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# THE SUPPORT FUNCTION OF THE HIGH-DIMENSIONAL POISSON POLYTOPE

PIERRE CALKA AND BENJAMIN DADOUN

ABSTRACT. Let  $K_\lambda^d$  be the convex hull of the intersection of the homogeneous Poisson point process of intensity  $\lambda$  in  $\mathbb{R}^d$ ,  $d \geq 2$ , with the Euclidean unit ball  $\mathbb{B}^d$ . In this paper, we study the asymptotic behavior as  $d \rightarrow \infty$  of the support function

$$h_\lambda^{(d)}(u) := \sup_{x \in K_\lambda^d} \langle u, x \rangle$$

in an arbitrary direction  $u \in \mathbb{S}^{d-1}$  of the Poisson polytope  $K_\lambda^d$ . We identify three different regimes (subcritical, critical, and supercritical) in terms of the intensity  $\lambda := \lambda(d)$  and derive in each regime the precise distributional convergence of  $h_\lambda^{(d)}$  after suitable scaling. We especially treat this question where the support function is considered over multiple directions at once. We finally deduce partial counterparts for the radius-vector function of the polytope.

KEYWORDS. Support function; random convex polytopes; phase transition in high dimension; covering of the sphere; special functions; extreme value distribution.

## 1. INTRODUCTION

The study of high-dimensional polytopes has attracted recent attention in stochastic geometry [6, 22, 28, 21]. This is due to several factors. First of all, several classical models of random polytopes can be defined in any dimension and as such, they provide natural examples which might confirm or deny several of the famous conjectures from high-dimensional convex geometry. For example, the work [23] deals with Poisson polytopes generated by random hyperplane tessellations in the context of the hyperplane conjecture which claims that any  $d$ -dimensional convex body with volume 1 should have a section by a hyperplane whose  $(d-1)$ -dimensional volume is bounded from below by a constant independent on the dimension  $d$ . Similarly, the Hirsch conjecture which asserts that the edge-vertex graph of the boundary of a  $d$ -dimensional convex polytope with  $n$  facets should have a graph-diameter at most  $n - d$  was recently denied [33] but since then, bounds in the form of polynomial functions of  $n$  and  $d$  have been investigated, in particular in the context of so-called random spherical polytopes [8]. The second motivation behind the study of high-dimensional polytopes has an information theory background in the context of Shannon's channel coding theory and of the one-bit compression source coding. This involves in particular studying high-dimensional cells from the Poisson hyperplane tessellation for different regimes connecting the intensity with the dimension [1, 29]. Lastly, another context in which high-dimensional random polytopes play a central role is percolation in high dimension. Since Gordon's breakthrough result on the equivalent of the critical probability for the percolation on  $\mathbb{Z}^d$  when  $d \rightarrow \infty$  [19], the high-dimensional setting is expected

to possibly help estimating the phase transition. In particular, Poisson-Voronoi percolation along Voronoi cells generated by homogeneous Poisson points has attracted interest [2, 14].

In this paper, we consider the Poisson polytope  $K_\lambda^d$  obtained as the convex hull of a Poisson point process with intensity  $\lambda := \lambda(d)$  sampled in the Euclidean,  $d$ -dimensional unit ball  $\mathbb{B}^d$ . Together with its binomial variant obtained when replacing the Poisson point process with a fixed number  $n$  of i.i.d. variables uniformly distributed in  $\mathbb{B}^d$ , this model of random polytope has proved to be one of the most intensively studied in the literature. In particular, several non-asymptotic results are known, including Wendel's calculation [38] of the probability that the origin belongs to the random polytope, Efron and Buchta's mean-value identities [17, 10] and the recent work due to Kabluchko [26, 27] which provides an explicit formula for the expectation of the  $f$ -vector constituted with the number of  $k$ -dimensional facets of  $K_\lambda^d$ . The asymptotic study of  $K_\lambda^d$ , when the dimension is fixed and the intensity or the number of input points is large, dates back to the seminal work due to Rényi and Sulanke in dimension two [31, 32] and has been then carried through many subsequent works which have made explicit limit expectations for several functionals [35, 36, 30], variance bounds and limit variances [4, 12] and functional limit theorems for the support function and radius-vector function [11].

In the continuation of the early work of Bárány and Füredi [3], who investigated the phase transition for the probability of having all sampled points in convex position, the high-dimensional study of  $K_\lambda^d$  and variants has been continued over the last decades [16, 18, 6, 13, 7], with more emphasis on the threshold for the emergence of significant volumes. Notably, in [7], Bonnet, Kabluchko and Turchi estimated the asymptotics for different intensity regimes of the mean volume of a more general polytope called the  $\beta$ -polytope which includes the case of  $K_\lambda^d$  (save for the fact that they consider a binomial version of it). In particular, they exhibit a critical phase when the logarithm of the number of input points is comparable to  $\frac{d}{2} \log d$ , i.e., they show that the expected normalized volume of the polytope vanishes when the number of input points increases slower than  $d^{d/2}$ , and persists when this number increases faster than  $d^{d/2}$ . At the critical threshold, when the number grows like  $(d/2x)^{d/2}$ , they prove a convergence to  $e^{-x}$ .

Associated with the (random) polytope  $K_\lambda^d$  are the random processes  $h_\lambda^{(d)}$  and  $\rho_\lambda^{(d)}$ , respectively called the *support function* and the *radius-vector function*, given for any direction  $u$  in the unit sphere  $\mathbb{S}^{d-1}$  by

$$h_\lambda^{(d)}(u) := \sup \left\{ \langle u, x \rangle : x \in K_\lambda^d \right\}, \quad (1)$$

and

$$\rho_\lambda^{(d)}(u) := \sup \left\{ t > 0 : tu \in K_\lambda^d \right\}. \quad (2)$$

The support function of  $K_\lambda^d$  corresponds to the radius-vector function of its Voronoi flower, see [Figure 1](#). Even though  $h_\lambda^{(d)}$  and  $\rho_\lambda^{(d)}$  are one-dimensional statistics, they are known to fully characterize the convex body [34, Theorem 1.7.1], so understanding  $h_\lambda^{(d)}$  or  $\rho_\lambda^{(d)}$  as  $d \rightarrow \infty$  already sheds significant light on the asymptotic geometry of  $K_\lambda^d$ . To the best of our knowledge, the study of these functionals in high dimension has so far been largely ignored. It appears however that the analysis of the radius-vector function is more delicate and we mostly focus on the support function in this work.

The distribution of the random variables  $h_\lambda^{(d)}(u)$  and  $\rho_\lambda^{(d)}(u)$  does not depend on  $u$  by rotational invariance; we let  $h_\lambda^{(d)} := h_\lambda^{(d)}(u)$  and  $\rho_\lambda^{(d)} := \rho_\lambda^{(d)}(u)$ , and consider that  $u$  is either fixed in  $\mathbb{S}^{d-1}$  or

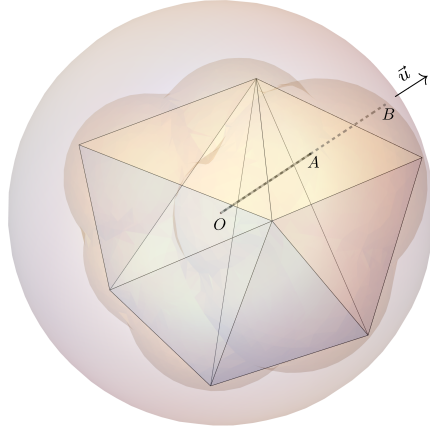


FIGURE 1. The support function  $h_\lambda^{(d)}(u) := OB$  and the radius-vector function  $\rho_\lambda^{(d)}(u) := OA$  of the polytope  $K_\lambda^d$  in dimension  $d = 3$ . The Voronoi flower is the union of the closed balls  $(\frac{x}{2} + \frac{|x|}{2}\mathbb{B}^d)$  where  $x$  runs over the vertices of  $K_\lambda^d$ .

chosen uniformly at random in  $\mathbb{S}^{d-1}$  (independently of the Poisson process). Given the following technical assumption on  $\lambda$ ,

$$\liminf_{d \rightarrow \infty} \frac{\lambda \kappa_d}{d} > 2, \quad (\text{H})$$

where  $\kappa_d := |\mathbb{B}^d|$  is the  $d$ -dimensional volume of the unit ball, the random variables  $h_\lambda^{(d)}$  and  $\rho_\lambda^{(d)}$  will take values in  $[0, 1]$  with high probability as  $d \rightarrow \infty$  (see [Lemma 2.1](#)). This hypothesis means that  $\lambda \kappa_d$ , i.e., the mean number of points in  $\mathbb{B}^d$  of the Poisson process, is asymptotically greater than twice the dimension  $d$ .

Our first result identifies the *subcritical*, *critical*, and *supercritical regimes* where  $h_\lambda^{(d)}$  tends either to 0, to a value in  $(0, 1)$ , or to 1 as  $d \rightarrow \infty$ , depending on whether  $\log \lambda \kappa_d$  grows slower than, comparably to, or faster than the dimension  $d$ .

**Theorem 1** (Asymptotic regimes of  $h_\lambda^{(d)}$ ). *Under [Assumption \(H\)](#), suppose that*

$$x := \lim_{d \rightarrow \infty} \frac{1}{d} \log \lambda \kappa_d$$

*exists in  $[0, \infty]$ . Then the following holds in probability as  $d \rightarrow \infty$ :*

$$\left\{ \begin{array}{ll} h_\lambda^{(d)} \sim \sqrt{\frac{2}{d} \log \lambda \kappa_d}, & \text{if } x = 0, \\ h_\lambda^{(d)} \rightarrow \sqrt{1 - e^{-2x}}, & \text{if } x \in (0, \infty), \\ 1 - h_\lambda^{(d)} \sim \frac{1}{2} (\lambda \kappa_d)^{-\frac{2}{d+1}}, & \text{if } x = \infty. \end{array} \right.$$

Note that [Assumption \(H\)](#) is automatically fulfilled in the critical and supercritical regime ( $x \in (0, \infty]$ ).

In addition, we determine the distributional fluctuations of  $h_\lambda^{(d)}$  around its limiting value.

**Theorem 2** (Convergence in distribution of the renormalized support function). *Under the assumption of Theorem 1, the random variable*

$$d \left( \log \frac{1}{\sqrt{1 - (h_\lambda^{(d)})^2}} - \frac{1}{d+1} \log \lambda \kappa_d \right) + \log \sqrt{\mathfrak{m}(d)}$$

converges towards the standard Gumbel distribution as  $d \rightarrow \infty$ , where

$$\mathfrak{m}(d) := \begin{cases} 4\pi \log \lambda \kappa_d, & \text{in the subcritical regime } \log \lambda \kappa_d \ll d, \\ 2\pi d(1 - e^{-2x}), & \text{in the critical regime } \log \lambda \kappa_d \sim dx \text{ with } x \in (0, \infty), \\ 2\pi d \left(1 - (\lambda \kappa_d)^{-\frac{2}{d+1}}\right), & \text{in the supercritical regime } \log \lambda \kappa_d \gg d. \end{cases}$$

**Remark.** *In the critical regime where  $\log \lambda \kappa_d = dx + y + o(1)$  with  $x \in (0, \infty)$  and  $y \in \mathbb{R}$ , Theorem 2 reads*

$$d \left( \log \frac{1}{\sqrt{1 - (h_\lambda^{(d)})^2}} - x \right) + \log \sqrt{2\pi d(1 - e^{-2x})} \xrightarrow{d \rightarrow \infty} G + y - x,$$

that is,

$$\sqrt{2\pi d} \left( \frac{e^{-2x}}{1 - (h_\lambda^{(d)})^2} \right)^{-\frac{d}{2}} \xrightarrow{d \rightarrow \infty} \frac{e^{G+y-x}}{\sqrt{1 - e^{-2x}}},$$

for some standard Gumbel variable  $G$ .

We now state analogous versions of the previous theorems when the support function is considered over several directions at once. Namely, for a fixed integer  $m \geq 2$ , we define

$$h_\lambda^{(d,m)} := \inf_{u \in \mathbb{S}^{d-1} \cap \mathbb{R}^m} h_\lambda^{(d)}(u).$$

Again, by rotational invariance, the distribution of this infimum does not depend on the direction of the linear  $m$ -dimensional section of  $\mathbb{S}^{d-1}$ . In particular, we expect  $h_\lambda^{(d,m)}$  to behave like  $h_\lambda^{(d)}$ , i.e., satisfy the exact same conclusions as Theorem 1 with the same threshold. This is confirmed by Theorem 3 below which provides a Gumbel limit distribution when  $\log \lambda \kappa_d$  belongs to the asymptotic range  $(\log d, d^{\frac{3}{2}})$ .

**Theorem 3** (Distributional limit). *Let  $m \geq 2$  be a fixed integer, and let  $\lambda := \lambda(d) > 0$  satisfy either one of the three assumptions  $(A_{\text{sub}})$ ,  $(A_{\text{crit}})$ , or  $(A_{\text{sup}})$  given in Lemma 4.4. Then there exist two explicit sequences  $\mathfrak{a}(d; m)$  and  $\mathfrak{b}(d; m)$  (given at (43) and (44)) such that*

$$\mathfrak{a}(d; m) - \mathfrak{b}(d; m) \log \frac{1}{\sqrt{1 - (h_\lambda^{(d,m)})^2}}$$

converges in law as  $d \rightarrow \infty$  towards the standard Gumbel distribution.

The proof of Theorem 3 relies essentially on the *ad hoc* application of a remarkable result due to S. Janson on random coverings of a set [24]. As a corollary, we extend Theorem 1 to  $h_\lambda^{(d,m)}$ .

