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On the chaotic expansion for counting processes*

Caroline Hillairet[†]

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Abstract

We introduce and study an alternative form of the chaotic expansion for Poisson functionals, using a very specific (non-equivalent) Girsanov transformation; we name this alternative form *pseudo-chaotic expansion*. This pseudo-chaotic expansion is derived on a general phase space and we analyze the link between this expansion and the classical chaotic decomposition available for random Poisson measures. When combined with the Poisson imbedding representation for point processes this expansion provides a new decomposition for this process and in the specific case of a linear Hawkes process, the coefficients of the pseudo-chaotic expansion are derived in closed form whereas those of the classical chaotic decomposition are defined through the distribution function of the marginal laws of the Hawkes process or of its intensity that cannot be obtained explicitly. Finally, this expansion enables us to study further the structure of linear Hawkes processes by constructing an example of a process in a pseudo-chaotic form that satisfies the stochastic self-exciting intensity equation which determines a Hawkes process (in particular its expectation equals the one of a Hawkes process) but which fails to be a counting process.

Keywords: Counting processes; Poisson imbedding representation; Malliavin calculus; Hawkes processes.

MSC2020 subject classifications: 60G55; 60G57; 60H07.

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1 Introduction

A large literature dealing with Gaussian and Poisson functionals has investigated the so-called chaotic expansion (also called Wiener-Itô expansion) which refers to an infinite dimensional version of the use of orthonormal polynomials associated to a given probability distribution. This notion finds many relations with the Malliavin calculus

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which has been largely developed in the Gaussian framework (see for instance the dedicated monograph [13]) and in the Poisson framework (see for e.g. [11]) or to more general frameworks that go beyond the Gaussian and the Poisson framework (see [14]). This paper contributes to this literature by introducing an alternative chaotic expansion, in the framework of a Poisson random measure \mathbf{N} defined on a complete separable metric space $(\mathbf{X}, \mathfrak{X})$ equipped with a non-atomic σ -finite measure denoted ρ . The standard chaotic expansion of a Poisson functional F in $L^2(\Omega)$ with respect to \mathbf{N} writes down as (see e.g. Theorem 2 in [10])

$$F = \mathbb{E}[F] + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n(T_n F) \tag{1.1}$$

where $\mathbf{I}_n(T_n F)$ is the n th iterated integral of $T_n F$ against the compensated Poisson measure \mathbf{N} defined as

$$\mathbf{I}_n(T_n F) := \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} T_n F(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \rho^{n-|J|}(d\mathbf{x}_{J^c}),$$

with $J^c := \{1, \dots, n\} \setminus J$ and $d\mathbf{x}_J := (d\mathbf{x}_j)_{j \in J}$. Furthermore, for any n , the symmetric coefficient $(T_n F)$ in $L^2(\mathbf{X}^n)$ is given as the expectation of the n^{th} -Malliavin derivative of F :

$$T_n F = \mathbb{E}[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F].$$

This expression requires some details on its definition that will be given in Section 2 below; but roughly speaking it allows one to expand the random variable F into iterated integrals with respect to the compensated Poisson measure $\mathbf{N}(d\mathbf{x}_j) - \rho(d\mathbf{x}_j)$. Such decomposition is proved to be useful (for example in the context of Brownian SPDEs) provided that the coefficients $T_n F$ can be computed or can be characterized by an equation. Unfortunately these coefficients are often far from being explicit, as for example in the case of a linear Hawkes process as discussed in Section 4.

In this paper we prove as Theorem 3.6 that any Poisson functional F in $L^2(\Omega)$ (assuming $\rho(\mathbf{X}) < +\infty$ such that F is also in $L^1(\Omega)$) satisfies an alternative representation of the chaotic expansion that we name pseudo-chaotic representation. This pseudo-chaotic expansion takes the form of

$$F = \mathbb{E}[LF] + \sum_{n=1}^{\mathbf{N}(\mathbf{X})} \frac{1}{n!} \mathcal{I}_n(\mathcal{T}_n F), \tag{1.2}$$

where $\mathcal{I}_n(\mathcal{T}_n F)$ is the n th iterated integral of $\mathcal{T}_n F$ against the Poisson measure \mathbf{N} only (and not its compensated version) defined as

$$\mathcal{I}_n(\mathcal{T}_n F) := \int_{\mathbf{X}^n} \mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n),$$

Furthermore, for any n , the symmetric coefficient $\mathcal{T}_n F$ in $L^1(\mathbf{X}^n)$ is given as the expectation of the n^{th} -Malliavin derivative of F but under a (non-equivalent) Girsanov transformation

$$\mathcal{T}_n F := \mathbb{E}[L\mathbf{D}^n F], \quad \text{with } L := \exp(\rho(\mathbf{X})) \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}.$$

The link between the coefficients $T_n F$ appearing in the chaotic expansion (1.1) with coefficients $\mathcal{T}_n F$ in the pseudo-chaotic one (1.2) is studied in Proposition 3.14.

The pseudo-chaotic expansion can be coupled with the Poisson imbedding (see [9, Chapter 4] or [3]) to provide a representation of general counting processes. This paper investigates in detail the particular case of the linear Hawkes process. The so-called linear Hawkes process is a counting process H with intensity process λ satisfying :

$$\lambda_t = \mu + \int_{(0,t)} \Phi(t-s)dH_s, \quad t \geq 0, \tag{1.3}$$

where the constant $\mu > 0$ is the baseline intensity and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is modeling the self-exciting feature of the process. Naturally conditions on Φ are required for a well-posed formulation (typically $\|\Phi\|_1 < 1$). Well-posedness here is accurate as, thanks to the Poisson imbedding, the linear Hawkes processes can be seen as a system of weakly coupled SDEs with respect to N

$$\begin{cases} H_t = \int_{(0,t] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta) \\ \lambda_t = \mu + \int_{(0,t)} \Phi(t-s) \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta). \end{cases} \quad t \geq 0,$$

Under mild condition on Φ the second equation (and so the system) can be proved to be well-posed (see e.g. [3, 4, 8]). Theorem 4.6 gives the coefficients $\mathcal{T}_n H_T$ of the pseudo-chaotic expansion (1.2) of the linear Hawkes process H , which are explicit, in contradistinction to the coefficients $T_n H_T$ in the classical chaotic expansion (1.1) which can not be computed (see Discussion 4.9). This provides then a closed-form expression to linear Hawkes processes. Finally, we study further in Section 5 the structure of linear Hawkes processes by constructing an example of a process in a pseudo-chaotic form that satisfies the stochastic intensity equation (1.3) but which fails to be a counting process (see Theorem 5.3 and Discussion 5.4).

The paper is organized as follows. Notations and the description of the Poisson space and elements of Malliavin’s calculus are presented in Section 2, as well as the operators and iterated integrals we will make use in the derivation of the expansions. The pseudo-chaotic expansion is derived in Section 3, and is then coupled with the Poisson imbedding to detail the case of counting processes. As an application, Section 4 is dedicated to linear Hawkes processes for which we give their explicit pseudo-chaotic representation. Finally, Section 5 investigates the construction, in a pseudo-chaotic form, of a process which is not a counting process while satisfying the same stochastic intensity equation than the Hawkes.

2 Elements of Malliavin calculus on the Poisson space

We introduce notions of stochastic analysis on general Poisson space. All the elements presented in this section are taken from [10, 12].

We set $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ the set of positive integers. For $a, b \in \mathbb{Z}$ with $a > b$, and for any map $\rho : \mathbb{Z} \rightarrow \mathbb{R}$, $\prod_{i=a}^b \rho(i) := 1$; $\sum_{i=a}^b \rho(i) := 0$. Finally, for any finite set S , we set $|S|$ its cardinal.

In this section we fix $(\mathbf{X}, \mathfrak{X})$ a complete separable metric space. Let ρ a non-atomic σ -finite measure on $(\mathbf{X}, \mathfrak{X})$. As a consequence, there exists an increasing (with respect to the inclusion) sequence of subsets $(\mathcal{R}_k)_{k \geq 1}$ of \mathbf{X} such that $\cup_{k \in \mathbb{N}} \mathcal{R}_k = \mathbf{X}$ and $\rho(\mathcal{R}_k) < +\infty$. For $n \geq 1$, ρ^n denotes the product measure on $(\mathbf{X}^n, \mathfrak{X}^{\otimes n})$.

We define Ω the space of configurations on \mathbf{X} as

$$\Omega := \left\{ \omega = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in \mathbf{X}, j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\}. \quad (2.1)$$

Let \mathcal{F} be the σ -field associated to the vague topology on Ω . Let \mathbb{P} the Poisson measure on Ω under which the canonical evaluation \mathbf{N} defines a Poisson random measure with intensity measure ρ . To be more precise given any element A in \mathfrak{X} with $\rho(A) > 0$, the random variable

$$(\mathbf{N}(\omega))(A) := \omega(A)$$

is a Poisson random variable with intensity $\rho(A)$. We denote by $\mathcal{F}^{\mathbf{N}} := \sigma\{\mathbf{N}(A); A \in \mathfrak{X}\}$ the σ -field generated by \mathbf{N} .

Add-points operators and the Malliavin derivative

We introduce some elements of Malliavin calculus on Poisson processes. We set

$$L^0(\Omega) := \{F : \Omega \rightarrow \mathbb{R}, \mathcal{F}^{\mathbf{N}} - \text{measurable}\},$$

$$L^2(\Omega) := \{F \in L^0(\Omega), \mathbb{E}[|F|^2] < +\infty\}$$

and

$$L^\infty(\Omega) := \{F \in L^0(\Omega), \exists C > 0, |F| \leq C, \mathbb{P} - a.s.\}.$$

Similarly

$$L^0(\mathbf{X}^j) := \{f : \mathbf{X}^j \rightarrow \mathbb{R}, \mathfrak{X}^{\otimes j} - \text{measurable}\}$$

and for $p \in \{1, 2\}$, for $j \in \mathbb{N}^*$

$$L^p(\mathbf{X}^j) := \left\{ f \in L^0(\mathbf{X}^j), \int_{\mathbf{X}^j} |f(\mathbf{x}_1, \dots, \mathbf{x}_j)|^p \rho^j(d\mathbf{x}_1 \dots d\mathbf{x}_j) < +\infty \right\}. \quad (2.2)$$

Besides,

$$L_s^p(\mathbf{X}^j) := \{f \in L^p(\mathbf{X}^j) \text{ and } f \text{ is symmetric}\}, \quad (2.3)$$

is the space of square-integrable symmetric mappings where we recall that $f : \mathbf{X}^j \rightarrow \mathbb{R}$ is said to be symmetric if for any element σ in \mathcal{S}_j (the set of all permutations of $\{1, \dots, j\}$),

$$f(\mathbf{x}_1, \dots, \mathbf{x}_j) = f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(j)}), \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_j) \in \mathbf{X}^j.$$

For f in $L^0(\mathbf{X}^j)$, we set \tilde{f} the symmetrization of f that is the map $\tilde{f} : \mathbf{X}^j \rightarrow \mathbb{R}$, with

$$\tilde{f}(\mathbf{x}_1, \dots, \mathbf{x}_j) := \frac{1}{j!} \sum_{\sigma \in \mathcal{S}_j} f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(j)}). \quad (2.4)$$

For $h \in L^0(\mathbf{X})$ and $j \geq 1$ we set the symmetric function $h^{\otimes j} \in L_s^0(\mathbf{X}^j)$ defined as

$$h^{\otimes j}(\mathbf{x}_1, \dots, \mathbf{x}_j) := \prod_{i=1}^j h(\mathbf{x}_i), \quad (\mathbf{x}_1, \dots, \mathbf{x}_j) \in \mathbf{X}^j. \quad (2.5)$$

The main ingredient we will make use of are the add-points operators on the Poisson space Ω .

Definition 2.1. [Add-points operators] Given $n \in \mathbb{N}^*$, and $J := \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \mathbf{X}$ a subset of \mathbf{X} with $|J| = n$, we set the measurable mapping :

$$\begin{aligned} \varepsilon_J^{+,n} : \Omega &\longrightarrow \Omega \\ \omega &\longmapsto \omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}}. \end{aligned}$$

Note that by definition

$$\omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}} = \omega + \sum_{j=1}^n \delta_{\mathbf{x}_j} \mathbf{1}_{\{\omega(\{\mathbf{x}_j\})=0\}}$$

that is we add the atoms \mathbf{x}_j to the path ω unless they already were part of it (which is the meaning of the term $\mathbf{1}_{\{\omega(\{\mathbf{x}_j\})=0\}}$). Note that since ρ is assumed to be atomless, given a set J as above, $\mathbb{P}[\mathbf{N}(J) = 0] = 1$ hence in what follows we will simply write $\omega + \sum_{j=1}^n \delta_{\mathbf{x}_j}$ for $\varepsilon_{\mathbf{x}}^{+,n}(\omega)$.

We now recall the Malliavin derivative operator.

Definition 2.2. For F in $L^0(\Omega)$, $n \in \mathbb{N}^*$, $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$, we set

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F := \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F \circ \varepsilon_J^{+,|J|}, \tag{2.6}$$

where we recall that $\emptyset \subset \mathbf{X}$.

For instance when $n = 1$, we write $D_{\mathbf{x}} F := D_{\mathbf{x}}^1 F = F(\cdot + \delta_{\mathbf{x}}) - F(\cdot)$ which is the difference operator (also called add-one cost operator¹). Relation (2.2) rewrites as

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F(\omega) = \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} F \left(\omega + \sum_{j \in J} \delta_{\mathbf{x}_j} \right), \quad \text{for a.e. } \omega \in \Omega.$$

Note that with this definition, for any ω in Ω , the mapping

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F(\omega)$$

belongs to $L_s^0(\mathbf{X}^j)$ defined as (2.3) and in addition the mapping

$$T_n F : (\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbb{E}[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F] \tag{2.7}$$

is well-defined and belongs to $L_s^2(\mathbf{X}^j)$ for any F in $L^2(\Omega)$ (see [10, 12]).

We recall the following property (see e.g. [10, Relation (15)]) : for F in $L^2(\Omega)$, $n \in \mathbb{N}^*$, and $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$, the n th iterated Malliavin derivative operator \mathbf{D}^n satisfies

$$\mathbf{D}^n F = \mathbf{D}(\mathbf{D}^{n-1} F), \quad n \geq 1; \quad \mathbf{D}^0 F := F. \tag{2.8}$$

Remark 2.3. If F is deterministic, then by definition $\mathbf{D}^n F = 0$ for any $n \geq 1$.

Evaluation points operator and the pathwise derivative

Here we introduce a purely deterministic operator that appears in our analysis and that will be at the core of the representation under interest.

¹see [10, p. 5]

Definition 2.4. For $F \in L^0(\Omega)$, we define the deterministic operators:

$$\mathcal{T}_0 F := F(\emptyset),$$

and for $n \in \mathbb{N}^*$, $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$,

$$\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n) := \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F(\varpi_J),$$

where if $J = \{\mathbf{y}_1, \dots, \mathbf{y}_k\}$, $\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_k\}} := \sum_{i=1}^k \delta_{\mathbf{y}_i} \in \Omega$.

In particular, even though F is a random variable, $\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is a real number as each term $F(\varpi_J)$ is the evaluation of F at the outcome ϖ_J . Besides, given the event $\{\mathbf{N}(\mathbf{X}) = 0\}$, $\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n)$ coincides with $\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F$ and $\mathcal{T}_0 F$ coincides with F .

