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Abstract. We analyze the properties of the spectral shift function (SSF) for some dispersive self-adjoint operators under suitable compact perturbations. New mechanisms allowing the SSF to have singularities at the thresholds are exhibited, based on the degeneracy of the spectrum of the unperturbed operator. In particular, we show how any desired type of singularities can be engineered in this situation. These results are completed by Levinson type results.

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Contents



## 1. Introduction

1.1. **General setting.** This work aims to improve our understanding of the mechanisms involved in the creation of eigenvalues clusters under (relatively) compact perturbations in the vicinity of some special points of the spectrum, called here thresholds. So far, such mechanisms have mainly been studied when a threshold coincides or is induced by an eigenvalue of infinite multiplicity. This has been performed in the self-adjoint perturbation setting (see [25, 31, 13, 7, 8, 28, 21] and references therein), as well as in the non-self-adjoint perturbation case (see [27, 29]). Such phenomena are also related to long range perturbations at the threshold of the absolutely continuous component of the spectrum, as is the case of the hydrogen atom model [30]. Technically, the formation of cluster of eigenvalues is somehow encoded in the behavior of the *Spectral Shift Function* (SSF) [20, 18], a useful notion for spectral analysis and scattering theory of quantum system, when it is well defined. However, in contrast to the scattering matrix, the spectral shift function is meaningful both on the continuous and discrete spectra.

Our purpose here is to analyze the behavior of the SSF when the infinitely degenerated eigenvalue is replaced by a threshold of a highly degenerated absolutely continuous component. The model introduced hereafter elaborates on previous results based on the discrete Laplace operator on the 1D lattice. Fibered versions of the 1D discrete Laplacian are considered in [4, 5] and we proved the finiteness of the discrete spectrum under (non)-self-adjoint exponential decay matrix-valued perturbations. We have also exhibited some LAP properties that complete some previous results obtained by [11, 12] for Jacobi matrices. Techniques developed and used in [4, 5] are based on resonances theory and complex scaling arguments for non necessarily self-adjoint operators. However, this approach is not adapted to study (matrix-valued) perturbations that decay polynomially at infinity. For such perturbations, the spectral analysis can be done by means of the SSF. Note also that there are few results showing spectral cluster phenomena near the essential spectrum for discrete models. In particular, the present paper is an attempt to fill this gap among the many results obtained so far.

Adapted to our context, let us recall briefly the abstract setting in which the SSF for a pair $(\mathcal{H}, \mathcal{H}_0)$ of self-adjoint operators occurs. We assume that the operators $\mathcal{H}$ and $\mathcal{H}_0$ act in the same

Hilbert space and moreover that

$$\mathcal{H}_0 - \mathcal{H} \in \mathfrak{S}_1, \tag{1.1}$$

where $\mathfrak{S}_1$ denotes the trace class operators. Then, there exists (see [18] or e.g. [32, Theorem 8.3.3]) a unique function $\xi(\cdot;\mathcal{H},\mathcal{H}_0) \in \mathrm{L}^1(\mathbb{R})$ such that the *Lifshits-Krein trace formula*

$$\mathrm{Tr}(f(\mathcal{H}) - f(\mathcal{H}_0)) = \int_{\mathbb{R}} \xi(\lambda;\mathcal{H},\mathcal{H}_0) f'(\lambda) d\lambda, \tag{1.2}$$

holds for every $f \in C_0^\infty(\mathbb{R})$. The function $\xi(\cdot;\mathcal{H},\mathcal{H}_0) \in \mathrm{L}^1(\mathbb{R})$ is called the *Spectral Shift Function* (SSF) for the pair $(\mathcal{H},\mathcal{H}_0)$. If we assume that $\sigma(\mathcal{H}_0) = \sigma_{\mathrm{ac}}(\mathcal{H}_0)$ such that $\sigma_{\mathrm{ess}}(\mathcal{H}) = \sigma(\mathcal{H}_0)$, then the SSF can be related to the eigenvalue counting function of the operator $\mathcal{H}$ outside the essential spectrum. In scattering theory, it is related to the scattering matrix $S(\lambda;\mathcal{H},\mathcal{H}_0)$ for the pair $(\mathcal{H},\mathcal{H}_0)$ through the Birman-Krein [3] formula

$$\det S(\lambda;\mathcal{H},\mathcal{H}_0) = e^{2i\pi\xi(\lambda;\mathcal{H},\mathcal{H}_0)}, \quad a.e.\ \lambda \in \sigma_{\mathrm{ac}}(\mathcal{H}_0). \tag{1.3}$$

1.2. **The model.** We consider the lattice $\mathbb{Z}_h := \{hn : n \in \mathbb{Z}\}$ with mesh size $h > 0$ and let $\mathcal{G}$ be a separable Hilbert space and $\ell^2(\mathbb{Z}_h,\mathcal{G})$ be the Hilbert space endowed with the scalar product

$$\langle \varphi, \phi \rangle := \sum_{n\in\mathbb{Z}} \langle \varphi(hn), \phi(hn) \rangle_{\mathcal{G}}.$$

Namely,

$$\ell^2(\mathbb{Z}_h,\mathcal{G}) := \big\{ \varphi \in \mathcal{G}^{\mathbb{Z}_h} : \|\varphi\|^2 = \sum_{n\in\mathbb{Z}} \|\varphi(hn)\|_{\mathcal{G}}^2 < \infty \big\}.$$

For $\varphi \in \ell^2(\mathbb{Z}_h,\mathcal{G})$, we define the finite-difference bounded operator

$$(\partial_h \varphi)(hn) := \frac{1}{h^2}\big(\varphi(h(n+1)) - \varphi(hn)\big),$$

whose adjoint $\partial_h^*$ is given by

$$(\partial_h^* \varphi)(hn) = \frac{1}{h^2}\big(\varphi(h(n-1)) - \varphi(hn)\big).$$

We consider the bounded self-adjoint Laplacian type operator

$$H_h = -\partial_h - \partial_h^* \quad \text{on} \quad \ell^2(\mathbb{Z}_h,\mathcal{G}). \tag{1.4}$$

The spectral analysis of the operator $H_h$ is described in details in Section 2 where it is shown that its spectrum is purely absolutely continuous and satisfies

$$\sigma(H_h) = \sigma_{\mathrm{ac}}(H_h) = \sigma_{\mathrm{ess}}(H_h) = [0, \tfrac{4}{h^2}]. \tag{1.5}$$

Here, the points $\{0, \frac{4}{h^2}\}$ are the thresholds of this spectrum.

Let $\mathcal{B}(\ell^2(\mathbb{Z}_h,\mathcal{G}))$ be the set of bounded linear operators in $\ell^2(\mathbb{Z}_h,\mathcal{G})$. We consider the perturbed operators

$$H_h^\pm := H_h \pm V_h, \tag{1.6}$$

where $V_h \in \mathcal{B}(\ell^2(\mathbb{Z}_h,\mathcal{G}))$ is a positive matrix-valued electric potential such that

$$V_h = \{v_h(n,m)\}_{(n,m)\in\mathbb{Z}^2}, \quad v_h(n,m) \in \mathcal{B}(\mathcal{G}), \tag{1.7}$$

where $\mathcal{B}(\mathcal{G})$ denotes the set of bounded linear operators in $\mathcal{G}$. This perturbation $V_h$ can be interpreted as a summation kernel operator whose kernel is given by the operator-valued function $(n,m) \in \mathbb{Z}^2 \mapsto v_h(n,m) \in \mathcal{B}(\mathcal{G})$. Namely, for any $\varphi \in \ell^2(\mathbb{Z}_h,\mathcal{G})$, one has

$$(V_h\varphi)(hn) = \sum_{m\in\mathbb{Z}} v_h(n,m)\varphi(hm), \quad n \in \mathbb{Z}. \tag{1.8}$$

We also assumes that $\|v_h(n,m)\|_{\mathcal{B}(\mathcal{G})}$ decays more rapidly than $\|(n,m)\|^{-3}$ as $\|(n,m)\| \to +\infty$ (see Assumption 3.1, (3) of Remark 3.5, (6.7) for more details) and we show that (1.1) holds for the pair $(\mathcal{H},\mathcal{H}_0) = (H_h^\pm, H_h)$. Therefore, there exists a unique function

$$\xi(\cdot; H_h^\pm, H_h) \in \mathrm{L}^1(\mathbb{R}), \tag{1.9}$$

such that the *Lifshits-Krein trace formula* (1.2) holds. The function $\xi(\cdot; H_h^{\pm}, H_h)$ is the SSF for the pair $(H_h^{\pm}, H_h)$. It can be related to the number of eigenvalues of the operators $H_h^{\pm}$ in $(-\infty, 0) \cup (\frac{4}{h^2}, +\infty)$ (see formula (4.8)). By the Birman-Krein formula (1.3), it is also related to the scattering matrix $S(\lambda; H_h^{\pm}, H_h)$ for the pair $(H_h^{\pm}, H_h)$, for a.e. $\lambda \in [0, \frac{4}{h^2}]$.

