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RATIONAL AND LACUNARY ALGEBRAIC CURVES

GEORGES COMTE AND SÉBASTIEN TAVENAS

ABSTRACT. We give a bound on the number \mathcal{Z} of intersection points in a ball of the complex plane, between a rational curve and a lacunary algebraic curve $Q = 0$. This bound depends only on the lacunarity diagram of Q , and in particular is uniform in the coefficients of Q . Our bound shows that $\mathcal{Z} = O(dm)$, where d is the degree of Q and m is the number of its monomials.

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INTRODUCTION

By the classical Bézout theorem (see [4] for the original article and, for instance, [5]) for any positive integer n , the number of isolated complex solutions of a polynomial system $F_1 = \dots = F_n = 0$, $F_1, \dots, F_n \in \mathbb{C}[X_1, \dots, X_n]$, is at most the product $d_1 \cdots d_n$ of the degrees of the F_i 's. Requiring the polynomials have prescribed support, one can improve on this bound: for instance in case all polynomials F_1, \dots, F_n have the same given support $S \subset \mathbb{N}^n$, by Kushnirenko's theorem the number of isolated complex solutions in $(\mathbb{C}^*)^n$ is at most $n! \text{Vol}_n(\Delta_S)$, where Δ_S is the convex hull of S in \mathbb{R}^n (see [16]). More accurately, in case S_i is the support of F_i , $i = 1, \dots, n$, by Bernstein's theorem, often called BKK's theorem, the number of isolated complex solutions in $(\mathbb{C}^*)^n$ is at most the Minkowski mixed volume of $\Delta_{S_1}, \dots, \Delta_{S_n}$ (see [3, 16, 17]). In some sense those improvements allow to measure how far the system is from the simplest situation where it can be solved by composition in order to eliminate one by one the variables, to eventually give a univariate polynomial of degree $d_1 \cdots d_n$.

The case of real polynomial systems and their real solutions has been extensively investigated in particular since the seminal work of A. Khovanskii, (see [12], and for instance [22] for an overview, and [18] for historical information), although some basic problems still resist, even in the lowest degree. For instance, there is still no answer to the question: for a fixed number $n \geq 2$ of variables, is the number of isolated solutions of the system polynomially bounded with respect to the sizes of the supports of the polynomials F_1 and F_2 (see for instance [14] on this question, for $n = 2$)?

We consider in what follows the complex bivariate case, that is the case of the intersection of two complex algebraic plane curves. More specifically we consider the case of a given and fixed rational curve (P_1, P_2) in \mathbb{C}^2 (containing the origin of \mathbb{C}^2) and of an algebraic curve $Q(X, Y) = 0$, the coefficients $\lambda \in \mathbb{C}^m$ and the degree d of Q being parameters of the problem. Actually, we are interested in the asymptotics of bounds on the number of intersection points of the two curves, uniformly with respect to λ , as $d \rightarrow +\infty$ (see Example 5.20 and Remark 5.21).

For simplicity, assume here that $P_1, P_2 \in \mathbb{C}[X]$. Then, of course in this situation, since one can compose the rational parametrization (P_1, P_2) with Q , the number of intersection points of the two curves $(P_1, P_2)(\mathbb{C})$ and $Q(X, Y) = 0$ in \mathbb{C}^2 is simply the degree of the univariate polynomial $f = Q(P_1, P_2)$, whatever the specific forms of P_1, P_2 and Q are. To obtain nontrivial bounds we thus count the number of intersection points only in a ball of \mathbb{C}^2 centred at the origin and of fixed radius, for instance in a ball $B(0, \rho)$, $\rho > 0$, such that $(P_1, P_2)^{-1}(B(0, \rho)) \subset D_{R_\rho}$, for some $R_\rho > 0$. Such a positive number R_ρ exists in the case P_1 and P_2 are (non constant) polynomials, since $|P_i(z)| \rightarrow +\infty$ when $|z| \rightarrow +\infty$. We then consider, instead of parameters in D_{R_ρ} , parameters in the unit disc D_1 (see Remark 5.8). In the general rational case, we may have $(P_1, P_2)(z) \rightarrow 0$ when $|z| \rightarrow +\infty$. In this case we compute the additional number of intersection points between $Q(X, Y) = 0$ and the branch of the rational curve (P_1, P_2) corresponding to parameters going to infinity, in the same way, using the transformation $z \mapsto 1/z$ of the parameters.

Denoting $\max(\deg(P_1), \deg(P_2))$ by D , the number of zeros $\mathcal{Z} = \mathcal{Z}_{P_1, P_2, Q}$ of f in \bar{D}_1 may still be dD , the maximal expected number of zeros, depending on the choice of P_1, P_2 and Q (see Remark 5.19). On the other side, considering the maximal multiplicity b of $f = f_\lambda$ at the origin, with respect to all possible choices of coefficients $\lambda \in \mathbb{C}^m$ of Q such that f is not identically 0 (b is called the *Bautin index of the family* f_λ , see Definition 1.2 and Proposition 1.4), any small variation of the constant term of Q will create b distinct roots of f , as closed as desired from the origin. In consequence, the question of a bound for \mathcal{Z} , uniform with respect to the coefficients λ of Q , is the question of how many zeros of f we create in \bar{D}_1 in addition to the amount of b imposed zeros, when freely moving the parameters of the system, and how far can this number be under dD .

We answer this question for lacunary curves $Q(X, Y) = 0$: we show in Theorem 5.15 that \mathcal{Z} is bounded from above by an expression depending on the *lacunarity diagram* of Q (see Definition 4.1) and on α_0, a_0 , the initial coefficients of P_1 and P_2 (Remark 5.19 shows that the dependency on some coefficients of P_1 and P_2 of our bound is unavoidable). This bound lies in $]b, dD[$, with $b < dD$, in case Q is sufficiently sparse and α_0 and a_0 are not too closed to 0.

The lacunarity of Q is a necessary condition for having nontrivial bounds on \mathcal{Z} , and the reason is easy to explain. Indeed, m being the dimension of the space of parameters of Q , by solving $\min(m - 1, dD)$ linear equations with m variables, one can cancel the first $\min(m - 1, dD)$ monomials of f , and thus $b \geq \min(m - 1, dD)$. It follows that m , which could be at most $(d + 1)(d + 2)/2$, has in fact to be less than dD when one searches better bounds than dD for \mathcal{Z} .

It is worth noticing, on the other side, that contrary to what happens in the real case, \mathcal{Z} cannot depend only on the number of monomials of P and Q , but has to depend also on d . Indeed, the lacunarity conditions for Q we consider (see conditions (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{3a}) and (\mathcal{L}_{3b}) in Section 5) are based on a fast growth of the degrees of the homogeneous parts of Q (such as geometric progressions) and allow us to compute¹ the exact value of b (see Proposition 5.12), showing that $b \geq d \min(\nu_1, \nu_2)$ (where ν_1 and ν_2 are the multiplicities of P_1 and P_2 respectively in 0). This lower bound for b in turn shows that \mathcal{Z} cannot be independent on d , whatever the number of monomials P may have.

This observation leaves little scope between b and dD to find better bounds than dD ; A goal which is however achieved in our main Theorem 5.15. For instance (see Remark 5.21), for d large enough a condition allowing $\mathcal{Z} \in]b, dD[$ is

$$\tau \log \left(\frac{1}{|\alpha_0 a_0|} \right) \leq \frac{D}{5} - \max(\nu_1, \nu_2) \log 2,$$

where τ is the maximal amount of monomials of degree d in Q .

Notice in particular that our lacunarity conditions allow in such examples (Example 5.20 and Remark 5.21) the total number of monomials in Q to be unbounded with respect to the degree of Q , contrary to the lacunarity conditions usually considered in the theory of fewnomials. Moreover the dependency of our bounds in the number of monomials of Q is not exponential, contrary to the case of most of the bounds so far obtained for polynomial systems with a fixed number of monomials (see again [22, Section 6.2] for bounds in the real case). In the real plane, a polynomial bound in $\deg(F)$ and t is obtained in [14] for the number of solutions of the polynomial system $F(X, Y) = G(X, Y) = 0$, with G having t monomials. This bounds turns

¹This computation is otherwise generally not tractable, specially in the analytic case. Our lacunarity conditions guarantee here that for any $\lambda \in \mathbb{C}^m$, the two curves have isolated intersections points.

out to be $O(\deg(F)^3t + \deg(F)^2t^3)$, in contrast to the bounds obtained in this article that may be linear in d and τ (see (45) and (46)).

One can next wonder whether, for two given algebraic curves $Q_1(X, Y) = 0$ and $Q_2(X, Y) = 0$, it is possible to bound the number of their intersection points in a small ball (centred at a point outside the coordinate axes²) by a polynomial function in the number of monomials of Q_1 and Q_2 . This would imply in particular a bound on the multiplicity of the roots outside the coordinate axes. In this direction a polynomial bound of the mixed form $O(\deg(F)^2t^2)$ is obtained in [15] for the multiplicity of an isolated solution (with nonzero coordinates) of the system $F(x, y) = G(x, y) = 0$, with G having again t monomials. Moreover, this would bring us closer to proving the τ -conjecture for multiplicities defined in [15] (Hrubeš noticed [11] that it derives from the *real* τ -conjecture introduced in [13]). This family of questions and conjectures on lacunary bivariate polynomials are deep, since for instance both the real τ and τ -conjectures have dramatic consequences in algebraic complexity; They imply almost the separation of both main algebraic complexity classes VP and VNP (for an introduction to the VP versus VNP see for example [21, 6]).

An other phenomenon of importance, arising from part *B* of Hilbert's 16th problem, is the question of the accumulation of limit cycles of trajectories of planar polynomial vector fields. In this context, the Bautin index of the difference, between the Poincaré first return map of a polynomial deformation of the planar field and the identity map, is a bound on the number of cycles of the vector field. Localizing those cycles therefore consists in knowing the maximal radius of the disc where only b zeros occur, the number of zeros that one necessarily finds, as observed above, in any ball centred at zero, say (see for instance among many other references [1, 2, 8, 10, 19, 24]). We provide a lower bound for this radius in our algebraic context, that is, we give in Theorem 5.15 a lower bound in terms of the lacunarity diagram of Q , α_0 and a_0 , for the radius ρ of a disc $\bar{D}_\rho \subset \bar{D}_1$ in which f_λ has only b zeros, uniformly in λ .

In [7, Section 3.4] the case of a transcendental analytic lacunary curve $(x, h(x))$ has been considered, in order to obtain a bound for the number of zeros of $Q(x, h(x))$ on \bar{D}_1 , polynomial in d . As explained above, we consider instead here the quite opposite case where the curve is rational and the polynomial family Q is lacunary. The challenge being here to find some room between b and dD for \mathcal{Z} , and we indeed show that there is some. In both cases (analytic in [7] and rational here), we start from a Jensen-Nevanlinna type estimate of zeros in Bernstein classes (see [20, Section 2.2], [24], and Remark 5.22), all the information we need to obtain an estimation of \mathcal{Z} is contained in the *Bautin matrix* M of f_λ (see Section 4), defined as the matrix of the linear map sending the parameters λ of Q to the coefficients

²The multiplicity can be made arbitrarily large on the coordinate axes without altering the number of monomials by multiplying Q_1 and Q_2 by a large monomial.

of the polynomial f_λ . In particular b appears as the rank of the lowest row in M independent of the above rows, and \mathcal{Z} may be estimated from b , the module δ of a non zero minor of maximal size in M , and from the dimension σ of the Bautin ideal. Consequently our method consists in the reduction of M : we provide in Section 2 and 3 an explicit Gauß reduction of M , that we can perform by blocks thanks to our lacunarity conditions. Crucial technical tools to achieve the reduction are the combinatorial Lemmas 2.5 and 2.7. A special case occurs in this reduction, enlightening a geometrical degeneration condition on the parametrization (P_1, P_2) , the case where P_1 and P_2 have proportional initial jets at the origin, up to some non trivial order > 1 . The reduction of the Bautin matrix in the singular case is specifically treated in Section 3. In this singular case, b and δ differ from their values in the regular case, where P_1 and P_2 have no proportional jets.

In Section 6 we show how to reduce the general case of a rational curve to the case of $(P_1, P_2)(\mathbb{C})$, with $P_1, P_2 \in \mathbb{C}[X]$, under our lacunarity conditions, and therefore from now on and up to the final Section 6, we only consider the polynomial case.

1. INTERSECTION OF A CURVE AND A FAMILY OF CURVES

For the convenience of the reader, we recall here the notation of [25, 7], in order to adapt it to our context.

In what follows, for a real number $R > 0$, \bar{D}_R is the closed disc $\{z \in \mathbb{C}, |z| \leq R\}$ in \mathbb{C} . For $D_1, D_2 \in \mathbb{N} \setminus \{0\}$ and complex numbers $a_0^i, \dots, a_{D_i}^i$, $i = 1, 2$, we consider

$$(1) \quad P(X) = (P_1(X), P_2(X)) = \left(\sum_{k=0}^{D_1} a_k^1 X^k, \sum_{k=0}^{D_2} a_k^2 X^k \right),$$

a nonzero polynomial mapping of degree $D = \max\{D_1, D_2\}$, sending $0 \in \mathbb{C}$ to $0 \in \mathbb{C}^2$ (thus $a_0^1 = a_0^2 = 0$), and we denote by M_R the real number

$$(2) \quad M_R = \max_{z \in \bar{D}_R, i=1,2} |P_i(z)|.$$

Let $m \in \mathbb{N}$, and Q_i , $i = 1, \dots, m$, be a family of monomials of $\mathbb{C}[X, Y]$ of degree at most d , for some integer $d \in \mathbb{N}$. For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$, we denote by Q_λ the following polynomial of degree at most d of $\mathbb{C}[X, Y]$

$$Q_\lambda = \sum_{i=1}^m \lambda_i Q_i.$$

In what follows we are interested in the set of zeros in \bar{D}_R (actually in $\bar{D}_{R/4}$) of the linear family (with parameters $\lambda_1, \dots, \lambda_m$) of polynomial functions

$$(3) \quad f_\lambda(z) = Q_\lambda(P(z)) = \sum_{i=1}^m \lambda_i Q_i(P(z)).$$

Geometrically this is the number of intersection points, for $z \in \bar{D}_R$, between the plane algebraic curves $z \mapsto P(z)$ of degree D of \mathbb{C}^2 and $Q_\lambda = 0$ of degree d of \mathbb{C}^2 .

Remark 1.1. If $Q_\lambda(P(z)) = 0$ for any $\lambda \in \mathbb{C}^m$ and any $z \in \bar{D}_R$, then for any $i = 1, \dots, m$, $Q_i(P) = 0$ on \bar{D}_R . We will always consider families Q_1, \dots, Q_m such that for at least one parameter $\lambda \in \mathbb{C}^m$, $f_\lambda(P)$ is not the zero polynomial. We can easily achieve this condition for instance by taking the monomial of constant value 1 in the family, or by allowing no zero polynomial among the P_i 's. Moreover we will consider zeros of f_λ only for such parameters, and in this case the number of zeros of f_λ is a finite number.

Denoting, for $i = 1, \dots, m$,

$$Q_i(P(z)) = \sum_{k=0}^{dD} c_k^i z^k, \quad \text{and} \quad B_R = \sup_{z \in \bar{D}_R, i=1, \dots, m} |Q_i(P(z))|,$$

by Cauchy estimate we have for any $i = 1, \dots, m$ and any $k = 0, \dots, dD$

$$(4) \quad |c_k^i| \leq \frac{B_R}{R^k}.$$

We write f_λ as a polynomial with coefficients linear forms v_0, \dots, v_{dD} of parameters $\lambda_1, \dots, \lambda_m$. Namely

$$(5) \quad f_\lambda(z) = \sum_{k=0}^{dD} v_k(\lambda) z^k, \quad \text{where} \quad v_k(\lambda) = \sum_{i=1}^m c_k^i \lambda_i.$$

We thus have by (4), for any $k = 0, \dots, dD$

$$(6) \quad |v_k(\lambda)| \leq \frac{mB_R |\lambda|}{R^k},$$

where $|\lambda| = \max\{|\lambda_1|, \dots, |\lambda_m|\}$.

