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DARK SUBSPACES AND INVARIANT MEASURES OF QUANTUM TRAJECTORIES

T. BENOIST, C. PELLEGRINI, AND A. SZCZEPANEK

ABSTRACT. Quantum trajectories are Markov processes describing the evolution of a quantum system subject to indirect measurements. They can be viewed as place dependent iterated function systems or the result of products of dependent and non identically distributed random matrices. In this article, we establish a complete classification of their invariant measures.

The classification is done in two steps. First, we prove a Markov process on some linear subspaces called *dark subspaces*, defined in (Maassen, Kümmerer 2006), admits a unique invariant measure. Second, we study the process inside the dark subspaces. Using a notion of minimal family of isometries from a reference space to dark subspaces, we prove a set of measures indexed by orbits of a unitary group is the set of ergodic measures of quantum trajectories.

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1. INTRODUCTION

1.1. **Model.** Consider the canonical complex d -dimensional space \mathbb{C}^d and its projective space $P(\mathbb{C}^d)$ equipped with its Borel σ -algebra. For a non-zero vector $x \in \mathbb{C}^d$, \hat{x} stands for the equivalence class of x in $P(\mathbb{C}^d)$. For $\hat{x} \in P(\mathbb{C}^d)$, by $x \in \mathbb{C}^d$ we denote an arbitrary unit norm representative of \hat{x} , and by $\pi_{\hat{x}}$ the orthogonal projector onto $\mathbb{C}x$. By \mathcal{M}_d we denote the set of linear maps on \mathbb{C}^d , and for a linear map $v \in \mathcal{M}_d$, $v \cdot \hat{x}$ is the element of the projective space represented by vx whenever $vx \neq 0$. We equip \mathcal{M}_d with its Borel σ -algebra and consider a measure μ on \mathcal{M}_d such that $v \mapsto \|v\|$ is square integrable, i.e.,

$$\int_{\mathcal{M}_d} \|v\|^2 d\mu(v) < \infty,$$

and the following stochasticity condition holds:

$$\int_{\mathcal{M}_d} v^*v d\mu(v) = \text{Id}_{\mathbb{C}^d}.$$

A quantum trajectory is a Markov chain on $P(\mathbb{C}^d)$ with transition kernel

$$\Pi(\hat{x}, S) = \int_{\mathcal{M}_d} \mathbf{1}_S(v \cdot \hat{x}) \|vx\|^2 d\mu(v),$$

where $\hat{x} \in P(\mathbb{C}^d)$ and S is a Borel subset of $P(\mathbb{C}^d)$. A realization of this Markov chain $(\hat{x}_n)_n$ can be described by

$$\hat{x}_n \rightsquigarrow \hat{x}_{n+1} = V_{n+1} \cdot \hat{x}_n,$$

where V_{n+1} is a matrix-valued random variable with law $\|vx_n\|^2 d\mu(v)$. Every \hat{x}_n can be written using a random product of matrices as

$$\hat{x}_n = V_n \cdots V_1 \cdot \hat{x}_0,$$

where $(V_n)_n$ is distributed along the density $\|v_n \cdots v_1 x_0\|^2 d\mu^{\otimes n}(v_1, \dots, v_n)$.

In the context of quantum physics, a realization of the process $(\hat{x}_n)_n$ is called a *quantum trajectory*. Its definition originates in quantum optics – see [Car93]. A prominent example of an experimental realization of a quantum trajectory is the Nobel prize-winning experiment of Serge Haroche’s group [Gue+07]. For a mathematical exposition of the theory of quantum measurements, we direct the interested reader to the books [Hol01; Bus+16] and the foundational article [DL70].

Since \hat{x}_n can be defined as the result of the action of some random product of matrices on the initial element of the projective space \hat{x}_0 , the study of quantum trajectories shares some similarity with that of products of independent and identically distributed (i.i.d.) random matrices – see e.g. [BL85b] for an introduction. For example, as stated in [Ben+19], uniqueness of the Π -invariant measure is a consequence of [GLP16, Theorem 2.6] applied to the case where their parameter s is equal to 2, albeit for invertible matrices and under a strong irreducibility assumption. Uniqueness of the Π -invariant measure was proved in [Ben+19] for possibly non-invertible matrices and under the standard irreducibility assumption. This extension relied on the fact that $\|V_n \cdots V_1 x_0\| \neq 0$ almost surely by definition. Leveraging this property, an alternative proof of the fact that $(V_n \cdots V_1 / \|V_n \cdots V_1\|)_n$ concentrates almost surely on a set of rank-one matrices with common pre-image could be found. Recently, a generalization of this new proof lead to the extension of the results from [GLP16] to irreducible non-invertible matrices for any $s > 0$ – see [Hau24].

Alas, both [GLP16] and [Ben+19] rely on an assumption of contractivity (called purification in [Ben+19] following the terminology of [MK06]). It ensures that, almost surely, any accumulation point of $(V_n \cdots V_1 / \|V_n \cdots V_1\|)_n$ is of rank one. In the present article we propose to lift the assumption of purification.

As shown with an example in [Ben+19, Appendix C], without purification, uniqueness of the Π -invariant measure may or may not be true. Our main result is thus a complete classification of the set of Π -invariant measures, assuming only that μ is irreducible.

Note that quantum trajectories can also be viewed as place-dependent iterated function systems (IFS). However, the standard assumptions on IFS are not satisfied by quantum trajectories – see [BFP23, Proposition 8.3]. Similarly, the usual approach of Markov chain using the notion of ϕ -irreducibility fails – see [BFP23, Proposition 8.1]. Note also that after the publication of [Ben+19], the authors of this reference found that some elements of their proofs already appeared with a different formulation in [Kus89] albeit for invertible matrices.

1.2. Irreducibility. Our only assumption throughout the article is an irreducibility one for the measure μ . We say a subspace E of \mathbb{C}^d is μ -invariant if $vE \subset E$ for μ -almost all v . We say μ is irreducible if there does not exist a non-trivial μ -invariant subspace. From now on we always assume μ is irreducible without necessarily mentioning it.

This assumption is the standard irreducibility assumption for families of matrices. In the theory or random products of matrices a strong irreducibility assumption is often made. It states that there does not exist a finite set of non trivial subspaces $\{E_i\}_i$ such that for μ -almost all v , $v \cup_i E_i \subset \cup_i E_i$. Strong irreducibility implies the standard one. From now on, when we evoke the irreducibility assumption, we mean the standard irreducibility.

The definition of irreducibility we require looks stronger than the (ϕ -**Erg**) assumption used in [Ben+19]. Indeed, under (ϕ -**Erg**), $\{0\}$ and \mathbb{C}^d are not necessarily the unique μ -invariant subspaces. Actually, as discussed at the end of Section 2.1, this apparent restriction is an illusion.

1.3. Random walk on Dark subspaces. As mentioned above, compared to [Ben+19], we forgo assuming purification. In [MK06], the authors show purification is equivalent to the absence of so-called dark subspaces of dimension greater than or equal to two. Dark subspaces are thus instrumental objects in our proofs. We especially study a Markov process on maximal dark subspaces introduced in [MK06, Section 5]. Denoting $\mathcal{SU}(d)$ the group of $d \times d$ special unitary matrices, the definition of dark subspaces goes as follows.

Definition. A non-zero linear subspace $D \subset \mathbb{C}^d$ is called a dark subspace if for all $n \in \mathbb{N}^*$ and $\mu^{\otimes n}$ -almost all $(v_1, \dots, v_n) \in \mathcal{M}_d^{\times n}$ there exists $\lambda \in \mathbb{C}$ and $u \in \mathcal{SU}(d)$ such that

$$v_n \cdots v_1|_D = \lambda u|_D.$$

Note that any one-dimensional subspace satisfies the definition. Since on dark subspaces any product of matrices $v \in \text{supp } \mu$ is equivalent to a mapping proportional to a unitary matrix, the evolution induced by the product of matrices on the dark subspace can be reverted almost surely according to the postulates of quantum mechanics. That makes dark subspaces of high dimension potential candidates for quantum error correcting codes – see [MMŻ10; BK+08; KLV00].

A first step towards the classification of Π -invariant measures and a first relevant result on its own is the proof that a Markov process on maximal dark subspaces admits a unique invariant measure.

Let \mathcal{D}_m be the set of dark subspaces of maximal dimension. We call them the maximal dark subspaces. Let r_m be their dimension.

Let $(D_n)_n$ be the Markov chain on \mathcal{D}_m defined by

$$D_n \rightsquigarrow D_{n+1} = V_{n+1}D_n$$

with V_{n+1} distributed along $\text{tr}(v \frac{\pi_{D_n}}{r_m} v^*) d\mu(v)$, where π_{D_n} is the orthogonal projector onto D_n . This is the Markov chain on dark subspaces introduced in [MK06, Section 5].

We denote K the associated Markov kernel. It is given by

$$K(D, B) = \int_{\mathcal{M}_d} \mathbf{1}_B(vD) \text{tr}(v \frac{\pi_D}{r_m} v^*) d\mu(v)$$

for any $D \in \mathcal{D}_m$ and B a Borel subset of \mathcal{D}_m . Interpreting K as an operator on the Banach space of continuous functions of \mathcal{D}_m , we recall that for any Borel probability measure χ , χK is the Borel probability measure defined by $\chi K(B) = \int_{\mathcal{D}_m} K(D, B) d\chi(D)$.

Our first relevant result is that, assuming μ is irreducible, there exists a unique K -invariant probability measure and the convergence towards it is exponential in Wasserstein metric.

Theorem 1. *There exists a unique K -invariant Borel probability measure on \mathcal{D}_m .*

Moreover, denoting this unique K -invariant Borel probability measure χ_{inv} , there exist an integer m and two positive constants C and $\lambda < 1$ such that for any Borel probability measure χ on \mathcal{D}_m ,

$$W_1 \left(\frac{1}{m} \sum_{r=0}^{m-1} \chi K^{mn+r}, \chi_{\text{inv}} \right) \leq C \lambda^n$$

for every $n \in \mathbb{N}^$, where W_1 denotes the 1-Wasserstein metric.*

The proof of this theorem is provided in Section 3. On top of it, in Theorem 4.1 we show that the process $(\hat{x}_n)_n$ converges exponentially fast to $\cup_{D \in \mathcal{D}_m} P(D)$.

We will rely on Theorem 1 to construct the whole set of Π -invariant measure. In that endeavor we need to track the evolution inside the dark subspaces induced by the matrices in $\text{supp } \mu$. Schematically, the matrices have two actions, they map a maximal dark subspace to another one and, thanks to the definition of dark subspaces, they induce a unitary transformation *inside* the maximal dark subspaces. To separate these two evolutions, we introduce a notion of *minimal family of isometries*.

1.4. Minimal families of isometries. We split \mathbb{C}^d into $\mathcal{D}_m \times \mathbb{C}^{r_m}$, where \mathbb{C}^{r_m} plays the part of a reference space where the evolution inside the dark subspaces is recorded. For that, we establish a correspondence between each $D \in \mathcal{D}_m$ and \mathbb{C}^{r_m} by fixing a family of maps $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$, where each $J_D: \mathbb{C}^{r_m} \rightarrow D$ is a linear isometry. This allows us to turn partial isometries on \mathbb{C}^d into special unitary matrices on the common reference space \mathbb{C}^{r_m} . Then we define a group $G_{\mathcal{J}}$ as the smallest closed subgroup of $\mathcal{SU}(r_m)$ containing the set

$$S := \{u_{v,D} \in \mathcal{SU}(r_m) \mid v \in \text{supp } \mu, D \in \text{supp } \chi_{\text{inv}}, \text{tr}(v \pi_D v^*) > 0\}.$$

where $u_{v,D} \propto J_{vD}^{-1} v J_D$ with the phase adjusted so that $\det u_{v,D} = 1$, provided $\text{tr}(v \pi_D v^*) > 0$. That is, $u_{v,D}$ is a special unitary matrix acting on \mathbb{C}^{r_m} when v acts on the dark subspace D . Recall that χ_{inv} is the unique K -invariant Borel probability measure.

The choice of \mathcal{J} is far from unique. Amongst all the possible choices we singularize some minimal ones.

Definition. We say that $\mathcal{J} = \{J_D : \mathbb{C}^{r_m} \rightarrow D\}_{D \in \mathcal{D}_m}$ is a minimal family if the group $G_{\mathcal{J}}$ is (up to a unitary equivalence) minimal as a subgroup of the special unitary group, i.e., for any other family $\tilde{\mathcal{J}} = \{\tilde{J}_D : \mathbb{C}^{r_m} \rightarrow D\}_{D \in \mathcal{D}_m}$ there exists $Q \in \mathcal{U}(r_m)$ such that $G_{\mathcal{J}} \subset QG_{\tilde{\mathcal{J}}}Q^{-1}$.

Based on this definition we build the set of Π -invariant Borel probability measures.

1.5. Classification of invariant measures. Let $\mathbb{P}(\mathbb{C}^{r_m})$ be the projective space of \mathbb{C}^{r_m} . We use the same notational conventions as for $\mathbb{P}(\mathbb{C}^d)$. Let \mathcal{J} be a family of isometries as defined in the previous subsection. Fix $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$. Let $m_{\hat{x}, \mathcal{J}}$ be the uniform measure on the $G_{\mathcal{J}}$ -orbit of \hat{x} , i.e., the image of the $G_{\mathcal{J}}$ -Haar measure by $G_{\mathcal{J}} \ni u \mapsto u \cdot \hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$. Let the map back from $\mathcal{D}_m \times \mathbb{P}(\mathbb{C}^{r_m})$ to $\mathbb{P}(\mathbb{C}^d)$ be

$$\Psi_{\mathcal{J}} : \mathcal{D}_m \times \mathbb{P}(\mathbb{C}^{r_m}) \ni (D, \hat{z}) \mapsto J_D \cdot \hat{z} \in \mathbb{P}(\mathbb{C}^d).$$

We define $\nu_{\hat{x}, \mathcal{J}}$ as the image measure of $\chi_{\text{inv}} \otimes m_{\hat{x}, \mathcal{J}}$ by $\Psi_{\mathcal{J}}$.

Direct computation shows that $\nu_{\hat{x}, \mathcal{J}}$ is Π -invariant for any $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$. We denote the set of all such measures by

$$\Upsilon_{\mathcal{J}} = \{\nu_{\hat{x}, \mathcal{J}} : \hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})\}.$$

Importantly, all minimal families lead to the same set of measures. Indeed, if \mathcal{J}_1 and \mathcal{J}_2 are both minimal families of isometries, then by definition, $G_{\mathcal{J}_1}$ and $G_{\mathcal{J}_2}$ are unitary conjugate. Hence, as proved in Theorem 5.17,

$$\Upsilon_{\mathcal{J}_1} = \Upsilon_{\mathcal{J}_2}.$$

Our main result is that, when \mathcal{J} is minimal, these measures are exactly the Π -ergodic measures. The following theorem is a direct consequence of Theorem 6.10 and Corollary 6.11.

Theorem 2. *If \mathcal{J} is minimal, then $\Upsilon_{\mathcal{J}}$ is the set of Π -ergodic measures. In other words, a probability measure ν is Π -ergodic if and only if $\nu = \nu_{\hat{x}, \mathcal{J}}$ for some $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$.*

As a consequence, we have a complete description of Π -invariant measures: a probability measure ν on $\mathbb{P}(\mathbb{C}^d)$ is Π -invariant if and only if

$$\nu = (\Psi_{\mathcal{J}})_* \left(\chi_{\text{inv}} \otimes \int_{\mathbb{P}(\mathbb{C}^{r_m})} m_{\hat{x}, \mathcal{J}} \, d\lambda(\hat{x}) \right) = \int_{\mathbb{P}(\mathbb{C}^{r_m})} \nu_{\hat{x}, \mathcal{J}} \, d\lambda(\hat{x})$$

for some probability measure λ on $\mathbb{P}(\mathbb{C}^{r_m})$.

Moreover, we provide a necessary and sufficient condition under which there exists a unique Π -invariant Borel probability measure. Let ν_{unif} be the Borel probability measure defined by,

$$\nu_{\text{unif}} = \int_{\mathbb{P}(\mathbb{C}^{r_m})} \nu_{\hat{x}, \mathcal{J}} \, d\text{Unif}(\hat{x}),$$

where Unif is the uniform measure over $\mathbb{P}(\mathbb{C}^{r_m})$, i.e., the image measure of the $\mathcal{SU}(r_m)$ -Haar measure by the map $u \mapsto u \cdot \hat{y}$ for some $\hat{y} \in \mathbb{P}(\mathbb{C}^{r_m})$. Theorem 2 implies it is always Π -invariant. The necessary and sufficient condition for its uniqueness reads as follows.

Theorem 3. *There exists a unique Π -invariant Borel probability measure on $\mathbb{P}(\mathbb{C}^d)$ if and only if the group $G_{\text{min}} = G_{\mathcal{J}}$ induced by a minimal family of isometries \mathcal{J} acts transitively on $\mathbb{P}(\mathbb{C}^{r_m})$. Namely, if and only if,*

- $G_{\text{min}} = \mathcal{SU}(r_m)$ if r_m is odd,

- $G_{\min} = \mathcal{S}U(r_m)$ or G_{\min} is unitarily equivalent to $\mathrm{Sp}(r_m/2)$ if r_m is even. Here $\mathrm{Sp}(r_m/2)$ denotes the subgroup of $\mathcal{S}U(r_m)$ called the compact symplectic group (or the hyperunitary group).

If there is a unique Π -invariant Borel probability measure, it is equal to ν_{unif} .

This theorem is a consequence of Corollary 6.12.

1.6. Outline of the proof of Theorem 2. Theorem 1 is the first step of the proof. It is proved in Section 3. The proof is similar to the one of [Ben+19] with non-trivial adaptations to take into account the possibility that $r_m > 1$. It not only shows there exists a unique K -invariant Borel probability measure, but its proof also allows us to show the process $(\hat{x}_n)_n$ in $\mathbb{P}(\mathbb{C}^d)$ concentrates exponentially fast on $\cup_{D \in \mathcal{D}_m} \mathbb{P}(D)$. This is the content of Theorem 4.1. Hence, we can restrict our search of Π -invariant measures to measures supported on this union of projective spaces.

It remains to show the measures $\nu_{\hat{x}, \mathcal{J}}$ are ergodic and are the only Π -ergodic Borel probability measures. For that we define and study the minimal families of isomorphisms in Section 5. In particular, Theorem 5.16 provides an equivalence between minimal families and so-called *smart* families, which are families of isometries constructed from a reference maximal dark subspace and products of matrices from $\mathrm{supp} \mu$.

Then, in Theorem 6.5, using the smartness property, we prove that any closed minimal Π -invariant set is of the form $\mathrm{supp} \nu_{\hat{x}, \mathcal{J}}$ with $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$ and \mathcal{J} an arbitrary fixed minimal family of isometries. These last two intermediary results are completely new to our knowledge and can be seen as the main technical difficulties of the proof.

For equicontinuous Markov kernels (i.e., Markov kernels T such that for any continuous function f , $(T^n f)_n$ is equicontinuous) there is a one-to-one correspondence between minimal sets and ergodic measures – see [Rau92, Proposition 3.2] and [BQ14, Proposition 2.9]. In Theorem 6.9 we show Π is equicontinuous. Then, leveraging the identification of Π -minimal sets, we conclude on the classification of Π -ergodic measures.

The structure of the proof is summarized Figure 1.1.

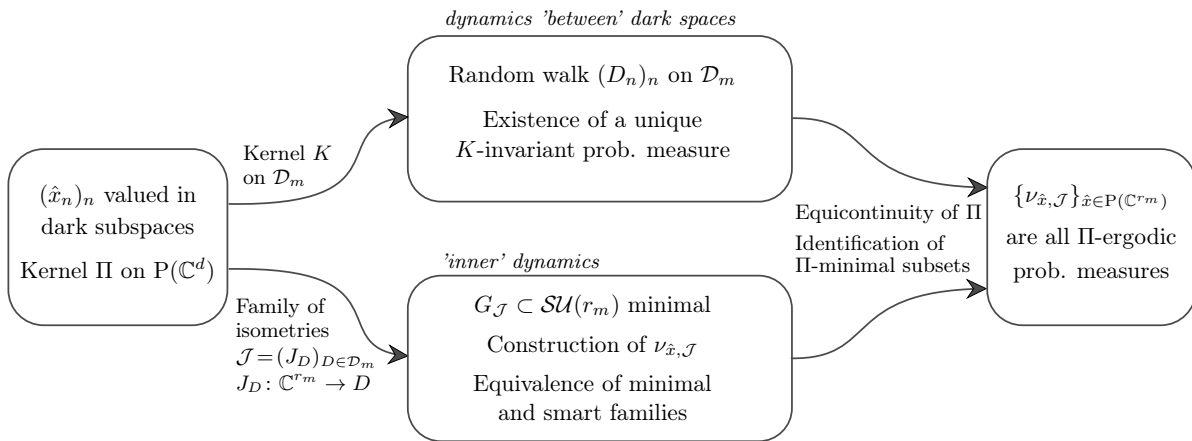


FIGURE 1.1. Structure of the proof of Theorem 2

1.7. Structure of the paper. In Section 2, we give a precise definition of quantum trajectories and, in particular, the probability space on which we work. The notion of dark subspaces is then described as well as that of maximal dark subspaces. In Section 3, we introduce the Markov chain on dark subspaces and show that it admits a unique invariant measure. Furthermore, we show that the convergence towards this measure is exponentially fast. This section concerns Theorem 1. Next, in Section 4 we show that quantum trajectories converge exponentially fast towards maximal dark subspaces. Section 5 discusses the isometries between the maximal dark subspaces and \mathbb{C}^{r_m} and introduces the notion of minimal families of isometries. Then Section 6 provides the classification of invariant measures for quantum trajectories, i.e., Theorem 2 and Theorem 3 after proving the equicontinuity of Π and the characterization of Π -minimal closed subsets. In the final Section 7 we discuss some enlightening examples that provide an illustration of the notions we use and various subtleties that can arise. Note that all these results concern the invariant measures for quantum trajectories on $P(\mathbb{C}^d)$. In Appendix A, we provide the classification of all invariant measures of quantum trajectories on density matrices.

2. QUANTUM TRAJECTORIES, DARK SUBSPACES

We begin this section by presenting the general mathematical model of quantum trajectories and the Markov chain on dark subspaces. Then, we introduce a process which has been one of the key tools in [Ben+19] and which turns out to be intricately linked to maximal dark subspaces. In particular, we use it in Section 4 to show that quantum trajectories converge exponentially fast towards maximal dark subspaces.

2.1. Framework and main assumption. Let us first introduce some notation. For a fixed dimension $d \geq 2$ we consider the complex vector space \mathbb{C}^d as a Hilbert space equipped with the usual Hermitian inner product anti-linear in the first coordinate. We put \mathcal{M}_d for the set of linear maps on \mathbb{C}^d equipped with the operator norm, $\mathcal{U}(d)$ for the group of $d \times d$ unitary matrices, $\mathcal{SU}(d)$ for the associated special unitary group, and $\mathcal{M}_d^{1,+}$ for the set of *density matrices* (also called *states*), i.e.,

$$\mathcal{M}_d^{1,+} = \{\rho \in \mathcal{M}_d \mid \rho \geq 0, \text{tr}(\rho) = 1\}.$$

The extreme points of $\mathcal{M}_d^{1,+}$ are the rank-one orthogonal projectors on \mathbb{C}^d . They constitute the set of *pure states*, which is isomorphic to the complex projective space $P(\mathbb{C}^d)$, i.e., the space of rays in \mathbb{C}^d . Throughout the paper we use the following identifications. For a non-zero vector $x \in \mathbb{C}^d$ we denote by $\pi_{\hat{x}}$ the orthogonal projection on $\mathbb{C}x$, and by \hat{x} the equivalence class of x in $P(\mathbb{C}^d)$. In particular, if $v \in \mathcal{M}_d$, then $v \cdot \hat{x}$ stands for the equivalence class of vx , provided that $vx \neq 0$; i.e., $v \cdot \hat{x} = \widehat{vx}$. For $\hat{x} \in P(\mathbb{C}^d)$ we denote by x an arbitrary normalized representative of \hat{x} in \mathbb{C}^d . Also, we put \mathbb{N}^* for the set of positive integers and $\mathbb{N} = \mathbb{N}^* \cup \{0\}$.

Now we introduce more precisely the model of quantum trajectories. Let μ be a Borel measure on \mathcal{M}_d that has a finite second moment, $\int_{\mathcal{M}_d} \|v\|^2 d\mu(v) < \infty$, and satisfies the stochasticity condition

$$(1) \quad \int_{\mathcal{M}_d} v^* v d\mu(v) = \text{Id}.$$

Such a measure μ defines a *quantum trajectory*. It is a Markov chain that takes values in $\mathcal{M}_d^{1,+}$ and has the following transition kernel

$$(2) \quad \Pi(\rho, B) = \int_{\mathcal{M}_d} \mathbf{1}_B \left(\frac{v\rho v^*}{\text{tr}(v\rho v^*)} \right) \text{tr}(v\rho v^*) \, d\mu(v),$$

where $\rho \in \mathcal{M}_d^{1,+}$ and B is a Borel subset of $\mathcal{M}_d^{1,+}$. In terms of stochastic processes, one can define the Markov chain $(\rho_n)_{n \in \mathbb{N}}$ valued in $\mathcal{M}_d^{1,+}$ by

$$(3) \quad \rho_n \longmapsto \rho_{n+1} = \frac{V_{n+1}\rho_n V_{n+1}^*}{\text{tr}(V_{n+1}\rho_n V_{n+1}^*)},$$

where V_{n+1} is a matrix-valued random variable whose law is $\text{tr}(v\rho_n v^*) \, d\mu(v)$. Note that if $\rho_0 = \pi_{\hat{x}_0}$ for some unit vector $x_0 \in \mathbb{C}^d$, we have

$$\frac{v\pi_{\hat{x}_0} v^*}{\text{tr}(v\pi_{\hat{x}_0} v^*)} = \pi_{v \cdot \hat{x}_0}$$

for every $v \in \mathcal{M}_d^{1,+}$ such that $v x_0 \neq 0$. That is, if the initial state is a rank-one projector, the quantum trajectory remains valued in rank-one projectors. Using the identification between rank-one projectors and equivalence classes of vectors in $\mathbb{P}(\mathbb{C}^d)$, we recover the description of (pure) quantum trajectories provided in the Introduction. Recall that the transition kernel then reads

$$(4) \quad \Pi(\hat{x}, B) = \int_{\mathcal{M}_d} \mathbf{1}_B(v \cdot \hat{x}) \|vx\|^2 \, d\mu(v),$$

where $\hat{x} \in \mathbb{P}(\mathbb{C}^d)$ and B is a Borel subset of $\mathbb{P}(\mathbb{C}^d)$, and the Markov chain $(\hat{x}_n)_{n \in \mathbb{N}}$ on $\mathbb{P}(\mathbb{C}^d)$ is given by

$$\hat{x}_n \longmapsto \hat{x}_{n+1} = V_{n+1} \cdot \hat{x}_n,$$

where V_{n+1} has law $\|vx_n\|^2 \, d\mu(v)$. Our main goal is to classify the invariant measures of $(\hat{x}_n)_{n \in \mathbb{N}}$. Analogous results for general quantum trajectories, which for the most part follow via a straightforward adaptation of the methods and results derived for pure trajectories, are relegated to Appendix A.

For a topological space \mathcal{T} , we denote $\mathcal{C}(\mathcal{T})$ the set of continuous functions $\mathcal{T} \rightarrow \mathbb{C}$. Then, on $\mathcal{C}(\mathcal{M}_d^{1,+})$ and $\mathcal{C}(\mathbb{P}(\mathbb{C}^d))$, the kernel Π acts as

$$\Pi f(\rho) = \int_{\mathcal{M}_d} f \left(\frac{v\rho v^*}{\text{tr}(v\rho v^*)} \right) \text{tr}(v\rho v^*) \, d\mu(v)$$

and

$$\Pi f(\hat{x}) = \int_{\mathcal{M}_d} f(v \cdot \hat{x}) \|vx\|^2 \, d\mu(v),$$

respectively. Also, by duality, the kernel acts on measures as

$$\nu \Pi(B) = \int_{\mathcal{M}_d^{1,+}} \Pi(\rho, B) \, d\nu(\rho) \quad \text{or} \quad \nu \Pi(B) = \int_{\mathbb{P}(\mathbb{C}^d)} \Pi(\hat{x}, B) \, d\nu(\hat{x}),$$

where ν is a Borel measure on $\mathcal{M}_d^{1,+}$ or $\mathbb{P}(\mathbb{C}^d)$, and B is a Borel subset of the respective set. An *invariant probability measure* for Π is a probability measure ν that satisfies $\nu \Pi = \nu$. The set of invariant probability measures is convex and its extreme points are called *ergodic measures*.

Since the kernel Π is Feller (see [BFP23, Prop 2.2]) and $\mathcal{M}_d^{1,+}$ is compact, quantum trajectories always admit an invariant measure. An important question is the uniqueness of this measure and

convergence towards it. A key notion in this context is that of a *dark subspace*, which was introduced by Kümmerer and Maassen in [MK06]. Let us recall its definition.