**Corollary 4** (Asymptotic regimes of  $h_\lambda^{(d,m)}$ ). *Under the assumptions of [Theorem 3](#), let*

$$x := \lim_{d \rightarrow \infty} \frac{1}{d} \log \lambda \kappa_d \in [0, \infty].$$

*Then the following holds in probability as  $d \rightarrow \infty$ :*

$$\left\{ \begin{array}{ll} h_\lambda^{(d,m)} \sim \sqrt{\frac{2}{d} \log \lambda \kappa_d}, & \text{if } x = 0, \\ h_\lambda^{(d,m)} \rightarrow \sqrt{1 - e^{-2x}}, & \text{if } x \in (0, \infty), \\ 1 - h_\lambda^{(d,m)} \sim \frac{1}{2} (\lambda \kappa_d)^{-\frac{2}{d+1}}, & \text{if } x = \infty. \end{array} \right.$$

Though the idea is quite natural, the study of the asymptotics of the support function as a tool for describing the geometry of high-dimensional random polytopes seems original. We provide here an extensive analysis of its minimum on a subspace of fixed dimension with the help of essentially two ingredients: on one hand, a careful application of known estimates of the incomplete beta function and on the other hand, a reformulation of the problem in terms of the probability to cover the sphere with random spherical caps. The latter requires in particular a slight extension of a precise estimate due to Janson [[24](#)] of the covering probability with i.i.d. rescaled random spherical caps when the common distribution of the geodesic radii of the caps is allowed to depend on the scaling parameter. This result is interesting on its own and is deduced by a coupling technique which avoids rewriting the details of Janson's original proof in our context. Besides, our work is a possible first step towards a more systematic study of the whole support function process along the  $d$ -dimensional sphere, see the discussion below. The radius-vector function process is certainly worthy of investigation as well and we state a minor first result in this direction deduced from Wendel's probability calculation [[38](#)].

[Theorem 1](#) and [Corollary 4](#) are clearly reminiscent of [[7](#), [Theorem 3.1](#)]. If  $\lambda$  is taken so that  $\log \lambda \kappa_d$  belongs to the asymptotic range  $[d, d \log d)$ , then the expected volume ratio  $\mathbb{E} |K_\lambda^d| / \kappa_d$  vanishes as observed in [[7](#)], while according to [Theorem 1](#) and [Corollary 4](#),  $K_\lambda^d$  still has long 'arms' in any finite number of independent directions. This confirms the well-known picture of a high-dimensional convex body which was popularized by Vitali Milman and which looks like a 'star-shaped body with a lot of points very far from the origin and lot of points very close to the origin' [[20](#), [Section 2](#)]. Studying the minimum of the support function over a section of  $K_\lambda^d$  may provide a way of quantifying the size of the 'holes', i.e., estimating the critical dimension under which the support function is close to one in every direction of the section of  $K_\lambda^d$  and above which we expect to see directions almost unoccupied by the section of  $K_\lambda^d$ . Unsurprisingly, [Corollary 4](#) suggests that as soon as we reach the threshold for the one-dimensional section given in [Theorem 1](#), we expect every section with fixed dimension  $m$  to look like the  $m$ -dimensional unit ball. The rest of the study should then lead us to consider  $m$  tend to infinity with  $d$  in the supercritical case of [Theorem 1](#) in order to decide when exactly the function  $h_\lambda^{(d,m)}$  switches from being almost equal to 1 to being almost equal to 0. This requires a serious revision of the covering methods used in the proof of [Theorem 3](#) that we leave for further work.

We expect that our results can be dePoissonized and that, as long as the asymptotics of the measure of a spherical cap is accessible, our methods should extend to other models with rotationally-invariant

points, such as the  $\beta$ -polytope studied in [6] which covers our model when  $\beta = 0$  and the case of the convex hull of points uniformly distributed on the unit sphere when  $\beta = -1$ . In the latter setting, Bonnet and O'Reilly [9] have recently studied the typical height of a facet, which, incidentally, is proved to have the same first-order asymptotics as the support function in a fixed direction, as can be seen from a comparison between Theorem 4 therein and our [Theorem 1](#).

The paper is structured as follows. We start in [Section 2](#) with some asymptotic preliminaries, notably for the incomplete beta function to which the tail probabilities of  $h_\lambda^{(d)}$  and  $h_\lambda^{(d,m)}$  are related. As a warm-up, in [Section 3](#), we establish [Theorems 1](#) and [2](#) by elementary means. [Section 4](#) is devoted to the use of Janson's covering techniques for proving [Theorem 3](#) and [Corollary 4](#). Finally, in [Section 5](#), we transfer some of our results to the radius-vector function  $\rho_\lambda^{(d)}$  of  $K_\lambda^d$ .

**Notation.** Unless otherwise specified, all asymptotic estimates are w.r.t.  $d \rightarrow \infty$ . The relation  $g \gg f$  (or  $f \ll g$ , or  $f = o(g)$ ) for nonnegative  $f := f(d)$  and  $g := g(d)$  means that  $f(d) \leq \varepsilon g(d)$  holds for all  $d$  sufficiently large and any  $\varepsilon > 0$ , while  $f = O(g)$  or  $f \lesssim g$  indicate that  $f(d) \leq Cg(d)$  holds for all  $d \geq 1$  and some constant  $C > 0$ . We also write  $f \sim g$  if  $|f - g| \ll g$ , and  $f \asymp g$  if both  $f \lesssim g$  and  $g \lesssim f$  hold.

## 2. PRELIMINARIES

This section aims at providing an explicit formula for the distribution function of  $h_\lambda^{(d)}$  in terms of the incomplete beta function. It also paves the way for the asymptotic study of the tail probability of  $h_\lambda^{(d)}$  and  $h_\lambda^{(d,m)}$ , which is the focus of [Sections 3](#) and [4](#).

Let  $\mathcal{P}_\lambda^d$ ,  $d \geq 2$ , be Poisson point processes (embedded in a common abstract probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ ) with intensities  $\lambda := \lambda(d) > 0$  in  $\mathbb{R}^d$ . The polytope  $K_\lambda^d$  is defined as the convex hull of  $\mathcal{P}_\lambda^d \cap \mathbb{B}^d$ , where  $\mathbb{B}^d := \{x \in \mathbb{R}^d : |x| \leq 1\}$  is the Euclidean unit ball of  $(\mathbb{R}^d, \|\cdot\|)$ , with the Euclidean norm  $\|\cdot\|$  derived from the usual inner product  $\langle \cdot, \cdot \rangle$ . We recall that  $|\cdot|$  denotes the  $d$ -dimensional Lebesgue measure of  $\mathbb{R}^d$ . In both notations  $\|\cdot\|$  and  $|\cdot|$ , the dependency on  $d$  is implicit.

A priori, the support function  $h_\lambda^{(d)}$  introduced in [\(1\)](#) takes values in  $\{-\infty\} \cup (-1, 1)$ , but if  $\lambda$  is not chosen too small, then the polytope  $K_\lambda^d$  will likely contain the origin, which means that  $h_\lambda^{(d)} \geq 0$ .

**Lemma 2.1** (Containing the origin). *If [Assumption \(H\)](#) holds, then  $\mathbb{P}(0 \in K_\lambda^d) \rightarrow 1$  as  $d \rightarrow \infty$ .*

*If however  $\limsup_{d \rightarrow \infty} \frac{\lambda \kappa_d}{d} < 2$ , then  $\mathbb{P}(0 \in K_\lambda^d) \rightarrow 0$  as  $d \rightarrow \infty$ .*

*Proof of Lemma 2.1.* The number  $N := N(d)$  of points in  $K_\lambda^d$  has a Poisson distribution with mean  $\lambda \kappa_d$ . Further, conditional on  $N$ , those points have a symmetric distribution in  $\mathbb{R}^d$ . It follows from Wendel's formula [\[38\]](#) that

$$\begin{aligned} \mathbb{P}(0 \notin K_\lambda^d \mid N) &= \mathbb{1}_{\{N \leq d\}} + \mathbb{1}_{\{N > d\}} 2^{-(N-1)} \sum_{k=0}^{d-1} \binom{N-1}{k} \\ &= 1 - \mathbb{1}_{\{N > d\}} 2^{-(N-1)} \sum_{k=d}^{N-1} \binom{N-1}{k} \\ &= 1 - \mathbb{P}(S_{N-1} \geq d \mid N), \end{aligned}$$

where  $S_n$ ,  $n \geq 0$ , are Binomial( $n, \frac{1}{2}$ ) random variables independent of  $N$ . Hence

$$\mathbb{P}(0 \in K_\lambda^d) = \mathbb{P}(S_{N-1} \geq d).$$

By the law of large numbers,  $\mathbb{P}(S_{N-1} \geq d) \rightarrow 1$  as  $d \rightarrow \infty$  if

$$\liminf_{d \rightarrow \infty} \frac{1}{d} \mathbb{E}[S_{N-1}] > 1,$$

that is (since  $\mathbb{E}[S_{N-1}] = \frac{\lambda \kappa_{d-1}}{2}$ ), if  $\lambda \kappa_d \geq (2 + \varepsilon)d$  holds for some  $\varepsilon > 0$  and all  $d$  sufficiently large; similarly,  $\mathbb{P}(S_{N-1} \geq d) \rightarrow 0$  if  $\lambda \kappa_d \leq (2 - \varepsilon)d$  holds for some  $\varepsilon > 0$  and all  $d$  sufficiently large. (In fact, by classical large deviation theory, these two convergences occur exponentially fast.)  $\square$

Taking [Assumption \(H\)](#) for granted, we thus have  $0 \leq h_\lambda^{(d)} \leq 1$  w.h.p. as  $d \rightarrow \infty$ . Now,

$$\mathbb{P}\left(h_\lambda^{(d)} \leq r\right) = \mathbb{P}\left(h_\lambda^{(d)}(u) \leq r\right), \quad 0 \leq r \leq 1,$$

for, e.g.,  $u := (1, 0, \dots) \in \mathbb{S}^{d-1}$ ; we compute this probability as

$$\mathbb{P}\left(\mathcal{P}_\lambda^d \cap \mathcal{C}^d(r; u) = \emptyset\right) = e^{-\lambda |\mathcal{C}^d(r; u)|},$$

where the spherical cap  $\mathcal{C}^d(r; u) := \{x \in \mathbb{B}^d : \langle u, x \rangle > r\}$  has volume

$$|\mathcal{C}^d(r; u)| = \kappa_{d-1} \int_r^1 (1-t^2)^{\frac{d-1}{2}} dt = \frac{\kappa_{d-1}}{2} \int_0^{1-r^2} v^{\frac{d-1}{2}} (1-v)^{-\frac{1}{2}} dv, \quad (3)$$

with the last integral resulting from the change of variable  $v \leftarrow 1 - t^2$ . Hence

$$\mathbb{P}\left(h_\lambda^{(d)} \leq r\right) = \exp\left(-\frac{\lambda \kappa_{d-1}}{2} B\left(1 - r^2; \frac{d+1}{2}, \frac{1}{2}\right)\right), \quad (4)$$

where

$$B(x; p, q) := \int_0^x v^{p-1} (1-v)^{q-1} dv, \quad x \in [0, 1], p, q > 0,$$

is the *lower incomplete beta function* (the complete beta function is  $B(p, q) := B(1; p, q)$ ). We will see in [Section 4](#) that when considering the support function over  $m \geq 2$  directions at once, the distribution function of the infimum  $h_\lambda^{(d, m)}$  also involves this special function (with the third parameter  $q = \frac{1}{2}$  replaced by  $q = \frac{m}{2}$ ) as well as the volume of unit balls. Thus, the asymptotic behavior of  $h_\lambda^{(d)}$  and of  $h_\lambda^{(d, m)}$  will depend on the interplay between the intensity  $\lambda := \lambda(d)$  and the two quantities  $\kappa_n$  and  $B(x; p, q)$ , where  $n, x$  and  $p$  may depend on the dimension  $d$ .