1.3. **Description of the main results.** The main results of this article concern the asymptotic behavior of $\xi(\lambda; H_h^{\pm}, H_h)$ as $\lambda \to z_0 \in \{0, \frac{4}{h^2}\}$ for matrix-valued perturbations $V_h$ of definite sign.

We will first identify $\xi(\cdot; H_h^{\pm}, H_h)$ with a representative of its equivalence class described explicitly in Section 4, assuming that the electric matrix-valued potential $V_h$ has a definite sign. Then, we prove the boundedness of $\xi(\cdot; H_h^{\pm}, H_h)$ on compact subsets of $\mathbb{R} \smallsetminus \{0, \frac{4}{h^2}\}$ (see Theorem 5.1). In Theorem 5.2, we establish the asymptotic behavior of $\xi(\lambda; H_h^{\pm}, H_h)$ as $\lambda \to z_0$, $\lambda \in (-\infty, 0) \cup (\frac{4}{h^2}, +\infty)$, and in Theorems 5.4, 5.6, we determine the asymptotic behavior of $\xi(\lambda; H_h^{\pm}, H_h)$ as $\lambda \to z_0$, $\lambda \in (0, \frac{4}{h^2})$. Several consequences, described below, can be deduced from Theorems 5.1, 5.2, 5.4, and 5.6.

In the finite-dimensional case where $\dim(\mathcal{G}) < \infty$, Theorem 5.7 implies that the SSF $\xi(\cdot; H_h^{\pm}, H_h)$ is bounded in $\mathbb{R} \smallsetminus \{0, \frac{4}{h^2}\}$. Notice that Theorem 5.7 is stronger than Theorem 5.1. In particular, if $\mathcal{G} = \mathbb{C}$, then the operator $H_h = -\Delta_h$ is the 1D discrete Laplacian on the lattice $h\mathbb{Z}$ and Corollary 5.8 shows the finiteness of the discrete spectrum of $-\Delta_h \pm V_h$ for polynomial decay perturbations at infinity. This extends results of [4, 5] where the finiteness of the discrete spectrum of $-\Delta_1 + V$ has been proved for exponential decay perturbations.

On the other hand, if $\dim(\mathcal{G}) = +\infty$, then we show that the SSF can have singularities at the spectral thresholds $\{0, \frac{4}{h^2}\}$ under generic assumptions on $V_h$ (see Theorems 5.11, 5.15 and Corollaries 5.12, 5.14, 5.16). More precisely, if the perturbation is positive, one has

$$\begin{cases} \xi(\lambda; H_h^+, H_h) = \mathcal{O}(1) & as \quad \lambda \nearrow 0, \\ \xi(\lambda; H_h^+, H_h) \to +\infty & as \quad \lambda \searrow 0, \end{cases} \quad \text{while } \xi(\lambda; H_h^+, H_h) \to +\infty \quad as \quad \lambda \to \tfrac{4}{h^2}.$$

If the perturbation is negative, we have

$$\begin{cases} \xi(\lambda; H_h^-, H_h) = \mathcal{O}(1) & as \quad \lambda \searrow \frac{4}{h^2}, \\ \xi(\lambda; H_h^-, H_h) \to -\infty & as \quad \lambda \nearrow \frac{4}{h^2}, \end{cases} \quad \text{while } \xi(\lambda; H_h^-, H_h) \to -\infty \quad as \quad \lambda \to 0.$$

Actually, the singularities of the SSF at the spectral thresholds are described in terms of some explicit effective "Berezin-Toeplitz" type operators (see (5.6) for a precise definition). Hence and under suitable condition, we give the main terms of the asymptotics expansions of $\xi(\lambda; H_h^{\pm}, H_h)$ as $\lambda \to z_0$ (see Corollary 5.12 and Theorem 5.15 for the general case, and Corollaries 5.14, 5.16 for power-like and exponential decay perturbations). In particular, if the perturbation is positive then the limits

$$\lim_{\lambda \searrow 0} \frac{\xi(\lambda; H_h^-, H_h)}{\xi(-\lambda; H_h^-, H_h)} \quad \text{and} \quad \lim_{\lambda \searrow 0} \frac{\xi(\frac{4}{h^2} - \lambda; H_h^+, H_h)}{\xi(\frac{4}{h^2} + \lambda; H_h^+, H_h)},$$

exist and are equal to positive constants depending on the decay rate of $V_h$ at infinity (see Theorem 5.18 and Corollary 5.19). This can be interpreted as generalized Levinson formulae (see the original work [19] or the survey article [26]).

1.4. **Comments on the literature.** Our results extend to a class of discrete Laplacians the results in [25, 31, 13] established for continuous models. More precisely, in [25, 31] the asymptotic behavior of the SSF has been considered near the low ground energy and near $\pm m$ for 2D Pauli and 3D Dirac operators with non-constant magnetic fields. In [13] the asymptotic behavior of the SSF has been considered near the Landau levels for 3D Schrödinger operators with constant magnetic fields. Similar results can be also found in [7, 8]. However, in the discrete case, there are few results concerning the asymptotics expansions of the SSF at spectral thresholds. We quote to the recent paper [21] where similar asymptotics to ours are obtained at thresholds for a discrete Dirac-type operator on $\mathbb{Z}^2$. It is important to highlight that in the above papers, the singularities of the SSF near the spectral thresholds are produced by infinitely degenerated eigenvalues. This is rather different compared to our situation where the singularities of the SSF near the thresholds $\{0, \frac{4}{h^2}\}$ are produced by a highly degenerated absolutely continuous component. In both situations,

the singularities of the SSF are probably due to an accumulation of resonances near the spectral thresholds. However, this aspect of the problem will not be treated here and will be consider in a further work. Some trace formulas have been also obtained for periodic graphs in [14, 17]. Asymptotics expansions of resolvents at spectral thresholds of discrete Laplacians can be found in [15, 16], see also references therein.

The article is organized as follows. In Section 2, we perform in details the spectral analysis of the free operator $H_h$ and introduce some standard tools needed so far. In Section 3, we state and discuss our main assumptions concerning the perturbed operators $H_h^{\pm}$. In Section 4, we recall some abstract results due to A. Pushnitski on the representation of the spectral shift function for a pair of self-adjoint operators. Section 5 is devoted to the formulation of our main results, some corollaries of them, as well as examples of explicit eigenvalues asymptotics. In Section 6, we compute a suitable factorisation of the perturbation $V_h$ satisfying a main assumption given in Section 3. Section 7 contains auxiliary material such as extensions of the convolution kernel of $-\Delta_h$, and estimates of appropriate sandwiched resolvents. Finally, in Section 8 we prove Theorems 5.2 and 5.4, while Section 9 establishes the one of the asymptotics identities (5.22) and (5.25).