Definition 2.1. *A non-zero linear subspace $D \subset \mathbb{C}^d$ is called a dark subspace when for all $n \in \mathbb{N}^*$ and $\mu^{\otimes n}$ -almost all $(v_1, \dots, v_n) \in \mathcal{M}_d^{\times n}$ there exists $\lambda \geq 0$ and $U \in \mathcal{U}(d)$ such that*

$$(5) \quad v_n \cdots v_1 x = \lambda U x \quad \forall x \in D$$

Equivalently, $D \subset \mathbb{C}^d$ is dark if and only if

$$(6) \quad \pi_D v_1^* \cdots v_n^* v_n \cdots v_1 \pi_D \propto \pi_D$$

for all $n \in \mathbb{N}^$ and $\mu^{\otimes n}$ -almost all (v_1, \dots, v_n) , where π_D stands for the orthogonal projection on D . The orthogonal projection on a dark subspace is called a dark projection.*

Maximal dark subspaces are those with highest dimension. We denote this dimension by r_m and the set of maximal dark subspaces by \mathcal{D}_m .

Note that every one-dimensional subspace satisfies Equation (5) trivially, so r_m is well defined. In the case $r_m = 1$ we say that purification occurs. This condition is equivalent to contractivity in the theory of product of random matrices – see [Ben+19, Proposition A.1].

Remark 2.2. *According to the original definition in [MK06], dark subspaces are required to have dimension of at least two; in particular, dark subspaces may not exist for a given μ . However, for the sake of consistency, in the present paper we consider rays in \mathbb{C}^d as (trivial) dark subspaces and rank-one projections as (trivial) dark projections.*

The following straightforward proposition gathers basic properties of dark subspaces. In particular, it states explicitly that the probability of the system undergoing a given update depends only on the dark subspace which the system occupies, i.e., it is constant for all states supported on this dark subspace. This means in particular that if two dark subspaces intersect non-trivially, they share all transition probabilities. We do not provide a proof since it follows directly from the definition – see also [MK06, Sec. 5].

Proposition 2.3. *Let D be a dark subspace.*

- (i) *Let $v \in \text{supp } \mu$. If $\lambda \geq 0$ and $U \in \mathcal{U}(d)$ are such that $v|_D = \lambda U|_D$, which is equivalent to $\pi_D v^* v \pi_D = \lambda^2 \pi_D$, then for every unit vector $x \in D$,*

$$\|vx\|^2 = \lambda^2 = \text{tr} \left(v \frac{\pi_D}{\text{tr}(\pi_D)} v^* \right).$$

Furthermore, if $\lambda > 0$, then vD is a dark subspace and $\dim vD = \dim D$.

- (ii) *Since every point $(v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}$ can be approximated by a sequence of points from $\text{supp } \mu^{\otimes n}$ that satisfy Equation (6), the continuity of matrix multiplication implies that Equation (6) holds for every $(v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}$.*

One of the main results of [MK06] is that the asymptotic behaviour of quantum trajectories is to perform a random walk between dark subspaces. More precisely, the transition rule

$$D_n \longmapsto D_{n+1} = vD_n$$

with probability $\mathrm{tr}\left(v\frac{\pi_{D_n}}{\mathrm{tr}(\pi_{D_n})}v^*\right)d\mu(v)$ defines a Markov chain $(D_n)_{n\in\mathbb{N}}$ in \mathcal{D}_m . This Markov chain is well defined by Proposition 2.3(1). The corresponding kernel is

$$K(D, B) = \int_{\mathcal{M}_d} \mathbf{1}_B(vD) \mathrm{tr}\left(v\frac{\pi_D}{\mathrm{tr}(\pi_D)}v^*\right) d\mu(v),$$

where $D \in \mathcal{D}_m$ and B is a Borel subset of \mathcal{D}_m . Let us point out that in terms of normalized dark projectors this transition reads

$$\rho_n = \frac{\pi_{D_n}}{\mathrm{tr}(\pi_{D_n})} \mapsto \rho_{n+1} = \frac{v\pi_{D_n}v^*}{\mathrm{tr}(v\pi_{D_n}v^*)} = \frac{\pi_{D_{n+1}}}{\mathrm{tr}(\pi_{D_{n+1}})},$$

and so it coincides with the Markov chain defined by Π on density matrices.

As already mentioned in Introduction, our only assumption all along the paper is that μ satisfies the following irreducibility assumption.

Irreducibility. There does not exist a non-trivial subspace $E \subset \mathbb{C}^d$ (i.e. $E \notin \{\mathbb{C}^d, \{0\}\}$) such that μ -almost surely we have $vE \subset E$.

In [Ben+19] assumption (ϕ -**Erg**) corresponded to the existence of a unique minimal non-zero μ -invariant subspace E . Here the irreducibility assumption moreover enforces $E = \mathbb{C}^d$. Both assumptions are related to the uniqueness of the fixed point of some map on $\mathcal{M}_d^{1,+}$ called a quantum channel.

Let

$$\phi : X \mapsto \int_{\mathcal{M}_d} vXv^* d\mu(v).$$

This map is positive and Equation (1) implies $\mathcal{M}_d^{1,+}$ is invariant for ϕ . Then, following Perron–Frobenius theorem for positive maps on matrix algebras – see [EHK78] – the irreducibility assumption is equivalent to ϕ having a unique fixed point ρ_∞ in $\mathcal{M}_d^{1,+}$ and this matrix is full rank. Following [CP16, Proposition 5.7], assumption (ϕ -**Erg**) in [Ben+19] is equivalent to ϕ having a unique fixed point ρ_∞ in $\mathcal{M}_d^{1,+}$; however, this matrix can be rank deficient.

When one is only concerned with invariant measures, the apparently stronger condition of irreducibility is actually not a restriction. Indeed, assume (ϕ -**Erg**) from [Ben+19] holds and let $E = \mathrm{supp}(\rho_\infty)$. Then, following [BPT17, Theorem 1.3], quantum trajectories $(\rho_n)_n$ are almost surely asymptotically supported on E and the convergence to it is exponentially fast. Hence, all invariant measures only charge states with support in E . Now, since ϕ restricted to states that have support in E has a unique fixed point in $\mathcal{M}_d^{1,+}$ which is full rank on E , $\phi|_E$ is irreducible. Hence, assuming irreducibility is not restrictive and our result can be extended assuming only (ϕ -**Erg**) from [Ben+19].

2.2. Probability space. We work with a specific probability space on which we define a process that has the same law as the quantum trajectory defined in Equation (2). It is essentially the one used in [Ben+19]. Let Ω be the set of infinite sequences of maps $\mathcal{M}_d^{\mathbb{N}^*}$. For any $\omega \in \Omega$, let $\omega = (v_1, v_2, \dots)$, and let $\pi_n : \Omega \rightarrow \mathcal{M}_d^{\times n}$ be the canonical projection on the first n components, i.e. $\pi_n(\omega) = (v_1, \dots, v_n)$. The usual cylinder σ -algebra on Ω is denoted \mathcal{O} . That is, denoting by \mathfrak{M} the Borel σ -algebra on \mathcal{M}_d , for $n \in \mathbb{N}^*$ we define the σ -algebra \mathcal{O}_n as $\pi_n^{-1}(\mathfrak{M}^{\otimes n})$, i.e., \mathcal{O}_n is the smallest σ -algebra on Ω such that all n -cylinder sets $\{(v_1, \dots, v_n, \dots) \in \Omega : v_i \in A_i, i = 1, \dots, n\}$, where $A_1, \dots, A_n \in \mathfrak{M}$, are measurable. Then \mathcal{O} is the smallest σ -algebra such that $\mathcal{O}_n \subset \mathcal{O}$ for any $n \in \mathbb{N}^*$.

Throughout the paper we use a series of identifications which improve the readability even if they constitute small abuses of notation. Namely, a cylinder set $O_n \in \mathcal{O}_n$ is identified with its base $\pi_n(O_n)$, a function f on $\mathcal{M}_d^{\times n}$ with $f \circ \pi_n$ on Ω , and a measure $\mu^{\otimes n}$ on $(\mathcal{M}_d^{\times n}, \mathfrak{M}^{\otimes n})$ with $\mu^{\otimes n} \circ \pi_n$ on (Ω, \mathcal{O}_n) . Note that we cannot extend μ to the whole Ω as it is not assumed to be finite.

To any $\rho \in \mathcal{M}_d^{1,+}$ we define a law \mathbb{P}^ρ for the sequences in Ω . It is defined as

$$\mathbb{P}^\rho(O_n) = \int_{O_n} \text{tr}(v_n \cdots v_1 \rho v_1^* \cdots v_n^*) d\mu^{\otimes n}(v_1, \dots, v_n),$$

where $O_n \in \mathcal{O}_n$, $n \in \mathbb{N}^*$. The stochasticity condition, see Equation (1), guarantees this sequence of measures is a consistent family of probability measures, so via Kolmogorov's extension theorem it defines a unique probability measure \mathbb{P}^ρ on Ω . In quantum mechanics, this law is the one of some repeated quantum measurements modeled by μ .

For the maximally mixed (chaotic) state \mathbb{I}/d we put $\mathbb{P}^{\text{ch}} = \mathbb{P}^{\mathbb{I}/d}$. We often use \mathbb{P}^{ch} as a reference measure. Importantly, since $\rho \mapsto \mathbb{P}^\rho$ is positive and extends to a linear map from \mathcal{M}_d to finite complex measures, for every $\rho \in \mathcal{M}_d^{1,+}$,

$$\mathbb{P}^\rho \ll \mathbb{P}^{\text{ch}}.$$

Indeed, for any $\rho \in \mathcal{M}_d^{1,+}$, $\rho \leq \|\rho\| \mathbb{I}$. Hence, $\mathbb{P}^\rho \leq d \|\rho\| \mathbb{P}^{\text{ch}}$.

Let \mathcal{B} be the Borel σ -algebra on $\mathcal{M}_d^{1,+}$ and ν a Borel probability measure on $\mathcal{M}_d^{1,+}$. We define the probability measure \mathbb{P}_ν on $(\mathcal{M}_d^{1,+} \times \Omega, \mathcal{B} \otimes \mathcal{O})$ by

$$(7) \quad \mathbb{P}_\nu(B \times O) = \int_B \mathbb{P}^\rho(O) d\nu(\rho)$$

for $B \in \mathcal{B}$ and $O \in \mathcal{O}$. Note that the restriction of \mathbb{P}_ν to $\mathcal{B} \otimes \{\emptyset, \Omega\}$ is, by construction, equal to ν .

Let \mathbb{E}_ν denote the expectation with respect to \mathbb{P}_ν . Then, convexity of $\mathcal{M}_d^{1,+}$ implies $\rho_\nu = \mathbb{E}_\nu(\rho)$ is an element of $\mathcal{M}_d^{1,+}$. Moreover, since $\rho \mapsto \mathbb{P}^\rho$ is affine by definition, the marginal of \mathbb{P}_ν on \mathcal{O} is \mathbb{P}^{ρ_ν} , i.e.

$$\mathbb{P}_\nu(\mathcal{M}_d^{1,+} \times O) = \mathbb{P}^{\rho_\nu}(O)$$

for any $O \in \mathcal{O}$.

We identify the sub- σ -algebra $\{\emptyset, \mathcal{M}_d^{1,+}\} \otimes \mathcal{O}$ with \mathcal{O} and an \mathcal{O} -measurable function f on Ω with the $\mathcal{B} \otimes \mathcal{O}$ -measurable function f on $\mathcal{M}_d^{1,+} \times \Omega$ defined as $f(\rho, \omega) = f(\omega)$ for all $(\rho, \omega) \in \mathcal{M}_d^{1,+} \times \Omega$. This construction verifies $\mathbb{E}_\nu(f) = \mathbb{E}^{\rho_\nu}(f)$ for any f \mathcal{O} -measurable.

We now construct on $(\mathcal{M}_d^{1,+} \times \Omega, \mathbb{P}_\nu)$ a process whose law is the one of the Markov chain defined by Π and initial law ν . Let $V_i: \Omega \rightarrow \mathcal{M}_d$, $i \in \mathbb{N}^*$ be the i^{th} coordinate random variable

$$V_i(v_1, v_2, \dots, v_i, \dots) = v_i,$$

and let $(W_n)_n$ be the $(\mathcal{O}_n)_n$ -adapted sequence of products of them:

$$W_n = V_n \cdots V_1$$

for $n \in \mathbb{N}^*$. For the sake of consistency we set $W_0 = \text{Id}$. Next, we define the $(\mathcal{B} \otimes \mathcal{O}_n)_n$ -adapted sequence of random states $(\rho_n)_n$ on $(\mathcal{M}_d^{1,+} \times \Omega, \mathcal{B} \otimes \mathcal{O}, \mathbb{P}_\nu)$ as $\rho_0(\rho, \omega) = \rho$ for every $\rho \in \mathcal{M}_d^{1,+}$ and $\omega \in \Omega$, and

$$\rho_n = \frac{W_n \rho_0 W_n^*}{\text{tr}(W_n \rho_0 W_n^*)}$$

for $n \in \mathbb{N}^*$. Note that $\mathbb{P}_\nu(\text{tr}(W_n \rho_0 W_n^*) = 0) = 0$ for every $n \in \mathbb{N}$, so $(\rho_n)_n$ is \mathbb{P}_ν -almost surely well defined. Moreover, $(\rho_n)_n$ has the same distribution as the Markov chain with the initial distribution ν and transition kernel Π .

3. INVARIANT MEASURE ON DARK SUBSPACES, PROOF OF THEOREM 1

In this section we prove Theorem 1, i.e., the exponential convergence towards the unique invariant measure on maximal dark subspaces. The proof has the same structure as the one in [Ben+19]. After defining some topological structure on the set of Dark subspaces \mathcal{D}_m and defining an explicit realization of the Markov chain, we prove two geometric convergences. First, that there exists an \mathcal{O} -measurable estimator \widehat{E}_n of D_n with their relative distance decreasing exponentially fast. Second, that for any \mathcal{O} -measurable random variable, some shift of \mathbb{P}^ρ converges exponentially fast in total variation to \mathbb{P}^ρ . We conclude by piecing together these two convergences.

3.1. The structure of \mathcal{D}_m . Dark subspaces are linear subspaces of \mathbb{C}^d , hence elements of a Grassmannian. We denote $\mathcal{G}_{k,d}(\mathbb{C})$ the set of k -dimensional subspaces of \mathbb{C}^d and equip it with the *Grassmann topology*, which is the final topology induced by the function mapping a complex matrix of size $d \times k$ with linearly independent columns to the linear span of these columns.

With respect to this topology, $\mathcal{G}_{k,d}(\mathbb{C})$ is a compact space – see [FGP94, I-1-7]. Moreover, the *gap metric* defined by

$$d_G(Q_1, Q_2) = \|\pi_{Q_1} - \pi_{Q_2}\|,$$

for $Q_1, Q_2 \in \mathcal{G}_{k,d}(\mathbb{C})$, metrizes this topology – see [FGP94, I-2-6]. Later on, we shall exploit the fact that the gap metric can be expressed as

$$(8) \quad d_G(Q_1, Q_2) = \max \left\{ \sup_{x \in Q_1, \|x\|=1} \Delta(x, Q_2), \sup_{x \in Q_2, \|x\|=1} \Delta(x, Q_1) \right\},$$

where $\Delta(x, Q) = \|x - \pi_Q x\|$ is the distance between the vector $x \in \mathbb{C}^d$ and the subspace $Q \subset \mathbb{C}^d$ – see [GLR06, Theorem 13.1.1]. In fact, the two suprema in Equation (8) coincide – see [Mor10, Lemma 3.2] – so

$$(9) \quad d_G(Q_1, Q_2) = \sup_{x \in Q_1, \|x\|=1} \Delta(x, Q_2) = \sup_{x \in Q_2, \|x\|=1} \Delta(x, Q_1).$$

The last structural result we need is the compactness of \mathcal{D}_m .

Proposition 3.1. *The set \mathcal{D}_m is compact.*

Proof. Since $\mathcal{G}_{r_m,d}(\mathbb{C})$ is compact and $\mathcal{D}_m \subset \mathcal{G}_{r_m,d}(\mathbb{C})$, it suffices to prove that \mathcal{D}_m is closed. To this end, consider a sequence of maximal dark projectors $(\pi_n)_{n \in \mathbb{N}}$ that converges to some projector π . We will show that π is a dark projector of rank r_m . It is clear that π is an orthogonal projector, because the set of orthogonal projectors is closed. As for the rank of π , we just have to check that $\text{tr}(\pi) = r_m$, which is straightforward since $\text{tr}(\pi_n) = r_m$ for all n the trace is continuous.

It remains to show that π satisfies Equation (6). Fix $k \in \mathbb{N}^*$. Since every π_n is dark, for $\mu^{\otimes k}$ -almost all (v_1, \dots, v_k) ,

$$\pi_n v_k^* \cdots v_1^* v_1 \cdots v_k \pi_n - \frac{1}{r_m} \text{tr}(\pi_n v_k^* \cdots v_1^* v_1 \cdots v_k) \pi_n = 0.$$

As a function of π_n , the left hand side is continuous. Therefore, for $\mu^{\otimes k}$ -almost all (v_1, \dots, v_k) ,

$$\pi v_k^* \cdots v_1^* v_1 \cdots v_k \pi = \frac{1}{r_m} \pi$$

and the proposition is proved. \square

Remark 3.2. *This proposition implies \mathcal{D}_m is Borel measurable as a closed set.*

3.2. Probability space. As in the case of quantum trajectory, we define a probability space on which the Markov chain is explicitly realized.

Recall that the transition kernel for the Markov chain on maximal dark subspaces reads

$$(10) \quad K(D, B) = \int_{\mathcal{M}_d} \mathbf{1}_B(vD) \operatorname{tr} \left(v \frac{\pi_D}{r_m} v^* \right) d\mu(v),$$

where $D \in \mathcal{D}_m$ and B is a Borel subset of \mathcal{D}_m .

We put \mathcal{Q} for the Borel σ -algebra on \mathcal{D}_m and consider a Borel probability measure χ on \mathcal{D}_m . We define the probability measure \mathbb{P}_χ on $(\mathcal{D}_m \times \Omega, \mathcal{Q} \otimes \mathcal{O})$ analogously to how we defined \mathbb{P}_ν from a measure ν on the set of states $\mathcal{M}_d^{1,+}$, i.e., for $B \in \mathcal{Q}$ and $O \in \mathcal{O}$ we put

$$\mathbb{P}_\chi(B \times O_n) = \int_B \mathbb{P}^{\pi_D/r_m}(O) d\chi(D).$$

We write \mathbb{E}_χ for the expectation with respect to \mathbb{P}_χ and denote $\rho_\chi := \mathbb{E}_\chi(\pi_D/r_m) \in \mathcal{M}_d^{1,+}$. Recall that $\mathbb{P}^{\rho_\chi} \ll \mathbb{P}^{\text{ch}}$. Also, as before, an \mathcal{O} -measurable map f on Ω is identified with the $\mathcal{Q} \otimes \mathcal{O}$ -measurable map f on $\mathcal{D}_m \times \Omega$ satisfying $f(D, \omega) = f(\omega)$ for all $D \in \mathcal{D}_m$ and $\omega \in \Omega$. Then $\mathbb{E}_\chi(f) = \mathbb{E}^{\rho_\chi}(f)$.

Now, on $(\mathcal{D}_m \times \Omega, \mathcal{Q} \otimes \mathcal{O}, \mathbb{P}_\chi)$ we define a $(\mathcal{Q} \otimes \mathcal{O})_n$ -adapted process $(D_n)_n$ by setting $D_0(D, \omega) = D$ for every $(D, \omega) \in \mathcal{D}_m \times \Omega$ and $D_n = W_n D_0$ for $n \geq 1$. This process has the same distribution as the Markov chain on \mathcal{D}_m with the initial distribution χ and transition kernel K , see Equation (10). In terms of dark projectors, we have

$$\frac{\pi_{D_n}}{r_m} = \frac{W_n \pi_{D_0} W_n^*}{\operatorname{tr}(W_n \pi_{D_0} W_n^*)}.$$

Observe that $\mathbb{P}_\chi(W_n D_0 \in \mathcal{D}_m) = \mathbb{P}_\chi(\operatorname{tr}(W_n \pi_{D_0} W_n^*) > 0) = 1$, so the process under consideration is \mathbb{P}_χ -almost surely well defined.

3.3. Some useful process in $\mathcal{M}_d^{1,+}$. In [Ben+19], a key object in the proofs was the $(\mathcal{O}_n)_n$ -adapted sequence of random variables $(M_n)_n$ taking values in $\mathcal{M}_d^{1,+}$ and defined by

$$M_n = \frac{W_n^* W_n}{\operatorname{tr}(W_n^* W_n)}.$$

Note that M_n is \mathbb{P}_χ -almost surely well defined since it is \mathcal{O} -measurable, $\mathbb{P}^{\rho_\chi} \ll \mathbb{P}^{\text{ch}}$ and $\mathbb{P}^{\text{ch}}(\operatorname{tr}(W_n^* W_n) = 0) = 0$. The following key proposition was proved in [Ben+19]:

Proposition 3.3 ([Ben+19], Proposition 2.2). *There exists an \mathcal{O} -measurable $\mathcal{M}_d^{1,+}$ -valued random variable M_∞ such that*

$$\lim_{n \rightarrow \infty} M_n = M_\infty,$$

where the limit holds \mathbb{P}_χ -almost surely and in $L^1(\mathbb{P}_\chi)$ -norm for any Borel probability measure χ over \mathcal{D}_m .

Moreover, for any $\rho \in \mathcal{M}_d^{1,+}$,

$$\frac{d\mathbb{P}^\rho}{d\mathbb{P}^{\text{ch}}} = d \operatorname{tr}(\rho M_\infty).$$

Proof. Proposition 2.2 in [Ben+19] was expressed with respect to \mathbb{P}_ν with ν a probability measures over $\mathbb{P}(\mathbb{C}^d)$ instead of \mathbb{P}_χ . However, the proof of convergence is done with respect to \mathbb{P}^{ch} , which implies it holds with respect to \mathbb{P}^ρ for any $\rho \in \mathcal{M}_d^{1,+}$ since $\mathbb{P}^\rho \leq d \mathbb{P}^{\text{ch}}$. Hence, the limit claims hold with respect to \mathbb{P}_χ by definition of this measure.

The Radon-Nikodym derivative expression is proved in [Ben+19, Proposition 2.2]. \square

From the polar decomposition of W_n follows the existence of a unitary-valued random variable $U_n: \Omega \rightarrow \mathcal{U}(d)$ satisfying

$$W_n = U_n \sqrt{W_n^* W_n},$$

where $n \in \mathbb{N}$. We can choose $(U_n)_n$ so that it is $(\mathcal{O}_n)_n$ -adapted. Note that by definition of M_n , \mathbb{P}^{ch} -almost surely, $W_n \propto U_n \sqrt{M_n}$. Since $U_n(\omega) \in \mathcal{U}(d)$ and $\mathcal{U}(d)$ is compact, the sequence $(U_n(\omega))_n$ almost surely has an accumulation point. Let us point out that, when ω is fixed, both the accumulation points and the extraction subsequence leading to one depend on ω .

The following two propositions, essential in the sequel, will lead to the concentration of quantum trajectories on maximal dark subspaces.

Proposition 3.4. *For any $\omega \in \Omega$, let $\mathcal{U}_{\text{acc}}(\omega)$ be the set of accumulation points of $(U_n(\omega))_n$. Then, for any $\rho \in \mathcal{M}_d^{1,+}$, \mathbb{P}^ρ -almost surely, for any $U_\infty \in \mathcal{U}_{\text{acc}}$, the range of $U_\infty M_\infty U_\infty^*$ is a dark subspace.*

Proof. This proof mirrors the second part of [Ben+19, Proposition 2.2] proof. For the reader's convenience, we repeat the key steps. Remark that since $\mathbb{P}^\rho \ll \mathbb{P}^{\text{ch}}$, it is sufficient to prove the result \mathbb{P}^{ch} -almost surely.

Let $p \in \mathbb{N}^*$. As proved in [Ben+19, Proposition 2.2], $(M_n)_{n \in \mathbb{N}}$ is a martingale with respect to \mathbb{P}^{ch} . Since it is bounded, it converges \mathbb{P}^{ch} -almost surely and in $L^2(\mathbb{P}^{\text{ch}})$. Then, a lengthy direct computation leads to

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\text{ch}}((M_{n+p} - M_n)^2 \mid \mathcal{O}_n) = 0.$$

(For the intermediate steps leading to this result, see [Ben+19, Eq. (17)]). Hence, using Jensen inequality,

$$(11) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\text{ch}}(\|M_{n+p} - M_n\| \mid \mathcal{O}_n) = 0.$$

For any $n \in \mathbb{N}^*$, using the polar decomposition of W_n , M_{n+p} becomes

$$M_{n+p} = \frac{M_n^{\frac{1}{2}} U_n^* V_{n+1}^* \cdots V_{n+p}^* V_{n+p} \cdots V_{n+1} U_n M_n^{\frac{1}{2}}}{\text{tr}(V_{n+1}^* \cdots V_{n+p}^* V_{n+p} \cdots V_{n+1} U_n M_n U_n^*)},$$

and so

$$(12) \quad \mathbb{E}^{\text{ch}}(\|M_{n+p} - M_n\| \mid \mathcal{O}_n) = \int_{\mathcal{M}_d^{\times p}} \|M_n^{\frac{1}{2}} U_n^* v_1^* \cdots v_p^* v_p \cdots v_1 U_n M_n^{\frac{1}{2}} - M_n \text{tr}(v_1^* \cdots v_p^* v_p \cdots v_1 U_n M_n U_n^*)\| d\mu^{\otimes p}(v_1, \dots, v_p).$$

Now fix a realization ω and let $(U_{n_k}(\omega))_k$ be a subsequence of $(U_n(\omega))_n$ that is convergent to some unitary operator $U_\infty(\omega)$ as k tends to infinity. Dropping the notation ω , taking the limit of the right hand side of Equation (12) along $(n_k)_k$ and using Equation (11), we have

$$\int_{\mathcal{M}_d^{\times p}} \|M_\infty^{\frac{1}{2}} U_\infty^* v_1^* \cdots v_p^* v_p \cdots v_1 U_\infty M_\infty^{\frac{1}{2}} - M_\infty \text{tr}(v_1^* \cdots v_p^* v_p \cdots v_1 U_\infty M_\infty U_\infty^*)\| d\mu^{\otimes p}(v_1, \dots, v_p) = 0$$

for any $U_\infty \in \mathbf{U}_{\text{acc}}$. Since the expression under the integral is non-negative,

$$U_\infty M_\infty^{\frac{1}{2}} U_\infty^* v_1^* \cdots v_p^* v_p \cdots v_1 U_\infty M_\infty^{\frac{1}{2}} U_\infty^* \propto U_\infty M_\infty U_\infty^*$$

for $\mu^{\otimes p}$ -almost all (v_1, \dots, v_p) . Thus, denoting by π_∞ the orthogonal projection on the range of $U_\infty M_\infty U_\infty^*$ (which depends on ω),

$$\pi_\infty v_1^* \cdots v_p^* v_p \cdots v_1 \pi_\infty \propto \pi_\infty$$

for $\mu^{\otimes p}$ -almost all (v_1, \dots, v_p) . This implies π_∞ is a dark projection, as claimed. \square

The following result implies that the dark subspaces of the form $\text{range}(U_\infty M_\infty U_\infty^*)$ considered in the previous proposition are (almost surely) maximal. This proposition is that same as [Kus89, Theorem 2.8], however our proof is different and we do not assume the matrices are invertible. Recall that r_m is the dimension of maximal dark subspaces.

Proposition 3.5. *For all $\rho \in \mathcal{M}_d^{1,+}$, the rank of M_∞ is \mathbb{P}^ρ -almost surely equal to r_m .*

Proof. Again, using $\mathbb{P}^\rho \ll \mathbb{P}^{\text{ch}}$ we only need to prove the result \mathbb{P}^{ch} -almost surely. Let U_∞ be an accumulation point of $(U_n)_n$. Since $\text{rank } M_\infty = \text{rank } U_\infty M_\infty U_\infty^*$, Proposition 3.4 implies

$$\text{rank } M_\infty \leq r_m.$$

For the inverse inequality, let π be the orthogonal projector on some $D \in \mathcal{D}_m$. For every $n \in \mathbb{N}^*$ and for $\mu^{\otimes n}$ -almost every (v_1, \dots, v_n) there exists $\lambda \geq 0$ such that

$$\pi v_1^* \cdots v_n^* v_n \cdots v_1 \pi = \lambda \pi,$$

so taking the trace on both sides of this equality yields

$$\lambda = \frac{1}{r_m} \text{tr}(\pi v_1^* \cdots v_n^* v_n \cdots v_1).$$

In what follows we adopt the usual convention that $v_n \cdots v_1$ for $n = 0$ is Id. For all $n, p \in \mathbb{N}$

$$\pi v_1^* \cdots v_n^* W_p^* W_p v_n \cdots v_1 \pi = \frac{1}{r_m} \text{tr}(\pi v_1^* \cdots v_n^* W_p^* W_p v_n \cdots v_1) \pi$$

from which, dividing both sides by $\text{tr}(W_p^* W_p)$ leads to

$$\pi v_1^* \cdots v_n^* M_p v_n \cdots v_1 \pi = \frac{1}{r_m} \text{tr}(\pi v_1^* \cdots v_n^* M_p v_n \cdots v_1) \pi$$

\mathbb{P}^{ch} -almost surely. Consequently, taking the limit p goes to infinity, for every $n \in \mathbb{N}$ we obtain

$$(13) \quad \pi v_1^* \cdots v_n^* M_\infty v_n \cdots v_1 \pi = \frac{1}{r_m} \text{tr}(\pi v_1^* \cdots v_n^* M_\infty v_n \cdots v_1) \pi$$

\mathbb{P}^{ch} -almost surely and, if $n > 0$, for $\mu^{\otimes n}$ -almost all (v_1, \dots, v_n) . In fact, by continuity (see Proposition 2.3(2)), Equation (13) holds for every $(v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}$. Assume for a moment that for \mathbb{P}^{ch} -almost all M_∞ there exists $(v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}$ such that $\text{tr}(\pi v_1^* \cdots v_n^* M_\infty v_n \cdots v_1) \neq 0$. Then, Equation (13) and the rank of product inequality yield

$$\text{rank } M_\infty \geq \text{rank } \pi = r_m$$

and the proposition is proved.

