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# ON THE PRODUCT OF TWO GENERIC $q$ -GEVREY SERIES

THOMAS DREYFUS AND CHANGGUI ZHANG

ABSTRACT. In this paper, we consider a  $q$ -analog of the Borel-Laplace summation process introduced by Fabienne Marotte and the second author. We specifically examine two power series solutions of linear  $q$ -difference equations whose Newton polygon admits only positive slopes equal to 1. These series, known as the generic  $q$ -Gevrey series, are shown to have the property that the product of two such series is  $Gq$ -summable at double level  $(1, 2)$ . Furthermore, we prove that the  $Gq$ -sum of this product equals the product of the  $Gq$ -sums of the original two series.

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## 1. INTRODUCTION

Through the whole paper,  $q$  denotes a real number in the open interval  $(1, +\infty)$ . Let  $\mathbb{C}[[x]]$  be the  $\mathbb{C}$ -vector space of power series of the variable  $x$ , and let  $\mathbb{C}[[x]]_{q;1}$  its sub-space of  $q$ -Gevrey series of the first order, *i.e.*:

$$(1.1) \quad \sum_{n \geq 0} c_n x^n \in \mathbb{C}[[x]]_{q;1} \iff \exists C, A > 0, \forall n \geq 0, |c_n| \leq C A^n q^{n(n-1)/2}.$$

Let  $\mathbb{C}\{x\}$  be the  $\mathbb{C}$ -vector space of convergent power series, which can be identified with the set of germs of analytic functions at  $x = 0$  in the complex plane. Thus, it is clear that the inclusion relation  $\mathbb{C}\{x\} \subset \mathbb{C}[[x]]_{q;1} \subset \mathbb{C}[[x]]$  holds.

As in [4, §4], we call a *generic  $q$ -Gevrey series* a power series  $\hat{f} \in \mathbb{C}[[x]]$  that satisfies a linear  $q$ -difference equation  $\Delta \hat{f} = g \in \mathbb{C}\{x\}$  whose Newton polygon admits only one finite slope equal to one. More explicitly, one can and one will assume that  $\Delta$  represents a certain  $n$ -th order  $q$ -difference operator of the following form :

$$(1.2) \quad \Delta = \sum_{k=0}^n a_{n-k}(x)(x\sigma_q)^k \in \mathbb{C}\{x\}[x\sigma_q], \quad a_0(0) a_n(0) \neq 0,$$

where  $\sigma_q$  denotes the  $q$ -difference operator  $\sigma_q \hat{f}(x) = \hat{f}(qx)$ .

In view of [9, Theorem 5.3], it is known that every generic  $q$ -Gevrey series  $\hat{f}$  is  $Gq$ -summable, that is, we can apply a  $q$ -Borel-Laplace summation process to it to obtain a meromorphic solution  $\mathcal{S}_{q;1}^d(\hat{f})$  having the series as an asymptotic expansion in the  $q$ -Gevrey sense. Note that generic  $q$ -Gevrey series are  $q$ -Gevrey of the first order.

Before going further, it is worthwhile to recollect some facts about one of the simplest generic  $q$ -Gevrey series, that will be seen as a  $q$ -analog of the well-known Euler series  $\sum_{n \geq 0} (-1)^n n! x^{n+1}$ :

$$(1.3) \quad \hat{E}_q(x) = \sum_{n \geq 0} (-1)^n q^{n(n-1)/2} x^n.$$

By replacing  $x$  with  $-x$  in [19, 4.3.8], one finds that the square of this last series,  $\hat{E}_q(x)^2$ , is **not**  $Gq$ -summable in any direction. Besides, it is easy to see that  $\hat{E}_q(x)^2$  satisfies a non-homogeneous second order  $q$ -difference equation; for instance, see [9, §2.2.2] or [20, (1.3)]. Thus, by invoking [3, Theorem 3.3.5], it can be found that the square  $\hat{E}_q(x)^2$  is  $Gq$ -multisummable, specifically  $Gq$ -summable of double level  $(1, 2)$ ; further details on this definition will be provided later.

With this established, a pertinent inquiry arises: does the  $Gq$ -multisum of the square of the  $q$ -Euler series equate to the square of the  $Gq$ -sum of the initial series? This paper will address this question in a broader context as follows.

**Theorem 1.1.** *Let  $(\hat{f}_1, \hat{f}_2)$  be a pair of generic  $q$ -Gevrey series, Then the product  $\hat{f}_1 \hat{f}_2$  belongs to the class  $\mathbb{C}\{x\}_{q;(1,2)}$ . Furthermore, there exists a finite set of directions  $\{\delta_1, \dots, \delta_M\} \subset [0, 2\pi)$ , which can be explicitly describe, such that for all directions  $d \in \mathbb{R} \setminus (\cup_{j=1}^M (\delta_j + 2\pi\mathbb{Z}))$ , the following holds:*

$$(1.4) \quad \mathcal{S}_{q;(1,2)}^d(\hat{f}_1 \hat{f}_2) = \mathcal{S}_{q;1}^d(\hat{f}_1) \mathcal{S}_{q;1}^d(\hat{f}_2),$$

where  $\mathcal{S}_{q;(1,2)}^d$  denotes the  $Gq$ -multisum in the direction  $d$  and  $\mathcal{S}_{q;1}^d$  denotes the  $Gq$ -sum in the same direction.

The remainder of the paper is structured as follows:

**Section 2:** We will review the concepts of  $Gq$ -summation and its variant of double-level extension as introduced in [19] and [9], respectively. Additionally, we will define the so-called generalized  $q$ -Euler series by differentiating a variant of the  $q$ -Euler series with respect to a parameter. This allows us to state a new characterization for any generic  $q$ -Gevrey series; see Theorem 2.5.

**Section 3:** We will first present several preliminary lemmas concerning the generalized  $q$ -Euler series. By revisiting a result from [4], we will give a proof of Theorem 2.5.

**Section 4:** This section is dedicated to establishing Theorem 1.1 for products of generalized  $q$ -Euler series; see Theorem 4.1. For achieve this, particular attention will be given for the study of such products in two successive  $q$ -Borel planes.

**Section 5:** We will focus on proving the main result of the paper, namely Theorem 1.1. This will be done in a somewhat technical manner, specifically combining Theorems 2.5 and 4.1.

Since the historical paper [3] by G. D. Birkhoff in 1913,  $q$ -difference equations and usual difference equations are often considered as sister theories to ordinary differential equations. Thanks to the work of many mathematicians throughout the 1980s and 1990s, it is now well-known that any power series solution to a linear or non-linear analytic ODE is multi-summable and, moreover, the set of multi-summables power series forms a differential algebra ([12, 1, 2, 6, 8, 10] ...). At the end of our previous paper [9], a similar but weaker statement was given for the  $Gq$ -multi-summable series. The present paper can be seen as a first proof of one part of this statement. We hope that this work can be continued to develop a complete theory for the summability of formal solutions of any non-linear  $q$ -difference equations, including  $q$ -Painlevé cases. For related topics, see [7, 11, 13, 14, 15, 16, 17, 18], for instance.

## 2. REVISITING $Gq$ -SUMMATION AND GENERIC $q$ -GEVREY SERIES

Throughout this section, we will adhere to the notation for functional transformations and spaces introduced in [19, 9], applying it wherever relevant. As usual,  $\tilde{\mathbb{C}}^*$  denotes the Riemann surface associated with the logarithm function, while  $\log$  signifies the principal branch of this function. Specifically, for any real number  $a > 0$ , the notation  $\log a$  corresponds to the natural logarithm  $\ln a$ .

**2.1. Space of  $Gq$ -summable series  $\mathbb{C}\{x\}_{q;1}$ .** To simplify, we define

$$(2.1) \quad e_q(x) = \sqrt{2\pi \ln q} e^{(\log(\sqrt{q}x))^2 / (2 \ln(q))} \quad (x \in \tilde{\mathbb{C}}^*),$$

and

$$(2.2) \quad \tilde{D}_R = \{x \in \tilde{\mathbb{C}}^* : |x| < R\}, \quad V_\varepsilon^d = \left\{ \xi \in \tilde{\mathbb{C}}^* : |\arg(\xi) - d| < \varepsilon \right\},$$

where  $R > 0$ ,  $d \in \mathbb{R}$  and  $\varepsilon > 0$ . Here,  $d$  represents the direction of the argument, and we will refer to  $d$  simply as the direction. In addition, we will denote by  $\mathcal{C}^d$  the analytic continuation along the direction  $d$  in some sector  $V_\varepsilon^d$ .

**Definition 2.1.** A power series  $\hat{f} = \sum_{n \geq 0} a_n x^n$  is said to be  $Gq$ -summable in a given direction  $d$  and one writes  $\hat{f} \in \mathbb{C}\{x\}_{q;1}^d$ , if one has  $\hat{\mathcal{B}}_{q;1} \hat{f} \in \mathbb{H}_{q;1}^d$ , where  $\hat{\mathcal{B}}_{q;1} \hat{f}$  and  $\mathbb{H}_{q;1}^d$  are defined as follows:

- $\hat{\mathcal{B}}_{q;1} \hat{f}(\xi) = \sum_{n \geq 0} a_n q^{-n(n-1)/2} \xi^n$ , and this is what we call the (first order) formal  $q$ -Borel transform of  $\hat{f}$ ;
- $\mathbb{H}_{q;1}^d$  denotes the  $\mathbb{C}$ -vector space of germs of analytic functions at  $\xi = 0 \in \mathbb{C}$  that can be continued into an analytic function  $\phi$  possessing a  $q$ -exponential growth of order one at the infinity in some sector  $V_\epsilon^d$  in the following sense:

$$(2.3) \quad \exists A > 0, \quad \phi(\xi) = O(e_q(A\xi)) \quad \text{as } |\xi| \rightarrow \infty \quad \text{with } \xi \in V_\epsilon^d.$$

In this case, the  $Gq$ -sum of  $\hat{f}$  related to the direction  $d$  is, by definition, the  $q$ -Laplace transform of  $\phi$  in the following manner:  $\phi = \mathcal{C}^d \circ \hat{\mathcal{B}}_{q;1}^d \hat{f}$ ,  $\mathcal{S}_{q;1}^d \hat{f} = \mathcal{L}_{q;1}^d \phi$ , and

$$(2.4) \quad \mathcal{L}_{q;1}^d \phi(x) = \int_0^{\infty e^{id}} \frac{\phi(\xi)}{e_q(\xi/x)} \frac{d\xi}{\xi},$$

where  $x$  belongs to some domain  $\tilde{D}_R$  ( $R > 0$ ) of the Riemann surface  $\tilde{\mathbb{C}}^*$ .

Furthermore, if there exists a finite subset  $S \subset [0, 2\pi]$  such that  $\hat{f} \in \mathbb{C}\{x\}_{q;1}^d$  for any  $d \in ([0, 2\pi] \setminus S)$ , we will say that  $\hat{f}$  is  $Gq$ -summable and simply write  $\hat{f} \in \mathbb{C}\{x\}_{q;1}$ .

In view of [19, Proposition 4.3.1 & Théorème 5.3], one has the following result.

**Proposition 2.2.** Let  $d \in \mathbb{R}$ .

- (1) Given  $\hat{f} \in \mathbb{C}\{x\}_{q;1}^d$  and  $h \in \mathbb{C}\{x\}$ , one has  $h\hat{f} \in \mathbb{C}\{x\}_{q;1}^d$  and  $\mathcal{S}_{q;1}^d(h\hat{f}) = h\mathcal{S}_{q;1}^d \hat{f}$ .
- (2) Any generic  $q$ -Gevrey power series is  $Gq$ -summable.

**2.2. Space of  $Gq$ -summable series  $\mathbb{C}\{x\}_{q;(1,2)}$ .** In order to simplify the presentation, we will only recall the definition of the double-level  $Gq$ -summable series ; for the more general definition, see [9].

**Definition 2.3.** A power series  $\hat{f} = \sum_{n \geq 0} a_n x^n$  is said to be  $Gq$ -summable of double-level (1, 2) in a given direction  $d$  and one writes  $\hat{f} \in \mathbb{C}\{x\}_{q;(1,2)}^d$ , if  $\hat{\mathcal{B}}_{q;1} \hat{f} \in \mathbb{H}_{q;2}^d$  and  $\mathcal{L}_{q;2}^d \circ \mathcal{C}^d \circ \hat{\mathcal{B}}_{q;1} \hat{f} \in \tilde{\mathbb{H}}_{q;2}^d$ , where  $\mathbb{H}_{q;2}^d$ ,  $\mathcal{L}_{q;2}^d$  and  $\tilde{\mathbb{H}}_{q;2}^d$  are defined as follows:

- $\mathbb{H}_{q;2}^d = \mathbb{H}_{q^{1/2};1}^d$  and  $\mathcal{L}_{q;2}^d = \mathcal{L}_{q^{1/2};1}^d$ , that are obtained by replacing  $q$  with  $q^{1/2}$  ( $= \sqrt{q}$ ) in (2.3) and (2.4), respectively;
- $\tilde{\mathbb{H}}_{q;2}^d$  denotes the  $\mathbb{C}$ -vector space of analytic functions  $\phi$  in some sector  $V_\epsilon^d$  ( $\epsilon > 0$  being arbitrary) such that  $\phi(\xi) = O(1)$  as  $\xi \rightarrow 0$  in  $V_\epsilon^d$  and that the condition in (2.3) is fulfilled for  $q^{1/2}$  instead of  $q$ .

If this is the case, the corresponding  $Gq$ -sum of  $\hat{f}$ ,  $\mathcal{S}_{q;(1,2)}^d \hat{f}$ , is defined as follows:  $\phi_0 = \mathcal{C}^d \circ \hat{\mathcal{B}}_{q;1} \hat{f}$ ,  $\phi_1^d = \mathcal{C}^d \circ \mathcal{L}_{q;2}^d \phi_0$ , and  $\mathcal{S}_{q;(1,2)}^d \hat{f} = \mathcal{L}_{q;2}^d \phi_1^d$ .

Furthermore, if  $\hat{f} \in \mathbb{C}\{x\}_{q;(1,2)}^d$  for any value of  $d$  in  $[0, 2\pi]$  but eventually excepted one finite number, we will say that  $\hat{f}$  is  $Gq$ -summable of level (1, 2) and simply write  $\hat{f} \in \mathbb{C}\{x\}_{q;(1,2)}$ .

By considering [9, Lemme 2.4.1 & Théorème 2.4.3], one gets the following statement.