Now we define the Bautin index $b = b(f_\lambda)$ of the family f_λ . Let, for $i \in \mathbb{N}$, L_i be the linear subspace of \mathbb{C}^m defined by $v_0(\lambda) = \dots = v_i(\lambda) = 0$. We have

$$L_0 \supseteq L_1 \supseteq \dots \supseteq L_i \supseteq \dots \supseteq L_{dD}.$$

Hence on a certain step $b \leq dD$ this sequence stabilizes

$$L_{b-1} \supsetneq L_b = L_{b+1} = \dots = L_{dD}.$$

Definition 1.2. We call the integer b defined above the *Bautin index of the family f_λ* (see [1, 2]).

Remarks 1.3.

- Note that $\lambda \in L_b \iff v_k(\lambda) = 0$, for $k = 0, 1, \dots, dD$.
- Equivalently, b is the biggest index k such that $v_k \notin \text{Span}(v_0, \dots, v_{k-1})$.
- Following Remark 1.1, for a given polynomial mapping P , we choose a family $Q_i, i = 1, \dots, m$ such that $L_b \neq \mathbb{C}^m$.

- The curve $P(\mathbb{C})$ is an irreducible algebraic subset of \mathbb{C}^2 . The vector space L_b is the space of parameters $\lambda \in \mathbb{C}^m$ such that $Q_\lambda \in I$ where I is the ideal of $P(\mathbb{C})$ in $\mathbb{C}[X, Y]$.

Since $L_b \neq \mathbb{C}^m$, for $\lambda \notin L_b$, the multiplicity of $f_\lambda(z)$ at the origin is well defined. The following proposition is a variant of [7, Proposition 2.3].

Proposition 1.4. *Let us denote by μ the maximal multiplicity at the origin of $f_\lambda(z)$, with respect to the parameters $\lambda \notin L_b$. Then $\mu = b \geq m - 1$.*

Proof. There exists a parameter $\lambda \notin L_b$ such that $f_\lambda(z)$ has multiplicity μ , that is such that

$$f_\lambda(z) = v_\mu(\lambda)z^\mu + v_{\mu+1}(\lambda)z^{\mu+1} + \dots,$$

with $v_\mu(\lambda) \neq 0$. Therefore $\lambda \in L_{\mu-1} \setminus L_\mu$. It follows that $L_\mu \subsetneq L_{\mu-1}$, and $b \geq \mu$. On the other hand, since no parameter $\lambda \notin L_b$ can cancel v_0, \dots, v_μ in the same time, $L_\mu \subseteq L_b$. But since $L_{b-1} \supsetneq L_b = \dots = L_{dD}$, we have $b \leq \mu$.

Using this characterization of b as μ , the bound $b \geq m - 1$ comes from the fact that the linear system $v_0 = \dots = v_{m-2}$ with m parameters and $m - 1$ equations always has a nonzero solution. \square

In Section 4, we bound from above the number of zeros of f_λ in \bar{D}_R . For this, all the information we need is encoded in the $b + 1$ first rows of the rank σ matrix $M(f_\lambda) = (c_k^i)$, $k = 0, \dots, dD$, $i = 1, \dots, m$, called the *Bautin matrix of the family* f_λ . With notation (5), the Bautin matrix $M(f_\lambda)$ is defined by

$$\begin{pmatrix} v_0(\lambda) \\ \vdots \\ v_{dD}(\lambda) \end{pmatrix} = M_\lambda \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}.$$

2. TRIANGULATION OF BAUTIN BLOCKS

Since the $(k + 1)$ -th line of the Bautin matrix $M(f_\lambda)$ is by definition the line of coefficients of the linear form v_k , by (5) we have to compute the coefficients of X^k in $f_\lambda(X) = Q_\lambda(P(X))$ to find this $(k + 1)$ -th line.

For a monomial $X^i Y^j \in \mathbb{C}[X, Y]$, of degree $i + j$, we denote by $\lambda_{i,j}$ the corresponding parameter coefficient in the family Q_λ . We also denote by $n_0 < \dots < n_{\ell_d} = d$ the total degrees of monomials occurring in Q_λ . In the case where the family Q_λ corresponds to all monomials $X^i Y^{n_\ell - i}$ for all $0 \leq \ell \leq \ell_d$, we simply denote by M the Bautin matrix of f_λ . Notice that if we choose a subfamily for Q_λ , the associated Bautin matrix is a submatrix of M (by selecting the columns corresponding to the selected monomials).

We fix in this section a degree n_ℓ , $\ell \in \{0, \dots, \ell_d\}$.

Remark 2.1. We first give hereafter the particular form of the block, denoted M_{n_ℓ} , appearing in the Bautin matrix M as the contribution of all possible monomials of $X^I Y^{n_\ell - I}$, $I = 0, \dots, n_\ell$, of fixed degree n_ℓ . In general

in our family $Q_\lambda(X, Y)$ only a certain amount of those monomials appears, this is why we next introduce (see Notation 2.8) the block $M_{n_\ell, T}$ corresponding to a certain choice T of $\tau \leq n_\ell + 1$ particular columns in M_{n_ℓ} , that is, to a certain choice of τ parameters $\lambda_{I, n_\ell - I}$ in the family Q_λ .

Remark 2.2. As already mentioned, up to some translation, we assume that $(P_1, P_2)(0) = 0$, and therefore that the multiplicity at the origin of P_1 and P_2 is bigger than 1.

As before, let $D = \max(D_1, D_2)$. Let ν_1 and ν_2 be the multiplicities at the origin of P_1 and P_2 respectively. Up to permutation of the variables, we assume in what follows that $\nu_1 \geq \nu_2 \geq 1$, and we will use the notation

$$\nu = \min(\nu_1, \nu_2) = \nu_2.$$

We write P_1 and P_2 in the following way

$$P_1(X) = X^{\nu_1} \sum_{i=0}^{D_1 - \nu_1} \alpha_i X^i, \quad P_2(X) = X^{\nu_2} \sum_{i=0}^{D_2 - \nu_2} a_i X^i,$$

and

$$Q(X_1, X_2) = X_1^I X_2^{n_\ell - I},$$

for $I = 0, \dots, n_\ell$, where α_0 and a_0 are nonzero. If needed, we will adopt the convention that coefficients above the degree are set to 0. Denoting $N_I = (n_\ell - I)D_2 + ID_1 - n_\ell$ and $\sum_{i_1 + \dots + i_I + (\nu_1 - \nu_2)I + j_1 + \dots + j_{n_\ell - I} = k}$ by $\sum_{*=k}$, for $k = 0, \dots, N_I$, the parameter $\lambda_{I, n_\ell - I}$ in f_λ appears in front of the polynomial

$$P_1(X)^I P_2(X)^{n_\ell - I} = \sum_{k=0}^{ID_1 + (n_\ell - I)D_2 - \nu n_\ell} \sum_{*=k} \alpha_{i_1} \cdots \alpha_{i_I} a_{j_1} \cdots a_{j_{n_\ell - I}} X^{\nu n_\ell + k}.$$

Therefore all monomials Q of degree n_ℓ contribute in f_λ only for v_k with $k = \nu n_\ell, \dots, Dn_\ell$, and this contribution is the coefficient

$$(7) \quad \sum_{I=0}^{n_\ell} \lambda_{I, n_\ell - I} \sum_{*=k - \nu n_\ell} \alpha_{i_1} \cdots \alpha_{i_I} a_{j_1} \cdots a_{j_{n_\ell - I}}.$$

It follows that the monomials of Q of degree n_ℓ contribute in M for a block M_{n_ℓ} with $(n_\ell + 1)$ columns and $(n_\ell(D - \nu) + 1)$ lines, corresponding to the $(n_\ell + 1)$ parameters $\lambda_{0, n_\ell}, \dots, \lambda_{n_\ell, 0}$ appearing in $v_k(\lambda)$, for $k = \nu n_\ell, \dots, Dn_\ell$.

In this block M_{n_ℓ} the $(I + 1)$ -th column, for $I = 0, \dots, n_\ell$, starts at row 1 with entries

$$\sum_{*=0} \alpha_{i_1} \cdots \alpha_{i_I} a_{j_1} \cdots a_{j_{n_\ell - I}}, \sum_{*=1} \alpha_{i_1} \cdots \alpha_{i_I} a_{j_1} \cdots a_{j_{n_\ell - I}}, \text{ etc.},$$

and ends at row $N_I + 1$, with entry $\sum_{*=N_I} \alpha_{i_1} \cdots \alpha_{i_I} a_{j_1} \cdots a_{j_{n_\ell - I}}$.

Notation 2.3. For $i \geq 0$, we denote by $C_{\geq i}$ the data

$$C_{\geq i} = (\lambda_i, \lambda_{i+1}, \dots, \lambda_{D_1-\nu_1}, l_i, l_{i+1}, \dots, l_{D_2-\nu_2}) \in \mathbb{N}^{D_1+D_2-\nu_1-\nu_2-2i+2},$$

and

$$\Lambda = \lambda_1 + \dots + \lambda_{D_1-\nu_1}, \quad L = l_1 + \dots + l_{D_2-\nu_2},$$

$$\Lambda! = \lambda_1! \cdots \lambda_{D_1-\nu_1}!, \quad L! = l_1! \cdots l_{D_2-\nu_2}!.$$

Remark 2.4. A monomial \mathbf{m} , in $\alpha_0, \dots, \alpha_{D_1-\nu_1}, a_0, \dots, a_{D_2-\nu_2}$, in column $I+1 \in \llbracket 1, n_\ell + 1 \rrbracket$ and at row $r \in \llbracket 1, n_\ell(D-\nu) + 1 \rrbracket$ in M_{n_ℓ} is of form

$$(8) \quad \alpha_0^{\lambda_0} \alpha_1^{\lambda_1} \cdots \alpha_{D_1-\nu_1}^{\lambda_{D_1-\nu_1}} a_0^{l_0} a_1^{l_1} \cdots a_{D_2-\nu_2}^{l_{D_2-\nu_2}},$$

with

$$(9) \quad \lambda_0 + \dots + \lambda_{D_1-\nu_1} = I, \quad l_0 + \dots + l_{D_2-\nu_2} = n_\ell - I,$$

and

$$(10) \quad 0\lambda_0 + \dots + (D_1 - \nu_1)\lambda_{D_1-\nu_1} + (\nu_1 - \nu_2)I + 0l_0 + \dots + (D_2 - \nu_2)l_{D_2-\nu_2} = r - 1.$$

Notice that the data $C_{\geq 0}$ fix the monomial \mathbf{m} . However, from (9), the data $C_{\geq 1}$ with the column \bar{I} already fix \mathbf{m} . The coefficient in front of such a monomial \mathbf{m} in column I is

$$\binom{I}{\lambda_0, \dots, \lambda_{D_1-\nu_1}} \cdot \binom{n_\ell - I}{l_0, \dots, l_{D_2-\nu_2}},$$

where the notation $\binom{i}{i_1, \dots, i_p}$ stands for the multinomial $\frac{i!}{i_1! \cdots i_p!}$. Therefore this coefficient is

$$(11) \quad \begin{aligned} P_{C_{\geq 1}, I} &= \frac{(\lambda_0 + 1) \cdots I}{\Lambda!} \cdot \frac{(l_0 + 1) \cdots (n_\ell - I)}{L!} \\ &= \frac{1}{\Lambda! L!} \prod_{k=0}^{I-(\lambda_0+1)} (I-k) \prod_{k=0}^{n_\ell-I-(l_0+1)} (n_\ell - I - k). \end{aligned}$$

Notice that, according to Notation 2.3, we have $I - \lambda_0 = \Lambda$ and $n_\ell - I - l_0 = L$. So for a monomial \mathbf{m} of given and fixed exponents $C_{\geq 1}$, the coefficient $P_{C_{\geq 1}, I}$ is a polynomial $P_{C_{\geq 1}}$ in I of degree

$$(12) \quad \deg(P_{C_{\geq 1}}) = \Lambda + L.$$

Furthermore, the leading coefficient of $P_{C_{\geq 1}}$ is

$$(13) \quad \frac{(-1)^L}{\Lambda! L!}.$$

Lemma 2.5. Let $c \in \mathbb{N}$ and $T = \{t_1, \dots, t_{c+1}\} \subseteq \mathbb{C}$, with $t_i < t_j$ for $i < j$.

(1) *There exists a finite sequence $(K_{T,j})_{j \in T}$ depending only on T , such that for any polynomial $P \in \mathbb{C}[X]$ with $\deg(P) \leq c$, we have*

$$\sum_{j \in T} P(j)K_{T,j} = \text{coeff}_{X^c}(P).$$

In particular when $c > \deg(P)$ we have $\sum_{j \in T} P(j)K_{T,j} = 0$.

(2) *On the other side, for $c \leq \deg(P)$, we have*

$$(14) \quad \sum_{j \in T} P(j)K_{T,j} = \text{coeff}_{X^c}(P) + \text{coeff}_{X^{c+1}}(P) \sum_{j=1}^{c+1} t_j^{c+1} K_{T,t_j} \\ + \cdots + \text{coeff}_{X^{\deg(P)}}(P) \sum_{j=1}^{c+1} t_j^{\deg(P)} K_{T,t_j}.$$

Proof. Let V_T be the Vandermonde matrix related to T , that is to say

$$V_T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_{c+1} \\ \vdots & \vdots & & \vdots \\ t_1^c & t_2^c & \cdots & t_{c+1}^c \end{pmatrix}$$

Let us denote by v_T its determinant. We have

$$v_T = \prod_{1 \leq i < j \leq c+1} (t_j - t_i),$$

Let us first prove (1). For this let C be the row matrix of coefficients of P , completed at its right with $c - \deg(P)$ zeros. Notice that

$$C \cdot V_T = (P(t_1), P(t_2), \dots, P(t_{c+1})).$$

Let $V_{T,c+1}^{-1}$ be the last column of the inverse matrix of V_T . We have

$$\text{coeff}_{X^c}(P) = C \cdot V_T \cdot V_{T,c+1}^{-1} = (P(t_1), P(t_2), \dots, P(t_{c+1})) \cdot V_{T,c+1}^{-1}.$$

Consequently, it is sufficient to take for K_{T,t_p} the p^{th} element of $V_{T,c+1}^{-1}$, which is by Cramer's formula

$$(15) \quad K_{T,t_p} = \frac{(-1)^{c+p+1}}{v_T} \det(V_{T \setminus \{t_p\}}) = \frac{1}{\prod_{i \neq p} (t_p - t_i)}.$$

This define the sequence $K_{T,t_1}, \dots, K_{T,t_{c+1}}$, and proves (1).

To prove (2), assuming $c \leq \deg(P)$, we consider \tilde{V}_T the square matrix of size $\deg(P) + 1$, defined as the matrix V_T first completed at its bottom by the matrix

$$\begin{pmatrix} t_1^{c+1} & \cdots & t_{c+1}^{c+1} \\ \vdots & & \vdots \\ t_1^{\deg(P)} & \cdots & t_{c+1}^{\deg(P)} \end{pmatrix},$$

and then at its right by the matrix with $\deg(P) + 1$ rows and $\deg(P) - c$ columns having only zeros for coefficients. Let us also consider $\tilde{V}_{T,c+1}^{-1}$ the last column of the inverse matrix of V_T , but this time completed at its bottom by $\deg(P) - c$ zeros, in order to obtain a column with $\deg(P) + 1$ rows. With this notation we have

$$\tilde{V}_T \cdot \tilde{V}_{T,c+1}^{-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \sum_{j=1}^{c+1} t_j^{c+1} K_{T,t_j} \\ \vdots \\ \sum_{j=1}^{c+1} t_j^{\deg(P)} K_{T,t_j} \end{pmatrix},$$

where the unit appears at row $c + 1$. Then the same computation as above, involving \tilde{V}_T and $\tilde{V}_{T,c+1}^{-1}$ instead of V_T and $V_{T,c+1}^{-1}$, shows that

$$\begin{aligned} \sum_{j \in T} P(j) K_{T,j} &= \text{coeff}_{X^c}(P) + \text{coeff}_{X^{c+1}}(P) \sum_{j=1}^{c+1} t_j^{c+1} K_{T,t_j} \\ &+ \cdots + \text{coeff}_{X^{\deg(P)}}(P) \sum_{j=1}^{c+1} t_j^{\deg(P)} K_{T,t_j}. \quad \square \end{aligned}$$

Remark 2.6. In case $T = \{0, 1, \dots, c\}$ in Lemma 2.5, we have

$$K_{T,p} = \frac{(-1)^{c+p}}{c!} \binom{c}{p}.$$

Therefore, for any polynomial P with $\deg(P) \leq c$, the following classical formula follows from Lemma 2.5 (1)

$$(16) \quad \sum_{j=0}^c (-1)^j \binom{c}{j} P(j) = c! (-1)^c \text{coeff}_{X^c}(P).$$

We now generalize formula (16) to the case of multivariate polynomials.