Regarding  $\kappa_d$ , we will essentially use the asymptotic relation

$$\kappa_{d-1} = \kappa_d \sqrt{\frac{d}{2\pi}} \left[1 + O\left(\frac{1}{d}\right)\right], \quad (5)$$

which is easily derived from the well-known formula

$$\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1 + \frac{d}{2}\right)} \quad (6)$$

and Stirling's formula for Euler's gamma function  $\Gamma$  (see, e.g., [\[37, \(3.24\)\]](#)). As for the incomplete beta function, we can derive basic first-order estimates:



**Lemma 2.2** (Estimates of the incomplete beta function). *Let  $p, q > 0$ . Then for any  $x \in (0, 1)$  such that  $(p + q)x < p + 1$ ,*

$$\text{the ratio } B(x; p, q) / \frac{x^p(1-x)^{q-1}}{p} \text{ lies between 1 and } \frac{1}{1 - \frac{q-1}{p+1} \cdot \frac{x}{1-x}}.$$

*In particular, when  $x \in (0, 1)$  and  $p, q > 0$  are three sequences indexed by  $d$ :*

1. *If  $p \gg \frac{|q-1|x}{1-x}$ , then*

$$B(x; p, q) = \frac{x^p(1-x)^{q-1}}{p} \left[ 1 + O\left(\frac{|q-1|x}{p(1-x)}\right) \right].$$

2. *If  $p \gg \frac{|q-1|}{1-x}$ , then*

$$\frac{B(x; p, q)}{B(p, q)} = \frac{x^p[(1-x)p]^{q-1}}{\Gamma(q)} \left[ 1 + O\left(\frac{|q-1|}{p(1-x)}\right) \right].$$

*Proof.* For  $x \in (0, 1)$  we have from [37, (11.33)] the series representation

$$B(x; p, q) = \frac{x^p(1-x)^q}{p} \sum_{n=0}^{\infty} \frac{(p+q)_n}{(p+1)_n} x^n,$$

where the ratio of Pochhammer symbols

$$\frac{(p+q)_n}{(p+1)_n} := \frac{p+q}{p+1} \cdot \frac{p+q+1}{p+2} \cdots \frac{p+q+n-1}{p+n}$$

belongs to  $\left[1, \left(\frac{p+q}{p+1}\right)^n\right]$  if  $q \geq 1$ , and to  $\left[\left(\frac{p+q}{p+1}\right)^n, 1\right]$  if  $0 < q < 1$ . Now if  $(p+q)x < p+1$ , then

$$\sum_{n=0}^{\infty} \left(\frac{p+q}{p+1}\right)^n x^n = \frac{1}{1 - \frac{p+q}{p+1}x} = \frac{1}{1-x} \cdot \frac{1}{1 - \frac{q-1}{p+1} \cdot \frac{x}{1-x}},$$

hence the lower and upper bounds on  $B(x; p, q) / \frac{x^p(1-x)^{q-1}}{p}$ . The first stated asymptotic estimate is an immediate consequence of these bounds, while the second one follows from the first one combined with

$$B(p, q) = \Gamma(q) \cdot \frac{\Gamma(p)}{\Gamma(p+q)} = \frac{\Gamma(q)}{p^q} \left[ 1 + O\left(\frac{|q-1|}{p}\right) \right],$$

see, e.g., [37, (3.31)]. □

### 3. THE SUPPORT FUNCTION IN ONE DIRECTION

We are now ready to establish [Theorems 1](#) and [2](#): we do so by recalling the distribution function [\(4\)](#) of the support function  $h_\lambda^{(d)}$ , then plug in asymptotics for the incomplete beta function ([Lemma 2.2](#)) and for the volume of the Euclidean unit balls [\(5\)](#).

To start with, we identify the limit of  $h_\lambda^{(d)}$  in probability.

**Lemma 3.1** (Limit in probability of the support function). *Under the assumption of [Theorem 1](#),*

$$\lim_{d \rightarrow \infty} h_\lambda^{(d)} = \sqrt{1 - e^{-2x}}, \quad \text{in probability}$$

(with the convention  $e^{-\infty} = 0$ ).

*Proof.* We start by applying [Lemma 2.2](#) with only the second argument of the incomplete beta function depending on  $d$ : for any fixed  $r \in (0, 1)$ ,

$$B(1 - r^2; \frac{d+1}{2}, \frac{1}{2}) = \frac{2(1 - r^2)^{\frac{d+1}{2}}}{rd} \left[ 1 + O\left(\frac{1}{d}\right) \right]. \quad (7)$$

Inserting this and the other estimate [\(5\)](#) into [\(4\)](#) then yields

$$\begin{aligned} -\log \mathbb{P}\left(h_\lambda^{(d)} \leq r\right) &= \frac{\lambda \kappa_d (1 - r^2)^{\frac{d+1}{2}}}{r \sqrt{2\pi d}} \left[ 1 + O\left(\frac{1}{d}\right) \right] \\ &= \exp \left\{ \log \lambda \kappa_d + \frac{d+1}{2} \log(1 - r^2) - \log r \sqrt{2\pi d} + O\left(\frac{1}{d}\right) \right\} \\ &= \exp \left\{ d \left( \frac{\log \lambda \kappa_d}{d} + \log \sqrt{1 - r^2} + o(1) \right) \right\}. \end{aligned} \quad (8)$$

Since  $\frac{\log \lambda \kappa_d}{d} \rightarrow x \in [0, \infty]$  as  $d \rightarrow \infty$ , the change of sign in this exponent provides the required threshold, i.e.,

$$\lim_{d \rightarrow \infty} \mathbb{P}\left(h_\lambda^{(d)} \leq r\right) = \begin{cases} 0 & \text{if } r < \sqrt{1 - e^{-2x}}, \\ 1 & \text{if } r > \sqrt{1 - e^{-2x}}. \end{cases}$$

This proves the convergence in distribution of  $h_\lambda^{(d)}$  towards the constant  $\sqrt{1 - e^{-2x}}$ , which is equivalent to the convergence in probability to the same limit.  $\square$

By also letting the first argument of the incomplete beta function depend on  $d$ , a deeper application of [Lemma 2.2](#) enables us to complete the proof of [Theorems 1 and 2](#).

*Proof of Theorem 2.* We apply [Lemma 2.2](#) again by letting  $r \in (0, 1)$  in the previous proof depend on  $d$ . Whenever  $dr^2 \gg 1$  holds, we may still write (instead of [\(7\)](#))

$$B(1 - r^2; \frac{d+1}{2}, \frac{1}{2}) \sim \frac{2(1 - r^2)^{\frac{d+1}{2}}}{rd},$$

which leads to the following weakening of [\(8\)](#):

$$-\log \mathbb{P}\left(h_\lambda^{(d)} \leq r\right) = \exp \left\{ \log \lambda \kappa_d + \frac{d+1}{2} \log(1 - r^2) - \log r \sqrt{2\pi d} + o(1) \right\}. \quad (9)$$

Fix  $\tau \in \mathbb{R}$  and choose  $r := r(d; \tau)$  as the unique solution to the equation

$$\log \lambda \kappa_d + \frac{d+1}{2} \log(1 - r^2) - \log r \sqrt{2\pi d} = -\tau, \quad \text{namely } \frac{(1 - r^2)^{\frac{d+1}{2}}}{r} = \frac{\sqrt{2\pi d}}{\lambda \kappa_d} e^{-\tau}. \quad (10)$$

In particular, under [Assumption \(H\)](#),

$$-\frac{d+1}{2} \log(1 - r^2) \geq \log \frac{\lambda \kappa_d}{\sqrt{2\pi d}} + \tau \rightarrow \infty,$$

so that, indeed,  $dr^2 \gg 1$  and [\(9\)](#) is true. Inserting [\(10\)](#) there then yields

$$\lim_{d \rightarrow \infty} \mathbb{P}\left(h_\lambda^{(d)} \leq r\right) = e^{-e^{-\tau}},$$

which is the c.d.f. of the standard Gumbel distribution. Now, on the one hand, we observe that

$$\begin{aligned} \mathbb{P}\left(h_\lambda^{(d)} \leq r\right) &= \mathbb{P}\left[-\frac{d+1}{2} \log\left(1 - (h_\lambda^{(d)})^2\right) - \log \frac{\lambda \kappa_d}{r \sqrt{2\pi d}} \leq \tau\right] \\ &= \mathbb{P}\left[d \left( \log \frac{1}{\sqrt{1 - (h_\lambda^{(d)})^2}} - \frac{1}{d+1} \log \lambda \kappa_d \right) + \frac{\log r \sqrt{2\pi d}}{1 + O\left(\frac{1}{d}\right)} \leq \tau + O\left(\frac{1}{d}\right)\right]. \end{aligned}$$

On the other hand, we have from (10),

$$1 - r^2 = r^{\frac{2}{d+1}} (\lambda \kappa_d)^{-\frac{2}{d+1}} \left[1 + O\left(\frac{\log d}{d}\right)\right],$$

and

$$\frac{1}{d} \log r = \frac{1}{d} \log \lambda \kappa_d + O(-\log(1 - r^2)),$$

from which it follows that

$$\begin{cases} r^2 \sim \frac{2}{d} \log \lambda \kappa_d, & \text{if } \log \lambda \kappa_d \ll d, \\ r^2 \rightarrow 1 - e^{-2x}, & \text{if } \log \lambda \kappa_d \sim dx \text{ with } x \in (0, \infty), \\ 1 - r^2 \sim (\lambda \kappa_d)^{-\frac{2}{d+1}}, & \text{if } \log \lambda \kappa_d \gg d. \end{cases}$$

This allows us to conclude that

$$\mathbb{P}\left[d \left( \log \frac{1}{\sqrt{1 - (h_\lambda^{(d)})^2}} - \frac{1}{d+1} \log \lambda \kappa_d \right) + \log \sqrt{\mathbf{m}(d)} \leq \tau\right] \xrightarrow{d \rightarrow \infty} e^{-e^{-\tau}},$$

with  $\mathbf{m}(d)$  as stated. □

*Proof of Theorem 1.* It follows from [Theorem 2](#) that

$$d \left( \log \frac{1}{\sqrt{1 - (h_\lambda^{(d)})^2}} - \frac{1}{d+1} \log \lambda \kappa_d \right) + \log \sqrt{\mathbf{m}(d)} = O_{\mathbb{P}}(1),$$

where  $X(d) = O_{\mathbb{P}}(1)$  means that  $\lim_{A \rightarrow \infty} \limsup_{d \rightarrow \infty} \mathbb{P}(|X(d)| > A) = 0$ . Multiplying this equation by  $-\frac{2}{d}$  and taking the exponential function, we deduce that

$$(\lambda \kappa_d)^{\frac{2}{d+1}} \left(1 - (h_\lambda^{(d)})^2\right) = 1 + \frac{1}{\log d} \cdot O_{\mathbb{P}}(1),$$

where we discarded the  $\mathbf{m}(d)^{-\frac{1}{d}}$  term because  $\mathbf{m}(d) = O(d)$ . In the subcritical regime  $\log \lambda \kappa_d \ll d$ , we obtain that in probability as  $d \rightarrow \infty$ ,

$$h_\lambda^{(d)} \sim \sqrt{\frac{2}{d} \log \lambda \kappa_d}$$

(because then  $1 - (\lambda \kappa_d)^{\frac{2}{d+1}} \sim -\frac{2}{d} \log \lambda \kappa_d$ ). In the supercritical regime  $\log \lambda \kappa_d \gg d$ , we get instead

$$1 - h_\lambda^{(d)} \sim \frac{1}{2} (\lambda \kappa_d)^{-\frac{2}{d+1}}$$

because, by [Lemma 3.1](#),  $1 + h_\lambda^{(d)} \rightarrow 2$  in probability. This completes the proof of [Theorem 1](#). □

## 4. THE INFIMUM OF THE SUPPORT FUNCTION OVER MULTIPLE DIRECTIONS

In this section, we extend the study of the asymptotic behavior of the support function when considered over several directions simultaneously. Namely, we consider

$$h_\lambda^{(d,m)} := \inf_{u \in \mathbb{S}^{d-1} \cap \mathbb{R}^m} h_\lambda^{(d)}(u),$$

where  $m \geq 2$  is a fixed integer. [Section 4.1](#) consists in reinterpreting the distribution of  $h_\lambda^{(d,m)}$  in terms of a covering probability. We then present in [Section 4.2](#) a remarkable covering technique due to Janson [24] which we specialize to our setting. This finally allows us to prove [Theorem 3](#).

**4.1. Reduction to a covering problem.** We start by relating the tail event  $h_\lambda^{(d,m)} \geq r$  to the event of covering the sphere  $\mathbb{S}^{m-1} := \{y \in \mathbb{R}^m : \|y\| = 1\}$  with i.i.d. geodesic balls. In this direction, we write  $v_{\mathbb{S}^{m-1}}(dx)$  for the  $(m-1)$ -dimensional surface measure on  $\mathbb{S}^{m-1}$  (so that  $v_{\mathbb{S}^{m-1}}(\mathbb{S}^{m-1}) = m\kappa_m$ ), and

$$B_{\mathbb{S}^{m-1}}(x, \theta) := \left\{ y \in \mathbb{R}^m : \|y\| = 1 \text{ and } \langle x, y \rangle > \cos \theta \right\}$$

for the geodesic ball in  $\mathbb{S}^{m-1}$  with center  $x \in \mathbb{S}^{m-1}$  and radius  $\theta \in (0, \pi]$ .

