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Ciro Ciliberto, Mikhail Zaidenberg

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# GROMOV ELLIPTIC RESOLUTIONS OF QUARTIC DOUBLE SOLIDS

CIRO CILIBERTO AND MIKHAIL ZAIDENBERG

ABSTRACT. We construct examples of nodal quartic double solids that admit uniformly rational, and so elliptic in Gromov' sense, small algebraic resolutions.

## 1. INTRODUCTION

We work over an algebraically closed field  $\mathbb{K}$  of characteristic zero. A smooth algebraic variety  $X$  is called *algebraically elliptic in Gromov' sense*, or simply *Gromov elliptic*, if it admits a *dominating spray*  $(E, p, s)$ , where  $p: E \rightarrow X$  is an algebraic vector bundle with a zero section  $Z$  and  $s: E \rightarrow X$  is a morphism such that  $s|_Z = p|_Z$  and for each fiber  $E_x = p^{-1}(x)$ ,  $x \in X$ , the restriction  $ds|_{T_{0_x}E_x}: T_{0_x}E_x \rightarrow T_xX$  is surjective, where  $0_x \in E_x$  is the origin of the vector space  $E_x$ . The Gromov ellipticity of a variety ensures some important properties of approximation and interpolation, see e.g. [Gro89], [For17], [For23] and [Zai24]. Furthermore, a Gromov elliptic variety  $X$  of dimension  $n$  admits a morphism  $f: \mathbb{A}^{n+1} \rightarrow X$  such that the restriction  $f|_U: U \rightarrow X$  to some open subset  $U \subset \mathbb{A}^{n+1}$  is smooth and surjective, see [Kus22]. For  $\mathbb{K} = \mathbb{C}$  there exists a morphism  $f: \mathbb{A}^n \rightarrow X$  with similar properties provided that  $X$  is complete, see [For17a, Theorem 1.1].

The following projective varieties are Gromov elliptic, see e.g. [AKZ24], [Zai24] and the literature therein:

- rational smooth projective surfaces;
- flag varieties  $G/P$  of semisimple algebraic groups;
- smooth complete spherical varieties, in particular,
- smooth complete toric varieties;
- smooth Fano threefolds of index 2 and genus 10, etc.

All varieties in this list are uniformly rational. In the terminology of [BB14], an algebraic variety  $X$  of dimension  $n$  is called *uniformly rational* if it is covered by open charts isomorphic to open subsets in  $\mathbb{A}^n$ . According to [BB14, Example 2.4] every rational smooth cubic hypersurface in  $\mathbb{P}^{n+1}$ ,  $n \geq 2$ , is uniformly rational, and so is Gromov elliptic due to the following result.

**Theorem 1.1** ([AKZ24, Theorem 1.3]). *A uniformly rational complete algebraic variety  $X$  is Gromov elliptic.*

In fact, every smooth cubic hypersurface in  $\mathbb{P}^{n+1}$ ,  $n \geq 2$ , is Gromov elliptic, see [KZ24]. In particular, smooth cubic threefolds in  $\mathbb{P}^4$  provide example of irrational Gromov elliptic projective varieties.

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It is plausible that also every smooth quartic double solid is Gromov elliptic, cf. [Zai24, Question 4.27]. Recall that a nodal quartic double solid with at most six nodes is irrational, see [CPS19, Theorem 1.2]. In the present paper we construct examples of rational nodal quartic double solids with  $6 + s$  ( $4 \leq s \leq 10$ ) nodes, which admit small algebraic resolutions that are uniformly rational. Due to Theorem 1.1 these resolutions are Gromov elliptic. So, our main result is the following theorem.

**Theorem 1.2.** *For any  $s = 4, \dots, 10$ , and  $s \neq 5$ , there are quartic double solids with  $6 + s$  nodes which admit algebraic small resolutions that are uniformly rational, and so they are Gromov elliptic.*

## 2. CUBIC THREEFOLDS AND QUARTIC DOUBLE SOLIDS WITH BRANCH SURFACE HAVING A TROPE

Let  $X \subset \mathbb{P}^4$  be an irreducible, reduced cubic hypersurface with  $s$  ordinary double points or nodes, i.e., singularities of type  $A_1$ . The projection from a node shows that  $X$  is rational provided  $s \geq 1$ . It is well known that  $s \leq 10$  (see, e.g., [Seg1886], [Kal86]). In what follows we concentrate on the case of a rational singular  $X$ , and so  $s \geq 1$ .