It remains to prove that for \mathbb{P}^{ch} -almost all M_∞ there exists $(v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}$ such that $\text{tr}(\pi v_1^* \cdots v_n^* M_\infty v_n \cdots v_1) \neq 0$. Let

$$E = \text{linspan}\{v_n \cdots v_1 \pi x : x \in \mathbb{C}^d, (v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}, n \in \mathbb{N}\}.$$

Then for μ -almost every v , $vE \subset E$. Hence, $E = \mathbb{C}^d$ as it is non-zero by definition. Assume $\text{tr}(\pi v_1^* \cdots v_n^* M_\infty v_n \cdots v_1 \pi) = 0$ for any $(v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}$ with $n \in \mathbb{N}$ arbitrary. Then Equation (13) implies all the elements of the generating family of E are in $\ker M_\infty$. Hence, $E \subset \ker M_\infty$. Since $E = \mathbb{C}^d$, that implies $M_\infty = 0$. This contradicts $M_\infty \in \mathcal{M}_d^{1,+}$ and the proposition is proved. \square

Remark 3.6. For any quantum trajectory $(\rho_n)_n$, $\rho_n = U_n \frac{\sqrt{M_n} \rho_0 \sqrt{M_n}}{\text{tr}(\rho_0 M_n)} U_n^*$. Then all the accumulation points ρ_∞ of $(\rho_n)_n$ can be written as $\rho_\infty = U_\infty \frac{\sqrt{M_\infty} \rho_0 \sqrt{M_\infty}}{\text{tr}(\rho_0 M_\infty)} U_\infty^*$ with $U_\infty \in \mathbf{U}_{\text{acc}}$. Hence, Propositions 3.4 and 3.5 imply that any accumulation point ρ_∞ of $(\rho_n)_n$ is supported in a maximal dark subspace. Hence, asymptotically in n , ρ_n is almost surely supported in a maximal dark subspace. We will come back to this property with more details in Section 4.

3.4. Markov chain \mathcal{O} -measurable estimator. Let $n \in \mathbb{N}^*$ and consider the following max-likelihood estimator

$$\widehat{E}_n = \text{argmax}_{E \in \mathcal{G}_{r_m, d}(\mathbb{C})} \text{tr}(W_n \pi_E W_n^*).$$

Remark that by definition \widehat{E}_n is an r_m -dimensional subspace spanned by eigenvectors of $W_n^* W_n$ of respective eigenvalues $a_1^2(W_n), \dots, a_{r_m}^2(W_n)$ with $A \mapsto a_i(A)$ the map giving the singular values of A in decreasing order with multiplicity so that $a_1(A) \geq \dots \geq a_d(A)$. The choice of such subspace can be made in a $(\mathcal{O}_n)_n$ -adapted way.

Then define

$$\widehat{D}_n = W_n \widehat{E}_n.$$

We consider this $(\mathcal{O}_n)_n$ -adapted process as an estimator of D_n according to the sampling W_n , given the max-likelihood estimation \widehat{E}_n of the initial space D_0 . Note however that \widehat{D}_n is not necessarily a dark subspace. However, by definition of r_m , \widehat{D}_n is an element of $\mathcal{G}_{r_m, d}(\mathbb{C})$. Indeed, by definition of \widehat{E}_n , $\dim \widehat{D}_n < r_m$ would imply $\text{rank } W_n < r_m$, which contradicts the definition of r_m .

We now upper bound the expectation of $d_G(D_n, \widehat{E}_n)$. This estimation relies on the expression of Δ using the exterior algebra of \mathbb{C}^d .

Let $p \in \mathbb{N}^*$, we denote by $x_1 \wedge \cdots \wedge x_p$ the exterior product of vectors x_1, \dots, x_p in \mathbb{C}^d , and by $\bigwedge^p \mathbb{C}^d$ the vector space generated by the p -wedge products $x_1 \wedge \cdots \wedge x_p$. More precisely, $x_1 \wedge \cdots \wedge x_p$ stands for the alternating multilinear form

$$(\mathbb{C}^d)^p \ni (y_1, \dots, y_p) \mapsto \det(\langle x_i, y_j \rangle)_{i,j=1}^p.$$

Then, $\{x_1 \wedge \cdots \wedge x_p : x_1, \dots, x_p \in \mathbb{C}^d\}$ is a generating family for the space $\bigwedge^p \mathbb{C}^d$ of alternating multilinear forms on \mathbb{C}^d . We define a hermitian inner product on $\bigwedge^p \mathbb{C}^d$ as

$$\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p \rangle = \det(\langle x_i, y_j \rangle)_{i,j=1}^p,$$

and denote the associated norm by $\|x_1 \wedge \cdots \wedge x_p\|$. By linearity and from the properties of determinants, if $\{x_1, \dots, x_p\}$ is a linearly dependent family of vectors, $x_1 \wedge \cdots \wedge x_p = 0$. That implies the following standard proposition. For the reader's convenience, we provide a short proof.

Proposition 3.7. For any linearly independent family of vectors $\{y_1, \dots, y_p\}$ and any $x \in \mathbb{C}^d$,

$$\Delta(x, \text{span}\{y_1, \dots, y_p\}) = \frac{\|x \wedge y_1 \wedge \cdots \wedge y_p\|}{\|y_1 \wedge \cdots \wedge y_p\|}.$$

Proof. Let $F = \text{span}\{y_1, \dots, y_p\}$ and $x = \pi_F x + x^\perp$ with π_F the orthogonal projector onto F . Then, $\|x \wedge y_1 \wedge \dots \wedge y_p\| = \|x^\perp \wedge y_1 \wedge \dots \wedge y_p\|$ and $\Delta(x, F) = \|x^\perp\|$. Since x^\perp is orthogonal to F , setting $y_0 = x^\perp / \|x^\perp\|$, the first row and column of $(\langle y_i, y_j \rangle)_{i,j=0}^p$ are 0 for any $i = 0$ or $j = 0$ except for $i = j = 0$ where it is equal to 1. Therefore, by the properties of determinants, $\|y_0 \wedge y_1 \wedge \dots \wedge y_p\| = \|y_1 \wedge \dots \wedge y_p\|$. Hence,

$$\|x \wedge y_1 \wedge \dots \wedge y_p\| = \Delta(x, F) \|y_1 \wedge \dots \wedge y_p\|$$

and the proposition follows. \square

The last notion we need is the action of a linear map on exterior products: for $A \in \mathcal{L}(\mathbb{C}^d)$ we write $\bigwedge^p A$ for the operator acting on the basis elements of $\bigwedge^p \mathbb{C}^d$ as

$$\bigwedge^p A(x_1 \wedge \dots \wedge x_p) = Ax_1 \wedge \dots \wedge Ax_p$$

and then extended linearly to the whole space. Since $\bigwedge^p(AB) = \bigwedge^p A \bigwedge^p B$,

$$(14) \quad \|\bigwedge^p(AB)\| \leq \|\bigwedge^p A\| \|\bigwedge^p B\|.$$

Moreover, for any $p \leq d$,

$$(15) \quad \|\bigwedge^p A\| = a_1(A) \cdots a_p(A).$$

See e.g. [BL85a, Lemma III.5.3].

That set, we introduce a sequence $(s(n))_n$ that play the part of $(f(n))_n$ in [Ben+19]. For $r_m = 1$ they are equal as expected. For any $n \in \mathbb{N}^*$, let

$$s(n) = \int_{\mathcal{M}_d^{x_n}} \left\| \bigwedge^{r_m+1} W_n \right\|^{\frac{2}{r_m+1}} d\mu^{\otimes n}.$$

First we show this sequence decreases exponentially fast to 0.

Lemma 3.8. *There exist positive constants C and $\lambda < 1$ such that for every $n \in \mathbb{N}$*

$$s(n) \leq C\lambda^n.$$

Proof. First, by definition of \mathbb{P}^{ch} , for any $n \in \mathbb{N}^*$,

$$s(n) = d \mathbb{E}^{\text{ch}} \left[\left(\frac{\|\bigwedge^{r_m+1} W_n\|}{\left(\text{tr}(W_n^* W_n)\right)^{\frac{r_m+1}{2}}} \right)^{\frac{2}{r_m+1}} \right].$$

Then, Equation (15) implies

$$\frac{\|\bigwedge^{r_m+1} W_n\|}{\left(\text{tr}(W_n^* W_n)\right)^{\frac{r_m+1}{2}}} = \left\| \bigwedge^{r_m+1} \left(\frac{W_n}{\sqrt{\text{tr}(W_n^* W_n)}} \right) \right\| = \prod_{i=1}^{r_m+1} a_i(U_n \sqrt{M_n}).$$

Then, since $\|M_n\| \leq 1$, $a_i(U_n \sqrt{M_n}) \leq 1$ for any $i \in \{1, \dots, d\}$ and

$$\prod_{i=1}^{r_m+1} a_i(U_n \sqrt{M_n}) \leq 1.$$

From Proposition 3.5, $\lim_{n \rightarrow \infty} \prod_{i=1}^{r_m+1} a_i(U_n \sqrt{M_n}) = 0$. Hence, Lebesgue's dominated convergence theorem implies $\lim_{n \rightarrow \infty} s(n) = 0$.

Second, Equation (14) implies $(s(n))_n$ is submultiplicative. Thus, there exists $n_0 \in \mathbb{N}^*$ such that $s(n_0) < 1$ and $s(n) \leq s(l) s^{\lfloor n/n_0 \rfloor}(n_0)$ with $l \in \{0, \dots, n_0 - 1\}$ the remainder of the Euclidean division

of n by n_0 . Then fixing $1 > \lambda \geq s^{\frac{1}{n_0}}(n_0)$ non-null and $C = \max_{l \in \{0, \dots, n_0-1\}} s(l)\lambda^{-l-1}$ yields the lemma. \square

We now show $\sqrt{s(n)}$ upper bounds the expectation of $d_G(D_n, \widehat{D}_n)$.

Lemma 3.9. *For any Borel probability measure χ over \mathcal{D}_m ,*

$$\mathbb{E}_\chi(d(D_n, \widehat{D}_n)) \leq \sqrt{s(n)}.$$

Proof. Let (y_1, \dots, y_n) be an orthonormal basis of \widehat{E}_n such that $W_n y_i = a_i(W_n) U_n y_i$ for each $i \in \{1, \dots, r_m\}$. Then, for any $x \in \mathbb{C}^d$, from Proposition 3.7

$$\Delta(W_n x, \widehat{D}_n) = \frac{\|\bigwedge^{r_m+1} W_n x \wedge y_1 \wedge \dots \wedge y_{r_m}\|}{\prod_{i=1}^{r_m} a_i(W_n)} \leq a_{r_m+1}(W_n) \Delta(x, \widehat{E}_n).$$

Then, for any $D \in \mathcal{D}_m$ such that $\|W_n \pi_D\| \neq 0$, using $\|W_n x\|^2 = \frac{1}{r_m} \text{tr}(W_n \pi_D W_n^*)$ for any $x \in D$ of norm 1 and $\Delta(x, \widehat{E}_n) \leq 1$,

$$d_G^2(W_n D, \widehat{D}_n) \leq \frac{a_{r_m+1}^2(W_n)}{\frac{1}{r_m} \text{tr}(W_n \pi_D W_n^*)} \leq \frac{\|\bigwedge^{r_m+1} W_n\|^{\frac{2}{r_m+1}}}{\frac{1}{r_m} \text{tr}(W_n \pi_D W_n^*)}.$$

Taking the expectation with respect to χ , Jensen's inequality applied to $t \mapsto t^2$ implies

$$\mathbb{E}_\chi(d_G(D_n, \widehat{D}_n))^2 \leq \mathbb{E}_\chi(d_G^2(D_n, \widehat{D}_n)) \leq \mathbb{E}_\chi \left(\frac{\|\bigwedge^{r_m+1} W_n\|^{\frac{2}{r_m+1}}}{\frac{1}{r_m} \text{tr}(W_n \pi_{D_0} W_n^*)} \right) = s(n)$$

and the lemma is proved. \square

In what follows, θ stands for the left-shift operator on Ω , i.e., $\theta(v_1, v_2, \dots) = (v_2, v_3, \dots)$.

Proposition 3.10. *There exist positive constants C and $\lambda < 1$ such that for every Borel probability measure χ over \mathcal{D}_m and every $n, l \in \mathbb{N}$*

$$\mathbb{E}_\chi(d_G(D_{n+l}, \widehat{D}_n \circ \theta^l)) \leq C \lambda^n$$

Proof. Markov property implies that for any $n, l \in \mathbb{N}$,

$$\mathbb{E}_\chi(d_G(D_{n+l}, \widehat{D}_n \circ \theta^l)) = \mathbb{E}_{\chi_{K^l}}(d_G(D_n, \widehat{D}_n)).$$

Then, Lemma 3.9 implies

$$\mathbb{E}_\chi(d_G(D_{n+l}, \widehat{D}_n \circ \theta^l)) \leq \sqrt{s(n)}.$$

Hence, Lemma 3.8 yields the proposition. \square

3.5. Convergence for \mathcal{O} -measurable functions. The integer m appearing in Theorem 1 is the *period* of the channel ϕ , see [Ben+19, Definition 3.1]. For the reader's convenience, we recall its definition adapted to the current context, i.e., with ϕ assumed to be irreducible.

Definition 3.11. *Let (E_1, \dots, E_k) be an orthogonal partition of \mathbb{C}^d , i.e., the subspaces E_1, \dots, E_k are non zero and mutually orthogonal and $E_1 \oplus \dots \oplus E_k = \mathbb{C}^d$. We say that (E_1, \dots, E_k) is a k -cycle of ϕ if $v E_j \subset E_{j+1}$ for μ -almost all $v \in \mathcal{M}_d$ and for every $j \in \{1, \dots, k\}$, with the convention $E_{k+1} = E_1$. The period m of ϕ is the largest $k \in \mathbb{N}^*$ such that there exists a k -cycle of ϕ .*

The full space \mathbb{C}^d is a 1-cycle for ϕ and $m \leq d$ since $\dim \mathbb{C}^d = d$. Hence, $m \in \{1, \dots, d\}$. From Perron–Frobenius theorem for Schwartz positive¹ maps – [EHK78, Theorem 4.4] – there exists two positive constants C and $\lambda < 1$ such that for every $\rho \in \mathcal{M}_d^{1,+}$ and every $n \in \mathbb{N}$,

$$\left\| \frac{1}{m} \sum_{r=0}^{m-1} \phi^{mn+r}(\rho) - \rho_\infty \right\| \leq C\lambda^n.$$

This bound was used in [Ben+19] to show that \mathbb{P}^ρ converges in total variation under the shift θ towards \mathbb{P}^{ρ_∞} and that the rate of this convergence is exponential.

Proposition 3.12 ([Ben+19], Proposition 3.4). *There exist positive constants C and $\lambda < 1$ such that for any \mathcal{O} -measurable essentially bounded function f , any $\rho \in \mathcal{M}_d^{1,+}$, and any $n \in \mathbb{N}$*

$$\left| \mathbb{E}^\rho \left(\frac{1}{m} \sum_{r=0}^{m-1} f \circ \theta^{mn+r} \right) - \mathbb{E}^{\rho_\infty}(f) \right| \leq C \|f\|_\infty \lambda^n$$

where $\|f\|_\infty$ is the essential supnorm of f and ρ_∞ is the unique fixed point of $\phi : \mathcal{M}_d^{1,+} \rightarrow \mathcal{M}_d^{1,+}$.

3.6. Uniqueness of the invariant measure on Dark subspaces. We are now ready to prove the convergence towards a unique K -invariant measure χ_{inv} on \mathcal{D}_m . Recall that the Wasserstein distance of order 1 between two Borel probability measures μ, ν on \mathcal{D}_m can be expressed using Kantorovich–Rubinstein duality as

$$W_1(\mu, \nu) = \sup \{ \mathbb{E}_\mu(f) - \mathbb{E}_\nu(f) \mid f \in \text{Lip}_1(\mathcal{D}_m) \},$$

where $\text{Lip}_1(\mathcal{D}_m)$ denotes the set of all Lipschitz functions on \mathcal{D}_m with Lipschitz constant equal to 1.

The following theorem directly implies uniqueness of the K -invariant measure and the rest of Theorem 1.

Theorem 3.13. *There exist positive constants C and $\lambda < 1$ such that for any Borel probability measure χ over \mathcal{D}_m , any K -invariant Borel probability measure χ_{inv} over \mathcal{D}_m and $n \in \mathbb{N}^*$*

$$W_1 \left(\frac{1}{m} \sum_{r=0}^{m-1} \chi K^{mn+r}, \chi_{\text{inv}} \right) \leq C\lambda^n.$$

Proof. We follow the strategy of proof of [Ben+19, Theorem 1.1]. We need to show that

$$\sup \left\{ \mathbb{E}_\chi \left(\frac{1}{m} \sum_{r=0}^{m-1} f(D_{mn+r}) \right) - \mathbb{E}_{\chi_{\text{inv}}}(f(D_0)) \mid f \in \text{Lip}_1(\mathcal{D}_m) \right\}$$

is exponentially decreasing as n is growing. The expression being maximized is invariant under adding an arbitrary constant to f . Hence we can restrict the supremum to functions $f \in \text{Lip}_1(\mathcal{D}_m)$ such that $f(D^*) = 0$ for an arbitrary $D^* \in \mathcal{D}_m$. Then, since $\sup_{D, D' \in \mathcal{D}_m} d_G(D, D') \leq 1$, any $f \in \text{Lip}_1(\mathcal{D}_m)$ such that $f(D^*) = 0$ has its range contained in $[-1, 1]$. Hence, for the remainder of the proof we can restrict our attention to functions $f \in \text{Lip}_1(\mathcal{D}_m)$ such that $\|f\|_\infty \leq 1$.

Let $n \in \mathbb{N}^*$ and denote $p = \lfloor n/2 \rfloor$, $q = \lceil n/2 \rceil$ such that $p + q = n$. From the K -invariance of χ_{inv} ,

$$(16) \quad \mathbb{E}_{\chi_{\text{inv}}}(f(D_0)) = \mathbb{E}_{\chi_{\text{inv}}}(f(D_{mn+r})) \quad \text{for every } r \in \{0, \dots, m-1\}.$$

¹A completely positive map $T : \mathcal{M}_d \rightarrow \mathcal{M}_d$ preserving the identity is necessarily Schwartz in the sense that for any $A \in \mathcal{M}_d$, $T(A^*A) \geq T(A)^*T(A)$. That follows from the positivity of $T \otimes \text{Id}_{\mathcal{M}_2}$ applied to $\begin{pmatrix} A^*A & A^* \\ A & \text{Id}_{\mathbb{C}^d} \end{pmatrix}$.

Using this invariance and triangular inequality,

$$\begin{aligned}
(17) \quad & \left| \mathbb{E}_\chi \left(\frac{1}{m} \sum_{r=0}^{m-1} f(D_{mn+r}) \right) - \mathbb{E}_{\chi_{\text{inv}}} (f(D_0)) \right| = \left| \frac{1}{m} \sum_{r=0}^{m-1} (\mathbb{E}_\chi (f(D_{mn+r})) - \mathbb{E}_{\chi_{\text{inv}}} (f(D_{mn+r}))) \right| \\
& \leq \frac{1}{m} \sum_{r=0}^{m-1} |\mathbb{E}_\chi (f(D_{m(p+q)+r})) - \mathbb{E}_\chi (f(\widehat{D}_{mp} \circ \theta^{mq+r}))| \\
& \quad + \frac{1}{m} \sum_{r=0}^{m-1} |\mathbb{E}_{\chi_{\text{inv}}} (f(\widehat{D}_{mp} \circ \theta^{mq+r})) - \mathbb{E}_{\chi_{\text{inv}}} (f(D_{m(p+q)+r}))| \\
& \quad + \left| \frac{1}{m} \sum_{r=0}^{m-1} (\mathbb{E}_\chi (f(\widehat{D}_{mp} \circ \theta^{mq+r})) - \mathbb{E}_{\chi_{\text{inv}}} (f(\widehat{D}_{mp} \circ \theta^{mq+r}))) \right|.
\end{aligned}$$

We first bound the first two terms of the sum on the right hand side. Since f is 1-Lipschitz, for every $r \in \{0, \dots, m-1\}$,

$$|f(D_{m(p+q)+r}) - f(\widehat{D}_{mp} \circ \theta^{mq+r})| \leq d_G(D_{m(p+q)+r}, \widehat{D}_{mp} \circ \theta^{mq+r}),$$

so Proposition 3.10 implies there exist $C > 0$ and $\lambda \in [0, 1)$ independent of χ and χ_{inv} such that

$$|\mathbb{E}_\chi (f(D_{m(p+q)+r})) - \mathbb{E}_\chi (f(\widehat{D}_{mp} \circ \theta^{mq+r}))| \leq C\lambda^p$$

and

$$|\mathbb{E}_{\chi_{\text{inv}}} (f(D_{m(p+q)+r})) - \mathbb{E}_{\chi_{\text{inv}}} (f(\widehat{D}_{mp} \circ \theta^{mq+r}))| \leq C\lambda^p.$$

The third term in the sum on the right-hand side of Equation (17) is bounded using Proposition 3.12. Since $\widehat{D}_{mp} \circ \theta^{mq+r}$ is \mathcal{O} -measurable,

$$\mathbb{E}_\chi (f(\widehat{D}_{mp} \circ \theta^{mq+r})) = \mathbb{E}^{\rho_\chi} (f(\widehat{D}_{mp} \circ \theta^{mq+r}))$$

Then, Proposition 3.12 and $\|f\|_\infty \leq 1$ imply there exists $C > 0$ and $\lambda \in [0, 1)$ independent on χ and χ_{inv} such that

$$\left| \frac{1}{m} \sum_{r=0}^{m-1} (\mathbb{E}_\chi (f(\widehat{D}_{mp} \circ \theta^{mq+r})) - \mathbb{E}^{\rho_\infty} (f(\widehat{D}_{mp}))) \right| \leq C\lambda^q.$$

Following the same argument, the same bound holds with χ replaced by χ_{inv} . Hence, triangular inequality leads to

$$\left| \frac{1}{m} \sum_{r=0}^{m-1} (\mathbb{E}_\chi (f(\widehat{D}_{mp} \circ \theta^{mq+r})) - \mathbb{E}_{\chi_{\text{inv}}} (f(\widehat{D}_{mp} \circ \theta^{mq+r}))) \right| \leq 2C\lambda^q.$$

Combining the bounds on the three terms in the sum on the right hand side of Equation (17), there exist $C > 0$ and $\lambda \in [0, 1)$ independent of χ and χ_{inv} such that

$$\left| \mathbb{E}_\chi \left(\frac{1}{m} \sum_{r=0}^{m-1} f(D_{mn+r}) \right) - \mathbb{E}_{\chi_{\text{inv}}} (f(D_0)) \right| \leq C\lambda^n$$

and the theorem is proved. \square

4. EXPONENTIAL CONVERGENCE TOWARDS DARK SUBSPACES

In this section we show that quantum trajectories converge exponentially fast towards dark subspaces. Note that for $\rho \in \mathcal{M}_d^{1,+}$ and $D \in \mathcal{D}_m$, we have $\text{tr}(\rho\pi_D) = 1$ if and only if $\text{supp } \rho \subset D$. Then, the exponential convergence towards dark subspaces is expressed in next theorem.

Theorem 4.1. *There exist two positive constants C and $\lambda < 1$ such that for any Borel probability measure ν over $\mathcal{M}_d^{1,+}$,*

$$\mathbb{E}_\nu\left(\inf_{D \in \mathcal{D}_m} (1 - \text{tr}(\pi_D \rho_n))\right) \leq C\lambda^n.$$

Before proving this result we shall need a few lemmas. First, Proposition 3.10 implies D can be replaced by $(\widehat{D}_n)_n$.

Lemma 4.2. *There exists two positive constants C and $\lambda < 1$ such that for any Borel probability measure ν over $\mathcal{M}_d^{1,+}$ and any $n \in \mathbb{N}$,*

$$\mathbb{E}_\nu\left(\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)\right) \leq C\lambda^n.$$

Proof. Since $\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)$ is \mathcal{O} -measurable,

$$\mathbb{E}_\nu\left(\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)\right) = \mathbb{E}^{\rho_\nu}\left(\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)\right).$$

Let χ_{inv} be the unique K -invariant Borel probability measure over \mathcal{D}_m from Theorem 1. Then $\rho_{\chi_{\text{inv}}} = \mathbb{E}_{\chi_{\text{inv}}}(\pi_{D_0}/r_m) = \mathbb{E}_{\chi_{\text{inv}}}(\pi_{D_1}/r_m) = \phi(\rho_{\chi_{\text{inv}}})$. Hence $\rho_{\chi_{\text{inv}}} = \rho_\infty$ with ρ_∞ the unique fixed point of $\phi : \mathcal{M}_d^{1,+} \rightarrow \mathcal{M}_d^{1,+}$. Since we assume μ is irreducible, ρ_∞ is positive definite and there exist $c > 0$ such that for any $\rho \in \mathcal{M}_d^{1,+}$, $\rho \leq c\rho_\infty$. The linear extension of the map $\rho \mapsto \mathbb{E}^\rho$ to \mathcal{M}_d is positive, thus

$$\mathbb{E}^{\rho_\nu}\left(\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)\right) \leq c\mathbb{E}^{\rho_\infty}\left(\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)\right)$$

with c independent of ν . Then, choosing $D = D_n$, we obtain

$$\mathbb{E}_\nu\left(\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)\right) \leq c\mathbb{E}_{\chi_{\text{inv}}}(d_G(D_n, \widehat{D}_n))$$

and Proposition 3.10 yields the lemma. \square

One can remark that $\inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n)$ corresponds to the usual distance to \mathcal{D}_m , the set of maximal dark subspaces that is $d_G(\mathcal{D}_m, \widehat{D}_n)$. Second, ρ_n concentrates exponentially fast on \widehat{D}_n .

Lemma 4.3. *There exist two positive constants C and $\lambda < 1$ such that for any Borel probability measure ν over $\mathcal{M}_d^{1,+}$ and any $n \in \mathbb{N}$,*

$$\mathbb{E}_\nu\left(1 - \text{tr}(\pi_{\widehat{D}_n} \rho_n)\right) \leq C\lambda^n.$$

Proof. Let $n \in \mathbb{N}^*$. Since \widehat{E}_n is the span of eigenvectors of $W_n^*W_n$ with non-zero corresponding eigenvalues, and $\widehat{D}_n = W_n\widehat{E}_n$, $\widehat{D}_n = U_n\widehat{E}_n$ with U_n the unitary in the polar decomposition of W_n . Using trace cyclicity, it follows that

$$\begin{aligned} \text{tr}\left(\pi_{\widehat{D}_n} \rho_n\right) \text{tr}(W_n \rho_0 W_n^*) &= \text{tr}\left(\pi_{\widehat{D}_n} W_n \rho_0 W_n^*\right) \\ &= \text{tr}\left(U_n \pi_{\widehat{E}_n} U_n^* U_n \sqrt{W_n^* W_n} \rho_0 \sqrt{W_n^* W_n} U_n^*\right) \\ &= \text{tr}\left(\pi_{\widehat{E}_n} \sqrt{W_n^* W_n} \rho_0 \sqrt{W_n^* W_n}\right). \end{aligned}$$

Then, since $\pi_{\widehat{E}_n} W_n^* W_n = W_n^* W_n \pi_{\widehat{E}_n}$,

$$\left(1 - \operatorname{tr}(\pi_{\widehat{D}_n} \rho_n)\right) \operatorname{tr}(W_n \rho_0 W_n^*) = \operatorname{tr}((\operatorname{Id}_{\mathbb{C}^d} - \pi_{\widehat{E}_n}) W_n^* W_n \rho_0).$$

Since $\operatorname{Id}_{\mathbb{C}^d} - \pi_{\widehat{E}_n}$ is the orthogonal projector on a subspace spanned by eigenvectors of $W_n^* W_n$ corresponding to eigenvalues $a_{r_m+1}^2(W_n), \dots, a_d^2(W_n)$, $\|(\operatorname{Id}_{\mathbb{C}^d} - \pi_{\widehat{E}_n}) W_n^* W_n\| = a_{r_m+1}^2(W_n)$ and

$$\left(1 - \operatorname{tr}(\pi_{\widehat{D}_n} \rho_n)\right) \operatorname{tr}(W_n \rho_0 W_n^*) \leq a_{r_m+1}^2(W_n).$$

Hence, using the ordering of singular values and Equation (15),

$$\left(1 - \operatorname{tr}(\pi_{\widehat{D}_n} \rho_n)\right) \operatorname{tr}(W_n \rho_0 W_n^*) \leq \left\| \bigwedge^{r_m+1} W_n \right\|^{\frac{2}{r_m+1}}.$$

Then, by definition of $s(n)$ and \mathbb{P}_ν ,

$$\mathbb{E}_\nu \left(1 - \operatorname{tr}(\pi_{\widehat{D}_n} \rho_n)\right) \leq s(n)$$

and Lemma 3.8 yields the lemma. \square

We can finally prove the theorem.