Lemma 2.7. *Let $P \in \mathbb{C}[X_1, \dots, X_k]$ of total degree d . Let $(w_1, \dots, w_k) \in \mathbb{N}^k$ such that $d < \sum_{i=1}^k w_i$. Then,*

$$\sum_{(i_1, \dots, i_k) \in \llbracket 0, w_1 \rrbracket \times \cdots \times \llbracket 0, w_k \rrbracket} (-1)^{i_1 + \cdots + i_k} \binom{w_1}{i_1} \cdots \binom{w_k}{i_k} P(i_1, \dots, i_k) = 0$$

Proof. By linearity it is sufficient to prove the formula in the case where P is a monomial $X_1^{\gamma_1} \cdots X_k^{\gamma_k}$ of degree at most d . Since $d < \sum_{i=1}^k w_i$, it means

there exists $l \in \llbracket 1, k \rrbracket$ such that $\gamma_l < w_l$. Therefore

$$\begin{aligned} & \sum_{(i_1, \dots, i_k) \in \llbracket 0, w_1 \rrbracket \times \dots \times \llbracket 0, w_k \rrbracket} (-1)^{i_1 + \dots + i_k} \binom{w_1}{i_1} \dots \binom{w_k}{i_k} i_1^{\gamma_1} \dots i_k^{\gamma_k} \\ &= \prod_{j=1}^k \left(\sum_{i_j \in \llbracket 0, w_j \rrbracket} (-1)^{i_j} \binom{w_j}{i_j} i_j^{\gamma_j} \right) = 0, \end{aligned}$$

since the l^{th} factor of the product is zero by Remark 2.6. \square

Notation 2.8. For $\tau \leq n_\ell + 1$ and $T = T_\ell = \{t_1, \dots, t_\tau\} \subset \{1, \dots, n_\ell + 1\}$, with $t_1 < \dots < t_\tau$, we consider $M_{n_\ell, T}$, the submatrix of M_{n_ℓ} obtained from M_{n_ℓ} by only keeping the columns indexed by elements of T . And in the same way, we denote by $M_{T_0, \dots, T_{\ell_d}}$ the submatrix of the complete Bautin matrix M where we only keep the blocks M_{n_ℓ, T_ℓ} , $\ell = 0, \dots, \ell_d$, corresponding to some degrees $n_0, \dots, n_{\ell_d} = d$, and for each of them, corresponding to a specific choice of columns T_0, \dots, T_{ℓ_d} . The matrix $M_{n_\ell, T}$ has

$$r_T = \max_{1 \leq i \leq \tau} ((n_\ell - (t_i - 1))D_2 + (t_i - 1)D_1) - \nu n_\ell + 1$$

rows and τ columns.

With the notation of Lemma 2.5 we denote by \mathcal{K}_T the following upper triangular matrix of size $\tau \times \tau$

$$(\mathcal{K}_T)_{i,j} = \begin{cases} \left(\frac{\alpha_0}{a_0} \right)^{t_j - t_i} \frac{K_{\{t_1, \dots, t_j\}, t_i}}{K_{\{t_1, \dots, t_j\}, t_j}} & \text{if } i \leq j \leq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Notice that all diagonal elements of the matrix \mathcal{K}_T are equal to one.

Remark 2.9. In the case $\nu_1 \neq \nu_2$ the matrix $M_{n_\ell, T}$ is in column echelon form. Indeed, in this case, each column is shifted down by a strictly monotone number of zeros, from one column to the next one, and each shifted column has for upper coefficient a (nonzero) monomial in a_0 and α_0 . In particular the corresponding block $M_{n_\ell, T}$ has maximal rank τ .

Following Remark 2.9, in what follows, we only deal with the case $\nu_1 = \nu_2 = \nu$.

Proposition 2.10. *With the notation above*

- (1) the (upper minor of size τ of the) matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$ is lower triangular,
- (2) the diagonal coefficients of (the upper minor of size τ of) this matrix at row $r \in \llbracket 1, \tau \rrbracket$ is

$$\frac{\prod_{i < r} (t_r - t_i)}{(r-1)!} \alpha_0^{t_r - r} a_0^{n_\ell - (t_r + r - 2)} (\alpha_1 a_0 - \alpha_0 a_1)^{r-1}.$$

Proof. For $r \in \llbracket 1, \tau \rrbracket$ and $c \geq r$, we compute the coefficient of the matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$ at its row r and its column c . Under the action of the multiplication by \mathcal{K}_T , a monomial

$$\mathbf{m}_i = \alpha_0^{\lambda_0 - t_c + t_i} \alpha_1^{\lambda_1} \cdots \alpha_{D_1 - \nu}^{\lambda_{D_1 - \nu}} a_0^{l_0 + t_c - t_i} a_1^{l_1} \cdots a_{D_2 - \nu}^{l_{D_2 - \nu}}$$

from row r and a column $i \leq c$ of the matrix $M_{n_\ell, T}$ is sent to the monomial

$$\mathbf{m} = \alpha_0^{\lambda_0} \alpha_1^{\lambda_1} \cdots \alpha_{D_1 - \nu}^{\lambda_{D_1 - \nu}} a_0^{l_0} a_1^{l_1} \cdots a_{D_2 - \nu}^{l_{D_2 - \nu}}$$

in the same row r and the column c of the matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$. Consequently, using the notation $C_{\geq 1} = (\lambda_1, \dots, \lambda_{D_1 - \nu}, l_1, \dots, l_{D_2 - \nu})$ and $P_{C_{\geq 1}}$ of Remark 2.4 we have

$$\begin{aligned} \text{coeff}_{\mathbf{m}}((M_{n_\ell, T} \cdot \mathcal{K}_T)_{r, c}) &= \sum_{i=1}^c \frac{K_{\{t_1, \dots, t_c\}, t_i}}{K_{\{t_1, \dots, t_c\}, t_c}} \text{coeff}_{\mathbf{m}_i}((M_{n_\ell, T})_{r, i}) \\ (17) \qquad \qquad \qquad &= \frac{1}{K_{\{t_1, \dots, t_c\}, t_c}} \sum_{i=1}^c K_{\{t_1, \dots, t_c\}, t_i} P_{C_{\geq 1}}(t_i - 1). \end{aligned}$$

where, by (12) $P_{C_{\geq 1}}$ is a polynomial of degree $\sum_{i \geq 1} \lambda_i + l_i$. By (10) $\deg(P_{C_{\geq 1}}) \leq r - 1 \leq c - 1$, and $\deg(P_{C_{\geq 1}}) < r - 1$ in case $\sum_{i \geq 2} l_i + \lambda_i \neq 0$. By Lemma 2.5 (1), we deduce from (17) that

$$\text{coeff}_{\mathbf{m}}((M_{n_\ell, T} \cdot \mathcal{K}_T)_{r, c}) = \frac{1}{K_{\{t_1, \dots, t_c\}, t_c}} \text{coeff}_{X^{c-1}}(P_{C_{\geq 1}}(X - 1)).$$

By (13), $\text{coeff}_{\mathbf{m}}((M_{n_\ell, T} \cdot \mathcal{K}_T)_{r, c})$ equals

$$\frac{(-1)^{l_1}}{\lambda_1! l_1! K_{\{t_1, \dots, t_r\}, t_r}}$$

in case $c = r = \deg(P_{C_{\geq 1}}) + 1$, and zero otherwise. This proves in particular statement (1), and shows moreover that for the diagonal coefficients of the matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$, where $c = r$, only monomials of the form $\alpha_0^{\lambda_0} \alpha_1^{\lambda_1} a_0^{l_0} a_1^{l_1}$ occur, each one with coefficient

$$\frac{(-1)^{l_1}}{(\lambda_1 + l_1)! K_{\{t_1, \dots, t_r\}, t_r}} \binom{\lambda_1 + l_1}{l_1}.$$

Considering that for monomials where $\sum_{i \geq 2} \lambda_i + l_i = 0$ we have from (9) and (10) that $\lambda_0 + \lambda_1 = t_r - 1$, $\lambda_1 + l_1 = r - 1$ and $l_0 + l_1 = n_\ell - t_r + 1$, the diagonal coefficient at row r of $M_{n_\ell, T} \cdot \mathcal{K}_T$ is

$$\begin{aligned} &\sum_{l_1=0}^{r-1} \frac{(-1)^{l_1}}{(r-1)! K_{\{t_1, \dots, t_r\}, t_r}} \binom{r-1}{l_1} \alpha_0^{\lambda_0} \alpha_1^{\lambda_1} a_0^{l_0} a_1^{l_1} \\ &= \frac{1}{(r-1)! K_{\{t_1, \dots, t_r\}, t_r}} \alpha_0^{t_r - r} a_0^{n_\ell - (t_r + r - 2)} \sum_{l_1=0}^{r-1} (-1)^{l_1} \binom{r-1}{l_1} \alpha_0^{l_1} \alpha_1^{r-1-l_1} a_0^{r-1-l_1} a_1^{l_1} \\ &= \frac{1}{(r-1)! K_{\{t_1, \dots, t_r\}, t_r}} \alpha_0^{t_r - r} a_0^{n_\ell - (t_r + r - 2)} (\alpha_1 a_0 - \alpha_0 a_1)^{r-1}. \end{aligned}$$

This completes the proof of statement (2). \square

Remark 2.11. Note that in case $T = \{1, \dots, n_\ell + 1\}$ the diagonal coefficient at row r of the (upper minor of size $n_\ell + 1$ of the) triangular matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$ is $a_0^{n_\ell - 2(r-1)} (\alpha_1 a_0 - \alpha_0 a_1)^{r-1}$, since in this case $t_r = r$ and $K_{\{t_1, \dots, t_r\}, t_r} = 1/(r-1)!$.

In Proposition 2.10 above we proved that the matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$ is lower triangular and we computed its diagonal coefficients. Let us now compute the other coefficients of this matrix. For this, in Proposition 2.12 below, we first compute the coefficient of a monomial $\mathbf{m} = \alpha_0^{\lambda_0} \alpha_1^{\lambda_1} \dots \alpha_{D_1 - \nu}^{\lambda_{D_1 - \nu}} a_0^{l_0} a_1^{l_1} \dots a_{D_2 - \nu}^{l_{D_2 - \nu}}$ appearing at row $r \in \llbracket 1, \tau \rrbracket$ and column $c \leq r - 1$ in $M_{n_\ell, T} \cdot \mathcal{K}_T$. Notice that this triangulation provides monomials with negative exponent for a_0 , since in \mathcal{K}_T denominators $a_0^{t_j - t_i}$, for $j > i$, appear. So from now, we will work with monomials where we formally allow exponent l_0 to be negative, that is to say, we consider monomials appearing in $M_{n_\ell, T} \cdot \mathcal{K}_T$ as elements of the Laurent ring $\mathbb{C}[\alpha_0, \alpha_1, \dots, \alpha_{D_1 - \nu}, a_0^{-1}, a_0, a_1, \dots, a_{D_2 - \nu}]$.

Also notice that equations (9) and (10) are satisfied for monomials appearing in $M_{n_\ell, T}$, as well as for monomials virtually appearing in $M_{n_\ell, T} \cdot \mathcal{K}_T$, since multiplication by factors $\left(\frac{\alpha_0}{a_0}\right)^{t_j - t_i}$, $j \geq i$, occurring in \mathcal{K}_T preserves those equations.

Proposition 2.12. *With the notation above, at row $r \in \llbracket 1, r_T \rrbracket$ and column $c \leq \min(r - 1, \tau)$ in $M_{n_\ell, T} \cdot \mathcal{K}_T$, for any exponents*

$$(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{D_1 - \nu}, l_0, l_1, \dots, l_{D_2 - \nu}) \in \mathbb{N}^{D_1 - \nu + 1} \times \mathbb{Z} \times \mathbb{N}^{D_2 - \nu}$$

satisfying (10), and satisfying (9), that is to say

$$\lambda_0 + \Lambda = t_c - 1, \quad l_0 + L = n_\ell + 1 - t_c,$$

the coefficient of the monomial

$$\mathbf{m} = \alpha_0^{\lambda_0} \alpha_1^{\lambda_1} \dots \alpha_{D_1 - \nu}^{\lambda_{D_1 - \nu}} a_0^{l_0} a_1^{l_1} \dots a_{D_2 - \nu}^{l_{D_2 - \nu}}$$

is 0 in case $c \geq \Lambda + L + 2$, and in case $c \leq \Lambda + L + 1$, this coefficient is

$$(18) \quad \sigma_{\Lambda + L - c + 1} \cdot k_{c-1} + \sigma_{\Lambda + L - c} \cdot k_c + \dots + \sigma_0 \cdot k_{\Lambda + L}$$

where

$$k_{c-1} = \frac{1}{K_{\{t_1, \dots, t_c\}, t_c}}$$

and for $p = c, \dots, \Lambda + L$,

$$k_p = \frac{\sum_{j=1}^c t_j^p K_{\{t_1, \dots, t_c\}, t_j}}{K_{\{t_1, \dots, t_c\}, t_c}},$$

depend only on c and of the choice of T , and where

$$\sigma_{\Lambda + L - p} = \frac{(-1)^{\Lambda - p}}{\Lambda! L!} \sigma_{\Lambda + L - p}(1, 2, \dots, \Lambda, n_\ell + 2 - L, \dots, n_\ell + 1),$$

with $\sigma_k(x_1, \dots, x_{\Lambda+L}) = \sum_{1 \leq i_1 < \dots < i_k \leq \Lambda+L} x_{i_1} \cdots x_{i_k}$ be the k -th elementary symmetric polynomial.

Proof. We fix a row $r \in \{1, \dots, \tau\}$, a column $c \leq r - 1$, and a monomial $\mathbf{m} = \alpha_0^{\lambda_0} \cdots \alpha_{D_1-\nu}^{\lambda_{D_1-\nu}} a_0^{l_0} \cdots a_{D_2-\nu}^{l_{D_2-\nu}}$ in $M_{n_\ell, T} \cdot \mathcal{K}_T$ verifying the hypotheses of the proposition.