Factorial measures and iterated integrals

Proposition 2.5. (Factorial measures; See e.g. [10, Proposition 1]). There exists a unique sequence of counting random measures $(\mathbf{N}^{(m)})_{m \in \mathbb{N}^*}$ where for any m , $\mathbf{N}^{(m)}$ is a counting random measure on $(\mathbf{X}^m, \mathfrak{X}^{\otimes m})$ with

$$\begin{aligned} \mathbf{N}^{(1)} &:= \mathbf{N} \quad \text{and for } A \in \mathfrak{X}^{m+1}, \\ \mathbf{N}^{(m+1)}(A) &:= \int_{\mathbf{X}^m} \left[\int_{\mathbf{X}} \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_{m+1}) \in A\}} \mathbf{N}(d\mathbf{x}_{m+1}) - \sum_{j=1}^m \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x}_j) \in A\}} \right] \mathbf{N}^{(m)}(d\mathbf{x}_1, \dots, d\mathbf{x}_m); \end{aligned}$$

With this definition at hand we introduce the notion of iterated integrals. In particular for $A \in \mathfrak{X}$,

$$\mathbf{N}^{(n)}(A^{\otimes n}) = \mathbf{N}(A)(\mathbf{N}(A) - 1) \times \dots \times (\mathbf{N}(A) - n + 1).$$

Note that by definition $\mathbf{N}^{(n)}(A) \mathbf{1}_{\{\mathbf{N}(A) < n\}} = 0$. We now turn to the definition of two types of iterated integrals.

Definition 2.6. Let $n \in \mathbb{N}^*$ and $f_n \in L^1(\mathbf{X}^n)$.

- $\mathcal{I}_n(f_n)$ the n th iterated integral of f_n against the Poisson measure \mathbf{N} defined as

$$\mathcal{I}_n(f_n) := \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n),$$

where each of the integrals above is well-defined pathwise for \mathbb{P} -a.e. $\omega \in \Omega$.

- $\mathbf{I}_n(f_n)$ the n th iterated integral of f_n against the compensated Poisson measure \mathbf{N} defined as

$$\mathbf{I}_n(f_n) := \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \rho^{n-|J|}(d\mathbf{x}_{J^c}),$$

where $J^c := \{1, \dots, n\} \setminus J$ and $d\mathbf{x}_J := (d\mathbf{x}_j)_{j \in J}$ and where each of the integrals above is well-defined pathwise for \mathbb{P} -a.e. $\omega \in \Omega$. We adopt the convention for $J = \emptyset$ that $\int \dots \mathbf{N}^{(0)} := 1$. As explained in Remark 2.7 below, the operator \mathbf{I}_n can be extended to any function $f \in L^2(\mathbf{X}^n)$ (chaotic expansion).

Note that for any $f_n : \mathbf{I}_n(f_n) = \mathbf{I}_n(\tilde{f}_n)$ and $\mathcal{I}_n(f_n) = \mathcal{I}_n(\tilde{f}_n)$, \mathbb{P} -a.s. where we recall that \tilde{f}_n denotes the symmetrisation of f_n as defined in (2.4).

Remark 2.7. It might be of interest to discuss on the use of compensated Poisson integrals that are operators \mathbf{I}_n rather than the non-compensated ones \mathcal{I}_n . By definition we defined these two types of integrals as pathwise integrals under an L^1 integrability condition. However, the compensated integrals benefit from an additional probabilistic structure. Indeed for $f \in L^1(\mathbf{X}^n) \cap L^2(\mathbf{X}^n)$ and $g \in L^1(\mathbf{X}^p) \cap L^2(\mathbf{X}^p)$ and $n, p \in \mathbb{N}^*$ it is classical to get that $\mathbb{E}[\mathbf{I}_n(f)\mathbf{I}_p(g)] = \mathbf{1}_{\{n=p\}} \int \tilde{f}\tilde{g}d\rho^n$, where \tilde{f} stands again for the symmetrization of f . In other words operators \mathbf{I}_n can be defined as orthogonal operators in $L^2(\Omega)$. This orthogonality opens the way to extend the random variables $\mathbf{I}_n(f)$ for f an element of $L^2(\mathbf{X}^n)$ only. It will coincide with its pathwise definition as above only on the additional condition of belonging to $L^1(\mathbf{X}^n) \cap L^2(\mathbf{X}^n)$. Naturally, $L^1(\mathbf{X}^n) \cap L^2(\mathbf{X}^n)$ reduces to $L^2(\mathbf{X}^n)$ only when $\rho(\mathbf{X}) < +\infty$. Since our approach will consist in giving an alternative expression to the chaotic expansion which is by essence a L^2 -decomposition our main results regarding these decompositions will take their specific form in case of spaces \mathbf{X} for which $\rho(\mathbf{X}) < +\infty$.

We end this section by recalling some classical results between the Malliavin derivatives and the iterated integrals that can be found for example in [10, 14].

Proposition 2.8. (i) For any $n, p \geq 1$, $f_n \in L^1_s(\mathbf{X}^n) \cap L^2_s(\mathbf{X}^n)$ and $g_p \in L^1_s(\mathbf{X}^p) \cap L^2_s(\mathbf{X}^p)$

$$\begin{aligned} \mathbb{E}[\mathbf{I}_n(f_n)] &= 0, \\ \mathbb{E}[\mathbf{I}_n(f_n)\mathbf{I}_p(g_p)] &= n! \mathbf{1}_{\{n=p\}} \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) g_p(\mathbf{x}_1, \dots, \mathbf{x}_n) \rho^n(d\mathbf{x}_1, \dots, d\mathbf{x}_n). \end{aligned}$$

(ii) Let $j \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, with $k \leq j$ and $h \in L^2_s(\mathbf{X}^j)$. Then :

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k \mathbf{I}_j(h) = \frac{j!}{(j-k)!} \mathbf{I}_{j-k}(h(\cdot, \mathbf{x}_1, \dots, \mathbf{x}_k)), \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbf{X}^k.$$

In particular if $h = g^{\otimes j}$ with $g \in L^2(\mathbf{X})$ then

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k \mathbf{I}_j(g^{\otimes j}) = \frac{j!}{(j-k)!} \mathbf{I}_{j-k}(g^{\otimes(j-k)}) \prod_{i=1}^k g(\mathbf{x}_i), \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbf{X}^k.$$

(iii) Under the assumptions of (ii), for $k > j$,

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k \mathbf{I}_j(h) = 0, \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbf{X}^k.$$

We conclude with a particular case of Mecke’s formula (see e.g. [10, Relation (11)]) and the so-called integration by parts formula².

Lemma 2.9 (A particular case of Mecke’s formula). Let $F \in L^0(\Omega)$, $k \in \mathbb{N}$ and $h \in L^0(\mathbf{X}^k)$ such that

$$\int_{\mathbf{X}^k} |h(\mathbf{x}_1, \dots, \mathbf{x}_k)| \mathbb{E} [|F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_k}^{+,k}|] \rho^k(d\mathbf{x}_1, \dots, d\mathbf{x}_k) < +\infty.$$

Then

$$\mathbb{E} \left[F \int_{\mathbf{X}^k} h d\mathbf{N}^{(k)} \right] = \int_{\mathbf{X}^k} h(\mathbf{x}_1, \dots, \mathbf{x}_k) \mathbb{E} [F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_k}^{+,k}] \rho^k(d\mathbf{x}_1, \dots, d\mathbf{x}_k),$$

If in addition F belongs to $L^2(\Omega)$ and h in $L^2_s(\mathbf{X}^k)$ then the so-called integration by parts formula holds :

$$\mathbb{E} [F \mathbf{I}_k(h)] = \int_{\mathbf{X}^k} h(\mathbf{x}_1, \dots, \mathbf{x}_k) \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right] \rho^k(d\mathbf{x}_1, \dots, d\mathbf{x}_k). \tag{2.9}$$

²as we did not find an exact correspondance of the integration by parts formula we make use of in the literature we provide a statement with *ad hoc* restrictive assumptions that fit with our application of it in the proof of Theorem 3.12 and in the proof of Proposition 3.14.

Proof. We only make the proof of the integration by parts formula (2.9). The proof relies on the chaotic expansion recalled in Theorem 3.1 below. Indeed, by Proposition 2.8 Relation (2.9) if F is deterministic or if $F = \mathbf{I}_p(g_p)$ for $p \geq 1$ and $g_p \in L^2_s(\mathbf{X}^p)$. Theorem 3.1 gives that for any F in $L^2(\Omega)$, $F = \mathbb{E}[F] + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n(T_n F)$ where the limit holds in L^2 and each $T_n F$ belongs to in $L^2_s(\mathbf{X}^n)$ which gives the result. \square

3 Pseudo-chaotic expansion

According to Remark 2.7, we carry out our analysis mostly under the assumption that $\rho(\mathbf{X}) < +\infty$ and provide some extension to the infinite case.

3.1 Main result

The first main ingredient in our analysis is the so-called chaotic expansion on the Poisson space that we recall below.

Theorem 3.1 (See e.g. Theorem 2 in [10]). *Let F in $L^2(\Omega)$. Then there exists a unique sequence $(f_n^F)_{n \geq 1}$ with $f_n^F \in L^2_s(\mathbf{X}^n)$ such that*

$$F = \mathbb{E}[F] + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n(f_n^F),$$

where the convergence of the series holds in $L^2(\Omega)$. In addition coefficients $(f_n^F)_n$ are given as

$$f_n^F = T_n F, \quad n \geq 1$$

where $T_n F$ is defined by (2.7) and where the equality is understood in $L^2(\mathbf{X}^n)$. In addition Theorem 1 in [10] provides the convergence of the series

$$\sum_{n=1}^{+\infty} \int_{\mathbf{X}^n} |T_n F(\mathbf{x}_1, \dots, \mathbf{x}_n)|^2 \rho^n(d\mathbf{x}_1, \dots, d\mathbf{x}_n).$$

This representation involves the operators \mathbf{I}_n , we aim in re-writting such a representation with the pathwise and non-compensated integrals \mathcal{I}_n . Before going into our analysis we would like to mention the related result given in [11, Proposition 12.11] where the authors performed the reverse version of the one we aim at. Indeed they provided the chaotic expansion of random variables (named U-statistics) of the form $U := \mathcal{I}_n(h)$ for some symmetric element $h \in L^1(\mathbf{X}^n)$ by computing the operators $T_k U$ involved in the decomposition. We are interested in transforming compensated iterated integrals \mathbf{I}_n into their non-compensated counterpart \mathcal{I}_n but mostly by determining these coefficients (that will coincide with \mathcal{T}_n) and by giving their form in terms of the Malliavin derivative and a Girsanov transform (see Theorem 3.6).

The second main ingredient is the combination of Malliavin operators with a very specific Girsanov transformation.

Proposition 3.2. *Assume $\rho(\mathbf{X}) < +\infty$ and set*

$$L := \exp(\rho(\mathbf{X})) \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}. \tag{3.1}$$

L defined as above admits the following chaotic expansion :

$$L = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n((-\mathbf{1}_{\{\mathbf{X}\}})^{\otimes n}),$$

where $(-\mathbf{1}_{\{\mathbf{X}\}})^{\otimes n}$ stands for the constant map equal to $(-1)^n$. In particular for any $n \geq 1$, $T_n L = (-\mathbf{1}_{\{\mathbf{X}\}})^{\otimes n}$.

Proof. Note first that as $\rho(\mathbf{X}) < +\infty$, $\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0] = e^{-\rho(\mathbf{X})} > 0$ and so $\mathbb{E}[L] = 1$. Let $n \geq 1$, $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbf{X}$. By definition

$$\begin{aligned} L \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_n\}}^{+,n} &= e^{-\rho(\mathbf{X})} \mathbf{1}_{\{\mathbf{N} \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_n\}}^{+,n}(\mathbf{X})=0\}} \\ &= e^{-\rho(\mathbf{X})} \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=-n\}} \\ &= 0, \end{aligned}$$

where the above equalities hold P-a.s. as the elements $\mathbf{x}_1, \dots, \mathbf{x}_n$ are fixed and thus $\mathbb{P}[\mathbf{N}(\{\mathbf{x}_1, \dots, \mathbf{x}_n\}) > 0] = 0$. Hence by definition of the Malliavin operator \mathbf{D}^n , the sum in its definition reduces to the term where $J = \emptyset$, that is

$$\begin{aligned} \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n L &= e^{-\rho(\mathbf{X})} \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}} \circ \varepsilon_J^{+,|J|} = e^{-\rho(\mathbf{X})} (-1)^n \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}} = (-1)^n L, \end{aligned}$$

leading to $T_n L = (-1)^n \mathbb{E}[L] = (-1)^n$ and to the result by Theorem 3.1. □

Remark 3.3. L is exactly the (non-equivalent) Girsanov transformation under which the Poisson measure \mathbf{N} (with intensity 1 on \mathbf{X}) becomes a Poisson measure with intensity 0 on \mathbf{X} by the integrable and constant shift $\mathbf{x} \mapsto -1$ on \mathbf{X} which belongs to $L^1(\mathbf{X})$ as once again $\rho(\mathbf{X}) < +\infty$. This corresponds here to a conditioning with respect to the set $\mathbf{N}(\mathbf{X}) = 0$ of positive probability. The role of this (quite surprising) conditioning will appear in our main result Theorem 3.6.

Proposition 3.4. Assume $\rho(\mathbf{X}) < +\infty$. For $F \in L^2(\Omega)$ and $n \in \mathbb{N}^*$, the symmetric deterministic operator on $L_s^1(\mathbf{X}^n)$ defined in Definition 2.4 can be re-expressed as :

$$\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbb{E}[L \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F], \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n.$$

In addition

$$\mathcal{T}_0 F = \mathbb{E}[LF].$$

Proof. This is a simple consequence of Definitions 2.4 and 2.2 of \mathcal{T}_n and \mathbf{D}^n and of the fact that L has support $\{\mathbf{N}(\mathbf{X}) = 0\}$. For $n = 0$ we simply have that

$$\mathcal{T}_0 F = F(\emptyset) = \mathbb{E}[e^{\rho(\mathbf{X})} \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}} F].$$

Remark that since $\mathbf{D}^n F$ belongs to $L_s^2(\mathbf{X}^n)$, $\mathcal{T}_n F$ belongs to $L^1(\mathbf{X}^n)$ by Cauchy Schwarz's inequality. □

We now adapt an approximation lemma from [12, Lemma 2.3], which in our framework is valid in L^1 and not only in L^2 .

