c_{(5.2)} L_f) |x - y|^\varepsilon \sqrt{T - r} \\ &\quad + \int_r^T \|\nabla u(s, B_s^{t,x}) - \nabla u(s, B_s^{t,y})\|_{L^2} \frac{L_f}{\sqrt{s - r}} ds. \end{aligned}$$

By [Lemma 5.2](#) we have

$$1 = \frac{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)} = \frac{(T - t)^{\frac{1-\varepsilon}{2}} \Psi_{2\beta}((T - t)^\varepsilon)^{\frac{1}{2}}}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)} \leq \frac{c_{(5.22)}}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)} \quad (5.22)$$

with $c_{(5.22)} = T^{\frac{1-\varepsilon}{2}} \Psi_{2\beta}(\hat{r})^{\frac{1}{2}}$, implying

$$1 \leq c_{(5.22)} \frac{\Phi_\beta(|x - y|^{2\varepsilon})}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)} \quad (5.23)$$

since $1 \leq \Phi_\beta(|x - y|^{2\varepsilon})$. This gives

$$\begin{aligned} &\|\nabla u(r, B_r^{t,x}) - \nabla u(r, B_r^{t,y})\|_{L^2} \\ &\leq 2C_1 \frac{\Psi_{2\beta}(|x - y|^{2\varepsilon})^{\frac{1}{2}}}{(T - r)^{\frac{1}{2}} \Phi_\beta((T - r)^\varepsilon)} + \int_r^T \|\nabla u(s, B_s^{t,x}) - \nabla u(s, B_s^{t,y})\|_{L^2} \frac{L_f}{\sqrt{s - r}} ds \end{aligned}$$

for $C_1 := c_\varepsilon + 2\sqrt{T}(|f|_{\varepsilon;x} + c_{(5.2)} L_f) c_{(5.22)}$. Because of

$$|\nabla u(t, x)| \leq \frac{c_{(5.16)}}{(T - t)^{(1-\varepsilon)/2}} \quad \text{on } [t_0, T] \times \mathbb{R}, \quad (5.24)$$

we have an integrable bound

$$\|\nabla u(s, B_s^{t,x}) - \nabla u(s, B_s^{t,y})\|_{L^2} \leq 2 \frac{c_{(5.16)}}{(T - t)^{(1-\varepsilon)/2}},$$

so that we may apply Gronwall's lemma ([Lemma 9.2](#)) and get

$$\|\nabla u(r, B_r^{t,x}) - \nabla u(r, B_r^{t,y})\|_{L^2} \leq c_0 C_1 \frac{\Psi_{2\beta}(|x - y|^{2\varepsilon})^{\frac{1}{2}}}{(T - r)^{\frac{1}{2}} \Phi_\beta((T - r)^\varepsilon)},$$

for some $c_0 = c_0(T, L_f) > 0$. Especially, for $r = t$ this implies

$$|\nabla u(t, x) - \nabla u(t, y)| \leq C \frac{\Psi_{2\beta}(|x - y|^{2\varepsilon})^{\frac{1}{2}}}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)}$$

for some $C = C(T, \varepsilon, \beta, |g|_\varepsilon, |f|_{\varepsilon;x}, L_f, c_{f_0}, f_0) > 0$.

(iv) Recalling that $H(t_0, t_1) = \frac{B_{t_1} - B_{t_0}}{t_1 - t_0}$ for $t_0 \leq t_0 < t_1 \leq T$, we use (5.14) to write

$$\begin{aligned} \nabla u(r, x) - \nabla u(t, x) &= \mathbb{E} \left[g(B_T^{r,x}) H(r, T) - g(B_T^{t,x}) H(t, T) \right] \\ &\quad + \mathbb{E} \int_t^T F(s, B_s^{r,x}) H(s, r) - F(s, B_s^{t,x}) H(s, t) ds \\ &\quad + \mathbb{E} \int_r^t (F(s, B_s^{r,x}) - F(s, x)) H(r, s) ds. \end{aligned}$$

Let us bound the three terms of the right hand side. By applying Lemma 5.4(i) we get for $\bar{\beta} \in [\frac{1}{2}, 1[$

$$\left| \mathbb{E} \left[g(B_T^{r,x}) H(r, T) - g(B_T^{t,x}) H(t, T) \right] \right| \leq |g|_\varepsilon \frac{\Psi_{2\bar{\beta}}(|t-r|^\varepsilon)^{\frac{1}{2}}}{(T-t)^{\frac{1}{2}} \Phi_{\bar{\beta}}(|T-t|^\varepsilon)}.$$

Let us bound the second term. Using (5.21) for $s = t$ combined with Proposition 5.5 (iii), and (5.23) leads to $(C_F = |f|_{\varepsilon;x} + c_{(5.2)} L_f + c_{(5.17)} L_f)$

$$\begin{aligned} |F(s, x) - F(s, y)| &\leq C_F \left(|x-y|^\varepsilon + \frac{\Psi_{2\beta}(|x-y|^{2\varepsilon})^{\frac{1}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta(|T-s|^\varepsilon)} \right) \\ &\leq c_{(5.25)} \frac{\Psi_{2\beta}(|x-y|^{2\varepsilon})^{\frac{1}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta(|T-s|^\varepsilon)}, \end{aligned} \quad (5.25)$$

where $c_{(5.25)} = C_F(c_{(5.22)} + 1)$. Hence we may apply Lemma 5.4(ii) and get

$$\begin{aligned} &\mathbb{E} \int_t^T |F(s, B_s^{r,x}) H(r, s) - F(s, B_s^{t,x}) H(t, s)| ds \\ &\leq c_{(5.25)} \Psi_{2\beta}(|t-r|^\varepsilon b_\varepsilon)^{\frac{1}{2}} \int_t^T \frac{\Phi_\beta(|s-t|^\varepsilon b_\varepsilon)}{(T-s)^{\frac{1}{2}} \Phi_\beta(|T-s|^\varepsilon) (s-t)^{\frac{1}{2}}} ds \\ &\leq c_{(5.25)} \Psi_{2\beta}(|t-r|^\varepsilon b_\varepsilon)^{\frac{1}{2}} \int_t^T \frac{(s-t)^{\frac{\varepsilon}{2}} \Phi_\beta(|s-t|^\varepsilon b_\varepsilon)}{(T-s)^{\frac{1}{2}} (s-t)^{\frac{1+\varepsilon}{2}}} ds \\ &\leq C \Psi_{2\beta}(|t-r|^\varepsilon b_\varepsilon)^{\frac{1}{2}}, \end{aligned}$$

where $C > 0$ depends on $(T, \varepsilon, \beta, |g|_\varepsilon, |f|_{\varepsilon;x}, L_f, c_{f_0}, f_0)$. Indeed, the integral in the last inequality is bounded: By Lemma 5.2(iii) we have

$$(s-t)^{\frac{\varepsilon}{2}} \Phi_\beta(|s-t|^\varepsilon b_\varepsilon) = \frac{\Psi_{2\beta}(|s-t|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{\sqrt{b_\varepsilon}} \leq \frac{e^{-\beta} (2\beta)^\beta}{\sqrt{b_\varepsilon}},$$

and

$$\int_t^T \frac{1}{(T-s)^{\frac{1}{2}} (s-t)^{\frac{1+\varepsilon}{2}}} ds < (T-t)^{-\frac{\varepsilon}{2}} \mathbf{B} \left(\frac{1}{2}, \frac{1-\varepsilon}{2} \right) < \infty,$$

where \mathbf{B} denotes the beta function.

Let us bound the third term. Using (5.25), Hölder's inequality, (5.9) and Lemma 5.2(iii) we obtain

$$\begin{aligned}
\left| \mathbb{E} \int_r^t (F(s, B_s^{r,x}) - F(s, x)) \frac{B_s - B_r}{s - r} ds \right| &\leq c_{(5.25)} \int_r^t \frac{\Psi_{2\beta}(|s - r|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T - s)^{\frac{1}{2}} \Phi_\beta((T - s)^\varepsilon) (s - r)^{\frac{1}{2}}} ds \\
&\leq c_{(5.25)} \frac{\Psi_{2\beta}(|t - r|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)} \int_r^t \frac{1}{(s - r)^{\frac{1}{2}}} ds \\
&= 2c_{(5.25)} \frac{\Psi_{2\beta}(|t - r|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)} (t - r)^{\frac{1}{2}},
\end{aligned}$$

where we used Lemma 5.2(ii) for the second inequality. Choosing now $\bar{\beta} = \beta$, collecting all terms and recalling (5.22) we have

$$|\nabla u(t, x) - \nabla u(r, x)| \leq C \frac{\Psi_{2\beta}(|t - r|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T - t)^{\frac{1}{2}} \Phi_\beta((T - t)^\varepsilon)}.$$

□

5.2 Local Hölder continuity of the generator with frozen mean field terms

In this section we show that if the measure components are frozen, then the obtained generator still satisfies (2.3).

Lemma 5.6. *Suppose Assumption 2.2, define*

$$f(t, x, y, z) := f(t, x, y, z, [Y_t^{t_0, \xi}], [Z_t^{t_0, \xi}]) \quad (5.26)$$

and recall (1.6), i.e.

$$f_n(t, x, y, z) = f(t \wedge t_{n-1}, x, y, z, [Y_t^{t_0, \xi}], [Z_{t \wedge t_{n-1}}^{t_0, \xi}]).$$

Then $\mathbf{d} \in \{f, f_n\}$ satisfies (2.3) provided an L^2 -continuous version of $(Z_t^{t_0, \xi})_{t \in [t_0, T]}$ is taken: For (t, x, y, z) and (t', x', y', z') in $[t_0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ one has

$$\begin{aligned}
|\mathbf{d}(t, x, y, z) - \mathbf{d}(t', x', y', z')| &\leq c_{(5.27)} (|f|_{\alpha; t}^{loc} + (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon) \frac{|t - t'|^{\alpha \wedge \frac{\varepsilon}{2}}}{(T - (t \vee t'))^{\frac{1}{2}}} + |f|_{\varepsilon; x} |x - x'|^\varepsilon \\
&\quad + L_f (|y - y'| + |z - z'|) \quad (5.27)
\end{aligned}$$

$$\int_{t_0}^T |\mathbf{d}(t, 0, 0, 0)| dt \leq c_{(5.28)} \quad (5.28)$$

for some $c_{(5.27)} > 0$ and $c_{(5.28)} > 0$ depending at most on $(T, \varepsilon, \alpha, |g|_\varepsilon, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, \xi)$.

Proof. (a) We show that there is a $c_{(5.29)} = c_{(5.29)}(T, \varepsilon, \alpha, |g|_\varepsilon, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, \xi) > 0$ such that

$$\left\| Y_t^{t_0, \xi} - Y_{t'}^{t_0, \xi} \right\|_{L^2} \leq c_{(5.29)} |t - t'|^{\frac{\varepsilon}{2}} \quad \text{and} \quad \left\| Z_t^{t_0, \xi} - Z_{t'}^{t_0, \xi} \right\|_{L^2} \leq c_{(5.29)} \frac{|t - t'|^{\frac{\varepsilon}{2}}}{(T - t \vee t')^{\frac{1}{2}}}, \quad (5.29)$$

provided we choose an appropriate version for the Z -process. Having these inequalities, (5.27) and (5.28) follow from the properties of the generator f .

(b) To show (5.29) we start with $\mu_t^0 = \nu_t^0 := \delta_0$ and consider the iteration

$$Y_s^{k+1, t_0, \xi} = g\left(B_T^{t_0, \xi}\right) + \int_s^T f\left(r, B_r^{t_0, \xi}, Y_r^{k+1, t_0, \xi}, Z_r^{k+1, t_0, \xi}, \mu_r^k, \nu_r^k\right) dr - \int_s^T Z_r^{k+1, t_0, \xi} dB_r, \\ t_0 \leq s \leq T,$$

By [Li18, Theorem A.1] we get for an appropriate $\beta > 0$ that

$$\|(Y^{t_0, \xi, k} - Y^{t_0, \xi}, Z^{t_0, \xi, k} - Z^{t_0, \xi})\|_\beta \rightarrow 0,$$

where

$$\|(\mathcal{Y}, \mathcal{Z})\|_\beta := \sqrt{\mathbb{E} \int_{t_0}^T e^{\beta s} (\mathcal{Y}_s^2 + \mathcal{Z}_s^2) ds}.$$

This implies

$$\lim_{k \rightarrow \infty} \int_{t_0}^T |f(t, 0, 0, 0, [Y_t^{t_0, \xi, k}], [Z_t^{t_0, \xi, k}])| dt = \int_{t_0}^T |f(t, 0, 0, 0, [Y_t^{t_0, \xi}], [Z_t^{t_0, \xi}])| dt < \infty$$

and

$$c_{(5.30)} := \sup_{k \geq 0} \int_{t_0}^T |f(t, 0, 0, 0, [Y_t^{t_0, \xi, k}], [Z_t^{t_0, \xi, k}])| dt < \infty. \quad (5.30)$$

(c) Now we use Lemma 5.1 for

$$Y_t^{k, t_0, \xi} = u^k(t, B_t^{t_0, \xi}) \quad \text{and} \quad Z_t^{k, t_0, \xi} = \nabla u^k(t, B_t^{t_0, \xi})$$

to get

$$\|Y_t^{k, t_0, \xi} - Y_{t'}^{k, t_0, \xi}\|_{L^2} \leq \|u^k(t, B_t^{t_0, \xi}) - u^k(t', B_{t'}^{t_0, \xi})\|_{L^2} \\ + \|u^k(t', B_{t'}^{t_0, \xi}) - u^k(t', B_{t'}^{t_0, \xi})\|_{L^2} \\ \leq c_{(5.31)} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon |t - t'|^{\frac{\varepsilon}{2}}, \quad (5.31)$$

where $c_{(5.31)} = c_{(5.31)}(T, \varepsilon, |g|_\varepsilon, |f|_{\varepsilon; x}, L_f, c_{(5.30)}) > 0$ and similarly by Proposition 5.5(iii) and (iv) (recall that $\Psi_{2\beta}(r) = r$ for $\beta = 0$)

$$\|Z_t^{k, t_0, \xi} - Z_{t'}^{k, t_0, \xi}\|_{L^2} \\ \leq \|\nabla u^k(t, B_t^{t_0, \xi}) - \nabla u^k(t', B_{t'}^{t_0, \xi})\|_{L^2} \\ \leq \|\nabla u^k(t, B_t^{t_0, \xi}) - \nabla u^k(t', B_{t'}^{t_0, \xi})\|_{L^2} + \|\nabla u^k(t', B_{t'}^{t_0, \xi}) - \nabla u^k(t', B_{t'}^{t_0, \xi})\|_{L^2} \\ \leq c_{(5.32)} \frac{|t - t'|^{\frac{\varepsilon}{2}}}{(T - t \vee t')^{\frac{1}{2}}}, \quad (5.32)$$

where $c_{(5.32)} = c_{(5.32)}(T, \varepsilon, |g|_\varepsilon, |f|_{\varepsilon; x}, L_f, c_{(5.30)}) > 0$. From the convergence $\|(Y^{t_0, \xi, k} - Y^{t_0, \xi}, Z^{t_0, \xi, k} - Z^{t_0, \xi})\|_\beta^2 \rightarrow 0$ we conclude that there is a subsequence $(k_l)_{l=1}^\infty$ which converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for a.e. $s \in [t_0, T]$. As for this subsequence we have the relations

$$\|Y_t^{k_l, t_0, \xi} - Y_{t'}^{k_l, t_0, \xi}\|_{L^2} \leq c|t - t'|^{\frac{\varepsilon}{2}} \quad \text{and} \quad \|Z_t^{k_l, t_0, \xi}, Z_{t'}^{k_l, t_0, \xi}\|_{L^2} \leq c \frac{|t - t'|^{\frac{\varepsilon}{2}}}{(T - t \vee t')^{\frac{1}{2}}}$$

with a common constant $c > 0$ depending at most on $(T, \varepsilon, L_f, |f|_{\varepsilon; x}, |g|_{\varepsilon}, c_{(5.30)}, \xi)$, this implies that (for an L^2 -continuous version of $(Z^{t_o, \xi})_{t \in [t_o, T]}$) the relations (5.29) are satisfied. \square

Convention 5.7. In the following we assume that (5.29) is satisfied.