As already noticed at (17) in the proof of Proposition 2.10, this coefficient comes from the coefficients of monomials \mathbf{m}_i in $M_{n_\ell, T}$ which appear at the same row r as \mathbf{m} , at columns $i = \{1, \dots, c\}$, with the same exponents $\lambda_1, \dots, \lambda_{D_1-\nu}, l_1, \dots, l_{D_2-\nu}$ as in \mathbf{m} , but with exponent $\lambda_0 - t_c + t_i$ for α_0 , and exponent $l_0 + t_c - t_i$ for a_0 . Notice that this monomial exists in $M_{n_\ell, T}$ if and only if $\lambda_0 - t_c + t_i \geq 0$ and $l_0 + t_c - t_i \geq 0$. This gives rise to the following expression for the coefficient of \mathbf{m}

$$\text{coeff}_{\mathbf{m}} = \frac{1}{K_{\{t_1, \dots, t_c\}, t_c}} \sum_{\substack{i \in \llbracket 1, c \rrbracket \text{ st} \\ \lambda_0 - t_c + t_i \geq 0 \text{ and } l_0 + t_c - t_i \geq 0}} K_{\{t_1, \dots, t_c\}, t_i} P_{C_{\geq 1}}(t_i - 1)$$

with $P_{C_{\geq 1}}$ the polynomial of degree $\Lambda + L$ defined in Remark 2.4. Since $P_{C_{\geq 1}}$ is defined by the expression (11), this polynomial cancels at $0, \dots, \Lambda - 1$ and $n_\ell - L + 1, \dots, n_\ell$. Moreover assuming $\lambda_0 - t_c + t_i < 0$, it implies that $t_i - 1 \leq t_c - \lambda_0 - 2 = \Lambda - 1$. Consequently $P_{C_{\geq 1}}(t_i - 1) = 0$. Similarly, if $l_0 + t_c - t_i < 0$, then $t_i - 1 \geq l_0 + t_c = n_\ell - L + 1$ which again implies $P_{C_{\geq 1}}(t_i - 1) = 0$. Consequently, the sum above is unchanged if we sum over all $\llbracket 1, c \rrbracket$:

$$(19) \quad \text{coeff}_{\mathbf{m}} = \frac{1}{K_{\{t_1, \dots, t_c\}, t_c}} \sum_{i=1}^c K_{\{t_1, \dots, t_c\}, t_i} P_{C_{\geq 1}}(t_i - 1).$$

- In case $c \geq \Lambda + L + 2$, by Lemma 2.5 (1), $\text{coeff}_{\mathbf{m}} = 0$.
- In case $c \leq \Lambda + L + 1$, by (14) of Lemma 2.5 (2), we have

$$\begin{aligned} \text{coeff}_{\mathbf{m}} = & \frac{1}{K_{\{t_1, \dots, t_c\}, t_c}} \left(\text{coeff}_{X^{c-1}}(P_{C_{\geq 1}}(X - 1)) \right. \\ & + \text{coeff}_{X^c}(P_{C_{\geq 1}}(X - 1)) \sum_{j=1}^c t_j^c K_{\{t_1, \dots, t_c\}, t_j} \\ & + \cdots + \text{coeff}_{X^{\Lambda+L}}(P_{C_{\geq 1}}(X - 1)) \sum_{j=1}^c t_j^{\Lambda+L} K_{\{t_1, \dots, t_c\}, t_j} \left. \right). \end{aligned}$$

We recall from (11) that

$$P_{C_{\geq 1}}(X - 1) = \frac{(-1)^L}{\Lambda! L!} \prod_{k=1}^{\Lambda} (X - k) \prod_{k=1}^L (X - (n_\ell + 2 - k)).$$

It follows that for $p = c - 1, \dots, \Lambda + L$

$$(20) \quad \text{coeff}_{X^p}(P_{C_{\geq 1}}(X-1)) = \frac{(-1)^{\Lambda-p}}{\Lambda!L!} \sigma_{\Lambda+L-p}(1, 2, \dots, \Lambda, n_{\ell}+2-L, \dots, n_{\ell}, n_{\ell}+1).$$

□

Lemma 2.13. *Let $N, R, p \in \mathbb{N}$ with $p \leq R < N$ and $f_p^{R,N} : \llbracket 0, R \rrbracket \rightarrow \mathbb{N}$ be the function defined by $f_p^{R,N}(n) = \sigma_p(1, 2, \dots, R-n, N-n, \dots, N)$, with the notation of Proposition 2.12. There exists a polynomial $P_p^{R,N}$ of degree p such that $f_p^{R,N}(n) = P_p^{R,N}(n)$, for all $n \in \llbracket 0, R \rrbracket$.*

Proof. The proof is by induction on $p \geq 0$. Notice first that $f_0^{R,N} \equiv 1$ is a constant function. Then for $p \geq 1$ a direct computation shows that for all $n \in \llbracket 0, R-1 \rrbracket$

$$f_p^{R,N}(n+1) - f_p^{R,N}(n) = (N-R-1)\sigma_{p-1}(1, 2, \dots, R-1-n, N-n, \dots, N).$$

Since by induction hypothesis the right hand side of the above equation equals the polynomial $(N-R-1)P_{p-1}^{R-1,N}$ of degree $p-1$ on $\llbracket 0, R-1 \rrbracket$, so does $f_p^{R,N}(n+1) - f_p^{R,N}(n)$.

We just need to show this implies there exists a polynomial $P_p^{R,N}$ which equals $f_p^{R,N}$ on $\llbracket 0, R \rrbracket$. To do that, we follow for instance [9, Proposition I.7.3 (b)]. Let us consider the family of Newton polynomials $\left(\binom{X}{r}\right)_{r \in \mathbb{N}}$ defined by $\binom{X}{r} = \frac{1}{r!} \prod_{i=0}^{r-1} (X-i)$. We can easily notice that $\left(\binom{X}{r}\right)_{r \leq d}$ is a basis of polynomials of degree at most d . Moreover, a simple calculation shows that for $r > 0$, $\binom{X+1}{r} - \binom{X}{r} = \binom{X}{r-1}$. Writing $(N-R-1)P_{p-1}^{R-1,N}$ in this basis, we obtain the coefficients (c_{p-1}, \dots, c_0)

$$(N-R-1)P_{p-1}^{R-1,N}(X) = c_{p-1} \binom{X}{p-1} + c_{p-2} \binom{X}{p-2} + \dots + c_0.$$

Let us define

$$P_p^{R,N}(X) = c_{p-1} \binom{X}{p} + c_{p-2} \binom{X}{p-1} + \dots + c_0 \binom{X}{1} + K$$

where the constant K is chosen to get $P_p^{R,N}(R) = f_p^{R,N}(R)$. It follows that for any $n \in \llbracket 0, R-1 \rrbracket$, $P_p^{R,N}(n+1) - P_p^{R,N}(n) = f_p^{R,N}(n+1) - f_p^{R,N}(n)$. An easy induction finishes the proof. □

3. TRIANGULATION OF BAUTIN BLOCKS - THE SINGULAR CASE

In this section we continue to focus on the case $\nu = \nu_1 = \nu_2$. We assume that the upper $\tau \times \tau$ minor of the lower triangular matrix $M_{n_{\ell}, T} \cdot \mathcal{K}_T$ has determinant zero, where $T = \{t_1, \dots, t_{\tau}\} \subset \{1, \dots, n_{\ell} + 1\}$, with $t_1 < \dots < t_{\tau}$, is a choice of τ columns in the Bautin block $M_{n_{\ell}}$, according to Notation 2.8. By Proposition 2.10, the condition that the determinant of the upper $\tau \times \tau$ minor of $M_{n_{\ell}, T} \cdot \mathcal{K}_T$ is zero means that $\alpha_0 a_1 = \alpha_1 a_0$. There exists

$\mu \in \mathbb{C} \setminus \{0\}$ such that $\alpha_0 = \mu a_0, \alpha_1 = \mu a_1$, and we denote by k the largest integer in $\llbracket 0, \min(D_1, D_2) - \nu \rrbracket$ satisfying

$$(21) \quad (\alpha_0, \dots, \alpha_k) = \mu \cdot (a_0, \dots, a_k),$$

assuming in this section that $k \geq 1$. In the lower triangular matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$ we fix a row $r \in \{1, \dots, r_T\}$ and a column $c \leq r - 1$. The entry in $M_{n_\ell, T} \cdot \mathcal{K}_T$ at row r and column c is a polynomial in $\alpha_0, \dots, \alpha_{D_1-\nu}, a_0, \dots, a_{D_2-\nu}$ (actually a Laurent polynomial, since a_0 may have negative power l_0 after multiplication of $M_{n_\ell, T}$ by \mathcal{K}_T), which is a linear combination of monomials of the form

$$\mathbf{m} = \alpha_0^{\lambda_0} \alpha_1^{\lambda_1} \dots \alpha_{D_1-\nu}^{\lambda_{D_1-\nu}} a_0^{l_0} a_1^{l_1} \dots a_{D_2-\nu}^{l_{D_2-\nu}}$$

with, according to (9),

$$(22) \quad \lambda_0 + \Lambda = t_c - 1, \quad l_0 + L = n_\ell - t_c + 1.$$

If, according to condition (21), we replace for $i \in \llbracket 0, k \rrbracket$, α_i by μa_i in the monomial \mathbf{m} we get a monomial of the form

$$(23) \quad \tilde{\mathbf{m}} = \mu^{\lambda_0 + \dots + \lambda_k} \cdot a_0^{\lambda_0 + l_0} \dots a_k^{\lambda_k + l_k} \cdot \alpha_{k+1}^{\lambda_{k+1}} \dots \alpha_{D_1-\nu}^{\lambda_{D_1-\nu}} a_{k+1}^{l_{k+1}} \dots a_{D_2-\nu}^{l_{D_2-\nu}}.$$

Notation 3.1. For $i = 0, \dots, D - \nu$, let us denote

$$\Lambda_{\geq i} = \lambda_i + \dots + \lambda_{D_1-\nu}, \quad L_{\geq i} = l_i + \dots + l_{D_2-\nu},$$

$$W_{\geq i} = i\lambda_i + \dots + (D_1 - \nu)\lambda_{D_1-\nu} + il_i + \dots + (D_2 - \nu)l_{D_2-\nu}.$$

Notice that up to now we have used Λ instead of $\Lambda_{\geq 1}$ and L instead of $L_{\geq 1}$ (see Notation 2.3); In what follows we use those two notations.

Remark 3.2. Using Notation 2.3, in case $C_{\geq k+1}$ is fixed, so is $\lambda_0 + \dots + \lambda_k$, since

$$(24) \quad \lambda_0 + \dots + \lambda_k = t_c - 1 - \Lambda_{\geq k+1}.$$

On the other hand, since $\lambda_0 + l_0 = n_\ell - L_{\geq 1} - \Lambda_{\geq 1}$, monomials \mathbf{m} contribute, after the substitution of α_i by μa_i for $i \in \llbracket 0, k \rrbracket$ in \mathbf{m} , to a monomial $\tilde{\mathbf{m}}$ defined only by the data $T, c, (w_1, \dots, w_k)$, and $C_{\geq k+1}$, where for all $i \geq 0$, $w_i = \lambda_i + l_i$, and according to (10), the sequence (w_1, \dots, w_k) verifies

$$w_1 + 2w_2 + \dots + kw_k = r - 1 - W_{\geq k+1}.$$

Also notice that

$$w_0 = \lambda_0 + l_0 = n_\ell - w_1 - \dots - w_k - \Lambda_{\geq k+1} - L_{\geq k+1}.$$

Remark 3.3. By Proposition 2.12, in case $\Lambda + L - c + 1 < 0$, the coefficient of \mathbf{m} is zero, so we may assume that $\Lambda + L - c + 1 \geq 0$.

Remark 3.4. By Proposition 2.10 (2) the entry at row 1 and column 1 of $M_{n_\ell, T} \cdot \mathcal{K}_T$ is

$$(25) \quad \alpha_0^{t_1-1} a_0^{n_\ell - (t_1-1)} = \mu^{t_1-1} a_0^{n_\ell},$$

and by (23) a monomial $\tilde{\mathbf{m}}$ at row r and column c in $M_{n_\ell, T} \cdot \mathcal{K}_T$ is of form

$$(26) \quad \tilde{\mathbf{m}} = \mu^{\lambda_0 + \dots + \lambda_k} a_0^{w_0} \dots a_k^{w_k} \alpha_{k+1}^{\lambda_{k+1}} \dots \alpha_{D_1 - \nu}^{\lambda_{D_1 - \nu}} a_{k+1}^{l_{k+1}} \dots a_{D_2 - \nu}^{l_{D_2 - \nu}},$$

coming, after the use of (21), from monomials (at same row r and same column c) $\mathbf{m} = \alpha_0^{\lambda_0} \dots \alpha_{D_1 - \nu}^{\lambda_{D_1 - \nu}} a_0^{l_0} \dots a_{D_2 - \nu}^{l_{D_2 - \nu}}$, where $w_i = \lambda_i + l_i$, and where

$$(27) \quad (l_1, \dots, l_k) \in \llbracket 0, w_1 \rrbracket \times \dots \times \llbracket 0, w_k \rrbracket.$$

Proposition 3.5. *With the notation above, the entry at column c and row $r \in \llbracket 1, (k+1)c - k \rrbracket$ in the matrix $M_{n_\ell, T} \cdot \mathcal{K}_T$ is zero for $1 \leq r < (k+1)c - k$, and for $r = (k+1)c - k$ this entry is the nonzero constant*

$$(28) \quad \frac{\mu^{t_c - c}}{(c-1)! K_{\{t_1, \dots, t_c\}, t_c}} a_0^{n_\ell - c + 1} (\alpha_{k+1} - \mu a_{k+1})^{c-1}.$$

Proof. A monomial $\tilde{\mathbf{m}}$ as in (26) of Remark 3.4 depends only on the choice of w_1, \dots, w_k and $C_{\geq k+1}$, as pointed out in Remark 3.2. So we fix a choice of data $w_1, \dots, w_k, C_{\geq k+1}$ with $w_1 + 2w_2 + \dots + kw_k = r - 1 - W_{\geq k+1}$. On the other hand, the monomial $\tilde{\mathbf{m}}$ comes from all monomials \mathbf{m} as in Remark 3.4, parameterized by $(l_1, \dots, l_k) \in \llbracket 0, w_1 \rrbracket \times \dots \times \llbracket 0, w_k \rrbracket$, and Proposition 2.12 gives the coefficient $\text{coeff}_{\mathbf{m}}$ of \mathbf{m} at row r and column c in $M_{n_\ell, T} \cdot \mathcal{K}_T$.

Two cases may then happen:

- (1) We first consider the case $w_1 = \dots = w_k = 0$, that is to say that the monomial $\tilde{\mathbf{m}}$ is given by a monomial \mathbf{m} , where $\lambda_1 = \dots = \lambda_k = l_1 = \dots = l_k = 0$. It implies in particular that

$$(29) \quad \Lambda + L = \Lambda_{\geq k+1} + L_{\geq k+1} \leq W_{\geq k+1} / (k+1) = (r-1) / (k+1).$$

So, since $c \geq (r+k)/(k+1)$,

$$(30) \quad 0 \leq \Lambda + L - c + 1 \leq \frac{r-1}{k+1} - \frac{r+k}{k+1} + 1 = 0.$$

Consequently inequalities (29) and (30) are equalities. In particular $\lambda_{k+1} + l_{k+1} = c - 1$ and for all $i \geq 1$, if $i \neq k+1$ we have $\lambda_i = l_i = 0$. Moreover there is a unique monomial \mathbf{m} which gives $\tilde{\mathbf{m}}$. Its coefficient is by (18) of Proposition 2.12

$$\mu^{\lambda_0} \sigma_0 k_{c-1} = \mu^{t_c - 1 - \lambda_{k+1}} \frac{(-1)^{l_{k+1}}}{l_{k+1}! \lambda_{k+1}! K_{\{t_1, \dots, t_c\}, t_c}}.$$

There are exactly c such monomials $\tilde{\mathbf{m}}$ (we can choose the values of λ_{k+1} and l_{k+1}). The sum of these monomials is

$$\begin{aligned} & \sum_{l_{k+1}=0}^{c-1} \mu^{t_c-c+l_{k+1}} \frac{(-1)^{l_{k+1}}}{l_{k+1}!(c-1-l_{k+1})!K_{\{t_1, \dots, t_c\}, t_c}} a_0^{n_\ell-c+1} \alpha_{k+1}^{c-1-l_{k+1}} a_{k+1}^{l_{k+1}} \\ &= \frac{\mu^{t_c-c}}{(c-1)!K_{\{t_1, \dots, t_c\}, t_c}} a_0^{n_\ell-c+1} \sum_{l_{k+1}=0}^{c-1} \binom{c-1}{l_{k+1}} \alpha_{k+1}^{c-1-l_{k+1}} (-\mu a_{k+1})^{l_{k+1}} \\ &= \frac{\mu^{t_c-c}}{(c-1)!K_{\{t_1, \dots, t_c\}, t_c}} a_0^{n_\ell-c+1} (\alpha_{k+1} - \mu a_{k+1})^{c-1}. \end{aligned}$$

- (2) Otherwise, some w_i is not zero where $1 \leq i \leq k$. By fixing the monomial $\tilde{\mathbf{m}}$, we already fixed $C_{\geq k+1}$, while $(l_1, \dots, l_k) \in \llbracket 0, w_1 \rrbracket \times \dots \times \llbracket 0, w_k \rrbracket$. The coefficient of a monomial \mathbf{m} which gives $\tilde{\mathbf{m}}$ is given by (18) of Proposition 2.12, and is

$$\text{coeff}_{\mathbf{m}} = \sigma_{\Lambda+L-c+1} k_{c-1} + \dots + \sigma_0 k_{\Lambda+L}$$

where

$$k_p = \frac{K_p}{K_{\{t_1, \dots, t_c\}, t_c}}$$

and

$$\sigma_p = \frac{(-1)^{p+L}}{\Lambda! L_1} \sigma_p(1, 2, \dots, \Lambda, n_\ell + 2 - L, \dots, n_\ell + 1)$$

with

$$K_{c-1} = 1 \text{ and } K_p = \sum_{j=1}^c t_j^p K_{\{t_1, \dots, t_c\}, t_j} \text{ when } p \geq c.$$

Since (l_1, \dots, l_k) is going through the box $\mathcal{B} = \llbracket 0, w_1 \rrbracket \times \dots \times \llbracket 0, w_k \rrbracket$, $\text{coeff}_{\tilde{\mathbf{m}}}$ is

$$(31) \quad \tilde{K} \sum_{l \in \mathcal{B}} (-1)^{l_1 + \dots + l_k} \binom{w_1}{l_1} \dots \binom{w_k}{l_k} \sum_{p=0}^{\Lambda+L-c+1} (-1)^p \times K_{\Lambda+L-p} \sigma_p(1, \dots, \Lambda, n_\ell + 2 - L, \dots, n_\ell + 1),$$

with

$$\tilde{K} = \frac{(-1)^{L_{\geq k+1}}}{\Lambda_{\geq k+1}! L_{\geq k+1}! (w_1!) \dots (w_k!) K_{\{t_1, \dots, t_c\}, t_c}}.$$

By Lemma 2.13, for any i , $\sigma_i(1, \dots, \Lambda, n_\ell + 2 - L, \dots, n_\ell + 1)$ is a polynomial in $l_1 + \dots + l_k$ of degree at most i . So,

$$\sum_{p=0}^{\Lambda+L-c+1} (-1)^p K_{\Lambda+L-p} \sigma_p(1, \dots, \Lambda, n_\ell + 2 - L, \dots, n_\ell + 1)$$

is a k -variate polynomial in (l_1, \dots, l_k) of total degree at most $\Lambda + L - c + 1$.