Lemma 3.5. Assume $\rho(\mathbf{X}) < +\infty$ and let $F \in L^1(\Omega)$. Assume there exists $(G_p)_p \subset L^1(\Omega)$ converging in $L^1(\Omega)$ to F . Then for any $k \geq 1$,

$$\lim_{p \rightarrow +\infty} \int_{\mathbf{X}^k} |\mathcal{T}_k G_p(\mathbf{x}_1, \dots, \mathbf{x}_k) - \mathcal{T}_k F(\mathbf{x}_1, \dots, \mathbf{x}_k)| \rho^k(d\mathbf{x}_1, \dots, d\mathbf{x}_k) = 0.$$

Proof. We adapt the proof of [12, Lemma 2.3] and give only the main arguments. By definition of the Malliavin derivative (see Definition 2.2) it is enough to prove that (after simplifying the expression by $e^{\rho(\mathbf{X})}$ which is a finite constant):

$$\lim_{p \rightarrow +\infty} \int_{\mathbf{X}^m} \mathbb{E}[|F \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m} - G_p \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m}| \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}] \rho^m(d\mathbf{x}_1, \dots, d\mathbf{x}_m) = 0,$$

for every $m \in \{1, \dots, k\}$ and $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathbf{X}^m$. Note first that for any $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathbf{X}^m$, $\mathbb{P}[\mathbf{N}(\{\mathbf{x}_1, \dots, \mathbf{x}_m\}) > 0] = 0$ since ρ is atomless. Hence

$$\mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}} = \mathbf{1}_{\{\mathbf{N} \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m}(\mathbf{X})=m\}}, \mathbb{P} - a.s..$$

Hence using Mecke's formula (see Lemma 2.9)

$$\begin{aligned} & \int_{\mathbf{X}^m} \mathbb{E} \left[|F \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m} - G_p \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m}| \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}} \right] \rho^m(d\mathbf{x}_1, \dots, d\mathbf{x}_m) \\ &= \int_{\mathbf{X}^m} \mathbb{E} \left[|F \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m} - G_p \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m}| \mathbf{1}_{\{\mathbf{N} \circ \varepsilon_{\{\mathbf{x}_1, \dots, \mathbf{x}_m\}}^{+,m}(\mathbf{X})=m\}} \right] \rho^m(d\mathbf{x}_1, \dots, d\mathbf{x}_m) \\ &= \mathbb{E} \left[|F - G_p| \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=m\}} \mathcal{I}_m(\mathbf{1}_{\mathbf{X}^{\otimes m}}) \right] \\ &\leq m! \mathbb{E} [|F - G_p|] \end{aligned}$$

which concludes the proof. □

We are now in position to state and prove the main result of this section.

Theorem 3.6. Assume $\rho(\mathbf{X}) < +\infty$ and F in $L^2(\Omega)$. Then

$$F = \mathbb{E}[LF] + \sum_{k=1}^{\mathbf{N}(\mathbf{X})} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F); \quad \mathbb{P} - a.s. \tag{3.2}$$

with $\mathcal{T}_k F = \mathbb{E}[LD^k F]$ belongs to $L^1_s(\mathbf{X}^k)$. We name Representation (3.2) the pseudo-chaotic expansion of F .

Proof. The proof is performed in several steps. We set

$$\mathbf{G} := \text{Span}\{e^{-\mathcal{I}_1(f)}; f : \mathbf{X} \rightarrow \mathbb{R}_+\}.$$

Step 1: We prove that for any $M \in \mathbb{N}$ and $F \in \mathbf{G}$,

$$F^M := F \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} = \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(\mathbb{E}[LF] + \sum_{k=1}^M \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F) \right), \mathbb{P} - a.s.. \tag{3.3}$$

By linearity of \mathbf{G} it is enough to consider $F = e^{-\mathcal{I}_1(f)} \in \mathbf{G}$. It holds that $\mathbb{E}[F] = e^{-\int_{\mathbf{X}} (1-e^{-f}) d\rho} = 1 + \sum_{n=1}^{+\infty} \frac{(\int_{\mathbf{X}} (e^{-f}-1) d\rho)^n}{n!}$ and by induction $\mathbf{D}^n F = F(e^{-f} - 1)^{\otimes n}$ so that $T_n F = \mathbb{E}[F](e^{-f} - 1)^{\otimes n}$ and $\mathcal{T}_n F = \mathbb{E}[LF](e^{-f} - 1)^{\otimes n}$. Hence for any $n \geq 1$,

$$\begin{aligned} \mathbf{I}_n(T_n F) &= \mathbb{E}[F] \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} \prod_{i=1}^n (e^{-f(\mathbf{x}_i)} - 1) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \rho^{n-|J|}(d\mathbf{x}_{J^c}) \\ &= \mathbb{E}[F] \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} \int_{\mathbf{X}^{|J|}} \left(\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \rho(d\mathbf{y}) \right)^{n-|J|} \prod_{i \in J} (e^{-f(\mathbf{x}_i)} - 1) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \\ &= \mathbb{E}[F] \sum_{k=0}^n \frac{n!}{k!(n-k)!} \left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \rho(d\mathbf{y}) \right)^{n-k} \int_{\mathbf{X}^k} \prod_{i=1}^k (e^{-f(\mathbf{x}_i)} - 1) \mathbf{N}^{(k)}(d\mathbf{x}_1, \dots, d\mathbf{x}_k). \end{aligned}$$

Hence for any $M \in \mathbb{N}^*$, using $\mathcal{I}_k((e^{-f} - 1)^{\otimes k}) = 0$ on $\{\mathbf{N}(\mathbf{X}) = M\}$ for $k > M$

$$\begin{aligned} & \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \mathbf{I}_n(T_n F) \\ &= \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \mathbb{E}[F] \left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \rho(d\mathbf{y}) \right)^n \end{aligned}$$

$$+ \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \mathbb{E}[F] \sum_{k=1}^{n \wedge M} \frac{n! \left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \boldsymbol{\rho}(d\mathbf{y})\right)^{n-k}}{k!(n-k)!} \mathcal{I}_k((e^{-f} - 1)^{\otimes k}).$$

Recall that $\sum_{n=0}^{+\infty} \frac{1}{n!} \left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \boldsymbol{\rho}(d\mathbf{y})\right)^n = 1/\mathbb{E}(F)$, therefore

$$\begin{aligned} F^M &:= F \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \\ &= \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(\mathbb{E}[F] + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n(T_n F) \right) \\ &= \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \mathbb{E}[F] \left(1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \boldsymbol{\rho}(d\mathbf{y})\right)^n \right) \\ &+ \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(\mathbb{E}[F] \sum_{n=1}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n \wedge M} \frac{n! \left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \boldsymbol{\rho}(d\mathbf{y})\right)^{n-k}}{k!(n-k)!} \mathcal{I}_k((e^{-f} - 1)^{\otimes k}) \right) \\ &= \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(1 + \mathbb{E}[F] \sum_{n=1}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n \wedge M} \frac{n! \left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \boldsymbol{\rho}(d\mathbf{y})\right)^{n-k}}{k!(n-k)!} \mathcal{I}_k((e^{-f} - 1)^{\otimes k}) \right) \\ &= \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(1 + \mathbb{E}[F] \sum_{k=1}^M \frac{1}{k!} \mathcal{I}_k((e^{-f} - 1)^{\otimes k}) \sum_{n=k}^{+\infty} \frac{\left(-\int_{\mathbf{X}} (e^{-f(\mathbf{y})} - 1) \boldsymbol{\rho}(d\mathbf{y})\right)^{n-k}}{(n-k)!} \right) \\ &= \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(1 + \sum_{k=1}^M \frac{1}{k!} \mathcal{I}_k((e^{-f} - 1)^{\otimes k}) \right). \end{aligned}$$

Thus by linearity, Relation (3.3) is in force for any F in \mathbf{G} and any $M \geq 0$.

Step 2: We extend Relation (3.3) to any F in $L^2(\Omega)$ and any $M \geq 0$.

Fix $M \geq 1$ and let F in $L^2(\Omega)$ and let $F^M := F \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}}$. As \mathbf{G} is dense in $L^2(\Omega)$ (see e.g. [10, Lemma 2]) there exists a sequence $(G_p)_p \subset \mathbf{G}$ such that $(G_p)_p$ converges in $L^2(\Omega)$ to F . Let also $G_p^M := G_p \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}}$. By the previous step we have that

$$G_p^M = \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(\mathbb{E}[L G_p] + \sum_{k=1}^M \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k G_p) \right).$$

In addition by Lemma 3.5, for any $k \in \{1, \dots, M\}$, $\mathcal{T}_k F$ is well-defined in $L^1_s(\mathbf{X}^k)$ and is the L^1 -limit of $(\mathcal{T}_k G_p)_p$. Hence

$$\begin{aligned} &\mathbb{E} \left[\left| F^M - \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left(\mathbb{E}[L F] + \sum_{k=1}^M \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F) \right) \right| \right] \\ &\leq \lim_{p \rightarrow +\infty} \mathbb{E}[|F^M - G_p^M|] + \mathbb{E} \left[\mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} \left| \mathbb{E}[L(G_p - F)] + \sum_{k=1}^M \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k G_p - \mathcal{T}_k F) \right| \right] \\ &\leq \lim_{p \rightarrow +\infty} \sum_{k=1}^M \frac{1}{k!} \mathbb{E} \left[\mathbf{1}_{\{\mathbf{N}(\mathbf{X})=M\}} |\mathcal{I}_k(\mathcal{T}_k G_p - \mathcal{T}_k F)| \right] \\ &\leq \lim_{p \rightarrow +\infty} \sum_{k=1}^M \frac{1}{k!} \mathbb{E} [\mathcal{I}_k(|\mathcal{T}_k G_p - \mathcal{T}_k F|)] \\ &\leq \sum_{k=1}^M \frac{1}{k!} \lim_{p \rightarrow +\infty} \int_{\mathbf{X}^k} |\mathcal{T}_k G_p - \mathcal{T}_k F| d\boldsymbol{\rho}^k = 0. \end{aligned}$$

Thus

$$F^M = \mathbf{1}_{\{N(\mathbf{X})=M\}} \left(\mathbb{E}[LF] + \sum_{k=1}^M \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F) \right); \mathbb{P} - a.s..$$

Step 3: Consider F in $L^2(\Omega)$. As $N(\mathbf{X}) < +\infty$ \mathbb{P} -a.s. we have that from the step before (by noting that the coefficients in the decomposition of each F^M only depend on F) that

$$F = \sum_{M=0}^{+\infty} F^M = \mathbb{E}[LF] + \sum_{k=1}^{N(\mathbf{X})} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F); \quad \mathbb{P} - a.s..$$

and so $\mathbb{E} \left[\left| \sum_{k=1}^{N(\mathbf{X})} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F) \right| \right] < +\infty$. □

3.2 Discussion and properties of the pseudo-chaotic representation

Remark 3.7 (Case when $\rho(\mathbf{X}) < +\infty$). Note that since $N(\mathbf{X}) < +\infty$ \mathbb{P} -a.s. and the fact that $N^{(n)}(\mathbf{1}_{\{N(\mathbf{X}) < n\}}) = 0$, Representation (3.2) naturally re-writes as

$$F = \mathbb{E}[LF] + \sum_{k=1}^{+\infty} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F); \quad \mathbb{P} - a.s..$$

Remark 3.8 (Case when $\rho(\mathbf{X}) = +\infty$). We have performed our analysis on the case where $\rho(\mathbf{X}) < +\infty$. The reason for this restriction lies in the strong use in our proof of the chaotic expansion which involves a convergence in $L^2(\Omega)$ of the series in Theorem 3.1. The isometry of the operators \mathbf{I}_n calls for kernels $T_n F$ in $L^2(\mathbf{X}^n)$. However the pseudo-chaotic expansion given as Representation (3.2) is an equality \mathbb{P} -a.s. involving a sum with a finite number of terms \mathbb{P} -a.s. and for which any term $\mathcal{I}_k(\mathcal{T}_k F)$ is defined as a $L^1(\Omega)$ random variable with $\mathcal{T}_k F \in L^1(\mathbf{X}^k)$. As we mentioned in Remark 2.7 only the case $\rho(\mathbf{X}) < +\infty$ allows one to unify the L^1 and L^2 -integrability conditions on \mathbf{X} (or spaces \mathbf{X}^k) as a L^2 -condition. This opens the way to investigate a pure $L^1(\Omega)$ theory not relying on the chaotic expansion. In other words :

In the case $\rho(\mathbf{X}) = +\infty$, is it possible to find necessary and sufficient conditions on F so that for any $n \geq 1$, $\mathcal{T}_n F \in L^1(\mathbf{X}^n)$ and

$$F = F(\emptyset) + \sum_{k=1}^{N(\mathbf{X})} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F)$$

where the convergence holds in $L^1(\Omega)$ (and where the term $\mathbb{E}[LF]$ needs to be defined properly) ? We leave this question for future research.

In the previous remark we alluded to a possible extension of our result to the case where $\rho(\mathbf{X}) = +\infty$ however we naturally wrote the expansion using the operators \mathcal{T}_n and the change of measure L (that needs to be made precise). We give some insights of both accounts.

Proposition 3.9. Let $F \in L^2(\Omega)$ such that there exists $c \in \mathbb{R}$ and $(g_n)_{n \geq 1}$ with $g_n \in L^1_s(\mathbf{X}^n)$ for any n and such that

$$F = c + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathcal{I}_n(g_n),$$

where the series converges in $L^1(\Omega)$. Then

$$c = F(\emptyset); \quad g_n = \mathcal{T}_n F \text{ in } L^1(\mathbf{X}^n), \quad \forall n \geq 1.$$

Proof. We deal separately the cases $\rho(\mathbf{X}) < +\infty$ and $\rho(\mathbf{X}) = +\infty$.

Case 1: $\rho(\mathbf{X}) < +\infty$

In this case it is first worth mentioning that $\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0] > 0$. Hence by definition

$$c = F(\emptyset) = \mathbb{E}[LF], \text{ where as in (3.1) } L = e^{\rho(\mathbf{X})} \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}.$$

The other identifications follow the same lines. Fix $p, n \geq 1$. Definition 2.4 entails for any $(\mathbf{x}_1, \dots, \mathbf{x}_p) \in \mathbf{X}^p$

$$\begin{aligned} & \mathcal{T}_p \mathcal{I}_n(g_n)(\mathbf{x}_1, \dots, \mathbf{x}_p) \\ &= \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}} (-1)^{p-|J|} \mathcal{I}_n(g_n) \left(\sum_{\mathbf{y} \in J} \delta_{\mathbf{y}} \right) \\ &= \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}; |J| \geq n} (-1)^{p-|J|} \mathcal{I}_n(g_n) \left(\sum_{\mathbf{y} \in J} \delta_{\mathbf{y}} \right), \end{aligned}$$

hence $\mathcal{T}_p \mathcal{I}_n(g_n) = 0$ if $p < n$. In case $p = n$

$$\mathcal{T}_p \mathcal{I}_p(g_p)(\mathbf{x}_1, \dots, \mathbf{x}_p) = \mathcal{I}_p(g_p) \left(\sum_{i=1}^p \delta_{\mathbf{x}_i} \right) = p! g_p(\mathbf{x}_1, \dots, \mathbf{x}_p).$$

We now deal with the case $p > n$. To simplify the notations, note that any subset $U \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ of cardinal n writes as $U = \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ with $\mathbf{y}_i \in \{\mathbf{x}_1, \dots, \mathbf{x}_p\}$. We thus write $g_n(U) := g_n(\mathbf{y}_1, \dots, \mathbf{y}_n)$. From the previous computations we have that

$$\begin{aligned} \mathcal{T}_p \mathcal{I}_n(g_n)(\mathbf{x}_1, \dots, \mathbf{x}_p) &= \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}; |J| \geq n} (-1)^{p-|J|} \mathcal{I}_n(g_n) \left(\sum_{\mathbf{y} \in J} \delta_{\mathbf{y}} \right) \\ &= \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}; |J| \geq n} (-1)^{p-|J|} \sum_{U \subset J; |U|=n} g_n(U) \\ &= \sum_{U \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}; |U|=n} g_n(U) \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}; U \subset J} (-1)^{p-|J|} \\ &= \sum_{U \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}; |U|=n} g_n(U) \sum_{k=n}^p \frac{(p-n)!}{(k-n)!(p-n-(k-n))!} (-1)^{p-k} \\ &= \sum_{U \subset \{\mathbf{x}_1, \dots, \mathbf{x}_p\}; |U|=n} g_n(U) (-1)^n \sum_{k'=0}^{p-n} \frac{(p-n)!}{k'!(p-n-k')!} (-1)^{p-k'} = 0. \end{aligned}$$