**Lemma 4.1** (Covering the sphere). *For every  $r \in (0, 1)$ , we have*

$$h_\lambda^{(d,m)} \geq r \iff \mathbb{S}^{m-1} = \bigcup_i B_{\mathbb{S}^{m-1}}(x_i, a\rho_i), \quad (11)$$

where

$$a := \frac{1}{\sqrt{d}} \frac{\sqrt{1-r^2}}{r}, \quad (12)$$

and the centers and radii  $(x_i, \rho_i)$  arise as the atoms of a Poisson point process on  $\mathbb{S}^{m-1} \times (0, \infty)$  whose intensity measure  $\Lambda_r^{(d,m)} v_{\mathbb{S}^{m-1}}(dx) \otimes \mathbb{P}(R_r^{(d,m)} \in d\rho)$  is given by

$$\Lambda_r^{(d,m)} := \lambda \kappa_d \cdot \frac{B(1-r^2; 1 + \frac{d-m}{2}, \frac{m}{2})}{B(1 + \frac{d-m}{2}, \frac{m}{2})}, \quad (13)$$

and, for every  $\rho > 0$ ,

$$\mathbb{P}(R_r^{(d,m)} > \rho) := \frac{B(1-r^2 \cos^{-2}(a\rho); 1 + \frac{d-m}{2}, \frac{m}{2})}{B(1-r^2; 1 + \frac{d-m}{2}, \frac{m}{2})} \mathbb{1}_{\{\rho < \frac{\arccos r}{a}\}}. \quad (14)$$

*Proof.* Indeed,  $h_\lambda^{(d,m)} \geq r$  if and only if the sphere  $r\mathbb{S}^{m-1}$  is entirely covered by the Voronoi flower associated with the projection of  $K_\lambda^d$  onto  $\mathbb{R}^m$ , that is, by the spherical patches

$$\left( \frac{X'}{2} + \frac{\|X'\|}{2} \mathbb{B}^m \right) \cap r\mathbb{S}^{m-1} = rB_{\mathbb{S}^{m-1}} \left( \frac{X'}{\|X'\|}, R_{X'} \right),$$

where  $R_{X'} := \arccos(\frac{r}{\|X'\|})$  and the points  $X'$  are the orthogonal projections of the points in  $\mathcal{P}_\lambda^d \cap \mathcal{R}_r^{d,m}$ , with

$$\mathcal{R}_r^{d,m} := \left\{ x \in \mathbb{B}^d : x_1^2 + \dots + x_m^2 \geq r^2 \right\};$$

see [Figure 2](#). Considering a point  $X \in \mathcal{P}_\lambda^d \cap \mathcal{R}_r^{d,m}$ , the square of the norm of its orthogonal projection  $X'$

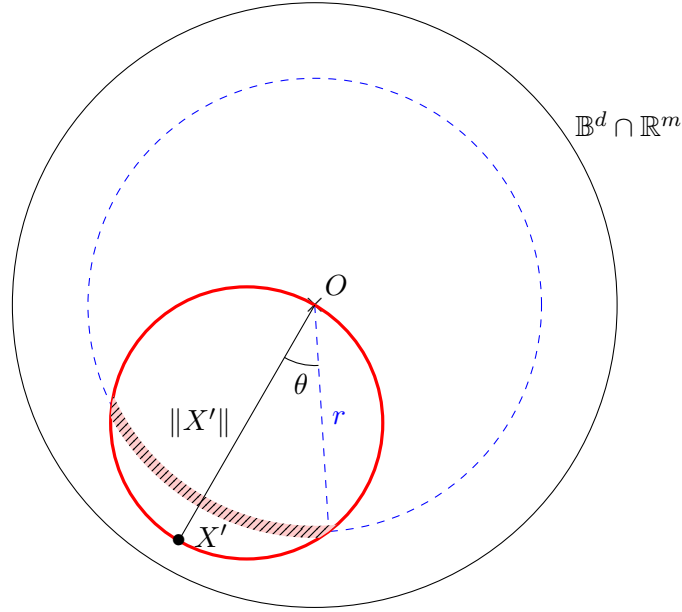


FIGURE 2. Projection onto  $\mathbb{R}^m$ . We have  $h_\lambda^{(d,m)} \geq r$  if and only if the sphere  $r\mathbb{S}^{m-1}$  (dashed) is covered by the union of its intersection with each petal (in red) of the Voronoi flower whose corresponding vertex lies in  $\mathcal{R}_r^{d,m}$ . Each projected vertex  $X'$  yields a geodesic ball of  $\mathbb{S}^{m-1}$  (hatched) of radius  $\theta := \arccos(r/|X'|)$ .

onto  $\mathbb{R}^m$  has a law given by

$$\mathbb{E}\left[g(|X'|^2)\right] = \frac{\int_{r^2}^1 t^{\frac{m}{2}-1}(1-t)^{\frac{d-m}{2}} g(t) dt}{\mathbb{B}(1-r^2; 1 + \frac{d-m}{2}, \frac{m}{2})}, \quad (15)$$

for any measurable function  $g: [0, \infty) \rightarrow [0, \infty)$ . We further note that the projections  $X'$  are identically distributed, with  $X'/\|X'\|$  uniform on  $\mathbb{S}^{m-1}$  (by rotational invariance). We let  $R_r^{(d,m)}$  denote a random variable with law

$$R_r^{(d,m)} \stackrel{(d)}{=} \frac{1}{a} R_{X'} = \frac{1}{a} \arccos \frac{r}{\|X'\|}, \quad \text{where } a := \frac{1}{\sqrt{d}} \frac{\sqrt{1-r^2}}{r} \quad (16)$$

(this rescaling with the quantity  $a$  may seem arbitrary for the time being but is in fact designed for the future convergence in distribution in [Lemma 4.2](#) and for the use of Janson's covering result in [Section 4.2](#)).

We deduce that  $h_\lambda^{(d,m)} \geq r$  if and only if  $\mathbb{S}^{m-1}$  is covered by the geodesic balls  $B_{\mathbb{S}^{m-1}}(x, a\rho)$ , whose centers and radii  $(x, \rho)$  arise from a Poisson point process on  $\mathbb{S}^{m-1} \times (0, \infty)$  with intensity measure

$$\lambda \cdot |\mathcal{R}_r^{d,m}| v_{\mathbb{S}^{m-1}}(dx) \otimes \mathbb{P}(R_r^{(d,m)} \in d\rho).$$

Now, the stated distribution function (14) of  $R_r^{(d,m)}$  easily follows from (15) and (16):

$$\begin{aligned} \mathbb{P}(R_r^{(d,m)} > \rho) &= \mathbb{P}\left(\|X'\|^2 > r^2 \cos^{-2}(a\rho)\right) \\ &= \frac{\int_{r^2}^1 t^{\frac{m}{2}-1} (1-t)^{\frac{d-m}{2}} \mathbb{1}_{\{t > r^2 \cos^{-2}(a\rho)\}} dt}{\mathbb{B}(1-r^2; 1 + \frac{d-m}{2}, \frac{m}{2})} \\ &= \frac{\mathbb{B}(1-r^2 \cos^{-2}(a\rho); 1 + \frac{d-m}{2}, \frac{m}{2})}{\mathbb{B}(1-r^2; 1 + \frac{d-m}{2}, \frac{m}{2})} \mathbb{1}_{\{\rho < \frac{\arccos r}{a}\}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \lambda \cdot |\mathcal{R}_r^{d,m}| &= \lambda \int \cdots \int \mathbb{1}_{\{x_1^2 + \cdots + x_m^2 \geq r^2\}} \kappa_{d-m} (1 - x_1^2 - \cdots - x_m^2)^{\frac{d-m}{2}} dx_1 \cdots dx_m \\ &= \frac{1}{2} \lambda m \kappa_m \kappa_{d-m} \int_{r^2}^1 t^{\frac{m}{2}-1} (1-t)^{\frac{d-m}{2}} dt \\ &= \lambda \kappa_d \cdot \frac{\mathbb{B}(1-r^2; 1 + \frac{d-m}{2}, \frac{m}{2})}{\mathbb{B}(1 + \frac{d-m}{2}, \frac{m}{2})} \\ &=: \Lambda_r^{(d,m)}, \end{aligned}$$

where the second equality comes from the use of spherical coordinates, and the third equality is due to the expression (6) for the volume of Euclidean balls and to the relation  $\mathbb{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  between the beta and gamma functions.  $\square$

We are therefore reduced to understanding the probability as  $d \rightarrow \infty$  of the covering event (11). As we will see in the next section, it turns out that Janson [24] derived precise estimates for the probability of covering a manifold of fixed dimension using a Poisson process of i.i.d. patches whose intensity  $\Lambda$  increases to infinity as the scale parameter  $a$  decreases to 0. We conclude this section by showing that the random radii  $R_r^{(d,m)}$ ,  $r \in (0, 1)$ , have a common limit distribution as  $d \rightarrow \infty$ . This convergence will hold with respect to the Monge-Kantorovitch-Wasserstein metric  $W_1$ , which for two real random variables  $X$  and  $Y$  is given by

$$W_1(X, Y) = \int_{\mathbb{R}} |\mathbb{P}(X > t) - \mathbb{P}(Y > t)| dt$$

(see [15, Problem 2 p. 425]). We recall that the  $W_1$ -convergence amounts to the convergence in distribution together with the convergence of the first moment.

**Lemma 4.2** (Convergence of the patch radii). *Let  $m \geq 2$  be a fixed integer, let  $r := r(d) \in (0, 1)$  and let  $a$  as in (12). Suppose that*

$$d \gg \frac{1}{r^2} + \log \frac{1}{1-r}, \quad \text{or equivalently, } a + \frac{1}{d} \log \frac{1}{a} \rightarrow 0. \quad (17)$$

Then

$$W_1\left(\log R_r^{(d,m)}, \log R\right) = O\left(\frac{1}{dr^2} \log^2(dr^2)\right), \quad (18)$$

where  $R$  is a standard Rayleigh random variable, with Lebesgue density  $\rho \mapsto \rho e^{-\rho^2/2}$  on  $(0, \infty)$  and moments

$$\mathbb{E} R^k = 2^{\frac{k}{2}} \Gamma\left(1 + \frac{k}{2}\right), \quad k \in \mathbb{Z}_+. \quad (19)$$

Moreover, the convergence holds with the following upper bound: for every  $w > 0$  and all  $d$  sufficiently large,

$$\sup_{\rho > 0} \frac{\mathbb{P}(R_r^{(d,m)} > \rho)}{\mathbb{P}\left(\left[1 + \frac{w}{\log \frac{1}{a}}\right] R > \rho\right)} \leq 1. \quad (20)$$

*Proof.* We observe from (14) that

$$\mathbb{P}(R_r^{(d,m)} > \rho) = \frac{\mathbb{B}(1 - r^2 \cos^{-2}(a\rho); 1 + \frac{d-m}{2}, \frac{m}{2})}{\mathbb{B}(1 - r^2; 1 + \frac{d-m}{2}, \frac{m}{2})} \mathbb{1}_{\{\rho < \frac{\arccos r}{a}\}}$$

is a positive and continuously differentiable function of  $x := \tan^{-2}(a\rho) = \cos^{-2}(a\rho) - 1$  on  $(0, r^{-2} - 1)$ . Letting  $F(x) := \log \mathbb{B}(1 - r^2 - r^2 x; 1 + \frac{d-m}{2}, \frac{m}{2})$  allows us to write  $\log \mathbb{P}(R_r^{(d,m)} > \rho) = F(x) - F(0)$ , so that by the mean value theorem there exists  $\bar{x} \in (0, x)$  with

$$\begin{aligned} \log \mathbb{P}(R_r^{(d,m)} > \rho) &= xF'(\bar{x}) \\ &= -r^2 x \frac{(1 - r^2 - r^2 \bar{x})^{\frac{d-m}{2}} r^{m-2} (1 + \bar{x})^{\frac{m}{2}-1}}{\mathbb{B}(1 - r^2 - r^2 \bar{x}; 1 + \frac{d-m}{2}, \frac{m}{2})}. \end{aligned}$$

Applying Lemma 2.2, part 1., for the denominator thanks to the assumption (17), we get (introducing  $r^{-2} - 1 = da^2$ )

$$\begin{aligned} \log \mathbb{P}(R_r^{(d,m)} > \rho) &= -\left(1 + \frac{d-m}{2}\right) \frac{r^2 x}{1 - r^2 - r^2 \bar{x}} \left[1 + O\left(\frac{1 - r^2 - r^2 \bar{x}}{dr^2(1 + \bar{x})}\right)\right] \\ &= -\frac{x}{2a^2} \left(1 - \frac{\bar{x}}{da^2}\right)^{-1} \left[1 + O\left(\frac{1}{dr^2}\right)\right], \end{aligned}$$

where the error term is uniform in  $x \in (0, r^{-2} - 1)$ . Recalling  $\bar{x} < x = \tan^2(a\rho)$  and using the inequality  $y < \tan y < y(1 - \frac{4}{\pi^2} y^2)^{-1}$  (see, e.g., [5]) with  $y := a\rho \in (0, \arccos r) \subset (0, \frac{\pi}{2})$ , we then obtain

$$-\frac{\rho^2}{2} \left[ \left(1 - \frac{4a^2 \rho^2}{\pi^2}\right)^2 - \frac{\rho^2}{d} \right]^{-1} \left[1 + O\left(\frac{1}{dr^2}\right)\right] \leq \log \mathbb{P}(R_r^{(d,m)} > \rho) \leq -\frac{\rho^2}{2} \left[1 + O\left(\frac{1}{dr^2}\right)\right], \quad (21)$$

uniformly in  $0 \leq \rho < A$ , where

$$A := \frac{\arccos r}{a} = r\sqrt{d} \frac{\arccos r}{\sqrt{1-r^2}} \asymp r\sqrt{d}$$

tends to infinity by (17). Recalling that the tail of the Rayleigh variable  $R$  fulfills  $\log \mathbb{P}(R > \rho) = -\frac{\rho^2}{2}$ , the upper bound in (21) entails that

$$\frac{\mathbb{P}(R_r^{(d,m)} > \rho)}{\mathbb{P}(R > \rho)} \mathbb{1}_{\{0 \leq \rho < A\}} = O(1). \quad (22)$$

Since

$$\left(1 - \frac{4a^2 \rho^2}{\pi^2}\right)^2 - \frac{\rho^2}{d} = \left[1 + \rho^2 \cdot O\left(a^2 + \frac{1}{d}\right)\right]^2 = \left[1 + \rho^2 \cdot O\left(\frac{1}{dr^2}\right)\right]^2,$$

we also obtain from (21) that, for any positive sequence  $\varepsilon := \varepsilon(d) \rightarrow 0$ ,

$$\left| \mathbb{P}(R_r^{(d,m)} > \rho) - \mathbb{P}(R > \rho) \right| \mathbf{1}_{\{0 \leq \rho < \varepsilon A\}} = \rho^2 \cdot O(\varepsilon^2). \quad (23)$$

Note that both error terms in (22) and (23) do not depend on  $\rho$ . We deduce that

$$\begin{aligned} W_1(\log R_r^{(d,m)}, \log R) &= \int_{\mathbb{R}} \left| \mathbb{P}(\log R_r^{(d,m)} > x) - \mathbb{P}(\log R > x) \right| dx \\ &= \int_0^\infty \left| \mathbb{P}(R_r^{(d,m)} > \rho) - \mathbb{P}(R > \rho) \right| \frac{d\rho}{\rho} \\ &\leq \int_0^{\varepsilon A} \rho^2 \cdot O(\varepsilon^2) \frac{d\rho}{\rho} + \int_{\varepsilon A}^\infty e^{-\frac{\rho^2}{2}} \cdot O(1) \frac{d\rho}{\rho} \\ &= O(\varepsilon^4 A^2) + O\left(\frac{e^{-\frac{1}{2}\varepsilon^2 A^2}}{\varepsilon^2 A^2}\right). \end{aligned}$$

Choosing  $\varepsilon := 2\sqrt{\log A}/A$ , we obtain

$$W_1(\log R_r^{(d,m)}, \log R) = O\left(\frac{\log^2 A}{A^2}\right) = O\left(\frac{1}{dr^2} \log^2(dr^2)\right),$$

as stated. Finally, for  $w > 0$ , the upper bound in (21) yields that, uniformly for all  $\rho > 0$ ,

$$\begin{aligned} \frac{\mathbb{P}(R_r^{(d,m)} > \rho)}{\mathbb{P}\left(\left[1 + \frac{w}{\log \frac{1}{a}}\right] R > \rho\right)} &\leq \exp\left(-\frac{\rho^2}{2} \left[1 - \left(1 + \frac{w}{\log \frac{1}{a}}\right)^{-2} + O\left(\frac{1}{dr^2}\right)\right]\right) \\ &= \exp\left(-\frac{w\rho^2}{\log \frac{1}{a}} [1 + o(1)]\right) \\ &\leq 1, \end{aligned}$$

provided  $d$  is chosen sufficiently large, where we used that  $\frac{1}{dr^2} \log \frac{1}{a} = (a^2 + \frac{1}{d}) \log \frac{1}{a} \rightarrow 0$ , by (17).