**Proposition 2.4.** *Given any  $d \in \mathbb{R}$ , the set  $\mathbb{C}\{x\}_{q;(1,2)}^d$  constitutes a  $\mathbb{C}\{x\}$ -module containing  $\mathbb{C}\{x\}_{q;1}^d$  as a sub-module.*

**2.3. Generic  $q$ -Gevrey series in terms of  $q$ -Euler series.** It is easy to see that every convergent power series is a generic  $q$ -Gevrey series. The  $q$ -Euler series  $\hat{E}_q$  given in (1.3) can be seen as one of the simplest generic  $q$ -Gevrey series whose radius of convergence is zero. Indeed, a direct calculation shows that

$$(2.5) \quad (x \sigma_q + 1) \hat{E}_q(x) = 1.$$

To any  $a \in \mathbb{C}^*$  will be associated the family  $(\hat{E}_q^{(a,\alpha)})_{\alpha \in \mathbb{Z}_{\geq 0}}$  of the so-called *generalized  $q$ -Euler series* as follows:  $\hat{E}_q^{(a,0)}(x) = -\frac{1}{a} \hat{E}_q(-\frac{x}{a})$ , and

$$(2.6) \quad \hat{E}_q^{(a,\alpha)}(x) = \frac{1}{\alpha!} \partial_a^\alpha \hat{E}_q^{(a,0)}(x)$$

for  $\alpha \geq 1$ . It follows that  $\hat{E}_q = \hat{E}_q^{(-1,0)}$ . Furthermore, for any  $\lambda \in \mathbb{C}^*$ , one gets from considering (2.6) that

$$(2.7) \quad \hat{E}_q^{(a/\lambda,\alpha)}(x) = \lambda^{\alpha+1} \hat{E}_q^{(a,\alpha)}(\lambda x).$$

**Theorem 2.5.** *A power series  $\hat{f} \in \mathbb{C}[[x]]$  is a generic  $q$ -Gevrey series if, and only if, there exist  $m \in \mathbb{Z}_{\geq 0}$ ,  $\{f_0, f_1, \dots, f_m\} \subset \mathbb{C}\{x\}$ ,  $\{a_1, \dots, a_m\} \subset \mathbb{C}^*$  and  $\{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}_{\geq 0}$  such that the following equality holds in  $\mathbb{C}[[x]]$ :*

$$(2.8) \quad \hat{f} = f_0 + f_1 \hat{E}_q^{(a_1,\alpha_1)} + \dots + f_m \hat{E}_q^{(a_m,\alpha_m)}.$$

The proof of Theorem 2.5, which is closely linked to [4, Theorem 4.20], will be given in §3.4. Let's emphasize that this result plays a central role in the proof of Theorem 1.1. Indeed, thanks to this result, it will suffice to verify the theorem for a pair of generalized  $q$ -Euler series.

### 3. GENERIC $q$ -GEVREY SERIES IN TERMS OF THE GENERALIZED $q$ -EULER SERIES

This section aims to prove Theorem 2.5 by first establishing properties for every family of generalized  $q$ -Euler series  $\{\hat{E}_q^{(a/q^k,\alpha)}\}_{k,\alpha \in \mathbb{Z}_{\geq 0}}$ . In §3.1, we will begin by examining the functional equation satisfied by each generalized  $q$ -Euler series  $\hat{E}_q^{(a,\alpha)}$ , noting that it is  $Gq$ -summable in any direction that does contain the point with affix  $a$ . Furthermore, it will be shown that every  $Gq$ -sum is bounded in any sector of finite openness; see Remark 3.2.

In §3.2, we will consider the linear relationships that exist among any finite family  $\{\hat{E}_q^{(a,\alpha)}\}$  where  $a \in a_0 q^{\mathbb{Z}}$  and  $\alpha \in (\alpha_0 + \mathbb{Z})$ . After that, in §3.3, we will explain how to construct a  $q$ -difference operator associated with a given linear combination of generalized  $q$ -Euler series; see Theorem 3.5. In this way, we will provide a new characterization for a power series to be generic  $q$ -Gevrey, thereby proving Theorem 2.5.

**3.1. Functional equations and  $Gq$ -sums of the generalized  $q$ -Euler series.** One recalls that the  $q$ -Euler series  $\hat{E}_q$ , which is precisely  $\hat{E}_q^{(-1,0)}$ , satisfies the  $q$ -difference equation given in (2.5). This result can be extended to any generalized  $q$ -Euler series  $\hat{E}_q^{(a,\alpha)}$ .

**Lemma 3.1.** *For any  $(a, \alpha) \in \mathbb{C}^* \times \mathbb{Z}_{\geq 0}$ , the generalized  $q$ -Euler series  $\hat{E}_q^{(a,\alpha)}$  defined in (2.6) satisfies the  $q$ -difference equation*

$$(3.1) \quad (x \sigma_q - a)^{\alpha+1} \hat{E}_q^{(a,\alpha)}(x) = 1,$$

and is  $Gq$ -summable for any direction  $d$  such that  $a \notin (0, \infty e^{id})$ . Furthermore, its associated  $Gq$ -sum is defined as follows:

$$(3.2) \quad \mathcal{S}_{q;1}^d \hat{E}_q^{(a,\alpha)}(x) = \mathcal{L}_{q;1}^d \left( \frac{1}{(\xi - a)^{\alpha+1}} \right) (x)$$

for all  $x \in \tilde{\mathbb{C}}^*$ .

Moreover, the following identity holds in  $\tilde{\mathbb{C}}^*$ :

$$(3.3) \quad \mathcal{S}_{q;1}^d \hat{E}_q^{(a,\alpha)}(x) = \frac{1}{\alpha!} \partial_a^\alpha \mathcal{S}_{q;1}^d \hat{E}_q^{(a,0)}(x).$$

*Proof.* First, we shall prove (3.1) by induction on  $\alpha$ . Replacing  $x$  with  $\left(-\frac{x}{a}\right)$  in (2.5) yields that

$$\left(-\frac{x}{a} \sigma_q + 1\right) \hat{E}_q\left(-\frac{x}{a}\right) = 1.$$

With  $\hat{E}_q^{(a,0)}(x) = -\frac{1}{a} \hat{E}_q\left(-\frac{x}{a}\right)$  we find (3.1) for  $\alpha = 0$ . Furthermore, let  $k \geq 1$ , and suppose that  $(x \sigma_q - a)^k \hat{E}_q^{(a,k-1)} = 1$ , where  $\alpha = k - 1$ . Expand this equation as follows:

$$\sum_{j=0}^k \binom{k}{j} (-a)^{k-j} (x \sigma_q)^j \hat{E}_q^{(a,k-1)}(x) = 1.$$

By taking the differentiation with respect to  $a$  in both sides in the above, using the identities  $(k-j) \binom{k}{j} = \frac{(k-j)k!}{j!(k-j)!} = k \binom{k-1}{j}$  yields that

$$-k \sum_{j=0}^{k-1} \binom{k-1}{j} (-a)^{k-j-1} (x \sigma_q)^j \hat{E}_q^{(a,k-1)}(x) + \sum_{j=0}^k \binom{k}{j} (-a)^{k-j} (x \sigma_q)^j \partial_a \hat{E}_q^{(a,k-1)}(x) = 0$$

or, equivalently,

$$-k(x \sigma_q - a)^{k-1} \hat{E}_q^{(a,k-1)}(x) + (x \sigma_q - a)^k \partial_a \hat{E}_q^{(a,k-1)}(x) = 0.$$

By (2.6), one knows that  $\partial_a \hat{E}_q^{(a,k-1)} = k \hat{E}_q^{(a,k)}$ . One gets from the above that

$$(x \sigma_q - a)^k \hat{E}_q^{(a,k)}(x) = (x \sigma_q - a)^{k-1} \hat{E}_q^{(a,k-1)}(x).$$

Multiplying both sides by the operator  $(x \sigma_q - a)$  and using  $(x \sigma_q - a)^k \hat{E}_q^{(a,k-1)} = 1$  implies (3.1) for  $\alpha = k$ .

By [5, Equation (2.1)] with  $\mu = 1$ , we find

$$(3.4) \quad \hat{\mathcal{B}}_{q;1} x^j \sigma_q^m = q^{-j(j-1)/2} \xi^j \sigma_q^{m-j} \hat{\mathcal{B}}_{q;1}$$

for all  $(j, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ . When  $j = m = 1$ , one gets that  $\hat{\mathcal{B}}_{q;1}(x \sigma_q) = \xi \hat{\mathcal{B}}_{q;1}$ . Thus, by applying the  $q$ -Borel transform  $\hat{\mathcal{B}}_{q;1}$  to both sides of (3.1), it follows that  $(\xi - a)^{\alpha+1} \hat{\mathcal{B}}_{q;1} \hat{E}_q^{(a,\alpha)} = 1$ , *i.e.*

$$(3.5) \quad \hat{\mathcal{B}}_{q;1} \hat{E}_q^{(a,\alpha)} = \frac{1}{(\xi - a)^{\alpha+1}}.$$

The right-hand side tends to 0 as  $|\xi|$  goes to  $\infty$ . Hence, for any direction  $d$  such that  $a \notin (0, \infty e^{id})$ ,  $\frac{1}{(\xi - a)^{\alpha+1}} \in \mathbb{H}_{q;1}^d$ . This implies the  $Gq$ -sum of  $\hat{E}_q^{(a,\alpha)}$  as being stated in (3.2).

Given any couple of compact sets  $K \subset \tilde{\mathbb{C}}^*$  and  $J \subset \mathbb{C}^*$ , the integral (3.2) is uniformly convergent for  $(x, a) \in K \times J$ , provided that  $J \cap (0, \infty e^{id}) = \emptyset$ . Thus, one can consider the differentiation with respect to  $a$  under the integral sign. In view of the elementary identity  $\partial_a^\alpha \frac{1}{\xi - a} = \frac{\alpha!}{(\xi - a)^{\alpha+1}}$ , one deduces easily (3.3) from (3.2). This finishes the proof of Lemma 3.1.  $\square$

By following [19, Théorème 3.3 & Définition 2.1.1], one knows that every  $Gq$ -sum  $\mathcal{S}_{q;1}^d \hat{E}_q^{(a,\alpha)}$  admits the power series  $\hat{E}_q^{(a,\alpha)}$  as a  $q$ -Gevrey asymptotic expansion of the first order in the direction  $d$ . It will be helpful to notice the following remark, due to the fact that  $|e_q(t)| = e_q(|t|) e^{-(\arg(t))^2/(2 \ln q)}$ ; see (2.1) for the definition of  $e_q$ .

**Remark 3.2.** *Let  $a, \alpha$  and  $d$  be as in Lemma 3.1. The  $Gq$ -sum function  $\mathcal{S}_{q;1}^d \hat{E}_q^{(a,\alpha)}$  is well-defined and analytic over the whole Riemann surface  $\tilde{\mathbb{C}}^*$ , and satisfies the following inequality for all  $x \in \tilde{\mathbb{C}}^*$ :*

$$(3.6) \quad \left| \mathcal{S}_{q;1}^d \hat{E}_q^{(a,\alpha)}(x) \right| \leq |a|_d^{-\alpha-1} e^{(\arg(e^{id}/x))^2/(2 \ln q)},$$

where  $|a|_d$  denotes the distance from  $a$  to the half-line  $(0, \infty e^{id})$  in the complex plane.

To see (3.6), one can notice that

$$\begin{aligned} \left| \mathcal{S}_{q;1}^d \hat{E}_q^{(a,\alpha)}(x) \right| &\leq \int_0^\infty \frac{1}{|\xi e^{id} - a|^{\alpha+1} |e_q(\xi e^{id}/x)|} \frac{d\xi}{\xi} \\ &\leq \frac{e^{(\arg(e^{id}/x))^2/(2 \ln q)}}{|a|_d^{\alpha+1}} \int_0^\infty \frac{1}{e_q(\xi/|x|)} \frac{d\xi}{\xi}, \end{aligned}$$

where the last integral is simply  $\mathcal{L}_{q;1}^0 1(|x|) = 1$ .

In what follows, we will make use of the following notation:

$$(3.7) \quad \forall k \geq 0, \quad \hat{E}_q^{(a,-k-1)} = (-a)^k \sum_{j=0}^k \binom{k}{j} q^{j(j-1)/2} \left(-\frac{x}{a}\right)^j.$$

Particularly, it follows that  $\hat{E}_q^{(a,-1)} = 1$ .



Note that when  $k < 0$ , we do not recover  $E_q^{(a, -k-1)}$ . On the other hand, the notation fits to obtain the following lemma.

**Lemma 3.3.** *Let  $(a, \alpha, k) \in \mathbb{C}^* \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ . Then:*

$$(3.8) \quad (x \sigma_q - a)^k \hat{E}_q^{(a, \alpha)} = \hat{E}_q^{(a, \alpha - k)}.$$

*Proof.* First, assume that  $\alpha \geq k$ . By applying (3.1), both sides of the equation in (3.8) are power series solutions of the linear  $q$ -difference equation  $(x \sigma_q - a)^{\alpha - k + 1} y = 1$ . Thus, their difference satisfies the homogeneous equation  $(x \sigma_q - a)^{\alpha - k + 1} y = 0$ , for which zero is the only power series solution. This implies (3.8).

Next, when  $k > \alpha$ , we set  $k' = k - \alpha - 1 > 0$ , noticing that

$$(3.9) \quad (x \sigma_q - a)^k \hat{E}_q^{(a, \alpha)} = (x \sigma_q - a)^{k'} \left( (x \sigma_q - a)^{\alpha + 1} \hat{E}_q^{(a, \alpha)} \right) = (x \sigma_q - a)^{k'} \cdot 1.$$

Since  $(x \sigma_q)^j \cdot 1 = q^{j(j-1)/2} x^j \sigma_q^j \cdot 1 = q^{j(j-1)/2} x^j$  for all  $j \in \mathbb{Z}$ , the last term in (3.9) becomes:

$$(x \sigma_q - a)^{k'} \cdot 1 = \sum_{j=0}^{k'} \binom{k'}{j} (-a)^{k'-j} q^{j(j-1)/2} x^j.$$

Thus, comparing (3.9) with (3.7) yields (3.8).  $\square$

**3.2. Linear dependence of the generalized  $q$ -Euler series.** We shall start with the following elementary result on binomial coefficients.

**Lemma 3.4.** *If  $\epsilon = 0$  or 1, then:*

$$(3.10) \quad \sum_{\ell=j}^{k+1-\epsilon} (-1)^{j+\ell} \binom{k+1}{\ell+\epsilon} \binom{\ell}{j} = \epsilon$$

for any integers  $k$  and  $j$  such that  $0 \leq j \leq k$ .

*Proof.* Let  $C_{k,j}(\epsilon)$  be the left-hand side of (3.10). One has

$$\begin{aligned} C_{k,j}(0) &= \sum_{\ell=j}^{k+1} (-1)^{j+\ell} \binom{k+1}{\ell} \binom{\ell}{j} = \binom{k+1}{j} \sum_{\ell=j}^{k+1} (-1)^{j+\ell} \binom{k+1-j}{\ell-j} \\ &= \binom{k+1}{j} \sum_{r=0}^{k+1-j} (-1)^r \binom{k+1-j}{r}. \end{aligned}$$

Since  $0 = (1-1)^{k+1-j} = \sum_{r=0}^{k+1-j} (-1)^r \binom{k+1-j}{r}$ , one obtains that (3.10) for  $\epsilon = 0$ .