## 2. The free hamiltonian

Consider an orthonormal basis $(\mathrm{e}_j)_{j\in\Lambda}$ of $\mathcal{G}$ where $\#\Lambda = \dim(\mathcal{G})$ if $\dim(\mathcal{G}) < +\infty$, and $\Lambda = \mathbb{Z}_+$ if $\dim(\mathcal{G}) = +\infty$. Let $(\delta_n^h)_{n\in\mathbb{Z}}$ be the canonical orthonormal basis of

$$\ell^2(\mathbb{Z}_h) := \ell^2(\mathbb{Z}_h, \mathbb{C}),$$

where $\delta_n^h(kh) = \delta_{nk}$, $k \in \mathbb{Z}$. Then, it is useful to identify the spaces $\ell^2(\mathbb{Z}_h, \mathcal{G})$ and $\ell^2(\mathbb{Z}_h) \otimes \mathcal{G}$: $\ell^2(\mathbb{Z}_h, \mathcal{G}) \cong \ell^2(\mathbb{Z}_h) \otimes \mathcal{G}$ and it follows that $(\delta_n^h \otimes \mathrm{e}_j)_{(n,j)\in\mathbb{Z}\times\Lambda}$ is an orthonormal basis of $\ell^2(\mathbb{Z}_h) \otimes \mathcal{G}$. For each $j \in \Lambda$, one defines the subspace $\mathcal{G}_j^h = \mathrm{span}\{x \otimes \mathrm{e}_j : x \in \ell^2(\mathbb{Z}_h)\}$, together with its corresponding orthogonal projection $P_j := I \otimes |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,|$, so that

$$\ell^2(\mathbb{Z}_h, \mathcal{G}) \cong \ell^2(\mathbb{Z}_h) \otimes \mathcal{G} = \bigoplus_{j\in\Lambda} \mathcal{G}_j^h.$$

Hence, one observes that for every $j \in \Lambda$, $\mathcal{G}_j^h$ is $H_h$-invariant so that $H_h$ rewrites

$$H_h = \bigoplus_{j\in\Lambda} P_j H_h P_j = \bigoplus_{j\in\Lambda} -\Delta_h \otimes |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,| = -\Delta_h \otimes I,$$

where $-\Delta_h$ is the 1D Schrödinger operator acting in $\ell^2(\mathbb{Z}_h)$ by

$$(-\Delta_h \phi)(n) = \frac{1}{h^2}\big(2\phi(hn) - \phi(h(n+1)) - \phi(h(n-1))\big).$$

Let $\tau > 0$ such that $h\tau = 2\pi$ and

$$\mathbb{T}_\tau = \mathbb{R}/\tau\mathbb{Z} \sim [-\tfrac{\tau}{2}, \tfrac{\tau}{2}].$$

In view of the bijection between $\ell^2(\mathbb{Z}_h)$ and $\mathrm{L}^2(\mathbb{T}_\tau) := \mathrm{L}^2(\mathbb{T}_\tau, \mathbb{C})$, one defines the discrete Fourier transform $\mathscr{F}_h : \ell^2(\mathbb{Z}_h) \to \mathrm{L}^2(\mathbb{T}_\tau)$ by

$$(\mathscr{F}_h \phi)(\theta) := \sum_{n\in\mathbb{Z}} \phi(hn) e^{-ihn\theta}, \quad \phi(hn) = \frac{1}{\tau}\int_{\mathbb{T}_\tau} (\mathscr{F}_h\phi)(\theta) e^{ihn\theta}\, d\theta. \tag{2.1}$$

The operator $\mathscr{F}_h$ is unitary. By using $\mathscr{F}_h \otimes I$ acting in $\ell^2(\mathbb{Z}_h) \otimes \mathcal{G}$, one shows that the operator $H_h$ is unitarily equivalent to the operator $\widehat{-\Delta_h} \otimes I$ acting in $\mathrm{L}^2(\mathbb{T}_\tau, \mathcal{G}) \cong \mathrm{L}^2(\mathbb{T}_\tau) \otimes \mathcal{G}$, where $\widehat{-\Delta_h}$ is the multiplication operator in $\mathrm{L}^2(\mathbb{T}_\tau)$ by the function $f_h$ defined by

$$f_h(\theta) := \frac{2 - 2\cos(h\theta)}{h^2} = \frac{4}{h^2}\sin^2\Big(\frac{h\theta}{2}\Big), \quad \theta \in \mathbb{T}_\tau. \tag{2.2}$$

Therefore

$$[0, \tfrac{4}{h^2}] = \sigma(-\Delta_h) = \sigma(H_h),$$

and the operators $-\Delta_h$ and $H_h$ are purely absolutely continuous so that (1.5) holds.

**Remark 2.1.** *If $\mathcal{G} = \mathbb{C}$, then the operator $H_h$ corresponds to $-\Delta_h$ the discrete Laplacian on the lattice $h\mathbb{Z}$.*

## 3. The perturbed hamiltonian

Recall that the perturbation $V_h$ is a bounded matrix-valued $V_h = \{v_h(n,m)\}_{(n,m)\in\mathbb{Z}^2}$ with coefficients $v_h(n,m) \in \mathcal{B}(\mathcal{G})$. In the basis $(\mathrm{e}_j)_{j\in\Lambda}$ of $\mathcal{G}$, for each $(n,m) \in \mathbb{Z}^2$, the operator $v_h(n,m)$ has the matrix representation

$$v_h(n,m) = \{v^h_{jk}(n,m)\}_{j,k\in\Lambda}, \qquad v^h_{jk}(n,m) := \langle \mathrm{e}_j, v_h(n,m)\,\mathrm{e}_k\rangle_{\mathcal{G}}. \tag{3.1}$$

Hence, one has

$$v_h(n,m) = \sum_{j\in\Lambda}\sum_{k\in\Lambda} v^h_{jk}(n,m)|\,\mathrm{e}_j\rangle\langle \mathrm{e}_k\,|. \tag{3.2}$$

The operator $v_h(n,m)$ viewed as a matrix $\{v^h_{jk}(n,m)\}_{(j,k)\in\Lambda^2}$ belongs to $\mathcal{M}_d(\mathbb{C})$ if $d = \dim(\mathcal{G}) < \infty$. In particular, if $d = 1$ (i.e. $\mathcal{G} = \mathbb{C}$), $\{v_h(n,m)\}_{(n,m)\in\mathbb{Z}^2}$ is the matrix representation of $V_h$ in the canonical basis of $\ell^2(\mathbb{Z}_h)$. In $\ell^2(\mathbb{Z}_h)\otimes\mathcal{G}$, $V_h$ has a canonical representation given by

$$V_h = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\delta^h_n\rangle\langle\delta^h_m| \otimes v_h(n,m) = \sum_{(n,m)\in\mathbb{Z}^2}\sum_{(j,k)\in\Lambda^2} |\delta^h_n\rangle\langle\delta^h_m| \otimes v^h_{jk}(n,m)|\,\mathrm{e}_j\rangle\langle \mathrm{e}_k\,|.$$

In the sequel, for $y = (y_1,\dots,y_d) \in \mathbb{R}^d$, one sets $\langle y\rangle := (1+|y|^2)^{1/2}$. Bearing in mind (1.7) and (3.1), we introduce the following polynomial decay assumption on $V_h$.

**Assumption 3.1.** *$V_h = \{v_h(n,m)\}_{(n,m)\in\mathbb{Z}^2}$ is of definite sign ($V_h \geq 0$) such that*

$$|v^h_{jk}(n,m)| \leq \mathrm{Const}_h \cdot G_1(j,k)\langle n\rangle^{-\nu_1}\langle m\rangle^{-\nu_2}, \quad (n,m)\in\mathbb{Z}^2, \tag{3.3}$$

*for some $\nu_1,\nu_2 > 1$, where $0 \leq G_1$ defined in $\Lambda^2$ satisfies*

$$\begin{cases} G_1 \in \mathrm{L}^\infty(\Lambda^2) \quad if \quad \dim(\mathcal{G}) < \infty, \\ G_1(j,k) \leq \mathrm{Const.}\langle j\rangle^{-\beta_1}\langle k\rangle^{-\beta_2} \quad if \quad \dim(\mathcal{G}) = +\infty, \end{cases}$$

*$(j,k)\in\Lambda^2$, for some constants $\beta_1,\beta_2 > 1$.*

Let us make some comments about Assumption 3.1.