As an immediate consequence we obtain regularity results for $u := u_f$ and $\nabla u := \nabla u_f$ related to f as well as for $u := u_{f_n}$ and $\nabla u := \nabla u_{f_n}$ related to f_n :

Proposition 5.8. *Lemma 5.1 and Proposition 5.5 hold for $u(t, x)$ and $\nabla u(t, x)$ corresponding to f given by (5.26) i.e. for the solution to (1.1) as well as for $u(t, x)$ and $\nabla u(t, x)$ corresponding to f_n given by (1.6).*

6 The discrete scheme

We provide the scheme giving $(Y^{n, t_o, \xi}, Z^{n, t_o, \xi})$. Assume $\underline{t_o} = \frac{jT}{n}$ for $j \in \{0, 1, \dots, n-1\}$. Then for $k = j, \dots, n-1$ one has

$$\begin{aligned} Y_{t_k}^{n, t_o, \xi} &= Y_{t_{k+1}}^{n, t_o, \xi} + hf \left(t_{k+1} \wedge t_{n-1}, B_{t_k}^{n, t_o, \xi}, Y_{t_k}^{n, t_o, \xi}, Z_{t_{k+1}}^{n, t_o, \xi}, [Y_{t_{k+1}}^{n, t_o, \xi}], [Z_{t_{k+1}}^{n, t_o, \xi}] \right) - \sqrt{h} Z_{t_{k+1}}^{n, t_o, \xi} \zeta_{k+1}, \\ Y_{t_n}^{n, t_o, \xi} &= g(B_T^{n, t_o, \xi}). \end{aligned}$$

Thus, if $Y_{t_{k+1}}^{n, t_o, \xi}$ is given, and $\mathcal{F}_{t_k}^{n, t_o, \xi} = \sigma(\zeta_{j+1}, \dots, \zeta_k, \xi)$

$$Z_{t_{k+1}}^{n, t_o, \xi} = h^{-1/2} \mathbb{E} \left[Y_{t_{k+1}}^{n, t_o, \xi} \zeta_{k+1} \mid \mathcal{F}_{t_k}^{n, t_o, \xi} \right], \quad (6.1)$$

$$Y_{t_k}^{n, t_o, \xi} = \mathbb{E} \left[Y_{t_{k+1}}^{n, t_o, \xi} \mid \mathcal{F}_{t_k}^{n, t_o, \xi} \right] + hf \left(t_{k+1} \wedge t_{n-1}, B_{t_k}^{n, t_o, \xi}, Y_{t_k}^{n, t_o, \xi}, Z_{t_{k+1}}^{n, t_o, \xi}, [Y_{t_{k+1}}^{n, t_o, \xi}], [Z_{t_{k+1}}^{n, t_o, \xi}] \right). \quad (6.2)$$

Translating (6.1) and (6.2) to $(Y_{f_n}^{n, t_o, \xi}, Z_{f_n}^{n, t_o, \xi})$ gives

$$Z_{f_n, t_{k+1}}^{n, t_o, \xi} = h^{-1/2} \mathbb{E} \left[Y_{f_n, t_{k+1}}^{n, t_o, \xi} \zeta_{k+1} \mid \mathcal{F}_{t_k}^{n, t_o, \xi} \right], \quad (6.3)$$

$$Y_{f_n, t_k}^{n, t_o, \xi} = \mathbb{E} \left[Y_{f_n, t_{k+1}}^{n, t_o, \xi} \mid \mathcal{F}_{t_k}^{n, t_o, \xi} \right] + hf \left(t_{k+1} \wedge t_{n-1}, B_{t_k}^{n, t_o, \xi}, Y_{f_n, t_k}^{n, t_o, \xi}, Z_{f_n, t_{k+1}}^{n, t_o, \xi}, [Y_{t_{k+1}}^{n, t_o, \xi}], [Z_{t_{k+1} \wedge t_{n-1}}^{n, t_o, \xi}] \right) \quad (6.4)$$

$$= \mathbb{E} \left[Y_{f_n, t_{k+1}}^{n, t_o, \xi} \mid \mathcal{F}_{t_k}^{n, t_o, \xi} \right] + hf_n \left(t_{k+1}, B_{t_k}^{n, t_o, \xi}, Y_{f_n, t_k}^{n, t_o, \xi}, Z_{f_n, t_{k+1}}^{n, t_o, \xi} \right). \quad (6.5)$$

Let the generator f_n be as given in (1.6). We define the finite difference scheme related to the random walk scheme for $t_k = kh$ by

$$\begin{cases} \partial_t^h U_{f_n}^n(t_k, x) + L^h U_{f_n}^n(t_{k+1}, x) + f_n(t_{k+1}, x, U_{f_n}^n(t_k, x), \nabla^h U_{f_n}^n(t_{k+1}, x)) = 0, \\ U_{f_n}^n(t_n, x) = g(x), \end{cases} \quad k = 0, \dots, n-1 \quad (6.6)$$

where

$$\begin{aligned} L^h u(x) &:= \frac{1}{2h} \left(u(x + \sqrt{h}) + u(x - \sqrt{h}) - 2u(x) \right), \\ \nabla^h u(x) &:= \frac{1}{2\sqrt{h}} \left(u(x + \sqrt{h}) - u(x - \sqrt{h}) \right), \\ \partial_t^h u(t) &:= \frac{1}{h} (u(t+h) - u(t)). \end{aligned}$$

Setting $\bar{s} := \min \{t_k \mid k \in \{0, \dots, n\} \text{ and } s \leq t_k\}$ we will exploit the relations (see [BGGL21])

$$Y_{f_n, s}^{n, t, x} = Y_{f_n, \underline{s}}^{n, t, x} = U_{f_n}^n(s, B_s^{n, t, x}), \quad s \in [t, T] \quad \text{and} \quad Z_{f_n, s}^{n, t, x} = Z_{f_n, \bar{s}}^{n, t, x} = \nabla^n U_{f_n}^n(\bar{s}, B_{s^-}^{n, t, x}) \quad s \in]\underline{t}, T]. \quad (6.7)$$

In particular,

$$Z_{f_n, s}^{n, t, x} = \nabla^h U_{f_n}^n(\underline{s} + h, B_s^{n, t, x}) = \Delta_{f_n}^n(s, B_s^{n, t, x}) \quad \text{whenever } s \neq \underline{s}. \quad (6.8)$$

7 Convergence rate

In this section we establish necessary results required to deduce the main result, [Theorem 1.1](#).

7.1 Comparing (Y, Z) and (Y_{f_n}, Z_{f_n})

We start by comparing the solution to (1.1) to the solution to (1.7).

Proposition 7.1. *Under [Assumption 2.2](#) for the solutions to (1.1) and (1.7) one has*

$$\|Z_s^{t_o, \xi} - Z_r^{t_o, \xi}\|_{L^2} \leq c_{(7.1)} \frac{|s - r|^{\frac{\varepsilon}{2}}}{|T - s|^{\frac{1}{2}}} \quad \text{for } t_o \leq r \leq s < T, \quad (7.1)$$

$$|u(t, x) - u_{f_n}(t, x)| + \left(\mathbb{E}_{t, x} \int_t^T |\nabla u(s, B_s) - \nabla u_{f_n}(s, B_s)|^2 ds \right)^{\frac{1}{2}} \leq c_{(7.2)} n^{-((\alpha \wedge \frac{\varepsilon}{2}) + \frac{1}{2})} \quad (7.2)$$

for $n \geq n_0$ and $(t, x) \in [t_o, T] \times \mathbb{R}$, where the constants $c_{(7.1)}, c_{(7.2)} > 0$ depend at most on $(T, \varepsilon, \alpha, |g|_\varepsilon, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, \xi)$. Moreover, there is an absolute constant $c_{(7.3)} > 0$ such that

$$\left| \int_t^T |\nabla u(s, B_s^{t, x}) - \nabla u_{f_n}(s, B_s^{t, x})|^2 ds \right|_{M_{\varphi_1}^0}^{\frac{1}{2}} \leq c_{(7.3)} c_{(7.2)} n^{-((\alpha \wedge \frac{\varepsilon}{2}) + \frac{1}{2})} \quad (7.3)$$

Proof. Inequality (7.1) is the same as (5.29). To show (7.2) we use an a priori estimate (see, for example, [GY21, Lemma 5.26]) and use (7.1) with $r = t_{n-1}$ to get, for some $c = c(T, L_f)$,

$$\begin{aligned} & \mathbb{E} \left[|Y_t^{t, x} - Y_{f_n, t}^{t, x}|^2 + \int_t^T |Z_s^{t, x} - Z_{f_n, s}^{t, x}|^2 ds \middle| \mathcal{F}_t \right] \\ & \leq c \mathbb{E} \left[\int_t^T |f(s, B_s^{t, x}, Y_s^{t, x}, Z_s^{t, x}, [Y_s^{t_o, \xi}], [Z_s^{t_o, \xi}]) \right. \\ & \quad \left. - f(s \wedge t_{n-1}, B_s^{t, x}, Y_s^{t, x}, Z_s^{t, x}, [Y_s^{t_o, \xi}], [Z_{s \wedge t_{n-1}}^{t_o, \xi}])| ds \right]^2 \middle| \mathcal{F}_t \end{aligned}$$

$$\begin{aligned}
&\leq c \left| \int_t^T \left[|f|_{\alpha;t}^{loc} \frac{|s-s \wedge t_{n-1}|^\alpha}{|T-s|^{\frac{1}{2}}} + L_f \mathcal{W}_2(Z_{s \wedge t_{n-1}}^{t_o, \xi}, Z_s^{t_o, \xi}) \right] ds \right|^2 \\
&\leq c \left| \int_t^T \left[|f|_{\alpha;t}^{loc} \frac{|s-s \wedge t_{n-1}|^\alpha}{|T-s|^{\frac{1}{2}}} + L_f c_{(7.1)} \frac{|s-t_{n-1}|^{\frac{\varepsilon}{2}}}{|T-s|^{\frac{1}{2}}} \right] ds \right|^2 \\
&\leq c' \left| \int_{t_{n-1}}^T \frac{|s-t_{n-1}|^{\frac{\varepsilon}{2} \wedge \alpha}}{|T-s|^{\frac{1}{2}}} ds \right|^2 \leq c'' n^{-2(\alpha \wedge \frac{\varepsilon}{2} + \frac{1}{2})}.
\end{aligned}$$

Finally we use that $Y_t^{t,x} = u(t,x)$, $Y_{f_n,t}^{t,x} = u_{f_n}(t,x)$, $Z_t^{t,x} = \nabla u(t,x)$, and $Z_{f_n,t}^{t,x} = \nabla u_{f_n}(t,x)$. Setting $\mathbb{E}_{t,x}[\cdot] := \mathbb{E}[\cdot | B_t = x]$ we conclude

$$\begin{aligned}
\mathbb{E} \left[\int_t^T |Z_s^{t,x} - Z_{f_n,s}^{t,x}|^2 ds | \mathcal{F}_t \right] &= \mathbb{E} \left[\int_t^T |\nabla u(s, B_s^{t,x}) - \nabla u_{f_n}(s, B_s^{t,x})|^2 ds | \mathcal{F}_t \right] \\
&= \mathbb{E}_{t,x} \int_t^T |\nabla u(s, B_s) - \nabla u_{f_n}(s, B_s)|^2 ds.
\end{aligned}$$

For the relation (7.3) we use (7.2) and Theorem 9.4. \square

7.2 Finite difference approximation - comparing (Y_{f_n}, Z_{f_n}) and $(Y_{f_n}^n, Z_{f_n}^n)$

We will use [BGGL21, Corollary 5] and reformulate it as follows:

Lemma 7.2. For $\beta > \frac{1}{2}$ there is a $c_{(7.4)} = c_{(7.4)}(T, \beta) > 0$ such that for $x \in \mathbb{R}$ and $t_o \leq t \leq s \leq T$,

$$|\mathbb{E}[g(B_s^{n,t,x})] - \mathbb{E}[g(B_s^{t,x})]| \leq c_{(7.4)} |g|_\varepsilon n^{-\varepsilon/2}, \quad (7.4)$$

and, setting $\delta(t, s) := \max(s - t, \underline{s} - \underline{t})$,

$$\begin{aligned}
&|\mathbb{E}[g(B_s^{n,t,x})(B_s^{n,t,x} - x)] - \mathbb{E}[g(B_s^{t,x})(B_s^{t,x} - x)]| \\
&\leq c_{(7.4)} |g|_\varepsilon \frac{\delta(t, s)^{\frac{1}{2}}}{\Phi_\beta((s-t)^\varepsilon)} n^{-\frac{\varepsilon}{2}} \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right).
\end{aligned}$$

Proof. The first result comes from [BGGL21, Corollary 5]. To prove the second result we adapt the proof of [BGGL21, Corollary 5]. We put $G(y) := (g(x+y) - g(x))y$. Then $\mathbb{E}[g(B_s^{n,t,x})(B_s^{n,t,x} - x)] - \mathbb{E}[g(B_s^{t,x})(B_s^{t,x} - x)] = \mathbb{E}(G(B_s^n - B_t^n)) - \mathbb{E}(G(B_s - B_t))$.

In case of $\underline{s} = \underline{t}$ we have $\mathcal{W}_1(G(B_s^n - B_t^n), G(B_s - B_t)) \leq \mathbb{E}(|G(B_s - B_t)|) \leq C|g|_\varepsilon (s-t)^{(1+\varepsilon)/2}$. Since $(s-t)^{(1+\varepsilon)/2} = (s-t)^{\frac{1}{2}}(s-t)^{\frac{\varepsilon}{2}} \leq \delta(s, t)^{\frac{1}{2}}(s-t)^{\frac{\varepsilon}{2}}$ and $s-t \leq h$ and the function $r \mapsto r\Phi_\beta(r^2)$ is non-decreasing by Lemma 5.2(ii), we get

$$\begin{aligned}
\mathcal{W}_1(G(B_s^n - B_t^n), G(B_s - B_t)) &\leq C|g|_\varepsilon \frac{\delta(t, s)^{\frac{1}{2}}(s-t)^{\frac{\varepsilon}{2}} \Phi_\beta((s-t)^\varepsilon)}{\Phi_\beta((s-t)^\varepsilon)} \\
&\leq C|g|_\varepsilon \frac{\delta(t, s)^{\frac{1}{2}}}{\Phi_\beta((s-t)^\varepsilon)} n^{-\frac{\varepsilon}{2}} \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right).
\end{aligned}$$

In case of $\underline{s} > \underline{t}$ we get from the proof of [BGGL21, Corollary 5] that $\mathcal{W}_1(G(B_s^n - B_t^n), G(B_s - B_t)) \leq C|g|_\varepsilon \left(\frac{T}{n}\right)^{\varepsilon/2} \delta(t, s)^{1/2}$. Furthermore, we have $h < (s - t)$, which implies that $\Phi_\beta((s - t)^\varepsilon) \leq \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right)$, and therefore

$$\mathcal{W}_1(G(B_s^n - B_t^n), G(B_s - B_t)) \leq C|g|_\varepsilon \frac{\delta(t, s)^{\frac{1}{2}}}{\Phi_\beta((s - t)^\varepsilon)} n^{-\frac{\varepsilon}{2}} \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right).$$

Now we finish the proof by choosing $f(x) = x$ in the Kantorovich-Rubinstein relation

$$\mathcal{W}_1(X, X') = \sup\{\mathbb{E}[f(X)] - \mathbb{E}[f(X')] : \|f\|_{\text{Lip}} \leq 1\}. \quad \square$$

We modify [BGGL21, Theorem 11] by introducing the function Φ_β with the aim to weaken the singularity in time to get square integrability at the expense of the convergence rate.