Consider the case of  $\epsilon = 1$ . It is easy to see that  $C_{j,j}(1) = 1$ . Let  $j \leq k$ . Since

$$C_{k+1,j}(1) = \sum_{\ell=j}^k (-1)^{j+\ell} \binom{k+2}{\ell+1} \binom{\ell}{j} + (-1)^{j+k+1} \binom{k+2}{k+2} \binom{k+1}{j},$$

using the equalities  $\binom{k+2}{\ell+1} = \binom{k+1}{\ell+1} + \binom{k+1}{\ell}$  and  $\binom{k+2}{k+2} = \binom{k+1}{k+1}$  gives that

$$\begin{aligned} C_{k+1,j}(1) &= \sum_{\ell=j}^k (-1)^{j+\ell} \binom{k+1}{\ell+1} \binom{\ell}{j} \\ &\quad + \sum_{\ell=j}^k (-1)^{j+\ell} \binom{k+1}{\ell} \binom{\ell}{j} + (-1)^{j+k+1} \binom{k+1}{k+1} \binom{k+1}{j}. \end{aligned}$$

Thus,  $C_{k+1,j}(1) = C_{k,j}(1) + C_{k,j}(0) = C_{k,j}(1)$  for  $0 \leq j \leq k$ . In this way, one gets (3.10) for  $\epsilon = 1$ .  $\square$

**Theorem 3.5.** *Let  $(a, \alpha, k) \in \mathbb{C}^* \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ . One has*

$$(3.11) \quad \hat{E}_q^{(a/q^k, \alpha)} = q^{-k(k-2\alpha-3)/2} \left(\frac{a}{x}\right)^k \sum_{j=0}^k \binom{k}{j} \left(\frac{1}{a}\right)^j \hat{E}_q^{(a, \alpha-j)}$$

and

$$(3.12) \quad \begin{aligned} \hat{E}_q^{(a, \alpha)} &= q^{k(k-2\alpha-3)/2} \left(\frac{x}{a}\right)^k \sum_{\ell=0}^{\alpha} \binom{k+\ell-1}{k-1} \left(-\frac{q^k}{a}\right)^{\ell} \hat{E}_q^{(a/q^k, \alpha-\ell)} \\ &\quad + \left(-\frac{1}{a}\right)^{\alpha+1} \sum_{j=0}^{k-1} q^{j(j-1)/2} \binom{\alpha+j}{\alpha} \left(\frac{x}{a}\right)^j. \end{aligned}$$

*Proof.* Replacing  $\lambda$  with  $q^k$  in (2.7) yields that  $\hat{E}_q^{(a/q^k, \alpha)} = q^{(\alpha+1)k} \sigma_q^k \hat{E}_q^{(a, \alpha)}$ . By multiplying this last equation by  $q^{k(k-1)/2} x^k$  and observing that  $(x \sigma_q)^k = q^{k(k-1)/2} x^k \sigma_q^k$ , one finds that

$$(3.13) \quad (x \sigma_q)^k \hat{E}_q^{(a, \alpha)} = q^{k(k-2\alpha-3)/2} x^k \hat{E}_q^{(a/q^k, \alpha)}.$$

On the other hand, by using the identity

$$(x \sigma_q)^k = ((x \sigma_q - a) + a)^k = \sum_{j=0}^k \binom{k}{j} a^{k-j} (x \sigma_q - a)^j,$$

one deduces from gathering (3.13) together with (3.8) the identity stated in (3.11).

Now, let  $\alpha = 0$  into (3.11), and separate  $j = 0$  and  $j = \ell + 1 > 0$  in the summation. This implies that

$$(3.14) \quad \hat{E}_q^{(a/q^k, 0)} = q^{-k(k-3)/2} \left(\frac{a}{x}\right)^k \left( \hat{E}_q^{(a, 0)} + A_k \right),$$

where  $A_0 = 0$  and for  $k \geq 1$ :

$$A_k = \sum_{\ell=0}^{k-1} \binom{k}{\ell+1} \left(\frac{1}{a}\right)^{\ell+1} \hat{E}_q^{(a, -\ell-1)}.$$

Assume  $k > 0$ , and apply (3.7) for each term  $\hat{E}_q^{(a, -\ell-1)}$  in the above. One gets that

$$A_k = \frac{1}{a} \sum_{\ell=0}^{k-1} \sum_{j=0}^{\ell} (-1)^\ell \binom{k}{\ell+1} \binom{\ell}{j} q^{j(j-1)/2} \left(-\frac{x}{a}\right)^j,$$

or, equivalently,

$$A_k = \frac{1}{a} \sum_{j=0}^{k-1} \sum_{\ell=j}^{k-1} (-1)^{j+\ell} \binom{k}{\ell+1} \binom{\ell}{j} q^{j(j-1)/2} \left(\frac{x}{a}\right)^j.$$

By considering (3.10) for  $\epsilon = 1$  and  $(k-1)$  instead of  $k$ , it follows that

$$A_k = \frac{1}{a} \sum_{j=0}^{k-1} q^{j(j-1)/2} \left(\frac{x}{a}\right)^j.$$

So, (3.14) can be put into the following form:

$$(3.15) \quad \hat{E}_q^{(a,0)}(x) = q^{k(k-3)/2} \left(\frac{x}{a}\right)^k \hat{E}_q^{(a/q^k,0)}(x) - \frac{1}{a} \sum_{j=0}^{k-1} q^{j(j-1)/2} \left(\frac{x}{a}\right)^j,$$

which is really (3.12) for  $\alpha = 0$ .

To obtain (3.12) for  $\alpha > 0$ , we will apply the definition of  $\hat{E}_q^{(a,\alpha)}$  in (2.6), considering the  $\alpha$ -th differential of  $\hat{E}_q^{(a,0)}$  with the help of (3.15). Indeed, given a couple of non-negative integers  $(\ell, j)$ , one has

$$\partial_a^\ell (a^{-j-1}) = (-1)^\ell (j+1)(j+2)\dots(j+\ell) a^{-j-\ell-1}$$

and, in addition, using (2.6) implies that  $\partial_a^\ell \hat{E}_q^{(a/q^j,0)} = \ell! \hat{E}_q^{(a/q^j,\ell)} q^{-j\ell}$ . Thus, by taking the  $\alpha$ -th differential with respect to  $a$  for both sides of (3.15), it follows that

$$\begin{aligned} \alpha! \hat{E}_q^{(a,\alpha)} &= q^{k(k-3)/2} \left(\frac{x}{a}\right)^k \sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} \frac{(k+\ell-1)!}{(k-1)!} \left(-\frac{1}{a}\right)^\ell (\alpha-\ell)! q^{-k(\alpha-\ell)} \hat{E}_q^{(a/q^k,\alpha-\ell)} \\ &\quad - \sum_{j=0}^{k-1} q^{j(j-1)/2} x^j (-1)^\alpha \frac{(j+\alpha)!}{j!} a^{-j-\alpha-1}. \end{aligned}$$

By dividing by  $\alpha!$  in the above, one gets the expression given in (3.12).  $\square$

**Corollary 3.6.** *Let  $a \in \mathbb{C}^*$ ,  $m \in \mathbb{Z}_{>0}$ ,  $(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m$ ,  $(\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m$ , and let  $(f_1, \dots, f_m) \in (\mathbb{C}\{x\})^m$ . If  $k = \max(k_1, \dots, k_m)$ ,  $a' = a/q^k$  and  $\alpha = \max(\alpha_1, \dots, \alpha_m)$ , then one can find  $(H, h_0, h_1, \dots, h_\alpha) \in (\mathbb{C}\{x\})^{\alpha+2}$  such that*

$$(3.16) \quad \sum_{j=1}^m f_j \hat{E}_q^{(a/q^{k_j}, \alpha_j)} = H + \sum_{j=0}^{\alpha} h_j \hat{E}_q^{(a', \alpha-j)}.$$

*Proof.* This comes directly from applying (3.12).  $\square$

**3.3. Generic irregular singular  $q$ -difference operators coming from combining several generalized  $q$ -Euler series.** For simplicity, let  $\mathfrak{D}_0 = \mathbb{C}^* \oplus (x \mathbb{C}\{x\})$ , let  $\mathfrak{D}_n$  be the set of the  $n$ -th order  $q$ -operators  $\Delta$  given in (1.2) for  $n \geq 1$ , and define

$$(3.17) \quad \mathfrak{D} = \cup_{n \geq 0} \mathfrak{D}_n.$$

We call a *generic irregular singular  $q$ -difference operator* any element  $\Delta \in \mathfrak{D}_n$  with  $n \geq 1$ .

By following [19, Prop. 5.1.4], each operator  $\Delta \in \mathfrak{D}_n$  can be factorized as follows:

$$(3.18) \quad \Delta = h_0 (x \sigma_q - a_1) h_1 \dots (x \sigma_q - a_n) h_n,$$

where  $a_j \in \mathbb{C}^*$  and  $h_j \in \mathfrak{D}_0$ . Let  $c \mapsto \bar{c}$  be the natural map from  $\mathbb{C}^*$  to the multiplicative elliptic curve  $\mathbb{C}^*/q^{\mathbb{Z}}$ . Define the set of (generalized) indices of monodromy  $IM(\Delta)$  as follows:  $IM(h) = \emptyset$  for  $h \in \mathfrak{D}_0$ , and

$$(3.19) \quad IM(\Delta) = \{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\} \subset \mathbb{C}^*/q^{\mathbb{Z}}$$

for  $\Delta \in \mathfrak{D}_n$ ,  $n \geq 1$ . One knows that  $IM(\Delta)$  constitutes a set of analytic invariants of  $\Delta$ ; for instance, see [19, §5.1.7]. In addition, given  $(\Delta_1, \Delta_2) \in \mathfrak{D}_{n_1} \times \mathfrak{D}_{n_2}$ , one has

$$(3.20) \quad \Delta_1 \Delta_2 \in \mathfrak{D}_{n_1+n_2}, \quad IM(\Delta_1 \Delta_2) = IM(\Delta_1) \cup IM(\Delta_2).$$

Furthermore, for any pair  $(\Delta, h) \in \mathfrak{D}_n \times (\mathbb{C}\{x\} \setminus \{0\})$  ( $n \geq 1$ ), if  $\Delta^{[h]}$  denotes the conjugation of  $\Delta$  by  $h$ , i.e.  $\Delta^{[h]} = (\sigma_q^n h)^{-1} \Delta h$ , where  $(\sigma_q^n h)^{-1}(x) = \frac{1}{h(q^n x)}$ , one has the following properties:

$$(3.21) \quad \Delta^{[h]} \in \mathfrak{D}_n, \quad IM(\Delta^{[h]}) = IM(\Delta)$$

and

$$(3.22) \quad \Delta(hy) = \sigma_q^n h \Delta^{[h]} y.$$

**Lemma 3.7.** *Given  $(a, n, \Delta) \in \mathbb{C} \times \mathbb{Z}_{>0} \times \mathfrak{D}_n$ , there exist  $b_0, b_1, \dots, b_n \in \mathbb{C}\{x\}$  such that the following expression holds:*

$$(3.23) \quad \Delta = b_0 (x \sigma_q - a)^n + b_1 (x \sigma_q - a)^{n-1} + \dots + b_n.$$

*Proof.* Let  $a_0, \dots, a_n$  be the coefficients of  $\Delta$  as given in (1.2). By writing  $x \sigma_q = (x \sigma_q - a) + a$ , applying the binomial expansion for  $(x \sigma_q)^k$  yields that

$$(x \sigma_q)^k = \sum_{j=0}^k \binom{k}{j} a^{k-j} (x \sigma_q - a)^j.$$

Putting this into (1.2) yields that

$$\Delta = \sum_{k=0}^n a_{n-k} \sum_{j=0}^k \binom{k}{j} a^{k-j} (x \sigma_q - a)^j.$$

Therefore, if

$$(3.24) \quad b_j = \sum_{\ell=0}^j \binom{n-\ell}{n-j} a^{j-\ell} a_\ell \quad (0 \leq j \leq n),$$

one gets immediately (3.23) and finishes the proof of Lemma 3.7.  $\square$

**Remark 3.8.** *The above relations in (3.24) lead us to the following elementary identity:*

$$(3.25) \quad \forall a \in \mathbb{C}, \quad \mathbb{C}\{x\}[x\sigma_q] = \mathbb{C}\{x\}[x\sigma_q - a].$$

**Theorem 3.9.** *Let  $(a, \alpha, m) \in \mathbb{C}^* \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{> 0}$ , and let  $(\hat{f}, \hat{g}, h_0, h_1, \dots, h_\alpha, \Lambda) \in \mathbb{C}[[x]] \times \mathbb{C}[[x]] \times (\mathbb{C}\{x\})^{\alpha+1} \times \mathfrak{D}_m$  to be such that*

$$(3.26) \quad \hat{f} = \hat{g} + \sum_{\ell=0}^{\alpha} h_{\alpha-\ell} \hat{E}_q^{(a,\ell)}, \quad \Lambda \hat{g} \in \mathbb{C}\{x\}.$$

*One supposes that  $\bar{a} \notin IM(\Lambda)$  and  $h_0 \neq 0$ . Then, there exists  $\Delta \in \mathfrak{D}_n$  such that  $0 \leq n \leq \frac{1}{2}(\alpha+1)(\alpha+2)$ ,*

$$(3.27) \quad \Delta \Lambda \hat{f} \in \mathbb{C}\{x\}, \quad IM(\Delta) = \{\bar{a}\}.$$

*Proof.* By considering (3.22), one knows that

$$\Lambda(h_{\alpha-\ell} \hat{E}_q^{(a,\ell)}) = \sigma_q^m h_{\alpha-\ell} \Lambda^{[h_{\alpha-\ell}]} \hat{E}_q^{(a,\ell)}$$

for  $\ell$  from 0 to  $\alpha$ . Applying Lemma 3.7 to  $\Lambda^{[h_{\alpha-\ell}]}$  gives that

$$(3.28) \quad \Lambda^{[h_{\alpha-\ell}]} = \sum_{j=0}^m b_{\alpha-\ell,j} (x\sigma_q - a)^{m-j}, \quad b_{\alpha-\ell,j} \in \mathbb{C}\{x\}.$$

From now on, define

$$(3.29) \quad h_{\ell,j} = b_{\alpha-\ell,m-j} \sigma_q^m h_{\alpha-\ell}, \quad \hat{Y}_{\ell,j} = (x\sigma_q - a)^j \hat{E}_q^{(a,\ell)}.$$

In view of (3.1), it follows that  $\hat{Y}_{\ell,j} \in \mathbb{C}\{x\}$  when  $j > \ell$ . Moreover, using (3.8) gives that  $\hat{Y}_{\ell,j} = \hat{Y}_{\ell-j,0} = \hat{E}_q^{(a,\ell-j)}$  if  $j \leq \ell$ . So, applying  $\Lambda$  to  $\hat{f}$  in (3.26) and gathering (3.28) together with (3.29) implies that

$$\Lambda \hat{f} = H_0 + \sum_{\ell=0}^{\alpha} \sum_{j=0}^{\ell} h_{\ell,j} \hat{E}_q^{(a,\ell-j)}, \quad H_0 \in \mathbb{C}\{x\}.$$

Letting  $h_k^* = \sum_{j=0}^{\alpha-k} h_{j+k,j}$ , it follows from the above that

$$(3.30) \quad \Lambda \hat{f} = H_0 + \sum_{k=0}^{\alpha} h_k^* \hat{E}_q^{(a,k)}, \quad H_0, h_k^* \in \mathbb{C}\{x\}.$$

In what follows, we will construct a family of  $q$ -difference operators to successively remove from (3.30) the divergent power series  $\hat{E}_q^{(a,0)}, \hat{E}_q^{(a,1)}, \dots, \hat{E}_q^{(a,\alpha)}$ . For doing this, define

$$(3.31) \quad \Delta_0 = \begin{cases} ((x\sigma_q - a)^{\alpha+1})^{[1/h_\alpha^*]} & \text{if } h_\alpha^* \neq 0; \\ 1 & \text{if } h_\alpha^* = 0. \end{cases}$$

By applying (3.22) to  $(\Delta, h) = ((x\sigma_q - a)^{\alpha+1}, 1/h_\alpha^*)$ , it follows from considering (3.1) that, in any case where  $h_\alpha^* = 0$  or not,

$$\Delta_0(h_\alpha^* \hat{E}_q^{(a,\alpha)}) = \sigma_q^{\alpha+1} h_\alpha^* (x\sigma_q - a)^{\alpha+1} \hat{E}_q^{(a,\alpha)} = \sigma_q^{\alpha+1} h_\alpha^*.$$

In this way, one obtains from (3.30) that

$$(3.32) \quad \Delta_0 \Lambda \hat{f} = H_1 + \sum_{k=0}^{\alpha-1} \Delta_0 \left( h_k^* \hat{E}_q^{(a,k)} \right),$$

where  $H_1 = \Delta_0 H_0 + \sigma_q^{\alpha+1} h_\alpha^*$  and where  $\sum_{k=0}^{\alpha-1} \dots = 0$  when  $\alpha = 0$ . Consequently, letting  $\Delta = \Delta_0$  gives (3.27) if  $\alpha = 0$ .