- If $\dim(\mathcal{G}) < \infty$, then typical examples of potentials satisfying (3.3) are $V_h$ such that

$$|v^h_{jk}(n,m)| \leq \mathrm{Const.}\langle (hn,hm)\rangle^{-\nu}, \tag{3.4}$$

  $(n,m)\in\mathbb{Z}^2$, $(j,k)\in\Lambda^2$, $\nu > 2$. Indeed (3.4) implies that for every $(n,m)\in\mathbb{Z}^2$,

$$|v^h_{jk}(n,m)| \leq \mathrm{Const.}\langle hn\rangle^{-\nu/2}\langle hm\rangle^{-\nu/2} \leq \mathrm{Const.}\frac{\langle n\rangle^{-\nu/2}\langle m\rangle^{-\nu/2}}{(\min(1,h^2))^\nu}.$$

- If $\dim(\mathcal{G}) = +\infty$, then (3.3) holds for instance if for $(j,k)\in\mathbb{Z}^2_+$,

$$|v^h_{jk}(n,m)| \leq \mathrm{Const}_h \cdot \langle (j,k)\rangle^{-\beta}\langle (n,m)\rangle^{-\nu}, \tag{3.5}$$

  $(n,m)\in\mathbb{Z}^2$, $\beta > 2$, $\nu > 2$. For example, (3.5) is satisfied if $\beta > 2$, $\nu > 2$ and the operators $v_h(n,m)$ given by

$$v_h(n,m) = \langle (hn,hm)\rangle^{-\nu}\sum_{j\in\Lambda}\sum_{k\in\Lambda}\langle (j,k)\rangle^{-\beta}|\,\mathrm{e}_j\rangle\langle \mathrm{e}_k\,|.$$

In the sequel, one sets

$$\nu_0 := \min(\nu_1,\nu_2) > 1 \quad \text{and} \quad \beta_0 := \min(\beta_1,\beta_2) > 1. \tag{3.6}$$

Consider the function

$$\langle (\cdot)h^{-1}\rangle^{-\nu_0/2} : hn \in \mathbb{Z}_h \mapsto \langle n\rangle^{-\nu_0/2} \in \mathbb{R}^*_+,$$

and define in $\ell^2(\mathbb{Z}_h)$ the multiplication operator by this function denoted again by $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$. Namely, for $\phi \in \ell^2(\mathbb{Z}_h)$, $(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\phi)(hn) = \langle n\rangle^{-\nu_0/2}\phi(hn)$ or

$$\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} = \sum_{n\in\mathbb{Z}} \langle n\rangle^{-\nu_0/2} |\delta_n^h\rangle\langle\delta_n^h|. \tag{3.7}$$

Similarly, one defines $p_{\beta_0}^{\mathcal{G}}$ the operator acting in $\mathcal{G}$ by

$$p_{\beta_0}^{\mathcal{G}} := \sum_{j\in\Lambda} \langle j\rangle^{-\beta_0/2} |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,|. \tag{3.8}$$

**Remark 3.2.**

(1) *The matrix representations of* $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$ *and* $p_{\beta_0}^{\mathcal{G}}$ *are diagonal. Moreover,* $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$ *and* $p_{\beta_0}^{\mathcal{G}}$ *belong to* $\mathfrak{S}_2$ *the Hilbert-Schmidt class, since* $\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0} < \infty$ *and* $\sum_{j\in\Lambda}\langle j\rangle^{-\beta_0} < \infty$. *So, they belong to* $\mathfrak{S}_\infty$ *the class of compact linear operators.*

(2) *Of course, if* $\dim(\mathcal{G}) < \infty$, *then* $p_{\beta_0}^{\mathcal{G}} \in \mathfrak{S}_2$.

In Lemma 6.1, one proves that if $V_h$ satisfies Assumption 3.1, then there exists $\mathscr{V}_h \in \mathcal{B}(\ell^2(\mathbb{Z}_h, \mathcal{G}))$, $\mathscr{V}_h \geq 0$, such that

$$V_h = (\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes p_{\beta_0}^{\mathcal{G}})\mathscr{V}_h(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes p_{\beta_0}^{\mathcal{G}}). \tag{3.9}$$

Moreover, $V_h = \mathscr{M}_{0,h}^*\mathscr{M}_{0,h}$ is trace class with $\mathscr{M}_{0,h} = \mathscr{V}_h^{1/2}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes p_{\beta_0}^{\mathcal{G}})$ and

$$\|V_h\|_{\mathfrak{S}_1} \leq \|\mathscr{M}_{0,h}\|_{\mathfrak{S}_2}^2.$$

**Remark 3.3.** *Under Assumption 3.1, the factorisation (3.9) of* $V_h$ *is not unique and other choices can be more convenient.*

(1) *For instance, if* $\dim(\mathcal{G}) < \infty$, *one can deal in our analysis with the factorisation (6.5) introduced in the proof of Lemma 6.1.*

(2) *If* $\dim(\mathcal{G}) = +\infty$, *suppose moreover that there exists* $N > 0$ *(fixed) such that for all* $(n, m) \in \mathbb{Z}^2$, $v_{jk}^h(n, m) = 0$ *for each* $j > N$ *or* $k > N$. *That is,* $v_h(n, m)$ *is of the form*

$$v_h(n,m) = \left(\begin{array}{c|c} M_N^h(n,m) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right). \tag{3.10}$$

*Then, by arguing as in the the proof of Lemma 6.1 part b), one can replace the operator* $p_{\beta_0}^{\mathcal{G}}$ *in (3.9) by the finite rank operator*

$$\widetilde{p}_{\beta_0}^{\mathcal{G}} := \sum_{j=0}^{N} \langle j\rangle^{-\beta_0/2} |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,|. \tag{3.11}$$

So, bearing in mind (3.9) and Remark 3.3, we will consider in the sequel the perturbed operators $H_h^{\pm}$ with self-adjoint perturbations $V_h$ satisfying the next assumption which generalizes Assumption 3.1.

**Assumption 3.4.** $V_h = (\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes K^*)\mathscr{V}_h(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes K)$, $\nu_0 > 1$, *where* $0 \leq \mathscr{V}_h \in \mathcal{B}(\ell^2(\mathbb{Z}_h, \mathcal{G}))$ *and* $K$ *acting in* $\mathcal{G}$ *satisfies* $K \in \mathfrak{S}_2(\mathcal{G})$.

In order to fix ideas, let us point out some important remarks on Assumption 3.4.

**Remark 3.5.**

(1) *If* $\dim(\mathcal{G}) = +\infty$, *then Assumption 3.1 implies Assumption 3.4 with* $K = K^* = p_{\beta_0}^{\mathcal{G}}$, *according to (3.9). Since* $p_{\beta_0}^{\mathcal{G}}$ *is of infinite rank, then Assumption 3.4 includes the class of finite rank operators* $K$ *in* $\mathcal{G}$.

(2) *Under Assumption 3.4, one has*

$$V_h = \mathscr{M}_h^*\mathscr{M}_h, \qquad \mathscr{M}_h := \mathscr{V}_h^{1/2}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes K) \in \mathfrak{S}_2. \tag{3.12}$$

(3) *Our main results will be formulated under a more restrictive assumption, namely under the condition* $\nu_0 > 3$.

Potentials $V_h$ satisfying Assumption 3.4 belong to the trace class $\mathfrak{S}_1$ with

$$\|V_h\|_{\mathfrak{S}_1} \leq \|\mathscr{M}_h\|^2_{\mathfrak{S}_2} \leq \|\mathscr{V}_h\| \|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes K\|^2_{\mathfrak{S}_2}. \tag{3.13}$$

Then, (1.1) holds with $(\mathcal{H}, \mathcal{H}_0) = (H_h^{\pm}, H_h)$. By (1.2), we know that there exists a unique function $\xi(\cdot; H_h^{\pm}, H_h) \in \mathrm{L}^1(\mathbb{R})$ such that the *Lifshits-Krein trace formula*

$$\mathrm{Tr}(f(H_h^{\pm}) - f(H_h)) = \int_{\mathbb{R}} \xi(\lambda; H_h^{\pm}, H_h) f'(\lambda) d\lambda, \quad \forall\, f \in C_0^{\infty}(\mathbb{R}), \tag{3.14}$$

called the SSF for the pair $(H_h^{\pm}, H_h)$.

## 4. Spectral shift function

In this section, one recalls some abstract results due to A. Pushnitski on the representation of the spectral shift function for a pair of self-adjoint operators.