For the moments of the standard Rayleigh distribution, see, e.g., [25, § 18.3].  $\square$

**4.2. Application of Janson's covering result.** In [24], Janson showed that the number of random “small sets” needed to cover a fixed “big set” converges, when properly normalized, to the Gumbel extreme value distribution as the “size” of the small sets tends to zero. More precisely, let the big set be a  $\mathcal{C}^2$ ,  $D$ -dimensional compact Riemannian manifold  $M$  with volume measure  $v_M$ , and suppose that the small sets are i.i.d. geodesic balls, that is, they are all of the form  $B_M(x_i, a\rho_i)$ ,  $i \geq 1$ , where  $a > 0$  is a vanishing scale parameter, and the centers and radii  $(x_i, \rho_i)$  arise as the atoms of a Poisson point process on  $M \times (0, \infty)$  with intensity  $\Lambda v_M(dx) \otimes \mathbb{P}(R \in d\rho)$ , for some positive random variable  $R$  and  $\Lambda := \Lambda(a) \rightarrow \infty$  as  $a \rightarrow 0$ . Then, denoting by  $\text{Cover}(\Lambda, R, a; M)$  the event

$$M = \bigcup_{i \geq 1} B_M(x_i, a\rho_i),$$

Janson [24, Lemma 8.1] proved that

$$\lim_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda, R, a; M)) = e^{-e^{-\tau}}, \quad (J)$$



under the following two conditions:

$$\mathbb{E} R^q < \infty \quad \text{for some } q > D, \quad (J_1)$$

and

$$J(\Lambda, R, a; M) := ba^D v_M(M) \Lambda - \log \frac{1}{ba^D} - D \log \log \frac{1}{ba^D} - \log \alpha(R) \xrightarrow{a \rightarrow 0} \tau \in \mathbb{R}, \quad (J_2)$$

with

$$b := b(R; M) := \frac{\pi^{D/2} \mathbb{E} R^D}{\Gamma(1 + \frac{D}{2}) v_M(M)}, \quad (J_b)$$

and [24, Eq. (9.24)]

$$\alpha(R) := \frac{1}{D!} \left( \frac{\sqrt{\pi} \Gamma(1 + \frac{D}{2})}{\Gamma(\frac{D+1}{2})} \right)^{D-1} \frac{(\mathbb{E} R^{D-1})^D}{(\mathbb{E} R^D)^{D-1}}. \quad (J_\alpha)$$

In view of [Lemma 4.1](#), we would then like to estimate the probability

$$\mathbb{P}\left(h_\lambda^{(d,m)} \geq r\right) = \mathbb{P}\left(\text{Cover}(\Lambda_r^{(d,m)}, R_r^{(d,m)}, a; \mathbb{S}^{m-1})\right), \quad (24)$$

where the scale parameter  $a := \sqrt{1 - r^2}/r\sqrt{d}$  vanishes as  $d \rightarrow \infty$ . Although Janson's original result is stated only when the distribution of the random radius does not depend on  $a$ , we show in [Lemma 4.3](#) below, under a minor reinforcement of the assumption given at (17), that it still holds when we use the radii  $R_r^{(d,m)}$  instead of their Rayleigh limit  $R$  in [Lemma 4.2](#). This is done thanks to a slight improvement of Janson's result, [Proposition A](#), which may be of independent interest (see [Appendix](#)).

**Lemma 4.3** (Application of the extension of Janson's result). *With the notation of [Lemmas 4.1](#) and [4.2](#), suppose [\(J<sub>2</sub>\)](#) holds with  $\Lambda := \Lambda_r^{(d,m)}$  and  $M := \mathbb{S}^{m-1}$  (i.e.,  $J(\Lambda_r^{(d,m)}, R, a; \mathbb{S}^{m-1}) \rightarrow \tau$ ), and suppose also that*

$$a + \frac{1}{d} \log^{2+\gamma} \frac{1}{a} \xrightarrow{d \rightarrow \infty} 0 \quad (25)$$

holds for some  $\gamma \in (0, \frac{1}{2})$ . Then

$$\lim_{d \rightarrow \infty} \mathbb{P}(\text{Cover}(\Lambda_r^{(d,m)}, R_r^{(d,m)}, a; \mathbb{S}^{m-1})) = e^{-e^{-\tau}}.$$

*Proof.* We apply [Proposition A](#) in [Appendix](#). We have [\(J<sub>2</sub>\)](#) for  $\Lambda := \Lambda^{(d,m)}$  and  $D := m - 1$ , and also [\(J<sub>1</sub>\)](#) because all moments of the Rayleigh distribution are finite. It remains to show that the two conditions of [Proposition A](#) related to  $R_a := R_r^{(d,m)}$  are satisfied. The first one is [\(20\)](#) given by [Lemma 4.2](#). For the second one, [\(18\)](#) gives

$$W_1\left(\log R_r^{(d,m)}, \log R\right) = O\left(\frac{1}{dr^2} \log^2(dr^2)\right) = o\left(\frac{1}{\log^2 \frac{1}{a}}\right), \quad (26)$$

since, by [\(25\)](#) and the relation  $\frac{1}{dr^2} = \frac{1}{d} + a^2$ ,

$$\frac{1}{dr^2} \log^2(dr^2) \log^2 \frac{1}{a} = \left[ \left( \frac{1}{dr^2} \right) \log^{\frac{4+2\gamma}{\gamma}}(dr^2) \right]^{\frac{\gamma}{2+\gamma}} \cdot \left[ \frac{1}{d} \log^{2+\gamma} \frac{1}{a} + a^2 \log^{2+\gamma} \frac{1}{a} \right]^{\frac{2}{2+\gamma}} \xrightarrow{d \rightarrow \infty} 0. \quad \square$$

The applicability of [Lemma 4.3](#) is done in [Lemma 4.4](#) below. Plugging in the expressions of  $\mathbb{E} R^k$  given by [\(19\)](#) into the expression  $(J_\alpha)$  on page [16](#), we can see that  $\alpha(R)$  simplifies to

$$\alpha := \alpha(R) = \frac{\pi^{\frac{m-1}{2}} \Gamma(\frac{m+1}{2})}{(m-1)!} = \frac{\pi^{\frac{m}{2}}}{2^{m-1} \Gamma(\frac{m}{2})}, \quad (27)$$

by an application of Legendre's duplication formula. Similarly, using also the expression of  $\kappa_m$  in [\(6\)](#) and the fundamental property  $\Gamma(1 + \frac{m}{2}) = \frac{m}{2} \Gamma(\frac{m}{2})$ , the expression of  $b(R; M)$  in  $(J_b)$  reduces to

$$b := b(R; \mathbb{S}^{m-1}) = \frac{(\sqrt{2\pi})^{m-1} \Gamma(\frac{m}{2})}{2\pi^{\frac{m}{2}}} = \frac{2^{\frac{m-3}{2}} \Gamma(\frac{m}{2})}{\sqrt{\pi}}. \quad (28)$$

**Lemma 4.4** (Verification of Janson's condition  $(J_2)$  and of [\(25\)](#)). *Let  $m \geq 2$  be a fixed integer, and suppose that one of the three assumptions  $(A_{\text{sub}})$ ,  $(A_{\text{crit}})$ , or  $(A_{\text{sup}})$  below occurs:*

$$\log d \ll \log \lambda \kappa_d \ll d, \quad (A_{\text{sub}})$$

$$\log \lambda \kappa_d \sim dx \text{ with } x \in (0, \infty), \quad (A_{\text{crit}})$$

$$d \ll \log \lambda \kappa_d \ll d^{\frac{3}{2}-\gamma} \text{ for some } \gamma \in \left(0, \frac{1}{2}\right). \quad (A_{\text{sup}})$$

Let  $\alpha$  and  $b$  as in [\(27\)](#) and [\(28\)](#). Then for every  $\tau \in \mathbb{R}$ , there exists  $r := r(d; \tau) > 0$  such that, for  $a := a(d; \tau)$  and  $\Lambda_r^{(d,m)}$  as in [\(12\)](#) and [\(13\)](#), condition  $(J_2)$  holds:

$$J(\Lambda_r^{(d,m)}, R, a; \mathbb{S}^{m-1}) = ba^{m-1} m \kappa_m \Lambda_r^{(d,m)} + \log(ba^{m-1}) - (m-1) \log[-\log(ba^{m-1})] - \log \alpha \rightarrow \tau. \quad (29)$$

Furthermore, [\(25\)](#) holds:  $a + \frac{1}{d} \log^{2+\gamma} \frac{1}{a} \rightarrow 0$  for some  $\gamma \in (0, \frac{1}{2})$ .

*Proof.* According to [\(12\)](#), we seek  $r := (1 + da^2)^{-\frac{1}{2}} \in (0, 1)$ . We start with the following asymptotics of [\(13\)](#), obtained by [Lemma 2.2](#), part 2.,

$$\begin{aligned} \Lambda_r^{(d,m)} &= \lambda \kappa_d \frac{(1-r^2)^{1+\frac{d-m}{2}} r^{m-2} (\frac{d}{2})^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2})} \left[ 1 + O\left(\frac{1}{dr^2}\right) \right] \\ &= \lambda \kappa_d d^{\frac{d}{2}} \frac{a^{d+2-m} (1+da^2)^{-\frac{d}{2}}}{2^{\frac{m}{2}-1} \Gamma(\frac{m}{2})} \left[ 1 + O\left(\frac{1}{d} + a^2\right) \right], \end{aligned}$$

provided that  $\frac{1}{dr^2} = \frac{1}{d} + a^2 \ll 1$ . In this case,

$$ba^{m-1} m \kappa_m \Lambda_r^{(d,m)} = \frac{\sqrt{2} \pi^{\frac{m-1}{2}}}{\Gamma(\frac{m}{2})} \lambda \kappa_d \frac{a}{(1 + \frac{1}{da^2})^{d/2}} \left[ 1 + O\left(\frac{1}{d} + a^2\right) \right], \quad (30)$$

Next,

$$\log(ba^{m-1}) = -(m-1) \log \frac{1}{a} + \log b, \quad (31)$$

and

$$-(m-1) \log[-\log(ba^{m-1})] = -(m-1) \log \log \frac{1}{a} - (m-1) \log(m-1) + o(1). \quad (32)$$

Adding (30), (31), (32) and  $-\log \alpha$  yields

$$ba^{m-1} m \kappa_m \Lambda_r^{(d,m)} + \log(ba^{m-1}) - (m-1) \log[-\log(ba^{m-1})] - \log \alpha \quad (33)$$

$$= \frac{\sqrt{2} \pi^{\frac{m-1}{2}}}{\Gamma(\frac{m}{2})} \lambda \kappa_d \frac{a}{(1 + \frac{1}{da^2})^{d/2}} \left[ 1 + O\left(\frac{1}{d} + a^2\right) \right] - (m-1) \log \frac{1}{a} - (m-1) \log \log \frac{1}{a} - \log B_m + o(1), \quad (34)$$

where

$$B_m := \frac{\alpha (m-1)^{m-1}}{b} = \frac{\pi^{\frac{m+1}{2}} (m-1)^{m-1}}{2^{\frac{3m-5}{2}} \Gamma(\frac{m}{2})^2} \quad (35)$$

follows from the expressions of  $\alpha$  in (27) and  $b$  in (28). We now observe by monotonicity and continuity in the variable  $a \in (0, 1)$  that the equation

$$\frac{\sqrt{2} \pi^{\frac{m-1}{2}}}{\Gamma(\frac{m}{2})} \lambda \kappa_d \frac{a}{(1 + \frac{1}{da^2})^{d/2}} = (m-1) \log \frac{1}{a} + (m-1) \log \log \frac{1}{a} + \log B_m + \tau \quad (36)$$

has a unique solution  $a := a(d; \tau)$ . Moreover, comparing both sides of this equation, we observe that any accumulation point of  $a(d; \tau)$  must equal 0 since  $\lambda \kappa_d \rightarrow \infty$ . Consequently,  $a(d; \tau) \rightarrow 0$  and we henceforth make this choice for the parameter  $a$  (where, for sake of clarity, we omit the dependency in  $d$  and  $\tau$ ). If we plug in (36) into (34), we obtain that (33) reduces to

$$ba^{m-1} m \kappa_m \Lambda_r^{(d,m)} + \log(ba^{m-1}) - (m-1) \log[-\log(ba^{m-1})] - \log \alpha = \tau + O\left(\left[\frac{1}{d} + a^2\right] \cdot \log \frac{1}{a}\right), \quad (37)$$

Note that, in order to get (29), it is enough to justify that the last error term in the right-hand side of (37) goes to zero. Actually, that last term will be negligible as soon as (17) and a fortiori (25) is satisfied. The remainder of the proof is then devoted to asymptotically relate  $\frac{1}{a}$  and  $\log \lambda \kappa_d$  in order to check that (25) holds in all three regimes.