Proof of Theorem 4.1. For any $D \in \mathcal{D}_m$,

$$\operatorname{tr}(\pi_D \rho_n) = \operatorname{tr}(\pi_{\widehat{D}_n} \rho_n) + \operatorname{tr}((\pi_D - \pi_{\widehat{D}_n}) \rho_n).$$

Hence, matrix Hölder's inequality in the form $|\operatorname{tr}(A^* B)| \leq \|A\|_{\operatorname{tr}} \|B\|$ for $A, B \in \mathcal{M}_d$ implies

$$\inf_{D \in \mathcal{D}_m} (1 - \operatorname{tr}(\pi_D \rho_n)) \leq 1 - \operatorname{tr}(\pi_{\widehat{D}_n} \rho_n) + \inf_{D \in \mathcal{D}_m} d_G(D, \widehat{D}_n).$$

Then, Lemmas 4.2 and 4.3 yield the theorem. \square

5. ISOMETRIES BETWEEN DARK SUBSPACES AND \mathbb{C}^{r_m}

In this section we focus on isometries between the maximal dark subspaces and the reference space \mathbb{C}^{r_m} . They are key to classifying the invariant measures for quantum trajectories, i.e., the Π -invariant measures on $\mathbb{P}(\mathbb{C}^d)$. To motivate their introduction, let us first consider a straightforward Π -invariant measure ν_{unif} coming from the K -invariant measure χ_{inv} on \mathcal{D}_m . Namely, ν_{unif} is constructed as the combination of uniform measures on the (projective) dark subspaces with weights given by χ_{inv} , i.e.,

$$\nu_{\text{unif}} = \int_{\mathcal{D}_m} \operatorname{Unif}_{\mathbb{P}(D)} d\chi_{\text{inv}}(D),$$

where $\mathbb{P}(D) = \{\hat{x} : x \in D, x \neq 0\}$ for $D \in \mathcal{D}_m$. We will verify its Π -invariance by direct computation. Recall that on pure states the action of Π reads

$$\Pi f(\hat{y}) = \int_{\mathcal{M}_d} f(v \cdot \hat{y}) \|vy\|^2 d\mu(v) \quad \text{for } \hat{y} \in \mathbb{P}(\mathbb{C}^d).$$

Proposition 5.1. ν_{unif} is Π -invariant.

Proof. For every $D \in \operatorname{supp} \chi_{\text{inv}}$ and $v \in \operatorname{supp} \mu$ the map $\mathbb{P}(D) \ni \hat{y} \mapsto v \cdot \hat{y} \in \mathbb{P}(vD)$ is bijective, provided that $\operatorname{tr}(v \pi_D v^*) > 0$, and so

$$\int_{\mathbb{P}(\mathbb{C}^d)} f(v \cdot \hat{y}) d\operatorname{Unif}_{\mathbb{P}(D)}(\hat{y}) = \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) d\operatorname{Unif}_{\mathbb{P}(vD)}(\hat{y}).$$

Thus, since $\|vy\|^2 = \text{tr}\left(v\frac{\pi D}{r_m}v^*\right)$ for every $\hat{y} \in \mathbb{P}(D)$, it follows that

$$\begin{aligned} \int_{\mathbb{P}(\mathbb{C}^d)} \Pi f(\hat{y}) \, d\nu_{\text{unif}}(\hat{y}) &= \int_{\mathcal{M}_d} \int_{\mathcal{D}_m} \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) \, d\text{Unif}_{\mathbb{P}(vD)}(\hat{y}) \, \text{tr}\left(v\frac{\pi D}{r_m}v^*\right) \, d\chi_{\text{inv}}(D) \, d\mu(v) \\ &= \int_{\mathcal{D}_m} \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) \, d\text{Unif}_{\mathbb{P}(D)}(\hat{y}) \, d(\chi_{\text{inv}}K)(D) \\ &= \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) \, d\nu_{\text{unif}}(\hat{y}), \end{aligned}$$

as desired. \square

A natural question now is whether ν_{unif} is the unique Π -invariant measure. In [Ben+19] it was shown that it is not always the case:

Proposition 5.2 ([Ben+19], Appendix C). *Assume $\text{supp } \mu \subset \mathcal{S}\mathcal{U}(d)$ and let G be the smallest closed subgroup of $\mathcal{S}\mathcal{U}(d)$ such that $\text{supp } \mu \subset G$. For any $\hat{x} \in \mathbb{P}(\mathbb{C}^d)$, put $[\hat{x}]_G$ for the orbit of \hat{x} with respect to G and the action $G \times \mathbb{P}(\mathbb{C}^d) \ni (u, \hat{x}) \mapsto u \cdot \hat{x}$, that is,*

$$[\hat{x}]_G := \{\hat{y} \in \mathbb{P}(\mathbb{C}^d) \mid \exists u \in G \text{ s.t. } \hat{y} = u \cdot \hat{x}\}.$$

Then, for any $\hat{x} \in \mathbb{P}(\mathbb{C}^d)$, there exists a unique Π -invariant probability measure $m_{\hat{x}}$ supported on $[\hat{x}]_G$, and this measure is uniform in the sense that it is invariant by the map $\hat{x} \mapsto u \cdot \hat{x}$ for any $u \in G$. More precisely, $m_{\hat{x}}$ is the image measure of the (normalized) Haar measure on G via the map $g_{\hat{x}}: G \rightarrow [\hat{x}]_G$ acting as $g_{\hat{x}}(u) = u \cdot \hat{x}$.

Note that, in particular, if $G = \mathcal{S}\mathcal{U}(d)$, then $[\hat{x}]_G = \mathbb{P}(\mathbb{C}^d)$ for every \hat{x} , so Π has a unique invariant probability measure and this measure is the uniform one on $\mathbb{P}(\mathbb{C}^d)$.

Let us point out that, within the setting of this proposition, if G is a proper subgroup of $\mathcal{S}\mathcal{U}(d)$, there may exist mutually singular Π -invariant measures on $\mathbb{P}(\mathbb{C}^d)$. Indeed, this is the case when there exist disjoint orbits under G , i.e., $[\hat{x}]_G \cap [\hat{y}]_G = \emptyset$ for some \hat{x} and \hat{y} . This happens, e.g., if G is finite, in which case there exist infinitely many disjoint trajectories, thus also infinitely many mutually singular Π -invariant measures. If, however, there exists a single orbit, i.e., $[\hat{x}]_G = \mathbb{P}(\mathbb{C}^d)$ for any \hat{x} (in other words, the action of G on $\mathbb{P}(\mathbb{C}^d)$ is transitive), then we have a unique Π -invariant probability measure on $\mathbb{P}(\mathbb{C}^d)$. A complete discussion and characterization is provided in Section 6.3.

Note that the assumption $\text{supp } \mu \subset \mathcal{S}\mathcal{U}(d)$ means that the whole space \mathbb{C}^d is the (unique) maximal dark subspace. A natural approach to the case of multiple maximal dark subspaces is to establish a correspondence between each $D \in \mathcal{D}_m$ and a common reference space \mathbb{C}^{r_m} by fixing a family of maps $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$, where

$$J_D: \mathbb{C}^{r_m} \rightarrow D$$

is a linear isometry. This allows us to turn (random) sequences of isometries acting between maximal dark subspaces into (random) sequences of unitary operators on \mathbb{C}^{r_m} , bringing us back to the setting of Proposition 5.2. Mirroring for $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$ the construction from Proposition 5.2, we will obtain a family of measures $m_{\hat{x}, \mathcal{J}}$ on $\mathbb{P}(\mathbb{C}^{r_m})$ invariant under the unitary dynamics from $G_{\mathcal{J}}$. Out of them we can then construct a family of Π -invariant measures on $\mathbb{P}(\mathbb{C}^d)$ by taking the product measures $\chi_{\text{inv}} \otimes m_{\hat{x}, \mathcal{J}}$ on $\mathcal{D}_m \times \mathbb{P}(\mathbb{C}^{r_m})$ and pushing them forward to $\mathbb{P}(\mathbb{C}^d)$ via the map $\Psi_{\mathcal{J}}: (D, \hat{z}) \mapsto J_D \cdot \hat{z}$, see also Figure 5.1.

Our ultimate aim, which we will reach in Section 6, is to show that the set of pushforward measures $(\Psi_{\mathcal{J}})_*(\chi_{\text{inv}} \otimes m_{\hat{x}, \mathcal{J}})$ over $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$ coincides with the set of all Π -ergodic measures on $\mathbb{P}(\mathbb{C}^d)$, provided that the family of isometries \mathcal{J} is chosen in an optimal way, i.e., it is a so-called minimal family. In the following subsections we first formalize the approach sketched above and then discuss what makes a family of isometries optimal for our construction.

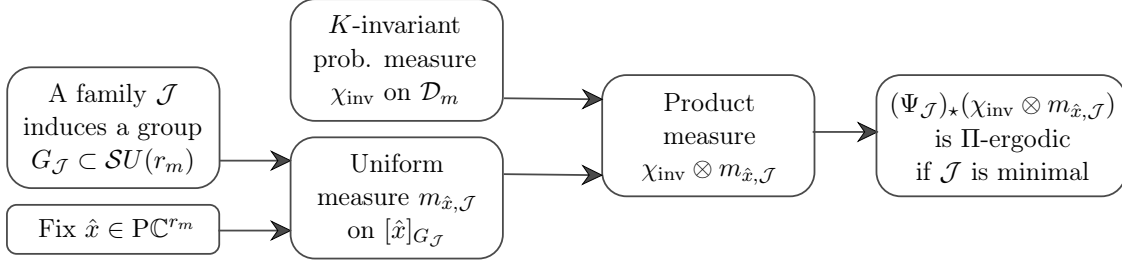


FIGURE 5.1. Diagram summarizing the construction of Π -ergodic measures

5.1. Minimal families of isometries. Fix $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$, where $J_D: \mathbb{C}^{r_m} \rightarrow D$ is a linear isometry. On $(\mathcal{D}_m \times \Omega, \mathcal{Q} \otimes \mathcal{O}, \mathbb{P}_{\chi_{\text{inv}}})$ we consider the random variable

$$U = \frac{J_{V_1 D_0}^{-1} V_1 J_{D_0}}{\sqrt{\text{tr}\left(V_1 \frac{\pi_{D_0}}{r_m} V_1^*\right)}} : \mathbb{C}^{r_m} \rightarrow D_0 \rightarrow V_1 D_0 \rightarrow \mathbb{C}^{r_m}.$$

Since $\mathbb{P}_{\chi_{\text{inv}}}(\text{tr}(V_1 \pi_{D_0} V_1^*) = 0) = \mathbb{P}_{\chi_{\text{inv}}}(V_1 D_0 = \{0\}) = 0$, we see that U is $\mathbb{P}_{\chi_{\text{inv}}}$ -almost surely well defined and takes values in the unitary group $\mathcal{U}(r_m)$ by definition of dark subspaces. Here and henceforth, if $\omega = (v, \dots)$ and $\text{tr}(v \pi_D v^*) > 0$, we let $u_{v, D}$ stand for $\exp(i\gamma)U(D, \omega)$, where γ is so chosen that $\det u_{v, D} = 1$. That is, $u_{v, D} \propto J_{vD}^{-1} v J_D$ is a special unitary operator induced on the reference space \mathbb{C}^{r_m} when the map v acts on the system occupying the dark subspace D . By $G_{\mathcal{J}}$ we denote the smallest closed subgroup of $\mathcal{SU}(r_m)$ containing the set

$$(18) \quad S_{\mathcal{J}} = \{u_{v, D} \in \mathcal{SU}(r_m) \mid v \in \text{supp } \mu, D \in \text{supp } \chi_{\text{inv}}, \text{tr}(v \pi_D v^*) > 0\}.$$

Let $H_{G_{\mathcal{J}}}$ be the normalized Haar measure on $G_{\mathcal{J}}$. Let $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$ and define $m_{\hat{x}, \mathcal{J}}$ as the pushforward measure of $H_{G_{\mathcal{J}}}$ by the map

$$\begin{aligned} g_{\hat{x}}: G_{\mathcal{J}} &\longrightarrow [\hat{x}]_{G_{\mathcal{J}}} \subset \mathbb{P}(\mathbb{C}^{r_m}) \\ u &\longmapsto u \cdot \hat{x} \end{aligned}.$$

We denote,

$$m_{\hat{x}, \mathcal{J}} = (g_{\hat{x}})_* H_{G_{\mathcal{J}}}.$$

Sending a copy of $m_{\hat{x}, \mathcal{J}}$ to each maximal dark subspace with a weight prescribed by χ_{inv} , we expect to obtain a Π -invariant measure on $\mathbb{P}(\mathbb{C}^d)$. Indeed, let

$$\begin{aligned} \Psi_{\mathcal{J}}: \mathcal{D}_m \times \mathbb{P}(\mathbb{C}^{r_m}) &\longrightarrow \mathbb{P}(\mathbb{C}^d) \\ (D, \hat{z}) &\longmapsto J_D \cdot \hat{z}. \end{aligned}$$

Then we define the measure $\nu_{\hat{x}, \mathcal{J}}$ as the push-forward measure of $\chi_{\text{inv}} \otimes m_{\hat{x}, \mathcal{J}}$ by $\Psi_{\mathcal{J}}$. Namely,

$$(19) \quad \nu_{\hat{x}, \mathcal{J}} = (\Psi_{\mathcal{J}})_*(\chi_{\text{inv}} \otimes m_{\hat{x}, \mathcal{J}}).$$

Explicitly,

$$\int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) d\nu_{\hat{x}, \mathcal{J}}(\hat{y}) = \int_{\mathcal{D}_m} \int_{G_{\mathcal{J}}} f(J_D u \cdot \hat{x}) d\mathbb{H}_{G_{\mathcal{J}}}(u) d\chi_{\text{inv}}(D),$$

where $f \in \mathcal{C}(\mathbb{P}(\mathbb{C}^d))$. As expected, we have the following

Proposition 5.3. $\nu_{\hat{x}, \mathcal{J}}$ is Π -invariant.

Proof. Recall that for every $u \in G_{\mathcal{J}}$ and $D \in \mathcal{D}_m$ we have

$$(20) \quad \|v J_D u x\|^2 = \text{tr}\left(v \frac{\pi_D}{r_m} v^*\right).$$

Also, recall that the invariance of Haar measure guarantees that for every $\tilde{u} \in G_{\mathcal{J}}$ and every $h \in \mathcal{C}(\mathbb{P}(\mathbb{C}^{r_m}))$,

$$(21) \quad \int_{G_{\mathcal{J}}} h(\tilde{u} u \cdot \hat{x}) d\mathbb{H}_{G_{\mathcal{J}}}(u) = \int_{G_{\mathcal{J}}} h(u \cdot \hat{x}) d\mathbb{H}_{G_{\mathcal{J}}}(u).$$

Fix $f \in \mathcal{C}(\mathbb{P}(\mathbb{C}^{r_m}))$. From Equations (20) and (21) and the K -invariance of χ_{inv} ,

$$\begin{aligned} & \int_{\mathbb{P}(\mathbb{C}^d)} \Pi f(\hat{y}) d\nu_{\hat{x}, \mathcal{J}}(\hat{y}) \\ &= \int_{\mathcal{M}_d} \int_{\mathcal{D}_m} \left[\int_{G_{\mathcal{J}}} f(v J_D u \cdot \hat{x}) d\mathbb{H}_{G_{\mathcal{J}}}(u) \right] \text{tr}\left(v \frac{\pi_D}{r_m} v^*\right) d\chi_{\text{inv}}(D) d\mu(v) \\ &= \int_{\mathcal{M}_d} \int_{\mathcal{D}_m} \left[\int_{G_{\mathcal{J}}} (f \circ J_{vD})(u_{v,D} u \cdot \hat{x}) d\mathbb{H}_{G_{\mathcal{J}}}(u) \right] \text{tr}\left(v \frac{\pi_D}{r_m} v^*\right) d\chi_{\text{inv}}(D) d\mu(v) \\ &= \int_{\mathcal{M}_d} \int_{\mathcal{D}_m} \left[\int_{G_{\mathcal{J}}} (f \circ J_{vD})(u \cdot \hat{x}) d\mathbb{H}_{G_{\mathcal{J}}}(u) \right] \text{tr}\left(v \frac{\pi_D}{r_m} v^*\right) d\chi_{\text{inv}}(D) d\mu(v) \\ &= \int_{\mathcal{D}_m} \int_{G_{\mathcal{J}}} (f \circ J_D)(u \cdot \hat{x}) d\mathbb{H}_{G_{\mathcal{J}}}(u) d[\chi_{\text{inv}} K](D) \\ &= \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) d\nu_{\hat{x}, \mathcal{J}}(\hat{y}), \end{aligned}$$

as desired. □

Remark 5.4. If $G_{\mathcal{J}} = \mathcal{S}U(r_m)$, then $m_{\hat{x}, \mathcal{J}} = \text{Unif}_{\mathbb{P}(\mathbb{C}^{r_m})}$ and $\nu_{\hat{x}, \mathcal{J}} = \nu_{\text{unif}}$ for all $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$.

Let us stress that all the objects related to the reference space $\mathbb{P}(\mathbb{C}^{r_m})$, in particular the group $G_{\mathcal{J}}$ and thus also the family of Π -invariant measures

$$\Upsilon_{\mathcal{J}} = \{\nu_{\hat{x}, \mathcal{J}} : \hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})\},$$

depend on the fixed family of isometries \mathcal{J} . It is no surprise that the choice of \mathcal{J} can heavily influence what Π -invariant measures we obtain in $\Upsilon_{\mathcal{J}}$. Namely, for two families \mathcal{J} and $\tilde{\mathcal{J}}$ it is possible that:

- $G_{\mathcal{J}}$ and $G_{\tilde{\mathcal{J}}}$ are different and neither of them is a subgroup of the other (even up to unitary equivalence), see Example 1(6a)
- $G_{\mathcal{J}}$ is uncountably infinite and $G_{\tilde{\mathcal{J}}}$ is finite, see Example 1(6b)
- $G_{\mathcal{J}}$ is the full special unitary group (so $\Upsilon_{\mathcal{J}} = \{\nu_{\text{unif}}\}$), while $G_{\tilde{\mathcal{J}}}$ is its proper subgroup (and $\Upsilon_{\tilde{\mathcal{J}}}$ is infinite), Example 1(5b).

As a consequence, there may exist a Π -invariant measure of the form $\nu_{\hat{x}, \tilde{\mathcal{J}}}$ which does not belong to the convex hull of $\Upsilon_{\mathcal{J}}$. This means that the set $\Upsilon_{\mathcal{J}}$ does not contain all Π -ergodic measures. Intuitively, this happens when the family \mathcal{J} is poorly chosen in the sense that the group $G_{\mathcal{J}}$ is too big compared to the dynamics actually taking place in $\mathbb{P}(\mathbb{C}^d)$. As a result, we end up with too few measures of the form $m_{\hat{x}, \mathcal{J}}$ on the reference space $\mathbb{P}(\mathbb{C}^{r_m})$, and so by transforming them to $\nu_{\hat{x}, \mathcal{J}}$ we are not able to recover all Π -invariant measures. We address this issue by introducing the notion of *minimal families*:

Definition 5.5. *A family of linear isometries $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ is called a minimal family if the group $G_{\mathcal{J}}$ is, up to a unitary equivalence, minimal as a subgroup of the special unitary group, i.e., for any other family $\tilde{\mathcal{J}}$ there exists $Q \in \mathcal{U}(r_m)$ such that $G_{\mathcal{J}} \subset QG_{\tilde{\mathcal{J}}}Q^{-1}$.*

We will see that the minimality of \mathcal{J} guarantees that $\Upsilon_{\mathcal{J}}$ is exactly the set of Π -ergodic measures. More precisely, as mentioned already in Theorem 2, if \mathcal{J} is a minimal family, then a probability measure ν on $\mathbb{P}(\mathbb{C}^d)$ is Π -ergodic if and only if $\nu = \nu_{\hat{x}, \mathcal{J}}$ for some $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$. As a first step towards proving this claim, in the next subsection we explore the properties of minimal families. In particular, we provide a characterization of minimal families in terms of the so-called smart families, which will play a key role in the results leading to the characterization of Π -invariant measures.

5.2. Properties of minimal families. First, we introduce and explore the notion of *smart families*, which we will then prove to be equivalent to minimal families. To this end, we need to introduce some notation. For $n \in \mathbb{N}^*$ we denote

$$\mathcal{V}_n = \{v_n \cdots v_1 \mid v_1, \dots, v_n \in \text{supp } \mu\}$$

and put $\mathcal{V}_0 = \{\text{Id}_{\mathbb{C}^d}\}$ for consistency. Then we define

$$\mathcal{V}_{\text{fin}} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n \quad \text{and} \quad \mathcal{W}_{\text{fin}} = \{v/\|v\| : v \in \mathcal{V}_{\text{fin}}, v \neq 0\},$$

and finally

$$\mathcal{W} = \overline{\mathcal{W}_{\text{fin}}}.$$

Note that, by construction, if $w_1, w_2 \in \mathcal{W}$ are such that $w_1 w_2 \neq 0$, then $w_1 w_2 / \|w_1 w_2\| \in \mathcal{W}$. Now, we can define smart families.

Definition 5.6. *Let $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ be a family of linear isometries such that $J_D \mathbb{C}^{r_m} = D$ and let $D_c \in \mathcal{D}_m$ be fixed. We say that \mathcal{J} is D_c -smart if for every $D \in \text{supp } \chi_{\text{inv}}$ there exists $w \in \mathcal{W}$ such that $J_D \propto w J_{D_c}$. We say that \mathcal{J} is smart if it is D_c -smart for all $D_c \in \text{supp } \chi_{\text{inv}}$.*

The idea behind D_c -smartness is the presence of the ‘central’ dark subspace D_c from which every other dark subspace in $\text{supp } \chi_{\text{inv}}$ can be reached with the help of some $w \in \mathcal{W}$ without inducing any non-trivial transformation on the reference space \mathbb{C}^{r_m} . This is because the property $J_D \propto w J_{D_c}$ is equivalent to $J_D^{-1} w J_{D_c} \propto \text{Id}$, and the following diagram is commuting

$$\begin{array}{ccc} D_c & \xrightarrow{\text{Id}} & D \\ J_{D_c} \uparrow & & \uparrow J_D \\ \mathbb{C}^{r_m} & \xrightarrow{w} & \mathbb{C}^{r_m} \end{array}$$

Our most immediate aim is to establish the existence of a D_c -smart family for an arbitrary $D_c \in \mathcal{D}_m$ and then to prove that D_c -smartness implies smartness whenever $D_c \in \text{supp } \chi_{\text{inv}}$. These results will be crucial in making the link between smart and minimal families, which is the main result of this section. To achieve these goals, we need a series of auxiliary results. Let us start by collecting a list of useful basic facts on isometries and dark subspaces (the proof is not provided as these properties follow directly from the definitions).

Proposition 5.7. *Let $D \in \mathcal{D}_m$, $\hat{y} \in \mathbb{P}(D)$ and $w \in \mathcal{W}$. Then*

- (i) $wy \neq 0$ iff $\text{tr}(w\pi_D w) > 0$ iff $wD \neq \{0\}$. In particular, $w \cdot \hat{y}$ and $u_{w,D}$ are well defined as soon as $\text{tr}(w\pi_D w) > 0$.
- (ii) $J_D \propto wJ_{D_c}$ implies that $wD_c = D$; in particular, $wD_c \neq \{0\}$.
- (iii) Since all J_D 's are isometric, the absolute value of the proportionality constant in $J_D \propto wJ_{D_c}$ is equal to $\text{tr}\left(w\frac{\pi_D}{r_m}w^*\right)^{-\frac{1}{2}}$.

Next, we establish an auxiliary result that concerns the support of χ_{inv} . It will allow us to prove the existence of a D_c -smart family for an arbitrary dark subspace D_c .

Proposition 5.8. *For any $D \in \mathcal{D}_m$ we have $\text{supp } \chi_{\text{inv}} \subset \mathcal{W}D$.*

Proof. Fix $D \in \mathcal{D}_m$. For $n \in \mathbb{N}$ we consider the empirical measure on \mathcal{D}_m , that is

$$\chi_n = \frac{1}{n+1} \sum_{k=0}^n \delta_D K^k = \frac{1}{n+1} \sum_{k=0}^n \int_{(\mathcal{M}_d)^k} \delta_{v_k \cdots v_1 D} \text{tr}\left(v_k \cdots v_1 \frac{\pi_D}{r_m} v_1^* \cdots v_k^*\right) d\mu^{\otimes k}(v_1, \dots, v_k).$$

Observe that $\text{supp } \chi_n \subset \mathcal{W}D$, because $\text{supp}(\delta_D K^k) = \overline{\{vD : v \in \mathcal{V}_k, \text{tr}(v\pi_D v^*) > 0\}}$ for every $k \in \{0, \dots, n\}$. Therefore,

$$(22) \quad \chi_n(\mathcal{W}D) = 1.$$

Since χ_{inv} is the unique K -invariant measure on \mathcal{D}_m , we have

$$(23) \quad \chi_n \xrightarrow[n \rightarrow \infty]{w^*} \chi_{\text{inv}}.$$

Indeed, by Prokhorov's theorem, $(\chi_n)_n$ has a weakly- $*$ convergent subsequence. Let χ be an accumulation point of $(\chi_n)_n$. Note that χ is necessarily K -invariant. Since χ_{inv} is the unique invariant probability measure, we have $\chi = \chi_{\text{inv}}$. Thus, the convergence of Equation (23) holds.

Now, using Portemanteau's theorem with the closed set $\mathcal{W}D$, we get

$$\limsup_{n \rightarrow \infty} \chi_n(\mathcal{W}D) \leq \chi_{\text{inv}}(\mathcal{W}D).$$

In consequence, from Equation (22) it follows that $\chi_{\text{inv}}(\mathcal{W}D) = 1$. Since $\mathcal{W}D$ is a closed set of full measure, we obtain $\text{supp } \chi_{\text{inv}} \subset \mathcal{W}D$, as desired. \square

Proposition 5.9. *Let $D_c \in \mathcal{D}_m$ and let $J : \mathbb{C}^{r_m} \rightarrow D_c$ be a linear isometry. There exists a D_c -smart family $(J_D)_{D \in \mathcal{D}_m}$ such that $J_{D_c} = J$.*

Proof. Set $J_{D_c} = J$. From Proposition 5.8, for every $D \in \text{supp } \chi_{\text{inv}}$ we have $D \in \mathcal{W}D_c$, so there exists $w \in \mathcal{W}$ such that $D = wD_c$. This guarantees that $\text{tr}(w\pi_{D_c} w^*) > 0$ and we set

$$J_D = \text{tr}\left(w\frac{\pi_{D_c}}{r_m}w^*\right)^{-\frac{1}{2}} wJ.$$

For every $D \notin \text{supp } \chi_{\text{inv}}$ we set J_D to be an arbitrary isometry between \mathbb{C}^{r_m} and D . By construction, $(J_D)_{D \in \mathcal{D}_m}$ satisfies the desired properties. \square

Next, we show that D_c -smartness is equivalent to smartness, provided that $D_c \in \text{supp } \chi_{\text{inv}}$. Although the following simple lemma can be considered folklore, its proof is included for the sake of completeness as we will invoke this result repeatedly.

Lemma 5.10. *Let $u \in \mathcal{U}(r_m)$ and $A, B \subset P(\mathbb{C}^{r_m})$.*

- (i) $(u^n)_{n=1}^\infty$ contains a subsequence convergent to Id .
- (ii) $uA \subset B \subset A$ implies that $A = B = uA$.

