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Line Ride-Sharing as a bi-sided mobility service with price schedule, transactional protocol and waiting policy: a logit traffic assignment model and its equilibrium

Abstract

A line ride-sharing service is supplied along a given roadway path by an operator that matches Users (riders) and Agents (drivers), under specific protocol that involves price schedule on both the U and A sides, waiting policy on either side and transaction times. The resulting time and money items add up over trip legs of Run / Stop / Transaction / Wait, yielding trip time and money cost depending on the service role, A or U, compared to Non-commitment, called role N for Neutral. The article brings about a traffic model of people involvement in the service. Service conditions of frequency φ and average number of users per car run ω are key factors of the time and money features of the alternative roles A, U and N. Individual choice of role is modeled as a rational behavior of maximizing the individual utility at the trip level. Aggregation over trip-makers according to a multinomial logit discrete choice model yields the respective role flows (y_A, y_U, y_N) , which in turn determine the macroscopic factors (φ, ω) . Traffic equilibrium is defined as a balance condition between the “supplied flows” and the “demanded flows” of the three roles. A computational scheme is provided, with graphical interpretation in the (ω, φ) plane, leading to properties of existence and uniqueness of equilibrium. A numerical experiment is conducted, exhibiting the influence of the demand volume on the model outcomes.

Keywords:

Ride-sharing service; bi-sided platform; traffic equilibrium; multi-sided equilibrium; equilibration algorithm

Highlights

H1/ Service featured out in terms of run frequency and average car load

H2/ Microeconomic behavior at the individual level to select a role concerning the service

H3/ Traffic equilibrium as balance conditions between supplied- and demanded- role flows

H4/ Equilibrium computation by graphical method

H5/ Specific domains of equilibrium states according to waiting policy

1/ Introduction

Background

Line ride-sharing (LRS) is a specific kind of platform-based RS service in which the rides take place along a specific roadway path, thus called the line or more specifically the line link. Rider access to cars may be restricted to special stop points called “stations” (cf. Ecov’s “Line” kind of service¹) or allowed at any point along the line path (cf. Ecov’s “Line+”).

Then, along the roadway path the car trip-makers fall into either one out of three classes: Service users as Riders (type U), car drivers involved in the service as Agents (type A) and other trip-makers

¹ Faghel & Ray, 2024

that do not take part to the service, called Neutral (type N). The three types can be seen as specific travel modes, say sub-modes of the car mode. By time period and flow direction, the respective trip flows of the three roles, denoted as y_r for each role $r \in \mathfrak{R} \equiv \{U, A, N\}$, determine both the service quantity and quality of the RS line:

- y_A amounts to the service frequency φ to potential users – a key component of service quality to them.
- Ratio $\omega = y_U/y_A$ is the expected passenger load by service-affiliated car – a basic indicator of money incomes to agents based on the principle of trip cost sharing.
- y_U and y_A are the basic quantities of service production, leading to its market share within the car mode, $(y_A + y_U)/q$ with $q = \sum_{r \in \mathfrak{R}} y_r$, and the overall occupancy rate in persons per car, $q/(y_A + y_N)$.

Service participation either on the User or Agent side depends on the attractiveness of the respective modal option to car trip-makers.

The modal specific money and time expenses per trip constitute option attributes in the modal decisions of the individual trip-makers. Money costs m_r are lower for service participants (A/U) than for non-participants (N). In contrast, times t_r are higher for service participants A and U than for N because of transaction operations, car dwelling and waiting on one side.

Research questions

This article addresses the following research questions:

RQ1/ Given the service conditions in terms of price schedule and usage protocol (including waiting conditions), what are the respective money and time attributes of the three modes?

RQ2/ What are the influences of service factors φ and ω onto the respective utility of modal options A, U and N?

RQ3/ Given the modal attributes and the (q, ε_r) pair of trip-maker flow and random variables in the modal utilities, what are the modal flows $[y_r: r \in \mathfrak{R}]$ and the related service performances (φ, ω) ?

RQ4/ Taking the determination of modal conditions from the (φ, ω) factors as a supply function, and that of trip flows from the modal conditions as a demand function, what is their joint outcome, is there a traffic equilibrium between supply and demand?

Article objectives

The article is aimed to devise a multinomial logit (MLN) discrete choice model of traffic assignment to a line ride-sharing service on a roadway link.

Under theoretical form, not only does the model shed light on service issues such as price schedule, waiting policies either on the User or Agent side and transactional protocol, but also it yields modal flows as analytical formulas of the various parameters and system state variables. It enables us to characterize the system state in the form of a fixed-point problem (FPP).

Two companion objectives are also addressed: (i) a specific, graphical approach is devised to compute equilibrium states, and (ii) a mock case study is conducted to provide a first understanding of potential situations.

Approach

The mode combines two sub-models of supply formation and demand formation. On the supply side, given the option flows, the time and money features of the options are derived from elementary properties of Poisson stochastic flows. On the demand side, given the option characteristics in money and leg times, the modal utility functions are constituted, yielding in turn the option modal shares in

the usual way of the MLN. Given the overall trip volume, or an elastic demand function relating that volume to the expected maximum utility, the demand sub-model yields the option flows.

Then, combining the two sub-models gives up a fixed-point problem (FPP) in the option flows, or equivalently an FPP in the service attributes $(\varphi, \omega, t_U^R, t_N^R)$ (denoting as t_U^R and t_N^R the link run time of service users or non-committed trip-makers, respectively). The FPP is shown to be equivalent to a simpler one with respect to ω only, yielding theoretical properties of equilibrium existence and uniqueness.

Article structure

The rest of the article is in five parts. Section Two represents the system of Roadway path, Line RS service and the population of car trip-makers. Section Three provides the traffic assignment model and studies its equilibrium state as a multi-sided traffic equilibrium, considering the two sides of Service Agents and Users and also the Neutral trips as a third tier. Section Four is devoted to a numerical experiment (mock case study). Section Five discusses the model outreach and limitations and also some issues of system design. Section Six concludes and points out to some directions of further research.

Tab.1: Notation.

<i>Service features</i>	<i>Territorial setting and link traffic</i>
ν number of riders per agent car run	L link length
ρ proportion of agent runs with riders	H period duration
ω average rider load per agent car run	T_r^{FP} run time function of role $r \in \{U, N\}$ under FP
φ service frequency	X_r^{FP} car flow rate on link of role $r \in \{U, N\}$
$r, s \in \mathfrak{R} \equiv \{A, U, N\}$ service role	<i>Mobility and behaviors</i>
$\ell \in \mathcal{L} \equiv \{R, S, T, W\}$ trip leg	q population size
δ_r^W indicator variable of Wait assignment to r	y_r person flow of role r
$\{IF, JF\}$ Isolated / Joint Flow Policy	x_r car trip flow of role r
<i>Time and money items</i>	β_r^ℓ Value-of-Time (VoT) by role r and leg ℓ
t_r trip time of role r , time t_r^ℓ by leg ℓ	g_r generalized cost of r to trip-maker
γ_r modal constant of role r	V_r deterministic utility of r
\underline{m} base car cost per trip	ε_r random perturbation of utility of role r
\dot{m} length factor of car cost	U_r stochastic utility of r
$\underline{\tau}_r$ base trip award ($r = A$) or fare ($r = U$)	θ concentration parameter of ε_r
$\hat{\tau}_r$ length factor of award ($r = A$) or fare ($r = U$)	p_r population proportion assigned to role r
$\hat{\tau}_r$ average trip award ($r = A$) or fare ($r = U$)	$V_{\mathfrak{R}}$ expected maximum utility
m_r money cost of role r	D demand function

2/ System representation

2.1/ Roadway path and traffic scene

A roadway link of network length denoted L , is considered in a given flow direction as a specific “traffic scene”. It is studied on a daily basis, by within-day time period. A specific period of interest is considered, e.g. morning peak or evening peak or in-between, with time duration denoted H .

2.2/ Service roles as travel modes

With respect to a line ride-sharing service on the roadway link, three roles are identified for car trip-makers: service user as Rider (role U), service agent as driver (role A) or non-participant i.e. neutral (role N). The set of roles is denoted $\mathfrak{R} \equiv \{U, A, N\}$.

On a per trip basis, the respective money expenses are denoted as m_r , and time expenses as t_r . These split into “leg times” t_r^ℓ according to trip phases or legs, $\ell \in \mathcal{L} \equiv \{R, S, T, W\}$:

- Leg R of car running on the link,
- Leg S of Stop or Dwelling for rider boarding and alighting, for Users and for those Agents that get “customers”,
- Leg T of Transaction operations: getting information and possibly being assigned, Paying on the User side or Being paid on the Agent side, both using the service digital platform,
- Leg W of Waiting: either the User is required to Wait for the next Agent under the UW policy, or the Agent is asked to wait for an incoming user under the AW policy.