Lemma 7.3. *For $n \geq n_0$, under Assumption 2.2 with \mathbf{f}_n satisfying (5.27), there are constants $c_{(7.5)}, c_{(7.6)} > 0$ depending at most on $c_{(5.2)} = c_{(5.2)}(T, \varepsilon, \alpha, |g|_\varepsilon, |f|_{\alpha; t}^{\text{loc}}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, \xi) > 0$, where $c_{(7.6)} > 0$ depends additionally on β , such that*

$$|u_{\mathbf{f}_n}(t, x) - U_{\mathbf{f}_n}^n(t, x)| \leq c_{(7.5)} (1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}}, \quad (7.5)$$

$$|\nabla u_{\mathbf{f}_n}(t, x) - \nabla^h U_{\mathbf{f}_n}^n(\underline{t} + h, x)| \leq c_{(7.6)} \frac{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}}}{(T - t)^{\frac{1-\varepsilon}{2}} (T - \underline{t})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{t})^\varepsilon)} \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right) \quad (7.6)$$

for all $(t, x) \in [t_0, T] \times \mathbb{R}$, and $\beta > \frac{1}{2}$ or $\beta = 0$.

Proof. Equation (7.5) is shown in [BGGL21, Theorem 11(i)] for the setting in [BGGL21]. For the setting of this article we have to check how our local Hölder continuity in time given in (5.27) for $\mathbf{d} = \mathbf{f}_n$ influences the calculations done in [BGGL21]. The proof of [BGGL21, Theorem 11(i)] contains only one estimate where the Hölder continuity in time is used. We need to change it as follows: Denoting $\Theta_s^{n, t, x} := (B_{s^-}^{n, t, x}, Y_{s^-}^{n, t, x}, Z_s^{n, t, x})$ we have, by Lemma 5.6,

$$\begin{aligned} & \left| \mathbb{E} \left[\int_{\underline{t}}^T (\mathbf{f}_n(\bar{s}, \Theta_s^{n, t, x}) - \mathbf{f}_n(s, \Theta_s^{n, t, x})) ds \right] \right| \\ & \leq c_{(5.27)} \int_{\underline{t}}^T (|f|_{\alpha; t}^{\text{loc}} + (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon) \frac{|\bar{s} - s|^{\alpha \wedge \frac{\varepsilon}{2}}}{(T - \bar{s} \wedge t_{n-1})^{\frac{1}{2}}} ds \\ & \leq c_{(5.27)} c (|f|_{\alpha; t}^{\text{loc}} + (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon) n^{-\alpha \wedge \frac{\varepsilon}{2}} \end{aligned}$$

for some $c = c(\alpha, \varepsilon, T) > 0$. Item (7.6) for the case $\beta = 0$ is shown in [BGGL21, Theorem 11]. The estimate concerning the Hölder continuity in time is changed as follows:

$$\begin{aligned} |R| &= \left| \int_{\underline{t}+h}^T \mathbb{E} \left[(\mathbf{f}_n(\bar{s}, \Theta_s^{n, t, x}) - \mathbf{f}_n(s, \Theta_s^{n, t, x})) \frac{B_s^{n, t, x} - x}{\underline{s} - \underline{t}} \right] ds \right| \\ &\leq c_{(5.27)} (|f|_{\alpha; t}^{\text{loc}} + (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon) \left(\frac{T}{n}\right)^{\alpha \wedge \frac{\varepsilon}{2}} \int_{\underline{t}+h}^T \frac{1}{(T - \bar{s} \wedge t_{n-1})^{\frac{1}{2}}} \frac{1}{\sqrt{s - (\underline{t} + h)}} ds \\ &\leq c' c_{(5.27)} (|f|_{\alpha; t}^{\text{loc}} + (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon) \left(\frac{T}{n}\right)^{\alpha \wedge \frac{\varepsilon}{2}}. \quad (7.7) \end{aligned}$$

Indeed, if $t < t_{n-2}$ we have

$$\begin{aligned} \int_{\underline{t}+h}^{t_{n-2}} \frac{1}{(T - \bar{s} \wedge t_{n-1})^{\frac{1}{2}}} \frac{1}{\sqrt{s - (\underline{t} + h)}} ds &= \int_{\underline{t}+h}^{t_{n-2}} \frac{1}{(T - \bar{s})^{\frac{1}{2}}} \frac{1}{\sqrt{s - (\underline{t} + h)}} ds \\ &\leq \sqrt{2} \int_{\underline{t}+h}^T \frac{1}{(T - s)^{\frac{1}{2}}} \frac{1}{\sqrt{s - (\underline{t} + h)}} ds = \sqrt{2} \mathbf{B}(\frac{1}{2}, \frac{1}{2}), \end{aligned}$$

where we used that $T + s = t_{n-2} + 2h + s \geq 2\bar{s}$ for $s < t_{n-2}$, implying $T - s \leq 2(T - \bar{s})$. Since $t < t_{n-2}$ implies $\underline{t} + h \leq t_{n-2}$ we have

$$\int_{t_{n-2}}^T \frac{1}{(T - \bar{s} \wedge t_{n-1})^{\frac{1}{2}}} \frac{1}{\sqrt{s - (\underline{t} + h)}} ds \leq \int_{t_{n-2}}^T \frac{1}{(T - t_{n-1})^{\frac{1}{2}}} \frac{1}{\sqrt{s - t_{n-2}}} ds = 2\sqrt{2}.$$

If $t \in [t_{n-1}, T]$ then $\underline{t} + h = T$ so it remains to check $t \in [t_{n-2}, t_{n-1}[$. In this case we have

$$\int_{\underline{t}+h}^T \frac{1}{(T - \bar{s} \wedge t_{n-1})^{\frac{1}{2}}} \frac{1}{\sqrt{s - (\underline{t} + h)}} ds = \int_{t_{n-1}}^T \frac{1}{(T - t_{n-1})^{\frac{1}{2}}} \frac{1}{\sqrt{s - t_{n-1}}} ds = 2.$$

To show (7.6) for $\beta > \frac{1}{2}$ we only need to make some slight changes in the proof of Theorem 11 where the statement was shown resulting in a singularity $\frac{1}{\sqrt{T-t}}$. The needed changes are as follows: As in [BGGL21, Theorem 11] we write

$$|\nabla u_{f_n}(t, x) - \nabla^h U_{f_n}^n(\underline{t} + h, x)| \leq |g \text{ difference}| + |f \text{ difference}|.$$

We will put

$$\Delta_{f_n}^n(t, x) := \nabla^h U_{f_n}^n(\underline{t} + h, x).$$

Using the representation (5.15) for $\nabla u_{f_n}(t, x)$ with $F(s, x) = f_n(s, x, u_{f_n}(s, x), \nabla u_{f_n}(s, x))$ and the representation

$$\begin{aligned} \nabla^h U_{f_n}^n(\underline{t} + h, x) &= \mathbb{E} \left[g(B_T^{n,t,x}) \frac{B_T^{n,t,x} - x}{T - \underline{t}} \right] \\ &\quad + \mathbb{E} \left[\int_{\underline{t}+h}^T f_n(\bar{s}, B_s^{n,t,x}, U_{f_n}^n(s, B_s^{n,t,x}), \Delta_{f_n}^n(s, B_s^{n,t,x})) \frac{B_s^{n,t,x} - x}{s - \underline{t}} ds \right], \end{aligned}$$

given in [BGGL21, equation (33)] and recalling that $B_s^{t,x} = x + B_s - B_t$ and $B_s^{n,t,x} = x + B_s^n - B_t^n$, we split the g difference into

$$\begin{aligned} |g \text{ difference}| &\leq \left| \mathbb{E} \left[g(B_T^{t,x})(B_T - B_t) \right] \left(\frac{1}{T - t} - \frac{1}{T - \underline{t}} \right) \right| \\ &\quad + \left| \frac{\mathbb{E} \left[g(B_T^{t,x})(B_T - B_t) \right] - \mathbb{E} \left[g(B_T^{n,t,x})(B_T^n - B_t^n) \right]}{T - \underline{t}} \right| \\ &=: S_1 + S_2. \end{aligned}$$

Then for S_1 we have

$$S_1 \leq |g|_\varepsilon \frac{(t - \underline{t})^{\frac{\varepsilon}{2}}}{(T - t)^{\frac{1-\varepsilon}{2}} (T - \underline{t})^{\frac{\varepsilon}{2}}} \leq |g|_\varepsilon \frac{(t - \underline{t})^{\frac{\varepsilon}{2}} \Phi_\beta((\bar{t} - \underline{t})^\varepsilon)}{(T - t)^{\frac{1-\varepsilon}{2}} (T - \underline{t})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{t})^\varepsilon)}.$$

For the second term we use [Lemma 7.2](#) and get

$$\begin{aligned} S_2 &\leq c_{(7.4)} |g|_\varepsilon \frac{\delta(t, T)^{\frac{1}{2}}}{(T - \underline{t}) \Phi_\beta((T - \underline{t})^\varepsilon)} n^{-\frac{\varepsilon}{2}} \Phi_\beta \left(\left(\frac{T}{n} \right)^\varepsilon \right) \\ &= c_{(7.4)} |g|_\varepsilon \frac{1}{(T - \underline{t})^{\frac{1}{2}} \Phi_\beta((T - \underline{t})^\varepsilon)} n^{-\frac{\varepsilon}{2}} \Phi_\beta \left(\left(\frac{T}{n} \right)^\varepsilon \right). \end{aligned}$$

Then, since $\bar{t} - \underline{t} = T/n$, we get

$$|g \text{ difference}| \leq \left[T^{\frac{\varepsilon}{2}} + c_{(7.4)} \right] |g|_\varepsilon \frac{1}{(T - t)^{\frac{1-\varepsilon}{2}} (T - \underline{t})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{t})^\varepsilon)} n^{-\frac{\varepsilon}{2}} \Phi_\beta \left(\left(\frac{T}{n} \right)^\varepsilon \right).$$

Analyzing the proof of [\[BGGL21, Theorem 11\]](#) it turns out that for

$$\beta_n(s) := \sup_{x \in \mathbb{R}} \left\{ \frac{|u_{f_n} - U_{f_n}^n|(s, x)}{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon} \right\} \quad \text{and} \quad \gamma_n(s) := \sup_{x \in \mathbb{R}} \left\{ \frac{|\nabla u_{f_n} - \Delta_{f_n}^n|(s, x)}{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon} \right\}$$

it holds (see [\[BGGL21, \(43\)\]](#) and [\[BGGL21, \(44\)\]](#) together with the estimates for $H(t)$ in this proof)

$$\gamma_n(t) \leq C \left(|g \text{ difference}| + \int_{(\underline{t}+h) \wedge T}^T (\beta_n(s) + \gamma_n(s)) \frac{ds}{\sqrt{s - \underline{t}}} + n^{-\alpha \wedge \frac{\varepsilon}{2}} \right),$$

where the estimate [\(7.7\)](#) for $|R|$ contributes to the last term. This implies, by the arguments in [\[BGGL21\]](#),

$$\gamma_n(t) \leq c_{(7.8)} \left[\frac{n^{-\alpha \wedge \frac{\varepsilon}{2}} \Phi_\beta \left(\left(\frac{T}{n} \right)^\varepsilon \right)}{(T - t)^{\frac{1-\varepsilon}{2}} (T - \underline{t})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{t})^\varepsilon)} + \int_{(\underline{t}+h) \wedge T}^T \gamma_n(s) \frac{ds}{\sqrt{s - \underline{t}}} \right] \quad \text{for } t \in [0, T[. \tag{7.8}$$

Observing that

$$\Phi_\beta \left(\left(\frac{T}{n} \right)^\varepsilon \right) \leq c_{(7.9)} |\log(n + 1)|^\beta \tag{7.9}$$

for $c_{(7.9)} = c_{(7.9)}(T, \varepsilon, \beta) > 0$, and inserting [\(7.8\)](#) into itself leads to

$$\begin{aligned} \gamma_n(t) &\leq c_{(7.8)} c_{(7.9)} \frac{|\log(n + 1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \frac{1}{(T - t)^{\frac{1-\varepsilon}{2}} (T - \underline{t})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{t})^\varepsilon)} \\ &\quad + c_{(7.8)}^2 c_{(7.9)} \frac{|\log(n + 1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \int_{(\underline{t}+h) \wedge T}^T \frac{1}{(T - s)^{\frac{1-\varepsilon}{2}} (T - \underline{s})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{s})^\varepsilon)} \frac{ds}{\sqrt{s - \underline{t}}} \\ &\quad + c_{(7.8)}^2 \int_{(\underline{t}+h) \wedge T}^T \int_{(\underline{s}+h) \wedge T}^T \gamma_n(r) \frac{dr}{\sqrt{r - \underline{s}}} \frac{ds}{\sqrt{s - \underline{t}}}. \end{aligned}$$

Since $T - \underline{s} \geq T - s$ and $\Phi_\beta((T - \underline{s})^\varepsilon) \geq 1$ by [Lemma 5.2](#), it holds that

$$(T - s)^{\frac{1-\varepsilon}{2}} (T - \underline{s})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{s})^\varepsilon) \geq (T - s)^{\frac{1}{2}}$$

and, in particular,

$$\int_{(\underline{t}+h) \wedge T}^T \frac{1}{(T - s)^{\frac{1-\varepsilon}{2}} (T - \underline{s})^{\frac{\varepsilon}{2}} \Phi_\beta((T - \underline{s})^\varepsilon)} \frac{ds}{\sqrt{s - \underline{t}}} \leq \int_{(\underline{t}+h) \wedge T}^T \frac{1}{(T - s)^{\frac{1}{2}}} \frac{ds}{\sqrt{s - \underline{t}}}.$$

In [BGGL21] (see the end of the proof) it is shown that

$$\int_{(t+h)\wedge T}^T \frac{1}{(T-s)^{\frac{1}{2}}} \frac{ds}{\sqrt{s-t}} \leq \mathbf{B}\left(\frac{1}{2}, \frac{1}{2}\right),$$

where \mathbf{B} denotes the beta function, and that

$$\int_{(t+h)\wedge T}^T \int_{(s+h)\wedge T}^T \gamma_n(r) \frac{dr}{\sqrt{r-s}} \frac{ds}{\sqrt{s-t}} \leq \int_{(t+2h)\wedge T}^T \gamma_n(r) dr.$$

Then Gronwall's lemma implies the assertion. \square

Proposition 7.4. *Suppose that Assumption 2.2 holds. Then for all $\beta > \frac{1}{2}$ and any $(t, x) \in [t_0, T] \times \mathbb{R}$, for all $n \geq n_0$, any continuous function $[t, T] \ni s \mapsto \mathbf{x}(s)$, and any function $[t, T] \ni s \mapsto \mathbf{x}^n(s)$ that is constant on the intervals $[t_{k-1}, t_k] \cap [t, T]$ one has*

$$|u_{f_n}(t, x) - U_{f_n}^n(t, y)| \leq c_{(7.10)} \left[|x - y|^\varepsilon + (1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}} \right], \quad (7.10)$$

$$\begin{aligned} \left(\int_t^T |\nabla u_{f_n}(s, \mathbf{x}(s)) - \Delta_{f_n}^n(s, \mathbf{x}^n(s))|^2 ds \right)^{\frac{1}{2}} &\leq c_{(7.11)} \left[\sup_{t \leq s \leq T} \Psi_{2\beta}(|\mathbf{x}(s) - \mathbf{x}^n(s)|^{2\varepsilon})^{\frac{1}{2}} \right. \\ &\quad \left. + \left(1 + \sup_{t \leq s \leq T} |\mathbf{x}^n(s)| + \|\xi\|_{L^{2\varepsilon}} \right)^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right], \end{aligned} \quad (7.11)$$

where $c_{(7.10)}, c_{(7.11)} > 0$ depend mostly on $(T, \varepsilon, \alpha, |g|_\varepsilon, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, L_f, c_{(5.28)}) > 0$ and $c_{(7.11)} > 0$ depends additionally on β .