Proof. (i) As a corollary to a more general theorem, this result can be found in [Ben06, Corollary 7]. There is also a very simple proof that exploits the unitarity of u . Namely, since $\mathcal{U}(r_m)$ is compact, $(u^n)_{n=1}^\infty$ has a convergent subsequence $(u^{n_p})_{p=1}^\infty$, which necessarily is also a Cauchy sequence, i.e., for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\forall p, \tilde{p} \geq N \quad \|u^{n_{\tilde{p}}} - u^{n_p}\| = \|u^{n_{\tilde{p}} - n_p} - \text{Id}\| < \varepsilon.$$

For $q \in \mathbb{N}^*$ we put $m_q = n_{\tilde{p}} - n_p$, where p, \tilde{p} are so chosen that $(m_q)_{q=1}^\infty$ is strictly increasing. Then $u^{m_q} \rightarrow \text{Id}$ as $q \rightarrow \infty$, as desired.

(ii) From $uA \subset B \subset A$ it follows that $u^n A \subset B \subset A$ for every $n \in \mathbb{N}^*$. Extracting from $(u^n)_{n=1}^\infty$ a subsequence $(u^{n_k})_{k=1}^\infty$ convergent to Id , we obtain

$$A = \lim_{k \rightarrow \infty} u^{n_k} A \subset B \subset A,$$

which gives $A = B$. To prove that $A = uA$, it suffices to show $A \subset uA$. Observe that $u^{n_k-1} A \subset A$ for every $k \in \mathbb{N}^*$, thus also

$$u^{-1} A = \lim_{k \rightarrow \infty} u^{n_k-1} A \subset A,$$

which concludes the proof. \square

Proposition 5.11. *Let $D_c \in \text{supp } \chi_{\text{inv}}$. If $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ is D_c -smart, then \mathcal{J} is smart.*

Proof. Fix $D, D' \in \text{supp } \chi_{\text{inv}}$. We need to show that $J_{D'} \propto w J_D$ for some $w \in \mathcal{W}$. Observe that the D_c -smartness of \mathcal{J} guarantees the existence of $w' \in \mathcal{W}$ such that $J_{D'} \propto w' J_{D_c}$, so it is sufficient to construct $\tilde{w} \in \mathcal{W}$ such that $J_{D_c} \propto \tilde{w} J_D$ as we will then have $J_{D'} \propto w' \tilde{w} J_D$.

Using again the D_c -smartness of \mathcal{J} , we deduce the existence of $w_1 \in \mathcal{W}$ such that $J_D^{-1} w_1 J_{D_c} \propto \text{Id}$. Also, via Proposition 5.8, there exists w_2 such that $D_c = w_2 D$, and for the corresponding unitary we put $u \propto J_{D_c}^{-1} w_2 J_D$. The following diagram summarizes the dark subspaces and operators introduced so far.

$$\begin{array}{ccccc}
 & & \tilde{w} & & \\
 & & \curvearrowright & & \\
 D & \xleftarrow{w_1} & D_c & \xrightarrow{w'} & D' \\
 & & \curvearrowleft & & \\
 & & w_2 & & \\
 \uparrow J_D & & \text{Id} & & \uparrow J_{D_c} \\
 \mathbb{C}^{r_m} & \xleftarrow{\text{Id}} & \mathbb{C}^{r_m} & \xrightarrow{\text{Id}} & \mathbb{C}^{r_m} \\
 & & \curvearrowright & & \\
 & & u & & \\
 & & \curvearrowleft & & \\
 & & J_D & & \uparrow J_{D'}
 \end{array}$$

It follows that $w_2 w_1 D_c = D_c$ and

$$J_{D_c}^{-1} w_2 w_1 J_{D_c} = (J_{D_c}^{-1} w_2 J_D)(J_D^{-1} w_1 J_{D_c}) \propto u;$$

thus, for every $k \in \mathbb{N}^*$ we have

$$J_{D_c}^{-1}(w_2 w_1)^{k-1} w_2 J_D \propto u^k.$$

By Lemma 5.10, we can extract a subsequence $(k_n)_{n \in \mathbb{N}}$ such that $u^{k_n} \rightarrow \text{Id}$ as $n \rightarrow \infty$. Hence, putting $\tilde{w}_k = (w_2 w_1)^{k-1} w_2 / \|(w_2 w_1)^{k-1} w_2\| \in \mathcal{W}$, $k \in \mathbb{N}^*$, we obtain

$$J_{D_c}^{-1} \tilde{w}_{k_n} J_D \propto u^{k_n}.$$

Since the right-hand side converges as n tends to infinity, so does the left hand-side and, since \mathcal{W} is compact, up to another extraction (keeping the same notation), there exists $\tilde{w} \in \mathcal{W}$ such that $\tilde{w} = \lim_{n \rightarrow \infty} \tilde{w}_{k_n}$. Hence, $J_{D_c}^{-1} \tilde{w} J_D \propto \text{Id}$, i.e., $J_{D_c} \propto \tilde{w} J_D$, as desired. \square

Proposition 5.11 states that if the central space D_c is in $\text{supp } \chi_{\text{inv}}$, then the defining property of a D_c -smart family, which we recall is the possibility of going from D_c to any other dark subspace in $\text{supp } \chi_{\text{inv}}$ via some map from \mathcal{W} without inducing any non-trivial dynamics on \mathbb{C}^{r_m} , propagates to the whole $\text{supp } \chi_{\text{inv}}$, i.e., for any $D, D' \in \text{supp } \chi_{\text{inv}}$ there exists $w \in \mathcal{W}$ such that

$$J_{D'}^{-1} w J_D \propto \text{Id}.$$

Equivalently, going from D to D' via the reference space, i.e., $D \xrightarrow{J_D^{-1}} \mathbb{C}^{r_m} \xrightarrow{J_{D'}} D'$, does not induce any extra dynamics besides what can be achieved with the help of maps from \mathcal{W} :

$$J_{D'} J_D^{-1} \propto w|_D.$$

This leads us to another auxiliary result about smart families. Namely, we identify a dense subset of $G_{\mathcal{J}}$ whose elements have a particularly nice form. To proceed, we need to introduce some notation.

Here and henceforth, for a family $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ we denote by $u_{w,D}$ a special unitary operator on \mathbb{C}^{r_m} proportional to $J_{wD}^{-1} w J_D$, provided that $\text{tr}(w \pi_D w^*) > 0$, and put

$$R_{\mathcal{J}} = \{u_{w,D} \in \text{SU}(r_m) \mid w \in \mathcal{W}, D \in \text{supp } \chi_{\text{inv}}, \text{tr}(w \pi_D w^*) > 0\}.$$

Also, we denote the set of finite products of generators of $G_{\mathcal{J}}$ as

$$S_{\mathcal{J}}^{\text{fin}} = \{s_1 \cdots s_n \mid s_i \in S_{\mathcal{J}}, \text{ for } i = 1, \dots, n \text{ and } n \in \mathbb{N}^*\},$$

where we recall that $S_{\mathcal{J}}$ is defined as

$$S_{\mathcal{J}} = \{u_{v,D} \in \text{SU}(r_m) \mid v \in \text{supp } \mu, D \in \text{supp } \chi_{\text{inv}}, \text{tr}(v \pi_D v^*) > 0\}.$$

The following proposition states that in the case of smart families the elements of the form $u_{w,D}$ are dense in $G_{\mathcal{J}}$. Note, however, that we need not have $R_{\mathcal{J}} \subset G_{\mathcal{J}}$.

Proposition 5.12. *Let \mathcal{J} be a family of isometries. Then*

- (i) $S_{\mathcal{J}}^{\text{fin}}$ is a dense subset of $G_{\mathcal{J}}$;
- (ii) If \mathcal{J} is smart, then $S_{\mathcal{J}}^{\text{fin}} \subset R_{\mathcal{J}}$.

As a consequence, if \mathcal{J} is smart, then $G_{\mathcal{J}} \cap R_{\mathcal{J}}$ is dense in $G_{\mathcal{J}}$.

Proof. (i) Recall that, by definition, $G_{\mathcal{J}}$ is the closure of

$$\langle S_{\mathcal{J}} | S_{\mathcal{J}} \rangle = \{s_1^{\epsilon_1} \cdots s_{\ell}^{\epsilon_{\ell}} \mid s_i \in S_{\mathcal{J}}, \epsilon_i = \pm 1 \text{ for } i = 1, \dots, \ell \text{ and } \ell \in \mathbb{N}^*\}.$$

We will show that

$$S_{\mathcal{J}}^{\text{fin}} \subset \langle S_{\mathcal{J}} | S_{\mathcal{J}} \rangle \subset \overline{S_{\mathcal{J}}^{\text{fin}}}.$$

The first inclusion holds trivially. As for the second one, consider $s = s_1^{\epsilon_1} \cdots s_\ell^{\epsilon_\ell} \in \langle S_{\mathcal{J}} | S_{\mathcal{J}} \rangle$. By Lemma 5.10 (i), there exists a subsequence $(n_k)_k \subset \mathbb{N}$ such that for every $i \in \{1, \dots, \ell\}$ we have $s_i^{n_k-1} \rightarrow s_i^{-1}$ as $k \rightarrow \infty$. For $k \in \mathbb{N}$ define

$$\tilde{s}_{i,k} = \begin{cases} s_i^{n_k-1} & \text{if } \epsilon_i = -1 \\ s_i & \text{if } \epsilon_i = 1 \end{cases}$$

and $\tilde{s}_k = \tilde{s}_{1,k} \cdots \tilde{s}_{\ell,k}$ (note that by extracting iteratively, up to ℓ times if needed, we have a common subsequence $(n_k)_k$). Then $\tilde{s}_k \in S_{\mathcal{J}}^{\text{fin}}$ and

$$\tilde{s}_k \xrightarrow{k \rightarrow \infty} s;$$

hence, $s \in \overline{S_{\mathcal{J}}^{\text{fin}}}$, as claimed. It follows, by taking closures, that

$$\overline{S_{\mathcal{J}}^{\text{fin}}} = \overline{\langle S_{\mathcal{J}} | S_{\mathcal{J}} \rangle} = G_{\mathcal{J}},$$

which concludes the proof of (i).

(ii) We show that every finite product of elements from $S_{\mathcal{J}}$ is of the form $u_{w,D} \propto J_{wD}^{-1} w J_D$ with some $w \in \mathcal{W}$ and $D \in \text{supp } \chi_{\text{inv}}$. Consider

$$\begin{aligned} \prod_{i=1}^n u_{v_i, D_i} &= (J_{v_1 D_1}^{-1} v_1 J_{D_1}) (J_{v_2 D_2}^{-1} v_2 J_{D_2}) \cdots (J_{v_{n-1} D_{n-1}}^{-1} v_{n-1} J_{D_{n-1}}) (J_{v_n D_n}^{-1} v_n J_{D_n}) \\ &= J_{v_1 D_1}^{-1} v_1 (J_{D_1} J_{v_2 D_2}^{-1}) v_2 J_{D_2} \cdots J_{v_{n-1} D_{n-1}}^{-1} v_{n-1} (J_{D_{n-1}} J_{v_n D_n}^{-1}) v_n J_{D_n}. \end{aligned}$$

From Proposition 5.11 we deduce that

$$\forall i \in \{2 \dots n\} \quad \exists w_i \in \mathcal{W} \quad J_{D_{i-1}} J_{v_i D_i}^{-1} \propto w_i |_{v_i D_i}$$

and so

$$\prod_{i=1}^n u_{v_i, D_i} = J_{v_1 D_1}^{-1} v_1 w_2 v_2 w_3 v_3 \cdots w_n v_n J_{D_n}.$$

Denoting $w = v_1 w_2 v_2 w_3 v_3 \cdots w_n v_n$ we have $v_1 D_1 = w D_n$ and

$$\prod_{i=1}^n u_{v_i, D_i} = J_{w D_n}^{-1} w J_{D_n} \propto u_{w, D_n},$$

as desired. \square

From next lemma we will conclude the invariance of $\text{supp } \chi_{\text{inv}}$ under \mathcal{W} . The lemma itself is formulated in a slightly more general setting, covering also the case of $\mathbb{P}(\mathbb{C}^d)$, which we will exploit later on. Recall that $F \subset \mathcal{M}_d^{1,+}$ is called Π -invariant if $\Pi(\rho, F) = 1$ for every $\rho \in F$.

Lemma 5.13. *Let $F \subset \mathcal{M}_d^{1,+}$ be a closed and Π -invariant subset of $\mathcal{M}_d^{1,+}$. Then for every $\rho \in F$ and $w \in \mathcal{W}$ such that $\text{tr}(w \rho w^*) > 0$ we have $\frac{w \rho w^*}{\text{tr}(w \rho w^*)} \in F$.*

Proof. Let $\rho \in F$. First we show that the claim holds for maps from $\text{supp } \mu$. By contradiction, suppose that $\frac{v_0 \rho v_0^*}{\text{tr}(v_0 \rho v_0^*)} \notin F$ for some $v_0 \in \text{supp } \mu$ such that $\text{tr}(v_0 \rho v_0^*) > 0$. By continuity and since F is closed, there exist $V \subset \mathcal{M}_d$ an open neighbourhood of v_0 such that $\text{tr}(v \rho v^*) > 0$ and $\frac{v \rho v^*}{\text{tr}(v \rho v^*)} \notin F$ for every $v \in V$. Therefore,

$$\int_V \mathbf{1}_F \left(\frac{v \rho v^*}{\text{tr}(v \rho v^*)} \right) \text{tr}(v \rho v^*) d\mu(v) = 0 \quad \text{and} \quad \int_V \text{tr}(v \rho v^*) d\mu(v) > 0.$$

Then,

$$\begin{aligned} 1 = \Pi(\rho, F) &= \int_{\mathcal{M}_d} \mathbf{1}_F\left(\frac{v\rho v^*}{\text{tr}(v\rho v^*)}\right) \text{tr}(v\rho v^*) \, d\mu(v) = \int_{\mathcal{M}_d \setminus V} \mathbf{1}_F\left(\frac{v\rho v^*}{\text{tr}(v\rho v^*)}\right) \text{tr}(v\rho v^*) \, d\mu(v) \\ &\leq \int_{\mathcal{M}_d \setminus V} \text{tr}(v\rho v^*) \, d\mu(v) < 1, \end{aligned}$$

which implies $\frac{v_0\rho v_0^*}{\text{tr}(v_0\rho v_0^*)} \in F$ by contradiction. In consequence, by repeated application of this result, we deduce that $\frac{v\rho v^*}{\text{tr}(v\rho v^*)} \in F$ for every $v \in \mathcal{W}_{\text{fin}}$ such that $\text{tr}(v\rho v^*) > 0$.

To conclude, let us consider $w \in \mathcal{W}$ such that $\text{tr}(w\rho w^*) > 0$. Since \mathcal{W} is the closure of \mathcal{W}_{fin} , F is closed and $\text{tr}(w\rho w^*) > 0$, by continuity $\frac{w\rho w^*}{\text{tr}(w\rho w^*)} \in F$, as desired. \square

To derive the \mathcal{W} -invariance of $\text{supp } \chi_{\text{inv}} \subset \mathcal{D}_m$ by means of this lemma, we consider ν_{ch} , the image measure of χ_{inv} by the map

$$\begin{aligned} q: \mathcal{D}_m &\longrightarrow \mathcal{M}_d^{1,+} \\ D &\longmapsto \frac{\pi_D}{r_m}. \end{aligned}$$

Proposition 5.14. *If $D \in \text{supp } \chi_{\text{inv}}$ and $w \in \mathcal{W}$ satisfy $wD \neq \{0\}$, then $wD \in \text{supp } \chi_{\text{inv}}$.*

Proof. Since \mathcal{D}_m is compact and K is Feller with a unique invariant probability measure, [BH22, Proposition 5.12] implies $\text{supp } \chi_{\text{inv}}$ is a K -invariant set. Then, since q is continuous and one to one, $\text{supp } \nu_{\text{ch}} = \{\frac{\pi_D}{r_m} \mid D \in \text{supp } \chi_{\text{inv}}\}$ is a Π -invariant subset of $\mathcal{M}_d^{1,+}$. Thus, we can apply Lemma 5.13, which guarantees that if $D \in \text{supp } \chi_{\text{inv}}$ and $w \in \mathcal{W}$ satisfy $\text{tr}(w\pi_D w^*) > 0$, then

$$\frac{\pi_{wD}}{r_m} = \frac{w\pi_D w^*}{\text{tr}(w\pi_D w^*)} \in \text{supp } \nu_{\text{ch}};$$

hence, $wD \in \text{supp } \chi_{\text{inv}}$. To conclude the proof, it remains to recall that the condition $\text{tr}(w\pi_D w^*) > 0$ is equivalent to $wD \neq \{0\}$. \square

We are now in position to prove the main result of this section, namely that the notions of minimality and smartness of families are equivalent. In what follows, we consider two families $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ and $\tilde{\mathcal{J}} = \{\tilde{J}_D\}_{D \in \mathcal{D}_m}$ and adopt the following notation. The (special) unitaries on the reference space are, respectively,

$$u_{w,D} \propto J_{wD}^{-1} w J_D \quad \text{and} \quad \tilde{u}_{w,D} \propto \tilde{J}_{wD}^{-1} w \tilde{J}_D,$$

where $w \in \mathcal{W}$ and $D \in \text{supp } \chi_{\text{inv}}$ satisfy $\text{tr}(w\pi_D w^*) > 0$. Also, we put

$$Q_D = J_D^{-1} \tilde{J}_D$$

for $D \in \mathcal{D}_m$. Note that $Q_D \in \mathcal{U}(r_m)$ and $\tilde{J}_{wD}^{-1} w \tilde{J}_D = Q_{wD}^{-1} (J_{wD}^{-1} w J_D) Q_D$, that is

$$(24) \quad \tilde{u}_{w,D} = Q_{wD}^{-1} u_{w,D} Q_D.$$

Lemma 5.15. *Let $D_c \in \text{supp } \chi_{\text{inv}}$. If $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ is D_c -smart and $\tilde{\mathcal{J}} = \{\tilde{J}_D\}_{D \in \mathcal{D}_m}$ is an arbitrary family, then $Q_D^{-1} Q_{D'} \in G_{\tilde{\mathcal{J}}}$ for all $D, D' \in \text{supp } \chi_{\text{inv}}$.*

Proof. Let $D, D' \in \text{supp } \chi_{\text{inv}}$. First, we show that $Q_D^{-1} Q_{D_c} \in G_{\tilde{\mathcal{J}}}$. By the D_c -smartness of \mathcal{J} , there exists $w \in \mathcal{W}$ such that $J_D \propto w J_{D_c}$, i.e., $u_{w,D_c} = \text{Id}$, so Equation (24) leads to $Q_D^{-1} Q_{D_c} = \tilde{u}_{w,D} \in G_{\tilde{\mathcal{J}}}$, as claimed. Analogously, $Q_{D'}^{-1} Q_{D_c} \in G_{\tilde{\mathcal{J}}}$. It remains to observe that

$$Q_D^{-1} Q_{D'} = (Q_D^{-1} Q_{D_c})(Q_{D'}^{-1} Q_{D_c})^{-1} \in G_{\tilde{\mathcal{J}}},$$

which concludes the proof. \square

Theorem 5.16. *A family of isometries is minimal iff it is smart.*

Proof. (\Leftarrow) Let \mathcal{J} be a smart family and $\tilde{\mathcal{J}}$ an arbitrary family. We show

$$(25) \quad G_{\mathcal{J}} \subset Q_{D'} G_{\tilde{\mathcal{J}}} Q_{D'}^{-1} \quad \forall D' \in \text{supp } \chi_{\text{inv}}$$

from which minimality follows immediately. Fix $D' \in \text{supp } \chi_{\text{inv}}$. From Equation (24),

$$u_{v,D} = Q_{vD} \tilde{u}_{v,D} Q_D^{-1} = Q_{D'} (Q_{D'}^{-1} Q_{vD}) \tilde{u}_{v,D} (Q_D^{-1} Q_{D'}) Q_{D'}^{-1}$$

for every $v \in \text{supp } \mu$ and $D \in \text{supp } \chi_{\text{inv}}$ such that $\text{tr}(v\pi_D v^*) > 0$. Proposition 5.14 guarantees $vD \in \text{supp } \chi_{\text{inv}}$ and from Lemma 5.15 it follows that both $Q_{D'}^{-1} Q_{vD}$ and $Q_D^{-1} Q_{D'}$ are in $G_{\tilde{\mathcal{J}}}$; hence, $u_{v,D} \in Q_{D'} G_{\tilde{\mathcal{J}}} Q_{D'}^{-1}$. This means $G_{\mathcal{J}} \subset Q_{D'} G_{\tilde{\mathcal{J}}} Q_{D'}^{-1}$, as desired.

(\Rightarrow) Let $\tilde{\mathcal{J}} = \{\tilde{J}_D\}_{D \in \mathcal{D}_m}$ be minimal and let $D_c \in \text{supp } \chi_{\text{inv}}$. We show that $\tilde{\mathcal{J}}$ is D_c -smart. It follows from Proposition 5.9 that there exists a D_c -smart family $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ such that $J_{D_c} = \tilde{J}_{D_c}$. First, we show that $G_{\mathcal{J}} = G_{\tilde{\mathcal{J}}}$. From the (\Leftarrow) proof above, \mathcal{J} is minimal and, since $Q_{D_c} = \text{Id}$, from Equation (25) it follows that $G_{\mathcal{J}} \subset G_{\tilde{\mathcal{J}}}$. On the other hand, the minimality of $\tilde{\mathcal{J}}$ implies that there exists $Q \in \mathcal{U}(r_m)$ such that $G_{\tilde{\mathcal{J}}} \subset Q G_{\mathcal{J}} Q^{-1}$. Therefore,

$$G_{\mathcal{J}} \subset G_{\tilde{\mathcal{J}}} \subset Q G_{\mathcal{J}} Q^{-1}.$$

It now suffices to proceed as in the proof of Lemma 5.10(ii), that is, for every $n \in \mathbb{N}^*$, $G_{\mathcal{J}} \subset Q^n G_{\mathcal{J}} Q^{-n}$ and there exists a subsequence $(Q^{n_k})_{k=1}^{\infty}$ convergent to Id , see Lemma 5.10(i). Hence,

$$G_{\mathcal{J}} \subset G_{\tilde{\mathcal{J}}} \subset Q^{n_k} G_{\mathcal{J}} Q^{-n_k} \xrightarrow{k \rightarrow \infty} G_{\mathcal{J}},$$

and so $G_{\mathcal{J}} = G_{\tilde{\mathcal{J}}}$, as claimed. In what follows, we denote this common group by G .

We now fix $D \in \text{supp } \chi_{\text{inv}}$ and prove the D_c -smartness of $\tilde{\mathcal{J}}$ by constructing $\tilde{w} \in \mathcal{W}$ such that $\tilde{J}_D = \tilde{w} \tilde{J}_{D_c}$. Since $Q_{D_c} = \text{Id}$, Lemma 5.15 implies $Q_D \in G$. Assume first that $Q_D \in R_{\mathcal{J}}$, i.e.,

$$Q_D = u_{w,D'} \times J_{wD'}^{-1} w J_{D'}$$

for some $D' \in \text{supp } \chi_{\text{inv}}$ and $w \in \mathcal{W}$ such that $\text{tr}(w\pi_{D'} w^*) > 0$. Since \mathcal{J} is D_c -smart, there exist $w_1, w_2 \in \mathcal{W}$ such that $J_D \times w_1 J_{D_c}$ and $J_{D'} \times w_2 J_{D_c}$, and from Proposition 5.11 there exists $w_3 \in \mathcal{W}$ such that $J_{D_c} \times w_3 J_{wD'}$. It follows that

$$\tilde{J}_D = J_D Q_D = J_D u_{w,D'} \times w_1 J_{D_c} J_{wD'}^{-1} w J_{D'} \times w_1 w_3 w w_2 J_{D_c};$$

thus, since $J_{D_c} = \tilde{J}_{D_c}$, for $\tilde{w} = \frac{w_1 w_3 w w_2}{\|w_1 w_3 w w_2\|} \in \mathcal{W}$, $\tilde{J}_D \times \tilde{w} \tilde{J}_{D_c}$, as desired.

If $Q_D \notin R_{\mathcal{J}}$, then from Proposition 5.12 it follows that $Q_D = \lim_{n \rightarrow \infty} u_n$ with $(u_n)_n \in R_{\mathcal{J}}^{\mathbb{N}}$. Then, by definition of $R_{\mathcal{J}}$, for each $n \in \mathbb{N}$ there exists $w_n \in \mathcal{W}$ such that $J_D u_n \times w_n J_{D_c}$. Therefore,

$$\tilde{J}_D = J_D Q_D = \lim_{n \rightarrow \infty} (J_D u_n) = \lim_{n \rightarrow \infty} (w_n J_{D_c}).$$

Since \mathcal{W} is compact, $(w_n)_n$ has an accumulation point $\tilde{w} \in \mathcal{W}$. Then it is such that

$$\tilde{J}_D = \tilde{w} J_{D_c} = \tilde{w} \tilde{J}_{D_c},$$

and the theorem is proved. \square

We can now also show that all minimal families produce the same set of Π -invariant measures on $\mathbb{P}(\mathbb{C}^d)$:

Theorem 5.17. *Let $\mathcal{J}, \tilde{\mathcal{J}}$ be two minimal families. Then $\Upsilon_{\mathcal{J}} = \Upsilon_{\tilde{\mathcal{J}}}$.*

Proof. Let $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$. By permutation symmetry between \mathcal{J} and $\tilde{\mathcal{J}}$, it is sufficient to show that $\nu_{\hat{x}, \tilde{\mathcal{J}}} \in \Upsilon_{\mathcal{J}}$. Since $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ and $\tilde{\mathcal{J}} = \{\tilde{J}_D\}_{D \in \mathcal{D}_m}$ are both minimal, the definitions of minimality and $Q_{D'}$ imply

$$G_{\mathcal{J}} = Q_{D'} G_{\tilde{\mathcal{J}}} Q_{D'}^{-1} \quad \forall D' \in \text{supp } \chi_{\text{inv}}.$$

Also, via Lemma 5.15, the minimality of \mathcal{J} guarantees that

$$Q_D^{-1} Q_{D'} \in G_{\tilde{\mathcal{J}}} \quad \forall D, D' \in \text{supp } \chi_{\text{inv}}.$$

Let $f \in \mathcal{C}(\mathbb{P}(\mathbb{C}^d))$. Using the unitary invariance of $\text{H}_{G_{\tilde{\mathcal{J}}}}$ and the substitution $u = Q_{D'} \tilde{u} Q_{D'}^{-1}$, it follows that

$$\begin{aligned} \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{z}) d\nu_{\hat{x}, \tilde{\mathcal{J}}}(\hat{z}) &= \int_{\mathcal{D}_m} \int_{G_{\tilde{\mathcal{J}}}} f(\tilde{J}_D Q_D^{-1} Q_{D'} \tilde{u} \cdot \hat{x}) d\text{H}_{G_{\tilde{\mathcal{J}}}}(\tilde{u}) d\chi_{\text{inv}}(D) \\ &= \int_{\mathcal{D}_m} \int_{G_{\tilde{\mathcal{J}}}} f(J_D Q_{D'} \tilde{u} Q_{D'}^{-1} Q_{D'} \cdot \hat{x}) d\text{H}_{G_{\tilde{\mathcal{J}}}}(\tilde{u}) d\chi_{\text{inv}}(D) \\ &= \int_{\mathcal{D}_m} \int_{G_{\mathcal{J}}} f(J_D u Q_{D'} \cdot \hat{x}) d\text{H}_{G_{\mathcal{J}}}(u) d\chi_{\text{inv}}(D) \\ &= \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{z}) d\nu_{Q_{D'} \cdot \hat{x}, \mathcal{J}}(\hat{z}). \end{aligned}$$

Hence, $\nu_{\hat{x}, \tilde{\mathcal{J}}} = \nu_{Q_{D'} \cdot \hat{x}, \mathcal{J}} \in \Upsilon_{\mathcal{J}}$, as desired. \square

6. INVARIANT MEASURES ON QUANTUM TRAJECTORIES, PROOF OF THEOREMS 2 AND 3

We recall that Π -ergodic measures are the extreme points of the convex set of Π -invariant Borel probability measures. The main result of this section is the characterization of Π -ergodic measures. It is obtained via a result of Raugi [Rau92, Proposition 3.2], see also [BQ14, Proposition 2.9], which, adapted to our context, reads:

Theorem 6.1. *If the kernel Π is equicontinuous, i.e., if $(\Pi^n f)_n$ is equicontinuous for any $f \in \mathcal{C}(\mathbb{P}(\mathbb{C}^d))$, then:*

- (i) *the support of any Π -ergodic measure on $\mathbb{P}(\mathbb{C}^d)$ is a Π -minimal subset of $\mathbb{P}(\mathbb{C}^d)$;*
- (ii) *any Π -minimal subset of $\mathbb{P}(\mathbb{C}^d)$ carries a unique Π -invariant probability measure.*

The two ingredients needed to obtain the complete characterization of Π -ergodic measures via Raugi's theorem are the characterization of Π -minimal subsets of $\mathbb{P}(\mathbb{C}^d)$ and the proof of Π equicontinuity. We provide them in Sections 6.1 and 6.2 respectively. In the final Section 6.3 we apply Theorem 6.1 to conclude that the set of Π -ergodic measures coincides with $\Upsilon_{\mathcal{J}}$, provided \mathcal{J} is minimal.