Finally, we have

$$(k+1)(c-1) \geq r-1 = W_{\geq 0} \\ \geq w_1 + 2w_2 + \dots + kw_k + (k+1)(\Lambda_{\geq k+1} + L_{\geq k+1}).$$

It implies

$$\Lambda + L - (c-1) \leq \frac{kw_1 + (k-1)w_2 + \dots + w_k}{k+1} < w_1 + \dots + w_k$$

since we assumed that one of the w_i is nonzero. By Lemma 2.7, the expression (31) above cancels. \square

Remark 3.6. In case $P_1 = \mu P_2$, all columns of $M_{n_\ell, T}$ are linearly dependent. In the first row, only the entry of the first column is nonzero. Therefore, all columns but the first one in $M_{n_\ell, T} \cdot \mathcal{K}_T$ have zero for entries.

4. LACUNARY CURVES

From now on, we consider that the polynomial $Q_\lambda = \sum_{i=1}^m \lambda_i Q_i$ is lacunary, in the sense that the family of parameters $\lambda = (\lambda_1, \dots, \lambda_m)$ has zeros at prescribed places, and of course we then omit those zero parameters. We still denote by d the degree of Q_λ , and by $m (= m_d \leq d+1)$ the number of (nonzero) parameters of the family λ . We still denote by $n_0 < n_1 < \dots < n_\ell < \dots < n_{\ell_d} = d$ the sequence of ℓ_d degrees of the monomials Q_i appearing in Q_λ . In addition, we return to the general case $\nu_1 \geq \nu_2 = \nu$.

We have already considered in the previous sections some general choice of coefficients appearing in Q_λ , by selecting, for $\ell \in \llbracket 0, \ell_d \rrbracket$, τ_ℓ columns $T_\ell = \{t_{\ell,1}, \dots, t_{\ell,\tau_\ell}\} \subset \{1, \dots, n_\ell + 1\}$, with $t_{\ell,1} < \dots < t_{\ell,\tau_\ell}$, in the block M_{n_ℓ} , corresponding to the specific monomials $X^{t_{\ell,i}-1} Y^{n_\ell - (t_{\ell,i}-1)}$, $i = 1, \dots, \tau_\ell$, of degree n_ℓ , a choice giving rise to the notation M_{n_ℓ, T_ℓ} for the blocks of the Bautin matrix $M_{T_0, \dots, T_{\ell_d}}$ (see Notation 2.8).

Definition 4.1. With this notation we say that the data

$$(d, \ell_d, n_0, \dots, n_{\ell_d}, \tau_0, \dots, \tau_{\ell_d}, t_{0,1}, \dots, t_{\ell_d, \tau_{\ell_d}}).$$

are the *lacunarity diagram* of Q_λ .

Notation 4.2. For any $\ell \in \llbracket 0, \ell_d \rrbracket$, we denote by Q_{λ, n_ℓ} the homogeneous part of degree n_ℓ of the polynomial Q_λ . With the notation above we have

$$Q_{\lambda, n_\ell}(X, Y) = \sum_{i=1}^{\tau_\ell} \lambda_{\ell, i} X^{t_{\ell, i}-1} Y^{n_\ell - (t_{\ell, i}-1)},$$

and $\ell_d + 1$ is the number of homogeneous polynomials in Q_λ . The number m of parameters of the family Q_λ is

$$m = \tau_0 + \dots + \tau_{\ell_d} \leq (n_0 + 1) + \dots + (n_{\ell_d} + 1).$$

We set for $\ell \in \llbracket 0, \ell_d \rrbracket$,

$$\begin{aligned}\bar{\nu}_\ell &= (t_{\ell,1} - 1)\nu_1 + (n_\ell - (t_{\ell,1} - 1))\nu_2, \\ \text{and } \tilde{\nu}_\ell &= (t_{\ell,\tau_\ell} - 1)\nu_1 + (n_\ell - (t_{\ell,\tau_\ell} - 1))\nu_2.\end{aligned}$$

The first lacunarity condition we consider is a condition insuring that the Bautin blocks $M_{n_\ell, T}$ do not encounter each other at some line in the Bautin matrix $M_{T_0, \dots, T_{\ell_d}}$. Since, for $\ell \in \llbracket 0, \ell_d \rrbracket$, the block M_{n_ℓ, T_ℓ} might contribute in f_λ at least to degree νn_ℓ and at most to degree $n_\ell D$, a first convenient condition is the following arithmetic progression for our degrees n_0, \dots, n_{ℓ_d}

$$(\mathcal{L}_1) \quad \forall \ell \in \llbracket 0, \ell_d - 1 \rrbracket, \quad \nu n_{\ell+1} > n_\ell D.$$

More accurately, with Notation 4.2, for $\ell \in \llbracket 0, \ell_d \rrbracket$, the block M_{n_ℓ, T_ℓ} contributes in f_λ at least to degree $\bar{\nu}_\ell$.

Moreover we allow that two blocks $M_{n_{\ell_1}, T}$ and $M_{n_{\ell_2}, T}$ (with $\ell_1 < \ell_2$) encounter at a same row r if the rows above r in $M_{n_{\ell_1}, T}$ are already of full rank. Consequently the sharper, but more involved, conditions insuring that our blocks do no intersect, are

$$(\mathcal{L}_{2a}) \quad \text{in the case } \nu_1 = \nu_2, \quad \forall \ell \in \llbracket 0, \ell_d - 1 \rrbracket, \quad \nu n_{\ell+1} > \nu n_\ell + (k+1)(\tau_\ell - 1),$$

$$(\mathcal{L}_{2b}) \quad \text{in the case } \nu_1 \neq \nu_2, \quad \forall \ell \in \llbracket 0, \ell_d - 1 \rrbracket, \quad \bar{\nu}_{\ell+1} > \tilde{\nu}_\ell$$

where we recall that k is defined as the largest integer such that $(\alpha_0, \dots, \alpha_k) = \mu \cdot (a_0, \dots, a_k)$ for some $\mu \in \mathbb{C} \setminus \{0\}$.

We can even refine hypotheses (\mathcal{L}_{2a}) and (\mathcal{L}_{2b}) , in the sense that we can allow more polynomials, under the following refined lacunarity conditions (\mathcal{L}_{3a}) and (\mathcal{L}_{3b}) .

Indeed in the case $\nu_1 = \nu_2$, after triangulation (obtained by a columns operation) of a Bautin block $M_{n_\ell, T_\ell} \cdot \mathcal{K}_{T_\ell}$, the leading entries of the column echelon form appear on rows $\{\bar{\nu}_\ell + i \cdot (k+1) \mid 0 \leq i < \tau_\ell\}$. Consequently, the following hypothesis is sufficient to ensure the different selected rows of each block do not overlap

$$(\mathcal{L}_{3a}) \quad \forall (\ell, \ell') \in \llbracket 0, \ell_d \rrbracket^2, \quad \ell > \ell' \text{ and } \bar{\nu}_\ell = \bar{\nu}_{\ell'} [\text{mod } k+1] \implies \bar{\nu}_\ell > \bar{\nu}_{\ell'} + (k+1)(\tau_{\ell'} - 1).$$

In the other case, $\nu_1 \neq \nu_2$, the rows which are linearly independent of the previous ones (see Remark 2.9) of a block M_{n_ℓ, T_ℓ} are the rows $\{(t_{\ell,i} - 1)\nu_1 + (n_\ell - (t_{\ell,i} - 1))\nu_2 \mid i \in \llbracket 1, \tau_\ell \rrbracket\}$. Given Q_λ and a pair of polynomials (P_1, P_2) , one can directly check whether two such rows of different blocks coincide, and the proof still works if it is not the case. Noticing that the gap between two such rows of a same block is always a multiple of $|\nu_1 - \nu_2|$, one can still set this sufficient hypothesis

$$(\mathcal{L}_{3b}) \quad \forall (\ell, \ell') \in \llbracket 0, \ell_d \rrbracket^2, \quad (\ell > \ell' \text{ and } \bar{\nu}_\ell = \bar{\nu}_{\ell'} [\text{mod } |\nu_1 - \nu_2|]) \implies (\bar{\nu}_\ell > \tilde{\nu}_{\ell'}).$$

Remark 4.3. Under condition (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{2b}) , (\mathcal{L}_{3a}) , or (\mathcal{L}_{3b}) , we have $\lambda \in L_b$, that is to say $Q_\lambda(P_1, P_2) = 0$, if and only if for any $\ell \in \llbracket 0, \ell_d \rrbracket$, $Q_{\lambda, n_\ell}(P_1, P_2) = 0$.

5. ZEROS OF FAMILIES OF LACUNARY CURVES

Using the notation of Section 2, let us consider a basis $(v_{i_1}, \dots, v_{i_\sigma})$ with $S = \{i_1, \dots, i_\sigma\} \subseteq \{0, \dots, b\}$, of the space of linear forms vanishing on L_b , chosen among the elements of the family (v_0, \dots, v_b) .

For any form $v_j(\lambda) = \sum_{i=1}^m c_j^i \lambda_i$, since $v_j \in \text{Span}((v_k)_{k \in S})$, there exist $\gamma_j^{i_1}, \dots, \gamma_j^{i_\sigma} \in \mathbb{C}$ and \tilde{c} such that

$$(32) \quad v_j(\lambda) = \sum_{k \in S} \gamma_j^k v_k(\lambda), \quad |\gamma_j^k| \leq \tilde{c} \|v_j\|.$$

where $\|v_j\| = \max_{i=1, \dots, m} |c_j^i|$.

Remark 5.1. We have $\sigma = \dim(\text{Span}((v_k)_{k \in S})) = m - \dim(L_b)$.

Notation 5.2. We denote by $c = c(f_\lambda, S) > 0$ the minimum of the constants \tilde{c} satisfying (32).

With the notation of Section 1, in this section we estimate from above the number of zeros of f_λ in a disc, following [20] and [7]. To fix notation, and for the convenience of the reader, we recall and adapt the method to our situation.

Definition 5.3 (Bernstein class). Let $r > 0$, $\alpha \in]0, 1[$, $K > 0$ and f be a nonzero analytic function on a neighbourhood of the closed disc \bar{D}_r . We say that f belongs to the *Bernstein class* $B_{r, \alpha, K}^1$ when

$$\frac{\max_{\bar{D}_r} |f|}{\max_{\bar{D}_{\alpha r}} |f|} \leq K.$$

Now let the family $Q_i, i = 1, \dots, m$ be given, and let the Bautin index b as in Section 1. For any $r < R$ and any $\alpha \in]0, 1[$, we denote by μ the maximum of $|f_\lambda|$ over the closed disk $\bar{D}_{\alpha r}$. By (4) we get for all i , $|v_i| \leq \mu / (\alpha r)^i$. Hence,

following the proof of Theorem 2.1.3 in [20], by (32) and using again (4),

$$\begin{aligned}
(33) \quad \max_{\bar{D}_r}(|f_\lambda|) &\leq \sum_{i=0}^b |v_i(\lambda)| r^i + \sum_{j \geq b+1} |v_j(\lambda)| r^j \\
&\leq \mu \sum_{i=0}^b \frac{r^i}{(\alpha r)^i} + \sum_{j \geq b+1} \sum_{k \in S} |\gamma_j^k| |v_k(\lambda)| R^j \left(\frac{r}{R}\right)^j \\
&\leq \frac{\mu}{\alpha^b} \frac{1 - \alpha^{b+1}}{1 - \alpha} + \sum_{j \geq b+1} \sum_{k \in S} c \left(R^j \max_{i=1, \dots, m} |c_j^i| \right) |v_k(\lambda)| \left(\frac{r}{R}\right)^j \\
&\leq \frac{\mu}{\alpha^b} \frac{1 - \alpha^{b+1}}{1 - \alpha} + B_{RC} \sum_{k \in S} |v_k(\lambda)| \sum_{j \geq b+1} \left(\frac{r}{R}\right)^j \\
&\leq \frac{\mu}{\alpha^b} \frac{1 - \alpha^{b+1}}{1 - \alpha} + B_{RC} \frac{(r/R)^{b+1}}{1 - r/R} \mu \sum_{k \in S} \frac{1}{(\alpha r)^k} \\
&\leq \frac{\mu}{\alpha^b} \left(1 + \frac{\alpha(1 - \alpha^b)}{1 - \alpha} + \frac{B_{RC} r}{R^{b+1}(1 - r/R)} \sum_{k \in S} (\alpha r)^{b-k} \right).
\end{aligned}$$

Let us introduce $\bar{\sigma}_\rho = \sum_{k \in S} \rho^{b-k}$. We have

$$(34) \quad \bar{\sigma}_\rho \leq \begin{cases} \frac{1 - \rho^\sigma}{1 - \rho} & \text{if } \rho < 1, \\ \sigma & \text{if } \rho = 1, \\ \rho^{b+1} \frac{1 - 1/\rho^\sigma}{\rho - 1} & \text{if } \rho > 1. \end{cases}$$

It follows that for $\lambda \notin L_b$, $f_\lambda \in B_{r, \alpha, K}^1$ with

$$(35) \quad K = \frac{1}{\alpha^b} \left(1 + \frac{\alpha(1 - \alpha^b)}{1 - \alpha} + \frac{B_{RC} r \bar{\sigma}_{\alpha r}}{R^{b+1}(1 - r/R)} \right).$$

We can now give in Theorem 5.5 (in the same way as [20, Lemma 2.2.3]) a bound from above on the number of zeros of f_λ in $\bar{D}_{R/4}$, using the following Jensen–Nevanlinna-type inequality (see [23, Lemma 1], or [24] for other proofs of statements comparable to (2) of Theorem 5.5 below).