2.3/ Service features and policy

Ride-sharing is a bi-sided form of transport service as it involves people in two different ways: not only the service-to-demand form of Users, but also the cooperator-to-service form of Agents. These cooperate with the service coordination (platform) by supplying seats in their vehicles, the driving function including dwelling for rider boarding and alighting, possibly waiting for incoming riders to get to the car and other assistance (e.g. putting a foldable two-wheeler in the car trunk).

Service coordination between the two sides, Agents and Users, relies upon a digital interaction platform that achieves user and agent matching (in other words, the assignment of riders to cars and that of cars to riders) by suitable information collection and delivery, as well as fare collection from users and money compensation to agents.

Transaction times. These transactional operations are assumedly performed efficiently owing to high level of automation and suitable platform customer interface, with total time per trip of t_A^T and t_U^T on the agent and user side respectively.

Stop times. Also given is the stop time per user trip, t_U^S on the user side, giving rise to a base stop time of t_A^S on the agent side if the agent car run is endowed with rides. Otherwise, the agent experiences no stop time.

The following Proposition is demonstrated in Appendix A on postulating that both flows of Users and Agents are independent Poisson processes with respective time rates $\psi_r \equiv y_r/H$.

Proposition 1: Car occupation and time items according to waiting policy.

1/ Car occupation: (i) The number of users per agent car run, a random variable denoted v , has a geometric distribution with parameter $\rho \equiv \frac{\psi_U}{\psi_{U+A}}$, and mean value $\frac{\rho}{1-\rho}$ denoted ω , so that $\rho = \frac{\omega}{1+\omega}$.

(ii) The respective probabilities of getting users or not are $\Pr\{v > 0\} = \rho$ and $\Pr\{v = 0\} = 1 - \rho$.

2/ Under User Waits policy (UW): (i) the wait times of agents are negligible: $t_A^W = 0$. (ii) The user wait times are distributed $\text{EXP}(\psi_A)$, i.e., exponentially distributed with parameter ψ_A , so that $t_U^W = H/\varphi$.

3/ Under Agent Waits policy (AW): (i) the wait times of users are negligible: $t_U^W = 0$. (ii) The wait times of agents make a random variable (RV) that is distributed as the probability mixture of 0 and $\text{EXP}(\psi_A)$, with respective coefficients $1 - \rho$ and ρ , yielding average agent wait time of $t_A^W = \rho H/\varphi$.

We thus have that

$$\varphi \equiv y_A, \quad (1a)$$

$$\omega \equiv E[v] = y_U/y_A, \quad (1b)$$

$$\rho \equiv \Pr\{v > 0\} = \frac{\omega}{1+\omega}. \quad (1c)$$

Letting δ_A^W be the binary indicator of AW and δ_U^W that of UW, the option wait times are formulated generically as follows:

$$t_N^W = 0, \quad (2-N)$$

$$t_A^W = \delta_A^W \rho H/\varphi, \quad (2-A)$$

$$t_U^W = \delta_U^W H/\varphi. \quad (2-U)$$

The average agent stop time satisfies that $t_A^S = \Pr\{v > 0\} \cdot \underline{t}_A^S + \Pr\{v = 0\} \cdot 0$, hence

$$t_A^S = \rho \underline{t}_A^S. \quad (3)$$

Link run times are denoted t_U^R for Users or t_N^R for role N to enable for possible distinction, using e.g. dedicated lanes for multi-occupied cars. An agent will get the same run time as a user if there is at least one user in the car, or the same run time as role N otherwise. Thus, on average,

$$t_A^R = \Pr\{v > 0\} \cdot t_U^R + \Pr\{v = 0\} \cdot t_N^R = \rho t_U^R + (1 - \rho)t_N^R. \quad (4)$$

Table 2 summarizes the average leg times by trip leg for each mode.

Tab. 2: Average trip leg time according to role.

Leg ℓ	Agent	User	Neutral
Run R	$\rho t_U^R + (1 - \rho)t_N^R$	t_U^R	t_N^R
Stop S	$\rho \underline{t}_A^S$	t_U^S	0
Transaction T	t_A^T	t_U^T	0
Wait W	$\delta_A^W \rho H/\varphi$	$\delta_U^W H/\varphi$	0

Beside the waiting policy and the right-of-way assignment, service policy also includes (i) roadway link selection in relation to the local Mobility Organizing Authority, (ii) Access conditions of Rides to Car runs, assumed here at link endpoints only, (iii) the price schedule.

By assumption, for each car trip offered by an agent as a service run, a twofold reward (fee) is awarded to the agent: a base fee denoted \underline{t}_A applies whatever the number of riders (including modality Zero), plus a per ride contribution that depends on ride length, say $L\hat{t}_A$ with \hat{t}_A the ride award rate per unit length. On average per service run, the money reward to the agent is thus

$$\hat{t}_A = \underline{t}_A + \omega L\hat{t}_A \quad (5-A)$$

On the user side, we similarly distinguish between a fixed fare \underline{t}_U and a length-variable fare $L\hat{t}_U$: the ride fare is thus

$$\hat{t}_U = \underline{t}_U + L\hat{t}_U \quad (5-U)$$

There certainly is a relationship between \hat{t}_A and \hat{t}_U : some financial balance between $y_A \hat{t}_A$ and $y_U \hat{t}_U$ may be expected, up to external subsidizes e.g. from the local mobility organizing authority on a per run basis, to taxes and of course to platform remuneration. Such issues, of obvious interest to the service business model, are not addressed in this article.

2.4/ Modal money items, generalized costs and utility functions

The ride fare constitutes the basic money cost of the link trip to a service user:

$$M_U \equiv \hat{t}_U = \tau_U + \omega L \hat{t}_U \quad (6-U)$$

Service-neutral trip-makers are assumed to spend a fixed money cost of \underline{m} plus a length-variable cost at unit length rate of \dot{m} , yielding link cost of

$$M_N \equiv \underline{m} + L \dot{m} \quad (6-N)$$

To a service agent, the link cost amounts to car expense minus run income, yielding

$$M_A \equiv M_N - \hat{t}_A = \underline{m} + L \dot{m} - \tau_A - \omega L \hat{t}_A \quad (6-A)$$

Some trip features remain unobserved in the model, notably the sub-paths up- and down-stream the roadway link. The unobserved features of the alternative modes r are called “modal constants” measured as money costs and denoted γ_r . We shall denote $m_r \equiv M_r + \gamma_r$.

Thus, each mode as a choice option is characterized by a set of attributes: leg times, money costs, modal constant, customized to the individual attributes and circumstances. For each individual, the own preferences (e.g. Value of Time, VoT) and conditions of exposures (walk speed, money expense depending on car type...) are called behavioral parameters. By mode r and trip leg ℓ , the VoT denoted β_r^ℓ is the change rate of leg time to money. Among the different legs, only the agent run leg can vary depending on car occupation: from no riders to one or several, not only may the run time change (from t_N^R to t_U^R) but also the agent VoT, say from β_{A0}^R without riders to β_{A1}^R with riders. On average, the agent value of run time is $\beta_A^R = (1 - \rho)\beta_{A0}^R + \rho\beta_{A1}^R$. Yet, as the amount of run time may vary depending on the load state, we denote loosely

$$(\beta t)_A^R = (1 - \rho)\beta_{A0}^R t_N^R + \rho\beta_{A1}^R t_U^R \quad (7)$$

At the trip level, the modal option r induces a “travel generalized cost” of

$$g_r^\# \equiv m_r + \sum_{\ell \in \mathcal{L}} \beta_r^\ell \cdot t_r^\ell \quad (8)$$

As cost is a disutility, the related deterministic utility function is a value, denoted

$$V_r \equiv -g_r^\# = -m_r - \sum_{\ell \in \mathcal{L}} \beta_r^\ell \cdot t_r^\ell \quad (9)$$

Beyond the modal constants, there are other option attributes, individual attributes and choice conditions that are unobserved: they are modeled as random variables $[\varepsilon_r; r \in \mathfrak{R}]$ that turn the deterministic utility function V_r into a “random utility function” denoted U_r and defined as

$$U_r \equiv V_r + \varepsilon_r. \quad (10)$$

2.5/ Mode choice and option flows

Having modeled the individual preferences in the utility functions, the microeconomic principle of individual choice behavior is that, on every occurrence, the individual selects the option of maximum utility to him or her. Then, by aggregation over the random occurrences behind the random variables ε_r , the probability of choosing option r comes out as

$$p_r \equiv \Pr\{U_r \geq U_s; \forall s \in \mathfrak{R}\}. \quad (11)$$

We shall resort to the multinomial logit model, in which the “perturbation” ε_r is a random variable that follows the Gumbel distribution, and such that between the options the random family

$[\varepsilon_r: r \in \mathfrak{R}]$ are independent and identically distributed (this assumption reduces to “homoscedastic” for Gumbel variables and modal constants). Denoting as θ the concentration parameter of every ε_r , the usage frequency of option r amounts to the following mathematical probability:

$$p_r = \frac{\exp(\theta.V_r)}{\exp(\theta.V_{\mathfrak{R}})} = \exp(\theta(V_r - V_{\mathfrak{R}})) \quad (12)$$

Wherein: $V_{\mathfrak{R}} \equiv \frac{1}{\theta} \ln \sum_{s \in \mathfrak{R}} \exp(\theta.V_s)$, called the satisfaction function, is the mean value of the maximum utility, $U_{\mathfrak{R}} \equiv \max\{U_s: \forall s \in \mathfrak{R}\}$. In the logit model, the maximum utility is also a Gumbel variable with concentration parameter θ .