To conclude we just remark that by definition the series $F = c + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathcal{I}_n(g_n)$ reduces to a finite sum $F = c + \sum_{n=1}^{\mathbf{N}(\mathbf{X})} \frac{1}{n!} \mathcal{I}_n(g_n)$. Hence we get that for any $p \geq 1$,

$$\mathcal{T}_p F = \mathcal{T}_p \left(c + \sum_{n=1}^{p-1} \frac{1}{n!} \mathcal{I}_n(g_n) \right) + \frac{1}{p!} \mathcal{T}_p \mathcal{I}_p(g_p) + \mathcal{T}_p \left(\sum_{n=p+1}^{\mathbf{N}(\mathbf{X})} \frac{1}{n!} \mathcal{I}_n(g_n) \right) = \frac{1}{p!} \mathcal{T}_p \mathcal{I}_p(g_p) = g_p.$$

Case 2: $\rho(\mathbf{X}) = +\infty$

Recall that as ρ is assumed to be σ -finite there exists an increasing (with respect to the inclusion) sequence of subsets $(\mathcal{R}_k)_{k \geq 1}$ of \mathbf{X} such that $\cup_{k \in \mathbb{N}} \mathcal{R}_k = \mathbf{X}$ and $\rho(\mathcal{R}_k) < +\infty$. For any $k \geq 1$ we let

$$\Omega^k := \left\{ \omega = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in \mathcal{R}_k, j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\}$$

and F^k the restriction of F to Ω^k . In addition, $\lim_{k \rightarrow +\infty} \Omega^k = \Omega$. In other words for any $k \geq 1$, F^k is a random variable in the situation of case 1 where the space \mathbf{X} is replaced with \mathcal{R}^k with $\rho(\mathcal{R}^k) < +\infty$. Hence reproducing the analysis performed in Case 1, we get using the definition of operators \mathcal{T}_p (see Definition 2.4) and Proposition 3.4,

$$\mathcal{T}_p F(\mathbf{x}_1, \dots, \mathbf{x}_p) = \mathcal{T}_p F^k(\mathbf{x}_1, \dots, \mathbf{x}_p) = g_p(\mathbf{x}_1, \dots, \mathbf{x}_p), \forall (\mathbf{x}_1, \dots, \mathbf{x}_p) \in (\mathcal{R}^k)^p.$$

Letting k go to $+\infty$ gives that $\mathcal{T}_p F = g_p$ for any $p \geq 1$. The identification of c follows the same lines as above. Indeed using the notation $L^k := e^{\rho(\mathcal{R}^k)} \mathbf{1}_{\{\mathbf{N}(\mathcal{R}^k)=0\}}$ and the fact that $F^k \mathbf{1}_{\{\mathbf{N}(\mathcal{R}^k)\}} = c$ we get that $\mathbb{E}[L^k F^k] = c$ for any $k \geq 1$ leading to $c = \lim_{k \rightarrow +\infty} \mathbb{E}[L^k F^k] = F \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}$. Note this is an example of a conditioning with respect to an event of probability 0 that can be defined through this approximation. \square

Remark 3.10. This result is related to Remark 3.8 as here we proved that if the expansion is valid in L^1 then there is some sort of uniqueness of the kernels and of the constant c . As for the classical chaotic expansion, the uniqueness is true only under the symmetric assumption on the kernels.

Remark 3.11. As mentioned in the proof, the representation of the constant term $F\mathbf{N}(\mathbf{X})$ in case $\rho(\mathbf{X}) = +\infty$ is an example of successful conditioning with respect to an event of probability 0. Indeed, probability theory is rich in situations where it is not possible to give sense to conditioning with respect to a set of probability 0 and on the contrary to other situations where it can be properly defined. The derivation of the constant c falls into the second category.

We aim in giving a better taste of the actual link between the coefficients $T_n F$ appearing in the chaotic expansion with coefficients $\mathcal{T}_n F$ in the pseudo-chaotic one. Since we work on this specific connexion we assume that $\rho(\mathbf{X}) < +\infty$ and to avoid additional technicalities we assume that F is bounded.

Theorem 3.12. Assume that $\rho(\mathbf{X}) < +\infty$. Let F in $L^\infty(\Omega)$. Then

$$\sum_{j=1}^{+\infty} \frac{(-1)^j}{j!} \int_{\mathbf{X}^j} T_j F(\mathbf{x}_1, \dots, \mathbf{x}_j) \rho^j(d\mathbf{x}_1, \dots, d\mathbf{x}_j) = \mathbb{E}[LF] - \mathbb{E}[F].$$

Proof. Let $G := 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n(\mathbf{1}_{\{\mathbf{X}\}}^{\otimes n})$ which is well defined as the constant mappings $(\mathbf{1}_{\{\mathbf{X}\}})^{\otimes n}(\mathbf{x}_1, \dots, \mathbf{x}_n) := 1$ belong to $L^2(\mathbf{X})$. Hence as F is an element of $L^2(\Omega)$, the series

$$\sum_{n=1}^{+\infty} \int_{\mathbf{X}^j} \frac{1}{j!} |T_j F| d\rho^j = \sum_{n=1}^{+\infty} \int_{\mathbf{X}^j} \frac{1}{j!} |T_j F T_j G| d\rho^j$$

converges absolutely (see e.g. [10, Theorem 1]). We have

$$\begin{aligned} & \sum_{j=1}^{+\infty} \frac{(-1)^j}{j!} \int_{\mathbf{X}^j} T_j F(\mathbf{x}_1, \dots, \mathbf{x}_j) \rho^j(d\mathbf{x}_1, \dots, d\mathbf{x}_j) \\ &= \sum_{j=1}^{+\infty} \frac{(-1)^j}{j!} \int_{\mathbf{X}^j} \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_j)}^j F \right] \rho^j(d\mathbf{x}_1, \dots, d\mathbf{x}_j) \\ &= \sum_{j=1}^{+\infty} \frac{1}{j!} \mathbb{E} \left[\int_{\mathbf{X}^j} \prod_{i=1}^j (-\mathbf{1}_{\{\mathbf{X}\}}(\mathbf{x}_i)) \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_j)}^j F \rho^j(d\mathbf{x}_1, \dots, d\mathbf{x}_j) \right]. \end{aligned}$$

The integration by parts formula recalled in (2.9) enables us to deduce that

$$\sum_{j=1}^{+\infty} \frac{(-1)^j}{j!} \int_{\mathbf{X}^j} T_j F(\mathbf{x}_1, \dots, \mathbf{x}_j) \rho^j(d\mathbf{x}_1, \dots, d\mathbf{x}_j)$$

$$\begin{aligned}
 &= \sum_{j=1}^{+\infty} \frac{1}{j!} \mathbb{E} [F \mathbf{I}_j ((-\mathbf{1}_{\{\mathbf{X}\}})^{\otimes j})] \\
 &= \mathbb{E} [F(L - 1)],
 \end{aligned}$$

where we identify the chaotic expansion of L according to Proposition 3.2 and where the exchange between expectation and the series is justified as follows

$$\begin{aligned}
 &\lim_{p \rightarrow +\infty} \mathbb{E} \left[\left| F(L - 1 - \sum_{j=1}^p \frac{1}{j!} \mathbf{I}_j ((-\mathbf{1}_{\{\mathbf{X}\}})^{\otimes j}) \right| \right] \\
 &= \lim_{p \rightarrow +\infty} \mathbb{E} \left[\left| F \sum_{j=p+1}^{+\infty} \frac{1}{j!} \mathbf{I}_j ((-\mathbf{1}_{\{\mathbf{X}\}})^{\otimes j}) \right| \right] \\
 &\leq \mathbb{E}[|F|^2]^{1/2} \lim_{p \rightarrow +\infty} \left(\sum_{j=p+1}^{+\infty} \frac{1}{j!} \mathbb{E} [|\mathbf{I}_j ((-\mathbf{1}_{\{\mathbf{X}\}})^{\otimes j})|^2] \right)^{1/2} \\
 &= \mathbb{E}[|F|^2]^{1/2} \lim_{p \rightarrow +\infty} \left(\sum_{j=p+1}^{+\infty} \frac{1}{j!} \rho(\mathbf{X})^j \right)^{1/2} = 0.
 \end{aligned}$$

□

Corollary 3.13. Assume that $\rho(\mathbf{X}) < +\infty$. Let F in $L^2(\Omega)$ such that $F \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}} = 0$ then

$$\sum_{j=1}^{+\infty} \frac{(-1)^j}{j!} \int_{\mathbf{X}^j} T_j F(\mathbf{x}_1, \dots, \mathbf{x}_j) \rho^j(d\mathbf{x}_1, \dots, d\mathbf{x}_j) = -\mathbb{E}[F].$$

In other words the chaotic expansion of F simplifies to involve only iterated integrals in which at least one integrand is the Poisson measure \mathbf{N} (all the terms involving only $d\rho^j$ cancel out with the expectation of F).

This observation calls then for a re-writing of the terms $T_n F$ in the chaotic expansion in view of an identification of the terms $\mathcal{T}_n F$ in the pseudo-chaotic expansion.

Proposition 3.14. Assume that $\rho(\mathbf{X}) < +\infty$. Let F in $L^\infty(\Omega)$ and fix $k \geq 1$. If $\int_{\mathbf{X}^k} \mathbb{E}[|\mathbf{D}^k F|^2] d\rho^k < +\infty$ then for ρ^k -a.e. $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ in \mathbf{X}^k

$$\begin{aligned}
 \mathcal{T}_k F(\mathbf{x}_1, \dots, \mathbf{x}_k) &= T_k F(\mathbf{x}_1, \dots, \mathbf{x}_k) \\
 &+ \sum_{j=k+1}^{+\infty} \frac{(-1)^{j-k}}{(j-k)!} \int_{\mathbf{X}^{j-k}} T_j F(\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_j) \rho^{j-k}(d\mathbf{x}_j, \dots, d\mathbf{x}_{k+1}).
 \end{aligned}$$

Proof. Recall property $\mathbf{D}^j = \mathbf{D}^{j-k} \mathbf{D}^k$ and the integration by parts formula (2.9). Fix $p \geq k$. We have

$$\begin{aligned}
 &\mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right] + \sum_{j=k+1}^p \frac{(-1)^{j-k}}{(j-k)!} \int_{\mathbf{X}^{j-k}} \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_j)}^j F \right] \rho^{j-k}(d\mathbf{x}_j, \dots, d\mathbf{x}_{k+1}) \\
 &= \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right] + \sum_{j=k+1}^p \frac{(-1)^{j-k}}{j!} \frac{j!}{(j-k)!} \int_{\mathbf{X}^{j-k}} \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_{k+1}, \dots, \mathbf{x}_j)}^{j-k} \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right] dx_j. \\
 &= \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right] + \sum_{j=k+1}^p \frac{1}{j!} \frac{j!}{(j-k)!} \mathbb{E} \left[\mathbf{I}_{j-k} \left((-\mathbf{1}_{\{\mathbf{X}\}} \right)^{\otimes (j-k)} \right) \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right] + \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \sum_{\ell=1}^{p-k} \frac{1}{\ell!} \mathbf{I}_\ell \left((-\mathbf{1}_{\{\mathbf{x}\}})^{\otimes \ell} \right) \right] \\
 &= \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \sum_{\ell=0}^{p-k} \frac{1}{\ell!} \mathbf{I}_\ell \left((-\mathbf{1}_{\{\mathbf{x}\}})^{\otimes \ell} \right) \right].
 \end{aligned}$$

Letting p go to $+\infty$ and using the fact that we assumed $\mathbb{E} \left[|\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F|^2 \right] < +\infty$ as F is supposed to be bounded, the last term converges to $\mathbb{E} \left[L\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F \right]$ and the result follows. \square

Remark 3.15. In the proposition we see the somehow artificial additional assumption $\int_{\mathbf{X}^k} \mathbb{E}[|\mathbf{D}^k F|^2] d\rho^k < +\infty$ even though we know that $\mathcal{T}_k F$ belongs to $L^1(\mathbf{X}^k)$. It would be of interest to get rid of this condition.

3.3 Pseudo-chaotic expansion for counting processes

The pseudo-chaotic expansion can be coupled with the Poisson imbedding to provide a representation of general counting processes. We first recall a particular case of the Poisson imbedding representation and refer to [9, Chapter 4] and [3] for additional material on this topic. Hereafter, $\mathbf{X} := \mathbb{R}_+^2$ and a generic element \mathbf{x} in \mathbb{R}_+^2 is denoted $\mathbf{x} := (t, \theta)$, with t and θ in \mathbb{R}_+ . We denote by $\mathbb{F} := (\mathcal{F}_t^{\mathbf{N}})_{t \geq 0}$ the natural history of \mathbf{N} with $\mathcal{F}_t^{\mathbf{N}} := \sigma(\mathbf{N}([0, s] \times B))$, $s \leq t$, $B \in \mathcal{B}(\mathbb{R}_+)$, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel σ -field on \mathbb{R}_+ .

Theorem 3.16 (Poisson imbedding). *Let $\mathbf{X} := \mathbb{R}_+^2$ and $\mathfrak{X} := \mathcal{B}(\mathbb{R}_+^2)$. Consider H a point process on \mathbb{R}_+ with stochastic intensity a \mathbb{F} -predictable stochastic process λ . These pair of processes can be represented on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random Poisson measure \mathbf{N} on \mathbf{X} and the following representation holds true :*

$$H_T = \int_{(0, T] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_t\}} \mathbf{N}(dt, d\theta); \quad \forall T > 0.$$

We now combine this result with our expansion for counting processes with bounded intensity processes.

Notation 3.17. *For $k \geq 1$ and any $(\mathbf{x}_1, \dots, \mathbf{x}_k) \in (\mathbb{R}_+^2)^k$ we denote by $(\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(k)})$ the ordering with respect to the t -coordinate of $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ so that $0 < t_{(1)} < \dots < t_{(k)}$.*

Theorem 3.18. *Let $T > 0$, H a counting process with intensity λ bounded by $M > 0$. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $F \in L^2(\Omega)$ with*

$$F := \int_{(0, T] \times [0, M]} \zeta(T - t) \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta).$$

Then F admits a pseudo-chaotic expansion with respect to \mathbf{N} with

$$F = \sum_{k=1}^{\mathbf{N}(\mathbf{X})} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F); \quad \mathbb{P} - a.s. \tag{3.4}$$

and for all $(\mathbf{x}_1, \dots, \mathbf{x}_k) \in ([0, T] \times [0, M])^k$

$$\mathcal{T}_k F(\mathbf{x}_1, \dots, \mathbf{x}_k) := \mathbb{E} \left[L\zeta(T - t_{(k)}) \mathbf{D}_{(\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(k-1)})}^{k-1} \mathbf{1}_{\{\theta_{(k)} \leq \lambda_{(t_{(k)})}\}} \right], \tag{3.5}$$

where we make use of Notation 3.17. In particular H_T admits the previous pseudo-chaotic expansion by choosing $\zeta \equiv 1$.