Proof. By Lemma 5.1 and Lemma 7.3 we have

$$\begin{aligned} &|u_{f_n}(s, x) - U_{f_n}^n(s, y)| \\ &\leq |u_{f_n}(s, x) - u_{f_n}(s, y)| + |u_{f_n}(s, y) - U_{f_n}^n(s, y)| \\ &\leq c_{(5.2)} |x - y|^\varepsilon + c_{(7.5)} (1 + |y| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}} \\ &\leq c_{(5.2)} |x - y|^\varepsilon + c_{(7.5)} (1 + |x| + |x - y| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}} \\ &\leq [c_{(5.2)} + c_{(7.5)}] |x - y|^\varepsilon + c_{(7.5)} (1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}}. \end{aligned}$$

while Proposition 5.5(iii) and Lemma 7.3 imply

$$\begin{aligned} &\int_t^T \left[|\nabla u_{f_n}(s, \mathbf{x}(s)) - \Delta_{f_n}^n(s, \mathbf{x}^n(s))|^2 \right] ds \\ &\leq 2 \int_t^T \left[|\nabla u_{f_n}(s, \mathbf{x}(s)) - \nabla u_{f_n}(s, \mathbf{x}^n(s))|^2 + |\nabla u_{f_n}(s, \mathbf{x}^n(s)) - \Delta_{f_n}^n(s, \mathbf{x}^n(s))|^2 \right] ds \\ &\leq 2 \int_t^T \left[c_{(5.17)}^2 \frac{\Psi_{2\beta}(|\mathbf{x}(s) - \mathbf{x}^n(s)|^{2\varepsilon})}{(T-s)\Phi_\beta((T-s)^\varepsilon)^2} \right. \\ &\quad \left. + c_{(7.6)}^2 \frac{(1 + |\mathbf{x}^n(s)| + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} n^{-(2\alpha) \wedge \varepsilon} \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right)^2}{(T-s)^{1-\varepsilon}(T-s)^\varepsilon \Phi_\beta((T-s)^\varepsilon)^2} \right] ds \end{aligned}$$

$$\begin{aligned}
&\leq \left[2c_{(5.17)}^2 \int_t^T \frac{ds}{(T-s)\Phi_\beta((T-s)^\varepsilon)^2} \right] \sup_{t \leq s \leq T} \Psi_{2\beta}(|\mathbf{x}(s) - \mathbf{x}^n(s)|^{2\varepsilon}) \\
&\quad + \left[2c_{(7.6)}^2 \int_t^T \frac{ds}{(T-s)^{1-\varepsilon}(T-s)^\varepsilon \Phi_\beta((T-s)^\varepsilon)^2} \right] \times \\
&\quad \quad \times \left(1 + \sup_{t \leq s \leq T} |\mathbf{x}^n(s)| + \|\xi\|_{L^{2\varepsilon}} \right)^{2\varepsilon} n^{-(2\alpha) \wedge \varepsilon} \Phi_\beta \left(\left(\frac{T}{n} \right)^\varepsilon \right)^2 \\
&\leq c \left[\sup_{t \leq s \leq T} \Psi_{2\beta}(|\mathbf{x}(s) - \mathbf{x}^n(s)|^{2\varepsilon}) + \left(1 + \sup_{t \leq s \leq T} |\mathbf{x}^n(s)| + \|\xi\|_{L^{2\varepsilon}} \right)^{2\varepsilon} n^{-(2\alpha) \wedge \varepsilon} (\log n)^{2\beta} \right].
\end{aligned}$$

□

7.3 Comparing (Y^n, Z^n) and $(Y_{t_n}^n, Z_{t_n}^n)$

We proceed with comparing the solutions to (1.5) and (1.8). To do so we use the following a priori estimates:

Lemma 7.5 (A priori estimates). *For data $(\eta_i, f^i)_{i=1,2}$ with $\eta_i \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and the corresponding solutions $((Y_{t_k}^{n,i})_{k=j}^n, (Z_{t_k}^{n,i})_{k=j+1}^n)$ to the random walk BSDEs*

$$Y_s^{n,i} = \eta_i + \int_{]s,T]} f^i(r, B_r^n, Y_r^{n,i}, Z_r^{n,i}) d\langle B^n \rangle_r - \int_{]s,T]} Z_r^{n,i} dB_r^n, \quad s \in [t, T]$$

where $n \geq n_0$ (and where $f = f_2$ in (i)),

(i) one has for continuous $f^i : [t_0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$|f^i(t, x, y, z) - f^i(t, x', y', z')| \leq |f^i|_{\varepsilon; x} |x - x'|^\varepsilon + L_{f^i} (|y - y'| + |z - z'|).$$

for some $L_{f^i} > 0$ and for $k = 1, \dots, n$ that

$$\begin{aligned}
&\mathbb{E}_{k-1} \left[\sup_{k-1 \leq m \leq n} |Y_{t_m}^{n,1} - Y_{t_m}^{n,2}|^2 + \int_{]t_{k-1}, T]} |Z_s^{n,1} - Z_s^{n,2}|^2 d\langle B^n \rangle_s \right] \\
&\leq c_{(7.12)} \mathbb{E}_{k-1} \left[|\eta_1 - \eta_2|^2 \right. \\
&\quad \left. + \int_{]t_{k-1}, T]} |f^1(s, B_s^n, Y_s^{n,1}, Z_s^{n,1}) - f^2(s, B_s^n, Y_s^{n,1}, Z_s^{n,1})|^2 d\langle B^n \rangle_s \right], \quad (7.12)
\end{aligned}$$

where $c_{(7.12)} > 0$ depends at most on (T, L_{f_2}) .

(ii) For the special case $\eta_1 = \eta_2$,

$$\begin{aligned}
f^1(s, x, y, z) &:= f(s \wedge t_{n-1}, x, y, z, \mu_s^1, \nu_s^1), \\
f^2(s, x, y, z) &:= f(s \wedge t_{n-1}, x, y, z, \mu_s^2, \nu_s^2)
\end{aligned}$$

with f satisfying (2.1) and $\mu^i, \nu^i : [t_0, T] \rightarrow \mathcal{P}_2(\mathbb{R}) : t \mapsto \mu_t^i, t \mapsto \nu_t^i$ measurable w.r.t. $(\mathcal{B}([0, T]), \mathcal{B}(\mathcal{P}_2(\mathbb{R})))$ we get

$$\left| Y_{t_{k-1}}^{n,1} - Y_{t_{k-1}}^{n,2} \right|^2 + \mathbb{E}_{k-1} \int_{]t_{k-1}, T]} |Z_s^{n,1} - Z_s^{n,2}|^2 d\langle B^n \rangle_s$$

$$\leq c_{(7.13)} \left[\int_{]t_{k-1}, T]} \mathcal{W}_2^2(\mu_s^1, \mu_s^2) d\langle B^n \rangle_s + \int_{]t_{k-1}, T]} \mathcal{W}_2^2(\nu_s^1, \nu_s^2) d\langle B^n \rangle_s \right], \quad (7.13)$$

where $c_{(7.13)} > 0$ depends at most on (T, L_f) .

(iii) For the special case $\eta_1 = \eta_2$,

$$\begin{aligned} f^1(s, x, y, z) &:= f(s \wedge t_{n-1}, x, y, z, [Y_s^{n,1}], [Z_s^{n,1}]), \\ f^2(s, x, y, z) &:= f(s \wedge t_{n-1}, x, y, z, \mu_s^2, \nu_s^2) \end{aligned}$$

with f satisfying (2.1) and μ^2, ν^2 like in (ii), we get

$$\begin{aligned} &\mathbb{E} \left[\sup_{k-1 \leq m \leq n} |Y_{t_m}^{n,1} - Y_{t_m}^{n,2}|^2 + \int_{]t_{k-1}, T]} |Z_s^{n,1} - Z_s^{n,2}|^2 d\langle B^n \rangle_s \right] \\ &\leq c_{(7.14)} \left[\int_{]t_{k-1}, T]} \mathcal{W}_2^2(Y_s^{n,2}, \mu_s^2) d\langle B^n \rangle_s + \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_s^{n,2}, \nu_s^2) d\langle B^n \rangle_s \right], \end{aligned} \quad (7.14)$$

where $c_{(7.14)} > 0$ depends at most on (T, L_f) .

Proof. (i) We use the short notation

$$\begin{aligned} \Delta Y_{t_k}^{n,i} &:= Y_{t_k}^{n,i} - Y_{t_{k-1}}^{n,i} = -h f^i(t_k) + Z_{t_k}^{n,i} \Delta B_{t_k}^n, \\ f^i(t_k) &:= f^i(t_k, B_{t_{k-1}}^n, Y_{t_{k-1}}^{n,i}, Z_{t_k}^{n,i}), \\ Y_{t_k} &:= Y_{t_k}^{n,1} - Y_{t_k}^{n,2} \quad \text{and} \quad Z_{t_k} := Z_{t_k}^{n,1} - Z_{t_k}^{n,2}, \\ \eta &:= |\eta_1 - \eta_2| \quad \text{and} \quad f_{t_k} := f^1(t_k) - f^2(t_k), \\ f_{t_k}^\# &:= f^1(t_k, B_{t_{k-1}}^n, Y_{t_{k-1}}^{n,1}, Z_{t_k}^{n,1}) - f^2(t_k, B_{t_{k-1}}^n, Y_{t_{k-1}}^{n,1}, Z_{t_k}^{n,1}). \end{aligned}$$

Then

$$\begin{aligned} |Y_{t_m}|^2 - |Y_{t_{m-1}}|^2 &= |Y_{t_{m-1}} + \Delta Y_{t_m}|^2 - |Y_{t_{m-1}}|^2 \\ &= |\Delta Y_{t_m}|^2 + 2Y_{t_{m-1}} \Delta Y_{t_m} \\ &\geq -h^2 f_{t_m}^2 + \frac{1}{2} Z_{t_m}^2 (\Delta B_{t_m}^n)^2 + 2Y_{t_{m-1}} \Delta Y_{t_m}, \end{aligned}$$

where we used that $|\Delta Y_{t_m}|^2 = (-h f_{t_m} + Z_{t_m} \Delta B_{t_m}^n)^2 \geq -h^2 f_{t_m}^2 + \frac{1}{2} Z_{t_m}^2 (\Delta B_{t_m}^n)^2$. From this we derive exploiting relations like

$$\sum_{m=k}^n Y_{t_{m-1}} \Delta Y_{t_m} = \int_{]t_{k-1}, T]} Y_{s-} dY_s$$

the estimates

$$\begin{aligned} |Y_{t_{k-1}}|^2 &= \eta^2 - \sum_{m=k}^n [|Y_{t_m}|^2 - |Y_{t_{m-1}}|^2] \\ &\leq \eta^2 - 2 \int_{]t_{k-1}, T]} Y_{s-} dY_s + \sum_{m=k}^n (h^2 |f_{t_m}|^2 - \frac{1}{2} Z_{t_m}^2 (\Delta B_{t_m}^n)^2) \end{aligned}$$

$$\begin{aligned} &\leq \eta^2 + 2 \int_{]t_{k-1}, T]} |Y_{s-}| |f_s| d\langle B^n \rangle_s - 2 \int_{]t_{k-1}, T]} Y_{s-} Z_s dB_s^n \\ &\quad + \int_{]t_{k-1}, T]} h |f_s|^2 d\langle B^n \rangle_s - \frac{1}{2} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s. \end{aligned}$$

Assume $\delta > 0$. By Young's inequality we have $2|Y_{s-}| |f_s| \leq \frac{|Y_{s-}|^2}{\delta} + \delta |f_s|^2$ which implies

$$\begin{aligned} &|Y_{t_{k-1}}|^2 + \frac{1}{2} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \\ &\leq \eta^2 + \frac{1}{\delta} \int_{]t_{k-1}, T]} |Y_{s-}|^2 d\langle B^n \rangle_s + (\delta + h) \int_{]t_{k-1}, T]} |f_s|^2 d\langle B^n \rangle_s \\ &\quad - 2 \int_{]t_{k-1}, T]} Y_{s-} Z_s dB_s^n. \end{aligned}$$

From $|f_{t_k}|^2 \leq 2L_{f^2}^2 |Z_{t_k}|^2 + 4|f_{t_k}^-|^2 + 4L_{f^2}^2 |Y_{t_{k-1}}|^2$ and for

$$L_{f^2}^2 (\delta + h) < \frac{1}{8} \quad (7.15)$$

we obtain

$$\begin{aligned} &|Y_{t_{k-1}}|^2 + \frac{1}{4} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \\ &\leq |Y_{t_{k-1}}|^2 + \left(\frac{1}{2} - 2L_{f^2}^2 (\delta + h) \right) \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \\ &\leq \eta^2 + \left(\frac{1}{\delta} + 4L_{f^2}^2 (\delta + h) \right) \int_{]t_{k-1}, T]} |Y_{s-}|^2 d\langle B^n \rangle_s + 4(\delta + h) \int_{]t_{k-1}, T]} |f_s^-|^2 d\langle B^n \rangle_s \\ &\quad - 2 \int_{]t_{k-1}, T]} Y_{s-} Z_s dB_s^n \\ &\leq \eta^2 + \left(\frac{1}{\delta} + 4L_{f^2}^2 (\delta + h) \right) \int_{]t_{k-1}, T]} |Y_{s-}|^2 d\langle B^n \rangle_s + 4(\delta + h) \int_{]t_{k-1}, T]} |f_s^-|^2 d\langle B^n \rangle_s \\ &\quad - 2 \int_{]t_{k-1}, T]} Y_{s-} Z_s dB_s^n. \end{aligned}$$