By assumption, each individual in the population of interest makes one and one trip only on the roadway link during the period under study. Thus the population size and the trip flow are equal and both are denoted as q . The choice model yields modal “demanded flows” $y_r^\#$ such that

$$y_r^\# = q \cdot p_r^\# \quad (13)$$

By construct, it holds that $y_r^\# \geq 0$ and $\sum_{r \in \mathfrak{R}} y_r^\# = q$.

2.6/ Elastic demand

The trip demand q may be related to the expected overall utility, $V_{\mathfrak{R}}$, through a demand function with respect to the expected overall generalized cost $g_{\mathfrak{R}} \equiv -V_{\mathfrak{R}}$:

$$q = D(g_{\mathfrak{R}}) \quad (14)$$

“Fixed demand” corresponds to a constant function D .

2.7/ Car flows and traffic issues

The modal flow y_r by travel mode $r \in \mathfrak{R}$ gives rise to a car flow x_r that is equal to y_r for $r \in \{A, N\}$ or null otherwise, i.e. $x_U = 0$. Total car flow amounts to

$$x = x_A + x_N = y_A + y_N \quad (15)$$

The organization of an LRS may be combined to a specific arrangement of car lanes on the roadway link by dedicating one lane to multi-occupied vehicles, that is, “Isolated Flow” (IF), or not, that is, “Joint Flow” (JF). As agent cars with users are multi-occupied, the run time on the dedicated lane would be t_U^R , whereas that on common lanes is just t_N^R : hence the indices for travel time functions. The car flow rate eligible to lane dedication is

$$X_U^{IF} \equiv y_A \cdot \Pr\{v > 0\} / H = \rho\varphi / H, \quad (15-U)$$

While that restricted to common lanes is

$$X_N^{IF} \equiv (y_N + y_A \cdot \Pr\{v = 0\}) / H = (y_N + (1 - \rho)\varphi) / H. \quad (15-N)$$

We model the influence of a car trip flow by flow policy $FP \in \{IF, JF\}$ as a travel time function T_r^{FP} with respect to the related flow time rate X_r^{FP} . Under Isolated Flow, then

$$t_U^R = T_U^{IF}(X_U^{IF}) \quad (16-I,U)$$

$$t_N^R = T_N^{IF}(X_N^{IF}) \quad (16-I,N)$$

While under Joint Flow, then $T_U^{JF} = T_N^{JF}$ and $X_U^{JF} = X_N^{JF} = x/H$, so that

$$t_U^R = t_N^R = T_U^{JF}(X_U^{JF}) = T_N^{JF}(X_N^{JF}) \quad (16-J)$$

Both flow volumes and run times are traffic conditions determining the local emissions of noise and of air pollutants, including GHG emissions of global outreach.

3/ Traffic equilibrium

3.1/ Supply sub-model

We select the vector $\mathbf{y}_{\mathfrak{R}} \equiv [y_r : r \in \mathfrak{R}]$ of modal flow volumes as “basic” subset of state variables from which all of the other exogenous variables are derived.

Vector $\mathbf{y}_{\mathfrak{R}}$ induces first the service conditions (φ, ω, ρ) in eqn. (1), then the leg time items $\mathbf{t}_{\mathfrak{R}}^{\mathcal{L}} \equiv [t_r^{\mathcal{L}} : \mathcal{L} \in \mathcal{L}, r \in \mathfrak{R}]$ in (2,3,4,15,16), next the service fee and fare $\hat{\mathbf{t}}_{\mathcal{S}} \equiv [\hat{t}_r : r \in \mathcal{S}]$ in (5) and the money items $\mathbf{m}_{\mathfrak{R}} \equiv [m_r : r \in \mathfrak{R}]$ in (6).

The supply sub-model is defined as the derivation of ρ , $\mathbf{t}_{\mathfrak{R}}^{\mathcal{L}}$ and $\mathbf{m}_{\mathfrak{R}}$ from vector $\mathbf{y}_{\mathfrak{R}}$. It is denoted as

$$(\rho, \mathbf{t}_{\mathfrak{R}}^{\mathcal{L}}, \mathbf{m}_{\mathfrak{R}}) = \mathbf{F}_{\mathcal{S}}(\mathbf{y}_{\mathfrak{R}}) \quad (17)$$

Only non-negative option flows are considered, making the domain $\Delta_{\mathbf{y}} \equiv \{\mathbf{y}_{\mathfrak{R}} \geq 0\}$. We also consider the sub-domain $\Delta_{\mathbf{y}}^* \equiv \{\mathbf{y}_{\mathfrak{R}} \geq 0 : y_A > 0\}$ where the supply function is continuous. On the contrary, if $y_A = 0$ then ω is indeterminate and so is ρ ; furthermore, $\varphi = 0$ leads to infinite wait times to users under UW. Under AW, if $\rho > 0$ then $\varphi = 0$ also entails infinite wait times to agents. We therefore extend the definition of the supply function on points such that $y_A = 0$ in the following way: $t_U^W = t_A^W = +\infty$, under both waiting policies AW and UW.

3.2/ Demand sub-model

The “demand sub-model” is defined as the derivation of “demanded flows” $\mathbf{y}_{\mathfrak{R}}^{\#} \equiv [y_r^{\#} : r \in \mathfrak{R}]$ from the vector $(\rho, \mathbf{t}_{\mathfrak{R}}^{\mathcal{L}}, \mathbf{m}_{\mathfrak{R}})$ through the formation of generalized costs in (7,8) and deterministic utilities in (9), expected maximum utility and demand volume in (14), option modal shares in (12), and finally demanded flows in (14). It is denoted as

$$\mathbf{y}_{\mathfrak{R}}^{\#} = \mathbf{F}_{\mathcal{D}}(\rho, \mathbf{t}_{\mathfrak{R}}^{\mathcal{L}}, \mathbf{m}_{\mathfrak{R}}) \quad (18)$$

The feasible domain of ρ is $\Delta_{\rho} \equiv [0,1]$ and that of $(\mathbf{t}_{\mathfrak{R}}^{\mathcal{L}}, \mathbf{m}_{\mathfrak{R}})$ is $\Delta_{\mathbf{t},\mathbf{m}} \equiv \mathbb{R}^{\mathcal{L} \times \mathfrak{R} + \mathfrak{R}}$. The composition of generalized cost is a linear combination, save for the multiplicative influence of ρ in g_A : these are well-defined and continuous functions on the joint domain $\Delta_{\rho} \times \Delta_{\mathbf{t},\mathbf{m}}$. By the properties of the exponential function, $V_{\mathfrak{R}}$ and the $p_r^{\#}$ are well-defined and continuous functions of $(\rho, \mathbf{t}_{\mathfrak{R}}^{\mathcal{L}}, \mathbf{m}_{\mathfrak{R}})$. Postulating a continuous demand function D , so are q and the demand flow functions $y_r^{\#}$.

Fig. 1 exhibits both the supply sub-model and the demand sub-model.

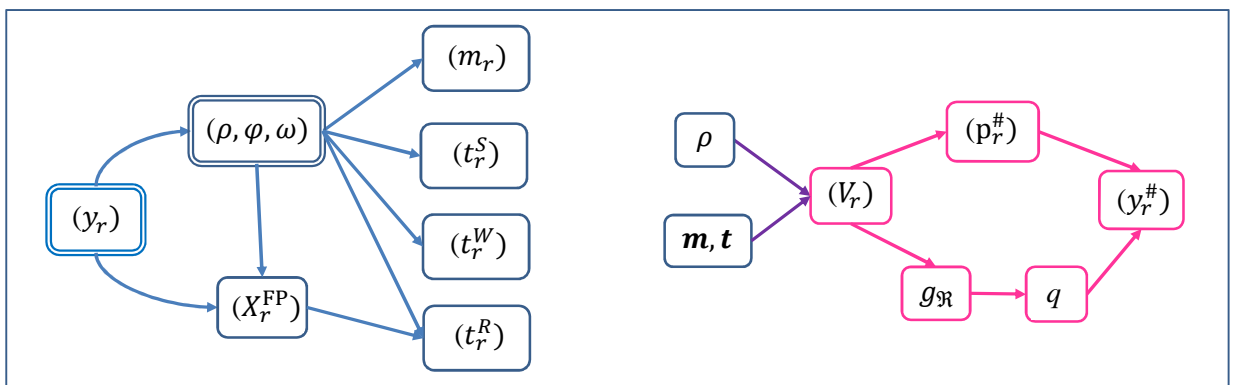


Fig. 1: (A) Supply sub-model, (B) Demand sub-model.