Proof. First note that using the Poisson imbedding given in Theorem 3.16 for any $T > 0$, H_T can be represented as :

$$H_T = \int_{(0,T] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta),$$

that coincides with F by choosing $\zeta \equiv 1$ as λ is supposed to be uniformly bounded by M leading to

$$H_T = \int_{(0,T] \times [0,M]} \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta).$$

To achieve the representation for F , Theorem 3.6 can be applied with $\mathbf{X} := [0, T] \times [0, M]$ and $\rho(dx) := dx = dt d\theta$ and we get that

$$F = \mathbb{E}[LF] + \sum_{k=1}^{\mathbf{N}(\mathbf{X})} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F); \quad \mathbb{P} - a.s.$$

with $L = \exp(TM) \mathbf{1}_{\{\mathbf{N}([0,T] \times [0,M])=0\}}$ and $\mathcal{T}_k F = \mathbb{E}[L \mathbf{D}^k F]$. Thus $\mathbb{E}[LF] = 0$. Recall also that \mathbf{D}^k is a symmetric operator so that $\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F = \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F$. Since by definition

$$F = \int_{(0,T] \times [0,M]} \zeta(T-t) \mathbf{1}_{\{\theta \leq \lambda_t\}} \mathbf{N}(dt, d\theta)$$

and λ is \mathbb{F} -predictable, we deduce that for any $k \geq 1$ and any $(\mathbf{x}_1, \dots, \mathbf{x}_k) \in (\mathbb{R}_+^2)^k$ with

$$\begin{aligned} \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k F &= \zeta(T-t_{(k)}) \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_{(k-1)})}^{k-1} \mathbf{1}_{\{\theta_{(k)} \leq \lambda_{t_{(k)}}\}} \\ &+ \int_{[0,T] \times \mathbb{R}_+} \zeta(T-t) \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_k)}^k \mathbf{1}_{\{\theta \leq \lambda_t\}} N(dt, d\theta). \end{aligned} \quad (3.6)$$

As L is supported on $\mathbf{1}_{\{\mathbf{N}([0,T] \times [0,M])=0\}}$

$$\mathcal{T}_k F(\mathbf{x}_1, \dots, \mathbf{x}_k) = \mathbb{E} \left[L \zeta(T-t_{(k)}) \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_{(k-1)})}^{k-1} \mathbf{1}_{\{\theta_{(k)} \leq \lambda_{t_{(k)}}\}} \right].$$

□

4 Application to linear Hawkes processes

Throughout this section $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ denotes a map in $L^1(\mathbb{R}_+; dt)$.

4.1 Generalities on linear Hawkes processes

Assumption 4.1. *The mapping $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to $L^1(\mathbb{R}_+; dt)$ with*

$$\|\Phi\|_1 := \int_{\mathbb{R}_+} \Phi(t) dt < 1.$$

For f, g in $L^1(\mathbb{R}_+; dt)$ we define the convolution of f and g by

$$(f * g)(t) := \int_0^t f(t-u)g(u)du, \quad t \geq 0.$$

Proposition 4.2 (See e.g. [1]). *Assume Φ enjoys Assumption 4.1. Let*

$$\Phi_1 := \Phi, \quad \Phi_n(t) := \int_0^t \Phi(t-s)\Phi_{n-1}(s)ds, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}^*. \quad (4.1)$$

For every $n \geq 1$, $\|\Phi_n\|_1 = \|\Phi\|_1^n$ and the mapping $\Psi := \sum_{n=1}^{+\infty} \Phi_n$ is well-defined as a limit in $L_1(\mathbb{R}_+; dt)$ and $\|\Psi\|_1 = \frac{\|\Phi\|_1}{1-\|\Phi\|_1}$.

Definition 4.3 (Linear Hawkes process, [5]). Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0})$ be a filtered probability space, $\mu > 0$ and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying Assumption 4.1. A linear Hawkes process $H := (H_t)_{t \geq 0}$ with parameters μ and Φ is a counting process such that

- (i) $H_0 = 0, \mathbb{P} - a.s.$,
- (ii) its $(\mathbb{F}$ -predictable) intensity process is given by

$$\lambda_t := \mu + \int_{(0,t)} \Phi(t-s) dH_s, \quad t \geq 0,$$

that is for any $0 \leq s \leq t$ and $A \in \mathcal{F}_s$,

$$\mathbb{E} [\mathbf{1}_{\{A\}}(H_t - H_s)] = \mathbb{E} \left[\int_{(s,t]} \mathbf{1}_{\{A\}} \lambda_r dr \right].$$

4.2 Pseudo-chaotic expansion of linear Hawkes processes and explicit representation

We aim at providing the coefficients in the pseudo-chaotic expansion of a linear Hawkes process. We start with some general facts regarding a linear Hawkes process. We consider the probability structure presented in Section 2 with a Poisson random measure \mathbf{N} on $\mathbf{X} = \mathbb{R}_+^2$ and with intensity 1 with respect to $\rho(dx) := dx := dt d\theta$ where we once again make use of the notation $\mathbf{x} := (t, \theta) \in \mathbb{R}_+^2$. With this setting the Hawkes process with parameters μ and Φ can be represented as follows.

Proposition 4.4. Let Φ as in Assumption 4.1 and $\mu > 0$ and (H, λ) the Hawkes process defined as the unique solution to the SDE

$$\begin{cases} H_t = \int_{(0,t] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta), \\ \lambda_t = \mu + \int_{(0,t) \times \mathbb{R}_+} \Phi(t-s) dH_s, \quad t \geq 0. \end{cases}$$

Let $n \in \mathbb{N}^*$ and $\{\mathbf{y}_1, \dots, \mathbf{y}_n\} = \{(s_1, \theta_1), \dots, (s_n, \theta_n)\} \subset \mathbf{X}$ with $0 < s_1 \leq \dots \leq s_n \leq t$. We set $(a_1^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}, \dots, a_n^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}})$ the solution to the system

$$\begin{cases} a_1^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}} = \mu + \mathbf{1}_{\{\theta_1 \leq \mu\}}, \\ a_j^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}} = \mu + \sum_{i=1}^{j-1} \Phi(s_j - s_i) \mathbf{1}_{\{\theta_i \leq a_i^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}\}}, \quad j \in \{2, \dots, n\}. \end{cases} \tag{4.2}$$

which is the triangular system

$$\begin{cases} a_1^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}} = \mu + \mathbf{1}_{\{\theta_1 \leq \mu\}}, \\ a_2^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}} = \mu + \Phi(s_2 - s_1) \mathbf{1}_{\{\theta_2 \leq a_1^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}\}}, \\ \vdots \\ a_n^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}} = \mu + \sum_{i=1}^{n-1} \Phi(s_n - s_i) \mathbf{1}_{\{\theta_i \leq a_i^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}\}}. \end{cases} \tag{4.3}$$

Let $\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}} := \sum_{i=1}^n \delta_{\mathbf{y}_i} \in \Omega$. Then the values of the deterministic path $\lambda(\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}})$ (resulting from the evaluation of λ at the specific $\omega = \varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}$) at times s_1, \dots, s_n is given by

$$(\lambda_{s_1}(\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}), \dots, \lambda_{s_n}(\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}})) = (a_1^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}, \dots, a_n^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}).$$

In addition

$$\lambda_t(\varpi_{\{y_1, \dots, y_n\}}) = \mu + \sum_{i=1}^n \Phi(t - s_i) \mathbf{1}_{\{\theta_i \leq a_i\}} \mathbf{1}_{\{s_i < t\}}, \quad \forall t \geq s_n. \quad (4.4)$$

Proof. Let $t \geq 0$. By definition of λ ,

$$\begin{aligned} & \lambda_t(\varpi_{\{y_1, \dots, y_n\}}) \\ & := \mu + \left(\int_{(0,t)} \Phi(t-u) dH_u \right) (\varpi_{\{y_1, \dots, y_n\}}) \\ & = \mu + \left(\int_{(0,t)} \Phi(t-u) \mathbf{1}_{\{\theta \leq \lambda_u\}} \mathbf{N}(du, d\theta) \right) (\varpi_{\{y_1, \dots, y_n\}}) \\ & = \mu + \int_{(0,t)} \Phi(t-u) \mathbf{1}_{\{\theta \leq \lambda_u(\varpi_{\{y_1, \dots, y_n\}})\}} (\mathbf{N}(du, d\theta)(\varpi_{\{y_1, \dots, y_n\}})) \\ & = \mu + \int_{(0,t)} \Phi(t-u) \mathbf{1}_{\{\theta \leq \lambda_u(\varpi_{\{y_1, \dots, y_n\}})\}} (\varpi_{\{y_1, \dots, y_n\}})(du, d\theta) \\ & = \mu + \int_{(0,s_1)} \Phi(t-u) \mathbf{1}_{\{\theta \leq \lambda_u(\varpi_{\{y_1, \dots, y_n\}})\}} \mathbf{1}_{\{u < t\}} (\varpi_{\{y_1, \dots, y_n\}})(du, d\theta) \\ & \quad + \sum_{i=1}^{n-1} \int_{[s_i, s_{i+1})} \Phi(t-u) \mathbf{1}_{\{\theta \leq \lambda_u(\varpi_{\{y_1, \dots, y_n\}})\}} \mathbf{1}_{\{u < t\}} (\varpi_{\{y_1, \dots, y_n\}})(du, d\theta) \\ & = \mu + \sum_{i=1}^{n-1} \int_{[s_i, s_{i+1})} \Phi(t-u) \mathbf{1}_{\{\theta \leq \lambda_u(\varpi_{\{y_1, \dots, y_n\}})\}} \mathbf{1}_{\{u < t\}} (\varpi_{\{y_1, \dots, y_n\}})(du, d\theta) \\ & = \mu + \sum_{i=1}^n \Phi(t - s_i) \mathbf{1}_{\{\theta_i \leq \lambda_{s_i}(\varpi_{\{y_1, \dots, y_n\}})\}} \mathbf{1}_{\{s_i < t\}}. \end{aligned}$$

In addition, by definition, of λ , for any i , $\lambda_{s_i}(\varpi_{\{y_1, \dots, y_n\}}) = \lambda_{s_i}(\varpi_{(y_1, \dots, y_{i-1})})$. Hence, the evaluation of λ at the specific path $\varpi_{\{y_1, \dots, y_n\}}$ is the deterministic path completely determined by its value at the dates s_1, \dots, s_n . Indeed,

$$\lambda_t(\varpi_{\{y_1, \dots, y_n\}}) = \mu, \quad \forall t \in [0, s_1],$$

in particular $a_1 := \lambda_{s_1}(\varpi_{\{y_1, \dots, y_n\}}) = \mu$. From this we deduce that for $t \in (s_1, s_2]$,

$$\lambda_t(\varpi_{\{y_1, \dots, y_n\}}) = \mu + \Phi(t - s_1) \mathbf{1}_{\{\theta_1 \leq \lambda_{s_1}(\varpi_{\{y_1, \dots, y_n\}})\}} = \mu + \Phi(t - s_1) \mathbf{1}_{\{\theta_1 \leq \mu\}}.$$

In particular $a_2 := \lambda_{s_2}(\varpi_{\{y_1, \dots, y_n\}}) = \mu + \Phi(s_2 - s_1) \mathbf{1}_{\{\theta_1 \leq \mu\}} = \mu + \Phi(s_2 - s_1) \mathbf{1}_{\{\theta_1 \leq a_2\}}$. By induction we get that for $t \in (s_j, s_{j+1}]$ ($j \in \{1, \dots, n-1\}$),

$$\lambda_t(\varpi_{\{y_1, \dots, y_n\}}) = \mu + \sum_{i=1}^j \Phi(t - s_i) \mathbf{1}_{\{\theta_i \leq a_i\}} \mathbf{1}_{\{s_i < t\}},$$

with $a_i := \lambda_{s_i}(\varpi_{\{y_1, \dots, y_n\}})$. In other words, (a_1, \dots, a_n) solves the triangular system of the statement. \square

The expansion will be performed on windows $[0, T] \times [0, M]$. Such approach is coherent since the decomposition is consistent as the windows enlarge ($T \rightarrow \infty$ and $M \rightarrow \infty$). To state the result rigorously let us introduce the following notation.

Definition 4.5. Let $T > 0$ and $M \geq \mu$. We set

(i) the measurable map $\varepsilon^{-,T,M}$ on Ω to itself :

$$\begin{aligned} \varepsilon^{-,T,M} : \Omega &\longrightarrow \Omega \\ \omega &\longmapsto \omega^{T,M}, \end{aligned}$$

where for any element $\omega = \sum_{j=1}^n \delta_{\mathbf{x}_j}$ of Ω ($\mathbf{x}_j \in \mathbf{X}$, $j = 1, \dots, n$, $n \in \mathbb{N} \cup \{+\infty\}$), $\omega^{T,M}$ is defined as :

$$\omega^{T,M} = \sum_{j=1}^n \delta_{\mathbf{x}_j} \mathbf{1}_{\{\mathbf{x}_j \in [0,T] \times [0,M]\}}.$$

In other words $\varepsilon^{-,T,M}$ removes all the atoms outside $[0, T] \times [0, M]$ or put it differently provides the restriction of \mathbf{N} on $[0, T] \times [0, M]$.

(ii) For any $F \in L^0(\Omega)$ we set $F^{T,M} := F \circ \varepsilon^{-,T,M}$.

(iii) $\mathbf{N}^{T,M} := \mathbf{N} \circ \varepsilon^{-,T,M}$ the restriction of \mathbf{N} on $[0, T] \times [0, M]$.

(iv) $\Omega^{T,M} := \varepsilon^{-,T,M}(\Omega)$ that is

$$\Omega^{T,M} := \left\{ \omega = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in [0, T] \times [0, M], j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\} \subset \Omega.$$

From these definitions, it follows that $\lim_{T,M \rightarrow +\infty} \varepsilon^{-,T,M} = \mathbf{Id}_\Omega$ (the identity map) and that $\lim_{T,M \rightarrow +\infty} \Omega^{T,M} = \Omega$ as a non-decreasing limit with respect to the inclusion. If F is a linear functional of the Poisson-measure \mathbf{N} , that is

$$F = c + \int_{\mathbb{R}_+^2} \zeta(s, \theta) \mathbf{N}(ds, d\theta),$$

for some $c \in \mathbb{R}$ and ζ a $\mathcal{F}^{\mathbf{N}}$ -measurable process with $\mathbb{E} \left[\int_{\mathbb{R}_+^2} |\zeta(s, \theta)| ds d\theta \right] < +\infty$, then

$$F^{T,M} = c + \int_{[0,T] \times [0,M]} \zeta \circ \varepsilon^{-,T,M}(s, \theta) \mathbf{N}(ds, d\theta) = c + \int_{[0,T] \times [0,M]} \zeta(s, \theta) \mathbf{N}(ds, d\theta)$$

and $\lim_{T,M \rightarrow +\infty} F^{T,M} = F$, \mathbb{P} -a.s..