If  $\alpha > 0$ , again by applying (3.22), one has

$$\Delta_0 (h_k^* \hat{E}_q^{(a,k)}) = \sigma_q^{\alpha+1} h_k^* \Delta_0^{[h_k^*]} \hat{E}_q^{(a,k)}$$

for  $0 \leq k < \alpha$ . One deduces from (3.32) that

$$(3.33) \quad \Delta_0 \Lambda \hat{f} = H_1 + \sum_{k=0}^{\alpha-1} \sigma_q^{\alpha+1} h_k^* \Delta_0^{[h_k^*]} \hat{E}_q^{(a,k)}.$$

By expanding each operator  $\Delta_0^{[h_k^*]}$  in  $\mathbb{C}\{x\}[x\sigma_q - a]$  (see Lemma 3.7) and observing that  $(x\sigma_q - a)^j \hat{E}_q^{(a,k)} = \hat{E}_q^{(a,k-j)}$  for  $k \geq j$  (see (3.8)), one can put (3.33) into the following form:

$$(3.34) \quad \Delta_0 \Lambda \hat{f} = H_1^* + \sum_{k=0}^{\alpha-1} h_k^{**} \hat{E}_q^{(a,k)}, \quad H_1^*, h_k^{**} \in \mathbb{C}\{x\}.$$

It is to be noted that the divergent series  $\hat{E}_q^{(a,\alpha)}$  does not appear in the right-hand side of the equation in (3.34). By a similar process as what done for the transformation from (3.30) into (3.34), one can define a series of operators  $\Delta_1, \Delta_2, \dots, \Delta_\alpha$  possessing the following properties:

- (1) either  $\Delta_k \in \mathfrak{D}_{k+1}$  and  $IM(\Delta_k) = \{\bar{a}\}$  or  $\Delta_k = 1$ ;
- (2) letting  $\Delta = \Delta_\alpha \dots \Delta_1 \Delta_0$ , one has  $\Delta \Lambda \hat{f} \in \mathbb{C}\{x\}$ .

Since the order of  $\Delta$  equals at most to the following sum:

$$\sum_{k=0}^{\alpha} (\alpha + 1 - k) = \frac{1}{2}(\alpha + 1)(\alpha + 2),$$

one gets (3.27) and finishes the proof of Theorem 3.9.  $\square$

### 3.4. Characterisation of generic $q$ -Gevrey series and proof of Theorem 2.5.

As in [4], let  $K$  be the set of rational functions bounded at zero, and let  $\mathbb{E}_{q;1}$  be the set of all entire functions admitting at most a  $q$ -exponential growth of order 1 at the infinity. It follows that

$$(3.35) \quad \mathbb{E}_{q;1} = \bigcap_{d \in [0, 2\pi]} \mathbb{H}_{q;1}^d = \hat{\mathcal{B}}_{q;1}(\mathbb{C}\{x\}).$$

Let  $\star_{q;1}$  be the bilinear map from  $\mathbb{C}[[\xi]] \times \mathbb{C}[[\xi]]$  to  $\mathbb{C}[[\xi]]$  such that  $\xi^m \star_{q;1} \xi^n = q^{-m.n} \xi^{m+n}$  for all  $(m, n) \in \mathbb{Z}_{\geq 0}^2$ . Thus, the  $\mathbb{C}$ -vector space  $\mathbb{C}[[\xi]]$  is equipped with two

*multiplications* that are the usual one and the so-called  $q$ -convolution product  $\star_{q;1}$ . Furthermore, the following identity holds for any  $(\hat{f}, \hat{g}) \in \mathbb{C}[[x]] \times \mathbb{C}[[x]]$ :

$$(3.36) \quad \hat{\mathcal{B}}_{q;1}(\hat{f}\hat{g}) = \hat{\mathcal{B}}_{q;1}\hat{f} \star_{q;1} \hat{\mathcal{B}}_{q;1}\hat{g}.$$

By following [4, §4], one defines the  $(K, \mathbb{E}_{q;1})$ -bimodule  $H \subset \mathbb{C}\{\xi\} \subset \mathbb{C}[[\xi]]$  by the relation  $H = \cup_{n \geq 0} H_n$ , where  $H_0 = K$  and, for any integer  $n \geq 0$ ,

$$H_{2n+1} = \mathbb{E}_{q;1} \star_{q;1} H_{2n}, \quad H_{2n+2} = K H_{2n+1}.$$

By [4, Theorem 4.20], one knows that  $\phi \in H$  if and only if

$$(3.37) \quad \phi = r_0 + \phi_0 + \phi_1 \star_{q;1} \frac{1}{(\xi - a_1)^{\alpha_1+1}} + \dots + \phi_n \star_{q;1} \frac{1}{(\xi - a_n)^{\alpha_n+1}},$$

where  $r_0 \in K$ ,  $\phi_j \in \mathbb{E}_{q;1}$ ,  $a_j \in \mathbb{C}^*$  and  $\alpha_j \in \mathbb{Z}_{\geq 0}$ . By noticing that both  $K$  and  $\mathbb{E}_{q;1}$  are stable by  $\sigma_q$  and that, in addition, either the multiplication or the  $q$ -convolution product is compatible with  $\sigma_q$ , one finds that  $H$  is a  $q$ -difference  $(K, \mathbb{E}_{q;1})$ -bimodule such that

$$(3.38) \quad \phi \in H \iff \sigma_q \phi \in H.$$

**Lemma 3.10.** *Given any  $\hat{f} \in \mathbb{C}[[x]]$ , one has  $\hat{\mathcal{B}}_{q;1}\hat{f} \in H$  if and only if  $\hat{f}$  can be put into the form of (2.8).*

*Proof.* First of all, suppose  $\hat{f}$  be as in (2.8). By considering (3.36), one finds that

$$(3.39) \quad \hat{\mathcal{B}}_{q;1}\hat{f} = \hat{\mathcal{B}}_{q;1}f_0 + \hat{\mathcal{B}}_{q;1}f_1 \star_{q;1} \hat{\mathcal{B}}_{q;1}\hat{E}_q^{(a_1, \alpha_1)} + \dots + \hat{\mathcal{B}}_{q;1}f_n \star_{q;1} \hat{\mathcal{B}}_{q;1}\hat{E}_q^{(a_n, \alpha_n)}.$$

Furthermore, by applying (3.5), one has  $\hat{\mathcal{B}}_{q;1}\hat{E}_q^{(a_j, \alpha_j)} = \frac{1}{(\xi - a_j)^{\alpha_j+1}}$ . Thus, letting  $\phi = \hat{\mathcal{B}}_{q;1}\hat{f}$ ,  $r_0 = 0$  and  $\phi_j = \hat{\mathcal{B}}_{q;1}f_j$  for  $0 \leq j \leq n$ , one deduces (3.37) from (3.39). This implies that  $\hat{\mathcal{B}}_{q;1}\hat{f} \in H$ .

Next, suppose  $\hat{\mathcal{B}}_{q;1}\hat{f} \in H$ , and assume (3.37) hold for  $\phi = \hat{\mathcal{B}}_{q;1}\hat{f}$ . By making use of the fractional decomposition of  $r_0$ , one can suppose that

$$r_0 = P + \sum_{\ell=1}^m \sum_{k=0}^{\mu_\ell-1} \frac{c_{\ell,k}}{(\xi - b_\ell)^{k+1}}, \quad P \in \mathbb{C}[\xi], \quad c_{\ell,k} \in \mathbb{C}, \quad b_\ell \in \mathbb{C}^*.$$

Therefore, by following (3.5), there exists  $Q \in \mathbb{C}[x]$  such that

$$r_0 = \hat{\mathcal{B}}_{q;1} \left( Q + \sum_{\ell=1}^m \sum_{k=0}^{\mu_\ell-1} c_{\ell,k} \hat{E}_q^{(b_\ell, k)} \right).$$

Thus, by a similar way as what done in the above, one gets (2.8) from considering (3.37), using (3.36) at the same time. We omit the detail.  $\square$

*Proof of Theorem 2.5.* Let  $\hat{f} = c_0 + c_1 x + c_2 x^2 + \dots \in \mathbb{C}[[x]]$ . We will proceed in two steps.

On the one hand, suppose  $\hat{f}$  be a generic  $q$ -Gevrey series, that is to say, there exists  $\Delta \in \mathfrak{D}$  such that  $\Delta\hat{f} \in \mathbb{C}\{x\}$ . Set  $\phi = \hat{\mathcal{B}}_{q;1}\hat{f}$ ,  $\hat{g} = \hat{f} - c_0$ , and observe that  $\Delta\hat{g} = \Delta\hat{f} - \Delta c_0 \in \mathbb{C}\{x\}$ . Now, consider the  $q$ -Borel transform  $B_q$  defined in [4, p. 382], which is the linear application  $x \mathbb{C}[[x]] \rightarrow \mathbb{C}[[\xi]]$  such that  $B_q(x^{n+1}) = q^{-n(n-1)/2} \xi^n$  for all

$n \in \mathbb{Z}_{\geq 0}$ . If  $\gamma = B_q \hat{g}$ , one finds that  $\gamma = \hat{\mathcal{B}}_{q;1} \left( (\hat{f} - c_0)/x \right)$ . By taking into account the relation  $\hat{f} = c_0 + x \left( (\hat{f} - c_0)/x \right)$ , applying  $\hat{\mathcal{B}}_{q;1}$  to  $\hat{f}$  and using (3.36) gives that

$$\phi = c_0 + \hat{\mathcal{B}}_{q;1} x \star_{q;1} \hat{\mathcal{B}}_{q;1} \left( \frac{\hat{f} - c_0}{x} \right) = c_0 + \xi \star_{q;1} \gamma.$$

Thus, the  $q$ -Borel transforms  $\phi$  and  $\gamma$  are linked as follows:

$$(3.40) \quad \phi = c_0 + \xi \sigma_q^{-1} \gamma.$$

By applying [4, Proposition 4.24], one knows that  $\gamma \in H$ . This together with considering both (3.38) and (3.40) implies that  $\phi \in H$ . By Lemma 3.10, it follows that  $\hat{f}$  can be put into the form as in (2.8).

On the other hand, suppose  $\hat{f}$  be such that the relation in (2.8) holds, *i.e.*

$$\hat{f} = f_0 + f_1 \hat{E}_q^{(a_1, \alpha_1)} + \dots + f_m \hat{E}_q^{(a_m, \alpha_m)}.$$

Let  $\mu$  be the number of different elements contained in the subset  $\{\bar{a}_1, \dots, \bar{a}_m\}$  of the elliptic curve  $\mathbb{C}^*/q^{\mathbb{Z}}$ . If  $\mu = 0$ , then  $\hat{f} = f_0 \in \mathbb{C}\{x\}$ . As the convergent power series  $\hat{f}$  is a generic  $q$ -Gevrey series, one can finish the proof of Theorem 2.5 by induction on  $\mu$ , using Theorem 3.9.  $\square$

#### 4. THE PRODUCT OF TWO GENERALIZED $q$ -EULER SERIES

This section aims to prove one special case of Theorem 1.1 for two generalized  $q$ -Euler series. Namely, we shall establish the following statement.

**Theorem 4.1.** *Given  $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$ , one has  $\hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)} \in \mathbb{C}\{x\}_{q;(1,2)}$ . Moreover, for any  $d \in \mathbb{R}$  such that  $\{a, b\} \cap (0, \infty e^{id}) = \emptyset$ , one has*

$$(4.1) \quad \mathcal{S}_{q;(1,2)}^d(\hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)}) = \mathcal{S}_{q;1}^d(\hat{E}_q^{(a,0)}) \mathcal{S}_{q;1}^d(\hat{E}_q^{(b,0)}).$$

The proof of Theorem 4.1 will be given in §4.4. To do that, we will start with a remarkable relation for special values of the  $Gq$ -sum functions of two generalized  $q$ -Euler series. This will permit us to obtain the  $Gq$ -summability of their product, after having considered their  $q$ -Borel transform and a first  $q$ -Laplace transform.

**4.1. Special values of a combination of two generalized  $q$ -Euler series.** By following (3.2), one knows that, given  $d \in \mathbb{R}$  and  $\alpha \in \mathbb{Z}_{\geq 0}$ , the associated sum-function  $\mathcal{S}_{q;1}^d(\hat{E}_q^{(a,\alpha)})$  represents an analytic function with respect to the parameter  $a \in \mathbb{C}^*$ , provided that  $a \notin (0, e^{id})$ . In Lemma 4.3 below, we shall consider sum-functions at some special points related to the parameter  $a$ . To simplify, we will make use of the following conventional hypothesis.

**Hypothesis 4.2.** *The triplet  $(a, b, d) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{R}$  satisfies the following conditions:*

$$(4.2) \quad \{\arg(a), \arg(b)\} \subset (d, d + 2\pi).$$

Note that (4.2) above implies  $\{a, b\} \cap (0, \infty e^{id}) = \emptyset$ .