3.3/ Supply-demand equilibrium

Definition. The system is in equilibrium if the system state $(\mathbf{y}_{\mathfrak{R}}, \rho, \mathbf{t}_{\mathfrak{R}}^{\ell}, \mathbf{m}_{\mathfrak{R}})$ satisfies jointly that

$$\begin{aligned} (\rho, \mathbf{t}_{\mathfrak{R}}^{\ell}, \mathbf{m}_{\mathfrak{R}}) &= \mathbf{F}_{\mathfrak{S}}(\mathbf{y}_{\mathfrak{R}}), \\ \mathbf{y}_{\mathfrak{R}} &= \mathbf{F}_{\mathfrak{D}}(\rho, \mathbf{t}_{\mathfrak{R}}^{\ell}, \mathbf{m}_{\mathfrak{R}}). \end{aligned}$$

Proposition 2: Equilibrium as a fixed-point problem. Vector $\mathbf{y}_{\mathfrak{R}}$ is a supply-demand equilibrium iff it satisfies the fixed-point problem associated to the mapping $\mathbf{F}_{\mathfrak{D}} \circ \mathbf{F}_{\mathfrak{S}}$, i.e.,

$$\mathbf{y}_{\mathfrak{R}} = \mathbf{F}_{\mathfrak{D}} \circ \mathbf{F}_{\mathfrak{S}}(\mathbf{y}_{\mathfrak{R}}). \quad (19)$$

The conjunction of (17) and (18) at $\mathbf{y}_{\mathfrak{R}}$ implies (19) straightforwardly. Conversely, if (19) holds true then the system state $(\mathbf{y}_{\mathfrak{R}}, \mathbf{F}_{\mathfrak{S}}(\mathbf{y}_{\mathfrak{R}}))$ satisfies the equilibrium definition.

The FPP states that the “demanded flows” should match the “supplied ones”.

By restricting the feasible set to $\Delta_{\mathbf{y}}^{\eta, Q} \equiv \{\mathbf{y}_{\mathfrak{R}} \geq 0: y_A \geq \eta, y_r \leq Q\}$ with some small $\eta > 0$ and some large $Q > 0$, and defining a modified demand function $\mathbf{F}_{\mathfrak{D}}^{\eta, Q}$ such that $\mathbf{F}_{\mathfrak{D}, r}^{\eta, Q} \equiv \max\{\eta, \min\{Q, y_r^{\#}\}\}$, then the mapping $\mathbf{F}_{\mathfrak{D}}^{\eta, Q} \circ \mathbf{F}_{\mathfrak{S}}$ is continuous on $\Delta_{\mathbf{y}}^{\eta, Q}$, and we can set parameters η and Q to values such that $\mathbf{F}_{\mathfrak{D}}^{\eta, Q} \circ \mathbf{F}_{\mathfrak{S}}(\Delta_{\mathbf{y}}^{\eta, Q}) \subset \Delta_{\mathbf{y}}^{\eta, Q}$. Then, by the Brouwer theorem, there exists a solution to the fixed point problem.

Under UW and AW, null agent flow value $y_A = 0$ yields infinite user wait time t_U^W : then $g_U^{\#} = +\infty$ if $\beta_U^W > 0$, $p_U^{\#} = 0$ and $y_U^{\#} = 0$. Similarly, $t_A^W = +\infty$, $g_A^{\#} = +\infty$ if $\beta_A^W > 0$, $p_A^{\#} = 0$ and $y_A^{\#} = 0$, which is consistent with $y_A = 0$. The point $\mathbf{y}_{\mathfrak{R}} = (0, 0, D(g_N^{\#}))$ is a trivial kind of equilibrium, along with the non-trivial kind that solves the truncated program with $\mathbf{F}_{\mathfrak{D}}^{\eta, Q}$.

3.4/ Mathematical characterizations

As $p_r^{\#} = \exp(\theta V_{r-\mathfrak{R}})$, the equality between $\mathbf{y}_{\mathfrak{R}}$ and $\mathbf{F}_{\mathfrak{D}} \circ \mathbf{F}_{\mathfrak{S}}(\mathbf{y}_{\mathfrak{R}})$ can also be put as

$$y_r = \exp(\theta V_{r-\mathfrak{R}}) \cdot D(g_{\mathfrak{R}}). \quad (20)$$

It is also equivalent to

$$\forall r \in \mathfrak{R} : g_r + \frac{1}{\theta} \ln y_r = g_{\mathfrak{R}} + \frac{1}{\theta} \ln D(g_{\mathfrak{R}}). \quad (21)$$

This is in essence the Variational Inequality formulation of a logit traffic assignment model, as stated by Caroline Fisk (1980) and extended to elastic demand by Akamatsu & Matsumoto (1989). See Appendix B.

This system is equivalent to just

$$g_A + \frac{1}{\theta} \ln y_A = g_U + \frac{1}{\theta} \ln y_U, \quad (22a)$$

$$g_U + \frac{1}{\theta} \ln y_U = g_N + \frac{1}{\theta} \ln y_N, \quad (22b)$$

$$y_A + y_U + y_N = D(g_{\mathfrak{R}}). \quad (22c)$$

As $y_U/y_A = \omega$, condition (22a) becomes

$$\frac{1}{\theta} \ln \omega = g_A - g_U. \quad (23a)$$

We call it the “inner service split” (ISS) condition since it compares the two options within the service. Given the option run times and ω (hence ρ), the ISS implies φ as

$$\frac{1}{\theta} \ln \omega = m_A + (\beta t)_A^{RST} + \delta_A^W \beta_A^W \frac{\rho H}{\varphi} - m_U - (\beta t)_U^{RST} - \delta_U^W \beta_U^W \frac{H}{\varphi}$$

So that

$$\frac{H}{\varphi} (\delta_U^W \beta_U^W - \delta_A^W \beta_A^W \rho) = m_{A-U} + (\beta t)_{A-U}^{RST} - \frac{1}{\theta} \ln \omega$$

The ISS gives $\varphi = y_A$ as a function $\hat{y}_A(\omega)$ with respect to ω (and the run times if variable):

$$\hat{y}_A(\omega) = \frac{H(\delta_U^W \beta_U^W - \delta_A^W \beta_A^W \rho)}{m_{A-U} + (\beta t)_{A-U}^{RST} - \frac{1}{\theta} \ln \omega}$$

The condition (22b) makes an “outer service split” (OSS) condition since it compares the outside option N to the inside option U. Knowing ω hence $\varphi = \hat{y}_A(\omega)$ and $\hat{y}_U(\omega) \equiv \omega \cdot \hat{y}_A(\omega)$, then the OSS gives $\hat{y}_N(\omega)$ with respect to ω :

$$\hat{y}_N(\omega) = \exp(\theta g_{U-N}) \hat{y}_U(\omega). \quad (23b)$$

Substituting in (22c), we obtain a condition that

$$\hat{q}(\omega) \equiv \hat{y}_A(\omega)(1 + \omega) + \hat{y}_N(\omega) = D(g_{\mathfrak{R}}). \quad (23c)$$

It is a condition in ω only if the run times are fixed.

Proposition 3. Under fixed run times and fixed demand, then there exists a unique value ω^* solving the equilibrium characterization.

Proof: Function $\hat{y}_A(\omega)$ is monotonous with respect to ω and so is $\hat{q}(\omega)$.

3.5/ Geometric properties

The ISS condition determines the sign of a function of ω , namely $F_\omega \equiv m_{A-U} + (\beta t)_{A-U}^{RST} - \frac{1}{\theta} \ln \omega$. As φ is non-negative, under UW it holds that $F_\omega > 0$, while under AW it holds that $F_\omega < 0$. Thus, UW equilibrium states belong to $\{F_\omega > 0\}$ while their AW counterparts belong to $\{F_\omega < 0\}$. The line $\{F_\omega = 0\}$ delimits the respective sub-domains.

Under fixed run times, condition $\{F_\omega = 0\}$ amounts to $\{\omega = \omega_0\}$, with $F_\omega < 0$ for $\omega > \omega_0$ and $F_\omega > 0$ for $\omega < \omega_0$. Thus, there exists a frontier load rate ω_0 such that AW equilibrium states lie in $\{\omega > \omega_0\}$ and UW ones in $\{\omega < \omega_0\}$.

3.6/ Computational scheme

Under fixed demand volume q , the obvious way to solve the LRS traffic equilibrium is to search for ω satisfying (23c) at q . But in fact, the variations of the system state with respect to the demand volume are of primary relevance to understand the service performance.

Then an efficient utilization of (23c) is to make q vary with ω , that is, as function $\hat{q}(\omega)$, and to analyze any system performance Z_q as a curve $(\hat{q}(\omega), Z_q)$ parameterized by ω .

4/ Numerical experiment

The experiment is aimed to demonstrate the model outcomes depending on specific conditions, most notably the Waiting policy and the effect of trip flow rate low or high.

4.1/ Case setting

An interurban road link with $L = 10$ km is considered.