Let us define the sandwiched resolvent

$$T_h(z) := \mathscr{M}_h (H_h - z)^{-1} \mathscr{M}_h^*, \quad z \in \mathbb{C} \setminus \sigma(H_h), \tag{4.1}$$

where $\mathscr{M}_h$ is given by (3.12). Denote by

$$A_h(z) := \mathrm{Re}\, T_h(z) \quad \text{and} \quad B_h(z) := \mathrm{Im}\, T_h(z), \tag{4.2}$$

respectively the real and the imaginary parts of the operator $T_h(z)$. Then, under (3.12), it is well known that for a.e. $\lambda \in \mathbb{R}$, the limit

$$T_h(\lambda + i0) := \lim_{\varepsilon \searrow 0} T_h(\lambda + i\varepsilon), \tag{4.3}$$

exists in the $\mathfrak{S}_2$-norm (and even in the $\mathfrak{S}_p$-norm for any $p > 1$). Moreover $0 \leq B_h(\lambda + i0) \in \mathfrak{S}_1$. See [32, 2] and [22] for the case $p > 1$. Let $T = T^* \in \mathfrak{S}_{\infty}(\mathcal{G})$. Define

$$\mathscr{N}_{\pm}(r, T) := \mathrm{Rank}\, \mathbb{1}_{(r,\infty)}(\pm T), \quad r > 0, \tag{4.4}$$

the counting functions of the positive eigenvalues of the operators $\pm T$. Therefore, by [24, Theorem 1.1], one has the following result:

**Theorem 4.1.** *Let Assumption 3.4 holds. Then, for a.e. $\lambda \in \mathbb{R}$, the SSF $\xi(\cdot; H_h^{\pm}, H_h)$ admits the representation via the converging integral*

$$\xi(\lambda; H_h^{\pm}, H_h) = \pm \int_{\mathbb{R}} \mathscr{N}_{\mp}(1, A_h(\lambda + i0) + tB_h(\lambda + i0)) \frac{dt}{\pi(1 + t^2)}. \tag{4.5}$$

For further use, let us recall the following estimates, useful in the study of the convergence of the r.h.s. of (4.5).

**Lemma 4.2** (Lemma 2.1 of [24])**.** *Let $T_1 = T_1^* \in \mathfrak{S}_{\infty}(\mathcal{G})$ and $T_2 = T_2^* \in \mathfrak{S}_1(\mathcal{G})$. Then, for any $0 < x < 1$, one has*

$$\frac{1}{\pi} \int_{\mathbb{R}} \mathscr{N}_{\pm}(1, T_1 + tT_2) \frac{dt}{1 + t^2} \leq \mathscr{N}_{\pm}((1 - x), T_1) + \frac{1}{\pi x} \|T_2\|_{\mathfrak{S}_1}.$$

In Corollary 7.9, one establishes that $T_h(\lambda + i0)$ belongs to $\mathfrak{S}_1$ for every $\lambda \in \mathbb{R} \setminus \{0, \frac{4}{h^2}\}$. It follows from Lemma 4.2 that the r.h.s. of (4.5) will turn out to be well-defined for each $\lambda \in \mathbb{R} \setminus \{0, \frac{4}{h^2}\}$. So, one can consider the function $\widetilde{\xi}(\cdot; H_h^{\pm}, H_h)$ defined in $\mathbb{R} \setminus \{0, \frac{4}{h^2}\}$ by

$$\lambda \in \mathbb{R} \setminus \{0, 4/h^2\} \mapsto \widetilde{\xi}(\lambda; H_h^{\pm}, H_h) = \pm \frac{1}{\pi} \int_{\mathbb{R}} \mathscr{N}_{\mp}(1, A_h(\lambda + i0) + tB_h(\lambda + i0)) \frac{dt}{1 + t^2}. \tag{4.6}$$

By Theorem 4.1,

$$\widetilde{\xi}(\lambda; H_h^{\pm}, H_h) = \xi(\lambda; H_h^{\pm}, H_h), \quad \text{a.e. } \lambda \in \mathbb{R}.$$

Then, in the sequel, we identify these two functions. If Assumption 3.4 is fulfilled, then the potential $V_h$ is relatively compact w.r.t. $H_h$ and by Weyl's criterion on the invariance of the essential spectrum, it follows that

$$\sigma_{\mathrm{ess}}(H_h^{\pm}) = \sigma_{\mathrm{ess}}(H_h) = [0, \tfrac{4}{h^2}]. \tag{4.7}$$

However in $(-\infty,0)\cup(\frac{4}{h^2},+\infty)$, $H_h$ has no spectrum and the spectrum of $H_h^\pm$ is purely discrete. Let $\lambda_1<\lambda_2$ with $[\lambda_1,\lambda_2]\subset\mathbb{R}\setminus[0,\frac{4}{h^2}]$ and $\lambda_1,\,\lambda_2\notin\sigma(H_h^\pm)$. Then, thanks to [23, Theorem 9.1], the SSF $\xi(\cdot;H_h^\pm,H_h)$ is related to the number of eigenvalues of $H_h^\pm$ through the formula

$$\xi(\lambda_1;H_h^\pm,H_h)-\xi(\lambda_2;H_h^\pm,H_h)=\operatorname{Rank}\mathbb{1}_{[\lambda_1,\lambda_2)}(H_h^\pm). \tag{4.8}$$

## 5. Main results

5.1. **Statement of the main results.** Our first theorem is the next simple result which is an immediate by-product of (4.6), Lemma 4.2, ii) of Proposition 7.2, (8.3), (8.31), Lemma 7.6 and Weyl's inequality (8.2).

**Theorem 5.1.** *Suppose that the potential $V_h$ satisfies Assumption 3.4. Then, on compact subsets $\Gamma\subset\mathbb{R}\setminus\{0,\frac{4}{h^2}\}$, one has*

$$\sup_{\lambda\in\Gamma}\xi(\lambda;H_h^\pm,H_h)<\infty.$$

*That is, the SSF $\xi(\cdot;H_h^\pm,H_h)$ is bounded on compact subsets of $\mathbb{R}\setminus\{0,\frac{4}{h^2}\}$.*

This result will be useful in Section 5.2.

For further use in the next sections, one introduces some notations. Let $\mathrm{p}_{\nu_0}:\ell^2(\mathbb{Z}_h)\to\mathbb{C}$ be the operator defined by

$$\mathrm{p}_{\nu_0}:=\big|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\big\rangle\quad\Longrightarrow\quad \mathrm{p}_{\nu_0}^*:\zeta\in\mathbb{C}\mapsto\mathrm{p}_{\nu_0}^*\zeta=\zeta\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\in\ell^2(\mathbb{Z}_h). \tag{5.1}$$

Define the compact operator $L_{0,h}:\ell^2(\mathbb{Z}_h)\otimes\mathcal{G}\to\mathbb{C}\otimes\mathcal{G}$ by

$$L_{0,h}:=(\mathrm{p}_{\nu_0}\otimes K^*)\mathscr{V}_h^{1/2}\implies L_{0,h}^*=\mathscr{V}_h^{1/2}(\mathrm{p}_{\nu_0}^*\otimes K):\mathbb{C}\otimes\mathcal{G}\to\ell^2(\mathbb{Z}_h)\otimes\mathcal{G}. \tag{5.2}$$

Let $J$ be the self-adjoint unitary operator defined in $\ell^2(\mathbb{Z}_h)$ by

$$(J\varphi)(hn):=(-1)^n\varphi(hn). \tag{5.3}$$

Note that $J$ commutes with any multiplication operator. Moreover, it relates both thresholds through the relation $J(-\Delta_h)J^*=-\Delta_h+\frac{4}{h^2}$. As above, one introduces the compact operators

$$L_{4/h^2,h}:=(\mathrm{p}_{\nu_0}J^*\otimes K^*)\mathscr{V}_h^{1/2}:\ell^2(\mathbb{Z}_h)\otimes\mathcal{G}\to\mathbb{C}\otimes\mathcal{G}, \tag{5.4}$$

and

$$L_{4/h^2,h}^*=\mathscr{V}_h^{1/2}(J\mathrm{p}_{\nu_0}^*\otimes K):\mathbb{C}\otimes\mathcal{G}\to\ell^2(\mathbb{Z}_h)\otimes\mathcal{G}.$$

Finally, for two real-valued functionals $F_1(V,\lambda)$ and $F_2(V,\lambda)$ of $V$ depending on $\lambda\in\mathbb{R}\setminus\{0,\frac{4}{h^2}\}$, one writes

$$F_1(V,\lambda)\sim F_2(V,\lambda),\quad \lambda\to z_0\in\{0,\tfrac{4}{h^2}\},$$

if for every $\varepsilon\in(0,1)$, one has

$$F_2((1-\varepsilon)^{-1}V,\lambda)+\mathcal{O}_\varepsilon(1)\le F_1(V,\lambda)\le F_2((1+\varepsilon)^{-1}V,\lambda)+\mathcal{O}_\varepsilon(1),$$

as $\lambda\to z_0$.