**Lemma 4.3.** *Let  $(a, b, d) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{R}$  to be such that  $(0, \infty e^{id}) \cap \{a, b\} = \emptyset$  and that (4.2) holds, and let  $e_q$  be as in (2.1). Then, the following equality holds for any  $\ell \in \mathbb{Z}$ :*

$$(4.3) \quad a \mathcal{S}_{q;1}^d \hat{E}_q^{(a,0)}(\sqrt{qab} e^{\pi i \ell}) + b \mathcal{S}_{q;1}^d \hat{E}_q^{(b,0)}(\sqrt{qab} e^{\pi i \ell}) = -1 + 2\pi i R_{q;\ell}^d(a, b),$$

where

$$(4.4) \quad R_{q;\ell}^d(a, b) = \begin{cases} -\sum_{j=\ell+1}^{-1} \frac{1}{e_q(\sqrt{b/(qa)} e^{(\ell-2j)\pi i})} & \text{for } \ell < -1; \\ 0 & \text{for } \ell = -1; \\ \sum_{j=1}^{\ell+1} \frac{1}{e_q(\sqrt{b/(qa)} e^{(\ell-2j)\pi i})} & \text{for } \ell > -1. \end{cases}$$

*Proof.* Let  $c = \sqrt{qab}$ ,  $c_\ell = c e^{\pi i \ell}$ ,  $a = |a| e^{ia'}$ ,  $b = |b| e^{ib'}$ , and let denote by  $A_\ell$  the left-hand side of (4.3). By replacing  $(a, \alpha, x)$  with  $(a, 0, c_\ell)$  and  $(b, 0, c_\ell)$  in (3.2) respectively, it follows from using (2.4) that

$$(4.5) \quad A_\ell = \mathcal{L}_{q;1}^d \left( \frac{a}{\xi - a} + \frac{b}{\xi - b} \right) (c_\ell) = I_{1,\ell} + I_{2,\ell},$$

where

$$I_{1,\ell} = \int_0^{\infty e^{id}} \frac{1}{(\xi/a) - 1} \frac{1}{e_q(\xi/c_\ell)} \frac{d\xi}{\xi}, \quad I_{2,\ell} = \int_0^{\infty e^{id}} \frac{1}{(\xi/b) - 1} \frac{1}{e_q(\xi/c_\ell)} \frac{d\xi}{\xi}.$$

Letting  $t = \frac{\xi}{a}$  and  $t = \frac{\xi}{b}$  respectively in  $I_{1,\ell}$  and  $I_{2,\ell}$  in the above gives that

$$(4.6) \quad I_{1,\ell} = \int_0^{\infty e^{i(d-a')}} \frac{1}{t-1} \frac{1}{e_q(at/c_\ell)} \frac{dt}{t}$$

and

$$(4.7) \quad I_{2,\ell} = \int_0^{\infty e^{i(d-b')}} \frac{1}{t-1} \frac{1}{e_q(bt/c_\ell)} \frac{dt}{t}.$$

By considering (2.1), one knows that  $e_q(x) = e_q(1/(qx))$ . Thus, the integral in (4.6) can be rewritten as follows:

$$I_{1,\ell} = \int_0^{\infty e^{i(d-a')}} \frac{1}{t-1} \frac{1}{e_q(c_\ell/(qat))} \frac{dt}{t} = \int_0^{\infty e^{i(a'-d)}} \frac{t}{1-t} \frac{1}{e_q(c_\ell t/(qa))} \frac{dt}{t},$$

the last integral being obtained by changing  $t$  with  $1/t$ . By using the fractional decomposition  $\frac{t}{1-t} = \frac{1}{1-t} - 1$  and the relation  $\mathcal{L}_{q;1}^d(1) = 1$ , one finds that

$$(4.8) \quad I_{1,\ell} = -1 + \int_0^{\infty e^{i(a'-d)}} \frac{1}{1-t} \frac{1}{e_q(c_\ell t/(qa))} \frac{dt}{t}.$$

Consider the case  $\ell = -1$ , and set  $c' = c_{-1}$ , i.e.  $c' = c e^{-\pi i}$ . By taking into account the hypothesis made on the triplet  $(a', b', d)$ , one finds that  $\frac{b}{c'} = \frac{c'}{qa} e^{2\pi i}$ . Thus, replacing

$(\ell, t)$  with  $(-1, t e^{-2\pi i})$  in (4.7) yields that

$$I_{2,-1} = \int_0^{\infty e^{i(d-b'+2\pi)}} \frac{1}{t-1} \frac{1}{e_q(c' t/(q a))} \frac{dt}{t}.$$

Gathering this together with (4.8) and (4.5) yields that

$$(4.9) \quad A_{-1} = -1 + \left( \int_0^{\infty e^{i(a'-d)}} - \int_0^{\infty e^{i(d-b'+2\pi)}} \right) \frac{1}{1-t} \frac{1}{e_q(c' t/(q a))} \frac{dt}{t}.$$

Furthermore, in view of the relation  $\{a'-d, d-b'+2\pi\} \subset (0, 2\pi)$ , the integration-contour composed of the half-lines  $(0, \infty e^{i(d-b'+2\pi)})$  and  $(0, \infty e^{i(a'-d)})$  does not contain any pole  $t = 1 = e^{2\pi j i}$ , where  $j \in \mathbb{Z}$ . So, by applying Cauchy Theorem, one gets from (4.9) that  $A_{-1} = -1$ , as being expected in (4.3) for  $\ell = -1$ .

Now, suppose  $\ell > -1$ , and observe that  $\frac{b}{c_\ell} = \frac{c_\ell}{q a} e^{-2\ell\pi i}$ . Similarly, it follows from combining (4.8) together with (4.7) and (4.5) that

$$(4.10) \quad A_\ell = -1 + \left( \int_0^{\infty e^{i(a'-d)}} - \int_0^{\infty e^{i(d-b'-2\ell\pi)}} \right) \frac{1}{1-t} \frac{1}{e_q(c_\ell t/(q a))} \frac{dt}{t}.$$

By applying the Residue Theorem, one obtains from (4.10) that

$$A_\ell = -1 + 2\pi i \sum_{j=1}^{\ell+1} \frac{1}{e_q(c_\ell e^{-2j\pi i}/(q a))} = -1 + 2\pi i \sum_{j=1}^{\ell+1} \frac{1}{e_q(c e^{(\ell-2j)\pi i}/(q a))}.$$

This is exactly what is stated in (4.3) combined with (4.4) for  $\ell > 0$ .

The case  $\ell < -1$  can be treated in a similar way, again using (4.10). Thus, one finishes the proof of Lemma 4.3.  $\square$

#### 4.2. The $q$ -Borel transform of the product of two generalized $q$ -Euler series.

In view of Lemma 3.1, we shall only consider any product of the form  $\hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)}$ , whose differentiations with respect to  $a$  and/or to  $b$  may lead us to every product  $\hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)}$  for all non-negative integers  $\alpha$  and  $\beta$ .

**Lemma 4.4.** *For any  $(a, b) \in \mathbb{C}^{*2}$ , the product series  $\hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)}$  satisfies the following  $q$ -difference equations:*

$$(4.11) \quad (x \sigma_q - a)(x \sigma_q - b)(x^2 \sigma_q - ab)y = qx^2 - ab.$$

*Proof.* This comes from considering (3.1). Indeed, letting  $\alpha = 0$  in (3.1) gives that

$$(x \sigma_q - a) \hat{E}_q^{(a,0)}(x) = 1,$$

or, equivalently,  $x \hat{E}_q^{(a,0)}(qx) = 1 + a \hat{E}_q^{(a,0)}(x)$ . This implies that

$$x^2 \hat{E}_q^{(a,0)}(qx) \hat{E}_q^{(b,0)}(qx) = (1 + a \hat{E}_q^{(a,0)}(x))(1 + b \hat{E}_q^{(b,0)}(x)),$$

which can be put into the following form:

$$(x^2 \sigma_q - a b) \left( \hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)} \right) (x) = 1 + a \hat{E}_q^{(a,0)}(x) + b \hat{E}_q^{(b,0)}(x).$$

Thus, (4.11) is obtained by applying the  $q$ -difference operator  $(x\sigma_q - a)(x\sigma_q - b)$  to both sides of the above equation.  $\square$

**Lemma 4.5.** *Given  $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$ , if  $\phi_0(a, b; \xi) = \hat{\mathcal{B}}_{q;1} \left( \hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)} \right)$ , then:*

$$(4.12) \quad \phi_0(a, b; \xi) = -\frac{1}{ab} \sum_{n \geq 0} \left( 1 + \frac{a q^n}{\xi - a q^n} + \frac{b q^n}{\xi - b q^n} \right) q^{-n^2} \left( \frac{\xi^2}{ab} \right)^n.$$

Consequently,  $\phi_0(a, b; \xi) \in \mathbb{H}_{q;2}^d$  for any  $d \in \mathbb{R}$  such that  $\{a, b\} \cap (0, \infty e^{id}) = \emptyset$ .

*Proof.* For simplicity, we will write  $\phi_0(\xi)$  instead of  $\phi_0(a, b; \xi)$ . By considering (3.4) for  $j = m = 1$ , applying the  $q$ -Borel transform  $\hat{\mathcal{B}}_{q;1}$  to both sides of (4.11) yields that

$$(\xi - a)(\xi - b)(q^{-1} \xi^2 \sigma_q^{-1} - ab)\phi_0(\xi) = \xi^2 - ab.$$

In other words,  $\phi_0$  satisfies the following  $q$ -difference equation:

$$(4.13) \quad (1 - X)u(\xi) = v(\xi),$$

where  $X = \frac{\xi^2}{abq} \sigma_q^{-1}$  and  $v(\xi) = \frac{ab - \xi^2}{ab(\xi - a)(\xi - b)}$ . By observing that

$$\xi^2 - ab = (\xi - a)(\xi - b) + a(\xi - b) + b(\xi - a),$$

one obtains that

$$(4.14) \quad v(\xi) = -\frac{1}{ab} \left( 1 + \frac{a}{\xi - a} + \frac{b}{\xi - b} \right).$$

Now, put (4.13) into the form  $u = v + Xu$ , and iterate this last relation. It follows that

$$u(\xi) = (1 + X + \dots + X^{N-1})v(\xi) + X^N u(\xi)$$

for any positive integer  $N$ . By considering the equality  $X^n = \left( \frac{\xi^2}{ab} \right)^n q^{-n^2} \sigma_q^{-n}$  for all  $n \in \mathbb{Z}$ , one deduces from (4.14) that

$$X^n v(\xi) = -\frac{q^{-n^2} \xi^{2n}}{(ab)^{n+1}} \left( 1 + \frac{a q^n}{\xi - a q^n} + \frac{b q^n}{\xi - b q^n} \right).$$

In this way, one can put (4.13) into the following form:

$$u(\xi) = -\sum_{n=0}^{N-1} \frac{q^{-n^2} \xi^{2n}}{(ab)^{n+1}} \left( 1 + \frac{a q^n}{\xi - a q^n} + \frac{b q^n}{\xi - b q^n} \right) + \frac{q^{-N^2} \xi^{2N}}{(ab)^N} u \left( \frac{\xi}{q^N} \right).$$

Letting  $N \rightarrow \infty$ , one arrives at the following expression deduced from (4.13):

$$(4.15) \quad u_0(\xi) = -\sum_{n \geq 0} \frac{q^{-n^2} \xi^{2n}}{(ab)^{n+1}} \left( 1 + \frac{a q^n}{\xi - a q^n} + \frac{b q^n}{\xi - b q^n} \right),$$

that is well-defined and analytic in  $\mathbb{C} \setminus (a q^{\mathbb{Z}_{\geq 0}} \cup b q^{\mathbb{Z}_{\geq 0}})$ .

By using the fact that  $\phi_0$  is the unique analytic function at  $\xi = 0$  that satisfies the functional equation in (4.13), one finds that  $\phi_0 = u_0$ . This gives (4.12), using (4.15).

To see the growth at the infinity of  $\phi_0$ , let  $V = V_\epsilon^d$  to be any sector that does not contain neither  $a$  nor  $b$ , and set

$$(4.16) \quad |a|_V = \min_{\xi \in V} |\xi - a|, \quad |b|_V = \min_{\xi \in V} |\xi - b|.$$

By taking into account the fact that

$$\left| 1 + \frac{a q^n}{\xi - a q^n} + \frac{b q^n}{\xi - b q^n} \right| \leq 1 + \frac{|a|}{|a|_V} + \frac{|b|}{|b|_V}$$

for all  $\xi \in V$ , one gets from (4.12) that, as  $\xi$  tends to  $\infty$  in  $V_\epsilon^d$ ,

$$(4.17) \quad \phi_0(\xi) = O(F(\xi)) \quad \text{where} \quad F(\xi) = \sum_{n \geq 0} q^{-n^2} \left( \frac{\xi^2}{ab} \right)^n.$$

Notice that  $F = \hat{\mathcal{B}}_{q;2} \hat{f}$ , where  $\hat{f}(x) = \sum_{n \geq 0} \left( \frac{x^2}{\sqrt{q} ab} \right)^n$ . By applying [19, Proposition 3.1.4]

for  $q^{1/2}$  instead of  $q$ , one find that  $F$  has a  $q^{1/2}$ -exponential growth of order one at the infinity. Thus, one finishes the proof of Lemma 4.5.  $\square$

**4.3. Towards the summability of the product of two generalized  $q$ -Euler series.** The next step is to consider the second order  $q$ -Laplace transform of the function  $\phi_0(a, b; \xi)$  as follows.

**Lemma 4.6.** *Let  $a, b, \phi_0(a, b; \xi)$  and  $d$  be as in Lemma 4.5, assume the relation in (4.2), define  $\zeta_{a,b}^d = q^{1/4} \sqrt{ab} e^{-\pi i}$  with*

$$(4.18) \quad \Lambda_{a,b}^d = \zeta_{a,b}^d e^{\pi i \mathbb{Z}} \quad \text{and} \quad \Lambda_{a,b}^{d*} = \Lambda_{a,b}^d \setminus \{\zeta_{a,b}^d\},$$

and set  $\phi_1^d(a, b; \zeta) = \mathcal{L}_{q;2}^d(\phi_0(a, b; \xi))(\zeta)$ .

- (1) *The function  $\phi_1^d(a, b; \zeta)$  is well-defined and analytic in  $\tilde{\mathbb{C}}^* \setminus \Lambda_{a,b}^{d*}$ , possessing a simple pole at each point of  $\Lambda_{a,b}^{d*}$  and satisfying the following equality for all  $\zeta \in \tilde{\mathbb{C}}^* \setminus \Lambda_{a,b}^{d*}$ :*

$$(4.19) \quad (q^{-1/2} \zeta^2 - ab) \phi_1^d(a, b; \zeta) = 1 + a \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta) + b \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,0)})(\zeta),$$

where  $\mathcal{S}_{q;2}^d = \mathcal{L}_{q;2}^d \circ \mathcal{C}^d \circ \hat{\mathcal{B}}_{q;2} = \mathcal{S}_{q^{1/2};1}^d$ .

- (2) *Given any  $d' \in \mathbb{R}$  and  $\epsilon > 0$ , one can find  $C > 0$  such that*

$$(4.20) \quad |\phi_1^d(a, b; \zeta)| \leq \frac{C}{|\zeta^2 - \sqrt{q} ab|}$$

for all  $\zeta \in V_\epsilon^{d'}$  with  $\zeta^2 \neq \sqrt{q} ab$ .

- (3) *One has  $\phi_1^d(a, b; \zeta) \in \tilde{\mathbb{H}}_{q;2}^{d'}$  for all  $d' \in \mathbb{R} \setminus \left( \frac{1}{2} (\arg(a) + \arg(b)) - \pi + \pi \mathbb{Z}^* \right)$ .*

*Proof.* (1) Remember that  $\phi_0(a, b; \xi)$  is solution to the functional equation stated in (4.13), which together with (4.14) implies that

$$(4.21) \quad \left( 1 - \frac{\xi^2}{ab q} \sigma_q^{-1} \right) \phi_0(a, b; \xi) = -\frac{1}{ab} \left( 1 + \frac{a}{\xi - a} + \frac{b}{\xi - b} \right).$$

From now on, let  $q' = q^{1/2}$ . By definition, one knows that  $\mathcal{L}_{q';2}^d = \mathcal{L}_{q';1}^d$ . A direct computation shows that

$$\mathcal{L}_{q';1}^d(\xi^2 \sigma_q^{-1}) = \mathcal{L}_{q';1}^d\left(q' (\xi \sigma_{q'}^{-1})^2\right) = q' \zeta^2 \mathcal{L}_{q';1}^d.$$

Thus, applying  $\mathcal{L}_{q';2}^d$  to both side of (4.21) yields that

$$(4.22) \quad \left(1 - \frac{\zeta^2}{abq'}\right) \phi_1^d(a, b; \zeta) = -\frac{1}{ab} \mathcal{L}_{q';1}^d\left(1 + \frac{a}{\xi - a} + \frac{b}{\xi - b}\right)(\zeta).$$

Note that  $\mathcal{L}_{q';1}^d(1) = 1$ . In view of (3.2) with  $\alpha = 0$ , it follows from (4.22) that

$$\left(1 - \frac{\zeta^2}{abq'}\right) \phi_1^d(a, b; \zeta) = -\frac{1}{ab} \left(1 + a \mathcal{S}_{q';1}^d(\hat{E}_{q'}^{(a,0)})(\zeta) + b \mathcal{S}_{q';1}^d(\hat{E}_{q'}^{(b,0)})(\zeta)\right).$$

Since  $\mathcal{S}_{q^{1/2};1}^d = \mathcal{S}_{q;2}^d$ , one deduces thus the equality stated in (4.19).