If

$$h \left(\frac{1}{\delta} + 4L_{f^2}^2 (\delta + h) \right) \leq \frac{1}{2} \quad (7.16)$$

we write

$$\int_{]t_{k-1}, T]} |Y_{s-}|^2 d\langle B^n \rangle_s = |Y_{t_{k-1}}|^2 h + \int_{]t_{k-1}, T]} |Y_s|^2 d\langle B^n \rangle_s$$

and continue to

$$\begin{aligned} &\frac{1}{2} |Y_{t_{k-1}}|^2 + \frac{1}{4} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \\ &\leq \eta^2 + \left(\frac{1}{\delta} + 4L_{f^2}^2 (\delta + T) \right) \int_{]t_{k-1}, T]} |Y_s|^2 d\langle B^n \rangle_s + 4(\delta + T) \int_{]t_{k-1}, T]} |f_s^-|^2 d\langle B^n \rangle_s \end{aligned}$$

$$-2 \int_{]t_{k-1}, T]} \Upsilon_{s-} Z_s dB_s^n. \quad (7.17)$$

Taking the conditional expectation we get for $j < l \leq k \leq n$ that

$$\begin{aligned} & \frac{1}{2} \mathbb{E}_{l-1} |\Upsilon_{t_{k-1}}|^2 + \frac{1}{4} \mathbb{E}_{l-1} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \\ & \leq \mathbb{E}_{l-1} \eta^2 + \left(\frac{1}{\delta} + 4L_{f^2}^2(\delta + T) \right) \int_{]t_{k-1}, T]} \mathbb{E}_{l-1} |\Upsilon_s|^2 d\langle B^n \rangle_s \\ & \quad + 4(\delta + T) \mathbb{E}_{l-1} \int_{]t_{l-1}, T]} |f_s^-|^2 d\langle B^n \rangle_s \end{aligned} \quad (7.18)$$

where we enlarged the RHS by extending $]t_{k-1}, T]$ to $]t_{l-1}, T]$. Gronwall's [Lemma 9.1](#) implies

$$\mathbb{E}_{l-1} |\Upsilon_{t_{k-1}}|^2 \leq c_{(7.19)} \left[\mathbb{E}_{l-1} \eta^2 + \mathbb{E}_{l-1} \int_{]t_{l-1}, T]} |f_s^-|^2 d\langle B^n \rangle_s \right] \quad (7.19)$$

with $c_{(7.19)} = c_{(7.19)}(T, L_{f^2}, \delta) > 0$. Denoting

$$K := \eta^2 + \int_{]t_{k-1}, T]} |f_s^-|^2 d\langle B^n \rangle_s$$

the relations (7.18) and (7.19) imply

$$\mathbb{E}_{k-1} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \leq c_{(7.20)} \mathbb{E}_{k-1} K \quad (7.20)$$

with $c_{(7.20)} = c_{(7.20)}(T, L_{f^2}, \delta) > 0$. Coming again back to (7.17), similarly as before we apply Gronwall's inequality, where we additionally estimate the stochastic integral by its supremum, and get

$$\Upsilon_*^2 \leq c_{(7.21)} \left(K + \sup_{k \leq m \leq n} \left| \int_{]t_{m-1}, T]} \Upsilon_{s-} Z_s dB_s^n \right| \right) \quad (7.21)$$

with $\Upsilon_* := \sup_{m=k-1, \dots, n} |\Upsilon_{t_m}|$ and $c_{(7.21)} = c_{(7.21)}(T, L_{f^2}, \delta) > 0$. By the conditional BDG inequality for $p = 1$ with constant $\beta_1 > 0$ and Young's inequality we have for $\lambda > 0$ that

$$\begin{aligned} \frac{1}{\beta_1} \mathbb{E}_{k-1} \sup_{k \leq m \leq n} \left| \int_{]t_{m-1}, T]} \Upsilon_{s-} Z_s dB_s^n \right| & \leq \mathbb{E}_{k-1} \left| \int_{]t_{k-1}, T]} |\Upsilon_{s-}|^2 Z_s^2 d\langle B^n \rangle_s \right|^{\frac{1}{2}} \\ & \leq \lambda \mathbb{E}_{k-1} \Upsilon_*^2 + \frac{1}{\lambda} \mathbb{E}_{k-1} \left| \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \right|. \end{aligned} \quad (7.22)$$

By (7.21), (7.22), and (7.20) we conclude

$$\mathbb{E}_{k-1} \Upsilon_*^2 \leq c_{(7.21)} \mathbb{E}_{k-1} \left(K + \sup_{k \leq m \leq n} \left| \int_{]t_{m-1}, T]} \Upsilon_{s-} Z_s dB_s^n \right| \right)$$

$$\begin{aligned}
&\leq c_{(7.21)} \left(\mathbb{E}_{k-1} K + \beta_1 \lambda \mathbb{E}_{k-1} \Upsilon_*^2 + \frac{\beta_1}{\lambda} \mathbb{E}_{k-1} \left| \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \right| \right) \\
&\leq c_{(7.21)} \left(\mathbb{E}_{k-1} K + \beta_1 \lambda \mathbb{E}_{k-1} \Upsilon_*^2 + \frac{\beta_1}{\lambda} c_{(7.20)} \mathbb{E}_{k-1} K \right).
\end{aligned}$$

For small enough $\lambda > 0$ we conclude

$$\mathbb{E}_{k-1} \Upsilon_*^2 \leq c_{(7.23)} \mathbb{E}_{k-1} K. \quad (7.23)$$

for $c_{(7.23)} = c_{(7.23)}(T, L_{f^2}, \delta) > 0$. Now (7.12) follows from (7.20) and from the last inequality.

(ii) By (2.1) we observe that

$$\begin{aligned}
&|f^1(s, x, y, z) - f^2(s, x, y, z)|^2 \\
&= |f(s \wedge t_{n-1}, x, y, z, \mu_s^1, \nu_s^1) - f(s \wedge t_{n-1}, x, y, z, \mu_s^2, \nu_s^2)|^2 \\
&\leq 2L_f^2 \left[\mathcal{W}_2^2(\mu_s^1, \mu_s^2) + \mathcal{W}_2^2(\nu_s^1, \nu_s^2) \right].
\end{aligned}$$

The assertion follows by (i).

(iii) We use

$$\begin{aligned}
|f_s^-| &\leq L_f (\mathcal{W}_2(Y_s^{n,1}, \mu_s^2) + \mathcal{W}_2(Z_s^{n,1}, \nu_s^2)) \\
&\leq L_f [\|\Upsilon_s\|_{L^2} + \|Z_s\|_{L^2} + \mathcal{W}_2(Y_s^{n,2}, \mu_s^2) + \mathcal{W}_2(Z_s^{n,2}, \nu_s^2)],
\end{aligned} \quad (7.24)$$

so that, with $|f_s^{\mathcal{W}}| := \mathcal{W}_2(Y_s^{n,2}, \mu_s^2) + \mathcal{W}_2(Z_s^{n,2}, \nu_s^2)$, we have

$$|f_s^-|^2 \leq 2L_f^2 \|Z_s\|_{L^2}^2 + 4L_f^2 (\|\Upsilon_s\|_{L^2}^2 + |f_s^{\mathcal{W}}|^2).$$

If we take the expectation in (7.17) and denote

$$K' := \eta^2 + \int_{]t_{k-1}, T]} |f_s^{\mathcal{W}}|^2 d\langle B^n \rangle_s$$

we get

$$\begin{aligned}
&\mathbb{E}|\Upsilon_{t_{k-1}}|^2 + \frac{1}{4} \mathbb{E} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \\
&\leq \mathbb{E}\eta^2 + \left(\frac{1}{\delta} + 4L_{f^2}^2(\delta + T) \right) \mathbb{E} \int_{]t_{k-1}, T]} \mathbb{E}|\Upsilon_s|^2 d\langle B^n \rangle_s + 16(\delta + T)L_f^2 \int_{]t_{k-1}, T]} \mathbb{E}|\Upsilon_s|^2 d\langle B^n \rangle_s \\
&\quad + 16L_f^2(\delta + T) \int_{]t_{k-1}, T]} |f_s^{\mathcal{W}}|^2 d\langle B^n \rangle_s \\
&\leq c_{(7.25)} \left[\int_{]t_{k-1}, T]} \mathbb{E}|\Upsilon_s|^2 d\langle B^n \rangle_s + \mathbb{E}K' \right].
\end{aligned} \quad (7.25)$$

with $c_{(7.25)} = c_{(7.25)}(T, L_f, \delta) > 0$. By Gronwall's inequality

$$\mathbb{E}|\Upsilon_{t_{k-1}}|^2 \leq c_{(7.26)} \mathbb{E}K' \quad (7.26)$$

for $c_{(7.26)} = c_{(7.26)}(T, L_f, \delta) > 0$. Therefore we get

$$\mathbb{E} \int_{]t_{k-1}, T]} Z_s^2 d\langle B^n \rangle_s \leq c_{(7.27)} \mathbb{E} K' \quad (7.27)$$

for $c_{(7.27)} = c_{(7.27)}(T, L_f, \delta) > 0$. Taking the expectation in (7.12), we may now conclude the statement as we have

$$\mathbb{E} \int_{]t_{k-1}, T]} |f_s^-|^2 d\langle B^n \rangle_s \leq c_{(7.28)} \left[\mathbb{E} \eta^2 + \int_{]t_{k-1}, T]} |f_s^{\mathcal{W}}|^2 d\langle B^n \rangle_s \right] \quad (7.28)$$

where $c_{(7.28)} = c_{(7.28)}(T, L_f, \delta) > 0$. Finally we choose $\delta > 0$ in (7.15) such that $L_{f^2}^2 \delta = \frac{1}{16}$ and can arrange some $n_0 = n_0(T, L_f) \in \mathbb{N}$ such that for $n \geq n_0$ the conditions (7.15) and (7.16) are satisfied. \square

Lemma 7.6. *Assume that Assumption 2.2 holds and let $\beta > \frac{1}{2}$ or $\beta = 0$. Then there exists a constant $c = c(T, \varepsilon, \alpha, \beta, |g|_\varepsilon, |f|_{\alpha; t}^{\text{loc}}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, \xi) > 0$ such that for all $x \in \mathbb{R}$ and $n \geq n_0$,*

- (i) $\mathcal{W}_2 \left(Y_{f_n, s}^{t, x}, Y_{f_n, s}^{n, t, x} \right) \leq c(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}}$ for all $t_0 \leq t \leq s \leq T$,
- (ii) $\mathcal{W}_2 \left(Z_{f_n, s}^{t, x}, Z_{f_n, s}^{n, t, x} \right) \leq c \frac{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)} \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}}$ for all $t_0 \leq t \leq s < T$.

Proof. Assertion (i) as well as (ii) for $\beta = 0$ are proved in [BGGL21, Theorem 12]. To show (ii) for $\beta > \frac{1}{2}$, we use that for $0 \leq t < s \leq T$ with $s \neq \underline{s}$ it holds (6.8), moreover we have $Z_{f_n, s}^{t, x} = \nabla u_{f_n}(s, B_s^{t, x})$, so that

$$\begin{aligned} \mathcal{W}_2 \left(Z_{f_n, s}^{t, x}, Z_{f_n, s}^{n, t, x} \right) &= \mathcal{W}_2 \left(\nabla u_{f_n}(s, B_s^{t, x}), \Delta_{f_n}^n(s, B_s^{n, t, x}) \right) \\ &\leq \mathcal{W}_2 \left(\nabla u_{f_n}(s, B_s^{t, x}), \nabla u_{f_n}(s, B_s^{n, t, x}) \right) + \mathcal{W}_2 \left(\nabla u_{f_n}(s, B_s^{n, t, x}), \Delta_{f_n}^n(s, B_s^{n, t, x}) \right). \end{aligned}$$

Using Proposition 5.5(iii) implies

$$\|\nabla u_{f_n}(s, B_s^{t, x}) - \nabla u_{f_n}(s, B_s^{n, t, x})\|_{L^2} \leq c_{(5.17)} \frac{\left(\mathbb{E} \Psi_{2\beta}(|B_s^{t, x} - B_s^{n, t, x}|^{2\varepsilon}) \right)^{\frac{1}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)}.$$

By Lemma 5.2(iii) the function $\Psi_{2\beta}$ is concave so that by Jensen's inequality

$$\mathbb{E} \Psi_{2\beta}(|\mathcal{B}_s^{t, x} - \mathcal{B}_s^{n, t, x}|^{2\varepsilon}) \leq \Psi_{2\beta}(\mathbb{E} |\mathcal{B}_s^{t, x} - \mathcal{B}_s^{n, t, x}|^{2\varepsilon}) \leq C \frac{|\log(n+1)|^{2\beta}}{n^\varepsilon}.$$

For the last inequality we have used [BGGL21, Proposition 4] which states that $\mathcal{W}_p(\mathcal{B}_s, \mathcal{B}_s^n) \leq c_p \left(\frac{T}{n}\right)^{\frac{1}{2}}$, hence there exists a couple $(\mathcal{B}_s, \mathcal{B}_s^n)$ such that $\mathbb{E} |\mathcal{B}_s^{t, x} - \mathcal{B}_s^{n, t, x}|^{2\varepsilon} \leq cn^{-\varepsilon}$ with $c = c(\varepsilon, T)$. This yields

$$\mathcal{W}_2 \left(\nabla u_{f_n}(s, B_s^{t, x}), \nabla u_{f_n}(s, B_s^{n, t, x}) \right) \leq \frac{C}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)} \frac{|\log(n+1)|^\beta}{n^{\frac{\varepsilon}{2}}}.$$

By Lemma 7.3,

$$\begin{aligned}
& \|\nabla u_{f_n}(s, B_s^{n,t,x}) - \Delta_{f_n}^n(s, B_s^{n,t,x})\|_{L^2} \\
& \leq c_{(7.6)} \frac{\left\| (1 + |B_s^{n,t,x}| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \right\|_{L^2} n^{-\alpha \wedge \frac{\varepsilon}{2}}}{(T-s)^{\frac{1-\varepsilon}{2}} (T-\underline{s})^{\frac{\varepsilon}{2}} \Phi_\beta((T-\underline{s})^\varepsilon)} \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right) \\
& \leq c \frac{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)} n^{\alpha \wedge \frac{\varepsilon}{2}} \Phi_\beta\left(\left(\frac{T}{n}\right)^\varepsilon\right)
\end{aligned}$$

since $T - \underline{s} \geq T - s$ and $r \mapsto r \Phi_\beta(r^2)$ is non-decreasing by Lemma 5.2(ii). Hence for $s \neq \underline{s}$,

$$\mathcal{W}_2\left(Z_{f_n,s}^{t,x}, Z_{f_n,s}^{n,t,x}\right) \leq C \frac{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)} \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}}. \quad (7.29)$$