To begin with, passing to the logarithm in (36) easily yields

$$\frac{d}{2} \log\left(1 + \frac{1}{da^2}\right) = \log \frac{A_m \lambda \kappa_d}{\frac{1}{a} \log \frac{1}{a}} - \frac{(m-1) \log \log \frac{1}{a} + \log B_m + \tau}{(m-1) \log \frac{1}{a}} + o\left(\frac{\log \log \frac{1}{a}}{\log \frac{1}{a}}\right),$$

where

$$A_m := \frac{\sqrt{2} \pi^{\frac{m-1}{2}}}{(m-1) \Gamma(\frac{m}{2})}. \quad (38)$$

Dropping the  $O(1)$  terms and dividing both sides by  $d$ , we have in particular

$$\frac{1}{d} \log \frac{1}{a} + \frac{1}{d} \log \log \frac{1}{a} + O\left(\frac{1}{d}\right) = \frac{1}{d} \log \lambda \kappa_d - \frac{1}{2} \log\left(1 + \frac{1}{da^2}\right). \quad (39)$$

We start with the subcritical case and assume that (A<sub>sub</sub>) holds. Since  $\log \lambda \kappa_d \ll d$ , the left-hand side of (39) is equivalent to  $\frac{1}{d} \log \frac{1}{a}$  (because  $a \rightarrow 0$ ), while the right-hand side is bounded from above by  $\frac{1}{d} \log \lambda \kappa_d$ . This entails that  $\frac{1}{d} \log \frac{1}{a} \rightarrow 0$ , and then, plugging this back into (39),  $\log(1 + \frac{1}{da^2}) \rightarrow 0$ . Thus  $da^2 \rightarrow \infty$ , so  $\log \frac{1}{a} = O(\log d)$ , and (39) becomes

$$\frac{1}{d} \log \lambda \kappa_d - \frac{1}{2da^2} (1 + o(1)) = O\left(\frac{\log d}{d}\right).$$

Because  $\log d \ll \log \lambda \kappa_d$ , this implies

$$\frac{1}{a} \sim \sqrt{2 \log \lambda \kappa_d} \quad (\text{subcritical regime}),$$

which results in  $\frac{1}{d} \log^2 \frac{1}{a} \rightarrow 0$  (so (25) holds for any  $\gamma \in (0, \frac{1}{2})$ ).

Next, for the critical case, let us now assume  $(A_{\text{crit}})$ , i.e.,  $\frac{1}{d} \log \lambda \kappa_d \rightarrow x \in (0, \infty)$ . Then (39) implies that  $\log \frac{1}{a} = O(d)$ , which in turn implies that  $\log(1 + \frac{1}{da^2}) = O(1)$ , i.e.,  $a \gtrsim \frac{1}{\sqrt{d}}$ . Hence, by (39) again,  $\frac{1}{d} \log \lambda \kappa_d - \frac{1}{2} \log(1 + \frac{1}{da^2}) \rightarrow 0$ , or in other words,

$$\frac{1}{a} \sim \sqrt{d(e^{2x} - 1)} \quad (\text{critical regime}),$$

which is stronger than (25).

Lastly, we consider the supercritical case and assume  $(A_{\text{sup}})$ . Then the inequality  $\frac{1}{d} \log \frac{1}{a} \leq \frac{1}{2} \log(1 + \frac{1}{da^2})$  (which follows from the concavity of the logarithm) and (39) give

$$\frac{1}{d} \log \lambda \kappa_d = O\left(\log\left(1 + \frac{1}{da^2}\right)\right).$$

Because  $\log \lambda \kappa_d \gg d$ , this forces  $da^2 \rightarrow 0$ , so (39) produces

$$(1 + o(1)) \log \frac{1}{a} = \log\left((\lambda \kappa_d)^{\frac{1}{d}} \sqrt{d}\right) + o(1),$$

that is

$$\frac{1}{a} \sim (\lambda \kappa_d)^{\frac{1}{d}} \sqrt{d} \quad (\text{supercritical regime}).$$

This will imply  $\frac{1}{d} \log^{2+\gamma} \frac{1}{a} \rightarrow 0$  and (25) (for some  $\gamma \in (0, \frac{1}{2})$ ) if we further assume that  $\log \lambda \kappa_d \ll d^{\frac{3}{2}-\gamma}$ , as stated in  $(A_{\text{sup}})$ .

Finally, we can see that the error term in (37) goes to zero because (25) and a fortiori (17) hold under each of the three stated conditions  $(A_{\text{sub}})$ ,  $(A_{\text{crit}})$  and  $(A_{\text{sup}})$ .  $\square$

**Scholium 4.5** (Summary of useful asymptotics derived in the proof of Lemma 4.4). *The sequence  $a$  fulfills the implicit asymptotic equality*

$$\frac{d}{2} \log\left(1 + \frac{1}{da^2}\right) = \log \frac{A_m \lambda \kappa_d}{\frac{1}{a} \log \frac{1}{a}} - \frac{(m-1) \log \log \frac{1}{a} + \log B_m + \tau}{(m-1) \log \frac{1}{a}} + o\left(\frac{1}{\log \frac{1}{a}}\right), \quad (40)$$

where we record

$$A_m := \frac{\sqrt{2} \pi^{\frac{m-1}{2}}}{(m-1) \Gamma(\frac{m}{2})}, \quad B_m := \frac{\pi^{\frac{m+1}{2}} (m-1)^{m-1}}{2^{\frac{3m-5}{2}} \Gamma(\frac{m}{2})^2}, \quad (41)$$

from (38) and (35). Furthermore we have, in all regimes,

$$\frac{1}{a} \sim \sqrt{d\left((\lambda \kappa_d)^{\frac{2}{d}} - 1\right)} \sim \begin{cases} \sqrt{2 \log \lambda \kappa_d}, & \text{under } (A_{\text{sub}}), \\ \sqrt{d(e^{2x} - 1)}, & \text{under } (A_{\text{crit}}), \\ (\lambda \kappa_d)^{\frac{1}{d}} \sqrt{d}, & \text{under } (A_{\text{sup}}). \end{cases} \quad (42)$$

We are now ready to prove Theorem 3 and Corollary 4.

*Proof of Theorem 3.* For  $\tau \in \mathbb{R}$ ,  $a := a(d; \tau)$ , and  $r := (1 + da^2)^{-\frac{1}{2}}$  as in [Lemma 4.4](#), the conditions to apply [Lemma 4.3](#) are in place, hence (recalling [\(24\)](#))

$$\lim_{d \rightarrow \infty} \mathbb{P}(h_\lambda^{(d,m)} \geq r) = e^{-e^{-\tau}}.$$

Now, notice that

$$\begin{aligned} \mathbb{P}(h_\lambda^{(d,m)} \geq r) &= \mathbb{P}\left(h_\lambda^{(d,m)} \geq (1 + da^2)^{-\frac{1}{2}}\right) \\ &= \mathbb{P}\left[d \log \frac{1}{\sqrt{1 - (h_\lambda^{(d,m)})^2}} \geq \frac{d}{2} \log\left(1 + \frac{1}{da^2}\right)\right]. \end{aligned}$$

Plugging in [\(40\)](#), we find

$$\mathbb{P}(h_\lambda^{(d,m)} \geq r) = \mathbb{P}\left[\log \frac{A_m \lambda \kappa_d}{\frac{1}{a} \log \frac{1}{a}} - d \log \frac{1}{\sqrt{1 - (h_\lambda^{(d,m)})^2}} \leq \frac{(m-1) \log \log \frac{1}{a} + \log B_m + \tau}{(m-1) \log \frac{1}{a}} + o\left(\frac{1}{\log \frac{1}{a}}\right)\right].$$

To conclude, it remains to express  $\log \frac{1}{a}$  and  $\log \log \frac{1}{a}$  in terms of  $\mathfrak{s}(d) = \log \sqrt{d\left((\lambda \kappa_d)^{\frac{2}{d}} - 1\right)}$ . According to [\(42\)](#),

$$\log \frac{1}{a} = \mathfrak{s}(d) + o(1), \quad \text{and also} \quad \log \log \frac{1}{a} = \log \mathfrak{s}(d) + o(1).$$

Hence

$$(m-1)d\mathfrak{s}(d) \left[ \frac{1}{d} \log \frac{A_m \lambda \kappa_d}{e^{\mathfrak{s}(d)} \mathfrak{s}(d)} - \log \frac{1}{\sqrt{1 - (h_\lambda^{(d,m)})^2}} \right] - (m-1) \log \mathfrak{s}(d) - \log B_m$$

converges in law to the standard Gumbel distribution. This establishes [Theorem 3](#) with

$$\mathfrak{a}(d; m) := (m-1)\mathfrak{s}(d) \log \frac{A_m \lambda \kappa_d}{\mathfrak{s}(d)} - (m-1)\mathfrak{s}(d)^2 - (m-1) \log \mathfrak{s}(d) - \log B_m, \quad (43)$$

and

$$\mathfrak{b}(d; m) := (m-1)d\mathfrak{s}(d), \quad (44)$$

where we recall that  $A_m$  and  $B_m$  have been defined at [\(41\)](#).  $\square$

*Proof of Corollary 4.* We get from [Theorem 3](#) that

$$\mathfrak{a}(d; m) - \mathfrak{b}(d; m) \log \frac{1}{\sqrt{1 - (h_\lambda^{(d,m)})^2}} = O_{\mathbb{P}}(1),$$

where  $X(d) = O_{\mathbb{P}}(1)$  means that  $\lim_{A \rightarrow \infty} \limsup_{d \rightarrow \infty} \mathbb{P}(|X(d)| > A) = 0$ . Multiplying by  $2\mathfrak{b}(d; m)^{-1}$  and taking the exponential function, it follows that

$$e^{\frac{2\mathfrak{a}(d; m)}{\mathfrak{b}(d; m)} \left(1 - (h_\lambda^{(d,m)})^2\right)} = 1 + \frac{1}{\mathfrak{b}(d; m)} \cdot O_{\mathbb{P}}(1),$$

where  $e^{\frac{2\mathfrak{a}(d; m)}{\mathfrak{b}(d; m)}} \sim (\lambda \kappa_d)^{\frac{2}{d}}$  from [\(43\)](#) and [\(44\)](#). Since  $\log \lambda \kappa_d \ll d^2$ , this is also equivalent to  $(\lambda \kappa_d)^{\frac{2}{d+1}}$ . We finish the proof similarly to that of [Theorem 1](#).  $\square$

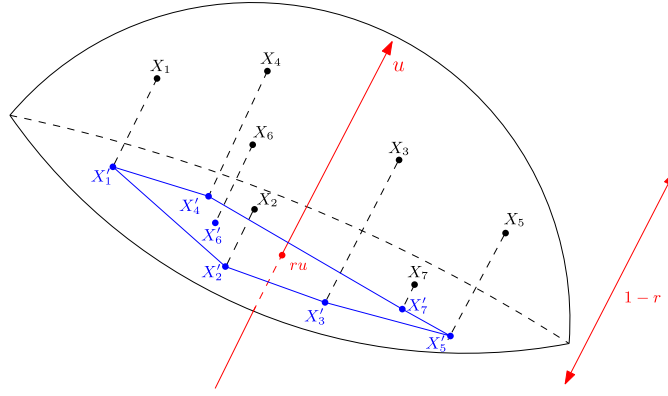


FIGURE 3. If  $ru \in \text{hull}(X'_1, \dots, X'_N)$ , then  $\rho_\lambda^{(d)}(u) > r$ .

### 5. CONSEQUENCES ON THE RADIUS-VECTOR FUNCTION

To conclude this work, we prove an easy consequence on the asymptotics of the radius-vector function  $\rho_\lambda^{(d)}$  given at (2). Recall that  $\rho_\lambda^{(d)} \leq h_\lambda^{(d)}$ , see Figure 1.