By noticing that  $\zeta^2 = \sqrt{q}ab$  if and only if  $\zeta \in \Lambda_{a,b}^d$  (see (4.18)), one deduces from (4.19) that  $\phi_1^d$  is well-defined and analytic in  $\tilde{\mathbb{C}}^* \setminus \Lambda_{a,b}^d$ , eventually admitting simple poles in  $\Lambda_{a,b}^d$ . Furthermore, if one replaces  $q$  with  $q^{1/2}$  in Lemma 4.3, one can easily deduce from (4.4) that, given  $\ell \in \mathbb{Z}$ , one has

$$1 + a \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta_{a,b}^d e^{\ell\pi i}) + b \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,0)})(\zeta_{a,b}^d e^{\ell\pi i}) = 0$$

if and only if  $\ell = -1$ . This implies that  $\phi_1^d(a, b; \zeta)$  is well-defined and analytic at  $\zeta_{a,b}^d$  and really admits simple poles in the complementary set  $\Lambda_{a,b}^{d*}$ .

(2) With regard to the relation expected in (4.20), one can make use of direct estimates on the  $q'$ -Laplace transforms appeared in (4.22), observing that both  $Gq$ -sums  $\mathcal{S}_{q';1}^d(\hat{E}_{q'}^{(a,0)})$  and  $\mathcal{S}_{q';1}^d(\hat{E}_{q'}^{(b,0)})$  are bounded in any sector  $V_\epsilon^{d'}$  of the Riemann surface  $\tilde{\mathbb{C}}^*$  (see (3.6)).

(3) This comes immediately from combining the relation in (4.20) with the fact that  $\phi_1^d(a, b; \zeta)$  is well-defined and analytic at  $\zeta_{a,b}^d$ .  $\square$

**Remark 4.7.** *By taking into account the condition stated in (4.2), one has*

$$(4.23) \quad d \notin \left(\frac{1}{2}(\arg(a) + \arg(b)) - \pi + \pi\mathbb{Z}^*\right).$$

*This together with Lemma 4.6 (3) implies that  $\phi_1^d(a, b; \zeta) \in \tilde{\mathbb{H}}_{q;2}^d$ .*

**4.4. Proof of Theorem 4.1.** Before starting the proof, we will establish one result about the product of two  $q$ -exponential functions given as in (2.1).

**Lemma 4.8.** *Let  $e_q$  as in (2.1), and let  $(\xi_1, \xi_2) \in \tilde{\mathbb{C}}^* \times \tilde{\mathbb{C}}^*$  and  $(\zeta_1, \zeta_2) \in \tilde{\mathbb{C}}^* \times \tilde{\mathbb{C}}^*$ . If*

$$(4.24) \quad \zeta_1 = q^{-1/4} \sqrt{\frac{\xi_1}{\xi_2}}, \quad \zeta_2 = q^{1/4} \sqrt{\xi_1 \xi_2},$$

*then the following identity holds for any  $x \in \tilde{\mathbb{C}}^*$ :*

$$(4.25) \quad e_q\left(\frac{\xi_1}{x}\right) e_q\left(\frac{\xi_2}{x}\right) = \frac{1}{2} e_{q^{1/2}}(\zeta_1) e_{q^{1/2}}\left(\frac{\zeta_2}{x}\right).$$

*Proof.* By applying the logarithm application to both sides of (4.25), considering (2.1) together with (4.24) implies that (4.25) is equivalent to the following one:

$$\left( \log \left( \frac{q^{1/2} \xi_1}{x} \right) \right)^2 + \left( \log \left( \frac{q^{1/2} \xi_2}{x} \right) \right)^2 = \frac{1}{2} \left( \log \left( \frac{\xi_1}{\xi_2} \right) \right)^2 + \frac{1}{2} \left( \log \left( \frac{q \xi_1 \xi_2}{x^2} \right) \right)^2.$$

This can be obtained by direct computation, noticing that

$$(\log A)^2 + (\log B)^2 = \frac{1}{2} \left( \left( \log \left( \frac{A}{B} \right) \right)^2 + (\log(AB))^2 \right)$$

for all  $A, B \in \tilde{\mathbb{C}}^*$ . Thus, one finishes the proof of Lemma 4.8.  $\square$

*Proof of Theorem 4.1.* Through the whole proof, fix  $d \in \mathbb{R}$  such that  $\{a, b\} \cap (0, \infty e^{id}) = \emptyset$ , also assuming the convention stated in (4.2).

By gathering Lemmas 4.5 and 4.6 together with Remark 4.7, one finds that the product  $\hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)}$  belongs to the space  $\mathbb{C}\{x\}_{q;(1,2)}^d$ . For simplify, we will make use of the following notation:  $f^d(c; x) = \mathcal{S}_{q;1}^d \hat{E}_q^{(c,0)}(x)$  for  $c = a$  or  $b$ , and  $f^d(a, b; x) = \mathcal{S}_{q;(1,2)}^d \hat{E}_q^{(a,0)} \hat{E}_q^{(b,0)}(x)$ . In this way, the relation stated in (4.1) of Theorem 4.1, that is what we shall prove, can be expressed as follows:

$$(4.26) \quad f^d(a, b; x) = f^d(a; x) f^d(b; x).$$

One the one hand, by applying Definition 2.3 and Lemma 4.6, it follows that, for all  $x \in \tilde{\mathbb{C}}^*$ ,

$$(4.27) \quad f^d(a, b; x) = \int_0^{\infty e^{id}} \frac{\phi_1^d(a, b; \zeta)}{e_{q^{1/2}}(\zeta/x)} \frac{d\zeta}{\zeta},$$

where  $\phi_1^d(a, b; \zeta)$  is given as in (4.19). On the other hand, by letting  $(a, \alpha) = (a, 0)$  or  $(b, 0)$  in (3.2), one can write

$$(4.28) \quad f^d(a; x) f^d(b; x) = \int_0^{\infty e^{id}} \int_0^{\infty e^{id}} \frac{1}{(\xi_1 - a)(\xi_2 - b)} \frac{1}{e_q(\xi_1/x) e_q(\xi_2/x)} \frac{d\xi_1 d\xi_2}{\xi_1 \xi_2}$$

for all  $x \in \tilde{\mathbb{C}}^*$ .

Let  $\Phi : (\xi_1, \xi_2) \mapsto (\zeta_1, \zeta_2)$  be the homeomorphism from  $(0, \infty e^{id}) \times (0, \infty e^{id})$  onto  $(0, +\infty) \times (0, \infty e^{id})$  defined by (4.24). By noticing  $\xi_1 = \zeta_1 \zeta_2$  and  $\xi_2 = \frac{q^{-1/2} \zeta_2}{\zeta_1}$ , the corresponding Jacobian is as follows:

$$\left| \frac{\partial(\xi_1, \xi_2)}{\partial(\zeta_1, \zeta_2)} \right| = \left| \begin{array}{cc} \zeta_2 & \zeta_1 \\ -q^{-1/2} \zeta_2 / \zeta_1^2 & q^{-1/2} / \zeta_1 \end{array} \right| = 2 q^{-1/2} \frac{\zeta_2}{\zeta_1}.$$

Thus, one gets that

$$\frac{d\xi_1 d\xi_2}{\xi_1 \xi_2} = 2 q^{-1/2} \frac{\zeta_2}{\zeta_1} \times \frac{d\zeta_1 d\zeta_2}{\xi_1 \xi_2} = 2 \frac{d\zeta_1 d\zeta_2}{\zeta_1 \zeta_2},$$

what together with (4.25) allows us to transform (4.28) into the following form:

$$(4.29) \quad f^d(a; x) f^d(b; x) = \int_0^{\infty e^{id}} \int_0^{+\infty} \frac{\eta(\zeta_1, \zeta_2)}{e_{q^{1/2}}(\zeta_1) e_{q^{1/2}}(\zeta_2/x)} \frac{d\zeta_1 d\zeta_2}{\zeta_1 \zeta_2},$$

where

$$\eta(\zeta_1, \zeta_2) = \frac{\zeta_1}{(\zeta_1 \zeta_2 - a)(q^{-1/2} \zeta_2 - b \zeta_1)}.$$

Therefore, letting

$$(4.30) \quad H(\zeta_2) = \int_0^{+\infty} \frac{\eta(\zeta_1, \zeta_2)}{e_{q^{1/2}}(\zeta_1)} \frac{d\zeta_1}{\zeta_1},$$

the relation in (4.29) can be expressed as follows:

$$(4.31) \quad f^d(a; x) f^d(b; x) = \int_0^{\infty e^{id}} \frac{H_2(\zeta_2)}{e_{1/2}(\zeta_2/x)} \frac{d\zeta_2}{\zeta_2}.$$

Note that the following fractional decomposition with respect to  $\zeta_1$  holds:

$$(4.32) \quad \eta(\zeta_1, \zeta_2) = \frac{1}{q^{-1/2} \zeta_2^2 - ab} \left( \frac{a}{\zeta_2 \zeta_1 - a} + \frac{\zeta_2}{\zeta_2 - q^{1/2} b \zeta_1} \right).$$

Letting (4.32) into (4.30) yields that

$$(4.33) \quad H(\zeta_2) = \frac{1}{q^{-1/2} \zeta_2^2 - ab} (I_1(\zeta_2) + I_2(\zeta_2)),$$

where

$$(4.34) \quad I_1(\zeta_2) = a \int_0^{+\infty} \frac{1}{(\zeta_2 \zeta_1 - a) e_{q^{1/2}}(\zeta_1)} \frac{d\zeta_1}{\zeta_1}$$

and

$$(4.35) \quad I_2(\zeta_2) = \int_0^{+\infty} \frac{\zeta_2}{(\zeta_2 - q^{1/2} b \zeta_1) e_{q^{1/2}}(\zeta_1)} \frac{d\zeta_1}{\zeta_1}.$$

Putting  $\zeta_1 = \xi/\zeta_2$  into (4.34) implies that

$$(4.36) \quad I_1(\zeta_2) = a \int_0^{\infty e^{id}} \frac{1}{(\xi - a) e_{q^{1/2}}(\xi/\zeta_2)} \frac{d\xi}{\xi} = a \mathcal{S}_{q^{1/2}; 1}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta_2).$$

Besides, by noticing that

$$\frac{\zeta_2}{\zeta_2 - q^{1/2} b \zeta_1} = 1 + \frac{q^{1/2} b \zeta_1}{\zeta_2 - q^{1/2} b \zeta_1} = 1 + \frac{b}{q^{-1/2} \zeta_2 / \zeta_1 - b}$$

and that  $e_{q^{1/2}}(\zeta_2/(q^{1/2} \xi)) = e_{q^{1/2}}(\xi/\zeta_2)$ , letting  $\zeta_1 = \zeta_2/(q^{1/2} \xi)$  into (4.35) yields that

$$(4.37) \quad I_2(\zeta_2) = 1 + b \int_0^{\infty e^{id}} \frac{1}{(\xi - b) e_{q^{1/2}}(\xi/\zeta_2)} \frac{d\xi}{\xi} = 1 + b \mathcal{S}_{q^{1/2}; 1}^d(\hat{E}_{q^{1/2}}^{(b,0)})(\zeta_2).$$

Hence, it follows from combining (4.33) with (4.36) and (4.37) that

$$(4.38) \quad H_2(\zeta_2) = \frac{1}{q^{-1/2} \zeta_2^2 - ab} \left( 1 + a \mathcal{S}_{q^{1/2}; 1}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta_2) + b \mathcal{S}_{q^{1/2}; 1}^d(\hat{E}_{q^{1/2}}^{(b,0)})(\zeta_2) \right).$$

Considering both (4.19) and (4.38) implies that  $H_2 = \phi_1^d$ . Finally, comparing (4.27) and (4.31) allows us to finish the proof of Theorem 4.1.  $\square$

## 5. PROOF OF THEOREM 1.1

This section aims to proving Theorem 1.1, by using Theorem 4.1 together with Theorem 2.5. For doing that, we will establish a generalization of Theorem 4.1 for any product power series of the form  $\hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)}$ ; see Theorem 5.7.