5.1.1. **The case $\lambda\in(-\infty,0)\cup(\frac{4}{h^2},+\infty)$.** Our second theorem concerns the asymptotic behavior of the SSF $\xi(\lambda;H_h^\pm,H_h)$ as $\lambda\to0$ from below and as $\lambda\to\frac{4}{h^2}$ from above. Define the operators

$$\mathscr{P}:=\mathrm{p}_{\nu_0}\otimes I:\ell^2(\mathbb{Z}_h)\otimes\mathcal{G}\to\mathbb{C}\otimes\mathcal{G},$$

$\mathrm{p}_{\nu_0}$ being defined by (5.1), and

$$\mathbf{V}_h:=(I\otimes K^*)\mathscr{V}_h(I\otimes K). \tag{5.5}$$

Our results are closely related to the trace class operators

$$\mathscr{P}\mathbf{V}_h\mathscr{P}^*=L_{0,h}L_{0,h}^*\quad\text{and}\quad\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*=L_{4/h^2,h}L_{4/h^2,h}^*, \tag{5.6}$$

acting from $\mathbb{C}\otimes\mathcal{G}\to\mathbb{C}\otimes\mathcal{G}$, where

$$\mathbf{V}_h^J:=(J\otimes I)^*\mathbf{V}_h(J\otimes I), \tag{5.7}$$

and $J$ is given by (5.3). Therefore, $\mathbf{V}_h^J$ is unitarily equivalent to $\mathbf{V}_h$. Next, one sets

$$\omega_0^h(\lambda) := \frac{h}{2}\frac{\mathscr{P}\mathbf{V}_h\mathscr{P}^*}{\sqrt{|\lambda|}} \quad \text{and} \quad \omega_{4/h^2}^h(\lambda) := -\frac{h}{2}\frac{\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*}{\sqrt{|4/h^2-\lambda|}}, \qquad \lambda \in \mathbb{R}\setminus\{0, \tfrac{4}{h^2}\}. \tag{5.8}$$

The following result holds true:

**Theorem 5.2.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$. Then:*

- *As $\lambda \nearrow 0$,*

$$\xi(\lambda; H_h^+, H_h) = \mathcal{O}(1), \tag{5.9}$$

$$\xi(\lambda; H_h^-, H_h) \sim -\mathrm{Tr}\, \mathbb{1}_{(1,+\infty)}(\omega_0^h(\lambda)). \tag{5.10}$$

- *As $\lambda \searrow \frac{4}{h^2}$,*

$$-\xi(\lambda; H_h^+, H_h) \sim -\mathrm{Tr}\, \mathbb{1}_{(1,+\infty)}(-\omega_{4/h^2}^h(\lambda)), \tag{5.11}$$

$$\xi(\lambda; H_h^-, H_h) = \mathcal{O}(1). \tag{5.12}$$

**Remark 5.3.** *Inequalities (5.9) and (4.8) imply that the discrete eigenvalues of $H_h + V_h$ do not accumulate at 0 from the left. Otherwise, the inequalities (5.10) tell us that the number of eigenvalues of $H_h - V_h$ in $(-\infty, 0)$, as $\lambda \nearrow 0$, scales up to $\mathcal{O}(1)$ terms, as*

$$\mathrm{Tr}\, \mathbb{1}_{(x,+\infty)}(\omega_0^h(\lambda)) = \mathrm{Tr}\, \mathbb{1}_{(\frac{2x}{h}\sqrt{-\lambda},+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*), \quad x \approx 1.$$

*So, the problem of counting the number of eigenvalues of the operator $H_h - V_h$ near 0 from the left, is reduced to the problem of counting the number of eigenvalues of the positive trace class operator $\mathscr{P}\mathbf{V}_h\mathscr{P}^*$ near 0. The inequalities (5.11) and (5.12) lead to similar conclusions on the number of eigenvalues of the operators $H_h \pm V_h$ in $(\frac{4}{h^2}, +\infty)$ near $\frac{4}{h^2}$. In particular, the number of eigenvalues of $H_h + V_h$ in $(\frac{4}{h^2}, +\infty)$, as $\lambda \searrow \frac{4}{h^2}$, scales up to $\mathcal{O}(1)$ terms, as*

$$\mathrm{Tr}\, \mathbb{1}_{(x,+\infty)}(-\omega_{4/h^2}^h(\lambda)) = \mathrm{Tr}\, \mathbb{1}_{(\frac{2x}{h}\sqrt{\lambda-4/h^2},+\infty)}(\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*), \quad x \approx 1.$$

*Similarly, the problem of counting the number of eigenvalues of the operator $H_h + V_h$ near $\frac{4}{h^2}$ from the right, is reduced to the problem of counting the number of eigenvalues of the positive trace class operator $\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*$ near 0.*

5.1.2. **The case $\lambda \in (0, \frac{4}{h^2})$.** Our third theorem concerns the asymptotic behavior of the SSF $\xi(\lambda; H_h^{\pm}, H_h)$ as $\lambda \to 0$ from above and as $\lambda \to \frac{4}{h^2}$ from below. One needs first to introduce some notations. Set

$$g_h(\lambda) := \arcsin\left(\frac{h}{2}\sqrt{\lambda}\right), \quad \lambda \in (0, \tfrac{4}{h^2}). \tag{5.13}$$

Introduce the operators $\cos_{\nu_0,h}$, $\sin_{\nu_0,h} : \ell^2(\mathbb{Z}_h) \to \mathbb{C}$ defined by

$$\cos_{\nu_0,h}(\lambda) := \big\langle \langle (\cdot)h^{-1}\rangle^{-\nu_0/2} \cos[2(\cdot)h^{-1}g_h(\lambda)]\big|, \tag{5.14}$$

and

$$\sin_{\nu_0,h}(\lambda) := \big\langle \langle (\cdot)h^{-1}\rangle^{-\nu_0/2} \sin[2(\cdot)h^{-1}g_h(\lambda)]\big|. \tag{5.15}$$

The adjoints $\cos_{\nu_0,h}^*$, $\sin_{\nu_0,h}^* : \mathbb{C} \to \ell^2(\mathbb{Z}_h)$ are the rank one operators given by

$$\cos_{\nu_0,h}(\lambda)^*\zeta = \zeta\langle (\cdot)h^{-1}\rangle^{-\nu_0/2} \cos[2(\cdot)h^{-1}g_h(\lambda)], \tag{5.16}$$

and

$$\sin_{\nu_0,h}(\lambda)^*\zeta = \zeta\langle (\cdot)h^{-1}\rangle^{-\nu_0/2} \sin[2(\cdot)h^{-1}g_h(\lambda)]. \tag{5.17}$$

Define $Y_h(\lambda) : \ell^2(\mathbb{Z}_h) \to \mathbb{C}^2$ the operator given by

$$Y_h(\lambda)\phi = \begin{pmatrix} \cos_{\nu_0,h}(\lambda)\phi \\ \sin_{\nu_0,h}(\lambda)\phi \end{pmatrix}, \tag{5.18}$$

so that the adjoint operator $Y_h(\lambda)^* : \mathbb{C}^2 \to \ell^2(\mathbb{Z}_h)$ is given by

$$Y_h(\lambda)^* \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \cos_{\nu_0,h}(\lambda)^* \zeta_1 + \sin_{\nu_0,h}(\lambda)^* \zeta_2. \tag{5.19}$$

The next result is closely related to the trace class positive operator

$$\Omega_h(\lambda) = \frac{1}{\sqrt{\lambda}\sqrt{4/h^2 - \lambda}} (Y_h(\lambda) \otimes I) \mathbf{V}_h (Y_h(\lambda)^* \otimes I) : \mathbb{C}^2 \otimes \mathcal{G} \to \mathbb{C}^2 \otimes \mathcal{G}, \tag{5.20}$$

where $\mathbf{V}_h$ is given by (5.5).