Theorem 4.6. [Pseudo-chaotic expansion for linear Hawkes processes]

Let Φ as in Assumption 4.1 and $\mu > 0$. Let (H, λ) be the unique solution of

$$\begin{cases} H_t = \int_{(0,t] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta), \\ \lambda_t = \mu + \int_{(0,t] \times \mathbb{R}_+} \Phi(t-s) dH_s, \quad t \geq 0 \end{cases} \quad (4.5)$$

Then H is a linear Hawkes process with intensity λ in the sense of Definition 4.3. Fix $T > 0$. Consider $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be $\zeta \equiv \Phi$ or $\zeta \equiv 1$ we set

$$F := \int_{(0,T] \times \mathbb{R}_+} \zeta(T-s) \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta).$$

We make use of the notations of Definition 4.5. For any $M \geq \mu$, it holds that

$$F^{T,M} = F \circ \varepsilon^{-,T,M} = \sum_{k=1}^{\mathbf{N}([0,T] \times [0,M])} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F \mathbf{1}_{\{([0,T] \times [0,M])^k\}}), \quad (4.6)$$

with

$$\begin{cases} \mathcal{T}_1 F(\mathbf{x}_1) = \mathbf{1}_{\{\theta_1 \leq \mu\}} \zeta(T - t_{(1)}), & \text{and for } k \geq 2 \\ \mathcal{T}_k F(\mathbf{x}_1, \dots, \mathbf{x}_k) \\ = \zeta(T - t_{(k)}) \left((-1)^{k-1} \mathbf{1}_{\{\theta_k \leq \mu\}} + \sum_{n=1}^{k-1} \sum_{\substack{\{\mathbf{y}_1, \dots, \mathbf{y}_n\} \\ \subset \{\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(k-1)}\}}} (-1)^{k-1-n} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}})\}} \right), \end{cases} \quad (4.7)$$

and $\lim_{T, M \rightarrow +\infty} F^{T, M} = F$, $\mathbb{P} - a.s.$. We recall for the previous expression

- Notation 3.17 for $(\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(k)})$
- Notation $\sum_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\} \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{k-1}\}}$ stands for the sum over all subsets $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ of cardinal n of $\{\mathbf{x}_1, \dots, \mathbf{x}_{k-1}\}$
- $\lambda_t(\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}})$ is given by (4.4) in Proposition 4.4.

Proof. First by [3, 4, 8] the system of SDEs (4.5) admits a unique solution which is a Hawkes process (we refer to [8] for more details on the construction with pathwise uniqueness). Hence by uniqueness $(H^{T, M} = H \circ \varepsilon^{-, T, M}, \lambda^{T, M} = \lambda \circ \varepsilon^{-, T, M})$ is solution to

$$\begin{cases} H_t^{T, M} = \int_{(0, t] \times [0, M]} \mathbf{1}_{\{\theta \leq \lambda_s^{T, M}\}} \mathbf{N}(ds, d\theta), \\ \lambda_t^{T, M} = \mu + \int_{(0, t) \times \mathbb{R}_+} \Phi(t - s) dH_s^{T, M}, \quad t \in [0, T], \end{cases} \quad (4.8)$$

and converges to the solution of the system (4.5), that is $\lim_{T, M \rightarrow +\infty} H^{T, M} = H$ and $\lim_{T, M \rightarrow +\infty} \lambda^{T, M} = \lambda$, $\mathbb{P} - a.s.$.

By Theorem 3.18, $F^{T, M}$ admits a pseudo-chaotic expansion

$$F^{T, M} = \sum_{k=1}^{+\infty} \mathcal{I}_k(\mathcal{T}_k F^{T, M} \mathbf{1}_{\{([0, T] \times [0, M])^k\}}), \quad (4.9)$$

where for all $(\mathbf{x}_1, \dots, \mathbf{x}_k) \in ([0, T] \times [0, M])^k$

$$\mathcal{T}_k F^{T, M}(\mathbf{x}_1, \dots, \mathbf{x}_k) := \mathbb{E} \left[L_{T, M}(\omega) \zeta(T - t_{(k)}) \mathbf{D}_{(\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(k-1)})}^{k-1} \mathbf{1}_{\{\theta_{(k)} \leq \lambda_{t_{(k)}}^{T, M}\}} \right], \quad (4.10)$$

with the ordering convention of Notation 3.17 and where³ $L_{T, M} := e^{TM} \mathbf{1}_{\{\mathbf{N}([0, T] \times [0, M]) = 0\}}$. By definition, $\lambda^{T, M} = \lambda$ on $\Omega^{T, M}$. For such k and $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ we assume for simplicity that $(\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(k)}) = (\mathbf{x}_1, \dots, \mathbf{x}_k)$. Using (2.8)

$$\begin{aligned} & L_{T, M}(\omega) (\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_{k-1})}^{k-1} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}\}})(\omega) \\ &= L_{T, M}(\omega) \sum_{J \subset \{1, \dots, k-1\}} (-1)^{k-1-|J|} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\omega + \sum_{j \in J} \delta_{\mathbf{x}_j})\}}, \end{aligned}$$

where the sum is over all subsets J of $\{1, \dots, k-1\}$ including the empty set which is of cardinal 0. The previous expression then rewrites as

$$\begin{aligned} & L_{T, M}(\omega) (\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_{k-1})}^{k-1} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}\}})(\omega) \\ &= \exp(MT) \mathbf{1}_{\{\mathbf{N}([0, T] \times [0, M]) = 0\}} \sum_{J \subset \{1, \dots, k-1\}} (-1)^{k-1-|J|} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\omega + \sum_{j \in J} \delta_{\mathbf{x}_j})\}} \end{aligned}$$

³Here we use the subscript notation for $L_{T, M}$ as because of the specific normalisation e^{TM} it is not the " $\circ \varepsilon^{-, T, M}$ " transformation of some random variable L .

$$\begin{aligned}
 &= \exp(MT) \mathbf{1}_{\{\omega([0,T] \times [0,M])=0\}} \sum_{J \subset \{1, \dots, k-1\}} (-1)^{k-1-|J|} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\omega + \sum_{j \in J} \delta_{\mathbf{x}_j})\}} \\
 &= \exp(MT) \mathbf{1}_{\{\omega([0,T] \times [0,M])=0\}} \sum_{J \subset \{1, \dots, k-1\}} (-1)^{k-1-|J|} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\sum_{j \in J} \delta_{\mathbf{x}_j})\}}.
 \end{aligned}$$

Recall that $\mathbb{E}[\exp(MT) \mathbf{1}_{\{\omega([0,T] \times [0,M])=0\}}] = 1$. In other words the effect of $L_{T,M}(\omega)$ is to freeze the evaluation of the intensity process λ on a specific outcome given by the atoms $(\mathbf{x}_1, \dots, \mathbf{x}_{k-1})$. Taking the expectation and reorganizing the sum above we get

$$\begin{aligned}
 &\mathcal{T}_k F^{T,M}(\mathbf{x}_1, \dots, \mathbf{x}_k) \\
 &= \mathbb{E} \left[L_{T,M}(\omega) \zeta(T - t_{(k)}) \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_{k-1})}^{k-1} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}\}} \right] \\
 &= \zeta(T - t_{(k)}) \sum_{J \subset \{1, \dots, k-1\}} (-1)^{k-1-|J|} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\sum_{j \in J} \delta_{\mathbf{x}_j})\}} \\
 &= \zeta(T - t_{(k)}) \left((-1)^{k-1} \mathbf{1}_{\{\theta_k \leq \mu\}} + \sum_{n=1}^{k-1} \sum_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\} \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{k-1}\}} (-1)^{k-1-n} \mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}})\}} \right) \\
 &= \zeta(T - t_{(k)}) \left((-1)^{k-1} \mathbf{1}_{\{\theta_k \leq \mu\}} + \sum_{n=1}^{k-1} \sum_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\} \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{k-1}\}} (-1)^{k-1-n} \mathbf{1}_{\{\theta_k \leq a_k^{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}\}} \right).
 \end{aligned}$$

Note that in each term $\mathbf{1}_{\{\theta_k \leq \lambda_{t_k}(\sum_{j \in J} \delta_{\mathbf{x}_j})\}}$, $\sum_{j \in J} \delta_{\mathbf{x}_j}$ is deterministic and $\lambda_{t_k}(\sum_{j \in J} \delta_{\mathbf{x}_j})$ is explicitly given by the triangular system in Notation 4.4 and this only dependency on (T, M) is the definition domain of points \mathbf{x}_i . More precisely, for (\tilde{T}, \tilde{M}) with $\tilde{T} \geq T$ and $\tilde{M} \geq M$, $\mathcal{T}_k F^{T,M} = \mathcal{T}_k F^{\tilde{T}, \tilde{M}}$ on $([0, T] \times [0, M])^k$. This explains why in the statement in (4.6) we replace the coefficients $\mathcal{T}_k F^{T,M}$ by $\mathcal{T}_k F$ given in system (4.7).

For $k = 1$, the previous expression just reduces to

$$\mathcal{T}_1 F(\mathbf{x}_1) = \mathbb{E} \left[L_{T,M}(\omega) \zeta(T - t_{(1)}) \mathbf{1}_{\{\theta_1 \leq \lambda_{t_1}\}} \right] = \mathbf{1}_{\{\theta_1 \leq \mu\}} \zeta(T - t_{(1)}).$$

□

Remark 4.7. Choosing $\zeta \equiv 1$ and $\zeta \equiv \Phi$ in Theorem 4.6 gives respectively the pseudo-chaotic expansion of H_T and λ_T .

Remark 4.8. To the price of cumbersome notations, the previous result can be extended to non-linear Hawkes processes; that is counting processes H with intensity process of the form

$$\begin{cases} H_t = \int_{(0,t] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_s\}} \mathbf{N}(ds, d\theta), \\ \lambda_t = h \left(\mu + \int_{(0,t] \times \mathbb{R}_+} \Phi(t-s) dH_s \right), \quad t \in [0, T]. \end{cases} \tag{4.11}$$

where $h : \mathbb{R} \rightarrow \mathbb{R}_+$, $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\|h\|_1 \|\Phi\|_1 < 1$. Indeed, when computing the coefficients in the expansion, the Poisson measure N is cancelled and involves evaluations of the intensity process at a specific configurations of the form $\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}$. This evaluation can be done by a straightforward extension of Proposition 4.4 for a non-linear Hawkes process; in other words for both linear or non-linear process the intensity process λ is a deterministic function of the fixed configuration of the form $\varpi_{\{\mathbf{y}_1, \dots, \mathbf{y}_n\}}$.

Remark 4.9. We would like to comment on the advantage of the pseudo-chaotic expansion compared to the usual one for the value H_T of a linear Hawkes process at any time T . Recall the two decompositions

$$H_T = \mathbb{E}[H_T] + \sum_{j=1}^{+\infty} \frac{1}{j!} \mathbf{I}_j(f_j^{H_T}), \quad \text{with} \quad f_j^{H_T}(\mathbf{x}_1, \dots, \mathbf{x}_j) = \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_j)}^j H_T \right]$$

$$= \sum_{k=1}^{+\infty} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k H_T)$$

For the chaotic expansion, in order to determine each coefficient f_j one has to compute $f_j^{H_T}(\mathbf{x}_1, \dots, \mathbf{x}_j) = \mathbb{E} \left[\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_j)}^j H_T \right]$ which turns out to be quite implicit for a general Φ kernel. Indeed, already for the first coefficient, using Relation (3.6) we have

$$\begin{aligned} f_j^{H_T}(\mathbf{x}_1) &= \mathbb{E}[\mathbf{D}_{\mathbf{x}_1} H_T] \\ &= \mathbb{E}[\mathbf{1}_{\{\theta_1 \leq \lambda_{t_1}\}}] + \int_{(t_1, T] \times \mathbb{R}_+} \mathbb{E}[\mathbf{D}_{\mathbf{x}_1} \mathbf{1}_{\{\theta \leq \lambda_s\}}] d\theta ds \\ &= \mathbb{P}[\theta_1 \leq \lambda_{t_1}] + \int_{(t_1, T]} \mathbb{E}[\mathbf{D}_{\mathbf{x}_1} \lambda_s] ds. \end{aligned}$$

The quantity $\int_{(t_1, T]} \mathbb{E}[\mathbf{D}_{\mathbf{x}_1} \lambda_s] ds$ has been computed in [7] however a closed form expression for $\mathbb{P}[\theta_1 \leq \lambda_{t_1}]$ for any kernel Φ satisfying Assumption 4.1 is unknown to the authors.

In contradistinction, Theorem 4.6 gives an explicit expression for the coefficients $\mathcal{T}_k H_T$. In that sense, the pseudo-chaotic expansion (4.6) is an exact representation and an explicit solution to the Hawkes equation formulation as given in Definition 4.3.

5 The pseudo-chaotic expansion and the Hawkes equation

The aim of this section is to investigate further the link between a decomposition of the form (4.9) that we named pseudo-chaotic expansion and the characterization of a Hawkes process as in Definition 4.3. First, let us emphasize that both the standard chaotic expansion and the pseudo-chaotic expansion characterize a given random variable and not a stochastic process. For instance in Theorem 4.6, the coefficients $\mathcal{T}_k H_T$ for the expansion of H_T depend on the time T . In this section, we consider once again the linear Hawkes process, which is essentially described as a counting process with a specific stochastic intensity like in (1.3), and we adopt a different point of view based on population dynamics and branching representation as in [6] or [2]. Inspired by this branching representation, we build in Theorem 5.3 below a stochastic process via its pseudo-chaotic expansion which is an integer-valued piecewise-constant and non-decreasing process with the specific intensity form of a Hawkes process. Nevertheless, although this stochastic process satisfies the stochastic self-exciting intensity equation which determines a Hawkes process, it fails to be a counting process as it may exhibit jumps larger than one. This leaves open further developments for studying the pseudo-chaotic expansion of processes; we refer to Discussion 5.4.

5.1 A pseudo-chaotic expansion and branching representation

Throughout this section we will make use of classical stochastic analysis tools we still describe elements of \mathbf{X} as $\mathbf{x} = (t, \theta)$. The branching representation viewpoint consists in counting the number of individuals in generation n , where generation 1 corresponds of the migrants. We therefore define a series of counting processes $X_t^{(n)}$ (where n stands for the generation) as follows. To this end we introduce some notations.

Definition 5.1. For a fixed $t > 0$ we define a family of non-negative maps $(b_n)_{n \geq 1}$, $b_n : \mathbf{X}^n \rightarrow \mathbb{R}_+$ symmetric with

$$\begin{aligned} b_1^t(\mathbf{x}_1) &:= \mathbf{1}_{\{\theta_1 \leq \mu\}} \mathbf{1}_{\{t_1 \leq t\}} \\ b_n^t(\mathbf{x}_1, \dots, \mathbf{x}_n) &:= \prod_{i=2}^n \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{1}_{\{t_n \leq t\}}; \quad n \geq 1 \end{aligned}$$

where as b_n are symmetric it is understood that $b_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = b_n(\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(n)})$ according to Notation 3.17.

Proposition 5.2. (i) Let for any $t \geq 0$

$$X_t^{(1)} := \int_{(0,t] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \mathbf{N}(dt_1, d\theta_1) = \mathcal{I}_1(b_1^t), \tag{5.1}$$

and for $n \geq 2$,

$$X_t^{(n)} := \frac{1}{n!} \mathcal{I}_n(b_n^t) \tag{5.2}$$

We set in addition

$$X_t := \sum_{n=1}^{+\infty} X_t^{(n)}, \tag{5.3}$$

where the series converges uniformly (in t) on compact sets; that is for any $T > 0$,

$$\lim_{p \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0, T]} \left| X_t - \sum_{n=1}^p X_t^{(n)} \right| \right] = 0.$$

(ii) We set the \mathbb{F}^X -predictable process

$$\ell_t := \mu + \int_{(0,t)} \Phi(t-r) dX_r = \mu + \sum_{n=1}^{+\infty} \int_{(0,t)} \Phi(t-r) dX_r^{(n)}, \quad t \geq 0 \tag{5.4}$$

where $\mathbb{F}^X := (\mathcal{F}_t^X)_{t \geq 0}$, with $\mathcal{F}_t^X := \sigma(X_s, s \leq t)$.