The ride-sharing service has the following price schedule:

- On the agent side, $\underline{t}_A = 0.5$ € and $\dot{t}_A = 0.10$ €/km,
- On the user side, $\underline{t}_U = 1$ € and $\dot{t}_U = 0.20$ €/km.

The base costs of car holding and utilization are set up to $\underline{m} = 1$ € per trip and $\dot{m} = 0.30$ €/km.

The modal constants are set up to $\gamma_N = 1$ €, $\gamma_A = 1$ € and $\gamma_U = 0$ €.

Thus, the per-trip costs amount to:

- Neutral: $M_N = 4$ € and $m_N = 5$ €,
- User: $M_U = 3$ € and $m_U = 3$ €,
- Agent: $M_A = 3.5 - \omega$ € and $m_A = 4.5 - \omega$ €.

As for time items, base stop times of $\underline{t}_A^S = 1$ min and $\underline{t}_U^S = 1$ min are assumed, together with transaction times of $\underline{t}_A^T = 1$ min and $\underline{t}_U^T = 1.5$ min.

In the population of trip-makers, homogenous VoT of 15€/h is assumed. During a period of $H = 1$ hour, contrasted flow levels are considered: “low flow” with $q = 20$ trips, versus “high flow” with $q = 200$ trips.

Link run time is fixed and homogenous among cars, under average speed of $q = 60$ km/h.

4.2/ Traffic equilibrium

By waiting policy either UW or AW, the OSS and ISS relationships in the (φ, ω) domain are depicted in Figure 2 under low flow (part A) and under high flow (part B). Intersection points between two function graphs of the same color (meaning the same policy) make states of traffic equilibrium.

At low flow under UW, about 10 individuals choose the service as agents, but only 5 as users, yielding relatively high frequency of $\varphi/H \approx 10/h$ but moderate car load of $\omega \approx 0.5$ riders per car. Taken together, the two service roles have a market share of about 3/4 in the population of trip-makers.

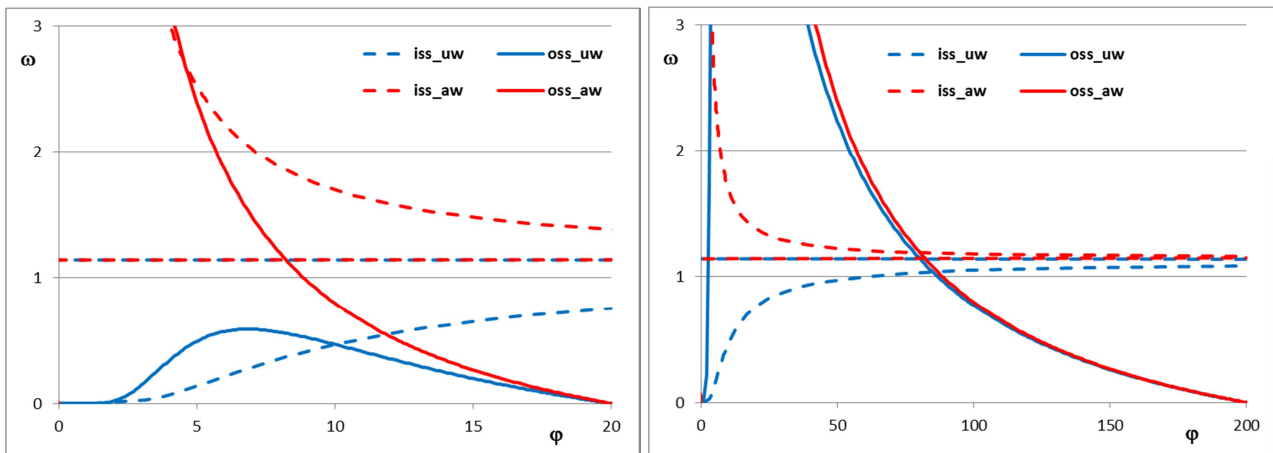


Fig.2: Equilibrium determination in the (φ, ω) domain: (A) under low flow, (B) under high flow.

Still at low flow but under AW the service is used quite differently (Table 3): about 5 individuals only choose the service as agents, but 12 as users, yielding moderate frequency of $\varphi/H \approx 5/h$ despite high car load of $\omega \approx 2.7$ riders per car. The reason is that relatively high Agent wait times counteract the money advantages.

Under high flow, both waiting policies yield relatively close results, with more than 80% of trip-makers involving themselves in the service, among them a larger half as users and a smaller half as agents. Under User waits, the gap between the numbers of agents and users is narrow. Both policies entail high frequency $\varphi/H \approx 80/h$ (very high indeed) and consistent car load of $\omega \approx 1 - 1.2$ riders per car. The money costs are greater for agents than for users, but both their costs are about half less than the money cost m_N to individuals N staying out of the service.

Tab. 3: Equilibrium states under low flow (left part) versus high flow (right part).

Indicator (unit)	LOW FLOW		HIGH FLOW	
	UW	AW	UW	AW
ω (riders/car)	0.468	2.73	1.038	1.189
$\varphi = y_A$ (trips)	10.021	4.52	85.05	80.32
y_U (trips)	4.685	12.35	88.32	95.53
y_N (trips)	5.293	3.12	26.64	24.15
m_A (€)	4.032	1.77	3.46	3.311
t_A (hour, with base run time)	0.189	0.357	0.192	0.199
t_U (hour, with base run time)	0.308	0.208	0.220	0.208
$V_R^\#$ (€)	-6.171	-5.643	-5.484	-5.386
Equilibrium involves LRS	YES	YES	YES	YES

4.3/ Cost-Benefit Analysis (COBA)

User surplus: out of speed improvement, the availability and usage of the service give rise to average utility $V_R^\#$, to be compared to initial neutral utility of $V_N = -7.5$ €/trip, yielding benefit of 1 to 2 €/trip (Table 4) depending on waiting policy and flow level. N.B. Part of this gain comes from modal constant γ_U that is assumedly lower than γ_N .

Given the flow level, policy AW yields more benefits to the trip-makers than policy UW: the per-trip difference between the respective policy gains is higher under low flow than under high flow.

Tab. 4: COBA according to Wait Policy under low flow (left part) versus high flow (right part).

Indicator (unit)	LOW FLOW		HIGH FLOW	
	UW	AW	UW	AW
Individual ΔU	1.329	1.857	2.016	2.114
People ΔU	26.58	37.14	403.2	422.8
User fares	14.06	37.05	264.96	286.59
Agent fees	9.70	14.60	130.81	135.66
Service net added value	4.35	22.45	134.15	150.93

5/ Discussion

5.1/ Lessons from the numerical experiment

The graphical determination of equilibrium constitutes a robust computational scheme. It is essentially a specific kind of fixed-point algorithm. Demand disaggregation underpinning the random utilities of the options is the source of computational robustness that is all the more useful here since the traffic effects between modal flows and modal conditions are much involved and highly intertwined, yielding asymmetry effects in addition to nonlinearities.

The numerical experiment demonstrates the different outcomes of different waiting policies. Yet, under high flow the respective equilibrium states are relatively close.

Lastly, the modal proportions varied greatly among the four instances: such variations call for truly behavioral modeling of ride-sharing services rather than making educated guesses on the respective modal proportions (e.g. Yin et al. 2018).

5.2/ About model assumptions: outreach and limitations

The temporal features of wait times and their assignment to either Users or Agents, of stop times and of transaction times are as much influential in service usage as the money items from the price schedule. Both the money items and the “time items” aside from car run times constitute key features in the “user experience” and the “customer journey” as evocated in marketing literature. Our model captures these features as well as the run times and the other physical times that are commonly modeled for passenger traffic in transit networks (dwell times, wait times, in addition to run times) and the transaction times specific to the roles.

On the demand side, the random utility functions capture not only the deterministic features in money and time but also random fluctuations around them among the population of individuals – some kind of population disaggregation. The individual behaviors, notably the trade-offs between time and money items, lie at the heart of the model, which also derives the aggregate consequences on the supply side in terms of service frequency φ and car occupancy rate ω .

The framework of random utility theory and discrete choice models enables one not only to model unobserved effects as residual random variables (and the modal constants) but also to represent comfort issues, notably VoT modulation according to service role and trip leg. Such modulation is especially important about waiting policies, since Agent wait times take place in-vehicles hence at sitting comfortably, whereas User wait times occur out of cars, possibly at standing, maybe so outdoor and subjected to meteorological circumstances.

Beyond comfort come the issues of individual mobility practices, routines and attitudes. Here, service awareness is postulated, so that parameter q standing for population size represents in fact the number of service-aware people. The prior mobility routines of individuals may also induce captivities of some sort: some car drivers have specific requirements of their private cars in their day activity programs, possibly with professional constraints. Such constraints may come with company rules about access to company cars and usage conditions, departing from our assumptions about the microeconomic situation of the trip-maker.

Yet, other features of individual travel routines may also align with our modeling assumptions. Home to work or study commuting is a recurrent purpose for the working people on a daily basis or so (at least on worked days, up to working from home). Frequent reiteration at the individual level of the travel situation is likely to even out the fluctuations between the occurrences, giving ground to considering the expected times and money outcomes of each role as statistical means over the population of occurrences. In other words, commuting frequency certainly is a factor of alignment between the real-world conditions and the model assumptions.