**5.1. Differentiating with respect to the parameters in the first Borel-plane.** The binomial coefficients  $\binom{m}{k}$  are usually defined for non-negative integers  $m$  and  $k$ . In what follows, it will be convenient to make use of the following extension:

$$(5.1) \quad \binom{m}{k} = \begin{cases} 1 & \text{for } m \in \mathbb{Z}_{<0} \text{ and } k = 0, \\ 0 & \text{for } m \in \mathbb{Z}_{<0} \text{ and } k > 0. \end{cases}$$

**Lemma 5.1.** *Given any  $(a, b, \alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ , one has*

$$(5.2) \quad \hat{\mathcal{B}}_{q;1} \left( \hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)} \right) (\xi) = \frac{(-1)^{\alpha+\beta}}{a^{\alpha+1} b^{\beta+1}} \sum_{n \geq 0} C_n(a, b; \xi) q^{-n^2} \left( \frac{\xi^2}{ab} \right)^n,$$

where the sequence  $\{C_n(a, b; \xi)\}_{n \geq 0}$  is defined as follows:

$$(5.3) \quad C_n(a, b; \xi) = - \binom{n+\alpha}{\alpha} \binom{n+\beta}{\beta} + \binom{n+\beta}{\beta} \sum_{\ell=1}^{\alpha+1} \binom{\alpha-\ell+n}{\alpha-\ell+1} \left( \frac{a q^n}{a q^n - \xi} \right)^\ell + \binom{n+\alpha}{\alpha} \sum_{\ell=1}^{\beta+1} \binom{\beta-\ell+n}{\beta-\ell+1} \left( \frac{b q^n}{b q^n - \xi} \right)^\ell.$$

*Proof.* By taking into account Lemma 4.5 and the fact that the  $q$ -Borel transform commutes with the derivation for the parameters, considering (2.6) yields that

$$(5.4) \quad \hat{\mathcal{B}}_{q;1} \left( \hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)} \right) (\xi) = \frac{1}{\alpha! \beta!} \partial_{a,b}^{\alpha,\beta} \phi_0(a, b; \xi).$$

Moreover, it is easy to see that (4.12) can be put into the following form:

$$(5.5) \quad \phi_0(a, b; \xi) = \sum_{n \geq 0} \left( -\frac{1}{(ab)^{n+1}} + \frac{q^n}{a^n b^{n+1} (a q^n - \xi)} + \frac{q^n}{a^n b^{n+1} (b q^n - \xi)} \right) q^{-n^2} \xi^{2n}.$$

Since

$$\partial_{a,b}^{\alpha,\beta} \frac{1}{(ab)^{n+1}} = \frac{(-1)^{\alpha+\beta}}{(ab)^{n+1}} \frac{(n+\alpha)! (n+\beta)!}{n! a^\alpha n! b^\beta},$$

one finds that

$$(5.6) \quad \frac{1}{\alpha! \beta!} \partial_{a,b}^{\alpha,\beta} \frac{1}{(ab)^{n+1}} = \frac{(-1)^{\alpha+\beta}}{ab} \frac{\binom{n+\alpha}{\alpha} \binom{n+\beta}{\beta}}{a^\alpha b^\beta} \frac{1}{(ab)^n}.$$

Besides, by noticing that

$$\partial_{a,b}^{\alpha,\beta} \frac{1}{a^n b^{n+1} (a q^n - \xi)} = \frac{(-1)^\beta}{b^{n+1}} \frac{(n+\beta)!}{n! b^\beta} \partial_a^\alpha \frac{1}{a^n (a q^n - \xi)}$$



and

$$\partial_a^\alpha \frac{1}{a^n (a q^n - \xi)} = \frac{(-1)^\alpha}{a^n (a q^n - \xi)} \sum_{k=0}^{\alpha} \binom{\alpha}{k} \frac{(n+k-1)! (\alpha-k)! q^{(\alpha-k)n}}{(n-1)! a^k (a q^n - \xi)^{\alpha-k}},$$

one gets that

$$(5.7) \quad \frac{1}{\alpha! \beta!} \partial_{a,b}^{\alpha,\beta} \frac{1}{a^n b^{n+1} (a q^n - \xi)} = \frac{(-1)^{\alpha+\beta}}{a b} \frac{\binom{n+\beta}{\beta}}{b^\beta} \sum_{k=0}^{\alpha} \frac{\binom{n+k-1}{k} q^{(\alpha-k)n}}{a^{k-1} (a q^n - \xi)^{\alpha-k+1}} \frac{1}{(a b)^n}.$$

Letting  $\ell = \alpha - k + 1$  into the summation of the right-hand side of (5.7) yields that

$$(5.8) \quad \frac{1}{\alpha! \beta!} \partial_{a,b}^{\alpha,\beta} \frac{1}{a^n b^{n+1} (a q^n - \xi)} = \frac{(-1)^{\alpha+\beta}}{a b} \frac{\binom{n+\beta}{\beta}}{b^\beta} \sum_{\ell=1}^{\alpha+1} \frac{\binom{\alpha+n-\ell}{\alpha+1-\ell} q^{(\ell-1)n}}{a^{\alpha-\ell} (a q^n - \xi)^\ell} \frac{1}{(a b)^n}.$$

Similarly, one can find that

$$(5.9) \quad \frac{1}{\alpha! \beta!} \partial_{a,b}^{\alpha,\beta} \frac{1}{a^{n+1} b^n (b q^n - \xi)} = \frac{(-1)^{\alpha+\beta}}{a b} \frac{\binom{n+\alpha}{\alpha}}{a^\alpha} \sum_{\ell=1}^{\beta+1} \frac{\binom{\beta+n-\ell}{\beta+1-\ell} q^{(\ell-1)n}}{b^{\beta-\ell} (b q^n - \xi)^\ell} \frac{1}{(a b)^n}.$$

By gathering (5.6), (5.8) and (5.9), considering both (5.4) and (5.5) implies (5.2).  $\square$

In what follows, we will write

$$(5.10) \quad \phi_0^{\alpha,\beta}(a, b; \xi) = \hat{\mathcal{B}}_{q;1} \left( \hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)} \right) (\xi).$$

**Proposition 5.2.** *Let  $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$ ,  $(\alpha, \beta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ , and let  $d \in \mathbb{R}$  be such that  $\{a, b\} \cap (0, \infty e^{id}) = \emptyset$ . One has  $\phi_0^{\alpha,\beta}(a, b; \xi) \in \mathbb{H}_{q;2}^d$ .*

*Proof.* Both  $\hat{E}_q^{(a,\alpha)}$  and  $\hat{E}_q^{(b,\beta)}$  belonging to the  $q$ -Gevrey space  $\mathbb{C}[[x]]_{q;1}$ , the  $q$ -Borel transform  $\phi_0^{\alpha,\beta}(a, b; \xi)$  given in (5.10) represents a germ of analytic function at the origin of the complex plane. In order to prove that  $\phi_0^{\alpha,\beta}(a, b; \xi) \in \mathbb{H}_{q;2}^d$ , let  $\epsilon > 0$ , suppose  $V = V_\epsilon^d$  be such that  $\{a, b\} \cap V = \emptyset$ , and consider the positive constants  $|a|_V$  and  $|b|_V$  defined in (4.16). Set

$$c = \max(|a|, |b|), \quad \gamma = \max(\alpha, \beta), \quad \delta = \min(|a|_V, |b|_V).$$

By using  $\frac{z q^n}{z q^n - \xi} = \frac{z}{z - q^{-n} \xi}$  for  $z = a$  or  $b$ , one can notice that

$$(5.11) \quad \max \left( \sup_{\xi \in V} \left| \frac{a q^n}{a q^n - \xi} \right|, \sup_{\xi \in V} \left| \frac{b q^n}{b q^n - \xi} \right| \right) \leq \frac{c}{\delta}$$

for all integer  $n$ .

In what follows, we will suppose that  $\gamma > 0$ , the case of  $\gamma = 0$  being already known with  $\alpha = \beta = 0$ . As  $\binom{n}{k} \leq n^k$  for any non-negative integers  $n$  and  $k$ , one deduces from combining (5.3) and (5.11) that for any  $\xi \in V$ ,

$$|C_n(a, b; \xi)| \leq (n + \gamma)^{2\gamma} + 2(n + \gamma)^{2\gamma} \sum_{\ell=1}^{\gamma+1} \left( \frac{c}{\delta} \right)^\ell.$$

This together with the expression given in (5.2) implies that there exists  $C > 0$  such that

$$(5.12) \quad \left| \phi_0^{\alpha,\beta}(a, b; \xi) \right| \leq C \sum_{n \geq 0} (n + \gamma)^{2\gamma} q^{-n^2} \left( \frac{|\xi|^2}{|a||b|} \right)^n$$

for all  $\xi \in V$ . So, by comparing (5.12) with (4.17), considering the same argument as what used at the end of the proof of Lemma 4.5 implies that  $\phi_0^{\alpha,\beta}(a, b; \xi) \in \mathbb{H}_{q;2}^d$ .  $\square$

**5.2. Fractional decomposition around a pole in the second Borel-plane.** Given  $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$  and  $(\alpha, \beta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ , we define  $\{P_j^{\alpha,\beta}\}_{0 \leq j \leq \alpha+\beta} \subset \mathbb{C}[X, Y]$  as follows:

$$(5.13) \quad \partial_{a,b}^{\alpha,\beta} \frac{1}{q^{-1/2}\zeta^2 - ab} = \frac{1}{q^{-1/2}\zeta^2 - ab} \sum_{j=0}^{\alpha+\beta} \frac{P_j^{\alpha,\beta}(a, b)}{(q^{-1/2}\zeta^2 - ab)^j}.$$

It will be convenient to write  $P_j^{\alpha,\beta} = 0$  when  $j > \alpha + \beta$  or  $j < 0$ . A direct computation shows that  $P_0^{0,0} = 1$ ,  $P_j^{0,0} = 0$  for  $j > 0$ , and  $P_0^{\alpha,\beta} = 0$  for  $\alpha + \beta > 0$ ; moreover,

$$P_j^{\alpha+1,\beta}(a, b) = \partial_a P_j^{\alpha,\beta}(a, b) + j b P_{j-1}^{\alpha,\beta}(a, b), \quad P_j^{\alpha,\beta+1}(a, b) = \partial_b P_j^{\alpha,\beta}(a, b) + j a P_{j-1}^{\alpha,\beta}(a, b).$$

**Lemma 5.3.** *Let  $\phi_1^d(a, b; \zeta)$  be as in Lemma 4.6. One has*

$$(5.14) \quad \partial_{a,b}^{\alpha,\beta} \phi_1^d(a, b; \zeta) = \frac{1}{q^{-1/2}\zeta^2 - ab} \sum_{j=0}^{\alpha+\beta} \frac{F_j(\zeta)}{(q^{-1/2}\zeta^2 - ab)^j},$$

where

$$(5.15) \quad \begin{aligned} F_j(\zeta) &= P_j^{\alpha,\beta}(a, b) + \sum_{k=j}^{\alpha} \frac{\alpha!}{k!} \left( k P_j^{k-1,\beta}(a, b) + a P_j^{k,\beta}(a, b) \right) \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,\alpha-k)})(\zeta) \\ &+ \sum_{k=j}^{\beta} \frac{\beta!}{k!} \left( k P_j^{\alpha,k-1}(a, b) + b P_j^{\alpha,k}(a, b) \right) \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,\beta-k)})(\zeta). \end{aligned}$$

*Proof.* By (4.19), it follows that

$$(5.16) \quad \phi_1^d(a, b; \zeta) = \frac{1}{q^{-1/2}\zeta^2 - ab} \left( 1 + a \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta) + b \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,0)})(\zeta) \right).$$

We will consider the expression of  $\phi_1^d(a, b; \zeta)$  given in (5.16), taking the differential  $\partial_{a,b}^{\alpha,\beta}$  of the members of its right-hand side. Since

$$\partial_{a,b}^{\alpha,\beta} \frac{a \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta)}{q^{-1/2}\zeta^2 - ab} = \alpha! \partial_b^\beta \sum_{k=0}^{\alpha} \frac{1}{k!} \partial_a^k \left( \frac{a}{q^{-1/2}\zeta^2 - ab} \right) \frac{1}{(\alpha - k)!} \partial_a^{\alpha-k} \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta),$$

using (3.3) gives that

$$(5.17) \quad \partial_{a,b}^{\alpha,\beta} \frac{a \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,0)})(\zeta)}{q^{-1/2}\zeta^2 - ab} = \sum_{k=0}^{\alpha} \frac{\alpha!}{k!} \partial_{a,b}^{k,\beta} \left( \frac{a}{q^{-1/2}\zeta^2 - ab} \right) \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,\alpha-k)})(\zeta).$$

Similarly, one has

$$(5.18) \quad \partial_{a,b}^{\alpha,\beta} \frac{b \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,0)})(\zeta)}{q^{-1/2}\zeta^2 - ab} = \sum_{k=0}^{\beta} \frac{\beta!}{k!} \partial_{a,b}^{\alpha,k} \left( \frac{b}{q^{-1/2}\zeta^2 - ab} \right) \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,\beta-k)})(\zeta).$$

In view of (5.17) and (5.18), taking the differential  $\partial_{a,b}^{\alpha,\beta}$  on both sides of (5.16) yields that

$$(5.19) \quad \begin{aligned} \partial_{a,b}^{\alpha,\beta} \phi_1^d(a, b; \zeta) &= \partial_{a,b}^{\alpha,\beta} \frac{1}{q^{-1/2}\zeta^2 - ab} + \sum_{k=0}^{\alpha} \frac{\alpha!}{k!} \partial_{a,b}^{k,\beta} \left( \frac{a}{q^{-1/2}\zeta^2 - ab} \right) \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,\alpha-k)})(\zeta) \\ &+ \sum_{k=0}^{\beta} \frac{\beta!}{k!} \partial_{a,b}^{\alpha,k} \left( \frac{b}{q^{-1/2}\zeta^2 - ab} \right) \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,\beta-k)})(\zeta). \end{aligned}$$

Since

$$\partial_{a,b}^{k,\beta} \frac{a}{q^{-1/2}\zeta^2 - ab} = k \partial_{a,b}^{k-1,\beta} \frac{1}{q^{-1/2}\zeta^2 - ab} + a \partial_{a,b}^{k,\beta} \frac{1}{q^{-1/2}\zeta^2 - ab},$$

replacing  $(\alpha, \beta)$  with  $(k, \beta)$  or  $(k-1, \beta)$  in (5.13) yields

$$(5.20) \quad \partial_{a,b}^{k,\beta} \frac{a}{q^{-1/2}\zeta^2 - ab} = \frac{1}{q^{-1/2}\zeta^2 - ab} \sum_{j=0}^{k+\beta} \frac{k P_j^{k-1,\beta}(a, b) + a P_j^{k,\beta}(a, b)}{(q^{-1/2}\zeta^2 - ab)^j}.$$

Similarly, one has

$$(5.21) \quad \partial_{a,b}^{\alpha,k} \frac{b}{q^{-1/2}\zeta^2 - ab} = \frac{1}{q^{-1/2}\zeta^2 - ab} \sum_{j=0}^{\alpha+k} \frac{b P_j^{\alpha,k}(a, b) + k P_j^{\alpha,k-1}(a, b)}{(q^{-1/2}\zeta^2 - ab)^j}.$$

Therefore, putting (5.13), (5.20) and (5.21) into (5.19) yields (5.14).  $\square$

As before, let  $(a, b, d) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{R}$  be such that  $\{a, b\} \cap (0, \infty e^{id}) = \emptyset$ . By following Proposition 5.2, if

$$(5.22) \quad \phi_1^{d;\alpha,\beta}(a, b; \zeta) = \int_0^{\infty e^{id}} \frac{\phi_0^{\alpha,\beta}(a, b; \xi)}{e_{q^{1/2}}(\xi/\zeta)} \frac{d\xi}{\xi},$$

then  $\phi_1^{d;\alpha,\beta}(a, b; \zeta)$  represents an analytic function for all  $\zeta \in \tilde{\mathbb{C}}^*$  such that  $|\zeta| < q^{1/4} \sqrt{|ab|}$ .

**Theorem 5.4.** *Let  $\phi_1^{d;\alpha,\beta}(a, b; \zeta)$  as in (5.22),  $d' \in \mathbb{R}$ , and consider the set  $\Lambda_{a,b}^d$  given in (4.18). The following assertions hold.*

(1) *If  $(0, \infty e^{id'}) \cap \Lambda_{a,b}^d = \emptyset$ , there exist  $\epsilon > 0$  and  $C > 0$  such that*

$$(5.23) \quad \forall \zeta \in V_\epsilon^{d'}, \quad \left| \phi_1^{d;\alpha,\beta}(a, b; \zeta) \right| \leq \frac{C}{|\zeta^2 - \sqrt{q}ab|}.$$

(2) *Otherwise, one has  $(0, \infty e^{id'}) \cap \Lambda_{a,b}^d = \{c\}$  and there exist a finite set  $\{\lambda_0, \dots, \lambda_{\alpha+\beta}\} \subset \mathbb{C}$  and an analytic function  $\psi$  in  $V_{\pi/2}^{d'}$  possessing the following properties.*

(a) For any  $\zeta \in V_{\pi/2}^{d'} \setminus \{c\}$ , one has

$$(5.24) \quad \phi_1^{d;\alpha,\beta}(a, b; \zeta) = \sum_{j=0}^{\alpha+\beta} \frac{\lambda_j}{(\zeta - c)^{j+1}} + \psi(\zeta).$$

(b) Given any  $\epsilon \in (0, \frac{\pi}{2})$ ,  $\psi$  remains bounded in  $V_\epsilon^{d'}$  and one has  $\psi(\zeta) = O(\frac{1}{\zeta})$  as  $\zeta \rightarrow \infty$  in  $V_\epsilon^{d'}$ .