6.1. Characterization of minimal sets. Here we prove that the Π -minimal subsets of $\mathbb{P}(\mathbb{C}^d)$ are exactly those of the form $\text{supp } \nu_{\hat{x}, \mathcal{J}}$, provided \mathcal{J} is a minimal family.

Recall that $F \subset \mathbb{P}(\mathbb{C}^d)$ is called Π -invariant if $\Pi(\hat{y}, F) = 1$ for every $\hat{y} \in F$, and that a Π -invariant, closed and non-empty set is called Π -minimal if it contains no proper subset enjoying these three properties. Since quantum trajectories converge (exponentially fast) towards dark subspaces,

see Theorem 4.1, we can restrict our search for Π -minimal subsets of $P(\mathbb{C}^d)$ to (projective) dark subspaces, as expressed in the following proposition, which is a corollary to Theorem 4.1.

Proposition 6.2. *If $F \subset P(\mathbb{C}^d)$ is Π -invariant, then $F \subset \bigcup_{D \in \mathcal{D}_m} P(D)$.*

In the context of pure trajectories, Lemma 5.13 can be restated as follows:

Proposition 6.3. *Let F be a closed and Π -invariant subset of $P(\mathbb{C}^d)$. If $\hat{y} \in F$ and $w \in \mathcal{W}$ are such that $w\hat{y} \neq 0$, then $w \cdot \hat{y} \in F$.*

Before we state and prove the main theorem of this section, we state and prove a standard lemma. Although it is standard, the formulation we require is not quite so. For example, we do not require the Markov kernel admits a unique invariant measure. That distinguishes it from [BH22, Proposition 5.12] for example. Hence, we provide a short proof.

Lemma 6.4. *Let \mathcal{T} be a topological space equipped with its Borel σ -algebra. Let T be a Feller Markov kernel on \mathcal{T} and ν a T -invariant probability measure over \mathcal{T} . Then $\text{supp } \nu$ is a T -invariant subset of \mathcal{T} . That is, for any $x \in \text{supp } \nu$, $T(x, \text{supp } \nu) = 1$.*

Proof. Assume there exist $x \in \text{supp } \nu$ such that $T(x, \text{supp } \nu) < 1$. Then $Tf(x) > 0$ for any continuous function f such that $f(y) = 0$ for every $y \in \text{supp } \nu$ and $f(y) > 0$ for every $y \notin \text{supp } \nu$. Since T is Feller, $x \mapsto Tf(x)$ is continuous. Therefore, there exists an open neighborhood A of x such that $Tf(y) > 0$ for any $y \in A$. By definition of $\text{supp } \nu$, $\nu(A) > 0$; hence, $\int_{\mathcal{T}} Tf(y) d\nu(y) > 0$. However, since ν is T -invariant, $\int_{\mathcal{T}} Tf(y) d\nu(y) = \int_{\mathcal{T}} f(y) d\nu(y) = 0$. Thus, by contradiction, $T(x, \text{supp } \nu) = 1$ for any $x \in \text{supp } \nu$. \square

We are now equipped to prove the main result of this section.

Theorem 6.5. *Let \mathcal{J} be a minimal family. Then every Π -minimal subset of $P(\mathbb{C}^d)$ is of the form $\text{supp } \nu_{\hat{x}, \mathcal{J}}$ for some $\hat{x} \in P(\mathbb{C}^{r_m})$.*

Proof. Fix a minimal family $\mathcal{J} = (J_D)_{D \in \mathcal{D}_m}$ and a set $F \subset P(\mathbb{C}^d)$. We will show that F is Π -minimal if and only if $F = \text{supp } \nu_{\hat{x}, \mathcal{J}}$ for some $\hat{x} \in P(\mathbb{C}^{r_m})$. To alleviate notation, we drop the subscript \mathcal{J} when referring to the measure $\nu_{\hat{x}, \mathcal{J}}$.

(\Leftarrow) Fix $\hat{x} \in P(\mathbb{C}^{r_m})$, we aim to show that $\text{supp } \nu_{\hat{x}} = \{J_D u \cdot \hat{x} \mid D \in \text{supp } \chi_{\text{inv}}, u \in G_{\mathcal{J}}\}$ is Π -minimal. Since Π is Feller and $\nu_{\hat{x}}$ is Π -invariant, Lemma 6.4 implies $\text{supp } \nu_{\hat{x}}$ is Π -invariant. It only remains to show that it has no non-trivial Π -invariant subsets. Let F be a non-empty, closed, and Π -invariant subset of $\text{supp } \nu_{\hat{x}}$. To conclude that $\text{supp } \nu_{\hat{x}} = F$, it suffices to show that $\text{supp } \nu_{\hat{x}} \subset F$. To this end, we fix

$$\hat{y}_0 = J_{D_0} u_0 \cdot \hat{x} \in F \subset \text{supp } \nu_{\hat{x}} \quad \text{and} \quad \hat{y}_1 = J_{D_1} u_1 \cdot \hat{x} \in \text{supp } \nu_{\hat{x}}.$$

In order to conclude that $\hat{y}_1 \in F$, we show that \hat{y}_1 can be reached from \hat{y}_0 with the help of a map from \mathcal{W} .

First, assume that $u_1 u_0^{-1} \in R_{\mathcal{J}}$. By Proposition 5.12(ii),

$$u_1 u_0^{-1} = u_{w,D} \circ J_w^{-1} w J_D$$

for some $D \in \text{supp } \chi_{\text{inv}}$ and $w \in \mathcal{W}$ such that $\text{tr}(w \pi_D w^*) > 0$. By Proposition 5.14, $wD \in \text{supp } \chi_{\text{inv}}$. Also, Proposition 5.11 and smartness imply there exist $w_0, w_1 \in \mathcal{W}$ such that $J_D \circ w_0 J_{D_0}$ and $J_{D_1} \circ w_1 J_{wD}$; that is,

$$J_D^{-1} w_0 J_{D_0} \circ \text{Id} \quad \text{and} \quad J_{D_1}^{-1} w_1 J_{wD} \circ \text{Id}.$$

The following diagram summarizes the dark subspaces and operations discussed so far.

$$\begin{array}{ccccccc}
D_0 & \xrightarrow{w_0} & D & \xrightarrow{w} & wD & \xrightarrow{w_1} & D_1 \\
\uparrow J_{D_0} & & \uparrow J_D & & \uparrow J_{wD} & & \uparrow J_{D_1} \\
\mathbb{C}^{r_m} & \xrightarrow{\text{Id}} & \mathbb{C}^{r_m} & \xrightarrow{u_1 u_0^{-1}} & \mathbb{C}^{r_m} & \xrightarrow{\text{Id}} & \mathbb{C}^{r_m}
\end{array}$$

It follows that $w_1 w w_0 D_0 = D_1$, so from Proposition 6.3 we get $w_1 w w_0 \cdot \hat{y}_0 \in F$. Moreover, by direct calculation we now show that $w_1 w w_0 \cdot \hat{y}_0 = \hat{y}_1$. Indeed, w_0 maps D_0 to D :

$$w_0 \cdot \hat{y}_0 = w_0 J_{D_0} u_0 \cdot \hat{x} = J_D J_D^{-1} w_0 J_{D_0} u_0 \cdot \hat{x} = J_D u_0 \cdot \hat{x}.$$

Next, under the action of w on D , the unitary $u_{w,D} = u_1 u_0^{-1}$ arises on $\text{P}(\mathbb{C}^{r_m})$:

$$w J_D u_0 \cdot \hat{x} = J_{wD} J_{wD}^{-1} w J_D u_0 \cdot \hat{x} = J_{wD} u_{w,D} u_0 \cdot \hat{x} = J_{wD} u_1 \cdot \hat{x}.$$

Finally, w_1 maps wD to the desired dark subspace D_1 while inducing Id on the reference space:

$$w_1 J_{wD} u_1 \cdot \hat{x} = J_{D_1} J_{D_1}^{-1} w_1 J_{wD} u_1 \cdot \hat{x} = J_{D_1} u_1 \cdot \hat{x} = \hat{y}_1,$$

as claimed.

If $u_1 u_0^{-1} \notin R_{\mathcal{J}}$, then $u_1 u_0^{-1} = \lim_{n \rightarrow \infty} u_n$, where $(u_n)_n \subset R_{\mathcal{J}}$, see Proposition 5.12. From the result above, for each $n \in \mathbb{N}$ there exists $w_n \in \mathcal{W}$ such that

$$w_n \cdot \hat{y}_0 = J_{D_1} u_n u_0 \cdot \hat{x}.$$

The right-hand side converges to \hat{y}_1 as $n \rightarrow \infty$. Hence, so does the left-hand side. Therefore,

$$\tilde{w} \cdot \hat{y}_0 = \hat{y}_1,$$

where \tilde{w} is an accumulation point of $(w_n)_n$, and the proof of the first implication is concluded.

(\Rightarrow) Fix a Π -minimal set $F \subset \text{P}(\mathbb{C}^d)$, we show there exists $\hat{x} \in \text{P}(\mathbb{C}^{r_m})$ such that $\text{supp } \nu_{\hat{x}} \subset F$. Then, by minimality, Lemma 6.4 implies $F = \text{supp } \nu_{\hat{x}}$. We need to investigate how the minimality of F is expressed in the dark subspaces and more precisely how it is expressed in $\text{P}(\mathbb{C}^{r_m})$. To this end, for $D \in \mathcal{D}_m$ we consider

$$X_D = J_D^{-1}(F \cap \text{P}(D)) \subset \text{P}(\mathbb{C}^{r_m}).$$

We break the proof into three main steps.

First step. It consists in showing that there exists a non-empty subset $X \subset \text{P}(\mathbb{C}^{r_m})$ such that $X_D = X$ for every $D \in \text{supp } \chi_{\text{inv}}$.

We start by showing that if $D_0 \in \text{supp } \chi_{\text{inv}}$, then $X_{D_0} \neq \emptyset$. Indeed, since F is a non-empty subset of $\bigcup_{D \in \mathcal{D}_m} \text{P}(D)$, see Proposition 6.2, we obtain

$$\bigcup_{D \in \mathcal{D}_m} J_D X_D = F \cap \bigcup_{D \in \mathcal{D}_m} \text{P}(D) = F \neq \emptyset,$$

and so there exists $D \in \mathcal{D}_m$ such that $X_D \neq \emptyset$. By Proposition 5.8, there exists $w \in \mathcal{W}$ such that $D_0 = wD$. Fix $\hat{x} \in X_D$. Then, $J_D \cdot \hat{x} \in F \cap \text{P}(D)$ and $\text{tr}(w \pi_D w^*) > 0$, i.e., $\|w J_D \hat{x}\| > 0$, which via Proposition 6.3 implies that $w J_D \cdot \hat{x} \in F \cap \text{P}(D_0)$. That is, $J_{D_0}^{-1} w J_D \cdot \hat{x} \in X_{D_0}$; hence $X_{D_0} \neq \emptyset$, as claimed.

Next, observe that if $D \in \text{supp } \chi_{\text{inv}}$ and $w \in \mathcal{W}$ satisfies $wD \neq \{0\}$, then $u_{w,D}X_D \subset X_{wD}$. Indeed, let $\hat{x} \in X_D$. We have $J_D \cdot \hat{x} \in F \cap P(D)$ and $wD \neq \{0\}$, so again via Proposition 6.3 we get $wJ_D \cdot \hat{x} \in F \cap P(wD)$; thus $J_{wD}^{-1}wJ_D \cdot \hat{x} = u_{w,D} \cdot \hat{x} \in X_{wD}$, as claimed.

We are now ready to conclude the first step. Fix $D, D' \in \text{supp } \chi_{\text{inv}}$. Proposition 5.11 guarantees that there exists $w' \in \mathcal{W}$ such that $J_D \propto w'J_{D'}$, i.e., $u_{w',D'} = \text{Id}$. This implies $X_{D'} \subset X_D$. On the other hand, by Proposition 5.8, there exists $w \in \mathcal{W}$ such that $wD = D'$, so $u_{w,D}X_D \subset X_{D'}$. Therefore,

$$u_{w,D}X_D \subset X_{D'} \subset X_D$$

and Lemma 5.10(ii) gives $X_D = X_{D'}$. For the remainder of the proof we denote this common set by X .

Second step. It consists in showing $X = uX$ for every $u \in G_{\mathcal{J}}$.

Since X is closed as the continuous pre-image of the intersection of two closed sets, by Proposition 5.12, it suffices to consider $u \in G_{\mathcal{J}} \cap R_{\mathcal{J}}$. Let $u = u_{w,D}$ for some $D \in \text{supp } \chi_{\text{inv}}$ and $w \in \mathcal{W}$ such that $\text{tr}(w\pi_D w^*) > 0$. Since $wD \in \text{supp } \chi_{\text{inv}}$, from the previous step we immediately get the inclusion $u_{w,D}X \subset X$, and so Lemma 5.10(ii) gives the desired equality $u_{w,D}X = X$. Hence, $X = uX$ for all $u \in G_{\mathcal{J}} \cap R_{\mathcal{J}}$. Note that, as a consequence, $[\hat{x}]_{G_{\mathcal{J}}} \subset X$ for any $\hat{x} \in X$.

Third step, conclusion. Fix any $\hat{x} \in X$ and observe that

$$F = \bigcup_{D \in \mathcal{D}_m} J_D X_D \supset \bigcup_{D \in \text{supp } \chi_{\text{inv}}} J_D X \supset \bigcup_{D \in \text{supp } \chi_{\text{inv}}} J_D [\hat{x}]_{G_{\mathcal{J}}} = \text{supp } \nu_{\hat{x}}.$$

Then, $\text{supp } \nu_{\hat{x}} \subset F$ and the theorem holds. \square

6.2. Equicontinuity of the kernel. In what follows we show that Π is equicontinuous, i.e., that the family $(\Pi^n f)_{n \in \mathbb{N}^*}$ is (uniformly) equicontinuous on $P(\mathbb{C}^d)$ for every $f \in \mathcal{C}(P(\mathbb{C}^d))$. Our approach consists in showing that this is the case when f is a Lipschitz function and then using a density argument.

On $P(\mathbb{C}^d)$ we use the metric $\delta(\hat{x}, \hat{y}) = \sqrt{1 - |\langle x, y \rangle|^2}$, which coincides with the gap metric $d_G(\mathbb{C}x, \mathbb{C}y) = \|\pi_{\hat{x}} - \pi_{\hat{y}}\| = \|x \wedge y\|$ (recall that x, y stand for unit norm representatives of \hat{x}, \hat{y} respectively). Also, recall the notation $\mathcal{V}_n = \{v_n \cdots v_1 : (v_1, \dots, v_n) \in \text{supp } \mu^{\otimes n}\}$ and observe that it can be rewritten as $\mathcal{V}_n = \{W_n(\omega) : \omega \in \Omega, \pi_n(\omega) \in \text{supp } \mu^{\otimes n}\}$. We repeatedly make use of the stochasticity condition, which in the current context is expressed as $\int_{\Omega} W_n^* W_n d\mu^{\otimes n} = \text{Id}_{\mathbb{C}^d}$.

The following three lemmas establish the equicontinuity property for Lipschitz functions. Similar estimations are made in [BHP24, Lemma 4.5] under the assumption of purification (i.e., $r_m = 1$). Since we work without this assumption, some bounds need to be adapted.

Lemma 6.6. *Let $f \in \mathcal{C}(P(\mathbb{C}^d))$ be Lipschitz. Let $\hat{x}, \hat{y} \in P(\mathbb{C}^d)$ and $n \in \mathbb{N}^*$. Then*

$$\int_{W_n \in A} |f(W_n \cdot \hat{x}) - f(W_n \cdot \hat{y})| \cdot \|W_n x\|^2 d\mu^{\otimes n} \leq L\delta(\hat{x}, \hat{y}),$$

where $A = \{w \in \mathcal{V}_n : \|wy\| \geq \|wx\|\}$ and L is the Lipschitz constant of f .

Proof. Fix \hat{x}, \hat{y} . Let $y^\perp = y - \langle x, y \rangle x$ and \tilde{y}^\perp its normalization. Then, $\|y^\perp\| = \delta(\hat{x}, \hat{y})$. Using the linearity and $x \wedge x = 0$, for any $w \in A$

$$\|wx \wedge wy\| = \|wx \wedge wy^\perp\| = \delta(\hat{x}, \hat{y}) \|wx \wedge w\tilde{y}^\perp\|.$$

Assume $wx \neq 0$. Then $\|wy\| \geq \|wx\| > 0$, so both $w \cdot \hat{x}$ and $w \cdot \hat{y}$ are well defined. It follows that

$$\delta(w \cdot \hat{x}, w \cdot \hat{y}) = \frac{\|wx \wedge wy\|}{\|wx\| \|wy\|} \leq \frac{\|wx \wedge w\tilde{y}^\perp\|}{\|wx\|^2} \delta(\hat{x}, \hat{y}).$$

Thus, using the Lipschitz property of f ,

$$(26) \quad |f(w \cdot \hat{x}) - f(w \cdot \hat{y})| \|wx\|^2 \leq L \delta(w \cdot \hat{x}, w \cdot \hat{y}) \|wx\|^2 \leq L \delta(\hat{x}, \hat{y}) \|wx \wedge w\tilde{y}^\perp\|.$$

Then, Cauchy-Schwartz inequality and the stochasticity condition imply

$$(27) \quad \begin{aligned} \int_{\Omega} \|W_n x \wedge W_n y^\perp\| d\mu^{\otimes n} &\leq \int_{\Omega} \|W_n x\| \|W_n y^\perp\| d\mu^{\otimes n} \\ &\leq \left(\int_{\Omega} \|W_n x\|^2 d\mu^{\otimes n} \right)^{\frac{1}{2}} \left(\int_{\Omega} \|W_n y^\perp\|^2 d\mu^{\otimes n} \right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega} \langle x, W_n^* W_n x \rangle d\mu^{\otimes n} \right)^{\frac{1}{2}} \left(\int_{\Omega} \langle \tilde{y}^\perp, W_n^* W_n \tilde{y}^\perp \rangle d\mu^{\otimes n} \right)^{\frac{1}{2}} = 1. \end{aligned}$$

It remains to note that both $W_n \cdot \hat{x}$ and $W_n \cdot \hat{y}$ are almost surely well defined with respect to the measure $\|W_n x\|^2 d\mu^{\otimes n}$ on A , so Equations (26) and (27) imply

$$\int_{W_n \in A} |f(W_n \cdot \hat{x}) - f(W_n \cdot \hat{y})| \|W_n x\|^2 d\mu^{\otimes n} \leq L \delta(\hat{x}, \hat{y}),$$

which concludes the proof. \square

Lemma 6.7. *Let $\hat{x}, \hat{y} \in \mathcal{P}(\mathbb{C}^d)$ and $n \in \mathbb{N}^*$. Then*

$$\int_{\Omega} \left| \|W_n x\|^2 - \|W_n y\|^2 \right| d\mu^{\otimes n} \leq 2\delta(\hat{x}, \hat{y}).$$

Proof. Note that $\left| \|wx\|^2 - \|wy\|^2 \right| = |\text{tr}(w^* w (\pi_{\hat{x}} - \pi_{\hat{y}}))| \leq \text{tr}(w^* w |\pi_{\hat{x}} - \pi_{\hat{y}}|)$ for all $w \in \mathcal{V}_n$. Then, using the stochasticity condition,

$$\int_{\Omega} \left| \|W_n x\|^2 - \|W_n y\|^2 \right| d\mu^{\otimes n} \leq \text{tr} \left(\int_{\Omega} W_n^* W_n d\mu^{\otimes n} |\pi_{\hat{x}} - \pi_{\hat{y}}| \right) = \|\pi_{\hat{x}} - \pi_{\hat{y}}\|_{\text{tr}}.$$

Since $\text{rank}(\pi_{\hat{x}} - \pi_{\hat{y}}) \leq 2$, the inequality $\|A\|_{\text{tr}} \leq \text{rank}(A) \|A\|$ concludes the proof. \square

Lemma 6.8. *Let $f \in \mathcal{C}(\mathcal{P}(\mathbb{C}^d))$ be Lipschitz. Let $\hat{x}, \hat{y} \in \mathcal{P}(\mathbb{C}^d)$ and $n \in \mathbb{N}^*$. Then*

$$|\Pi^n f(\hat{x}) - \Pi^n f(\hat{y})| < C \delta(\hat{x}, \hat{y}),$$

where C is a positive constant that depends only on f .

Proof. Let $A = \{w \in \mathcal{V}_n : \|wy\| \geq \|wx\|\}$ and consider the disjoint decomposition $\mathcal{V}_n = A \cup A^c$. Then

$$\begin{aligned} |\Pi^n f(\hat{x}) - \Pi^n f(\hat{y})| &\leq \int_{W_n \in A} |f(W_n \cdot \hat{x}) \|W_n x\|^2 - f(W_n \cdot \hat{y}) \|W_n y\|^2| d\mu^{\otimes n} \\ &\quad + \int_{W_n \in A^c} |f(W_n \cdot \hat{x}) \|W_n x\|^2 - f(W_n \cdot \hat{y}) \|W_n y\|^2| d\mu^{\otimes n}. \end{aligned}$$

The first term on the right-hand side above can be bounded as follows

$$\begin{aligned}
& \int_{W_n \in A} |f(W_n \cdot \hat{x}) \|W_n x\|^2 - f(W_n \cdot \hat{y}) \|W_n y\|^2| \, d\mu^{\otimes n} \\
& \leq \int_{W_n \in A} |f(W_n \cdot \hat{x}) - f(W_n \cdot \hat{y})| \cdot \|W_n x\|^2 \, d\mu^{\otimes n} \\
& \quad + \int_{W_n \in A} |f(W_n \cdot \hat{y})| \cdot |\|W_n x\|^2 - \|W_n y\|^2| \, d\mu^{\otimes n} \\
& \leq \int_{W_n \in A} |f(W_n \cdot \hat{x}) - f(W_n \cdot \hat{y})| \cdot \|W_n x\|^2 \, d\mu^{\otimes n} \\
& \quad + \|f\|_\infty \int_{W_n \in A} |\|W_n x\|^2 - \|W_n y\|^2| \, d\mu^{\otimes n}.
\end{aligned}$$

Bounding the other term in an analogous manner,

$$(28) \quad |\Pi^n f(\hat{x}) - \Pi^n f(\hat{y})| \leq \int_{W_n \in A} |f(W_n \cdot \hat{x}) - f(W_n \cdot \hat{y})| \cdot \|W_n x\|^2 \, d\mu^{\otimes n}$$

$$(29) \quad + \int_{W_n \in A^c} |f(W_n \cdot \hat{y}) - f(W_n \cdot \hat{x})| \cdot \|W_n y\|^2 \, d\mu^{\otimes n}$$

$$(30) \quad + \|f\|_\infty \int_{W_n \in \mathcal{V}_n} |\|W_n x\|^2 - \|W_n y\|^2| \, d\mu^{\otimes n}.$$

Let L denote the Lipschitz constant of f . Then Lemma 6.6 implies each of Equation (28) and Equation (29) is bounded by $L\delta(\hat{x}, \hat{y})$. Then, Lemma 6.7 implies Equation (30) is upper bounded by $2\|f\|_\infty \delta(\hat{x}, \hat{y})$. Therefore,

$$|\Pi^n f(\hat{x}) - \Pi^n f(\hat{y})| \leq 2(L + \|f\|_\infty) \delta(\hat{x}, \hat{y}),$$

which concludes the proof. \square

Theorem 6.9. *Let $f \in \mathcal{C}(\mathcal{P}(\mathbb{C}^d))$. Then $(\Pi^n f)_{n \in \mathbb{N}^*}$ is equicontinuous.*

Proof. Fix $\varepsilon > 0$. Let f_L be a Lipschitz function satisfying $\|f - f_L\|_\infty < \varepsilon/3$. Let $\hat{x}, \hat{y} \in \mathcal{P}(\mathbb{C}^d)$ and $n \in \mathbb{N}$. We have

$$|\Pi^n f(\hat{x}) - \Pi^n f(\hat{y})| \leq |\Pi^n(f - f_L)(\hat{x})| + |\Pi^n(f - f_L)(\hat{y})| + |\Pi^n f_L(\hat{x}) - \Pi^n f_L(\hat{y})|.$$

Since Markov kernels are contractions on the Banach space of bounded continuous functions, $\|\Pi\|_\infty = 1$, so

$$|\Pi^n(f - f_L)(\hat{x})| \leq \|\Pi^n(f - f_L)\|_\infty \leq \|f - f_L\|_\infty < \varepsilon/3,$$

and, analogously, $|\Pi^n(f - f_L)(\hat{y})| < \varepsilon/3$. Lemma 6.8 implies the last term is bounded as

$$|\Pi^n f_L(\hat{x}) - \Pi^n f_L(\hat{y})| \leq C\delta(\hat{x}, \hat{y}),$$

where the constant C depends only on f_L . Therefore, for $\delta = \varepsilon/(3C)$,

$$\delta(\hat{x}, \hat{y}) < \delta \implies |\Pi^n f(\hat{x}) - \Pi^n f(\hat{y})| < \varepsilon$$

for any $n \in \mathbb{N}$ and the theorem holds. \square

6.3. Characterization of ergodic measures. Finally, we are in position to prove the main result of this section, which is the content of Theorem 2. Recall that for $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$ and a family of isometries \mathcal{J} we denote $\nu_{\hat{x}, \mathcal{J}} = (\Psi_{\mathcal{J}})_*(\chi_{\text{inv}} \otimes m_{\hat{x}, \mathcal{J}})$ and $\Upsilon_{\mathcal{J}} = \{\nu_{\hat{x}, \mathcal{J}} : \hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})\}$.

Theorem 6.10. *Let \mathcal{J} be a minimal family. Then $\Upsilon_{\mathcal{J}}$ is the set of Π -ergodic measures.*

Proof. We show that a probability measure ν on $\mathbb{P}(\mathbb{C}^d)$ is Π -ergodic if and only if $\nu = \nu_{\hat{x}, \mathcal{J}}$ for some $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$. The result follows from Raugi's Theorem 6.1, which can be applied thanks to the equicontinuity of Π (Theorem 6.9), and relies on the characterization of Π -minimal subsets of $\mathbb{P}(\mathbb{C}^d)$ (Theorem 6.5).

- (\Rightarrow) Let ν be Π -ergodic. By Theorem 6.1, $\text{supp } \nu$ is a Π -minimal set that carries a unique Π -invariant probability measure. By Theorem 6.5, there exists $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$ such that $\text{supp } \nu = \text{supp } \nu_{\hat{x}, \mathcal{J}}$ and Proposition 5.3 implies $\nu_{\hat{x}, \mathcal{J}}$ is Π -invariant. Therefore, $\nu = \nu_{\hat{x}, \mathcal{J}}$, as desired.
- (\Leftarrow) Let $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$. By Theorem 6.5, $\text{supp } \nu_{\hat{x}, \mathcal{J}}$ is a Π -minimal subset of $\mathbb{P}(\mathbb{C}^d)$. Again by Theorem 6.1, this set carries a unique Π -invariant probability measure, and this measure is $\nu_{\hat{x}, \mathcal{J}}$, see Proposition 5.3. As a consequence, $\nu_{\hat{x}, \mathcal{J}}$ is uniquely invariant on its support, hence it is ergodic on $\mathbb{P}(\mathbb{C}^d)$, as desired. \square

From the characterization of Π -ergodic measures we can immediately deduce the following corollary.

Corollary 6.11. *Let \mathcal{J} be a minimal family. A probability measure ν on $\mathbb{P}(\mathbb{C}^d)$ is Π -invariant if and only if*

$$\nu = \int_{\mathbb{P}(\mathbb{C}^{r_m})} \nu_{\hat{x}, \mathcal{J}} \, d\lambda(\hat{x}) = (\Psi_{\mathcal{J}})_* \left(\chi_{\text{inv}} \otimes \int_{\mathbb{P}(\mathbb{C}^{r_m})} m_{\hat{x}, \mathcal{J}} \, d\lambda(\hat{x}) \right)$$

for some probability measure λ on $\mathbb{P}(\mathbb{C}^{r_m})$.

In what follows, we denote by G_{\min} the subgroup of $SU(d)$ induced by some minimal family. Let us recall that this group is unique up to unitary equivalence. Corollary 6.11 implies Π admits a unique invariant probability measure if and only if

$$[\hat{x}]_{G_{\min}} = \mathbb{P}(\mathbb{C}^{r_m})$$

for some $\hat{x} \in \mathbb{P}(\mathbb{C}^{r_m})$, which is equivalent to the transitivity of the action of G_{\min} on $\mathbb{P}(\mathbb{C}^{r_m})$.

By definition, $SU(r_m)$ acts transitively on $\mathbb{P}(\mathbb{C}^{r_m})$. There may exist proper subgroups of $SU(r_m)$ that act transitively on $\mathbb{P}(\mathbb{C}^{r_m})$. It was shown by Onishchik [Oni63, Theorem 6] (see also [Bes08; Sha01]) that, up to unitary equivalence, the only proper subgroup of $SU(r_m)$ that acts transitively on $\mathbb{P}(\mathbb{C}^{r_m})$ is the symplectic group $\text{Sp}(r_m/2)$, known also as the hyperunitary group, provided that r_m is even. Let us recall that $\text{Sp}(r_m/2)$ consists of $r_m \times r_m$ special unitary complex matrices U satisfying $U^T J U = J$, where $J = \begin{bmatrix} 0 & \text{Id}_{\mathbb{C}^{r_m/2}} \\ -\text{Id}_{\mathbb{C}^{r_m/2}} & 0 \end{bmatrix}$. From that follows next corollary, which is Theorem 3.