Theorem 5.4. *Let $f_\lambda \in B_{r, \alpha, K}^1$, then the number of zeros of f_λ in $\bar{D}_{\alpha r}$ is at most $\frac{\log K}{\log((1 + \alpha^2)/2\alpha)}$.*

Theorem 5.5. *Zeros being counted with multiplicity, we have the following uniform bounds.*

- (1) *The maximal number of zeros of f_λ , with respect to $\lambda \notin L_b$, in the disk $\bar{D}_{\frac{R}{4}}$, is at most*

$$5b \log 2 + 5 \log(2R^b + B_{RC} \bar{\sigma}_{R/4}) - 5b \log(R).$$

(2) The maximal number of zeros of f_λ , with respect to $\lambda \notin L_b$, is at most b in \bar{D}_ρ , where

$$\rho \leq \frac{R}{8^b \max(2, B_R c \bar{\sigma}_{R/4}/R^b)}.$$

Proof. Since for $\lambda \notin L_b$, $f_\lambda \in B_{r,\alpha,K}^1$ for any $r < R$ and any $\alpha \in]0, 1[$, and for K as in (35), choosing $r = R/2$ and $\alpha = 1/2$, one gets $K \leq 2^b(2 + B_R c \bar{\sigma}_{R/4}/R^b)$. By Theorem 5.4, one obtains that the number of zeros in $\bar{D}_{\frac{R}{4}}$ is at most

$$\begin{aligned} & \frac{1}{\log 5/4} \left(b \log 2 + \log \left(2 + \frac{B_R c \bar{\sigma}_{R/4}}{R^b} \right) \right) \\ & \leq 5b \log 2 + 5 \log(2R^b + B_R c \bar{\sigma}_{R/4}) - 5b \log(R). \end{aligned}$$

The second bound in the statement comes from [20, Lemma 2.2.3]. \square

The next step consists now in the search of a bound from above for the constants b, c and σ appearing in Theorem 5.5. All the information we need is encoded in the rank σ matrix $M(f_\lambda) = (c_k^i)$, $k = 0, \dots, b$, $i = 1, \dots, m$, with $b+1$ lines and m columns.

Notation 5.6. We will denote by $\delta > 0$ the maximum of the absolute value of all nonzero minor determinants of size σ of the Bautin matrix $M(f_\lambda)$.

We give below an estimate of the constant c , and then of $\mathcal{Z}(f_\lambda)$.

Proposition 5.7 ([7, Proposition 2.11]). *Let f_λ be given as above, with $\lambda \notin L_b$. Then*

$$c(f_\lambda) \leq \sigma \frac{(B_R \sqrt{\sigma})^{\sigma-1}}{\delta R^{\beta(\sigma-1)}},$$

where $\beta = b$ if $R \leq 1$ and $\beta = \sigma/2$ if $R \geq 1$.

In the disk $\bar{D}_{\frac{R}{4}}$, the maximal number $\mathcal{Z}(f_\lambda)$ of zeros of the family f_λ , with respect to the parameter λ , satisfies

$$\begin{aligned} \mathcal{Z}(f_\lambda) & \leq 5b \log 2 + 5 \log \left(2 + \frac{\sigma^{\frac{\sigma+1}{2}} B_R^\sigma \bar{\sigma}_{R/4}}{\delta R^{b\sigma}} \right) \quad \text{if } R \leq 1, \\ \text{and } \mathcal{Z}(f_\lambda) & \leq 5b \log 2 + 5 \log \left(2 + \frac{\sigma^{\frac{\sigma+1}{2}} B_R^\sigma \bar{\sigma}_{R/4}}{\delta R^{b+\frac{\sigma}{2}(\sigma-1)}} \right) \quad \text{if } R \geq 1. \end{aligned}$$

Proof. The proof of the bound on $c(f_\lambda)$ is the same as the one of [7, Proposition 2.11]. The proof of the bounds on $\mathcal{Z}(f_\lambda)$ is then a consequence of Theorem 5.5. \square

Remark 5.8. Since $f_\lambda(z) = 0$ for any $z \in \bar{D}_R$ is equivalent to $f_\lambda(Rz) = 0$ for any $z \in \bar{D}_1$, a uniform bound with respect to $\lambda \notin L_b(f_\lambda(Rz))$ on the zeros of $f_\lambda(Rz)$ in $\bar{D}_{\frac{1}{4}}$ gives a uniform bound with respect to $\lambda \notin L_b(f_\lambda(z))$ on the zeros of $f_\lambda(z)$ in $\bar{D}_{\frac{R}{4}}$.

If we denote by \tilde{P} the polynomial mapping $\frac{P}{M_R}$, with notation (1) and (2), we have a bijection between $L_b(\tilde{f}_\lambda = Q_\lambda(\tilde{P}))$ and $L_b(f_\lambda)$ sending $(\lambda_i)_{i=1,\dots,m}$ to $\left(\frac{\lambda_i}{M_R^{d_i}}\right)_{i=1,\dots,m}$, where d_i is the degree of the monomial Q_i corresponding to the parameter λ_i . In consequence a uniform bound, with respect to $\lambda \notin L_b(\tilde{f}_\lambda)$, on the zeros of $\tilde{f}_\lambda = Q_\lambda(P)$ in $\bar{D}_{\frac{R}{4}}$ gives a uniform bound, with respect to $\lambda \notin L_b(f_\lambda)$, on the zeros of f_λ in $\bar{D}_{\frac{R}{4}}$. Notice that in this situation since \tilde{P} has its components bounded by 1 on \bar{D}_R , for any monomial Q_i , $Q_i(P)$ is also bounded by 1 on \bar{D}_R .

In conclusion to bound the number of zeros of f_λ , with respect to $\lambda \notin L_b(f_\lambda)$, we can always consider that $R = B_R = 1$, up to changing the polynomial mapping $P(X)$ into the polynomial mapping (of same degree D) $P(RX)/\max_{z \in \bar{D}_1, i=1,2} |P_i(Rz)|$.

Remark 5.8 allows us to only investigate the case $R = B_R = 1$ without loss of generality. By Proposition 5.7 we obtain in this case (since by (34), we have $\bar{\sigma}_{1/4} \leq 4/3$) the following simple bound for $\mathcal{Z}(f_\lambda)$.

Corollary 5.9. *Let a family f_λ be given as in (3), with $R = B_R = 1$, then the maximal number $\mathcal{Z}(f_\lambda)$ of zeros of f_λ in $\bar{D}_{\frac{1}{4}}$, with respect to $\lambda \notin L_b$, satisfies*

$$\mathcal{Z}(f_\lambda) \leq 5b \log 2 + 5 \log \left(2 + \frac{4\sigma^{\frac{\sigma+1}{2}}}{3\delta} \right).$$

Furthermore, the maximal number of zeros of f_λ , with respect to $\lambda \notin L_b$, is at most b in \bar{D}_ρ , where

$$\rho = \frac{1}{8^b \max\{2, 4c/3\}} \geq \frac{1}{8^b} \min \left\{ \frac{1}{2}, \frac{3\delta}{4\sigma^{\frac{\sigma+1}{2}}} \right\}.$$

Remark 5.10. Under condition (\mathcal{L}_1) , we can improve on the bound on $\mathcal{Z}(f_\lambda)$ of Corollary 5.9 by giving a better estimate on the constants γ_j^k , where $j > b$, in (33). Indeed, we can improve on arguments of the proof of Proposition 5.7 in the following way. By condition (\mathcal{L}_1) , for $j > b$, the linear form v_j is a linear combination of the v_k where k is in the set S_{ℓ_d} which is the intersection of S (introduced in the beginning of this section) and of the indices k satisfying $k \geq \bar{\nu}_{\ell_d} - 1$ (that is to say, the coefficients of v_k are given by the last block $M_{n_{\ell_d}, T_{\ell_d}}$ of the Bautin matrix $M_{T_0, \dots, T_{\ell_d}}$, see Notation 2.8). Working with these parameters in the proof of Proposition 5.7, gives the same bound with σ replaced by τ_{ℓ_d} and δ replaced by the determinant of the upper $\tau_{\ell_d} \times \tau_{\ell_d}$ minor of $M_{n_{\ell_d}, T_{\ell_d}}$.

In view of Corollary 5.9, we compute in this section the Bautin index b , the dimension σ and the determinant δ . All those data are related to the Bautin matrix $M_{T_0, \dots, T_{\ell_d}}$ (see Notation 2.8), and thanks to (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{2b}) , (\mathcal{L}_{3a})

or (\mathcal{L}_{3b}) , they can be computed on the blocks M_{n_ℓ} , $\ell = 0, \dots, \ell_d$. Moreover for each block M_{n_ℓ} we can perform our computation on the reduced matrix $M_{n_\ell, T_\ell} \cdot \mathcal{K}_{T_\ell}$, since our triangulation process does not change the rank nor δ .

Proposition 5.11 (Value of σ). *Under condition (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{2b}) , (\mathcal{L}_{3a}) or (\mathcal{L}_{3b}) , we have*

$$\dim(L_b) = 0 \text{ and } \sigma = m = \tau_0 + \dots + \tau_{\ell_d},$$

unless P_1 and P_2 are proportional, in which case

$$\dim(L_b) = \tau_0 + \dots + \tau_{\ell_d} - (\ell_d + 1) \text{ and } \sigma = \ell_d + 1.$$

In particular, in case P_1 and P_2 are not proportional, condition (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{2b}) , (\mathcal{L}_{3a}) or (\mathcal{L}_{3b}) guarantees that $(P_1, P_2)(\mathbb{C}) \cap \{Q_\lambda = 0\}$ is a finite set.

Proof. In case P_1 and P_2 are proportional, by Remark 3.6, any block M_{n_ℓ, T_ℓ} contributes for only one linear form in a basis of linear forms of $\text{Span}(v_0, \dots, v_b)$. It follows that σ is the number of blocks, that is $\ell_d + 1$.

Finally, in case P_1 and P_2 are not proportional, no parameter $\lambda \neq 0$ can cancel $Q_\lambda(P_1, P_2)$, showing that $\dim(L_b) = 0$, and by Remark 5.1, that $\sigma = m - \dim(L_b) = \tau_0 + \dots + \tau_{\ell_d}$. \square

Proposition 5.12 (Value of b). *Under condition (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{2b}) , (\mathcal{L}_{3a}) or (\mathcal{L}_{3b}) , with the notation of Notation 4.2 and (21), we have*

$$(36) \quad \text{for } \nu_1 \neq \nu_2, \quad b = \max_{\ell \in \llbracket 0, \ell_d \rrbracket} (\tilde{\nu}_\ell),$$

$$(37) \quad \text{and for } \nu_1 = \nu_2 = \nu, \quad b = \max_{\ell \in \llbracket 0, \ell_d \rrbracket} (\nu n_\ell + (k+1)(\tau_\ell - 1)),$$

unless P_1 and P_2 are proportional, in which case

$$(38) \quad b = \bar{\nu}_{\ell_d} = \nu d.$$

Notice that under conditions (\mathcal{L}_1) , (\mathcal{L}_{2a}) , or (\mathcal{L}_{2b}) , both maxima are achieved for $\ell = \ell_d$.

Proof. The Bautin index b is the lowest row $b+1$ in the Bautin matrix $M_{T_0, \dots, T_{\ell_d}}$, corresponding to the coefficients of the linear form v_b , where $v_b \notin \text{Span}(v_0, \dots, v_{b-1})$. Under condition (\mathcal{L}_1) , (\mathcal{L}_{2a}) , or (\mathcal{L}_{2b}) this last row has to be found in the last block $M_{n_{\ell_d}, T_{\ell_d}}$ of $M_{T_0, \dots, T_{\ell_d}}$. This last block starts at row $\bar{\nu}_{\ell_d} + 1$ of $M_{T_0, \dots, T_{\ell_d}}$, corresponding to the coefficients of $v_{\bar{\nu}_{\ell_d}}$.

- Let us first assume that P_1 and P_2 are not proportional. In this situation, we have two sub-cases.

- The first one is the case where the multiplicities ν_1 and ν_2 of P_1 and P_2 are not the same. Then by Remark 2.9 the last row in $M_{n_{\ell_d}, T_{\ell_d}}$ independent of the highest ones is the row corresponding to $v_{\tilde{\nu}_{\ell_d}}$. So $b = \tilde{\nu}_{\ell_d}$.

- The second sub-case is the case where $\nu_1 = \nu_2$. In this situation, let us denote by $k \in \llbracket 0, \min(D_1, D_2) - \nu \rrbracket$ the largest integer such that the k -jets of P_1 and P_2 are proportional, accordingly to (21). By Proposition 3.5, the row $(k+1)\tau_{\ell_d} - k$ of the block $M_{n_{\ell_d}, T_{\ell_d}}$, corresponding to the coefficients of the linear form $v_{\bar{\nu}_{\ell_d} + (k+1)\tau_{\ell_d} - k - 1}$, is independent of the other highest rows, and the other lowest rows are not independent of those $(k+1)\tau_{\ell_d} - k$ first ones. In conclusion, $b = \bar{\nu}_{\ell_d} + (k+1)\tau_{\ell_d} - k - 1 = \nu n_{\ell_d} + (k+1)(\tau_{\ell_d} - 1)$.
- In case P_1 and P_2 are proportional, as noticed in Remark 3.6, all columns of $M_{n_{\ell_d}, T_{\ell_d}}$ but the first one have only zero for entries, meaning that in this situation $b = \bar{\nu}_{\ell_d} = n_{\ell_d} \nu = \nu d$.

Under condition (L3a) or (L3b), the proof is similar by replacing ℓ_d by respectively $\operatorname{argmax}_{\ell}(\bar{\nu}_{\ell})$ or $\operatorname{argmax}_{\ell}(\bar{\nu}_{\ell} + (k+1)\tau_{\ell})$. \square

We finally give a bound from above of δ , the maximum of the absolute value of all nonzero minor of size σ of $M_{T_0, \dots, T_{\ell_d}}$ (see Notation 5.6). For this we first fix some notation.

Notation 5.13. In the case where $\nu_1 = \nu_2$, with notation (21), we set

$$\bar{\delta}_k = \alpha_{k+1} - \mu a_{k+1}, \quad \ell_e = \operatorname{argmax}_{\ell}(\bar{\nu}_{\ell} + (k+1)\tau_{\ell}).$$

In the case where $\nu_1 \neq \nu_2$, we set $\ell_e = \operatorname{argmax}_{\ell}(\bar{\nu}_{\ell})$. And in both cases, we set, for $\ell \in \llbracket 0, \ell_d \rrbracket$,

$$\begin{aligned} \bar{t}_{\ell} &= \sum_{i \in \llbracket 1, \tau_{\ell} \rrbracket} t_{\ell, i}, \quad \bar{t} = \sum_{\ell \in \llbracket 0, \ell_d \rrbracket} \bar{t}_{\ell}, \quad \bar{t}'_1 = \sum_{\ell \in \llbracket 0, \ell_d \rrbracket} \bar{t}_{\ell, 1}, \\ \bar{n} &= \sum_{i \in \llbracket 0, \ell_d \rrbracket} n_{\ell}, \quad \bar{\tau} \bar{n} = \sum_{\ell \in \llbracket 0, \ell_d \rrbracket} \tau_{\ell} n_{\ell}, \\ C_{\ell} &= \frac{1}{\prod_{i \in \llbracket 1, \tau_{\ell} \rrbracket} (i-1)! \prod_{i \in \llbracket 1, \tau_{\ell} \rrbracket} |K_{\{t_{\ell, 1}, \dots, t_{\ell, i}, t_{\ell, i}\}}|}, \quad \bar{C} = \prod_{\ell \in \llbracket 0, \ell_d \rrbracket} C_{\ell}. \end{aligned}$$

We finally recall that, m being the number of parameters of our family Q_{λ} ,

$$m = \sum_{\ell=0}^{\ell_d} \tau_{\ell}, \quad \text{and we set } \bar{\tau} = \sum_{\ell \in \llbracket 0, \ell_d \rrbracket} \frac{\tau_{\ell}(\tau_{\ell} + 1)}{2}, \quad \bar{\tau}' = \sum_{\ell \in \llbracket 0, \ell_d \rrbracket} \frac{\tau_{\ell}(\tau_{\ell} - 1)}{2}.$$