**Corollary 5.1** (Subcritical and supercritical regimes for  $\rho_\lambda^{(d)}$ ). *Let Assumption (H) hold.*

(i) *In the subcritical regime, under  $\log \lambda \kappa_d \ll d$ ,*

$$\limsup_{d \rightarrow \infty} \sqrt{\frac{d}{2 \log \lambda \kappa_d}} \rho_\lambda^{(d)} \leq 1, \quad \text{in probability.}$$

(ii) *In the supercritical regime, under the condition  $\log \lambda \kappa_d \gg d$*

$$\lim_{d \rightarrow \infty} (\lambda \kappa_d)^{\frac{2}{d+1}} (1 - \rho_\lambda^{(d)}) = \frac{1}{2}, \quad \text{in probability.}$$

*Proof.* Since  $\rho_\lambda^{(d)} \leq h_\lambda^{(d)}$ , (i) is an immediate consequence of Theorem 1, and (ii) will also follow from Theorem 1 if we prove that, for every  $\varepsilon \in (0, 1)$ ,

$$\mathbb{P}\left((\lambda \kappa_d)^{\frac{2}{d+1}} (1 - \rho_\lambda^{(d)}(u)) > \frac{1}{2} + \varepsilon\right) \xrightarrow{d \rightarrow \infty} 0. \quad (45)$$

Recall the definition of the cap  $\mathcal{C}^d(r; u) := \{x \in \mathbb{B}^d : \langle u, x \rangle > r\}$ . Let  $\{X_1, \dots, X_N\} = \mathcal{P}_\lambda^d \cap \mathcal{C}^d(r; u)$  and denote by  $X'_i$ ,  $1 \leq i \leq n$ , their projections onto the  $(d-1)$ -dimensional hyperplane  $\{x \in \mathbb{R}^d : \langle u, x \rangle = r\}$ , with  $r \in (0, 1)$  arbitrary. The number  $N$  has a Poisson distribution with parameter  $\ell(r) := \lambda |\mathcal{C}^d(r; u)|$ , and conditional on  $N$ , the points  $X'_i - ru$  are i.i.d. according to a symmetric distribution on  $\mathbb{R}^{d-1}$ . First, Wendel's formula [38] allows us to write

$$\begin{aligned} \mathbb{P}\left(0 \notin \text{hull}\{X'_1 - ru, \dots, X'_N - ru\} \mid N\right) &= \mathbb{1}_{\{N < d\}} + \mathbb{1}_{\{N \geq d\}} 2^{-(N-1)} \sum_{k=0}^{d-2} \binom{N-1}{k} \\ &= 1 - \mathbb{1}_{\{N \geq d\}} 2^{-(N-1)} \sum_{k=d}^N \binom{N-1}{k-1}. \end{aligned}$$

Next, we observe that  $ru \notin \text{hull}\{X'_1, \dots, X'_N\}$  on the event  $\{\rho_\lambda^{(d)}(u) \leq r\}$  (see Figure 3), so we get

$$\begin{aligned}
\mathbb{P}\left(\rho_\lambda^{(d)}(u) \leq r\right) &\leq \mathbb{E}\left[\mathbb{P}\left(0 \notin \text{hull}\{X'_1 - ru, \dots, X'_N - ru\} \mid N\right)\right] \\
&= 1 - \mathbb{E}\left[\mathbf{1}_{\{N \geq d\}} 2^{-(N-1)} \sum_{k=d}^N \binom{N-1}{k-1}\right] \\
&= 1 - \mathbb{P}(S_{N-1} \geq d),
\end{aligned} \tag{46}$$

where as in the proof of [Lemma 2.1](#), the random variable  $S_{N-1}$  conditional on  $N$  has the Binomial( $N-1, \frac{1}{2}$ ) distribution. We now let  $r$  depend on  $d$  and observe that, for  $r := 1 - (\lambda\kappa_d)^{-\frac{2}{d+1}}(\frac{1}{2} + \varepsilon)$ , we have

$$\rho_\lambda^{(d)}(u) \leq r \iff (\lambda\kappa_d)^{\frac{2}{d+1}} \left(1 - \rho_\lambda^{(d)}(u)\right) > \frac{1}{2} + \varepsilon,$$

so in order to get [\(45\)](#) it suffices to prove that the upper bound in [\(46\)](#) goes to 0, where  $N$  is a Poisson r.v. with mean  $\ell(r)$ . But

$$\begin{aligned}
\ell(r) &= \frac{\lambda\kappa_{d-1}}{2} \text{B}(1 - r^2; \frac{d+1}{2}, \frac{1}{2}) \\
&= -\log \mathbb{P}\left(h_\lambda^{(d)} \leq r\right) \\
&= \exp\left\{\log \lambda\kappa_d + \frac{d+1}{2} \log(1 - r^2) - \log r \sqrt{2\pi d} + o(1)\right\},
\end{aligned}$$

by [\(3\)](#), [\(4\)](#), and [\(9\)](#), provided that  $d \gg r^{-2}$ . In fact,

$$1 - r^2 = (\lambda\kappa_d)^{-\frac{2}{d+1}}(1 + 2\varepsilon) \left[1 + O\left((\lambda\kappa_d)^{-\frac{2}{d+1}}\right)\right],$$

so

$$\frac{d+1}{2} \log(1 - r^2) = -\log \lambda\kappa_d + \frac{d+1}{2} \log(1 + 2\varepsilon) + o(d).$$

We can then see that  $\ell(r) = (1 + 2\varepsilon)^{\frac{d}{2} + o(d)} \gg d$  and so  $N \gg d$ , w.h.p. Hence  $\mathbb{P}(S_{N-1} \geq d) \rightarrow 1$  and this completes the proof.  $\square$

**Remark** (Asymptotics for the volume ratio). *By integrating the  $d$ -th power of the radius-vector function  $\rho_\lambda^{(d)}$  over the whole sphere  $\mathbb{S}^{d-1}$ , we obtain the  $d$ -dimensional volume of  $K_\lambda^d$ , namely*

$$|K_\lambda^d| = \int_{\mathbb{S}^{d-1}} \sigma_{d-1}(du) \int_0^{\rho_\lambda^{(d)}(u)} t^{d-1} dt,$$

which by Fubini's theorem leads to

$$\mathbb{E}|K_\lambda^d| = \kappa_d \int_0^{+\infty} \mathbb{P}\left(\rho_\lambda^{(d)} > t\right) dt^{d-1} dt = \kappa_d \mathbb{E}\left(\rho_\lambda^{(d)}\right)^d. \tag{47}$$

In particular, in the case  $\log \lambda \kappa_d = \frac{d}{2} \log \frac{d}{2x} + o(d)$  with  $x > 0$ , which corresponds to the critical regime considered by Bonnet, Kabluchko and Turchi [7], we can recover the asymptotics for the volume ratio stated in their Theorem 3.1. Indeed, for any subsequence of  $d \rightarrow \infty$  we have, almost surely along a further sub-subsequence,

$$\begin{aligned} (\rho_\lambda^{(d)})^d &= \exp \left[ d \log \left( 1 - \frac{1}{2} (\lambda \kappa_d)^{-\frac{2}{d+1}} (1 + o(1)) \right) \right] \\ &= \exp \left[ d \log \left( 1 - \frac{x}{d} (1 + o(1)) \right) \right] \\ &\sim e^{-x}, \end{aligned}$$

using [Corollary 5.1](#), part (ii). This together with the dominated convergence theorem yields

$$\lim_{d \rightarrow \infty} \frac{\mathbb{E} |K_\lambda^d|}{\kappa_d} = e^{-x}.$$

In view of (47) and [Corollary 5.1](#), part (ii), we can conjecture that when  $d \ll \log \lambda \kappa_d \ll \frac{d}{2} \log(d)$ ,

$$\log \frac{\mathbb{E} |K_\lambda^d|}{\kappa_d} \sim -\frac{d}{2} (\lambda \kappa_d)^{-\frac{2}{d+1}}.$$

Similarly, when  $\log(\lambda \kappa_d) \gg \frac{d}{2} \log(d)$ , we also predict that

$$1 - \frac{\mathbb{E} |K_\lambda^d|}{\kappa_d} \sim \frac{d}{2} (\lambda \kappa_d)^{-\frac{2}{d+1}}.$$

We thank Matthias Reitzner for his suggestion which led us to this observation.

## APPENDIX

In this appendix, we state and prove [Proposition A](#), which is instrumental in deriving [Theorem 3](#). As exposed in [Section 4.2](#), Janson's result [[24](#), Lemma 8.1] says that, under  $(J_1)$  and  $(J_2)$ , the probability  $\mathbb{P}(\text{Cover}(\Lambda(a), R, a; M))$  of covering the manifold  $M$  by a union of geodesic balls,  $\bigcup_i B_M(x_i, a\rho_i) = M$ , where the  $(x_i, \rho_i)$ 's arise as the atoms of a Poisson point process with intensity  $\Lambda(a)v_M(dx) \otimes \mathbb{P}(R \in d\rho)$ , converges to the Gumbel distribution function as  $a \rightarrow 0$ . Our aim is to prove the following extension where the random radius  $R$  is allowed to depend on  $a$ . Instead of adapting Janson's result and rewriting the whole proof, we rather reduce to it using a coupling argument. Namely, if  $R_a$  converges to  $R$  in such a way that we may construct  $(R, R_a)$  so that

$$(1 - \eta)R \leq R_a \leq (1 + \eta)R \tag{48}$$

holds with high probability for some positive sequence  $\eta := \eta(a)$  vanishing sufficiently fast, then we may approximate (from above and below)  $\mathbb{P}(\text{Cover}(\Lambda, R_a, a; M))$  with similar probabilities where  $R_a$  is changed to  $(1 \pm \eta)R$ , which after replacing  $a$  by  $a/(1 \pm \eta)$  leads to the case handled by Janson.

In [Proposition A](#) below, condition (49) allows us to achieve the upper bound in (48) and is of course satisfied if the stronger condition  $(\forall \rho > 0, \mathbb{P}(R_a > \rho) \leq \mathbb{P}(R > \rho))$  holds for  $a > 0$  sufficiently small (e.g., if the sequence of functions  $\rho \mapsto \mathbb{P}(R_a > \rho)$  is eventually non-decreasing as  $a \rightarrow 0$ ), while condition (50) allows us to obtain the lower bound in (48).



**Proposition A** (Extension of Janson's result). *Let  $M$  be a  $\mathcal{C}^2$ ,  $D$ -dimensional compact Riemannian manifold, and let  $\Lambda := \Lambda(a) > 0$  and  $R$  fulfill Janson's conditions  $(J_1)$  and  $(J_2)$ . Suppose  $R_a$ ,  $a > 0$ , are positive random variables such that for every  $w > 0$  and all  $a$  sufficiently small,*

$$\sup_{\rho > 0} \frac{\mathbb{P}(R_a > \rho)}{\mathbb{P}\left(\left[1 + \frac{w}{\log \frac{1}{a}}\right] R > \rho\right)} \leq 1. \quad (49)$$

Suppose further that, as  $a \rightarrow 0$ ,

$$W_1(\log R_a, \log R) = o\left(\frac{1}{\log^2 \frac{1}{a}}\right). \quad (50)$$

Then  $(J)$  also holds with  $R_a$  in place of  $R$ :

$$\lim_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda, R_a, a; M)) = e^{-e^{-\tau}}.$$

*Proof.* First, with the assumptions  $(J_1)$  and  $(J_2)$  of Janson's theorem [24, Lemma 8.1] fulfilled for  $R$ , the convergence  $(J)$  holds:

$$\lim_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda, R, a; M)) = e^{-e^{-\tau}}.$$

To prove the same for  $\text{Cover}(\Lambda, R_a, a; M)$ , we use a coupling argument. Let  $w > 0$  and

$$\eta_a := \frac{w}{\log \frac{1}{a}}. \quad (51)$$

By (49), it holds for all  $a$  sufficiently small that

$$\forall \rho > 0, \quad \mathbb{P}(R_a > \rho) \leq \mathbb{P}((1 + \eta_a)R > \rho).$$

Applying the inverse method, that is, considering the generalized inverses  $F_a^{-1}$  and  $F^{-1}$  of the distribution functions of  $R_a$  and  $R$  respectively, and setting  $R_a := F_a^{-1}(U)$  and  $R := F^{-1}(U)$  for a uniform variable  $U$  in  $[0, 1]$ , we may then suppose for every small  $a > 0$  that  $R_a$  and  $R$  are coupled so that

$$R_a \leq (1 + \eta_a)R, \quad \text{almost surely.}$$

We then introduce independent, uniformly distributed variables  $X_i$ ,  $i \geq 1$ , on  $M$  and, for every small  $a > 0$ , an independent Poisson-distributed variable  $N_a$  with mean  $\Lambda(a)v_M(M)$ , as well as a further independent family  $(R_{a,i}, R_i)_{i \geq 1}$  of i.i.d. copies of  $(R_a, R)$ . Thus, for every small  $a > 0$ , we have constructed a Poisson point process  $\Xi_a := \{X_i, R_{a,i}, R_i\}_{1 \leq i \leq N_a}$  on  $M \times (0, \infty)^2$  whose projections  $\Xi_{1,a} := (X_i, R_{a,i})_{1 \leq i \leq N_a}$  and  $\Xi_{2,a} := (X_i, R_i)_{1 \leq i \leq N_a}$  have intensity  $\Lambda v_M(dx) \otimes \mathbb{P}(R_a \in d\rho)$  and  $\Lambda v_M(dx) \otimes \mathbb{P}(R \in d\rho)$  respectively. In particular, with this construction the two covering events are given by

$$\text{Cover}(\Lambda, R_a, a; M) = \left\{ \bigcup_{\Xi_{1,a}} B_M(X_i, aR_{a,i}) = M \right\} \quad \text{and} \quad \text{Cover}(\Lambda, R, a; M) = \left\{ \bigcup_{\Xi_{2,a}} B_M(X_i, aR_i) = M \right\}.$$