Let  $x \in X$  be a general point and  $p: Y \rightarrow X$  be the blow-up of  $X$  at  $x$  with exceptional divisor  $E \cong \mathbb{P}^2$ . The projection of  $X$  from  $x$  to  $\mathbb{P}^3$  determines a surjective morphism  $\pi: Y \rightarrow \mathbb{P}^3$  that is a double cover branched along a quartic surface  $B \subset \mathbb{P}^3$ , see [Kre00, Sec. 3]. The image of  $E$  via  $\pi$  is a plane in  $\mathbb{P}^3$  touching  $B$  along a conic (that we will soon see to be irreducible), which passes through exactly six distinct nodes of  $B$ . In the terminology of [Hen11] and [Jes16], such a plane is called a *trope* of  $B$ .

More in detail, we can choose affine coordinates  $(x_1, \dots, x_4)$  in  $\mathbb{P}^4$  such that  $x$  is the origin and  $X$  is defined by an affine equation of the form  $x_1 + 2g_2 + g_3 = 0$  where  $g_k(x_1, x_2, x_3, x_4) \in \mathbb{K}[x_1, x_2, x_3, x_4]$  is homogeneous of degree  $k \in \{2, 3\}$ . The affine equation of the tangent hyperplane  $T_{X,x}$  is  $x_1 = 0$ . The surface  $B$  is defined in  $\mathbb{P}^3$  with homogeneous coordinates  $[x_1, x_2, x_3, x_4]$  by the homogeneous equation  $g_2^2 - x_1g_3 = 0$ . The intersection of  $X$  with the tangent hyperplane  $x_1 = 0$  to  $X$  at  $x$  is the cubic surface  $F$  in  $\mathbb{P}^3$  with equation  $2g_2(0, x_2, x_3, x_4) + g_3(0, x_2, x_3, x_4) = 0$ . It is smooth off  $x$  and has at  $x$  an ordinary double point (cf. [CC01, Thm. 1.4]). This implies that  $g_2(0, x_2, x_3, x_4)$  is a quadratic form of rank 3, and so the equation  $g_2(0, x_2, x_3, x_4) = 0$  defines a smooth conic  $C$  in  $\mathbb{P}^2$ . There are exactly six distinct lines on  $F$  passing through  $x$ . In the hyperplane  $x_1 = 0$  these lines are defined by the system  $g_2(0, x_2, x_3, x_4) = g_3(0, x_2, x_3, x_4) = 0$ .

The proper transforms on  $Y$  of the six lines on  $X$  through  $x$  are contracted by  $\pi$  to the six points in  $\mathbb{P}^3$ , with homogeneous coordinates  $[x_1, x_2, x_3, x_4]$ , defined by the system  $x_1 = g_2 = g_3 = 0$ . These points are nodes of the surface  $B$  lying on the conic  $C$ . The plane of  $\mathbb{P}^3$  with equation  $x_1 = 0$  is the trope containing these nodes. The trope is tangent to  $B$  along  $C$ . Finally,  $B$  has further  $s$  distinct nodes that are the images of the original  $s$  nodes of  $X$ ; for all this, see again [Kre00, Sec. 3].

One has the following:

**Proposition 2.1** (see [Kre00, Thm. 3.1] or [Had00, Sec. 3.2]). *Giving a double cover  $\phi: Z \rightarrow \mathbb{P}^3$  branched along a quartic surface  $B$ , with exactly  $6 + s$  nodes corresponding to the nodes of  $B$ , six of which lie on a trope, is equivalent to giving the following data:*

- a cubic hypersurface  $X$  in  $\mathbb{P}^4$  with  $s$  nodes;

- a smooth point  $x \in X$  with six distinct lines on  $X$  passing through  $x$ , lying on a quadric cone of rank 3, with the blow-up  $\sigma: Y \rightarrow X$  of  $X$  at  $x$ ;
- a morphism  $\varphi: Y \rightarrow Z$  such that the morphism  $\tilde{\pi}: Y \rightarrow \mathbb{P}^3$  determined by the projection  $\pi$  from  $x$  factors through  $\varphi$  and  $\phi$ , where
- $\varphi: Y \rightarrow Z$  is a partial small resolution of  $Z$  at the six nodes of  $Z$  lying on the trope.