5.3/ Some hints about service design

As an abstract representation, the model brings about concepts, state variables and notations to describe a real-world system, to indicate its conditions and to monitor its performances. It is inspired from early field experiments of ride-sharing lines along some roadway paths and also from pioneer implementations in France by the Ecov company. Our numerical experiment, though simple, reveals density economies and supports the techno-economic principle of lines for ride-sharing.

Path selection and identification is a primary topic of service design, along with station identification and the allowance or not to stop at any place along the path. Up to now, the waiting policy in Ecov's implementations is an "Agent waits" one. Our numerical experiment suggests that the alternative policy "User waits" may be fruitful on some path and demand conditions.

The model also emphasizes the importance of the price schedule and of the transactional conditions. Transaction times, though expectedly short, are nonetheless significant compared to dwell times and even to run times on one or two tens of km. While wait times are even more important under low flows, under high flows they are likely to fall down to a level similar to dwell times and transaction times. The ease and comfort of transaction operations are important, too, so as to make transaction times enjoyable rather than hard-felt. The "customer journey" has to be addressed in all of its respects by any LRS operator.

About the business model of the LRS operator, we have considered mainly the demand and traffic facets of the service as a system. Such model of demand and traffic can be a useful tool to design a business model, by considering path features (L, t_r^R) as well as population characteristics (q, β) and by specifying service scenarios in terms of Waiting policy, price schedule and transactional conditions, up to access conditions.

While the model is focused on service attractiveness compared to the Neutral option, we have not addressed the issue of service awareness. Awareness rising in the population living near the line corridor is especially important to develop the service basis both of users and of agents. Among the solutions under way, displaying RS lines in local MaaS systems of multimodal service information is a prominent one, since they are purported for general use by all of the local population. MaaS integration will likely not be limited to service and traffic information, it will also foster commercial integration and multimodal synergies between the different kinds of public transit services.

6/ Conclusion

To understand and simulate a line ride-sharing service, we devised a model that is both a traffic model and a microeconomic model of the service on its two sides of Users and Agents. The economic conditions in money and time of the User and Agent roles and also of the service-Neutral role, are modeled at the trip level from the expected conditions of trip legs (Run, Stop, Wait, Transaction), depending on the service protocol that includes price schedule, waiting policy and transaction times, together with macroscopic service conditions in run frequency φ and average car rider load ω . *This sensitive and parametric representation of service roles answers to the first Research Question.*

Then, on the demand side, role choice is modeled at the individual level as a rational microeconomic behavior of utility maximization, considering the utility functions of the roles according to the individual VoT by trip leg and role. This microeconomic model answers to the second Research Question. Individual assignment according to random utility maximization among the population of car trip-makers yields the modal flows, which in turn determine the (φ, ω) variables. This answers to the third RQ.

The "service facet" and the "people facet" of the LRS as a traffic system interact in two ways: from service to people (RQ1&2) and conversely from people to service (RQ2&3). The resulting equilibrium was defined mathematically as a set of balance conditions by role A, U and N, between "supplied

flow” (from ω and φ) and “demanded flow”. This answers to the fourth Research Question, while raising a further question of equilibrium determination. A computational scheme was devised that locates the two essential balance conditions as a pair of so-called Outer- and Inner- Split Service conditions in the (ω, φ) plane.

The formulation of traffic equilibrium as a fixed point problem ensures the existence of an equilibrium state, up to degenerate cases. The issue of uniqueness is still under study.

In its current stage, the model of LRS as a bi-sided mobility service with traffic equilibrium and configuration issues lies at the interface of economic modeling and traffic modeling. Further research may be invested along three avenues. The first avenue consists in developing the physical and economic content. On the demand side, multiple classes may be considered with different conditions regarding travel modes, for instance car dependency or transit dependency. The individual frequency of the travel situation in a multi-day timeframe also deserves specific investigation, as does the consideration of individual mobility at the level of the day chain of activities and trips. On the supply side, the interaction of people through and with the service operator may be modeled in a dynamic way rather than reduced to average conditions – cf. hyperpath motivation in traffic assignment to transit networks. Furthermore, the economic conditions of the service operator deserve to be modeled, not only commercial revenues but also specific production costs and the relationship between the operator and the mobility authority – the potential rewarding of runs and rides in line with the service impacts on the society and the environment. Potential applications may be targeted to the design of price schedules.

The second avenue of research is to develop the spatial scope of the model by introducing the LRS concept into OR problems of network design and service planning. Potential applications include (i) path selection for LRS design, (ii) optimal development of LRS network.

Empirical matters constitute the third avenue of research. The model may be applied to LRS experiments so as to characterize the service conditions and assess its performances. Conversely, such application would enable for econometric estimation of the model parameters, from transaction times to VoT depending on service role and trip leg.

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Appendix A: Car occupation & time items depending on waiting policy

It is postulated that the flows of Users and Agents are independent Poisson processes with respective time rates $\psi_r \equiv y_r/H$. On average over H , the expected number of individuals with role r is then $\psi_r.H = y_r$. In each flow, the individuals arrive successively, with inter-arrival times between two successive individuals that are independent and distributed $\text{EXP}(\psi_r)$: the average inter-arrival time is thus $1/\psi_r = H/y_r$.

A1/ Number of users per agent car run

Let us consider any agent conditionally to time θ taken either [AW] from his instant of arrival to that of the next agent, [UW] from the arrival time of the previous agent to his own one. The users taking the car run are those arriving during time lapse θ . Conditionally to θ , the number v_θ of user arrivals is an RV distributed Poisson with parameter $\psi_U\theta$, yielding

$$\Pr\{v_\theta = n\} = e^{-\psi_U\theta} \cdot \frac{(\psi_U\theta)^n}{n!}, \forall n \in \mathbb{N} \quad (\text{A.1})$$

Deconditioning over θ which is distributed $\text{EXP}(\psi_A)$, and denoting $\psi_{U+A} \equiv \psi_U + \psi_A$,

$$\begin{aligned} \Pr\{v = n\} &= \int_0^\infty \Pr\{v_\theta = n\} d \Pr\{\theta\} \\ &= \int_0^\infty e^{-\psi_U\theta} \cdot \frac{(\psi_U\theta)^n}{n!} e^{-\psi_A\theta} \psi_A d\theta \\ &= \left(\frac{\psi_U}{\psi_{U+A}}\right)^n \frac{\psi_A}{\psi_{U+A}} \int_0^\infty \frac{(\psi_{U+A}\theta)^n}{n!} e^{-\psi_{U+A}\theta} \psi_{U+A} d\theta \end{aligned}$$

The latter integral is easily integrated by parts, yielding say u_n such that

$$u_n \equiv \int_0^\infty \frac{(\psi\theta)^n}{n!} e^{-\psi\theta} \psi d\theta = \left[-\frac{(\psi\theta)^n}{n!} e^{-\psi\theta} \right]_0^\infty + \int_0^\infty \frac{(\psi\theta)^{n-1}}{(n-1)!} e^{-\psi\theta} \psi d\theta = u_{n-1}$$

So that $u_n = u_0 = 1$. Thus,

$$\Pr\{v = n\} = \left(\frac{\psi_U}{\psi_{U+A}}\right)^n \frac{\psi_A}{\psi_{U+A}} = (1 - \rho) \cdot \rho^n, \forall n \in \mathbb{N} \quad (\text{A.2})$$

We recognize a geometric distribution with parameter $\rho \equiv \frac{\psi_U}{\psi_{U+A}}$. Put in words, the number of users per agent is an RV that only depends on the ratio of their respective flows. The average value of such distribution is

$$E[v] = \frac{\rho}{(1-\rho)} = \frac{\psi_U}{\psi_A}. \quad (\text{A.3})$$

This implies the intuitive property that

$$E[v] = \frac{y_U}{y_A}.$$

Denoting $\omega \equiv \frac{y_U}{y_A}$, then

$$\rho = \frac{y_U}{y_{U+A}} = \frac{\omega}{1+\omega}. \quad (\text{A.4})$$

N.B. In the Poisson stochastic model, ω is just the average number of users per agent car. Its effect on the agent stop time is just to multiply the base stop time t_A^S by ρ that is a probability. In case of several users boarding the car, we can expect them to board simultaneously under UW, or under AW maybe to board in turn – yet, if so then the stop time associated to all of them save the last one are embedded in the agent wait time.