The proof of the convergence of the series (5.3) is postponed to Section 5.2. The resulting process X aims at counting the number of individuals in the population, while the predictable process ℓ is the candidate to be the self-exciting intensity of the process X . This intensity reads as follows

$$\begin{aligned} \ell_t &= \mu + \int_{(0,t)} \Phi(t-r) dX_r \\ &= \mu + \int_{(0,t) \times \mathbb{R}_+} \Phi(t-t_1) \mathbf{1}_{\{\theta_1 \leq \mu\}} \mathbf{N}(d\mathbf{x}_1) \\ &+ \sum_{n=2}^{+\infty} \int_{(0,t] \times \mathbb{R}_+} \Phi(t-t_n) \int_{(0,t_n] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^n \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\ &= \mu + \sum_{n=1}^{+\infty} \int_{(0,t] \times \mathbb{R}_+} \Phi(t-t_n) \int_{(0,t_n] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^n \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n). \end{aligned}$$

The main result of this section is stated below. Its proof based on Lemmata 5.8 and 5.9 is postponed to Section 5.3.

Theorem 5.3. Let $\mu > 0$ and Φ satisfying Assumption 4.1. Recall the stochastic process X and ℓ defined (in Definition-Proposition 5.2). We define the process $M := (M_t)_{t \geq 0}$ by

$$M_t := X_t - \int_{(0,t)} \ell_u du, \quad t \geq 0. \tag{5.5}$$

Then X is a \mathbb{N} -valued non-decreasing process piecewise constant with predictable intensity process ℓ , in the sense that the process M is a \mathbb{F}^N -martingale (and so a \mathbb{F}^H -martingale as M is \mathbb{F}^H -adapted and $\mathcal{F}^H \subset \mathcal{F}^N$).

Remark 5.4. In other words X would be a Hawkes process if it were a counting process, unless it has the same expectation of a Hawkes process. Indeed some atoms generates simultaneous jumps : any atom (t_0, θ_0) of \mathbb{N} with $\theta_0 \leq \mu$ will generate a jump for $X^{(1)}$ and for all $X^{(n)}$ who have t_0 as an ancestor so $X_{t_0} - X_{t_0-}$ may be larger than one. This example leads to a question. More specifically, the more intricate structure of the coefficients in Theorem 4.6 for the pseudo-chaotic expansion of a Hawkes process as a sum and differences of indicator functions suggests a necessary algebraic structure with respect to the time variable on the coefficients of the expansion to guarantee the counting-feature of the process. We leave this issue for future research.

Finally, we would like to mention that our approach contributes to building a bridge between the Poisson imbedding representation of a Hawkes process in terms of a Poisson measure on \mathbb{R}_+^2 and the Poisson cluster approach (branching approach) initially provided in [6]. The combination of this two points of view has been studied for instance in [2] where the concept of age pyramid is introduced and involves a branching approach together with Poisson SDEs.

Before handling in Section 5.3 the proof of Theorem 5.3, we start with some useful technical lemmata.

5.2 Technical results and proofs

Lemma 5.5. *Let f in $L_1(\mathbb{R}_+; dt)$. For any $n \in \mathbb{N}$ with $n \geq 3$, and for any $0 \leq s \leq t$,*

$$\int_s^t \int_s^u \Phi_{n-1}(t-r)f(r)drdu = \int_s^t \int_s^{t_n} \int_s^{t_{n-1}} \dots \int_s^{t_2} \prod_{i=2}^n \Phi(t_i - t_{i-1})f(t_1)dt_1 \dots dt_n. \tag{5.6}$$

Proof. For g a mapping, let $\mathfrak{F}(g)$ the Fourier transform of g . Fix $s \geq 0$ and $n \geq 3$. On the one hand let

$$F(u) := \int_s^u \Phi_{n-1}(u-r)f(r)dr, \quad u \geq s$$

so that $F = \Phi_{n-1} * \tilde{f}$, with $\tilde{f}(v) := f(v)\mathbf{1}_{\{v \geq s\}}$. We have that

$$\mathfrak{F}(F) = \mathfrak{F}(\Phi_{n-1})\mathfrak{F}(\tilde{f}) = \mathfrak{F}(\Phi)^{n-1}\mathfrak{F}(\tilde{f}),$$

since by definition of mappings Φ_i (see Relation (4.1)), each Φ_i is the i^{th} convolution of Φ with itself; hence $\mathfrak{F}(\Phi_{n-1}) = (\mathfrak{F}(\Phi))^{n-1}$. On the other hand let :

$$G(u) := \int_s^u \Phi(u - t_{n-1}) \int_s^{t_{n-1}} \Phi(t_{n-1} - t_{n-2}) \dots \int_s^{t_2} \Phi(t_2 - t_1)f(t_1)dt_1 \dots dt_{n-1},$$

we immediately get that $\mathfrak{F}(G) = (\mathfrak{F}(\Phi))^{n-1}\mathfrak{F}(\tilde{f}) = \mathfrak{F}(F)$. Using the inverse Fourier transform (on the left) we get that $F(u) = G(u)$ for a.e. u leading to $\int_s^t F(u)du = \int_s^t G(u)du$ which is Relation (5.6). □

Lemma 5.5 allows one to prove Proposition 5.2, namely to prove that the series $X_t = \sum_{n=1}^{+\infty} X_t^{(n)}$ converges uniformly (in t) on compact sets; that is for any $T > 0$,

$$\lim_{p \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0, T]} \left| X_t - \sum_{n=1}^p X_t^{(n)} \right| \right] = 0.$$

Proof. (Proposition 5.2) Set $T > 0$. For $p \geq 2$, let $S_p := \sum_{n=1}^p X^{(n)}$. As each counting process $X^{(n)}$ is non-negative and non-decreasing in t we have that :

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - S_p(t)| \right]$$

$$\begin{aligned}
 &= \mathbb{E} \left[\sup_{t \in [0, T]} \left| \sum_{n=p+1}^{+\infty} X_t^{(n)} \right| \right] \\
 &= \sum_{n=p+1}^{+\infty} \mathbb{E} \left[X_T^{(n)} \right] \\
 &= \mu \sum_{n=p+1}^{+\infty} \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} \prod_{i=2}^n \Phi(t_i - t_{i-1}) dt_1 \cdots dt_n \\
 &= \mu \sum_{n=p+1}^{+\infty} \int_0^T \int_0^t \Phi_{n-1}(T-r) dr dt, \quad \text{by Lemma 5.5} \\
 &\leq \mu T \sum_{n=p}^{+\infty} \|\Phi_n\|_1 = \mu T \frac{\|\Phi\|_1^p}{1 - \|\Phi\|_1} \xrightarrow{p \rightarrow +\infty} 0, \quad \text{by Proposition 4.2.}
 \end{aligned}$$

□

For simplicity we denote $\mathbb{E}_{s-}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_{s-}^N]$ and we introduce the following notation.

Notation 5.6. Let $s \geq 0, v \geq s$ and $n \geq 1$, we set

$$\begin{aligned}
 h_v^{s,(n)} &:= \int_{(0,s)} \Phi(v - t_n) dX_{t_n}^{(n)} \\
 &= \int_{(0,s) \times \mathbb{R}_+} \Phi(v - t_n) \int_{(0,t_n] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^n \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, \mathbf{x}_n).
 \end{aligned}$$

Lemma 5.7. For $n \geq 3, 0 \leq s \leq u$ we set

$$\begin{aligned}
 Q(n, u) &:= \mathbb{E}_{s-} \left[\int_{(0,u)} \Phi(u - r) dX_r^{(n-1)} \right] \\
 &= \mathbb{E}_{s-} \left[\int_{(0,u) \times \mathbb{R}_+} \Phi(u - t_{n-1}) \int_{(0,t_{n-1}] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-1} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n-1)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \right].
 \end{aligned}$$

We have

$$\begin{aligned}
 Q(n, u) &= h_u^{s,(n-1)} \\
 &+ \sum_{i=1}^{n-2} \int_s^u \cdots \int_s^{t_{n-i+1}^*} \Phi(u - t_{n-1}) \prod_{k=n-i+1}^{n-1} \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s,(n-i-1)} dt_{n-i} \cdots dt_{n-1} \\
 &+ \mu \int_s^u \int_s^{t_{n-1}} \cdots \int_s^{t_2} \Phi(u - t_{n-1}) \prod_{k=2}^{n-1} \Phi(t_k - t_{k-1}) dt_1 \cdots dt_{n-1}, \tag{5.7}
 \end{aligned}$$

where $t_{n-i+1}^* := t_{n-i+1}$ for $i \neq 1$ and $t_{n-i+1}^* := u$ for $i = 1$. An explicit computation gives that Relation (5.7) is valid for $n = 2$ using the convention that for $a, b \in \mathbb{Z}$ with $a > b$, and for any map $f : \mathbb{Z} \rightarrow \mathbb{R}, \prod_{i=a}^b f(i) := 1$.

Proof. Using Fubini's theorem⁴ we have

$$Q(n, u)$$

⁴Here Fubini's theorem is used pathwise as the integral against \mathbf{N} are finite-a.e.

$$\begin{aligned}
 &= \int_{(0,s] \times \mathbb{R}_+} \Phi(u - t_{n-1}) \int_{(0,t_{n-1}] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-1} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n-1)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \\
 &+ \int_{\mathbb{R}_+} \int_s^u \Phi(u - t_{n-1}) \mathbb{E}_{s-} \left[\int_{(0,t_{n-1}] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-1} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-2}) \right] dt_{n-1} d\theta_{n-1} \\
 &= \int_{(0,s] \times \mathbb{R}_+} \Phi(u - t_{n-1}) \int_{(0,t_{n-1}] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-1} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n-1)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \\
 &+ \int_s^u \Phi(u - t_{n-1}) \mathbb{E}_{s-} \left[\int_{(0,t_{n-1}] \times \mathbb{R}_+} \Phi(t_{n-1} - t_{n-2}) \int_{(0,t_{n-2}] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-2} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n-2)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-2}) \right] dt_{n-1} \\
 &= \int_{(0,s] \times \mathbb{R}_+} \Phi(u - t_{n-1}) \int_{(0,t_{n-1}] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-1} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n-1)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \\
 &+ \int_s^u \Phi(u - t_{n-1}) Q(n-1, t_{n-1}) dt_{n-1} \\
 &= h_u^{s,(n-1)} + \int_s^u \Phi(u - t_{n-1}) Q(n-1, t_{n-1}) dt_{n-1}.
 \end{aligned}$$

So we have proved that

$$Q(n, u) = h_u^{s,(n-1)} + \int_s^u \Phi(u - t_{n-1}) Q(n-1, t_{n-1}) dt_{n-1}.$$

We then deduce by induction that

$$\begin{aligned}
 Q(n, u) &= h_u^{s,(n-1)} \\
 &+ \sum_{i=1}^{n-2} \int_s^u \cdots \int_s^{t_{n-i+1}^*} \Phi(u - t_{n-1}) \prod_{k=n-i+1}^{n-1} \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s,(n-i-1)} dt_{n-i} \dots dt_{n-1} \\
 &+ \mu \int_s^u \int_s^{t_{n-1}} \cdots \int_s^{t_2} \Phi(u - t_{n-1}) \prod_{k=2}^{n-1} \Phi(t_k - t_{k-1}) dt_1 \dots dt_{n-1}.
 \end{aligned}$$

Indeed, assuming the previous relation is true for $Q(n, u)$ for a given $n \geq 3$ and for any u ; we have

$$\begin{aligned}
 &Q(n+1, u) \\
 &= h_u^{s,(n)} + \int_s^u \Phi(u - t_n) Q(n, t_n) dt_n \\
 &= h_u^{s,(n)} + \int_s^u \Phi(u - t_n) h_{t_n}^{s,(n-1)} dt_n \\
 &+ \int_s^u \Phi(u - t_n) \left[\sum_{i=1}^{n-2} \int_s^{t_n} \cdots \int_s^{t_{n-i+1}} \Phi(t_n - t_{n-1}) \prod_{k=n-i+1}^{n-1} \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s,(n-i-1)} dt_{n-i} \dots dt_{n-1} \right] dt_n \\
 &+ \mu \int_s^u \Phi(u - t_n) \int_s^{t_n} \cdots \int_s^{t_2} \Phi(t_n - t_{n-1}) \prod_{k=2}^{n-1} \Phi(t_k - t_{k-1}) dt_1 \dots dt_{n-1} dt_n \\
 &= h_u^{s,(n)} + \int_s^u \Phi(u - t_n) h_{t_n}^{s,(n-1)} dt_n \\
 &+ \sum_{i=1}^{n-2} \int_s^u \Phi(u - t_n) \int_s^{t_n} \cdots \int_s^{t_{n-i+1}} \prod_{k=n-i+1}^n \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s,(n-i-1)} dt_{n-i} \dots dt_{n-1} dt_n
 \end{aligned}$$

$$\begin{aligned}
 &+ \mu \int_s^u \int_s^{t_n} \cdots \int_s^{t_2} \Phi(u - t_n) \prod_{k=2}^n \Phi(t_k - t_{k-1}) dt_1 \dots dt_{n-1} dt_n \\
 &= h_u^{s,(n)} \\
 &+ \sum_{i=0}^{n-2} \int_s^u \Phi(u - t_n) \int_s^{t_n} \cdots \int_s^{t_{n-i+1}} \Phi(t_n - t_{n-1}) \prod_{k=n-i+1}^n \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s,(n-i-1)} dt_{n-i} \dots dt_{n-1} dt_n \\
 &+ \mu \int_s^u \int_s^{t_n} \cdots \int_s^{t_2} \Phi(u - t_n) \prod_{k=2}^n \Phi(t_k - t_{k-1}) dt_1 \dots dt_{n-1} dt_n,
 \end{aligned}$$

where for $i = 0$ we use the same convention as previously. Hence

$$\begin{aligned}
 &Q(n + 1, u) \\
 &= h_u^{s,(n)} \\
 &+ \sum_{j=1}^{(n+1)-2} \int_s^u \Phi(u - t_n) \int_s^{t_n} \cdots \int_s^{t_{(n+1)-j+1}^*} \prod_{k=(n+1)-j+1}^{n+1} \Phi(t_k - t_{k-1}) h_{t_{n+1-j}}^{s,((n+1)-j-1)} dt_{(n+1)-j} \dots dt_n \\
 &+ \mu \int_s^u \int_s^{t_n} \cdots \int_s^{t_2} \Phi(u - t_n) \prod_{k=2}^{(n+1)-1} \Phi(t_k - t_{k-1}) dt_1 \dots dt_{(n+1)-1},
 \end{aligned}$$

which gives the result. □

5.3 Proof of Theorem 5.3

The proof consists in showing that the process M defined by (5.5) is a \mathbb{F}^N -martingale, that is for any $0 \leq s \leq t$,

$$\mathbb{E}_{s-} [X_t - X_s] = \int_s^t \mathbb{E}_{s-} [\ell_r] dr, \tag{5.8}$$

where we recall that for simplicity $\mathbb{E}_{s-} [\cdot] := \mathbb{E}[\cdot | \mathcal{F}_{s-}^N]$. This result is a direct consequence of Lemma 5.8 and 5.9 below in which we compute both terms in (5.8) (recall Notation 5.6).