Proposition 5.14 (Value of δ). *Under condition (L1), (L2a), (L2b), (L3a) or (L3b), with the notation of Notation 4.2, 5.13 and (21), we have for $\nu_1 \neq \nu_2$,*

$$(39) \quad \delta \geq |\alpha_0|^{\bar{t}-m} |a_0|^{\bar{\tau} \bar{n} - (\bar{t}-m)},$$

and for $\nu_1 = \nu_2$,

$$(40) \quad \delta \geq \bar{C} |\mu|^{\bar{t}-\bar{\tau}} |a_0|^{\bar{\tau} \bar{n} - \bar{\tau}'} |\bar{\delta}_k|^{\bar{\tau}'},$$

unless P_1 and P_2 are proportional, in which case

$$(41) \quad \delta \geq |\mu|^{\bar{t}_1 - (\ell_d + 1)} |a_0|^{\bar{n}}.$$

Under condition \mathcal{L}_1 , more accurately, instead of δ one can take in Corollary 5.9 the following bounds: for $\nu_1 \neq \nu_2$,

$$(42) \quad |\alpha_0|^{\bar{t}_{\ell_d} - \tau_{\ell_d}} |a_0|^{\tau_{\ell_d} n_{\ell_d} + \tau_{\ell_d} - \bar{t}_{\ell_d}},$$

and for $\nu_1 = \nu_2$,

$$(43) \quad C_{\ell_d} |\mu|^{\bar{t}_{\ell_d} - \frac{\tau_{\ell_d}(\tau_{\ell_d} + 1)}{2}} |a_0|^{\tau_{\ell_d} n_{\ell_d} - \frac{\tau_{\ell_d}(\tau_{\ell_d} - 1)}{2}} |\bar{\delta}_k|^{\frac{\tau_{\ell_d}(\tau_{\ell_d} - 1)}{2}},$$

unless P_1 and P_2 are proportional, in which case

$$(44) \quad |\mu|^{t_{\ell_d, 1} - 1} |a_0|^{n_{\ell_d}}.$$

Proof. Under condition (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{2b}) , (\mathcal{L}_{3a}) or (\mathcal{L}_{3b}) , a nonzero minor of maximal size σ of the Bautin matrix $M_{T_0, \dots, T_{\ell_d}}$ is given as the product, over $\ell \in \llbracket 0, \ell_d \rrbracket$, of nonzero minors δ_ℓ of maximal size chosen in each block M_{n_ℓ, T_ℓ} (as long as the rows of the minor δ_ℓ do not overlap the blocks $M_{n_{\ell'}, T_{\ell'}}$ for $\ell' > \ell$). Moreover if needed, we can choose δ_ℓ in the reduced matrix $M_{n_\ell, T_\ell} \cdot \mathcal{K}_{T_\ell}$.

- Let us first assume that P_1 and P_2 are not proportional. In this situation, as in the proof of Proposition 5.12, we have two sub-cases.
 - The first one is the case where the multiplicities ν_1 and ν_2 of P_1 and P_2 are not the same. Let us fix $\ell \in \llbracket 0, \ell_d \rrbracket$. Then by Remark 2.9, the columns of the block M_{n_ℓ, T_ℓ} are shifted down by zero-columns, and the first upper nonzero coefficient in column $i \in \llbracket 1, \tau_\ell \rrbracket$, is $\alpha_0^{t_{\ell, i} - 1} a_0^{n_\ell - (t_{\ell, i} - 1)}$. We can therefore choose

$$\delta_\ell = |\alpha_0|^{\sum_{i \in \llbracket 1, \tau_\ell \rrbracket} t_{\ell, i} - \tau_\ell} |a_0|^{\tau_\ell n_\ell - \left(\sum_{i \in \llbracket 1, \tau_\ell \rrbracket} t_{\ell, i} - \tau_\ell \right)},$$

giving, with Notation 5.13, the following bound from below for δ

$$\prod_{\ell \in \llbracket 0, \ell_d \rrbracket} \delta_\ell = |\alpha_0|^{\bar{t} - m} |a_0|^{\bar{\tau} n - (\bar{t} - m)}.$$

- The second sub-case is the case where $\nu_1 = \nu_2$. In this situation, let us denote by $k \in \llbracket 0, \min(D_1, D_2) - \nu \rrbracket$ the largest integer such that the k -jets of P_1 and P_2 are proportional, accordingly to (21). Let us fix $\ell \in \llbracket 0, \ell_d \rrbracket$. By Proposition 3.5, in the block $M_{n_\ell, T_\ell} \cdot \mathcal{K}_{T_\ell}$ we can choose the rows $(k + 1)i - k$, for $i \in \llbracket 1, \tau_\ell \rrbracket$ to compute δ_ℓ . We obtain by (28), with Notation 5.13,

$$\delta_\ell = C_\ell |\mu|^{\bar{t}_\ell - \frac{\tau_\ell(\tau_\ell + 1)}{2}} |a_0|^{\tau_\ell n_\ell - \frac{\tau_\ell(\tau_\ell - 1)}{2}} |\bar{\delta}_k|^{\frac{\tau_\ell(\tau_\ell - 1)}{2}},$$

giving the following bound from below for δ

$$\prod_{\ell \in \llbracket 0, \ell_d \rrbracket} \delta_\ell = \bar{C}' |\mu|^{\bar{t} - \bar{\tau}'} |a_0|^{\bar{\tau} n - \bar{\tau}' n} |\bar{\delta}_k|^{\bar{\tau}' n}.$$

- In case P_1 and P_2 are proportional, using Remark 3.6, all columns of $M_{n_{\ell_d}, T_{\ell_d}}$ but the first one have only zero for entries. We then choose, for each $\ell \in \llbracket 0, \ell_d \rrbracket$, in each block M_{n_ℓ, T_ℓ} , the first row, having for only nonzero coefficient $\mu^{t_{\ell,1}-1} a_0^{n_\ell}$ by (25). This gives, with Notation 5.13, the following bound from below for δ

$$\prod_{\ell \in \llbracket 0, \ell_d \rrbracket} |\mu|^{t_{\ell,1}-1} a_0^{n_\ell} = |\mu|^{\bar{t}_1 - (\ell_d + 1)} |a_0|^{\bar{n}}.$$

Under condition \mathcal{L}_1 , we can take δ_{ℓ_d} in each case above as the final bound. \square

We now combine Propositions 5.11, 5.12 and 5.14 with Corollary 5.9 to give bounds for the number of intersection points, near the origin, between the zero set of a family of lacunary polynomials and a given polynomially parametrized algebraic curve. Those bounds are given in terms of the data that explicitly quantify the lacunarity: the number of monomials appearing in the family, the degree of those monomials, and the particular choices made among monomials of a given degree.

Theorem 5.15. *Let a family f_λ be given as in (3), with $R = B_R = 1$ and $m \geq 2$. Under condition (\mathcal{L}_1) , (\mathcal{L}_{2a}) , (\mathcal{L}_{2b}) , (\mathcal{L}_{3a}) or (\mathcal{L}_{3b}) , with the notation of Notation 4.2, 5.13 and (21), denoting by $\mathcal{Z}(f_\lambda)$ the maximal number of zeros of f_λ in $\bar{D}_{\frac{1}{4}}$, with respect to $\lambda \notin L_b$, we have, for P_1 and P_2 not proportional,*

- (1) *in the case where $\nu_1 \neq \nu_2$,*

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5(\tilde{\nu}_{\ell_e} + 1) \log 2 + \frac{5}{2}(m + 1) \log m \\ &\quad + 5(\bar{t} - m) \log \left(\frac{1}{|\alpha_0|} \right) + 5(\bar{\tau} \bar{n} - (\bar{t} - m)) \log \left(\frac{1}{|a_0|} \right) \\ &= O(dm), \end{aligned}$$

and the maximal number of zeros of f_λ , with respect to $\lambda \neq 0$, is at most $\tilde{\nu}_{\ell_e}$ in \bar{D}_ρ , where

$$\rho = \frac{3}{4 \cdot 8^{\tilde{\nu}_{\ell_e}}} \frac{|\alpha_0|^{\bar{t}-m} |a_0|^{\bar{\tau} \bar{n} - (\bar{t}-m)}}{m^{\frac{m+1}{2}}}.$$

- (2) *In the case where $\nu_1 = \nu_2$,*

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5(\bar{\nu}_{\ell_e} + (k + 1)(\tau_{\ell_e} - 1)) \log 2 \\ &\quad + 5 \log \left(2 + \frac{4m^{\frac{m+1}{2}}}{3\bar{C} |\mu|^{\bar{t}-\bar{\tau}} |a_0|^{\bar{\tau} \bar{n} - \bar{\tau}'} |\bar{\delta}_k|^{\bar{\tau}'}} \right) \\ &= O(dm), \end{aligned}$$

and the maximal number of zeros of f_λ , with respect to $\lambda \neq 0$, is at most νn_{ℓ_e} in \bar{D}_ρ , where

$$\rho = \frac{3}{4 \cdot 8^{\nu n_{\ell_e} + (k+1)(\tau_{\ell_e} - 1)}} \min \left\{ \frac{2}{3}, \frac{\bar{C} |\mu|^{\bar{t} - \bar{\tau}} |a_0|^{\bar{\tau} n - \bar{\tau}'}}{m^{\frac{m+1}{2}} |\bar{\delta}_k|^{\bar{\tau}'}} \right\}.$$

Remark 5.16. The rational curves (P_1, P_2) and (P_2, P_1) are symmetric in the plane. Given a lacunary polynomial $Q_\lambda(X, Y)$, the number of intersection points between (P_1, P_2) and $Q_\lambda = 0$ is the same as the number of intersection points between (P_2, P_1) and $Q_\lambda(Y, X) = 0$. Exchanging X and Y in Q_λ changes the numbering $(t_{0,1}, \dots, t_{\ell_d, \tau_{\ell_d}})$ of the columns in the following way: In a block M_{n_ℓ, T_ℓ} , the number \bar{t}_ℓ becomes $(n_\ell + 1)\tau_\ell - \bar{t}_\ell$. The bounds on $\mathcal{Z}(f_\lambda)$ and ρ in Theorem 5.15 are unaffected under this transformation.

Proof. We consider the two cases separately.

- (1) Let us first consider the case where $\nu_1 \neq \nu_2$.

By Proposition 5.11 we have $\sigma = m$, by (36) we have $b = \tilde{\nu}_{\ell_e}$, and by (39) $\delta \geq |\alpha_0|^{\bar{t} - m} |a_0|^{\bar{\tau} n - (\bar{t} - m)}$. By Corollary 5.9 we deduce that

$$\mathcal{Z}(f_\lambda) \leq 5\tilde{\nu}_{\ell_e} \log 2 + 5 \log \left(2 + \frac{4m^{\frac{m+1}{2}}}{3|\alpha_0|^{\bar{t} - m} |a_0|^{\bar{\tau} n - (\bar{t} - m)}} \right).$$

Notice that since $m \geq 2$ and by Cauchy estimate $|\alpha_0|$ and $|a_0|$ are bounded from above by 1, we have

$$\frac{m^{\frac{m+1}{2}}}{|\alpha_0|^{\bar{t} - m} |a_0|^{\bar{\tau} n - (\bar{t} - m)}} \geq 2\sqrt{2},$$

and therefore

$$2 + \frac{4}{3} \frac{m^{\frac{m+1}{2}}}{|\alpha_0|^{\bar{t} - m} |a_0|^{\bar{\tau} n - (\bar{t} - m)}} \leq \frac{3\sqrt{2} + 8}{6} \frac{m^{\frac{m+1}{2}}}{|\alpha_0|^{\bar{t} - m} |a_0|^{\bar{\tau} n - (\bar{t} - m)}}.$$

Recall that the constant 5 is in fact an approximation from above of $1/\log(5/4)$. Since we have

$$\frac{\log((3\sqrt{2} + 8)/6)}{\log(5/4)} < 5 \log 2,$$

we can get the following bound from above for $\mathcal{Z}(f_\lambda)$

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5 \log 2 (\tilde{\nu}_{\ell_e} + 1) + \frac{5}{2} (m + 1) \log m \\ &\quad + 5(\bar{t} - m) \log \left(\frac{1}{|\alpha_0|} \right) + 5(\bar{\tau} n - (\bar{t} - m)) \log \left(\frac{1}{|a_0|} \right). \end{aligned}$$

Applying the second part of Corollary 5.9 we immediately obtain that the maximal number of zeros of f_λ , with respect to $\lambda \neq 0$, is at most $\tilde{\nu}_{\ell_e}$ in \bar{D}_ρ , where

$$\rho = \frac{1}{8^{\tilde{\nu}_{\ell_e}}} \frac{3|\alpha_0|^{\bar{t} - m} |a_0|^{\bar{\tau} n - (\bar{t} - m)}}{4m^{\frac{m+1}{2}}}.$$

- (2) In the subcase where $\nu_1 = \nu_2$, by Proposition 5.11 we still have $\sigma = m$, by (37) we have $b = \bar{\nu}_{\ell_e} + (k+1)(\tau_{\ell_e} - 1)$, and by (40) $\delta \geq \bar{C}|\mu|^{\bar{t}-\bar{\tau}}|a_0|^{\bar{\tau}n-\bar{\tau}'}|\bar{\delta}_k|^{\bar{\tau}'}$. Then Corollary 5.9 gives immediately the bounds in our statement for $\mathcal{Z}(f_\lambda)$ and ρ . \square

In fact under condition \mathcal{L}_1 , we can directly improve the bounds from the previous Theorem by replacing σ by τ_{ℓ_d} and the bounds on δ by the bounds given in the second part of Proposition 5.14.

Theorem 5.17. *Let a family f_λ be given as in (3), with $R = B_R = 1$. Under condition (\mathcal{L}_1) , denoting by $\mathcal{Z}(f_\lambda)$ the maximal number of zeros of f_λ in $\bar{D}_{\frac{1}{4}}$, with respect to $\lambda \notin L_b$, we have, for P_1 and P_2 not proportional,*

- (1) *in the case where $\nu_1 \neq \nu_2$,*

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5\tilde{\nu}_{\ell_d} \log 2 + 5 \log \left(2 + \frac{4\tau_{\ell_d}^{\frac{\tau_{\ell_d}+1}{2}}}{3|\alpha_0|^{\bar{t}_{\ell_d}-\tau_{\ell_d}}|a_0|^{\tau_{\ell_d}d-(\bar{t}_{\ell_d}-\tau_{\ell_d})}} \right) \\ &= O(d\tau_{\ell_d}), \end{aligned}$$

and the maximal number of zeros of f_λ , with respect to $\lambda \neq 0$, is at most $\tilde{\nu}_{\ell_d}$ in \bar{D}_ρ , where

$$\rho = \frac{3}{4 \cdot 8^{\tilde{\nu}_{\ell_d}}} \frac{|\alpha_0|^{\bar{t}_{\ell_d}-\tau_{\ell_d}}|a_0|^{\tau_{\ell_d}d-(\bar{t}_{\ell_d}-\tau_{\ell_d})}}{\tau_{\ell_d}^{\frac{\tau_{\ell_d}+1}{2}}}.$$

- (2) *In the case where $\nu_1 = \nu_2$,*

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5(\nu d + (k+1)(\tau_{\ell_d} - 1)) \log 2 \\ &+ 5 \log \left(2 + \frac{4\tau_{\ell_d}^{\frac{\tau_{\ell_d}+1}{2}}}{3C_{\ell_d}|\mu|^{\bar{t}_{\ell_d}-\frac{\tau_{\ell_d}(\tau_{\ell_d}+1)}{2}}|a_0|^{\tau_{\ell_d}d-\frac{\tau_{\ell_d}(\tau_{\ell_d}-1)}{2}}|\bar{\delta}_k|^{\frac{\tau_{\ell_d}(\tau_{\ell_d}-1)}{2}}} \right) \\ &= O(d\tau_{\ell_d}), \end{aligned}$$

and the maximal number of zeros of f_λ , with respect to $\lambda \neq 0$, is at most νd in \bar{D}_ρ , where

$$\rho = \frac{3}{4 \cdot 8^{\nu d + (k+1)(\tau_{\ell_d}-1)}} \min \left\{ \frac{2}{3}, \frac{C_{\ell_d}|\mu|^{\bar{t}_{\ell_d}-\frac{\tau_{\ell_d}(\tau_{\ell_d}+1)}{2}}|a_0|^{\tau_{\ell_d}d-\frac{\tau_{\ell_d}(\tau_{\ell_d}-1)}{2}}|\bar{\delta}_k|^{\frac{\tau_{\ell_d}(\tau_{\ell_d}-1)}{2}}}{\tau_{\ell_d}^{\frac{\tau_{\ell_d}+1}{2}}} \right\}.$$

Remark 5.18. The case where P_1 and P_2 are proportional has no practical interest, since in this situation (P_1, P_2) parametrizes a line throughout the origin. Nevertheless, by sake of exhaustiveness, we give here the form of the

bound obtained when $\ell_d \geq 1$:

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5 \log 2(\nu d + 1) + \frac{5}{2}(\ell_d + 2) \log(\ell_d + 1) \\ &\quad + 5(\bar{\ell}'_1 - \ell_d - 1) \log \left(\frac{1}{|\mu|} \right) + 5\bar{n} \log \left(\frac{1}{|a_0|} \right). \end{aligned}$$

and the maximal number of zeros of f_λ , with respect to $\lambda \notin L_b$, is at most νd in \bar{D}_ρ , where

$$\rho = \frac{3}{4 \cdot 8^{\nu d}} \frac{|\mu|^{\bar{\ell}'_1 - (\ell_d + 1)} |a_0|^{\bar{n}}}{(\ell_d + 1)^{\frac{\ell_d + 2}{2}}}.$$

Indeed, in this case by Proposition 5.11 we have $\sigma = \ell_d + 1$, by (38) we have $b = \nu d$, and by (41) $\delta \geq |\mu|^{\bar{\ell}'_1 - (\ell_d + 1)} |a_0|^{\bar{n}}$. Since $|a_0| \leq 1$, $|\mu a_0| = |\alpha_0| \leq 1$, and $\bar{n} \geq \bar{\ell}'_1 - (\ell_d + 1)$, we have $\delta \leq 1$, and thus

$$\frac{(\ell_d + 1)^{\frac{\ell_d + 2}{2}}}{|\mu|^{\bar{\ell}'_1 - (\ell_d + 1)} |a_0|^{\bar{n}}} \geq 2\sqrt{2}.$$

Again, applying Corollary 5.9, the same computation as in the case where $\nu_1 \neq \nu_2$ gives the bounds for $\mathcal{Z}(f_\lambda)$ and ρ .