Under either waiting policy, conditionally to θ the probability of No users riding in the agent car is $\Pr\{v_\theta = 0\}$, i.e. $\exp(-\psi_U\theta)$. Deconditioning over θ , we get

$$\Pr\{v = 0\} = \frac{\psi_A}{\psi_{U+A}} = 1 - \rho = \frac{1}{1+\omega}. \quad (\text{A.5})$$

This outcome can be obtained more straightforwardly by considering the respective inter-arrival times θ_U and θ_A of users and agents: the probability of an agent not getting any user is the probability that there would be no user arriving during the inter-arrival time between himself and the next agent (AW) or between himself and the previous agent (UW), so that

$$\Pr\{v = 0\} = \Pr\{\theta_A \leq \theta_U\} = \Pr\{\theta_A = \min\{\theta_A, \theta_U\}\} = \frac{\psi_A}{\psi_A + \psi_U} = \frac{1}{1+\omega} = 1 - \rho. \quad (\text{A.6})$$

Thus, to the agent the probability of having to stop is

$$\Pr\{v > 0\} = 1 - \Pr\{v = 0\} = \rho. \quad (\text{A.7})$$

It yields an average Agent stop time of

$$E[t_A^S] = \rho t_A^S. \quad (\text{A.8})$$

A2/ Under User Waits: User wait time

The seat capacity in cars is neglected, so that every user can board any car as soon as it arrives after the user own time of arrival at the origin station. Whatever the user time of arrival, from it the wait time up to an agent arrival is distributed $\text{EXP}(\psi_A)$, from the memory-less property of the exponential distribution of agent inter-arrival times. Thus

$$E[t_U^W] = \frac{1}{\psi_A} = \frac{H}{y_A} = \frac{H}{\varphi}. \quad (\text{A.9})$$

As for Agents, there is no wait time on their side. In practice, an exception would be the arrival of additional users during the stop time of an agent dwelling for a first user to board in it.

A3/ Under Agent Waits: Agent wait time

Conditionally to agent inter-arrival time θ , letting $\{v_\theta = n\}$, from the properties of the Poisson process of user arrivals, the n instants of user arrivals are independent RVs denoted t_i with identical uniform distribution on $[0, \theta]$, hence with CDF as follows:

$$F_\theta(z) \equiv \Pr\{t_i \leq z \mid \theta\} = \min(1, \frac{z}{\theta}), \forall z \in \mathbb{R}^+.$$

The agent wait time $t_{W|\theta,n}$ conditionally to θ and n is the maximum of the n RVs t_i :

$$t_{\theta,n} \equiv \max\{t_i : i \in \{1, \dots, n\}\}$$

Thus $\{t_{W|\theta,n} \leq z \mid \theta\} = \cap_{i=1}^n \{t_i \leq z \mid \theta\}$. From the independence between the t_i , the CDF of $t_{W|\theta,n}$ is

$$\Pr\{t_{W|\theta,n} \leq z \mid \theta\} = \prod_{i=1}^n \Pr\{t_i \leq z \mid \theta\} = (F_{z\theta})^n, \text{ wherein } F_{z\theta} \equiv F_\theta(z).$$

Deconditioning over n ,

$$\begin{aligned}
\Pr\{t_{W|\theta} \leq z \mid \theta\} &= \sum_{n \geq 0} \Pr\{N_\theta = n\} \Pr\{t_{W|\theta,n} \leq z \mid \theta, n\} \\
&= e^{-\psi_U \theta} \sum_{n \geq 0} \frac{(\psi_U \theta F_{z\theta})^n}{n!} \\
&= \exp(-\psi_U \theta (1 - F_{z\theta})) = \exp(-\psi_U (\theta - z)^+)
\end{aligned}$$

Deconditioning now over θ ,

$$\begin{aligned}
\Pr\{t_W \leq z\} &= \int_0^\infty \Pr\{t_{W|\theta} \leq z \mid \theta\} d\Pr\{\theta\} \\
&= \int_0^\infty \exp(-\psi_U (\theta - z)^+) e^{-\psi_A \theta} \psi_A d\theta \\
&= \int_0^z e^{-\psi_A \theta} \psi_A d\theta + e^{\psi_U z} \int_z^\infty e^{-\psi_{U+A} \theta} \psi_A d\theta \\
&= 1 - e^{-\psi_A z} + e^{\psi_U z} e^{-\psi_{U+A} z} \frac{\psi_A}{\psi_{U+A}} \\
&= 1 - e^{-\psi_A z} \rho \\
&= (1 - \rho) + \rho(1 - e^{-\psi_A z})
\end{aligned}$$

This CDF is that of the probabilistic mixture of a Dirac variable at 0 and a variable $\text{EXP}(\psi_A)$, with respective probabilities $1 - \rho$ and ρ , i.e., of the probabilities of having null or strictly positive number of users on board the Agent's car. Thus, under AW,

$$E[t_A^W] = (1 - \rho) \cdot 0 + \rho \frac{1}{\psi_A} = \rho \frac{H}{\psi_A} = \rho \frac{H}{\varphi}. \quad (\text{A.10})$$

N.B. To satisfy the distributional assumptions, it requires that (i) user flow to be not so large compared to agent flow, so that the seat capacity in the car is mostly sufficient, (ii) the service coordination collects information about passage times from both the users and the agents, so that the platform can know θ and n "in advance" compared to the individuals: thus, the platform is able to tell to each agent whether to stop or not, and how many users will board the car.

Appendix B: Traffic equilibrium

B1/ System state and state vector

A line ride-sharing service as a system has a system state determined by the values of its endogenous variables. These include (cf. Figure 1):

- (i) people trip flows (y_r) according to service roles $r \in \mathfrak{R}$,
- (ii) service quality variables (φ, ω, ρ),
- (iii) the money and time items of the roles, denoted t_r^ℓ and m_r^ℓ by leg $\ell \in \{R, S, T, W\}$, depending on waiting policy $\# \in \{AW, UW\}$ and on the flow policy,
- (iv) the deterministic utility functions $V_r^\#$ according to role r and waiting policy $\#$,
- (v) the role probabilities $p_r^\#$ among the q individuals.

These probabilities sum up the preferences of the individuals, by assigning each of them to the optimal role according to deterministic features together with random circumstances.

B2/ Structure of influences and basic state vector

The endogenous variables are related by the following set of conditions:

$$a/ y_r \geq 0 \text{ and } \sum_{r \in \mathfrak{R}} y_r = q.$$

b/ quality formation: $\varphi = y_A$ and ω such that $y_U = \omega \cdot y_A$, denoted as $(\varphi, \omega) = (\hat{\varphi}, \hat{\omega})(\mathbf{y}_{\mathfrak{R}})$.

c/ traffic laws yielding link run times according to roles and depending on flow policy.

d/ time and money item composition on the basis of (φ, ω, v) and according to waiting policy.

e/ modal utility function composition from money and time items.

f/ optimal choices: choice probabilities $p_r^\#$ stem from random utility functions.

g/ flow assignment $y_r^\# = q \cdot p_r^\#, \forall r \in \mathfrak{R}$.

The flow vector $[y_r: r \in \mathfrak{R}]$ can make a basic state vector from which all of the other variables are derived. Such flow vector constitutes an equilibrium state if it satisfies that:

$$\forall r \in \mathfrak{R}: y_r = y_r^\#(\mathbf{y}_{\mathfrak{R}}) \quad (\text{B.1})$$

It is a condition of quantitative balance (market clearing) between supply (y_r) and demand ($y_r^\#$).

Yet, as basic state vector it is more convenient to consider the (φ, ω) pair, from which stem (\hat{y}_A, \hat{y}_U) , service flow $\hat{y}_S = \hat{y}_A + \hat{y}_U$, neutral flow $\hat{y}_N = q - \hat{y}_S$, hence v and the rest of endogenous variables. This basic state vector is feasible if $\varphi \geq 0$, $\omega \geq 0$ and $\varphi \leq q$.

A feasible vector (φ, ω) is an equilibrium state iff

$$\forall r \in \mathfrak{R}: \hat{y}_r(\varphi, \omega) = y_r^\#(\varphi, \omega) \quad (\text{B.2})$$

B3/ Fixed-point problem and iterative solution scheme

FPP in (φ, ω) . A basic state vector (φ, ω) is in equilibrium if it is a fixed point for the mapping $(\varphi, \omega) \mapsto (\varphi', \omega')$ such that $\varphi' = y_A^\#(\varphi, \omega)$ and ω' satisfies that $y_U^\#(\varphi, \omega) = \omega' y_A^\#(\varphi, \omega)$.

FPP in (y_A, y_U) . A basic state vector (y_A, y_U) is in equilibrium if it is a fixed point for the chained mapping $(y_A, y_U) \mapsto (\varphi, \omega)$ such that $\varphi = y_A$ and ω satisfies that $y_U = \omega y_A$, followed by $(\varphi, \omega) \mapsto (y_A^\#, y_U^\#)$.