Corollary 6.12. *The measure ν_{unif} is the unique Π -invariant probability measure on $\mathbb{P}(\mathbb{C}^d)$ if and only if*

- $G_{\min} = SU(r_m)$ if r_m is odd,
- $G_{\min} = SU(r_m)$ or $G_{\min} = Q\text{Sp}(r_m/2)Q^*$ for some $Q \in \mathcal{U}(r_m)$ if r_m is even.

Remark 6.13. *In principle, it is possible to obtain $\Upsilon_{\mathcal{J}}$ equal to the set of all Π -ergodic measures even if the family $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ used for the construction is not minimal. However, if the trajectories on $\mathbb{P}(\mathbb{C}^m)$ allow us to distinguish between $G_{\mathcal{J}}$ and $G_{\mathcal{J}_{\min}}$, where \mathcal{J}_{\min} is a minimal family such that $G_{\mathcal{J}_{\min}} \subset G_{\mathcal{J}}$, then $\Upsilon_{\mathcal{J}}$ contains a non-ergodic measure. More precisely, if $\hat{x}_0 \in \mathbb{P}(\mathbb{C}^m)$ is such that $[\hat{x}_0]_{G_{\mathcal{J}_{\min}}} \subsetneq [\hat{x}_0]_{G_{\mathcal{J}}}$, then $\nu_{\hat{x}_0, \mathcal{J}}$ is not Π -ergodic. To see this, consider*

$$\nu = \int_{G_{\mathcal{J}}} \nu_{u \cdot \hat{x}_0, \mathcal{J}_{\min}} dH_{G_{\mathcal{J}}}(u).$$

By direct calculation we can check that $\nu = \nu_{\hat{x}_0, \mathcal{J}}$. Indeed, using the invariance of Haar measure, for any $f \in \mathcal{C}(\mathbb{P}(\mathbb{C}^d))$ we obtain

$$\begin{aligned} \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) d\nu(\hat{y}) &= \int_{\mathcal{D}_m} \int_{G_{\mathcal{J}_{\min}}} \int_{G_{\mathcal{J}}} f(J_D \tilde{u} u \cdot \hat{x}_0) dH_{G_{\mathcal{J}}}(u) dH_{G_{\mathcal{J}_{\min}}}(\tilde{u}) d\chi_{\text{inv}}(D) \\ &= \int_{\mathcal{D}_m} \int_{G_{\mathcal{J}}} f(J_D u \cdot \hat{x}_0) dH_{G_{\mathcal{J}}}(u) d\chi_{\text{inv}}(D) \\ &= \int_{\mathbb{P}(\mathbb{C}^d)} f(\hat{y}) d\nu_{\hat{x}_0, \mathcal{J}}(\hat{y}). \end{aligned}$$

Therefore, $\nu_{\hat{x}_0, \mathcal{J}}$ can be non-trivially decomposed into a convex combination of Π -ergodic measures of the form $\nu_{u \cdot \hat{x}_0, \mathcal{J}_{\min}}$, which means that $\nu_{\hat{x}_0, \mathcal{J}}$ is not Π -ergodic, as claimed.

7. EXAMPLES

Remark 7.1. *The projective space $\mathbb{P}(\mathbb{C}^2)$ can be represented on the unit sphere of \mathbb{R}^3 using the mapping $\hat{x} \mapsto (\text{tr}(\pi_{\hat{x}} \sigma_x), \text{tr}(\pi_{\hat{x}} \sigma_y), \text{tr}(\pi_{\hat{x}} \sigma_z))$ with the three Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this context, the unit sphere in \mathbb{R}^3 is called the Bloch sphere. We will use it to illustrate our examples.

Example 1. (Two disjoint Bloch spheres.) Let us fix the unitaries $u_1, u_2, u_3, u_4 \in \mathcal{SU}(2)$ and consider the two matrices:

$$v_1 = \begin{bmatrix} 0 & \sqrt{\frac{1}{3}}u_1 \\ \sqrt{\frac{1}{4}}u_2 & 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}}u_3 \\ \sqrt{\frac{3}{4}}u_4 & 0 \end{bmatrix}.$$

(1) Irreducibility. We show that $\mu = \delta_{v_1} + \delta_{v_2}$ is irreducible if $u_1 u_2$ and $u_3 u_4$ do not share an eigenbasis. To this end, observe that $\text{Id}/4$ is a fixed point of the quantum channel $\phi(\rho) = v_1 \rho v_1^* + v_2 \rho v_2^*$, so from [Wol12, Theorem 6.13] we know that μ is irreducible iff the only linear operators on \mathbb{C}^4 that commute with both Kraus operators v_1 and v_2 and also with their adjoints are the multiples of identity, i.e.,

$$\{M \in \mathcal{M}_4 : [M, v_1] = [M, v_1^*] = [M, v_2] = [M, v_2^*] = 0\} = \{\alpha \text{Id}_4 : \alpha \in \mathbb{C}\}.$$

To see that this condition is satisfied if $u_1 u_2$ and $u_3 u_4$ do not share an eigenbasis, consider $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and assume that M commutes with v_1, v_2, v_1^*, v_2^* . Then

$$\sqrt{\frac{1}{4}}u_2 B = \sqrt{\frac{1}{3}}C u_1 \quad \text{and} \quad \sqrt{\frac{1}{3}}B u_1^* = \sqrt{\frac{1}{4}}u_2^* C,$$

which implies that $u_2 B = \sqrt{\frac{4}{3}} C u_1 = \sqrt{\frac{3}{4}} C u_1$, thus also $B = C = 0$. Moreover,

$$(31) \quad u_1 D = A u_1 \quad \text{and} \quad u_2 A = D u_2$$

which gives $u_1 u_2 A = u_1 D u_2 = A u_1 u_2$, that is $[A, u_1 u_2] = 0$. Analogously, $[A, u_3 u_4] = 0$.

Let us now assume that $u_1 u_2$ and $u_3 u_4$ do not have a common eigenbasis. Then both $u_1 u_2$ and $u_3 u_4$ have distinct eigenvalues (i.e., neither $u_1 u_2$ nor $u_3 u_4$ is a multiple of identity). From the commutativity relations $[A, u_1 u_2] = [A, u_3 u_4] = 0$, it follows that A shares an eigenbasis with both $u_1 u_2$ and $u_3 u_4$, which implies that $A = \alpha \text{Id}_2$ for some $\alpha \in \mathbb{C}$. From Equation (31) it then follows that $D = \alpha \text{Id}_2$, thus also $M = \alpha \text{Id}_4$, as desired.

For the remainder of this example we assume that u_1, u_2, u_3, u_4 are so chosen that the irreducibility of μ holds.

(2) Dark spaces. One can immediately identify the following two dark subspaces:

$$D_a = \{[x, y, 0, 0]^T : x, y \in \mathbb{C}\} \quad \text{and} \quad D_b = \{[0, 0, x, y]^T : x, y \in \mathbb{C}\}.$$

We see that D_a, D_b are orthogonal, so in particular they intersect trivially. The projective dark subspaces $\mathbb{P}(D_a), \mathbb{P}(D_b)$ are isomorphic to $\mathbb{P}(\mathbb{C}^2)$.

To see that D_a, D_b are maximal dark subspaces, assume $D \subset \mathbb{C}^4$ is a dark subspace of dimension 3. Then for any $A \in \text{span}\{v_1^* v_1, v_2^* v_2\} = \text{span}\{\pi_{D_a}, \pi_{D_b}\}$, $\pi_D A \pi_D = \frac{1}{3} \text{tr}(\pi_D A) \pi_D$. Taking $A = \pi_{D_a}$ or $A = \pi_{D_b}$, it follows from $\text{rank}(\pi_D \pi_{D_{a/b}} \pi_D) \leq 2$ and $\text{rank} \pi_D = 3$ that $\text{tr}(\pi_D \pi_{D_{a/b}}) = 0$ hence $\text{tr}(\pi_D) = 0$, which contradicts $\pi_D \neq 0$, so D is not dark.

(3) Invariant measure for dark spaces. The Markov chain on $\{D_a, D_b\}$ is deterministic: its transition matrix reads $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note that the period is 2. Hence, the uniform measure

$$\frac{1}{2} \delta_{D_a} + \frac{1}{2} \delta_{D_b}$$

is K -invariant. We have thus identified an invariant measure on \mathcal{D}_m and Theorem 1 implies it is unique. We can therefore restrict our attention to its support, i.e., to the subspaces D_a, D_b . However, in principle, this system may admit additional maximal dark subspaces.

(4) Minimal family. The family of isomorphisms \mathcal{J}_{emb} consisting of natural embeddings

$$\mathcal{J}_a^{\text{emb}}: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{J}_b^{\text{emb}}: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ x \\ y \end{bmatrix}$$

leads to the group generated by the four unitaries: $G_{\mathcal{J}_{\text{emb}}} = \overline{\langle u_1, u_2, u_3, u_4 \rangle}$. This group need not be minimal. To obtain a minimal group, let us construct a D_a -smart family $\mathcal{J} = \{J_a, J_b\}$ such that $J_a = \mathcal{J}_a^{\text{emb}}$. Taking $J_b = v_1 J_a = J_a u_2$, we obtain

$$u_{1, D_a} = u_2^{-1} u_2 = \text{Id}, \quad u_{2, D_a} = u_2^{-1} u_4, \quad u_{1, D_b} = u_1 u_2, \quad u_{2, D_b} = u_3 u_2,$$

and so $G_{\mathcal{J}} = G_{\min} = \overline{\langle u_2^{-1} u_4, u_1 u_2, u_3 u_2 \rangle}$.

(5) Let us take a closer look at a more concrete case. For $\theta \in \mathbb{R}$ let $R_x(\theta) = \exp(i\theta/2\sigma_x)$ and $R_z(\theta) = \exp(i\theta/2\sigma_z)$. On the Bloch sphere, these correspond to rotations of angle θ about the axes x and z respectively. Set

$$u_1 = \text{Id}, \quad u_2 = R_x(\theta_x), \quad u_3 = R_z(\theta_z), \quad u_4 = \text{Id}$$

with arbitrary rotation angles $\theta_x, \theta_z \in (0, 2\pi)$. The family \mathcal{J}_{emb} of canonical embeddings is now minimal (because it is smart) and

$$G_{\mathcal{J}_{\text{emb}}} = G_{\text{min}} = \overline{\langle R_x(\theta_x), R_z(\theta_z) \rangle}.$$

Let us recall that we do not adjust the phases explicitly but all unitary operators on the reference space are in fact special unitary matrices. There are three qualitatively different cases:

- (a) If $\theta_x, \theta_z \notin \pi\mathbb{Q}$, then, by a compactness argument, the minimal family \mathcal{J}_{emb} (thus also every other family) induces the full group: $G_{\text{min}} = \mathcal{SU}(2)$, and so $\nu_{\text{inv}} = \frac{1}{2}\text{Unif}_{\mathbb{P}(D_a)} + \frac{1}{2}\text{Unif}_{\mathbb{P}(D_b)}$ is the unique Π -invariant measure on $\mathbb{P}(\mathbb{C}^d)$.
- (b) If $\theta_x = \pi$ and $\theta_z \notin \pi\mathbb{Q}$, then $G_{\text{min}} = \overline{\langle i\sigma_x, R_z(\theta_z) \rangle} = \{R_z(t), i\sigma_x R_z(t) \mid t \in \mathbb{R}\}$. Hence, there are infinitely many mutually singular Π -ergodic measures. Orbits in $\mathbb{P}(\mathbb{C}^2)$ are in bijection with $[0, 1/2]$ where for a given $c \in [0, 1/2]$ the orbit is given by $\{\hat{x}_t\}_{t \in \mathbb{R}} \cup \{\sigma_x \cdot \hat{x}_t\}_{t \in \mathbb{R}}$ with $x_t = [ce^{it}, \sqrt{1-c^2}]^T$.
On the other hand, a non-minimal family $\tilde{\mathcal{J}} = \{J_a^{\text{emb}}, J_b^{\text{emb}} R_x(\psi_x)\}$ with $\psi_x \notin \pi\mathbb{Q}$ induces the full group: $G_{\tilde{\mathcal{J}}} = \mathcal{SU}(2)$; thus, the only Π -invariant measure that $\tilde{\mathcal{J}}$ can induce is ν_{unif} (which is non-ergodic).
- (c) If $\theta_x, \theta_z \in \pi\mathbb{Q}$, then setting $p \in \mathbb{N}^*$ such that $p\theta_x, p\theta_z \in \pi\mathbb{Z}$ and using $\mathcal{SU}(2)$ commutation relations, any product of $R_x(\theta_x)$ and $R_z(\theta_z)$ to the power p^2 is proportional to the identity. Thus G_{min} is finite. There are infinitely many mutually singular Π -ergodic measures and each of them is finitely supported. Note that both \mathcal{W}_{fin} and $\text{supp } \chi_{\text{inv}}$ are finite; hence, all the sets considered in Proposition 5.12 coincide: $S_{\mathcal{J}_{\text{emb}}}^{\text{fin}} = R_{\mathcal{J}_{\text{emb}}} = G_{\text{min}}$.

In particular, if we set $\theta_x = \theta_z = \pi$, i.e.,

$$u_1 = \text{Id}, \quad u_2 = i\sigma_x, \quad u_3 = i\sigma_z, \quad u_4 = \text{Id},$$

then $G_{\text{min}} = \{\pm \text{Id}, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$ and a generic Π -ergodic measure is supported on 8 points, which can degenerate to 4 points if an eigenvector of σ_x or σ_z is in the support. See also Figure 7.1.

(6) As in (5c), we set $u_1 = \text{Id}$, $u_2 = \sigma_x$, $u_3 = \sigma_z$, $u_4 = \text{Id}$, but consider now the following (non-minimal) families:

$$\tilde{\mathcal{J}}_1 = \{J_a^{\text{emb}}, J_b^{\text{emb}} R_z(\frac{\pi}{3})\}, \quad \tilde{\mathcal{J}}_2 = \{J_a^{\text{emb}}, J_b^{\text{emb}} R_z(\frac{\pi}{2})\}, \quad \tilde{\mathcal{J}}_3 = \{J_a^{\text{emb}}, J_b^{\text{emb}} R_z(\psi)\}$$

with $\psi \notin \pi\mathbb{Q}$. They induce the following groups:

$$\begin{aligned} G_{\tilde{\mathcal{J}}_1} &= \langle i\sigma_x, R_z(\frac{\pi}{3}) \rangle = \{R_z(\frac{j\pi}{3}), i\sigma_x R_z(\frac{j\pi}{3}) : j = 0, \dots, 5\} \\ G_{\tilde{\mathcal{J}}_2} &= \langle i\sigma_x, R_z(\frac{\pi}{2}) \rangle = \{R_z(\frac{j\pi}{2}), i\sigma_x R_z(\frac{j\pi}{2}) : j = 0, \dots, 3\} \\ G_{\tilde{\mathcal{J}}_3} &= \overline{\langle i\sigma_x, R_z(\psi) \rangle} = \{R_z(t), i\sigma_x R_z(t) : t \in \mathbb{R}\}, \end{aligned}$$

see also Figure 7.2 for a generic trajectory under each of these groups. Then:

- (a) Considering $G_{\tilde{\mathcal{J}}_1}$ and $G_{\tilde{\mathcal{J}}_2}$, we see that neither of them is a subgroup of the other. The minimal group $G_{\text{min}} = \langle i\sigma_x, i\sigma_z \rangle$ is a subgroup of both.
- (b) The group $G_{\tilde{\mathcal{J}}_3}$ is uncountably infinite, which contrasts with G_{min} being finite.

This illustrates how the choice of a family of isometries can influence what groups on \mathbb{C}^{r_m} , thus also what Π -invariant measures on $\mathbb{P}(\mathbb{C}^d)$, can be generated by the procedure described in Section 5.

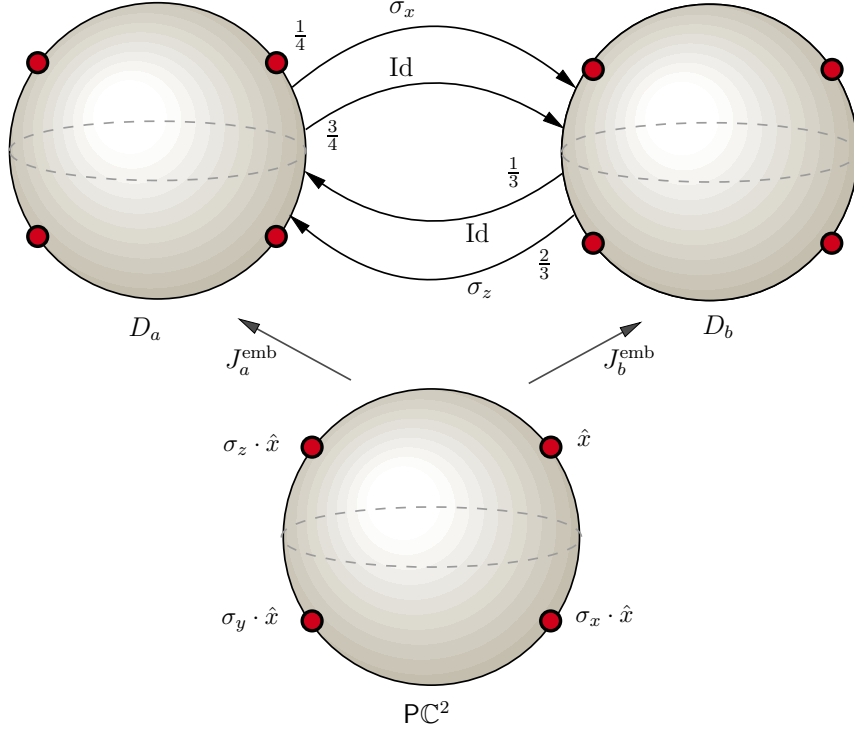


FIGURE 7.1. A generic trajectory under the action of a minimal group on the reference space $P(\mathbb{C}^2)$ consists of four points. A generic Π -ergodic measure on $P(\mathbb{C}^4)$ is uniform on eight atoms.

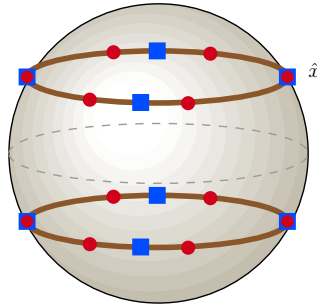


FIGURE 7.2. Trajectories of a generic point $\hat{x} \in P(\mathbb{C}^2)$ under the action of three non-minimal groups from Example 1(6). The red bullets are the 12-element set $[\hat{x}]_{G_{\mathcal{J}_1}}$, the blue squares are the 8-element set $[\hat{x}]_{G_{\mathcal{J}_2}}$, and the brown circles correspond to the uncountable set $[\hat{x}]_{G_{\mathcal{J}_3}}$. All three orbits contain $[\hat{x}]_{G_{\min}}$ as a proper subset, cf. Figure 7.1.

Example 2. (Adjacent Bloch spheres.) Consider two matrices:

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \theta & \sin \theta & \cos \theta \\ \sin \theta & -\cos \theta & \sin \theta \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ \cos \phi & -\sin \phi & -\cos \phi \\ \sin \phi & \cos \phi & -\sin \phi \end{bmatrix}$$

with $\theta, \phi \in \mathbb{R}$

(0) Intuition. To develop some intuition about the dynamics of this system, let us first discuss the action of v_1 and v_2 on some real vectors. Denote

$$S_\alpha = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix} \quad \text{and} \quad Q_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

When acting on a plane, S_α describes the reflection of this plane about a line through the origin which makes an angle $\alpha/2$ with the horizontal axis, while Q_α is the (counterclockwise) rotation about the origin through the angle α . Consider now the two planes

$$P_a = \{[x, y, 0]^T : x, y \in \mathbb{R}\} \quad \text{and} \quad P_b = \{[0, x, y]^T : x, y \in \mathbb{R}\}.$$

The action of v_1 and v_2 on P_a and P_b can be described as follows. A vector in P_a can either stay in P_a while being reflected using S_θ (if v_1 acted), or it can jump to P_b while being rotated by Q_ϕ (if v_2 acted). Similarly, a vector in P_b can either stay in P_b while being rotated by $Q_{\phi+\frac{\pi}{2}}$ (if v_2 acted) or jump to P_a while being rotated by $Q_{\theta-\frac{\pi}{2}}$ (if v_1 acted). The jumps between planes happen with constant probability $\frac{1}{2}$.

Reverting back to the complex field, let P_a and P_b denote now their complex extension, namely

$$P_a = \{[x, y, 0]^T : x, y \in \mathbb{C}\} \quad \text{and} \quad P_b = \{[0, x, y]^T : x, y \in \mathbb{C}\}.$$

(1) Irreducibility. To assure $\mu = \delta_{v_1} + \delta_{v_2}$ is irreducible and avoid trivial statements, we assume that neither θ nor ϕ are elements of $\mathbb{Z}\pi/2$. Indeed, since $v_1\mathbb{C}^3 = P_a$ and $v_2\mathbb{C}^3 = P_b$, μ is not irreducible only if $P_a \cap P_b$ is invariant for v_1 and v_2 , namely only if $e_1 = [0, 1, 0]^T$ is a common eigenvector of v_1 and v_2 . That is not the case either if $\theta \notin \mathbb{Z}\pi$ or $\phi \notin \pi/2 + \mathbb{Z}\pi$.

(2) Dark subspaces. Both P_a and P_b are dark subspaces. Therefore, we denote them D_a and D_b respectively. They share a one-dimensional subspace equal to $\mathbb{C}e_1$. Hence, the two projective dark subspaces $\mathbb{P}(D_a)$ and $\mathbb{P}(D_b)$ intersect at one ray and are both isomorphic to $\mathbb{P}(\mathbb{C}^2)$, so they can be visualized as two Bloch spheres tangent to each other.

To see D_a and D_b are maximal dark subspaces, it suffices to observe that the whole space \mathbb{C}^3 is not dark. To this end, we consider $(i_1, \dots, i_n) \in \{1, 2\}^n$, $n \in \mathbb{N}^*$, and observe that

$$(32) \quad v_{i_1}^* \cdots v_{i_n}^* v_{i_n} \cdots v_{i_1} = 2^{-n} \begin{bmatrix} 1 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 1 \end{bmatrix}.$$

Since $v_{i_1}^* \cdots v_{i_n}^* v_{i_n} \cdots v_{i_1}$ is not proportional to the identity matrix, \mathbb{C}^3 is not dark. Remark that Equation (32) implies $M_n = \frac{1}{3} \begin{bmatrix} 1 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 1 \end{bmatrix}$ for any $n \in \mathbb{N}^*$.

(3) Invariant measure for dark spaces. The Markov chain on $\{D_a, D_b\}$ has transition matrix $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; hence, its unique invariant measure is uniform:

$$\chi_{\text{inv}} = \frac{1}{2} \delta_{D_a} + \frac{1}{2} \delta_{D_b}.$$

As in [Example 1](#), [Theorem 1](#) implies it is the unique K -invariant measure on \mathcal{D}_m , and so we restrict our attention to the dark subspaces D_a, D_b .

(4) Minimal family. For the family of canonical embeddings $\mathcal{J}_{\text{emb}} = \{J_a^{\text{emb}}, J_b^{\text{emb}}\}$ with

$$J_a^{\text{emb}} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{and} \quad J_b^{\text{emb}} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}$$

the group $G_{\mathcal{J}_{\text{emb}}}$ is the closure of $\langle iS_\theta, Q_{\theta-\frac{\pi}{2}}, Q_\phi, Q_{\phi+\frac{\pi}{2}} \rangle$. As in the previous example, \mathcal{J}_{emb} need not be minimal. Let us construct a D_a -smart family $\mathcal{J} = \{J_a, J_b\}$ with $J_a = J_a^{\text{emb}}$. Setting $J_b = v_2 J_a = v_1 Q_\phi$, we obtain

$$u_{D_a,1} = iS_\theta, \quad u_{D_b,1} = Q_{\theta-\frac{\pi}{2}+\phi}, \quad u_{D_a,2} = \text{Id}, \quad u_{D_b,2} = Q_{\phi+\frac{\pi}{2}}.$$

and G_{\min} is the closure of $\langle iS_\theta, Q_\theta, Q_{\phi+\frac{\pi}{2}} \rangle$. If we revert to elements of $SU(2)$, since $Q(\alpha) = R_y(-2\alpha) = \exp(i\alpha\sigma_y)$ and $S_\alpha = i\sigma_z R_y(2\alpha)$ for any $\alpha \in \mathbb{R}$,

$$G_{\min} = \overline{\langle i\sigma_z, R_y(2\theta), R_y(2\phi + \pi) \rangle}.$$

We can then distinguish two cases:

(a) If either ϕ or θ is not in $\pi\mathbb{Q}$, then

$$G_{\min} = \{R_y(t), i\sigma_z R_y(t) : t \in \mathbb{R}\}.$$

Thus, denoting $G_{1,b}$ the minimal group of Example 1(5b),

$$G_{\min} = R_x(\pi/2)R_y(\pi/2)G_{1,b}R_y(-\pi/2)R_x(-\pi/2).$$

It follows the orbits of G_{\min} are the image by $R_x(\pi/2)R_y(\pi/2)$ of the orbits of Example 1(5b). For $c \in [0, 1/2]$, denoting $C_c = \{R_x(\pi/2)R_y(\pi/2)[ce^{it}, \sqrt{1-c^2}]^T : t \in \mathbb{R}\}$, and $A_c = C_c \cup \sigma_x C_c$, $\Upsilon_{\mathcal{J}} = \{\nu_c : c \in [0, 1/2]\}$, where

$$\nu_c = \frac{1}{2} \text{Unif}_{P(J_a A_c)} + \frac{1}{2} \text{Unif}_{P(J_b A_c)}.$$

Then, the only finitely-supported Π -ergodic measure is ν_0 , which is a uniform measure on the four atoms $\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$, corresponding to the y oriented poles of the two Bloch spheres. See Figure 7.3 for an illustration.

(b) If $\phi, \theta \in \pi\mathbb{Q}$, then G_{\min} is a finite group and every Π -ergodic measure is finitely supported. More precisely, if $\theta = \pi p_1/q_1$ and $\phi + \pi/2 = \pi p_2/q_2$, where $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$ with \gcd denoting the greatest common divisor, then

$$G_{\min} = \langle i\sigma_z, R_y(2\pi/q_1), R_y(2\pi/q_2) \rangle = \langle i\sigma_z, R_y(2\pi \gcd(q_1, q_2)/(q_1 q_2)) \rangle.$$

Hence, a generic trajectory under G_{\min} consists of $2q_1 q_2$ points, and a generic Π -ergodic measure is uniform over $4q_1 q_2$ atoms, see also Figure 7.4.

However, non-uniform Π -ergodic measure exist as well: $\nu_{\hat{x}}$ is non-uniform iff the tangency point of the two Bloch spheres belongs to $\text{supp } \nu_{\hat{x}}$. This happens, for instance, if $\hat{x} = [0, 1]^T$, i.e., the starting point on the reference Bloch sphere corresponds to the tangency point. Then $\nu_{\hat{x}}$ has $4q_1 q_2 - 1$ atoms and the atom at the tangency point, which is $\delta_{[0,1,0]^T}$, has double weight compared to all the other atoms.

Example 3. Here we recall a classical example of a measure μ with uncountably many dark subspaces [MK06, Example 2] and describe its ergodic measures by using our framework of smart families.

Let $\ell, p, r \in \mathbb{N}^*$. We fix a family of complex $p \times p$ matrices b_1, \dots, b_ℓ such that they admit only dark subspaces of dimension 1 and satisfy the stochasticity condition $\sum_{i=1}^{\ell} b_i^* b_i = \text{Id}$. We extend the system by fixing a family u_1, \dots, u_ℓ of special unitary operators on \mathbb{C}^r . We shall consider the joint space $\mathbb{C}^p \otimes \mathbb{C}^r$ with the operators

$$v_i = b_i \otimes u_i, \quad i \in \{1, \dots, \ell\}.$$

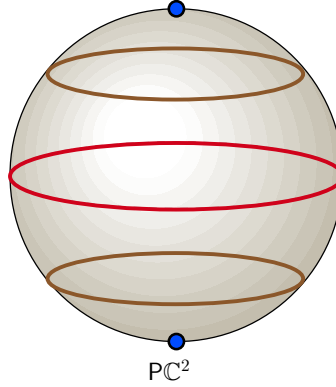


FIGURE 7.3. Invariant subsets of $P(\mathbb{C}^2)$. The Bloch sphere is so oriented that the y -axis is pointing upwards. A generic invariant subset consists of two circles of latitude symmetric with respect to the equator of the Bloch sphere (in brown). Degenerate invariant subsets are the equator (in red) and the poles (in blue). Recall that on the Bloch sphere $Q_\alpha = R_y(2\alpha)$ is the rotation around the y -axis by the angle 2α , while S_α is the rotation by the angle π around an axis determined by some real vector (so contained in the equator) whose exact coefficients depend on α . In particular, S_α maps the equator to itself, while the upper brown circle is mapped to the lower brown circle and vice versa.