**Theorem 5.4.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$. Then,*

$$\mp \xi(\lambda; H_h^{\pm}, H_h) \sim -\frac{1}{\pi} \operatorname{Tr} \arctan(\Omega_h(\lambda)), \tag{5.21}$$

*as $\lambda \searrow 0$ and $\lambda \nearrow \frac{4}{h^2}$.*

**Remark 5.5.** *Under the conditions of Theorem 5.4 and for $x > 0$, the operator $\Omega_h(\lambda)$ satisfies*

$$\operatorname{Tr} \arctan(x^{-1}\Omega_h(\lambda)) = \operatorname{Tr} \arctan(x^{-1}\Omega_{0,h}(\lambda)) + \mathcal{O}_h(1), \qquad \lambda \searrow 0, \tag{5.22}$$

*where for $\lambda > 0$, the operator $\Omega_{0,h}(\lambda)$ is given*

$$\Omega_{0,h}(\lambda) := \frac{h}{2\sqrt{\lambda}} (Y_{0,\nu_0} \otimes I) \mathbf{V}_h (Y_{0,\nu_0}^* \otimes I), \quad Y_{0,\nu_0} = \begin{pmatrix} \mathrm{p}_{\nu_0} \\ 0 \end{pmatrix} : \ell^2(\mathbb{Z}_h) \to \mathbb{C}^2, \tag{5.23}$$

*with $\mathrm{p}_{\nu_0}$ defined by (5.1). The estimate (5.22) follows by using the Lifshits-Krein trace formula (1.2). The details of the proof are given in Section 9 and the argument is analogous to the one of* [13, Corollary 2.2]. *Now, using (8.36), one gets*

$$\operatorname{Tr} \arctan(x^{-1}\Omega_{0,h}(\lambda)) = \operatorname{Tr} \arctan(x^{-1}\omega_0^h(\lambda)), \tag{5.24}$$

*the operator $\omega_0^h(\lambda)$ being defined by (5.8). In particular, it follows from (5.22) and (5.24) that Theorem 5.4 can be expressed as $\lambda \searrow 0$ in terms of the operator $\omega_0^h(\lambda)$, as in Theorem 5.2. In a similar way, one has for $x > 0$*

$$\operatorname{Tr} \arctan(x^{-1}\Omega_h(\lambda)) = \operatorname{Tr} \arctan(x^{-1}\Omega_{4/h^2,h}(\lambda)) + \mathcal{O}_h(1), \qquad \lambda \nearrow \frac{4}{h^2}, \tag{5.25}$$

*where the operator $\Omega_{4/h^2,h}(\lambda)$ is given*

$$\Omega_{4/h^2,h}(\lambda) := \frac{h}{2\sqrt{4/h^2 - \lambda}} (Y_{4/h^2,\nu_0} \otimes I) \mathbf{V}_h (Y_{4/h^2,\nu_0}^* \otimes I), \quad Y_{4/h^2,\nu_0} = \begin{pmatrix} \mathrm{p}_{\nu_0} J^* \\ 0 \end{pmatrix} : \ell^2(\mathbb{Z}_h) \to \mathbb{C}^2,$$

*with $J$ defined by (5.3). By using (8.36), one obtains*

$$\operatorname{Tr} \arctan(x^{-1}\Omega_{4/h^2,h}(\lambda)) = \operatorname{Tr} \arctan(-x^{-1}\omega_{4/h^2}^h(\lambda)), \tag{5.26}$$

*$\omega_{4/h^2}^h(\lambda)$ being defined by (5.8). In particular, it follows from (5.25) and (5.26) that Theorem 5.4 can be expressed as $\lambda \nearrow \frac{4}{h^2}$ in terms of the operator $\omega_{4/h^2}^h(\lambda)$, as in Theorem 5.2.*

It follows from Theorem 5.4 and Remark 5.5 the following result:

**Theorem 5.6.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$. Then,*

$$\mp \xi(\lambda; H_h^{\pm}, H_h) \sim -\frac{1}{\pi} \operatorname{Tr} \arctan(\omega_0^h(\lambda)), \quad \lambda \searrow 0,$$

*and*

$$\mp \xi(\lambda; H_h^{\pm}, H_h) \sim -\frac{1}{\pi} \operatorname{Tr} \arctan(-\omega_{4/h^2}^h(\lambda)), \quad \lambda \nearrow \frac{4}{h^2}.$$

5.2. **Corollaries.** In this section, one presents some consequences of the above results gathered in two parts. The first one concerns the case $\dim(\mathcal{G}) < \infty$ and the second one the case $\dim(\mathcal{G}) = +\infty$. We will see that in the first situation, the SSF is bounded at the spectral thresholds $\{0, \frac{4}{h^2}\}$ while it may have singularities in the second one.

5.2.1. **Boundedness of the SSF at the spectral thresholds.** Here, one assumes that $\dim(\mathcal{G}) < \infty$. Then, the operators $\mathscr{P}\mathbf{V}_h\mathscr{P}^*$ et $\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*$ acting from $\mathbb{C}\otimes\mathcal{G} \to \mathbb{C}\otimes\mathcal{G}$ are of finite rank. Otherwise, for $x > 0$ we have

$$\operatorname{Tr}\arctan(x^{-1}\omega_0^h(\lambda)) = \int_{\mathbb{R}_+} \mathscr{N}_+\big(x\sqrt{|\lambda|}t, (h/2)\mathscr{P}\mathbf{V}_h\mathscr{P}^*\big)\frac{dt}{1+t^2}, \tag{5.27}$$

$$\operatorname{Tr}\arctan(-x^{-1}\omega_{4/h^2}^h(\lambda)) = \int_{\mathbb{R}_+} \mathscr{N}_+\big(x\sqrt{|4/h^2-\lambda|}t, (h/2)\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*\big)\frac{dt}{1+t^2}. \tag{5.28}$$

Together with Theorems 5.2, 5.6, 5.1, this implies the following result:

**Theorem 5.7.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$ and $\dim(\mathcal{G}) < \infty$. Then,*

$$\sup_{\lambda\in\mathbb{R}\setminus\{0,\frac{4}{h^2}\}} \xi(\lambda; H_h^{\pm}, H_h) < \infty. \tag{5.29}$$

**Corollary 5.8.** *If $V_h$ satisfies Assumption 3.4 with $\nu_0 > 3$ and $\dim(\mathcal{G}) < \infty$, then:*

- $\sigma_{\rm ess}(H_h \pm V_h) = \sigma_{\rm ess}(H_h) = [0, \frac{4}{h^2}]$.
- *The discrete eigenvalues of the operators $H_h \pm V_h$ do not accumulate to $\{0, \frac{4}{h^2}\}$. In particular, it follows that*

$$\#\sigma_{\rm disc}(H_h \pm V_h) < \infty.$$

**Remark 5.9.** *Thanks to Lemma 6.1, it follows from Corollary 5.8 that the discrete eigenvalues of the perturbed operators $H_h \pm V_h$ do not accumulate to $\{0, \frac{4}{h^2}\}$, under Assumption 3.1 with $\nu_i > 3$, $i = 1, 2$. In particular, according to Remark 2.1, the perturbed Laplacians $-\Delta_h \pm V_h$ satisfy*

$$\#\sigma_{\rm disc}(-\Delta_h \pm V_h) < \infty. \tag{5.30}$$

*One can compare Corollary 5.8 and (5.30) with* [4, Corollary 2.1] *where we prove, for $h = 1$, the finiteness of $\sigma_{\rm disc}(-\Delta_h \pm V)$ under a more restrictive assumption, namely $V$ exponentially decaying at infinity.*

5.2.2. **Thresholds singularities and asymptotics behaviors of the SSF.** In this part, we assume that $\dim(\mathcal{G}) = +\infty$ and we start with by the following comments:

**Remark 5.10.** *If the positive trace class operators $\mathscr{P}\mathbf{V}_h\mathscr{P}^*$ and $\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*$ acting from $\mathbb{C}\otimes\mathcal{G} \to \mathbb{C}\otimes\mathcal{G}$ are of finite rank, then Theorem 5.7 and Corollary 5.8 remain valid. For instance this holds when the operator $K$ acting in $\mathcal{G}$ is finite rank. A non-trivial example is given in point (2) of Remark 3.3. In order to shorten the paper, one omits the statements of the corresponding results.*

Then, one focuses on the case where $\mathscr{P}\mathbf{V}_h\mathscr{P}^*$ and $\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*$ are of infinite rank.