Iterative solution scheme. The basic strategy to solve an FPP is to progressively adapt a current state vector by combining it with its image through the mapping so as to obtain the next value. At step k , current state (φ_k, ω_k) induces image $(\hat{\varphi}_k, \hat{\omega}_k)$: then the next step can be obtained as

$$\begin{aligned} \varphi_{k+1} &\leftarrow \text{Fn}[\varphi_k, \hat{\varphi}_k], \\ \omega_{k+1} &\leftarrow \text{Fn}[\omega_k, \hat{\omega}_k]. \end{aligned}$$

For instance, a convex combination scheme with step sizes ζ_k that decrease to zero:

$$\begin{aligned} \varphi_{k+1} &\leftarrow \varphi_k + \zeta_k(\hat{\varphi}_k - \varphi_k) = (1 - \zeta_k)\varphi_k + \zeta_k \hat{\varphi}_k \\ \omega_{k+1} &\leftarrow \omega_k + \zeta_k(\hat{\omega}_k - \omega_k) = (1 - \zeta_k)\omega_k + \zeta_k \hat{\omega}_k. \end{aligned}$$

A similar strategy on the (y_A, y_U) flow vector constitutes an equilibration algorithm well-known in network traffic assignment. The convex combination scheme is an instance of the ‘‘Method of Successive Averages’’. Yet the graphical scheme in Section 3 is a more straightforward strategy.

C/ Equilibrium properties

C.1/ Mathematical property of existence

On the demand side, model functions leading from (φ, ω) to $y_r^\#$ are continuous because they combine continuous elementary functions (e.g., deterministic utility V_r) in simple, continuous ways (e.g. $y_r^\#$ as a ratio of strictly positive functions $\exp(\theta \cdot V_r)$ that are continuous owing to the continuity of the exponential function and that of V_r).

On the supply side, the derivation of \hat{y}_r from (φ, ω) is continuous: $\hat{y}_A = \varphi$ is continuous everywhere, so are $\hat{y}_U = \varphi \cdot \omega$ and $\hat{y}_N = q - \hat{y}_A - \hat{y}_U$.

Now, for any small parameter $\eta > 0$, function $(\hat{\varphi}, \hat{\omega}) = (y_A, \frac{y_U}{y_A})$ is continuous on the parameterized set $\Delta_y^\eta \equiv \{y_{\mathfrak{R}} \in \Delta_y : y_A \geq \eta\}$. Defining $\hat{\varphi}_\eta \equiv \max\{\eta, \hat{\varphi}\}$ and $\hat{\omega}_\eta \equiv y_U / \max\{\eta, y_A\}$, the function $y_{\mathfrak{R}} \mapsto y_{\mathfrak{R}}^\# \circ (\hat{\varphi}_\eta, \hat{\omega}_\eta)(y_{\mathfrak{R}})$ is continuous on Δ_y^η where it takes its values, hence, owing to Brouwer's theorem, it admits a fixed point there which we denote as y_η^* .

Furthermore, any sequence of $(\eta_n > 0 : n \in \mathbb{N})$ such that $\eta_n \rightarrow 0$ for $n \rightarrow \infty$ yields an associated sequence of points $y_{\eta(n)}^*$ in the overall feasible domain Δ_y . As it is a compact set, the sequence $(y_{\eta(n)}^* : n \in \mathbb{N})$ admits at least one accumulation point, denoted y^* . If $y_A^* > 0$ then y^* satisfies all of the model equations if, making it a solution to the unrestricted FPP, hence a traffic equilibrium state. But otherwise, if $y_A^* = 0$ for all such accumulation points, then the accumulation principle shows that there is no non-degenerate solution to the problem of traffic equilibrium.

C.2/ Min-cost formulation of traffic equilibrium

In logit traffic assignment, each path or mode r with cost function g_r and flow y_r gives rise to an impedance function as follows:

$$I_r \equiv g_r + \frac{1}{\theta} \ln y_r$$

This is because $y_r \exp(-\theta V_r) = y_r \exp(\theta g_r)$ is equal between options r of positive flow y_r and finite cost g_r : taking the natural logarithm, $\ln[y_r \exp(\theta g_r)] = \theta g_r + \ln y_r$, leading to the impedance function.

A logit traffic equilibrium with demand volume q is a flow vector $[y_r : r \in \mathfrak{R}]$ such that

$$\begin{aligned} y_r &\geq 0 \\ \sum_{r \in \mathfrak{R}} y_r &= q \\ y_r \cdot (I_r - \hat{I}_{\mathfrak{R}}) &= 0 \end{aligned}$$

With $\hat{I}_{\mathfrak{R}} \equiv \min\{I_r : r \in \mathfrak{R}\}$.

The last condition states that only an option of minimum impedance can have positive flow.

C.3/ VIP formulation of traffic equilibrium

Defining impedance functions $I_r \equiv g_r + \frac{1}{\theta} \ln y_r$, the VIP associated to mapping $\mathbf{I}_{\mathfrak{R}} \equiv [I_r : r \in \mathfrak{R}]$ is

$$\begin{aligned} \text{“Find } y_{\mathfrak{R}}^* \geq 0 \text{ and satisfying } \sum_{r \in \mathfrak{R}} y_r^* = q : \text{ denoting } \mathbf{I}_{\mathfrak{R}}^* \equiv \mathbf{I}_{\mathfrak{R}}(y_{\mathfrak{R}}^*), \text{ such that} \\ \forall y_{\mathfrak{R}} \geq 0 \text{ with } \sum_{r \in \mathfrak{R}} y_r = q, \mathbf{I}_{\mathfrak{R}}^* \cdot (y_{\mathfrak{R}} - y_{\mathfrak{R}}^*) \geq 0.” \end{aligned}$$

It means that under option costs $\mathbf{I}_{\mathfrak{R}}^*$, no admissible assignment $y_{\mathfrak{R}}$ can get total cost $\mathbf{I}_{\mathfrak{R}}^* \cdot y_{\mathfrak{R}}$ lower than at $y_{\mathfrak{R}}^*$. Demand elasticity can be taken into account by considering domain $y_{\mathfrak{R}} \geq 0$ and mapping $I_r \equiv g_r + \frac{1}{\theta} \ln y_r - D^{(-1)}(\sum_{r \in \mathfrak{R}} y_r)$.

C.4/ Nesting the service options in the choice universe

The three options $\{U, A, N\}$ are put on the same level in the multinomial logit model. We may also think of a composite option “Service” $\mathcal{S} \equiv \{U, A\}$ and of a nested logit model with choice set $\{\mathcal{S}, N\}$ on the first level and, on the second level, choice subset $\{U, A\}$ at \mathcal{S} as a nest.

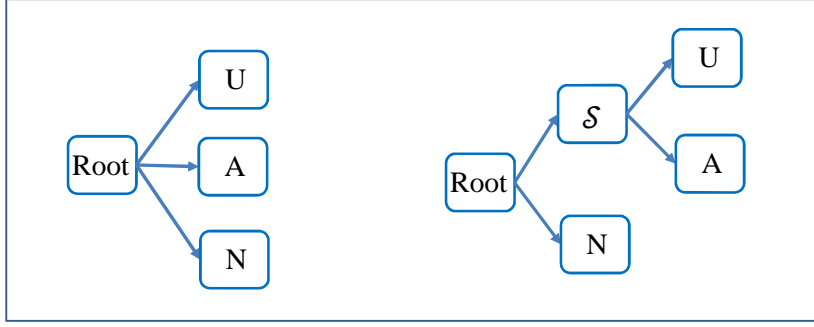


Fig. 3: (A) Multinomial choice model, (B) Nested choice model.

The combined utility function is $U_S \equiv \max\{U_U, U_A\}$.

In the logit framework, it is a Gumbel variable with concentration parameter θ and mean value

$$V_S \equiv \frac{1}{\theta} (\exp(\theta V_U) + \exp(\theta V_A))$$

As U_U and U_A are independent of U_N , so is U_S .

The impedance function associated with S is

$$I_S \equiv g_S + \frac{1}{\theta} \ln y_S$$

With $g_S \equiv -V_S$. Under demand choice in the S nest, the impedance functions satisfy that

$$I_S = I_U = I_A$$

Hence that $g_S + \frac{1}{\theta} \ln y_S = g_U + \frac{1}{\theta} \ln y_U = g_A + \frac{1}{\theta} \ln y_A$

Considering now the complementary probabilities $\rho = y_U/y_S$ and $1 - \rho = y_A/y_S$, we get that

$$g_S = g_U + \frac{1}{\theta} \ln \rho = g_A + \frac{1}{\theta} \ln(1 - \rho)$$

Thus, decomposing $g_S = \rho g_S + (1 - \rho) g_S = \rho(g_U + \frac{1}{\theta} \ln \rho) + (1 - \rho)(g_A + \frac{1}{\theta} \ln(1 - \rho))$,

$$g_S = \rho g_U + (1 - \rho) g_A - \frac{1}{\theta} \mathcal{H}_\rho$$

Wherein $\mathcal{H}_\rho \equiv -\rho \ln \rho - (1 - \rho) \ln(1 - \rho)$ is the entropy function of the probability distribution $[\rho, 1 - \rho]$. On changing signs, we get the equivalent relationship that

$$V_S = \rho V_U + (1 - \rho) V_A + \frac{1}{\theta} \mathcal{H}_\rho$$

The service deterministic utility is a convex combination of the U and A deterministic utilities, plus (up to factor $1/\theta$) the entropy function of their respective shares within the service (entropy is positive, making the service utility higher than the convex combination of its two constituents).