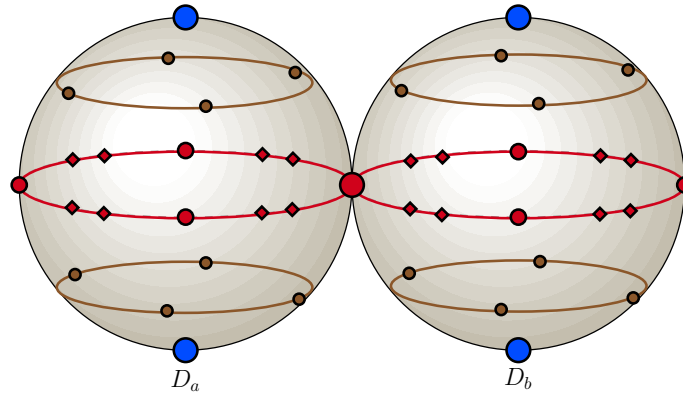


FIGURE 7.4. The Bloch sphere is so oriented that the y -axis is pointing upwards. Three Π -minimal sets in $P(\mathbb{C}^4)$: the set of four brown circles (generic), of two red equators, and of four poles (cf. Figure 7.3). If G is infinite, the uniform measures on these sets are the Π -ergodic measures. If G is finite, all Π -ergodic measures are finitely-supported (here we assume G has 8 elements, which corresponds to setting, e.g., $\theta = \pi/2$ and $\phi = \pi/4$). Brown bullets: a generic Π -ergodic measure (uniform, each atom with weight $1/16$). Blue bullets: the ergodic measure supported on the poles (uniform, each atom with weight $1/4$). Red bullets: a non-uniform Π -ergodic measure (the atom at the touching point has weight $1/4$, the other atoms have weight $1/8$). Red diamonds: a Π -ergodic measure (uniform, each atom with weight $1/16$).

Then $\sum_{i=1}^{\ell} v_i^* v_i = \text{Id}$. We assume that $\mu = \sum_{i=1}^{\ell} \delta_{v_i}$ is irreducible. Note that the irreducibility of $\sum_{i=1}^{\ell} \delta_{b_i}$ follows.

The maximal dark subspaces of the joint system are r -dimensional and contain the elements of the set

$$\mathcal{D}_{\infty} = \{y \otimes \mathbb{C}^r \mid y \in \mathbb{C}^p, y \neq 0\}.$$

Actually this set is \mathcal{W} -invariant and one on which the process of dark subspaces concentrates. To see that observe that the process $(M_n)_n$ can be written $M_n = M_n^1 \otimes M_n^2$ with M_n^1 and M_n^2 operators on respectively \mathbb{C}^p and \mathbb{C}^r . The assumption of triviality of the dark subspaces for the $\{b_1, \dots, b_{\ell}\}$ and Proposition 3.5 guaranty that M_n^1 converges to some rank-one projector π_z for a random $z \in \mathbb{C}^p$ as n tends to infinity, while on \mathbb{C}^r the operators are unitary, so $M_n^2 = \text{Id}_r$ for every n . As a consequence, $M_{\infty} = \pi_z \otimes \text{Id}_r$. Then, since for any product w of matrices from $\{v_1, \dots, v_{\ell}\}$, $w = b \otimes u$ for some $p \times p$ matrix b and a unitary $r \times r$ matrix u , U_n in the polar decomposition is of the form $\tilde{u}_n \otimes (u_{i_n} \cdots u_{i_1})$ with \tilde{u}_n a $p \times p$ unitary matrix. Hence, every maximal dark subspace that is charged asymptotically coincides with the range of some $\pi_{U \cdot \hat{z}_0} \otimes \text{Id}_r$ with $U \in \mathcal{SU}(p)$ thanks to Proposition 3.4.

We now restrict our discussion to dark subspaces from \mathcal{D}_{∞} .

By Theorem 1, the kernel K on \mathcal{D}_m , defined by

$$Kf(D) = \sum_{i=1}^{\ell} f(v_i D) \text{tr}(v_i \frac{\pi_D}{r} v_i^*),$$

where $f \in \mathcal{C}(\mathcal{D}_m)$, admits a unique invariant probability measure χ_{inv} . Also, by [Ben+19, Theorem 1.1], the transition kernel Π_b on $\mathcal{P}(\mathbb{C}^p)$, defined by

$$\Pi_b f(\hat{y}) = \sum_{i=1}^{\ell} f(b_i \cdot \hat{y}) \|b_i y\|^2,$$

where $f \in \mathcal{C}(\mathcal{P}(\mathbb{C}^p))$, admits a unique invariant measure, which we denote by γ_{inv} . We shall now establish the connection between χ_{inv} and γ_{inv} .

The dark subspaces of \mathcal{D}_{∞} can be associated with rays in \mathbb{C}^p via

$$\xi: y \otimes \mathbb{C}^r \mapsto \hat{y} \in \mathcal{P}(\mathbb{C}^p).$$

For $\hat{y} \in \mathcal{P}(\mathbb{C}^p)$ we denote $D_{\hat{y}} = y \otimes \mathbb{C}^r$. Observe that $v_i D_{\hat{y}} = D_{b_i \cdot \hat{y}}$ if $b_i y \neq 0$ and also

$$\text{tr}(v_i \pi_{D_{\hat{y}}} v_i^*) = \text{tr}(b_i \pi_{\hat{y}} b_i^* \otimes u_i \text{Id}_{\mathbb{C}^r} u_i^*) = r \|b_i y\|^2.$$

Thus, by direct calculation,

$$\Pi f(\hat{y}) = K[f \circ \xi](D_{\hat{y}})$$

for any $f \in \mathcal{C}(\mathcal{P}(\mathbb{C}^p))$. From the K -invariance of χ_{inv} we deduce the Π_b -invariance of $\xi_* \chi_{\text{inv}}$ and, since γ_{inv} is the unique Π_b -invariant probability measure, we conclude that

$$(33) \quad \gamma_{\text{inv}} = \xi_* \chi_{\text{inv}}.$$

Let us apply the minimal family approach to describe the set of all Π -ergodic measures on $\mathcal{P}(\mathbb{C}^p \otimes \mathbb{C}^r)$, where we recall that Π acts on $f \in \mathcal{C}(\mathcal{P}(\mathbb{C}^p \otimes \mathbb{C}^r))$ as

$$\Pi f(\hat{x}) = \sum_{i=1}^{\ell} f(v_i \cdot \hat{x}) \|v_i x\|^2.$$

The convergence of quantum trajectories towards dark subspaces allows us to restrict our attention to product states of the form $\hat{x} = \hat{y} \otimes \hat{z}$ with $y \in \mathbb{C}^p$ and $z \in \mathbb{C}^r$, where we denote $\hat{y} \otimes \hat{z}$ for $\widehat{y \otimes z}$. On this set of states,

$$\Pi f(\hat{y} \otimes \hat{z}) = \sum_{i=1}^{\ell} f(b_i \cdot \hat{y} \otimes u_i \cdot \hat{z}) \|b_i y\|^2.$$

Denote by G the smallest closed subgroup of $SU(r)$ containing $\{u_1, \dots, u_\ell\}$, and by H_G the normalized Haar measure on G . Consider the image through $\Psi = (\hat{y}, \hat{z}) \mapsto \hat{y} \otimes \hat{z}$ of the measures

$$\gamma_{\text{inv}} \otimes (g_{\hat{z}})_* H_G,$$

where $\hat{z} \in P(\mathbb{C}^r)$ and $g_{\hat{z}}: SU(r) \ni u \mapsto u \cdot \hat{z} \in P(\mathbb{C}^r)$. With a small abuse of notation we write a measure on $P(\mathbb{C}^p) \times P(\mathbb{C}^r)$ and its pushforward by Ψ the same way. Their Π -invariance can be verified by direct calculation. In what follows, we show that, quite counter-intuitively, they may or may not form the set of Π -ergodic measures.

First, remark that the family of isomorphisms $\tilde{\mathcal{J}}$ defined by $\tilde{J}_{\hat{y}}: x \mapsto y \otimes x$ with y denoting a normalized representative of \hat{y} satisfies $G_{\tilde{\mathcal{J}}} = G$. We show that this family may or may not be minimal through an example. Let $p = r = 2$, $q \in (0, \frac{1}{2})$ and

$$b_1 = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & \sqrt{q} \\ \sqrt{1-q} & 0 \end{pmatrix}, \quad u_1 = i\sigma_x \quad \text{and} \quad u_2 = i\sigma_z.$$

Then, $G = \{\pm \text{Id}, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$. Computing explicitly the eigenvectors of $v_1 = b_1 \otimes u_1$ and $v_2 = b_2 \otimes u_2$, one checks that μ is irreducible.

Following [BFP23, Section 8], the only maximal dark subspaces charged by χ_{inv} are $e_0 \otimes \mathbb{C}^2$ and $e_1 \otimes \mathbb{C}^2$ with $\{e_0, e_1\}$ the canonical basis of \mathbb{C}^2 . Let $J_{\hat{e}_0}: z \mapsto e_0 \otimes z$ and $J_{\hat{e}_1} = v_2 J_{\hat{e}_0}$. Then $J_{\hat{e}_1} z = e_1 \otimes i\sigma_z z$ and by construction $\mathcal{J} = \{J_{\hat{e}_0}, J_{\hat{e}_1}\}$ (completed with arbitrary isomorphisms for other maximal dark subspaces) is $D_{\hat{e}_0}$ -smart, thus smart and minimal thanks to Theorem 5.16. Using composition rules, to determine $G_{\mathcal{J}}$, we only need to compute four unitary matrices that will generate $R_{\mathcal{J}}$ and therefore $G_{\mathcal{J}}$. Direct computations lead to

$$\begin{aligned} J_{\hat{e}_0}^{-1} v_1 J_{\hat{e}_0} &\propto i\sigma_x & J_{\hat{e}_1}^{-1} v_2 J_{\hat{e}_0} &\propto \text{Id} \\ J_{\hat{e}_1}^{-1} v_1 J_{\hat{e}_1} &\propto -i\sigma_x & J_{\hat{e}_0}^{-1} v_2 J_{\hat{e}_1} &\propto -\text{Id}. \end{aligned}$$

Therefore $G_{\mathcal{J}} = \{\pm \text{Id}, \pm i\sigma_x\} \subsetneq G$. Hence, $\tilde{\mathcal{J}}$ is not minimal and $\Upsilon_{\tilde{\mathcal{J}}}$ contains non-ergodic Π -invariant measures.

Now let us introduce a new set of matrices $\{\frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{2}}v_2, \frac{1}{\sqrt{2}}v_3\}$ with the same v_1 and v_2 and adding $v_3 = \text{Id} \otimes i\sigma_y$. The group G generated by the unitary operators u_i is not changed by this addition. The new invariant measure χ_{inv} is supported on the same two maximally dark subspaces. Keeping the same definition of \mathcal{J} , it is still minimal since it is still $D_{\hat{e}_0}$ -smart. However, the generator $i\sigma_y$ is added to $G_{\mathcal{J}}$ since

$$J_{\hat{e}_0}^{-1} v_3 J_{\hat{e}_0} = J_{\hat{e}_1}^{-1} v_3 J_{\hat{e}_1} \propto i\sigma_y.$$

It follows that $G_{\mathcal{J}} = G$ and $\Upsilon_{\tilde{\mathcal{J}}} = \Upsilon_{\mathcal{J}}$.

APPENDIX A. INVARIANT MEASURES ON $\mathcal{M}_d^{1,+}$

In view of the results of Section 6, there naturally arises a question about Π -invariant measures on $\mathcal{M}_d^{1,+}$. We immediately see that uniqueness no longer can hold as we always have $\nu_{\text{unif}} = \int_{\mathcal{D}_m} \text{Unif}_{P(D)} d\chi_{\text{inv}}(D)$ on pure states and $\nu_{\text{ch}} = \int_{\mathcal{D}_m} \delta_{\pi_D/r_m} d\chi_{\text{inv}}(D)$ on (normalized) maximal dark projectors. In what follows we provide the characterization of Π -invariant probability measures on $\mathcal{M}_d^{1,+}$. Throughout the appendix we denote $\mathcal{S}(D) = \{\rho \in \mathcal{M}_d^{1,+} : \text{supp } \rho \subset D\}$ for $D \in \mathcal{D}_m$.

First, we observe that the convergence of quantum trajectories towards dark subspaces (Theorem 4.1) gives the following counterpart of Proposition 6.2:

Proposition A.1. *The support of any Π -invariant Borel probability measure on $\mathcal{M}_d^{1,+}$ is contained in $\bigcup_{D \in \mathcal{D}_m} \mathcal{S}(D)$.*

Now, we fix a family of linear isometries $\mathcal{J} = \{J_D\}_{D \in \mathcal{D}_m}$ such that $J_D: \mathbb{C}^{r_m} \rightarrow D$. For each $D \in \mathcal{D}_m$ we define the map $T_D: \mathcal{M}_{r_m}^{1,+} \rightarrow \mathcal{S}(D)$ as

$$T_D(\rho) = J_D \rho J_D^*.$$

Note that $\text{tr}(v T_D(\rho) v^*) = \text{tr}\left(v \frac{\pi_D}{r_m} v^*\right)$ and

$$(34) \quad T_{vD}^{-1} \left(\frac{v T_D(\rho) v^*}{\text{tr}\left(v \frac{\pi_D}{r_m} v^*\right)} \right) = u_{v,D} \rho u_{v,D}^*$$

for $v \in \text{supp } \mu$ and $D \in \mathcal{D}_m$ such that $\text{tr}(v \pi_D v^*) > 0$ since $u_{v,D} \propto J_{vD}^{-1} v J_D$.

We define the group $G_{\mathcal{J}}$ as before, i.e., as the smallest closed subgroup of $\mathcal{SU}(r_m)$ containing the set $\{u_{v,D} \in \mathcal{SU}(r_m) \mid v \in \text{supp } \mu, D \in \text{supp } \chi_{\text{inv}}, \text{tr}(v \pi_D v^*) > 0\}$. Then for $\rho \in \mathcal{M}_{r_m}^{1,+}$ we define

$$m_{\rho, \mathcal{J}} = (g_{\rho})_{\star} \mathbb{H}_{G_{\mathcal{J}}},$$

where $\mathbb{H}_{G_{\mathcal{J}}}$ is the normalized Haar measure on $G_{\mathcal{J}}$ and

$$\begin{aligned} g_{\rho}: G_{\mathcal{J}} &\longrightarrow \mathcal{M}_{r_m}^{1,+} \\ u &\longmapsto u \rho u^*. \end{aligned}$$

Then we set

$$\nu_{\rho, \mathcal{J}} = (\Psi_{\mathcal{J}})_{\star} (\chi_{\text{inv}} \otimes m_{\rho, \mathcal{J}}),$$

where

$$\begin{aligned} \Psi_{\mathcal{J}}: \mathcal{D}_m \times \mathcal{M}_{r_m}^{1,+} &\longrightarrow \bigcup_{D \in \mathcal{D}_m} \mathcal{S}(D) \subset \mathcal{M}_d^{1,+} \\ (D, \rho) &\longmapsto T_D(\rho). \end{aligned}$$

Explicitly, for $f \in \mathcal{C}(\mathcal{M}_d^{1,+})$,

$$\int_{\mathcal{M}_d^{1,+}} f(\sigma) d\nu_{\rho, \mathcal{J}}(\sigma) = \int_{\mathcal{D}_m} \int_{G_{\mathcal{J}}} f(T_D(u \rho u^*)) d\mathbb{H}_{G_{\mathcal{J}}}(u) d\chi_{\text{inv}}(D).$$

Proposition A.2. *$\nu_{\rho, \mathcal{J}}$ is Π -invariant*

Proof. Recall that the invariance of Haar measure guarantees that

$$\int_{G_{\mathcal{J}}} h(\tilde{u} u \rho u^* \tilde{u}^*) d\mathbb{H}_{G_{\mathcal{J}}}(u) = \int_{G_{\mathcal{J}}} h(u \rho u^*) d\mathbb{H}_{G_{\mathcal{J}}}(u)$$

for every $\tilde{u} \in G_{\mathcal{J}}$ and $h \in \mathcal{C}(\mathcal{M}_{r_m}^{1,+})$. Using also Equation (34) and the K -invariance of χ_{inv} , it follows that

$$\begin{aligned}
& \int_{\mathcal{M}_d^{1,+}} \Pi f(\sigma) \, d\nu_{\rho, \mathcal{J}}(\sigma) \\
&= \int_{\mathcal{M}_d} \int_{\mathcal{D}_m} \left[\int_{G_{\mathcal{J}}} (f \circ T_{vD})(u_{v,D} \rho u^* u_{v,D}^*) \, d\mathbf{H}_{G_{\mathcal{J}}}(u) \right] \text{tr}\left(v \frac{\pi_D}{r_m} v^*\right) \, d\chi_{\text{inv}}(D) \, d\mu(v) \\
&= \int_{\mathcal{M}_d} \int_{\mathcal{D}_m} \left[\int_{G_{\mathcal{J}}} (f \circ T_{vD})(u \rho u^*) \, d\mathbf{H}_{G_{\mathcal{J}}}(u) \right] \text{tr}\left(v \frac{\pi_D}{r_m} v^*\right) \, d\chi_{\text{inv}}(D) \, d\mu(v) \\
&= \int_{\mathcal{D}_m} \int_{G_{\mathcal{J}}} (f \circ T_D)(u \rho u^*) \, d\mathbf{H}_{G_{\mathcal{J}}}(u) \, d[\chi_{\text{inv}} K](D) \\
&= \int_{\mathcal{M}_d^{1,+}} f(\sigma) \, d\nu_{\rho, \mathcal{J}}(\sigma)
\end{aligned}$$

for every $f \in \mathcal{C}(\mathcal{M}_{r_m}^{1,+})$, as desired. \square

Therefore, $\Upsilon_{\rho} = \{\nu_{\rho, \mathcal{J}} : \rho \in \mathcal{M}_{r_m}^{1,+}\}$ is a set of Π -invariant measures. Following the same strategy as for $\mathbb{P}(\mathbb{C}^d)$, we can show via Raugi's theorem that Υ_{ρ} coincides with the set of all Π -ergodic measures on $\mathcal{M}_d^{1,+}$ if \mathcal{J} is a minimal family. To reach this conclusion, we need to check the equicontinuity of Π on $\mathcal{M}_d^{1,+}$ and characterize the Π -minimal subsets of $\mathcal{M}_d^{1,+}$.

For the latter task, the following result is a counterpart of Theorem 6.5. We do not provide the proof as it is identical as for pure states, up to writing $u \rho u^*$ instead of $u \cdot \hat{x}$ on the reference space and $w \tilde{\rho} w^* / \text{tr}(w \tilde{\rho} w^*)$ instead of $w \cdot \hat{y}$ on dark subspaces. In the (\Rightarrow) part of the proof the family of sets $X_D = T_D^{-1}(F \cap \mathcal{S}(D)) \subset \mathcal{M}_{r_m}^{1,+}$ over $D \in \mathcal{D}_m$ is to be examined, where F is a Π -minimal subset of $\mathcal{M}_d^{1,+}$.

Theorem A.3. *Let \mathcal{J} be a minimal family. Then every Π -minimal subset of $\mathcal{M}_d^{1,+}$ is of the form $\text{supp } \nu_{\rho, \mathcal{J}}$ for some $\rho \in \mathcal{M}_d^{1,+}$.*

For the equicontinuity of Π , we need the following two lemmas, which are counterparts of Lemmas 6.6 and 6.7, respectively.

Lemma A.4. *Let $f \in \mathcal{C}(\mathcal{M}_d^{1,+})$ be Lipschitz. Let $\rho_1, \rho_2 \in \mathcal{M}_d^{1,+}$ and $n \in \mathbb{N}^*$. Then*

$$\int_{W_n \in A} \left| f\left(\frac{W_n \rho_1 W_n^*}{\text{tr}(W_n \rho_1 W_n^*)}\right) - f\left(\frac{W_n \rho_2 W_n^*}{\text{tr}(W_n \rho_2 W_n^*)}\right) \right| \text{tr}(W_n \rho_1 W_n^*) \, d\mu^{\otimes n} \leq C \|\rho_1 - \rho_2\|_{\text{tr}},$$

where $A = \{w \in \mathcal{V}_n : \text{tr}(w \rho_2 w) \geq \text{tr}(w \rho_1 w^*)\}$ and $C > 0$ is a constant depending on the Lipschitz constant of f and the dimension d of the system.

Proof. Fix $w \in \mathcal{W}$ and observe that

$$\begin{aligned} & \left\| \frac{w\rho_1 w^*}{\text{tr}(w\rho_1 w^*)} - \frac{w\rho_2 w^*}{\text{tr}(w\rho_2 w^*)} \right\|_{\text{tr}} \text{tr}(w\rho_1 w^*) \text{tr}(w\rho_2 w^*) \\ & \leq \|w\rho_1 w^* \text{tr}(w\rho_2 w^*) - w\rho_2 w^* \text{tr}(w\rho_2 w^*)\|_{\text{tr}} \\ & \quad + \|w\rho_2 w^* \text{tr}(w\rho_1 w^*) - w\rho_2 w^* \text{tr}(w\rho_2 w^*)\|_{\text{tr}} \\ (35) \quad & = \|w(\rho_1 - \rho_2)w^*\|_{\text{tr}} \text{tr}(w\rho_2 w^*) \end{aligned}$$

$$(36) \quad + |\text{tr}(w^* w(\rho_1 - \rho_2))| \|w\rho_2 w^*\|_{\text{tr}}.$$

In what follows, we bound the two terms in the last sum above separately.

For Equation (35), Hölder's inequality for Schatten norms used twice implies $\|w(\rho_1 - \rho_2)w^*\|_{\text{tr}} \leq \|\rho_1 - \rho_2\|_{\text{tr}} \|w\|^2$. Then, the inequality $\|w\|^2 \leq \text{tr}(w^* w)$ implies

$$(37) \quad \|w(\rho_1 - \rho_2)w^*\|_{\text{tr}} \text{tr}(w\rho_2 w^*) \leq \|\rho_1 - \rho_2\|_{\text{tr}} \text{tr}(w^* w) \text{tr}(w\rho_2 w^*).$$

For Equation (36), matrix Hölder's inequality and the domination of the sup norm by the trace norm lead to

$$(38) \quad |\text{tr}(w^* w(\rho_1 - \rho_2))| \|w\rho_2 w^*\|_{\text{tr}} \leq \|\rho_1 - \rho_2\|_{\text{tr}} \text{tr}(w^* w) \text{tr}(w\rho_2 w^*).$$

As a consequence, for every $w \in \mathcal{W}$ such that $\text{tr}(w\rho_2 w^*) > 0$, merging bounds (37) and (38) and dividing by $\text{tr}(w\rho_2 w^*)$ lead to

$$\left\| \frac{w\rho_1 w^*}{\text{tr}(w\rho_1 w^*)} - \frac{w\rho_2 w^*}{\text{tr}(w\rho_2 w^*)} \right\|_{\text{tr}} \text{tr}(w\rho_1 w^*) \leq 2\|\rho_1 - \rho_2\|_{\text{tr}} \text{tr}(w^* w).$$

Therefore, using the Lipschitz property of f ,

$$\left| f\left(\frac{w\rho_1 w^*}{\text{tr}(w\rho_1 w^*)}\right) - f\left(\frac{w\rho_2 w^*}{\text{tr}(w\rho_2 w^*)}\right) \right| \text{tr}(w\rho_1 w^*) \leq 2L\|\rho_1 - \rho_2\|_{\text{tr}} \text{tr}(w^* w).$$

It remains to note that both $W_n \rho_1 W_n^* / \text{tr}(W_n \rho_1 W_n^*)$ and $W_n \rho_2 W_n^* / \text{tr}(W_n \rho_2 W_n^*)$ are almost surely well defined with respect to the measure $\text{tr}(W_n \rho_1 W_n^*) d\mu^{\otimes n}$ on A . The stochasticity condition assures that $\int_{W_n \in A} \text{tr}(W_n^* W_n) d\mu^{\otimes n} = d$, so it follows that

$$\int_{W_n \in A} \left| f\left(\frac{W_n \rho_1 W_n^*}{\text{tr}(W_n \rho_1 W_n^*)}\right) - f\left(\frac{W_n \rho_2 W_n^*}{\text{tr}(W_n \rho_2 W_n^*)}\right) \right| \text{tr}(W_n \rho_1 W_n^*) d\mu^{\otimes n} \leq 2Ld\|\rho_1 - \rho_2\|_{\text{tr}},$$

which concludes the proof. \square

Lemma A.5. *Let $\rho_1, \rho_2 \in \mathcal{M}_d^{1,+}$ and $n \in \mathbb{N}^*$. Then*

$$\int_{\Omega} |\text{tr}(W_n \rho_1 W_n^*) - \text{tr}(W_n \rho_2 W_n^*)| d\mu^{\otimes n} \leq \|\rho_1 - \rho_2\|_{\text{tr}}.$$

Proof. As in Lemma 6.7, $|\text{tr}(w\rho_1 w^*) - \text{tr}(w\rho_2 w^*)| \leq \text{tr}(w^* w |\rho_1 - \rho_2|)$ for every $w \in \mathcal{V}_n$, so

$$\int_{\Omega} |\text{tr}(W_n \rho_1 W_n^*) - \text{tr}(W_n \rho_2 W_n^*)| d\mu^{\otimes n} \leq \text{tr} \left(\int_{\Omega} W_n^* W_n d\mu^{\otimes n} |\rho_1 - \rho_2| \right) = \|\rho_1 - \rho_2\|_{\text{tr}},$$

as desired. \square

Estimations analogous to those in the the proof of Lemma 6.8 together with Lemmas A.4 and A.5 lead to the following result.

Theorem A.6. Π is equicontinuous on $\mathcal{M}_d^{1,+}$.

Therefore, applying Raugi’s Theorem 6.1, we arrive at the desired conclusion.

Theorem A.7. Let \mathcal{J} be a minimal family. Then $\Upsilon_{\mathcal{J}}$ coincides with the set of all Π -ergodic measures on $\mathcal{M}_d^{1,+}$. As a consequence, a probability measure ν on $\mathcal{M}_d^{1,+}$ is Π -invariant iff

$$\nu = \int_{\mathcal{M}_{r_m}^{1,+}} \nu_{\rho, \mathcal{J}} d\lambda(\rho)$$

for some probability measure λ on $\mathcal{M}_{r_m}^{1,+}$.

As already mentioned, for density matrices we always have multiple Π -invariant mutually singular measures. This is because the unitary dynamics that takes place on dark subspaces preserves the state spectrum. Thus, each set of states with the same non degenerate spectrum carries a Π -invariant probability measure and this measure is unique iff G_{\min} acts transitively on the set $\{(\hat{x}_1, \dots, \hat{x}_{r_m}) : (x_1, \dots, x_{r_m}) \text{ is an orthonormal basis of } \mathbb{C}^{r_m}\}$, e.g., if $G_{\min} = \mathcal{SU}(r)$.

Theorem A.8. Fix $\lambda_1 \geq \dots \geq \lambda_{r_m} \geq 0$ such that $\sum_{i=1}^{r_m} \lambda_i = 1$. If $G_{\min} = \mathcal{SU}(r_m)$, there exists a unique Π -invariant probability measure on $\{\rho \in \mathcal{M}_d^{1,+} \mid \text{ev}_i(\rho) = \lambda_i, i = 1, \dots, r_m\}$, where $\text{ev}_i(\rho)$ is the i -th largest eigenvalue of ρ (counted with multiplicities), and the formula for this measure reads

$$\int \text{Unif}_{\mathcal{S}_\lambda(D)} d\chi_{\text{inv}}(D),$$

where $\mathcal{S}_\lambda(D) = \{\rho \in \mathcal{S}(D) \mid \text{ev}_i(\rho) = \lambda_i, i = 1, \dots, r_m\}$. In particular, for $\lambda_1 = 1$ this measure is ν_{unif} , and for $\lambda_1 = \dots = \lambda_{r_m} = \frac{1}{r_m}$ it is ν_{ch} .

Remark A.9. In the case $r_m = 2$, the purity of a state (i.e. $\text{tr}(\rho^2)$) determines its spectrum. Assume that $G_{\min} = \mathcal{SU}(2)$. Then for every given level of purity there exists a unique Π -invariant probability measure. That is, if $s \in [\frac{1}{2}, 1]$, the unique Π -invariant probability measure on $\{\rho \in \mathcal{M}_d^{1,+} \mid \text{tr}(\rho^2) = s\}$ is

$$\int \text{Unif}_{\mathcal{S}_s(D)} d\chi_{\text{inv}}(D),$$

where $\mathcal{S}_s(D) = \{\rho \in \mathcal{S}(D) \mid \text{tr}(\rho^2) = s\}$. In particular, this remark applies to Examples 1 & 2.

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