Of course, we can again make these bounds more accurate under the condition (\mathcal{L}_1) using (44).

Remark 5.19. Let $P = (P_1, P_2)$ be given, let us fix a lacunarity diagram (see Definition 4.1) $(d, \ell_d, n_0, \dots, n_{\ell_d}, \tau_0, \dots, \tau_{\ell_d}, t_{0,1}, \dots, t_{\ell_d, \tau_{\ell_d}})$, and let us choose a particular polynomial Q_{λ_0} with this diagram. The polynomial $f_{\lambda_0} = Q_{\lambda_0}(P_1, P_2)$ being of degree dD , it has dD roots in a disc $\bar{D}_R \subset \mathbb{C}$, for some $R > 0$. Following Remark 5.8, the polynomial mapping

$$\widehat{P}(z) = P(4Rz) / \max_{z \in \bar{D}_1, i=1,2} |P_i(4Rz)|$$

is bounded by 1 on \bar{D}_1 , and has dD zeros in $\bar{D}_{\frac{1}{4}}$. It follows that the bounds given by Theorem 5.15 for $\mathcal{Z}(f_\lambda)$, relatively to the particular polynomial \widehat{P} and relative to the particular degree d , are necessarily bigger than the largest expected bound dD , and consequently, for the particular polynomial \widehat{P} and the particular degree d , have no interest. This transformation of P into \widehat{P} shows that in order to obtain an interesting bound for $\mathcal{Z}(f_\lambda)$, uniform in the coefficients λ of Q_λ , one cannot avoid a dependency on some coefficients of P_1 and P_2 .

We stress here the fact that our bounds are not only uniform with respect to the coefficients of the lacunary polynomial Q_λ , and explicitly depend on its lacunarity diagram, but also depend on $|\alpha_0|$ and $|a_0|$, quantities which are modified when one replaces P by \widehat{P} : considering P with $|a_0|$ or $|\alpha_0|$ too small gives too large bounds (see for instance Example 5.20 below).

The interest of the bounds provided by Theorem 5.15 lies in particular, for a fixed choice of P , in the asymptotics they provide with respect to

lacunarity diagrams, and in particular when d goes to ∞ . We give below such instance of interesting asymptotic bounds.

Example 5.20. Let us fix $P_1, P_2 \in \mathbb{C}[X]$, of degree D_1 and D_2 (with $D = \max(D_1, D_2)$) and of multiplicity at least $\nu \geq 1$ on zero. We then choose a sequence (parametrized by $d \in \mathbb{N}$) of lacunarity diagrams

$$(\mathcal{D}_d)_{d \in \mathbb{N}} = \left(d, \ell_d, n_0, \dots, n_{\ell_d} = d, \tau_0, \dots, \tau_{\ell_d}, t_{0,1}, \dots, t_{\ell_d, \tau_{\ell_d}} \right)_{d \in \mathbb{N}},$$

in the following way.

- First of all we choose the sequence of degrees $n_0, n_1, \dots, n_{\ell_d} = d$ so as to satisfy our simplest lacunarity condition (\mathcal{L}_1)

$$n_0 = 0, n_1 = 1, \dots, n_{\ell} = (D+1)^{\ell-1}, \dots, n_{\ell_d} = (D+1)^{\ell_d-1} = d.$$

$$\text{In particular } \ell_d = 1 + \frac{\log(d)}{\log(D+1)}.$$

- Then, we assume that τ_{ℓ_d} is bounded by an integer τ independent of d .

It follows, by Theorem 5.17 (1), in case $\nu_1 \neq \nu_2$, that

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5 \log 2 + 5d \max(\nu_1, \nu_2) \log 2 + \frac{5}{2} (\tau + 1) \log \tau \\ &\quad + 5 \left(\tau d - \frac{\tau(\tau-1)}{2} \right) \log \left(\frac{1}{|\alpha_0|} \right) + 5 \left(\tau d - \frac{\tau(\tau-1)}{2} \right) \log \left(\frac{1}{|a_0|} \right), \\ (45) \quad &\underset{d \rightarrow +\infty}{\sim} 5d \left(\max(\nu_1, \nu_2) \log 2 + \tau \log \left(\frac{1}{|\alpha_0 a_0|} \right) \right). \end{aligned}$$

As stated in Remark 5.16, we observe in this example the symmetric role of P_1 and P_2 .

Still by Theorem 5.17 (1), in case $\nu_1 \neq \nu_2$, the maximal number of zeros of f_λ is at most $d \max(\nu_1, \nu_2)$ in the disc \bar{D}_ρ , with

$$\rho \geq \frac{3}{4 \cdot 8^{d \max(\nu_1, \nu_2)}} \frac{|\alpha_0 a_0|^{\tau d - \frac{\tau(\tau-1)}{2}}}{\tau^{\frac{\tau+1}{2}}}.$$

Noticing that $C_{\ell_d} \geq 1$, in case $\nu_1 = \nu_2 = \nu$, it follows from Theorem 5.17 (2), that our bound on $\mathcal{Z}(f_\lambda)$ has the same asymptotics, as $d \rightarrow +\infty$, as in the case $\nu_1 \neq \nu_2$, namely

$$(46) \quad 5d \left(\nu \log 2 + \tau \log \left(\frac{1}{|\alpha_0 a_0|} \right) \right).$$

Remark 5.21. The asymptotics (45) and (46) for our bound on $\mathcal{Z}(f_\lambda)$, given in Example 5.20, are better than the maximal number of zeros dD , only in case

$$\tau \log \left(\frac{1}{|\alpha_0 a_0|} \right) < \frac{D}{5} - \max(\nu_1, \nu_2) \log 2.$$

As already noticed in Remark 5.19, in the situation where $|\alpha_0 a_0|$ is too small, those bounds have no interest; They depend on a specific choice of (P_1, P_2) .

Remark 5.22. The classical Jensen formula estimating from above the number of zeros on $\bar{D}(0, 1/4)$, for an analytic function f with initial Taylor monomial $w_0 z^k$, gives the bound

$$k \left(2 + \frac{\log(1/|w_0|)}{\log 4} \right) + \frac{1}{\log(4)} \log \left(\frac{\sup_{z \in \partial \bar{D}(0,1)} |f(z)|}{|w_0|} \right).$$

Applied to the family f_λ , from this estimate we cannot deduce a uniform bound with respect to λ , since, in this case, w_0 is a linear form in λ .

6. THE CASE OF RATIONAL CURVES

We show in this last section how to reduce the case of rational parametrizations, the only situation we considered so far, to the case of polynomial parametrizations, in order to also get a version of Theorem 5.15 in the general case of rational planar curves.

For this, let us consider now the general case of the intersection of the algebraic curve $Q = 0$ of \mathbb{C}^2 and of a rational curve is $(P_1/V, P_2/V)$, where P_1, P_2, V are in $\mathbb{C}[X]$, X divides P_1 and P_2 , $\gcd(P_1, V) = \gcd(P_2, V) = 1$, and P_1, P_2 are not proportional. In particular, V has a nonzero constant term, denoted β_0 .

For bounding the number of zeros of $Q(P_1/V, P_2/V)$ following Section 4, we are reduced to the case where the rational curve is defined on \bar{D}_1 and both functions P_1/V and P_2/V are bounded by 1 on \bar{D}_1 (see Remark 5.8). By Cauchy's estimate, $|a_0/\beta_0| \leq 1$ and $|\alpha_0/\beta_0| \leq 1$.

As above we set $f_\lambda(z) = Q_\lambda(P_1(z), P_2(z))$. Then the Taylor series at the origin of $g_\lambda(z) = Q_\lambda((P_1/V, P_2/V)(z))$ has for coefficients linear forms $v_k, k \in \mathbb{N}$, in the parameters $\lambda_1, \dots, \lambda_m$, and we write

$$g_\lambda(z) = \sum_{k \geq 0} v_k(\lambda) z^k, \quad \text{where } v_k(\lambda) = \sum_{i=1}^m c_k^i \lambda_i.$$

With the notation of the previous sections, for $\ell \in \llbracket 1, \ell_d \rrbracket$, let

$$P_1^I P_2^{n_\ell - I}(z) = \sum_{k \geq 0} r_k z^k \quad \text{and} \quad 1/V^{n_\ell}(z) = \sum_{k \geq 0} s_k z^k$$

the Taylor series at the origin of $P_1^I P_2^{n_\ell - I}$ and $1/V$. In particular $s_0 = 1/(\beta_0)^{n_\ell}$. We then have

$$(47) \quad \begin{aligned} \frac{P_1^I P_2^{n_\ell - I}}{V^{n_\ell}}(z) &= \sum_{k \geq 0} \sum_{j=0}^k s_j r_{k-j} z^k \\ &= \sum_{j \geq 0} s_j \sum_{k \geq 0} r_k z^{k+j}. \end{aligned}$$

Still with the notation of the beginning of Section 5, let us consider the blocks $M(f_\lambda) = M_{n_\ell, T_\ell}(f_\lambda)$ and $M(g_\lambda) = M_{n_\ell, T_\ell}(g_\lambda)$ of the Bautin matrix of f_λ and g_λ respectively. The number of rows of $M(g_\lambda)$ is potentially infinite, and we only consider the $b(g_\lambda)$ first rows of $M(g_\lambda)$. By (47), $M(g_\lambda)$ is a linear combination of several shifts of $M(f_\lambda)$. More precisely, by denoting by \widetilde{M}_j the j -th shift of $M(f_\lambda)$, obtained by adding j rows of zeros above $M(f_\lambda)$, we have that $M(g_\lambda) = \sum_{j \geq 0} s_j \widetilde{M}_j$. In particular, if we consider the triangular matrix

$$G_{b, n_\ell, T_\ell} = \begin{pmatrix} s_0 & 0 & \cdots & 0 \\ s_1 & s_0 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ s_b & s_{b-1} & \cdots & s_0 \end{pmatrix},$$

we get that $M(g_\lambda) = G_{b, n_\ell, T_\ell} \cdot \widetilde{M}(f_\lambda)$ where $\widetilde{M}(f_\lambda)$ is the matrix $M(f_\lambda)$ truncated or completed with sufficiently many rows of zeros. As s_0 is nonzero, the matrix G_{b, n_ℓ, T_ℓ} is invertible. Hence, the parameters b and σ are not altered by this multiplication and the minor δ is multiplied by $s_0^{T_\ell}$.

Therefore, the matrix obtained by multiplying the full Bautin matrix $M_{T_0, \dots, T_{\ell_d}}(g_\lambda)$, of the family g_λ , on the left by the matrix

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ G_{b, n_0, T_0} & & & \vdots \\ & G_{b, n_1, T_1} & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & G_{b, n_\ell, T_\ell} \\ & & & 0 \end{pmatrix},$$

where there are exactly $\bar{\nu}_\ell$ rows of zeros above the block G_{b, n_ℓ, T_ℓ} , have the same pertinent blocks, from the rank point of view, as $M_{T_0, \dots, T_{\ell_d}}(f_\lambda)$. We recall that $\bar{\nu}_\ell$ was defined in section 5 and represents the number of rows of zeros above the block M_{n_ℓ, T_ℓ} .

Consequently, under conditions (\mathcal{L}_1) , (\mathcal{L}_{2a}) , and (\mathcal{L}_{2b}) , Propositions 5.11 (value of σ) and 5.12 (value of b) still stand. Concerning Proposition 5.14, we now have that for $\nu_1 \neq \nu_2$

$$\delta \geq \left| \frac{\alpha_0}{\beta_0} \right|^{\bar{t}-m} \left| \frac{a_0}{\beta_0} \right|^{\overline{\tau n} - (\bar{t}-m)},$$

and for $\nu_1 = \nu_2$

$$\delta \geq \bar{C} |\mu|^{\bar{t}-\bar{\tau}} \left| \frac{a_0}{\beta_0} \right|^{\overline{\tau n} - \bar{\tau}'} \left| \frac{\bar{\delta}_k}{\beta_0} \right|^{\bar{\tau}'}$$

Similarly to Theorem 5.15, using $|a_0/\beta_0| \leq 1$ and $|\alpha_0/\beta_0| \leq 1$, we obtain that under conditions (\mathcal{L}_1) , (\mathcal{L}_{2a}) , and (\mathcal{L}_{2b}) , for $m \geq 2$, the maximal number of zeros of g_λ in $D_{1/4}$ is bounded

- in case $\nu_1 \neq \nu_2$, by

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5(\tilde{\nu}_{\ell_d} + 1) \log 2 + \frac{5}{2}(m + 1) \log m \\ &\quad + 5(\bar{t} - m) \log \left(\frac{|\beta_0|}{|\alpha_0|} \right) + 5(\overline{\tau n} - (\bar{t} - m)) \log \left(\frac{|\beta_0|}{|a_0|} \right), \end{aligned}$$

- and in case $\nu_1 = \nu_2$, by

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5(\nu d + (k + 1)\tau_{\ell_d} - k) \log 2 \\ &\quad + 5 \log \left(2 + \frac{4m^{\frac{m+1}{2}}}{3\bar{C}|\mu|^{\bar{t}-\bar{\tau}}|a_0/\beta_0|^{\overline{\tau n}-\bar{\tau}' }|\bar{\delta}_k/\beta_0|^{\bar{\tau}'}} \right). \end{aligned}$$

Finally, the bound can be again simplified under condition \mathcal{L}_1 when $\tau_{\ell_d} \geq 2$

$$\begin{aligned} \mathcal{Z}(f_\lambda) &\leq 5(\tilde{\nu}_{\ell_d} + 1) \log 2 + \frac{5}{2}(\tau_{\ell_d} + 1) \log(\tau_{\ell_d}) \\ &\quad + 5(\bar{t}_{\ell_d} - \tau_{\ell_d}) \log \left(\frac{|\beta_0|}{|\alpha_0|} \right) + 5(\tau_{\ell_d} d - (\bar{t}_{\ell_d} - \tau_{\ell_d})) \log \left(\frac{|\beta_0|}{|a_0|} \right). \end{aligned}$$

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