*Proof.* Since  $\phi_0^{\alpha,\beta}(a, b; \xi) = \partial_{a,b}^{\alpha,\beta} \phi_0(a, b; \xi)$ , equation in (5.22) takes the following form:

$$\phi_1^{d;\alpha,\beta}(a, b; \zeta) = \int_0^{\infty \epsilon^{id}} \frac{\partial_{a,b}^{\alpha,\beta} \phi_0(a, b; \xi) d\xi}{e_{q^{1/2}}(\xi/\zeta) \xi}.$$

This implies that

$$(5.25) \quad \phi_1^{d;\alpha,\beta}(a, b; \zeta) = \partial_{a,b}^{\alpha,\beta} \phi_1^d(a, b; \zeta),$$

what allows one to make use of the expression given in (5.14) for  $\partial_{a,b}^{\alpha,\beta} \phi_1^d(a, b; \zeta)$ . Furthermore, considering the relation stated in (3.6) implies that everyone of the members of both  $Gq$ -sums families  $\left\{ \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(a,k)}) \right\}_{0 \leq k \leq \alpha}$  and  $\left\{ \mathcal{S}_{q;2}^d(\hat{E}_{q^{1/2}}^{(b,k)}) \right\}_{0 \leq k \leq \beta}$  is bounded over any sector  $V_\epsilon^{d'}$  with  $\epsilon > 0$ . Thus, one obtains from (5.15) that there exists a constant  $C_1 > 0$ , depending of  $a, b, \alpha, \beta$  and  $V_\epsilon^{d'}$ , such that

$$(5.26) \quad \forall \zeta \in V_\epsilon^{d'}, \quad \max_{0 \leq j \leq \alpha+\beta} |F_j(\zeta)| \leq C_1.$$

(1) If  $(0, \infty e^{id'}) \cap \Lambda_{a,b}^d = \emptyset$ , one has  $q^{-1/2} \zeta^2 - ab \neq 0$  for all  $\zeta \in (0, \infty e^{id'})$ . Let  $\epsilon > 0$  be enough small in such way that  $q^{-1/2} \zeta^2 - ab \neq 0$  for all  $\zeta \in V_{2\epsilon}^{d'}$ , and define  $\delta = \inf_{\zeta \in V_\epsilon^{d'}} |\zeta^2 - \sqrt{q} ab|$ . By letting  $C = \sqrt{q} (1 + \delta + \dots + \delta^{\alpha+\beta}) C_1$  and taking into account (5.26), one deduces (5.23) from (5.14).

(2) By hypothesis,  $c$  is the only root of  $q^{-1/2} \zeta^2 - ab = 0$  in the half-line  $(0, \infty e^{id'})$ . One the one hand, gathering (5.26) together with (5.14) implies that, for any  $\epsilon > 0$ , one has

$$(5.27) \quad \phi_1^{d;\alpha,\beta}(a, b; \zeta) = O\left(\frac{1}{\zeta^2}\right), \quad V_\epsilon^{d'} \ni \zeta \rightarrow \infty.$$

One the other hand, letting  $q^{-1/2} \zeta^2 - ab = q^{-1/2} (\zeta - c)(\zeta + c)$  into (5.14) and considering the Taylor expansion of each function  $F_j(\zeta)$  at  $\zeta = c$  yields the fractional decomposition as (5.24), where  $\psi$  remains analytic and bounded in every relatively-compact subset of  $V_{\pi/2}^{d'}$ . Note that  $\sum_{j=0}^{\alpha+\beta} \frac{\lambda_j}{(\zeta - c)^{j+1}} = O\left(\frac{1}{\zeta}\right)$  as  $\zeta \rightarrow \infty$ . With the help of (5.27), one finds that  $\psi = O\left(\frac{1}{\zeta}\right)$  as  $\zeta \rightarrow \infty$ . This finishes the proof of Theorem 5.5.  $\square$

The expression given in (5.14) for the function  $\phi_1^{\alpha,\beta}(a, b; \zeta)$  shows that it may have a singularity at any square-root of  $\sqrt{q} ab$  in  $\tilde{\mathbb{C}}^*$ . The following result says that it remains well-defined and analytic at  $\zeta = q^{1/4} \sqrt{a} b e^{-\pi i}$ . This generalizes Lemma 4.6 (1) in which one had  $\alpha = \beta = 0$ .

**Remark 5.5.** Let  $(a, b, d) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{R}$  to be such that (4.2) holds. Given any couple  $(\alpha, \beta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ , the associated function  $\phi_1^{d; \alpha, \beta}(a, b; \zeta)$  remains analytic at  $\zeta = q^{1/4} \sqrt{ab} e^{-\pi i}$ .

This will be deduced from Lemma 5.6 ; for more details, see the end of the proof of Theorem 5.7 given in §5.3 below.

**5.3. One generalization of Theorem 4.1.** In order to generalize Theorem 4.1 for  $\hat{E}_q^{(a, \alpha)} \hat{E}_q^{(b, \beta)}$ , we shall establish the following result.

**Lemma 5.6.** Let  $\delta \in \mathbb{R}$ ,  $N \in \mathbb{Z}_{> 0}$ ,  $\{c_1, \dots, c_N\} \subset (0, \infty e^\delta)$ ,  $\{\lambda_1, \dots, \lambda_N\} \subset \mathbb{C}$ ,  $\{\mu_1, \dots, \mu_N\} \subset \mathbb{Z}_{\geq 0}$ , and let  $\psi \in \tilde{\mathbb{H}}_{q;1}^\delta$ . Set

$$(5.28) \quad \phi(\xi) = \sum_{j=1}^N \frac{\lambda_j}{(\xi - c_j)^{\mu_j+1}} + \psi(\xi).$$

If  $(c_j, \mu_j) \neq (c_k, \mu_k)$  for all  $(j, k)$  such that  $j \neq k$ , then the following conditions are equivalent.

- (1)  $\lambda_j = 0$  for  $1 \leq j \leq N$ .
- (2)  $\phi \in \tilde{\mathbb{H}}_{q;1}^\delta$ .
- (3) There exist  $\delta^- < \delta < \delta^+$  such that  $\phi \in \left( \tilde{\mathbb{H}}_{q;1}^{\delta^-} \cap \tilde{\mathbb{H}}_{q;1}^{\delta^+} \right)$  and, moreover, the identity  $\mathcal{L}_{q;1}^{\delta^-} \phi = \mathcal{L}_{q;1}^{\delta^+} \phi$  holds.

*Proof.* The equivalence (1)  $\Leftrightarrow$  (2) and implication (2)  $\Rightarrow$  (3) being trivial, it suffices to prove (3)  $\Rightarrow$  (2) or (3)  $\Rightarrow$  (1). This can be done by applying Residue Theorem to the contour integral  $\left( \mathcal{L}_{q;1}^{\delta^-} \phi - \mathcal{L}_{q;1}^{\delta^+} \phi \right)$ . Indeed, one deduces from considering (5.28) that

$$(5.29) \quad \mathcal{L}_{q;1}^{\delta^-} \phi(x) - \mathcal{L}_{q;1}^{\delta^+} \phi(x) = 2\pi i \sum_{j=0}^N \frac{\lambda_j}{\mu_j!} \partial_\xi^{\mu_j} \left( \frac{1}{\xi e_q(\xi/x)} \right) \Big|_{\xi=c_j}.$$

Since  $\xi e_q\left(\frac{\xi}{x}\right) = \frac{x}{q} e_q\left(\frac{q\xi}{x}\right)$ , one can observe that

$$\frac{1}{\xi e_q(\xi/x)} = \frac{q}{x \sqrt{2\pi \ln q}} e^{P(\xi)}, \quad \text{where } P(\xi) = \frac{(\log(q^{3/2} \xi/x))^2}{2 \ln q}.$$

If one defines the sequence  $\{P_k\}$  by the relation

$$(5.30) \quad \partial_\xi^k \left( \frac{1}{\xi e_q(\xi/x)} \right) = \frac{P_k(\xi)}{\xi e_q(\xi/x)},$$

then  $P_0 = 1$  and  $P_{k+1} = P' P_k + P'_k$  for all  $k \geq 0$ . One can find that

$$(5.31) \quad P_k(\xi) = \frac{Q_k(\log(q^{3/2} \xi/x))}{(2(\ln q) \xi)^k}, \quad Q_k \in \mathbb{C}[X], \quad \deg Q_k = k.$$

Thus, the relation in (5.29) yields that

$$(5.32) \quad \mathcal{L}_{q;1}^{\delta^-} \phi(x) - \mathcal{L}_{q;1}^{\delta^+} \phi(x) = 2\pi i \sum_{j=0}^N \frac{\lambda_j Q_{\mu_j}(\log(q^{3/2} c_j/x))}{\mu_j! (2(\ln q))^{\mu_j} c_j^{\mu_j+1} e_q(c_j/x)}.$$

Assume  $x \rightarrow 0$ , and consider the different asymptotic scales contained in the right-hand side of (5.32). One can easily obtain that the left-hand side is null if and only if  $\lambda_j = 0$  for all  $j$  from 1 to  $N$ . This means that (3)  $\Rightarrow$  (1).  $\square$

**Theorem 5.7.** *Let  $(a, b, d) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{R}$  be as in Theorem 4.1, and let  $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ . One has  $\hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)} \in \mathbb{C}\{x\}_{q;(1,2)}^d$  and*

$$(5.33) \quad \mathcal{S}_{q;(1,2)}^d(\hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)}) = \mathcal{S}_{q;1}^d(\hat{E}_q^{(a,\alpha)}) \mathcal{S}_{q;1}^d(\hat{E}_q^{(b,\beta)}).$$

*Proof.* Let  $d' \in \mathbb{R}$  be as in Theorem 5.4 (1), and define

$$(5.34) \quad f^{d',d;\alpha,\beta}(a, b; x) = \int_0^{\infty e^{id'}} \frac{\phi_1^{d;\alpha,\beta}(a, b; \zeta)}{e_{q^{1/2}}(\zeta/x)} \frac{d\zeta}{\zeta}$$

and

$$(5.35) \quad f^{d',d}(a, b; x) = \int_0^{\infty e^{id'}} \frac{\phi_1^d(a, b; \zeta)}{e_{q^{1/2}}(\zeta/x)} \frac{d\zeta}{\zeta}$$

for all  $x \in \tilde{\mathbb{C}}^*$ . By taking into account both (5.23) and (5.25), one gets that

$$(5.36) \quad f^{d',d;\alpha,\beta}(a, b; \zeta) = \partial_{a,b}^{\alpha,\beta} f^{d',d}(a, b; x).$$

Let  $\bar{d}_{a,b} = \frac{1}{2}(\arg(a) + \arg(b))$ . In view of (4.2), one has  $\bar{d}_{a,b} \in (d, 2\pi)$ . One remembers that  $f^d(a, b; x)$  was defined by means of the ingegral of (4.27). By following Lemma 4.6, the function  $\phi_1^d(a, b; \zeta)$  is analytic in the sector  $V_{\pi/2}^{\bar{d}_{a,b}}$ . Thus, comparing (4.27) with (5.36) implies that

$$(5.37) \quad f^{d',d}(a, b; x) = f^d(a, b; x)$$

for any  $d' \in (\bar{d}_{a,b} - \frac{\pi}{2}, \bar{d}_{a,b} + \frac{\pi}{2})$ . Besides, combining (5.37) with (5.35) yields that

$$(5.38) \quad f^{d',d;\alpha,\beta}(a, b; x) = \partial_{a,b}^{\alpha,\beta} f^d(a, b; x)$$

for any  $d' \in (\bar{d}_{a,b} - \frac{\pi}{2}, \bar{d}_{a,b}) \cup (\bar{d}_{a,b}, \bar{d}_{a,b} + \frac{\pi}{2})$ . Thus, applying Lemma 5.6 to the triplet  $(q^{1/2}, \phi_1^{d;\alpha,\beta}(a, b; \zeta), \bar{d}_{a,b})$  instead of  $(q, \phi, \delta)$  implies that  $\phi_1^{d;\alpha,\beta}(a, b; \zeta) \in \tilde{\mathbb{H}}_{q;2}^{\bar{d}_{a,b}}$ . One gets that  $\phi_1^{d;\alpha,\beta}(a, b; \zeta)$  is analytic at  $\zeta = q^{1/4} \sqrt{ab} e^{-\pi i}$ , as claimed by Remark 5.5. As by-product, it follows from combining this with Theorem 5.4 that  $\hat{E}_q^{(a,\alpha)} \hat{E}_q^{(b,\beta)} \in \mathbb{C}\{x\}_{q;(1,2)}^d$ . At the same time, letting  $d' = d$  into (5.36) and using (4.26) yields (5.33), what permits us to achieve the proof of Theorem 5.7.  $\square$

**5.4. End of the proof of Theorem 1.1.** Let  $\hat{f}_1$  and  $\hat{f}_2$  be two generic  $q$ -Gevrey series. In view of the notational convention given in (3.7), one knows that  $\hat{E}_q^{(a,-1)} = 1$  for any  $a \in \mathbb{C}^*$ . Thus, by applying (2.8) to  $\hat{f}_1$  and  $\hat{f}_2$ , one can find  $m, n \in \mathbb{Z}_{\geq 0}$ ,  $g_0, \dots, g_m, h_0, \dots, h_n \in \mathbb{C}\{x\}$ ,  $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{C}^*$  and  $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in \mathbb{Z}_{\geq 0}$  such that

$$\hat{f}_1 = \sum_{j=0}^m g_j \hat{E}_q^{(a_j, \alpha_j)}, \quad \hat{f}_2 = \sum_{k=0}^n h_k \hat{E}_q^{(b_k, \beta_k)},$$

where  $\alpha_0 = \beta_0 = -1$  and where  $a_0$  and  $b_0$  may be arbitrarily chosen in  $\mathbb{C}^*$ . This implies that

$$(5.39) \quad \hat{f}_1 \hat{f}_2 = \sum_{j=0}^m \sum_{k=0}^n g_j h_k \hat{E}_q^{(a_j, \alpha_j)} \hat{E}_q^{(b_k, \beta_k)}.$$

Set  $\mathbb{S} = \{\arg(a_j) \in [0, 2\pi) : 1 \leq j \leq m\} \cup \{\arg(b_k) \in [0, 2\pi) : 1 \leq k \leq n\}$ . By taking into account the fact that  $\mathbb{C}\{x\}_{q;(1,2)}^d$  constitutes a  $\mathbb{C}\{x\}$ -module (see Propositions 2.2 & 2.4), applying Theorem 5.7 to each term of the right-hand side of (5.39) yields that  $\hat{f}_1 \hat{f}_2$  is  $Gq$ -summable of order  $(1, 2)$  in any direction  $d \notin \mathbb{S} \bmod 2\pi$ . Finally, gathering (5.39) together with (5.33) allows one to obtain the equality expected in (1.4), with  $\mathbb{S} = \{\delta_1, \dots, \delta_M\}$ . In this way, we finish the proof of Theorem 1.1.

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