In the case where $s = \underline{s}$ we have

$$Z_{f_n,s}^{n,t,x} = \nabla^n U_{f_n}^n(s, B_{s^-}^{n,t,x}) = \Delta_{f_n}^n(s - h/2, B_{s-h/2}^{n,t,x})$$

which is not equal to $\Delta^n(s, B_s^{n,t,x})$ in general. We have similar to (7.37), by (7.29) and (5.9),

$$\begin{aligned}
\mathcal{W}_2\left(Z_{f_n,s}^{t,x}, Z_{f_n,s}^{n,t,x}\right) &= \mathcal{W}_2\left(Z_{f_n,s}^{t,x}, Z_{f_n,s-h/2}^{n,t,x}\right) \\
&\leq \mathcal{W}_2\left(Z_{f_n,s}^{t,x}, Z_{f_n,s-h/2}^{t,x}\right) + \mathcal{W}_2\left(Z_{f_n,s-h/2}^{t,x}, Z_{f_n,s-h/2}^{n,t,x}\right) \\
&\leq c_1 \left[\frac{(\mathbb{E} \Psi_{2\beta}(|B_s - B_{s-h/2}|^{2\varepsilon}))^{\frac{1}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)} + \frac{\Psi_{2\beta}(|h/2|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)} \right. \\
&\quad \left. + \frac{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon (\log n)^\beta n^{-\alpha \wedge \frac{\varepsilon}{2}}}{(T-(s-h/2))^{\frac{1}{2}} \Phi_\beta((T-(s-h/2))^\varepsilon)} \right] \\
&\leq c_2 \left[\frac{n^{-\frac{\varepsilon}{2}} |\log(n+1)|^\beta + \Psi_{2\beta}(|h/2|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)} \right. \\
&\quad \left. + \frac{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon |\log(n+1)|^\beta n^{-\alpha \wedge \frac{\varepsilon}{2}}}{(T-(s-h/2))^{\frac{1}{2}} \Phi_\beta((T-(s-h/2))^\varepsilon)} \right] \\
&\leq c_3 \frac{(1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon |\log(n+1)|^\beta n^{-\alpha \wedge \frac{\varepsilon}{2}}}{(T-s)^{\frac{1}{2}} \Phi_\beta((T-s)^\varepsilon)},
\end{aligned}$$

where we used for the denominator that $r \mapsto r^a \Phi_\beta(r^2)$ is non-decreasing for $a := \frac{1}{\varepsilon} \geq 1$. \square

Proposition 7.7. *Assume that Assumption 2.2 holds and assume that $t_{k-1} \geq t_\circ$ and $x \in \mathbb{R}$. Then for the solutions to (1.5) and (1.8) it holds for any $n \geq n_0$ that*

$$\mathbb{E} \left[\sup_{k-1 \leq m \leq n} \left| Y_{t_m}^{n,t_\circ,\xi} - Y_{f_n,t_m}^{n,t_\circ,\xi} \right|^2 + \int_{]t_{k-1}, T[} \left| Z_s^{n,t_\circ,\xi} - Z_{f_n,s}^{n,t_\circ,\xi} \right|^2 d\langle B^n \rangle_s \right]$$

$$\leq c_{(7.30)}^2 (1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} \frac{|\log(n+1)|^{2\beta}}{n^{2(\alpha \wedge \frac{\varepsilon}{2})}}, \quad (7.30)$$

where $c_{(7.30)} = c_{(7.30)}(T, \varepsilon, \alpha, \beta, |g|_\varepsilon, |f|_{\alpha;t}^{loc}, |f|_{\varepsilon;x}, L_f, c_{f_0}^{(2)}, f_0, \xi) > 0$, and

$$\begin{aligned} & |U^n(t_{k-1}, x) - U_{f_n}^n(t_{k-1}, x)| + \left(\mathbb{E}_{t_{k-1}, x} \int_{]t_{k-1}, T[} |\Delta^n(s, B_s^n) - \Delta_{f_n}^n(s, B_s^n)|^2 d\langle B^n \rangle_s \right)^{\frac{1}{2}} \\ & \leq c_{(7.31)} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}}, \quad (7.31) \end{aligned}$$

where $c_{(7.31)} = c_{(7.31)}(T, \varepsilon, \alpha, \beta, |g|_\varepsilon, |f|_{\alpha;t}^{loc}, |f|_{\varepsilon;x}, L_f, c_{f_0}^{(2)}, \xi) > 0$. Moreover, there is an absolute constant $c_{(7.32)} > 0$ such that

$$\begin{aligned} & \left| \int_{]t_{k-1}, T[} |\Delta^n(s, B_s^{n, t_{k-1}, x}) - \Delta_{f_n}^n(s, B_s^{n, t_{k-1}, x})|^2 d\langle B^n \rangle_s \right|_{M_{\varphi_1}}^{\frac{1}{2}} \\ & \leq c_{(7.32)} c_{(7.31)} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}}. \quad (7.32) \end{aligned}$$

Proof. Inequality (7.30): We use Lemma 7.5(iii) for $\eta_1 = \eta_2 = g(B_T^{n, t_0, \xi})$ and

$$\begin{aligned} f^1(s, x, y, z) &:= f(s \wedge t_{n-1}, x, y, z, [Y_s^{n, t_0, \xi}], [Z_s^{n, t_0, \xi}]) \\ f^2(s, x, y, z) &:= f(s \wedge t_{n-1}, x, y, z, [Y_s^{t_0, \xi}], [Z_{s \wedge t_{n-1}}^{t_0, \xi}]) \end{aligned}$$

to get, for $C = C(L_f, T)$,

$$\begin{aligned} & \mathbb{E} \sup_{k-1 \leq m \leq n} |Y_{t_m}^{n, t_0, \xi} - Y_{f_n, t_m}^{n, t_0, \xi}|^2 + \mathbb{E} \int_{]t_{k-1}, T[} |Z_s^{n, t_0, \xi} - Z_{f_n, s}^{n, t_0, \xi}|^2 d\langle B^n \rangle_s \\ & \leq c_{(7.14)} \left[\int_{]t_{k-1}, T[} \mathcal{W}_2^2(Y_s^{t_0, \xi}, Y_{f_n, s}^{n, t_0, \xi}) d\langle B^n \rangle_s + \int_{]t_{k-1}, T[} \mathcal{W}_2^2(Z_{s \wedge t_{n-1}}^{t_0, \xi}, Z_{f_n, s}^{n, t_0, \xi}) d\langle B^n \rangle_s \right] \quad (7.33) \end{aligned}$$

$$\leq 3c_{(7.14)} \left[\int_{]t_{k-1}, T[} \mathcal{W}_2^2(Y_s^{t_0, \xi}, Y_{f_n, s}^{t_0, \xi}) d\langle B^n \rangle_s + \int_{]t_{k-1}, T[} \mathcal{W}_2^2(Z_{s \wedge t_{n-1}}^{t_0, \xi}, Z_{f_n, s \wedge t_{n-1}}^{t_0, \xi}) d\langle B^n \rangle_s \right] \quad (7.34)$$

$$+ \int_{]t_{k-1}, T[} \mathcal{W}_2^2(Y_{f_n, s}^{t_0, \xi}, Y_{f_n, s}^{n, t_0, \xi}) d\langle B^n \rangle_s + \int_{]t_{k-1}, T[} \mathcal{W}_2^2(Z_{f_n, s \wedge t_{n-1}}^{t_0, \xi}, Z_{f_n, s \wedge t_{n-1}}^{n, t_0, \xi}) d\langle B^n \rangle_s \quad (7.35)$$

$$+ \int_{]t_{n-1}, T[} \mathcal{W}_2^2(Z_{f_n, s \wedge t_{n-1}}^{n, t_0, \xi}, Z_{f_n, s}^{n, t_0, \xi}) d\langle B^n \rangle_s. \quad (7.36)$$

We start with estimating (7.34). By Proposition 7.1, exploiting that $Y_s^{t_0, \xi} = u(s, B_s^{t_0, \xi})$ and $Y_{f_n, s}^{t_0, \xi} = u_{f_n}(s, B_s^{t_0, \xi})$,

$$\int_{]t_{k-1}, T[} \mathcal{W}_2^2(Y_s^{t_0, \xi}, Y_{f_n, s}^{t_0, \xi}) d\langle B^n \rangle_s \leq \int_{]t_{k-1}, T[} \mathbb{E} |Y_s^{t_0, \xi} - Y_{f_n, s}^{t_0, \xi}|^2 d\langle B^n \rangle_s \leq T c_{(7.2)}^2 n^{-2(\alpha \wedge \frac{\varepsilon}{2} + \frac{1}{2})}.$$

The second term of (7.34) we rewrite as integral w.r.t. ds and then decompose:

$$\begin{aligned}
& \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{s \wedge t_{n-1}}^{t_o, \xi}, Z_{f_{n, s \wedge t_{n-1}}}^{t_o, \xi}) d\langle B^n \rangle_s \\
&= \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{\bar{s} \wedge t_{n-1}}^{t_o, \xi}, Z_{f_{n, \bar{s} \wedge t_{n-1}}}^{t_o, \xi}) ds \\
&\leq 3 \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{\bar{s} \wedge t_{n-1}}^{t_o, \xi}, Z_{s \wedge t_{n-1}}^{t_o, \xi}) ds + 3 \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{s \wedge t_{n-1}}^{t_o, \xi}, Z_{f_{n, s \wedge t_{n-1}}}^{t_o, \xi}) ds \\
&\quad + 3 \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{f_{n, s \wedge t_{n-1}}}^{t_o, \xi}, Z_{f_{n, \bar{s} \wedge t_{n-1}}}^{t_o, \xi}) ds.
\end{aligned}$$

Proceeding like in (5.32), thanks to Proposition 5.8 we may apply Proposition 5.5 so that by (5.13) for any $s, s' \in [t_o, T[$

$$\begin{aligned}
\|Z_{s'}^{t_o, \xi} - Z_s^{t_o, \xi}\|_{L^2} &\leq \frac{c_{(5.17)} (\mathbb{E} \Psi_{2\beta}(|B_{s'} - B_s|^{2\varepsilon}))^{\frac{1}{2}}}{(T - s' \vee s)^{\frac{1}{2}} \Phi_\beta((T - s' \vee s)^\varepsilon)} + \frac{c_{(5.18)} \Psi_{2\beta}(|s' - s|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T - s' \vee s)^{\frac{1}{2}} \Phi_\beta((T - s' \vee s)^\varepsilon)} \\
&\leq c_{(7.37)} \frac{\Psi_{2\beta}(|s' - s|^\varepsilon b_\varepsilon)^{\frac{1}{2}}}{(T - s' \vee s)^{\frac{1}{2}} \Phi_\beta((T - s' \vee s)^\varepsilon)}. \tag{7.37}
\end{aligned}$$

This yields to

$$\begin{aligned}
\int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{\bar{s} \wedge t_{n-1}}^{t_o, \xi}, Z_{s \wedge t_{n-1}}^{t_o, \xi}) ds &\leq \int_{]t_{k-1}, t_{n-1}] } \|Z_{\bar{s}}^{t_o, \xi} - Z_s^{t_o, \xi}\|_{L^2}^2 ds \\
&\leq c_{(7.37)}^2 \Psi_{2\beta}(h^\varepsilon b_\varepsilon) \int_{]t_{k-1}, t_{n-1}] } \frac{1}{(T - \bar{s}) \Phi_\beta((T - \bar{s})^\varepsilon)^2} ds \\
&\leq C n^{-\varepsilon} |\log(n+1)|^{2\beta},
\end{aligned}$$

where for last inequality also Remark 5.3 was used. By the same arguments we get

$$\int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{f_{n, s \wedge t_{n-1}}}^{t_o, \xi}, Z_{f_{n, \bar{s} \wedge t_{n-1}}}^{t_o, \xi}) ds \leq C n^{-\varepsilon} |\log(n+1)|^{2\beta}.$$

The remaining term we split again into three terms,

$$\begin{aligned}
& \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{s \wedge t_{n-1}}^{t_o, \xi}, Z_{f_{n, s \wedge t_{n-1}}}^{t_o, \xi}) ds \\
&\leq 3 \int_{]t_{n-1}, T]} \|Z_{s \wedge t_{n-1}}^{t_o, \xi} - Z_s^{t_o, \xi}\|_{L^2}^2 ds + 3 \int_{]t_{k-1}, T]} \|Z_s^{t_o, \xi} - Z_{f_{n, s}}^{t_o, \xi}\|_{L^2}^2 ds \\
&\quad + 3 \int_{]t_{n-1}, T]} \|Z_{f_{n, s}}^{t_o, \xi} - Z_{f_{n, s \wedge t_{n-1}}}^{t_o, \xi}\|_{L^2}^2 ds \\
&\leq c_{(7.2)}^2 n^{-((\alpha \wedge \frac{\varepsilon}{2}) + \frac{1}{2})} + C' n^{-\varepsilon} |\log(n+1)|^{2\beta} \leq C n^{-\varepsilon} |\log(n+1)|^{2\beta},
\end{aligned}$$

where we applied (7.2) for the second term on the RHS, and (7.37) together with Remark 5.3 for the first and third term. Indeed, thanks to Proposition 5.8 we may use (7.37) for $Z_{f_{n, s}}^{t_o, \xi}$ as well.

Hence the resulting bound for (7.34) is $C n^{-\varepsilon} |\log(n+1)|^{2\beta}$.

Regarding (7.35), we apply Lemma 7.6 which gives

$$\begin{aligned} & \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Y_{f_n, s}^{t_o, \xi}, Y_{f_n, s}^{n, t_o, \xi}) d\langle B^n \rangle_s + \int_{]t_{k-1}, T]} \mathcal{W}_2^2(Z_{f_n, s \wedge t_{n-1}}^{t_o, \xi}, Z_{f_n, s \wedge t_{n-1}}^{n, t_o, \xi}) d\langle B^n \rangle_s \\ & \leq \frac{C(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon}}{n^{2(\alpha \wedge \frac{\varepsilon}{2})}} + C(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} \frac{(\log n)^{2\beta}}{n^{2(\alpha \wedge \frac{\varepsilon}{2})}} \int_{]t_{k-1}, t_{n-1}]} \frac{1}{(T - \bar{s}) \Phi_\beta((T - \bar{s})^\varepsilon)^2} ds \\ & \leq C(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} \frac{(\log n)^{2\beta}}{n^{2(\alpha \wedge \frac{\varepsilon}{2})}}, \end{aligned}$$

by using Remark 5.3, and where $C = C(T, \varepsilon, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, |g|_\varepsilon)$.