Since  $R_{a,i} \leq (1 + \eta_a)R_i$  almost surely for all  $i \geq 1$ , we get

$$\begin{aligned} \mathbb{P}(\text{Cover}(\Lambda, R_a, a; M)) &= \mathbb{P}\left(\bigcup_{\Xi_{1,a}} B_M(X_i, aR_{a,i}) = M\right) \\ &\leq \mathbb{P}\left(\bigcup_{\Xi_{2,a}} B_M(X_i, a(1 + \eta_a)R_i) = M\right) \\ &= \mathbb{P}(\text{Cover}(\Lambda, R, (1 + \eta_a)a; M)), \end{aligned}$$

and therefore,

$$\begin{aligned} \limsup_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda, R_a, a; M)) &\leq \limsup_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda, R, (1 + \eta_a)a; M)) \\ &= \limsup_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda^-, R, a; M)), \end{aligned}$$

with  $\Lambda^-(a) := \Lambda(\frac{a}{1+\eta_a})$  and the last two limsup being equal because  $a \mapsto a/(1 + \eta_a)$  can be inverted when  $a > 0$  is sufficiently small. This limsup is in fact a true limit, as we now show. Recalling  $J(\Lambda, R, a; M) \rightarrow \tau$  by (J<sub>2</sub>) and  $\eta_a := w/\log \frac{1}{a}$  in (51),

$$\begin{aligned} \Lambda^-(a) &= \frac{(1 + \eta_a)^D}{b(R; M)a^D v_M(M)} \left( \log \frac{(1 + \eta_a)^D}{b(R; M)a^D} + D \log \log \frac{(1 + \eta_a)^D}{b(R; M)a^D} + \log \alpha(R) + \tau + o(1) \right) \\ &= \frac{1 + D\eta_a + o(\eta_a)}{b(R; M)a^D v_M(M)} \left( \log \frac{1}{b(R; M)a^D} + D \log \log \frac{1}{b(R; M)a^D} + \log \alpha(R) + \tau + o(1) \right) \\ &= \left[ 1 + \frac{Dw}{\log \frac{1}{a}} + o\left(\frac{1}{\log \frac{1}{a}}\right) \right] \Lambda(a). \end{aligned}$$

This entails  $J(\Lambda^-, R, a; M) \rightarrow \tau + D^2w$ , so Janson's theorem applies with

$$\lim_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda^-, R, a; M)) = e^{-e^{-\tau - D^2w}}.$$

Letting  $w \rightarrow 0^+$ , we have proved

$$\limsup_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda, R_a, a; M)) \leq e^{-e^{-\tau}}.$$

To establish the other direction, we keep  $w > 0$  and  $\eta_a := w/\log \frac{1}{a}$ , as well as the Poisson point process  $\Xi_a := \{X_i, R_{a,i}, R_i\}_{1 \leq i \leq N_a}$  and its projections  $\Xi_{1,a}$  and  $\Xi_{2,a}$ , except that this time  $(R_{a,i}, R_i)_{i \geq 1}$  are i.i.d. copies of a different coupling of  $(R_a, R)$ . Namely, the Kantorovich–Rubinstein theorem [15, Theorem 11.8.2] allows us to choose  $(R_a, R)$  so that

$$W_1(\log R_a, \log R) = \mathbb{E} |\log R_a - \log R|. \quad (52)$$

We then restrict  $\Xi_a$  by keeping only the radii  $R_{a,i}$  such that  $R_{a,i} \geq (1 - \eta_a)R_i$ :

$$\tilde{\Xi}_a := \left\{ (X_i, R_{a,i}, R_i) \in \Xi_a : R_{a,i} \geq (1 - \eta_a)R_i \right\}.$$

Hence  $\tilde{\Xi}_{2,a} := \{(X_i, R_i) : (X_i, R_{a,i}, R_i) \in \tilde{\Xi}_a\} \subseteq \Xi_{2,a}$  has a smaller intensity,  $\tilde{\Lambda}v_M(dx) \otimes \mathbb{P}(R_a \in d\rho)$  with  $\tilde{\Lambda}(a) := \Lambda(a)\mathbb{P}(R_a \geq (1 - \eta_a)R)$ . Since, by Markov's inequality, (52) and (50),

$$\begin{aligned} \mathbb{P}(R_a < (1 - \eta_a)R) &= \mathbb{P}\left(\log \frac{R}{R_a} > \log \frac{1}{1 - \eta_a}\right) \\ &\leq \frac{1}{\log \frac{1}{1 - \eta_a}} \mathbb{E} |\log R_a - \log R| \\ &\leq \frac{\log \frac{1}{a}}{w} W_1(\log R_a, \log R) \\ &= o\left(\frac{1}{\log \frac{1}{a}}\right), \end{aligned}$$

we have  $\tilde{\Lambda}(a) = \Lambda(a)[1 + o(\log^{-1}(\frac{1}{a}))]$ , which by (J<sub>2</sub>) implies that  $J(\tilde{\Lambda}, R, a; D) \rightarrow \tau$ , i.e.,

$$\tilde{\Lambda}(a) = \frac{1}{b(R; M)a^D v_M(M)} \left( \log \frac{1}{b(R; M)a^D} + D \log \log \frac{1}{b(R; M)a^D} + \log \alpha(R) + \tau + o(1) \right). \quad (\tilde{J}_2)$$

Now,

$$\begin{aligned} \mathbb{P}(\text{Cover}(\Lambda, R_a, a; M)) &= \mathbb{P}\left(\bigcup_{\tilde{\Xi}_{1,a}} B_M(X_i, aR_{a,i}) = M\right) \\ &\geq \mathbb{P}\left(\bigcup_{\tilde{\Xi}_a} B_M(X_i, aR_{a,i}) = M\right) \\ &\geq \mathbb{P}\left(\bigcup_{\tilde{\Xi}_{2,a}} B_M(X_i, a(1 - \eta_a)R_i) = M\right) \\ &= \mathbb{P}(\text{Cover}(\Lambda^+, R_i, (1 - \eta_a)a; M)) \end{aligned}$$

and therefore (in the same way as above),

$$\liminf_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda, R_a, a; M)) \geq \liminf_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda^+, R, a; M)),$$

where, recalling ( $\tilde{J}_2$ ) and  $\eta_a := w/\log \frac{1}{a}$ ,

$$\begin{aligned} \Lambda^+(a) &:= \tilde{\Lambda}\left(\frac{a}{1 - \eta_a}\right) \\ &= \frac{(1 - \eta_a)^D}{b(R; M)a^D v_M(M)} \left( \log \frac{(1 - \eta_a)^D}{b(R; M)a^D} + D \log \log \frac{(1 - \eta_a)^D}{b(R; M)a^D} + \log \alpha(R) + \tau + o(1) \right) \\ &= \frac{1 - D\eta_a + o(\eta_a)}{b(R; M)a^D v_M(M)} \left( \log \frac{1}{b(R; M)a^D} + D \log \log \frac{1}{b(R; M)a^D} + \log \alpha(R) + \tau + o(1) \right) \\ &= \left[ 1 - \frac{Dw}{\log \frac{1}{a}} + o\left(\frac{1}{\log \frac{1}{a}}\right) \right] \Lambda(a). \end{aligned}$$

This entails  $J(\Lambda^+, R, a; M) \rightarrow \tau - D^2w$ , so Janson's theorem applies again with

$$\lim_{a \rightarrow 0} \mathbb{P}(\text{Cover}(\Lambda^+, R, a; M)) = e^{-e^{-\tau + D^2w}}.$$

It remains to let  $w \rightarrow 0^+$  to complete the proof.  $\square$

#### REFERENCES

- [1] F. L. Baccelli and E. O'Reilly, *The stochastic geometry of unconstrained one-bit data compression*, Electron. J. Probab. **24** (2019), Paper No. 138, 27 pp. [MR 4040998](#)
- [2] P. N. Balister and B. Bollobás, *Bond percolation with attenuation in high dimensional Voronoï tilings*, Random Structures Algorithms **36** (2010), no. 1, 5–10. [MR 2591043](#)
- [3] I. Bárány and Z. Füredi, *On the shape of the convex hull of random points*, Probab. Theory Related Fields **77** (1988), no. 2, 231–240. [MR 927239](#)
- [4] I. Bárány and M. Reitzner, *On the variance of random polytopes*, Adv. Math. **225** (2010), no. 4, 1986–2001. [MR 2680197](#)
- [5] M. Becker and E. L. Stark, *On a hierarchy of quolynomial inequalities for  $\tan x$* , Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. (1978), no. 602-633, 133–138. [MR 580435](#)
- [6] G. Bonnet, G. Chasapis, J. Grote, D. Temesvari, and N. Turchi, *Threshold phenomena for high-dimensional random polytopes*, Commun. Contemp. Math. **21** (2019), no. 5, 1850038, 30. [MR 3980692](#)
- [7] G. Bonnet, Z. Kabluchko, and N. Turchi, *Phase transition for the volume of high-dimensional random polytopes*, Random Structures Algorithms **58** (2021), no. 4, 648–663. [MR 4258130](#)
- [8] G. Bonnet, D. Dadush, U. Grupel, S. Huiberts and G. Livshyts, *Asymptotic bounds on the combinatorial diameter of random polytopes*, in *38th International Symposium on Computational Geometry*, Art. No. 18, 15 pp., LIPIcs. Leibniz Int. Proc. Inform., 224, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern (2022). [MR 4470897](#)
- [9] G. Bonnet and E. O'Reilly, *Facets of spherical random polytopes*, Math. Nachr. **295** (2022), no. 10, 1901–1933; [MR 4515438](#)
- [10] C. Buchta, *An identity relating moments of functionals of convex hulls*, Discrete Comput. Geom. **33** (2005), no. 1, 125–142. [MR 2105754](#)
- [11] P. Calka, T. Schreiber, and J. E. Yukich, *Brownian limits, local limits and variance asymptotics for convex hulls in the ball*, Ann. Probab. **41** (2013), no. 1, 50–108. [MR 3059193](#)
- [12] P. Calka and J. E. Yukich, *Variance asymptotics for random polytopes in smooth convex bodies*, Probab. Theory Related Fields **158** (2014), no. 1-2, 435–463. [MR 3152787](#)
- [13] D. Chakraborti, T. Tkocz, and B.-H. Vritsiou, *A note on volume thresholds for random polytopes*, Geom. Dedicata **213** (2021), 423–431. [MR 4278336](#)
- [14] R. Conijn, M. Irlbeck, Z. Kabluchko, and T. Müller, *Poisson-Voronoi percolation in high dimensions*. In preparation.
- [15] R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002, Revised reprint of the 1989 original. [MR 1932358](#)
- [16] M. E. Dyer, Z. Füredi, and C. McDiarmid, *Volumes spanned by random points in the hypercube*, Random Structures Algorithms **3** (1992), no. 1, 91–106. [MR 1139489](#)
- [17] B. Efron, *The convex hull of a random set of points*, Biometrika **52** (1965), 331–343. [MR 207004](#)
- [18] D. Gatzouras and A. Giannopoulos, *Threshold for the volume spanned by random points with independent coordinates*, Israel J. Math. **169** (2009), 125–153. [MR 2460902](#)
- [19] D. M. Gordon, *Percolation in High Dimensions*, J. London Math. Soc. **44** (1991), no. 2, 373–384. [MR 1136447](#)
- [20] O. Guédon, *Concentration phenomena in high dimensional geometry*, Journées MAS 2012, ESAIM Proc., vol. 44, EDP Sci., Les Ulis, 2014, pp. 47–60. [MR 3178607](#)
- [21] A. Gusakova, Z. Kabluchko, and C. Thäle, *Sectional Voronoi tessellations: Characterization and high-dimensional limits*, 2023. [arXiv:2301.03297](#)
- [22] A. Gusakova and C. Thäle, *The volume of simplices in high-dimensional Poisson-Delaunay tessellations*, Ann. H. Lebesgue **4** (2021), 121–153. [MR 4213157](#)

- [23] J. Hörrmann, D. Hug, M. Reitzner, and C. Thäle, *Poisson polyhedra in high dimensions*, Adv. Math. **281** (2015), 1–39. [MR 3366836](#)
- [24] S. Janson, *Random coverings in several dimensions*, Acta Math. **156** (1986), no. 1-2, 83–118. [MR 822331](#)
- [25] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous univariate distributions.*, second edition ed., Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, vol. 1, John Wiley & Sons, Inc., New York, 1994, A Wiley-Interscience Publication. [MR 1299979](#)
- [26] Z. Kabluchko, *Angles of random simplices and face numbers of random polytopes*, Adv. Math. **380** (2021), Paper No. 107612, 68. [MR 4205707](#)
- [27] ———, *Recursive scheme for angles of random simplices, and applications to random polytopes*, Discrete Comput. Geom. **66** (2021), no. 3, 902–937. [MR 4310599](#)
- [28] ———, *Face numbers of high-dimensional Poisson zero cells*, Proc. Amer. Math. Soc. **151** (2023), no. 1, 401–415. [MR 4504634](#)
- [29] E. O’Reilly, *Thin-shell concentration for zero cells of stationary Poisson mosaics*, Adv. in Appl. Math. **117** (2020), 102017, 24 pp.. [MR 4072799](#)
- [30] M. Reitzner, *Random polytopes and the Efron-Stein jackknife inequality*, Ann. Probab. **31** (2003), no. 4, 2136–2166. [MR 2016615](#)
- [31] A. Rényi and R. Sulanke, *Über die konvexe Hülle von  $n$  zufällig gewählten Punkten*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1963), 75–84. [MR 156262](#)
- [32] ———, *Über die konvexe Hülle von  $n$  zufällig gewählten Punkten. II*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3** (1964), 138–147. [MR 169139](#)
- [33] F. Santos, *A counterexample to the Hirsch conjecture*, Ann. of Math. (2) **176** (2012), no. 1, 383–412. [MR 2925387](#)
- [34] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, expanded ed., Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. [MR 3155183](#)
- [35] R. Schneider and J. A. Wieacker, *Random polytopes in a convex body*, Z. Wahrsch. Verw. Gebiete **52** (1980), no. 1, 69–73. [MR 568260](#)
- [36] C. Schütt, *Random polytopes and affine surface area*, Math. Nachr. **170** (1994), 227–249. [MR 1302377](#)
- [37] N. M. Temme, *Special functions*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996, An introduction to the classical functions of mathematical physics. [MR 1376370](#)
- [38] J. G. Wendel, *A problem in geometric probability*, Math. Scand. **11** (1962), 109–111. [MR 146858](#)

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