*Remark 2.2.* The varieties and morphisms from Proposition 2.1 fit in the following commutative diagram:

$$\begin{array}{ccccc}
 Y & \xrightarrow{\sigma} & X & \hookrightarrow & \mathbb{P}^4 \\
 \varphi \downarrow & & \searrow \tilde{\pi} & & \downarrow \pi \\
 & & \phi & & \swarrow \pi \\
 Z & \xrightarrow{\quad} & \mathbb{P}^3 & & 
 \end{array}$$

### 3. PROOF OF THEOREM 1.2

We recall first some results from [BB14] used in the proof.

**Theorem 3.1** (cf. [BB14, Prop. 2.6]). *Let  $X$  be a uniformly rational variety and let  $Z \subset X$  be a smooth subvariety. Then the blow-up of  $X$  along  $Z$  is uniformly rational.*

**Theorem 3.2** (cf. [BB14, Thm. 3.10]). *All small algebraic resolutions of nodal cubic threefolds  $X$  in  $\mathbb{P}^4$  are uniformly rational.*

**Theorem 3.3** (cf. [BB14, Lem. 3.4]). *Let  $X$  be a nodal cubic threefold in  $\mathbb{P}^4$ . There exists a projective small resolution of  $X$  if and only if there exists an algebraic small resolution of  $X$ . More precisely, if there exists a projective small resolution of  $X$ , then all small resolutions of  $X$  are algebraic<sup>1</sup>.*

One can find in [FW89] and [Wer87] a complete classification of nodal cubic threefolds  $X$  in  $\mathbb{P}^4$  that admit projective small resolutions. This classification uses a basic invariant  $d = d(X)$  called the *defect* of  $X$ . It can be defined as the non-negative integer  $D$  such that the vector space of linear forms that vanish on the  $s$  nodes of  $X$  has dimension  $5 + d - s \geq 0$  (see [Wer87, Thm. 4.2]). We have  $s \geq d$ , see [FW89, Sec. 1]. It turns out that if  $X$  admits a projective (hence algebraic) small resolution then it must have defect  $d \geq 1$ . Moreover, using the classification results in [FW89, Sec. 3] and [Wer87, Ch. VIII, Theorem 8.1 and Lemma 8.2] one can immediately deduce the following:

**Theorem 3.4.** *If a nodal cubic threefold  $X$  in  $\mathbb{P}^4$  admits a projective small resolution, then the pair  $(d, s)$  takes the values  $(1, 4), (1, 6), (2, 6), (2, 7), (3, 8), (4, 9), (5, 10)$ . Furthermore, for every pair  $(d, s)$  from this list there exists a nodal cubic threefold  $X$  in  $\mathbb{P}^4$  with precisely  $s$  nodes and with defect  $d$  that admits a projective small resolution.*

We are now ready to prove our main result.

*Proof of Theorem 1.2.* Let  $X$  be a cubic threefold in  $\mathbb{P}^4$  with  $s$  nodes and defect  $d$ , admitting an algebraic small resolution, see Theorem 3.4. Then all small resolutions of  $X$  are algebraic (see Theorem 3.3) and they are uniformly rational by Theorem 3.2. Let  $x \in X$  be a general

<sup>1</sup> However, these are in general only complete algebraic varieties which need not be projective.

point and let  $Y$  be the blow-up of  $X$  at  $x$ . By Theorem 3.1, all small resolutions of  $Y$  are also algebraic and uniformly rational.

As in Proposition 2.1, there is a double cover  $\phi : Z \rightarrow \mathbb{P}^3$  branched along a quartic surface  $B$  with  $6 + s$  nodes, six of which lying on a trope, with a morphism  $\varphi : Y \rightarrow Z$  such that the morphism  $\tilde{\pi} : Y \rightarrow \mathbb{P}^3$  determined by the projection  $\pi$  from  $x$  factors through  $\varphi$  and  $\phi$ . Moreover  $\varphi : Y \rightarrow Z$  is a partial small resolution of  $Z$  at the six nodes of  $Z$  lying on the trope. Then all small resolutions of  $Y$  determine small resolutions of  $Z$  that are algebraic and uniformly rational, and therefore Gromov elliptic. Now the assertion follows.  $\square$

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA “TOR VERGATA”, VIA DELLA RICERCA SCIENTIFICA, 00177 ROMA, ITALIA

*E-mail address:* cilibert@axp.mat.uniroma2.it

UNIV. GRENOBLE ALPES, CNRS, IF, 38000 GRENOBLE, FRANCE

*E-mail address:* mikhaïl.zaidenberg@univ-grenoble-alpes.fr