Finally we estimate (7.36):

$$\begin{aligned} \int_{]t_{n-1}, T]} \mathcal{W}_2^2(Z_{f_n, s \wedge t_{n-1}}^{n, t_o, \xi}, Z_{f_n, s}^{n, t_o, \xi}) d\langle B^n \rangle_s &= \mathcal{W}_2^2(Z_{f_n, t_{n-1}}^{n, t_o, \xi}, Z_{f_n, T}^{n, t_o, \xi}) h \\ &\leq h \left(\mathbb{E} |Z_{f_n, t_{n-1}}^{n, t_o, \xi} - Z_{f_n, T}^{n, t_o, \xi}|^2 \right) \\ &\leq 2h \left(\mathbb{E} |Z_{f_n, t_{n-1}}^{n, t_o, \xi}|^2 + \mathbb{E} |Z_{f_n, T}^{n, t_o, \xi}|^2 \right). \end{aligned} \quad (7.38)$$

For the solution to (1.8) we use the scheme (6.3)-(6.4). Hence by (6.3) we have (denoting by \tilde{E} the expectation w.r.t. $\tilde{\zeta}_n$, independent of $B_{t_{n-1}}^{n, t_o, \xi}$)

$$\begin{aligned} |Z_{f_n, t_n}^{n, t_o, \xi}| &= \left| h^{-1/2} \mathbb{E} \left[Y_{f_n, t_n}^{n, t_o, \xi} \zeta_n \mid \mathcal{F}_{t_{n-1}}^n \right] \right| = |h^{-1/2} \tilde{E} g(B_{t_{n-1}}^{n, t_o, \xi} + \sqrt{h} \tilde{\zeta}_n) \tilde{\zeta}_n| \\ &\leq \frac{|g(B_{t_{n-1}}^{n, t_o, \xi} + \sqrt{h}) - g(B_{t_{n-1}}^{n, t_o, \xi} - \sqrt{h})|}{2\sqrt{h}} \leq C |g|_\varepsilon h^{\frac{\varepsilon-1}{2}}. \end{aligned}$$

By (6.4) and (6.3)

$$\begin{aligned} Z_{f_n, t_{n-1}}^{n, t_o, \xi} &= h^{-1/2} \mathbb{E} \left[Y_{f_n, t_{n-1}}^{n, t_o, \xi} \zeta_{n-1} \mid \mathcal{F}_{t_{n-2}}^n \right] \\ &= h^{-1/2} \tilde{E} \left[g(B_{t_{n-2}}^{n, t_o, \xi} + \sqrt{h} \tilde{\zeta}_{n-1} + \sqrt{h} \tilde{\zeta}_n) \tilde{\zeta}_{n-1} \right] \\ &\quad + h^{1/2} \mathbb{E} \left[f_n \left(t_{n-1}, B_{t_{n-1}}^{n, t_o, \xi}, Y_{f_n, t_{n-1}}^{n, t_o, \xi}, Z_{f_n, t_{n-1}}^{n, t_o, \xi} \right) \zeta_{n-1} \mid \mathcal{F}_{t_{n-2}}^n \right]. \end{aligned}$$

Like above,

$$\left| h^{-1/2} \tilde{E} \left[g(B_{t_{n-2}}^{n, t_o, \xi} + \sqrt{h} \tilde{\zeta}_{n-1} + \sqrt{h} \tilde{\zeta}_n) \tilde{\zeta}_{n-1} \right] \right| \leq C |g|_\varepsilon h^{\frac{\varepsilon-1}{2}}.$$

For the second term we estimate the second moment. By Jensen's inequality, (2.1) and the above estimates,

$$\begin{aligned} & \mathbb{E} \left| h^{1/2} \mathbb{E} \left[f_n \left(t_{n-1}, B_{t_{n-1}}^{n, t_o, \xi}, Y_{f_n, t_{n-1}}^{n, t_o, \xi}, Z_{f_n, t_{n-1}}^{n, t_o, \xi} \right) \zeta_{n-1} \mid \mathcal{F}_{t_{n-2}}^n \right] \right|^2 \\ & \leq h \mathbb{E} \left| f \left(t_{n-1}, B_{t_{n-1}}^{n, t_o, \xi}, Y_{f_n, t_{n-1}}^{n, t_o, \xi}, Z_{f_n, t_{n-1}}^{n, t_o, \xi}, [Y_{t_n}^{t_o, \xi}], [Z_{t_{n-1}}^{t_o, \xi}] \right) \right|^2 \\ & \leq hC \left(f_0^2 + (|f|_{\alpha; t}^{loc} + (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon)^2 \frac{|t_{n-1}|^{(2\alpha) \wedge \varepsilon}}{(T - t_{n-1})} + (|f|_{\varepsilon; x})^2 \mathbb{E} |B_{t_{n-1}}^{n, t_o, \xi}|^{2\varepsilon} \right) \end{aligned}$$

$$\begin{aligned}
& + L_f^2 \mathbb{E}|Y_{f_n, t_{n-1}}^{n, t_0, \xi}|^2 + L_f^2 \mathbb{E}|Z_{f_n, t_n}^{n, t_0, \xi}|^2 + L_f^2 \mathbb{E}|Y_{t_n}^{t_0, \xi}|^2 + L_f^2 \mathbb{E}|Z_{t_{n-1}}^{t_0, \xi}|^2 \\
& \leq C(hf_0^2 + (|f|_{\alpha; t}^{loc} + (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon)^2 + h(|f|_{\varepsilon; x})^2(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} \\
& \quad + hL_f^2 \mathbb{E}|Y_{f_n, t_{n-1}}^{n, t_0, \xi}|^2 + hL_f^2 |g|_\varepsilon^2 h^{\varepsilon-1} + hL_f^2 \mathbb{E}|Y_{t_n}^{t_0, \xi}|^2 + hL_f^2 \mathbb{E}|Z_{t_{n-1}}^{t_0, \xi}|^2) \\
& \leq C(T, f_0, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, |g|_\varepsilon, L_f, \varepsilon) \left[(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} + hL_f^2 \mathbb{E}|Y_{t_{n-1}}^{n, t_0, \xi}|^2 \right],
\end{aligned}$$

where we used (5.1), Lemma 5.1 and Proposition 5.5 to estimate $\mathbb{E}|Y_{t_n}^{t_0, \xi}|^2$ and $|Z_{t_{n-1}}^{t_0, \xi}|^2$. The estimate for $\mathbb{E}|Y_{f_n, t_{n-1}}^{n, t_0, \xi}|^2$ follows from an a priori estimate which is an immediate consequence of Lemma 7.5(i) by setting $(\xi^1, f^1) := (g(B_T^{n, t_0, \xi}), f)$ and $(\xi^2, f^2) := (0, 0)$:

$$\begin{aligned}
\mathbb{E}|Y_{f_n, t_{n-1}}^{n, t_0, \xi}|^2 & \leq C(L_f, T) \left(\mathbb{E}g(B_T^{n, t_0, \xi})^2 + \mathbb{E} \int_{(t_{n-1}, T]} f(s, B_{s-}^{n, t_0, \xi}, 0, 0, [Y_s^{n, t_0, \xi}], [Z_s^{n, t_0, \xi}])^2 d\langle B^n \rangle_s \right) \\
& \leq C(L_f, T, f_0, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, \varepsilon) (1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon}.
\end{aligned}$$

Summarizing we may estimate (7.36) by $C(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} h^\varepsilon$.

Consequently, collecting all terms, we get using the estimates for (7.34), (7.35), and (7.36)

$$\begin{aligned}
& \mathbb{E} \sup_{k-1 \leq m \leq n} \left| Y_{f_n, t_m}^{n, t_0, \xi} - Y_{t_m}^{n, t_0, \xi} \right|^2 + \mathbb{E} \int_{]t_{k-1}, T[} \left| Z_{f_n, s}^{n, t_0, \xi} - Z_s^{n, t_0, \xi} \right|^2 d\langle B^n \rangle_s \\
& \leq C n^{-\varepsilon} (\log n)^{2\beta} + C(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} \frac{(\log n)^{2\beta}}{n^{2(\alpha \wedge \frac{\varepsilon}{2})}} + C(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} n^{-\varepsilon} \\
& \leq C(1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} \frac{(\log n)^{2\beta}}{n^{2(\alpha \wedge \frac{\varepsilon}{2})}}.
\end{aligned}$$

Inequality (7.31): We get from Lemma 7.5(ii) that

$$\begin{aligned}
& \mathbb{E}_{k-1} \left[\sup_{k-1 \leq m \leq n} \left| Y_{f_n, t_m}^{n, t, x} - Y_{t_m}^{n, t, x} \right|^2 + \int_{]t_{k-1}, T[} \left| Z_{f_n, s}^{n, t, x} - Z_s^{n, t, x} \right|^2 d\langle B^n \rangle_s \right] \\
& \leq c_{(7.12)} L_f \left[\int_{]t_{k-1}, T[} \left[\mathcal{W}_2(Y_s^{t_0, \xi}, Y_s^{n, t_0, \xi}) + \mathcal{W}_2(Z_{s \wedge t_{n-1}}^{t_0, \xi}, Z_s^{n, t_0, \xi}) \right]^2 d\langle B^n \rangle_s \right] \\
& \leq C(1 + \|\xi\|_{L^{2\varepsilon}})^{p\varepsilon} \frac{(\log n)^{p\beta}}{n^{p(\alpha \wedge \frac{\varepsilon}{2})}},
\end{aligned}$$

where we use in the last step that

$$\begin{aligned}
\mathcal{W}_2(Y_s^{t_0, \xi}, Y_s^{n, t_0, \xi}) & \leq \mathcal{W}_2(Y_s^{t_0, \xi}, Y_{f_n, s}^{n, t_0, \xi}) + \mathcal{W}_2(Y_{f_n, s}^{n, t_0, \xi}, Y_s^{n, t_0, \xi}), \\
\mathcal{W}_2(Z_{s \wedge t_{n-1}}^{t_0, \xi}, Z_s^{n, t_0, \xi}) & \leq \mathcal{W}_2(Z_{s \wedge t_{n-1}}^{t_0, \xi}, Z_{f_n, s}^{n, t_0, \xi}) + \mathcal{W}_2(Z_{f_n, s}^{n, t_0, \xi}, Z_s^{n, t_0, \xi})
\end{aligned}$$

and apply the estimates (7.33) and (7.30). Finally, we use Theorem 9.6 to deduce from (7.31) the relation (7.32). \square

8 Proof of Theorem 1.1

We start with a more general version of Theorem 1.1 which holds for *any coupling* (B, B^n) of the Brownian motion with the random walk. For the formulation we use Banach function norms, see [GG24, Chapter 18]:

Theorem 8.1. Let $\beta > \frac{1}{2}$. For $t \in [t_0, T)$ and $x, y \in \mathbb{R}$ one has

$$|u(t, x) - U^n(t, y)| \leq c_{(8.1)} \left[|x - y|^\varepsilon + (1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right]. \quad (8.1)$$

where $c_{(8.1)} = c_{(8.1)}(T, \varepsilon, \alpha, \beta, |g|_\varepsilon, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, f_0, \xi) > 0$. If $\|\cdot\|$ is a Banach function norm such that $\|\cdot\| \leq \kappa|\cdot|_{M_{\varphi_1}}$, then one has

$$\begin{aligned} & \frac{1}{c_{(8.2)}} \left\| \int_t^T |\nabla u(s, B_s^{t,x}) - \Delta^n(s, B_s^{n,t,y})|^2 ds \right\|^{\frac{1}{2}} \\ & \leq \left\| \sup_{t \leq s \leq T} \Psi_{2\beta}(|B_s^{t,x} - B_s^{n,t,y}|^{2\varepsilon}) \right\|^{\frac{1}{2}} + \left\| \sup_{t \leq s \leq T} |B_s^{n,t,y}|^{2\varepsilon} \right\|^{\frac{1}{2}} \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \\ & \quad + \sqrt{\kappa} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}}. \end{aligned} \quad (8.2)$$

for $n \geq n_0$, where $c_{(8.2)} = c_{(8.2)}(T, \varepsilon, \alpha, \beta, |g|_\varepsilon, |f|_{\alpha; t}^{loc}, |f|_{\varepsilon; x}, L_f, c_{f_0}^{(2)}, f_0, \xi) > 0$.

Proof. Proof of (8.1): We have by [Proposition 7.1](#), [Proposition 7.4](#), and [Proposition 7.7](#) that

$$\begin{aligned} |u(t, x) - U^n(t, y)| & \leq |u(t, x) - u_{f_n}(t, x)| + |u_{f_n}(t, x) - U_{f_n}^n(t, y)| + |U_{f_n}^n(t, y) - U^n(t, y)| \\ & \leq c_{(7.2)} n^{-((\alpha \wedge \frac{\varepsilon}{2}) + \frac{1}{2})} \\ & \quad + c_{(7.10)} \left[|x - y|^\varepsilon + (1 + |x| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon n^{-\alpha \wedge \frac{\varepsilon}{2}} \right] \\ & \quad + c_{(7.31)} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}}. \end{aligned}$$

Proof of (8.2): Again by [Proposition 7.1](#), [Proposition 7.4](#), and [Proposition 7.7](#) we get

$$\begin{aligned} & \frac{1}{3} \left\| \int_t^T |\nabla u(s, B_s^{t,x}) - \Delta^n(s, B_s^{n,t,y})|^2 ds \right\| \\ & \leq \left\| \int_t^T |\nabla u(s, B_s^{t,x}) - \nabla u_{f_n}(s, B_s^{t,x})|^2 ds \right\| \\ & \quad + \left\| \int_t^T |\nabla u_{f_n}(s, B_s^{t,x}) - \Delta_{f_n}^n(s, B_s^{n,t,y})|^2 ds \right\| \\ & \quad + \left\| \int_t^T |\Delta_{f_n}^n(s, B_s^{n,t,y}) - \Delta^n(s, B_s^{n,t,y})|^2 ds \right\| \\ & \leq \kappa \left[c_{(7.3)} c_{(7.2)} n^{-((\alpha \wedge \frac{\varepsilon}{2}) + \frac{1}{2})} \right]^2 \\ & \quad + 3c_{(7.11)}^2 \left[\left\| \sup_{t \leq s \leq T} \Psi_{2\beta}(|B_s^{t,x} - B_s^{n,t,y}|^{2\varepsilon}) \right\| \right. \\ & \quad \left. + \left\| \sup_{t \leq s \leq T} |B_s^{n,t,y}|^{2\varepsilon} \right\| \left\| \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right\|^2 \right] \end{aligned}$$

$$\begin{aligned}
& + (1 + \|\xi\|_{L^{2\varepsilon}})^{2\varepsilon} \left| \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right|^2 \Big] \\
& + \kappa \left[c_{(7.32)} c_{(7.31)} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right]^2.
\end{aligned}$$

□

To apply [Theorem 8.1](#) we use the following estimate:

Lemma 8.2. *Let $\beta > \frac{1}{2}$. Assume random variables such that $\sup_{p \in [1, \infty)} \sup_{n \geq 1} \frac{\|\theta_n\|_{L^p}}{p} \leq 1$ and constants $c_n > 0$. Then one has*

$$\left\| \Psi_{2\beta}(|c_n \theta_n|^{2\varepsilon})^{\frac{1}{2}} \right\|_{L^p} \leq c_{(8.3)} c_n^\varepsilon \left| \log \left(\frac{1}{c_n^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right|^\beta p^{\varepsilon + \beta} \quad (8.3)$$

for $p \geq 1$, where $c_{(8.3)} = c_{(8.3)}(\varepsilon, \beta) > 0$.