Lemma 5.8 first computes the left-hand side of in (5.8).

Lemma 5.8. *For any $0 \leq s \leq t$ we have that*

$$\mathbb{E}_{s-} \left[\int_{(s,t]} dX_r \right] = \int_s^t (\mu + h_u^s) du + \int_s^t \int_s^u \Psi(u - r) (\mu + h_r^s) dr du. \tag{5.9}$$

Proof. We have :

$$\mathbb{E}_{s-} \left[\int_{(s,t]} dX_r^{(1)} \right] = \mathbb{E}_{s-} \left[\int_{(s,t] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \mathbf{N}(d\mathbf{x}_1) \right] = \mu(t - s). \tag{5.10}$$

Let $n \geq 2$.

$$\begin{aligned}
 &\mathbb{E}_{s-} \left[\int_{(s,t]} dX_r^{(n)} \right] \\
 &= \mathbb{E}_{s-} \left[\int_{(s,t] \times \mathbb{R}_+} \int_{(0,t_n] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^n \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, \mathbf{x}_n) \right] \\
 &= \int_s^t \mathbb{E}_{s-} \left[\int_{(0,t_n] \times \mathbb{R}_+} \Phi(t_n - t_{n-1}) \int_{(0,t_{n-1}] \times \mathbb{R}_+} \cdots \int_{(0,t_2] \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-1} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n-1)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \right] dt_n \\
 &= \int_s^t Q(n, t_n) dt_n.
 \end{aligned}$$

Hence, by Lemma 5.7,

$$\begin{aligned} \mathbb{E}_{s-} \left[\int_{(s,t]} dX_r^{(n)} \right] &= \int_s^t h_{t_n}^{s,(n-1)} dt_n \\ &\quad + \sum_{i=1}^{n-2} \int_s^t \int_s^{t_n} \cdots \int_s^{t_{n-i+1}} \prod_{k=n-i+1}^n \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s,(n-i-1)} dt_{n-i} \cdots dt_n \\ &\quad + \mu \int_s^t \int_s^{t_n} \cdots \int_s^{t_2} \prod_{k=2}^n \Phi(t_k - t_{k-1}) dt_1 \cdots dt_n. \end{aligned} \tag{5.11}$$

Using Lemma 5.5,

$$\begin{aligned} &\sum_{i=1}^{n-2} \int_s^t \int_s^{t_n} \cdots \int_s^{t_{n-i+1}} \prod_{k=n-i+1}^n \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s,(n-i-1)} dt_{n-i} \cdots dt_n \\ &= \sum_{i=1}^{n-2} \int_s^t \int_s^u \Phi_i(u - r) h_r^{s,(n-i-1)} dr du, \end{aligned}$$

and

$$\mu \int_s^t \int_s^{t_n} \cdots \int_s^{t_2} \prod_{k=2}^n \Phi(t_k - t_{k-1}) dt_1 \cdots dt_n = \mu \int_s^t \int_s^u \Phi_{n-1}(u - r) dr du.$$

Plugging back these expressions in (5.11) we get

$$\begin{aligned} \mathbb{E}_{s-} \left[\int_{(s,t]} dX_r^{(n)} \right] &= \int_s^t h_u^{s,(n-1)} du + \sum_{i=1}^{n-2} \int_s^t \int_s^u \Phi_i(u - r) h_r^{s,(n-i-1)} dr du + \mu \int_s^t \int_s^u \Phi_{n-1}(u - r) dr du. \end{aligned} \tag{5.12}$$

We now sum the previous quantity over $n \geq 2$. The main term to be treated is the second one that we treat separately below. Note also that using the convention for $n = 2$ that $\sum_{i=1}^0 \cdots = 0$ we get

$$\begin{aligned} &\sum_{n=2}^{+\infty} \sum_{i=1}^{n-2} \int_s^t \int_s^u \Phi_i(u - r) h_r^{s,(n-i-1)} dr du \\ &= \sum_{n=3}^{+\infty} \sum_{i=1}^{n-2} \int_s^t \int_s^u \Phi_i(u - r) h_r^{s,(n-i-1)} dr du \\ &= \sum_{n=3}^{+\infty} \sum_{j=1}^{n-2} \int_s^t \int_s^u \Phi_{n-j-1}(u - r) h_r^{s,(j)} dr du \\ &= \sum_{j=1}^{+\infty} \int_s^t \int_s^u h_r^{s,(j)} \left(\sum_{n=3}^{+\infty} \mathbf{1}_{\{j \leq n-2\}} \Phi_{n-j-1}(u - r) \right) dr du \\ &= \int_s^t \int_s^u h_r^{s,(1)} \left(\sum_{n=3}^{+\infty} \Phi_{n-2}(u - r) \right) dr du + \sum_{j=2}^{+\infty} \int_s^t \int_s^u h_r^{s,(j)} \left(\sum_{n=j+2}^{+\infty} \Phi_{n-j-1}(u - r) \right) dr du \\ &= \int_s^t \int_s^u h_r^{s,(1)} \left(\sum_{k=1}^{+\infty} \Phi_k(u - r) \right) dr du + \sum_{j=2}^{+\infty} \int_s^t \int_s^u h_r^{s,(j)} \left(\sum_{k=1}^{+\infty} \Phi_k(u - r) \right) dr du \end{aligned}$$

$$\begin{aligned}
 &= \int_s^t \int_s^u h_r^{s,(1)} \Psi(u-r) dr du + \sum_{j=2}^{+\infty} \int_s^t \int_s^u h_r^{s,(j)} \Psi(u-r) dr du \\
 &= \sum_{j=1}^{+\infty} \int_s^t \int_s^u h_r^{s,(j)} \Psi(u-r) dr du \\
 &= \int_s^t \int_s^u \Psi(u-r) h_r^s dr du.
 \end{aligned} \tag{5.13}$$

With these computations at hand, Relations (5.11) and (5.12) lead to

$$\begin{aligned}
 &\mathbb{E}_{s-} \left[\int_{(s,t]} dX_r \right] \\
 &= \mathbb{E}_{s-} \left[\int_{(s,t]} dX_r^{(1)} \right] + \sum_{n=2}^{+\infty} \mathbb{E}_{s-} \left[\int_{(s,t]} dX_r^{(n)} \right] \\
 &= \int_s^t \mu du + \sum_{n=2}^{+\infty} \int_s^t h_u^{s,(n-1)} du + \sum_{n=2}^{+\infty} \sum_{i=1}^{n-2} \int_s^t \int_s^u \Phi_i(u-r) h_r^{s,(n-i-1)} dr du \\
 &\quad + \mu \sum_{n=2}^{+\infty} \int_s^t \int_s^u \Phi_{n-1}(u-r) dr du \\
 &= \int_s^t (\mu + h_u^s) du + \int_s^t \int_s^u \Psi(u-r) (\mu + h_r^s) dr du,
 \end{aligned}$$

which concludes the proof. □

We now compute the right-hand side in (5.8).

Lemma 5.9. *For any $0 \leq s \leq t$ we have that :*

$$\mathbb{E}_{s-} \left[\int_s^t \ell_r dr \right] = \int_s^t (\mu + h_u^s) du + \int_s^t \int_s^u \Psi(u-r) (\mu + h_r^s) dr du. \tag{5.14}$$

Proof. The proof is rather similar to the one of Lemma 5.8, we provide a proof for the sake of completeness.

Let $0 \leq s \leq r \leq t$. Recall Notation 5.6. We have

$$\begin{aligned}
 \mathbb{E}_{s-} [\ell_r] &= \mathbb{E}_{s-} \left[\mu + \int_{(0,r)} \Phi(r-u) dX_u \right] \\
 &= \mu + \sum_{n=1}^{+\infty} \mathbb{E}_{s-} \left[\int_{(0,r)} \Phi(r-u) dX_u^{(n)} \right] \\
 &= \mu + \sum_{n=1}^{+\infty} \int_{(0,s)} \Phi(r-u) dX_u^{(n)} + \sum_{n=1}^{+\infty} \mathbb{E}_{s-} \left[\int_{(s,r)} \Phi(r-u) dX_u^{(n)} \right] \\
 &= \mu + h_r^s + \sum_{n=1}^{+\infty} \mathbb{E}_{s-} \left[\int_{(s,r)} \Phi(r-u) dX_u^{(n)} \right].
 \end{aligned} \tag{5.15}$$

Let $n \geq 2$,

$$\begin{aligned}
 &\mathbb{E}_{s-} \left[\int_{(s,r)} \Phi(r-u) dX_u^{(n)} \right] \\
 &= \mathbb{E}_{s-} \left[\int_{(s,r)} \Phi(r-t_n) \int_{(0,t_n) \times \mathbb{R}_+} \cdots \int_{(0,t_2) \times \mathbb{R}_+} \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^n \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, \mathbf{x}_n) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \int_s^r \Phi(r - t_n) \mathbb{E}_{s-} \left[\int_{(0, t_n] \times \mathbb{R}_+} \Phi(t_n - t_{n-1}) \int_{(0, t_{n-1}] \times \mathbb{R}_+} \cdots \int_{(0, t_2] \times \mathbb{R}_+} \right. \\
 &\quad \left. \mathbf{1}_{\{\theta_1 \leq \mu\}} \prod_{i=2}^{n-1} \mathbf{1}_{\{\theta_i \leq \Phi(t_i - t_{i-1})\}} \mathbf{N}^{(n-1)}(d\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \right] dt_n \\
 &= \int_s^r \Phi(r - t_n) Q(n, t_n) dt_n \\
 &= \int_s^r \Phi(r - t_n) h_{t_n}^{s, (n-1)} dt_n \\
 &+ \sum_{i=1}^{n-2} \int_s^r \Phi(r - t_n) \int_s^{t_n} \cdots \int_s^{t_{n-i+1}^*} \prod_{k=n-i+1}^n \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s, (n-i-1)} dt_{n-i} \cdots dt_{n-1} dt_n \\
 &+ \mu \int_s^r \Phi(r - t_n) \int_s^{t_n} \int_s^{t_{n-1}} \cdots \int_s^{t_2} \prod_{k=2}^n \Phi(t_k - t_{k-1}) dt_1 \cdots dt_{n-1} dt_n
 \end{aligned}$$

where the last equality follows from Lemma 5.7. Integrating the previous expression in r on $(s, t]$ and using Lemma 5.5 one gets

$$\begin{aligned}
 &\int_s^t \mathbb{E}_{s-} \left[\int_{(s, r)} \Phi(r - u) dX_u^{(n)} \right] dr \\
 &= \int_s^t \int_s^{t_n} \Phi(t_n - t_{n+1}) h_{t_{n+1}}^{s, (n-1)} dt_{n+1} dt_n \\
 &+ \sum_{i=1}^{n-2} \int_s^t \int_s^{t_{n+1}} \int_s^{t_n} \cdots \int_s^{t_{n-i+1}^*} \prod_{k=n-i+1}^{n+1} \Phi(t_k - t_{k-1}) h_{t_{n-i}}^{s, (n-i-1)} dt_{n-i} \cdots dt_{n-1} dt_n dt_{n+1} \\
 &+ \mu \int_s^t \int_s^{t_{n+1}} \int_s^{t_n} \int_s^{t_{n-1}} \cdots \int_s^{t_2} \prod_{k=2}^{n+1} \Phi(t_k - t_{k-1}) dt_1 \cdots dt_{n-1} dt_n dt_{n+1} \\
 &= \int_s^t \int_s^u \Phi(u - r) h_r^{s, (n-1)} dr du \\
 &+ \sum_{i=1}^{n-2} \int_s^t \int_s^u \Phi_{i+1}(u - r) h_r^{s, (n-i-1)} dr du \\
 &+ \mu \int_s^t \int_s^u \Phi_n(u - r) dr du
 \end{aligned}$$

Using the same computations than (5.13) we deduce that

$$\begin{aligned}
 &\sum_{n=1}^{+\infty} \int_s^t \mathbb{E}_{s-} \left[\int_{(s, r)} \Phi(r - u) dX_u^{(n)} \right] dr \\
 &= \int_s^t \mathbb{E}_{s-} \left[\int_{(s, r)} \Phi(r - u) dX_u^{(1)} \right] + \sum_{n=2}^{+\infty} \int_s^t \mathbb{E}_{s-} \left[\int_{(s, r)} \Phi(r - u) dX_u^{(n)} \right] dr \\
 &= \mu \int_s^t \int_s^u \Phi(r - u) dr du + \sum_{n=2}^{+\infty} \int_s^t \int_s^u \Phi(u - r) h_r^{s, (n-1)} dr du \\
 &+ \sum_{n=3}^{+\infty} \sum_{i=1}^{n-2} \int_s^t \int_s^u \Phi_{i+1}(u - r) h_r^{s, (n-i-1)} dr du + \sum_{n=2}^{+\infty} \mu \int_s^t \int_s^u \Phi_n(u - r) dr du \\
 &= \mu \int_s^t \int_s^u \Psi(u - r) dr du
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=2}^{+\infty} \int_s^t \int_s^u \Phi(u-r) h_r^{s,(n-1)} dr du \\
 & + \sum_{n=3}^{+\infty} \sum_{j=2}^{(n+1)-2} \int_s^t \int_s^u \Phi_j(u-r) h_r^{s,(n+1-j-1)} dr du \\
 & = \mu \int_s^t \int_s^u \Psi(u-r) dr du + \int_s^t \int_s^u \Phi(u-r) h_r^s dr du \\
 & + \sum_{n=2}^{+\infty} \sum_{j=2}^n \int_s^t \int_s^u \Phi_j(u-r) h_r^{s,(n+1-j)} dr du \\
 & = \mu \int_s^t \int_s^u \Psi(u-r) dr du + \int_s^t \int_s^u \Phi(u-r) h_r^s dr du \\
 & + \int_s^t \int_s^u \sum_{j=2}^{+\infty} \Phi_j(u-r) \sum_{n=j}^{\infty} h_r^{s,(n+1-j)} dr du \\
 & = \mu \int_s^t \int_s^u \Psi(u-r) dr du + \int_s^t \int_s^u \Psi(u-r) h_r^s dr du \\
 & = \int_s^t \int_s^u \Psi(u-r) (\mu + h_r^s) dr du
 \end{aligned}$$

where we have used Relation (5.13). Thus we have proved that

$$\begin{aligned}
 \int_s^t \mathbb{E}_{s-} \left[\int_{(s,r)} \Phi(r-u) dX_u \right] dr & = \sum_{n=1}^{+\infty} \int_s^t \mathbb{E}_{s-} \left[\int_{(s,r)} \Phi(r-u) dX_u^{(n)} \right] dr \\
 & = \int_s^t \int_s^u \Psi(u-r) (\mu + h_r^s) dr du.
 \end{aligned}$$

Hence coming back to Relation (5.15) we obtain

$$\mathbb{E}_{s-} \left[\int_s^t \ell_r dr \right] = \int_s^t (\mu + h_u^s) du + \int_s^t \int_s^u \Psi(u-r) (\mu + h_r^s) dr du.$$

□

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