Proof. We recall that $\Psi_{2\beta}(r) = r \left(\log \left(\frac{1}{r \wedge \hat{r}} \right) \right)^{2\beta}$ and use for $\gamma, a, b > 0$ and $x \geq 1$ that it holds

$$\frac{1}{(ab) \wedge \hat{r}} \leq \frac{1}{a \wedge \sqrt{\hat{r}}} \frac{1}{b \wedge \sqrt{\hat{r}}} \quad \text{and} \quad \log x \leq \frac{1}{\gamma} x^\gamma.$$

Let $\gamma = \frac{1}{2\beta p}$. Then we get

$$\begin{aligned}
& \mathbb{E} |\Psi_{2\beta}(|c_n \theta_n|^{2\varepsilon})|^{\frac{p}{2}} \\
& = \mathbb{E} \left[|c_n \theta_n|^{\varepsilon p} \left(\log \left(\frac{1}{|c_n \theta_n|^{2\varepsilon} \wedge \hat{r}} \right) \right)^{\beta p} \right] \\
& \leq 2^{(p\beta-1)+} \left\{ \mathbb{E} \left[|c_n \theta_n|^{\varepsilon p} \left(\log \left(\frac{1}{c_n^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right)^{\beta p} \right] + \mathbb{E} \left[|c_n \theta_n|^{\varepsilon p} \left(\log \left(\frac{1}{|\theta_n|^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right)^{\beta p} \right] \right\} \\
& = c_n^{\varepsilon p} 2^{(p\beta-1)+} \left[\left| \log \left(\frac{1}{c_n^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right|^{\beta p} \mathbb{E} [|\theta_n|^{\varepsilon p}] + \mathbb{E} \left[|\theta_n|^{\varepsilon p} \left(\log \left(\frac{1}{|\theta_n|^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right)^{\beta p} \right] \right].
\end{aligned}$$

Now, using $\beta p \gamma = \frac{1}{2}$,

$$\begin{aligned}
\mathbb{E} \left[|\theta_n|^{\varepsilon p} \left(\log \left(\frac{1}{|\theta_n|^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right)^{p\beta} \right] & \leq \mathbb{E} \left[|\theta_n|^{\varepsilon p} \frac{1}{\gamma^{\beta p}} \left| \frac{1}{|\theta_n|^{2\varepsilon} \wedge \hat{r}} \right|^{\beta p \gamma} \right] \\
& \leq \mathbb{E} \left[|\theta_n|^{\varepsilon p} \frac{1}{\gamma^{\beta p}} \left| \frac{1}{|\theta_n|^{2\varepsilon}} + \frac{1}{\hat{r}} \right|^{\beta p \gamma} \right] \\
& \leq \frac{1}{\gamma^{\beta p}} \mathbb{E} \left[|\theta_n|^{\varepsilon p} \left[\frac{1}{|\theta_n|^\varepsilon} + \frac{1}{\hat{r}^{1/2}} \right] \right] \\
& = (2\beta p)^{\beta p} \left[\mathbb{E} |\theta_n|^{\varepsilon(p-1)} + \mathbb{E} |\theta_n|^{\varepsilon p} \frac{1}{\hat{r}^{1/2}} \right].
\end{aligned}$$

Finally, we observe for $\varepsilon(p-1) \geq 1$ that

$$\left(\mathbb{E} |\theta_n|^{\varepsilon(p-1)} \right)^{\frac{1}{\varepsilon(p-1)}} \leq (\varepsilon(p-1))^{\frac{\varepsilon(p-1)}{p}} \leq (\varepsilon p)^{\frac{\varepsilon(p-1)}{p}}.$$

So we get for $p \geq 1 + \frac{1}{\varepsilon}$ that

$$\begin{aligned} & \left\| \Psi_{2\beta}(|c_n \theta_n|^{2\varepsilon})^{\frac{1}{2}} \right\|_{L^p}^p \\ & \leq c_n^{\varepsilon p} 2^{(p\beta-1)^+} \left[\left| \log \left(\frac{1}{c_n^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right|^{\beta p} (\varepsilon p)^{\varepsilon p} + (2\beta p)^{\beta p} \left[(\varepsilon(p-1))^{\varepsilon(p-1)} + (\varepsilon p)^{\varepsilon p} \frac{1}{\hat{r}^{1/2}} \right] \right]. \end{aligned}$$

Taking the p -th root the result follows (as the L^p -norm is monotone in p the range $p \geq 1 + \frac{1}{\varepsilon}$ instead of $p \geq 2$ is sufficient). \square

Proof of Theorem 1.1. Here we assume the coupling of the Brownian motion with the random walk such that the RHS of (4.4) is satisfied. For $p \in [2, \infty)$ and $q := p/2 \in [1, \infty)$ we have that

$$\|\cdot\|_{L^q} \leq \kappa_0 q \|\cdot\|_{M_{\varphi_1}} \quad \text{i.e.} \quad \kappa = \kappa_0 q$$

for some $\kappa_0 > 0$. Then Theorem 8.1 implies that

$$\begin{aligned} & \frac{1}{c_{(8.2)}} \left\| \left(\int_t^T |\nabla u(s, B_s^{t,x}) - \Delta^n(s, B_s^{n,t,y})|^2 ds \right)^{\frac{1}{2}} \right\|_{L^p} \\ & = \frac{1}{c_{(8.2)}} \left\| \int_t^T |\nabla u(s, B_s^{t,x}) - \Delta^n(s, B_s^{n,t,y})|^2 ds \right\|_{L^q}^{\frac{1}{2}} \\ & \leq \left\| \sup_{t \leq s \leq T} \Psi_{2\beta}(|B_s^{t,x} - B_s^{n,t,y}|^{2\varepsilon}) \right\|_{L^q}^{\frac{1}{2}} + \left\| \sup_{t \leq s \leq T} |B_s^{n,t,y}|^{2\varepsilon} \right\|_{L^q}^{\frac{1}{2}} \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \\ & \quad + \sqrt{\kappa_0 q} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \\ & = \left\| \sup_{t \leq s \leq T} \Psi_{2\beta}(|B_s^{t,x} - B_s^{n,t,y}|^{2\varepsilon})^{\frac{1}{2}} \right\|_{L^p} + \left\| \sup_{t \leq s \leq T} |B_s^{n,t,y}|^\varepsilon \right\|_{L^p} \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \\ & \quad + \sqrt{p} \left[\sqrt{\frac{\kappa_0}{2}} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right]. \end{aligned}$$

Now let

$$c_n \theta_n = \sup_{t \leq s \leq T} |B_s^{t,x} - B_s^{n,t,y}|$$

where $\sup_{p \in [1, \infty)} \sup_{n \geq 1} \frac{\|\theta_n\|_{L^p}}{p} \leq 1$. By Theorem 4.2 we can arrange

$$c_n := c \frac{\log(n+1)}{\sqrt{n}} \quad \text{for some absolute constant } c > 0.$$

For $x = y$ we deduce by Lemma 8.2 that

$$\frac{1}{c_{(8.2)}} \left\| \left(\int_t^T |\nabla u(s, B_s^{t,x}) - \Delta^n(s, B_s^{n,t,x})|^2 ds \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$\begin{aligned} &\leq c_{(8.3)} c_n^\varepsilon \left| \log \left(\frac{1}{c_n^{2\varepsilon} \wedge \sqrt{\hat{r}}} \right) \right|^\beta p^{\varepsilon+\beta} + \left\| \sup_{t \leq s \leq T} |B_s^{n,t,x}|^\varepsilon \right\|_{L^p} \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \\ &\quad + \sqrt{p} \left[\sqrt{\frac{\kappa_0}{2}} (1 + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right]. \end{aligned}$$

Finally, we use Doob's maximal inequality and Hoeffding's inequality (3.3) to get

$$\left\| \sup_{t \leq s \leq T} |B_s^{n,t,x}|^\varepsilon \right\|_{L^p} \leq |x|^\varepsilon + \left\| \sup_{t \leq s \leq T} |B_s^{n,t,0}|^\varepsilon \right\|_{L^p} \leq |x|^\varepsilon + cp^{\frac{\varepsilon}{2}}$$

with $c = c(T, \varepsilon) > 0$.

Similarly, we observe that Theorem 8.1 implies that

$$\begin{aligned} &\left\| \sup_{s \in [t, T]} |u(s, B_s^{t,x}) - U^n(s, B_s^{n,t,x})| \right\|_{L^p} \\ &\leq c_{(8.1)} \left\| \sup_{s \in [t, T]} \left[|B_s^{t,x} - B_s^{n,t,x}|^\varepsilon + (1 + |B_s^{t,x}| + \|\xi\|_{L^{2\varepsilon}})^\varepsilon \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right] \right\|_{L^p} \\ &\leq c_{(8.1)} \left[\left\| \sup_{s \in [t, T]} |B_s^{t,x} - B_s^{n,t,x}|^\varepsilon \right\|_{L^p} + \left\| \sup_{s \in [t, T]} |B_s^{t,x}|^\varepsilon \right\|_{L^p} \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right. \\ &\quad \left. + [1 + \|\xi\|_{L^{2\varepsilon}}^\varepsilon] \frac{|\log(n+1)|^\beta}{n^{\alpha \wedge \frac{\varepsilon}{2}}} \right]. \end{aligned}$$

Finally we use the Theorem 4.2 (applied to εp , where we assume w.l.o.g. that $\varepsilon p \geq 1$) which gives

$$\left\| \sup_{s \in [t, T]} |B_s^{t,x} - B_s^{n,t,x}|^\varepsilon \right\|_{L^p} = \left\| \sup_{s \in [t, T]} |B_s^{t,x} - B_s^{n,t,x}| \right\|_{L^{\varepsilon p}}^\varepsilon \leq c_{(4.4)}^\varepsilon \left| \varepsilon p \frac{\log(n+1)}{n^{\frac{1}{2}}} \right|^\varepsilon,$$

and again

$$\left\| \sup_{s \in [t, T]} |B_s^{t,x}|^\varepsilon \right\|_{L^p} \leq |x|^\varepsilon + c' p^{\frac{\varepsilon}{2}}$$

for some $c' = c'(T, \varepsilon) > 0$. □

9 Appendix

9.1 Gronwall lemmas

Lemma 9.1. *Let $\theta \in [0, 1[$ and $A, B \geq 0$, and assume that $f : [0, T] \rightarrow [0, \infty[$ is a bounded, Borel measurable function. Then*

$$(i) \quad f(t) \leq A + B \int_t^T \frac{f(r)}{(T-r)^\theta} dr, \quad t \in [0, T], \quad \text{implies that } f(t) \leq A e^{B \int_t^T (T-s)^{-\theta} ds}, \quad t \in [0, T].$$

(ii) $f(t_{k-1}) \leq A + B \int_{]t_{k-1}, T[} \frac{f(r)}{(T-r)^\theta} d\langle B^n \rangle_r$ for $k = 1, \dots, n$ implies that

$$f(t_{k-1}) \leq A e^{B \int_{t_k}^T (T-s)^{-\theta} ds}$$

for $k = 1, \dots, n$.

Proof. We follow [EK86, Theorem 5.1, Appendix] claiming that for a finite Borel measure μ on $[0, T]$ the relation

$$f(t) \leq A + \int_{]t, T[} f(s) d\mu(s)$$

implies

$$\begin{aligned} f(t) &\leq A + \int_{]t, T[} A d\mu(s) + \int_{]t, T[} \int_{]s, T[} f(s_1) d\mu(s_1) d\mu(s) \\ &\leq A \sum_{n=0}^{\infty} \frac{(\mu(]t, T[))^n}{n!} = A e^{\mu(]t, T[)}, \end{aligned}$$

where we used that

$$\int_{]t, T[} \int_{]s_1, T[} \dots \int_{]s_{n-1}, T[} f(s_n) d\mu(s_n) \dots d\mu(s_2) d\mu(s_1) \rightarrow 0, \quad n \rightarrow \infty.$$

Then (i) follows from setting $\mu(]t, T[) := B \int_t^T (T-s)^{-\theta} ds$.

For (ii) we put $\mu(]t, T[) := B \int_{]t, T[} \frac{1}{(T-r)^\theta} d\langle B^n \rangle_r$ which yields to

$$f(t_{k-1}) \leq A e^{B \int_{]t_{k-1}, T[} \frac{1}{(T-r)^\theta} d\langle B^n \rangle_r}.$$

Using the definition of $d\langle B^n \rangle_r$ we get the assertion from

$$\int_{]t_{k-1}, T[} \frac{1}{(T-r)^\theta} d\langle B^n \rangle_r = \sum_{m=k}^{n-1} h \frac{1}{(T-t_m)^\theta} \leq \sum_{m=k}^{n-1} \int_{t_m}^{t_{m+1}} \frac{1}{(T-r)^\theta} dr = \int_{t_k}^T \frac{1}{(T-r)^\theta} dr.$$

□

We formulate a Gronwall lemma of Volterra type. It can be either proved directly by a convolution argument or one can use [Hen81, Exercise 4, page 190].

Lemma 9.2. *Assume a measurable $g : [0, T[\rightarrow [0, \infty[\in L_1([0, T[)$ and $\alpha, \beta > 0$ such that*

$$g(t) \leq \frac{\alpha}{(T-t)^\alpha} + \beta \int_t^T \frac{g(s)}{(s-t)^\beta} ds$$

for all $t \in [0, T[$. Then $g(t) \leq c \frac{\alpha}{(T-t)^\alpha}$ for $t \in [0, T[$ for a constant $c = c(T, \alpha, \beta) > 0$.

9.2 Bounded mean oscillation

We use the following notation: If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $F \in \mathcal{F}$ of positive measure, then $\mathbb{P}_F(A) := \mathbb{P}(A)/\mathbb{P}(F)$, for $A \subseteq F$ with $A \in \mathcal{F}$, is the restriction of \mathbb{P} to F equipped with the trace σ -algebra. Then, by [Section 3](#),

$$|f|_{M_{\varphi_1}(F, \mathbb{P}_F)} := \inf \left\{ c > 0 : \mathbb{P}_F(|f| > \lambda) \leq e^{1-\frac{\lambda}{c}} \text{ for } \lambda \geq c \right\}.$$

We will use the following two statements, where we remark that in [\[Gei97\]](#) the spaces M_φ are denoted by M_φ^0 :

Definition 9.3. For $p \in (0, \infty)$ and an adapted *continuous* process $(A_t)_{t \in [0, T]}$ with $A_0 = 0$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ sharing the usual conditions we define

$$\|A\|_{\text{BMO}_p([0, T])} := \inf c,$$

where the infimum is taken over all $c \geq 0$ such that for all $0 \leq a \leq t < T$ one has

$$\mathbb{E}[|A_t - A_a|^p | \mathcal{F}_a] \leq c^p \quad a.s.$$

The John-Nirenberg theorem for *continuous* adapted processes reads as follows:

Theorem 9.4 ([\[DM82, p. 176\]](#)). For $p \in (0, \infty)$ there is a constant $c_{(9.1)} = c_{(9.1)}(p) > 0$ such that for all $t \in [0, T]$ and $F \in \mathcal{F}_t$ with $\mathbb{P}(F) > 0$ one has

$$\|A_T - A_t\|_{M_{\varphi_1}(F, \mathbb{P}_F)} \leq c_{(9.1)} \|A\|_{\text{BMO}_p([0, T])}. \quad (9.1)$$

Definition 9.5. For $p \in (0, \infty)$ and an adapted process $(A_n)_{n=0}^N$ with $A_0 = 0$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{G}_n)_{n=0}^N)$ satisfying the usual conditions we define $\|A\|_{\text{BMO}_p} := \inf c$ where the infimum is taken over all $c \geq 0$ such that for all $1 \leq n \leq N$ one has

$$\mathbb{E}[|A_N - A_{n-1}|^p | \mathcal{G}_n] \leq c^p \quad a.s.$$

The John-Nirenberg theorem for *discrete time* adapted processes reads as follows:

Theorem 9.6 ([\[Ga73, Theorem III.1.1\]](#)). For $p \in (0, \infty)$ there is a constant $c_{(9.2)} = c_{(9.2)}(p) > 0$ such that for all $1 \leq n \leq N$ and $F \in \mathcal{G}_n$ with $\mathbb{P}(F) > 0$ one has

$$\|A_N - A_{n-1}\|_{M_{\varphi_1}(F, \mathbb{P}_F)} \leq c_{(9.2)} \|A\|_{\text{BMO}_p([0, T])}. \quad (9.2)$$

References

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