– ***Case*** $\lambda \in (-\infty, 0) \cup (\frac{4}{h^2}, +\infty)$: A direct consequence of Theorem 5.2 is the following result:

**Theorem 5.11.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$, and suppose that* $\operatorname{Rank}\mathscr{P}\mathbf{V}_h\mathscr{P}^* = \infty$ *and* $\operatorname{Rank}\mathscr{P}\mathbf{V}_h^J\mathscr{P}^* = \infty$. *Then:*

- *$H_h^-$ has infinitely many discrete eigenvalues below 0. They accumulate to 0 with*

$$\xi(\lambda; H_h^-, H_h) = -\operatorname{Tr}\mathbb{1}_{(-\infty,\lambda)}(H_h^-) \sim -\operatorname{Tr}\mathbb{1}_{(\frac{2}{h}\sqrt{-\lambda},+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*), \quad \lambda \nearrow 0.$$

- *$H_h^+$ has infinitely many discrete eigenvalues above $\frac{4}{h^2}$. They accumulate to $\frac{4}{h^2}$ with*

$$-\xi(\lambda; H_h^+, H_h) = -\operatorname{Tr}\mathbb{1}_{(\lambda,+\infty)}(H_h^+) \sim -\operatorname{Tr}\mathbb{1}_{(\frac{2}{h}\sqrt{\lambda-4/h^2},+\infty)}(\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*), \quad \lambda \searrow \frac{4}{h^2}.$$

Notice that it is not difficult to construct potentials $V$ such that $\operatorname{Rank}\mathscr{P}\mathbf{V}_h\mathscr{P}^* = \infty$ and $\operatorname{Rank}\mathscr{P}\mathbf{V}_h^J\mathscr{P}^* = \infty$. But, it is more interesting to investigate cases where we can obtain a more precise description of the asymptotic behavior of $\xi(\lambda; H_h^+, H_h)$ near the spectral thresholds. So, in what follows below, one sets for $r > 0$

$$\Phi_1(r) := \operatorname{Tr}\mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) \quad \text{and} \quad \Phi_2(r) := \operatorname{Tr}\mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h^J\mathscr{P}^*). \tag{5.31}$$

Actually, under additional conditions on the functions $\Phi_i$, $i = 1, 2$, Theorems 5.11 and 5.2 produce the next more precise result giving the main terms of the asymptotics of $\xi(\lambda; H_h^{\pm}, H_h)$, $\lambda \in (-\infty, 0) \cup (\frac{4}{h^2}, +\infty)$, near the spectral thresholds.

**Corollary 5.12.** *Under the assumptions of Theorem 5.11, suppose in addition that for any $\varepsilon \in (0,1)$ small, $\Phi_i(r(1\pm\varepsilon)) = \Psi_i(r)(1+o(1)+\mathcal{O}(\varepsilon))$ as $r \searrow 0$, $i=1,2$, with $\Psi_i(r) \to +\infty$ as $r \to 0$. Then, one has the asymptotics*

$$\xi(\lambda; H_h^-, H_h) = -\mathrm{Tr}\, \mathbb{1}_{(-\infty,\lambda)}(H_h^-) = \begin{cases} -\Psi_1((2/h)\sqrt{-\lambda})(1+o(1)) & as \quad \lambda \nearrow 0, \\ \mathcal{O}(1) & as \quad \lambda \searrow \frac{4}{h^2}, \end{cases}$$

*and*

$$\xi(\lambda; H_h^+, H_h) = \mathrm{Tr}\, \mathbb{1}_{(\lambda,+\infty)}(H_h^+) = \begin{cases} \mathcal{O}(1) & as \quad \lambda \nearrow 0, \\ \Psi_2((2/h)\sqrt{\lambda - 4/h^2})(1+o(1)) & as \quad \lambda \searrow \frac{4}{h^2}. \end{cases}$$

**Remark 5.13.** *Examples of such $\Psi_i$, $i=1,2$ of Corollary 5.12 are given by*

$$\Psi_i(r) = \Psi(r) = \begin{cases} C_0 r^{-\alpha}, \quad \alpha > 0 \\ C_0 |\ln r|^{\alpha}, \quad \alpha > 0 \\ C_0 (\ln|\ln r|)^{\alpha}, \quad \alpha > 0 \\ C_0 |\ln r| (\ln|\ln r|)^{-1}, \end{cases}, \quad r > 0, \tag{5.32}$$

*where $C_0 > 0$ is a constant (see* [9, Proof of Corollary 3.11]*). For more details, one gives examples of explicit computations of $\Phi_1(r)$, $\Phi_2(r)$ and $\Psi(r)$ in Section 5.3, including polynomial and (super)-exponential decay potentials along the component $\mathcal{G}$ of $\ell^2(\mathbb{Z}_h, \mathcal{G})$ (see Propositions 5.20 and 5.21).*

Taking into account the previous remark, the next result holds.

**Corollary 5.14.** *Let $\lambda \in (-\infty, 0) \cup (\frac{4}{h^2}, +\infty)$, $z_- = 0$, $z_+ = \frac{4}{h^2}$. Suppose that $V_h$ satisfies:*

(1) *The assumptions of Proposition 5.20 with $\nu_0 > 3$. Then, one has respectively*

$$\xi(z_\pm \pm \lambda; H_h^\pm, H_h) = \pm\Big(\sum_{n\in\mathbb{Z}} \langle n\rangle^{-\nu_0}\Big)^{1/\beta_0} (2/h)^{-1/\beta_0} (\sqrt{\lambda})^{-1/\beta_0} (1+o(1)), \quad \lambda \searrow 0.$$

(2) *The assumptions of Proposition 5.21 with $\nu_0 > 3$. Then, one has w.r.t. $\pm$,*

i) *If $\xi(j) = \eta j^\beta$, $\eta > 0$ and $\beta > 0$,*

$$\xi(z_\pm \pm \lambda; H_h^\pm, H_h) = \pm(2/\eta)^{1/\beta} |\ln\sqrt{\lambda}|^{1/\beta} (1+o(1)), \quad \lambda \searrow 0.$$

ii) *If $\xi(j) = e^{\eta j^\beta}$, $\eta > 0$ and $\beta > 0$,*

$$\xi(z_\pm \pm \lambda; H_h^\pm, H_h) = \pm\eta^{-1/\beta} (\ln|\ln\sqrt{\lambda}|)^{1/\beta} (1+o(1)), \quad \lambda \searrow 0.$$

iii) *If $\xi(j) = \chi_\eta^{-1}(j)$, $\eta > 0$,*

$$\xi(z_\pm \pm \lambda; H_h^\pm, H_h) = \pm 2\eta |\ln\sqrt{\lambda}| (\ln|\ln\sqrt{\lambda}|)^{-1} (1+o(1)), \quad \lambda \searrow 0.$$

– ***Case*** $\lambda \in (0, \frac{4}{h^2})$: For $\lambda \in (0, \frac{4}{h^2})$, formulas (5.27) and (5.28) can be rewritten as

$$\mathrm{Tr}\arctan(x^{-1}\omega_0^h(\lambda)) = \int_{\mathbb{R}_+} \Phi_1\big((2/h)x\sqrt{\lambda}t\big)\frac{dt}{1+t^2}, \tag{5.33}$$

$$\mathrm{Tr}\arctan(-x^{-1}\omega_{4/h^2}^h(\lambda)) = \int_{\mathbb{R}_+} \Phi_2\big((2/h)x\sqrt{4/h^2-\lambda}t\big)\frac{dt}{1+t^2}. \tag{5.34}$$

As above, if the functions $\Phi_i$, $i=1,2$ verify some asymptotics behaviors near 0, then Theorem 5.6 together with (5.33) ans (5.34) produces the next more precise result giving the main terms of the asymptotics of $\xi(\lambda; H_h^\pm, H_h)$, $\lambda \in (0, \frac{4}{h^2})$, near the spectral thresholds.

**Theorem 5.15.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$. Suppose in addition that $\Phi_i(r) = \Psi_i(r)(1+o(1))$ as $r \searrow 0$, $i=1,2$, with $\Psi_i(r)$ given by (5.32). Set $z_1 = 0$ and $z_2 = \frac{4}{h^2}$. Then, one has the following asymptotics for $\lambda \in (0, \frac{4}{h^2})$:*

i) *If $\Psi_i(r) = C_0 r^{-\alpha}$, $0 < \alpha < 1$, then*

$$\xi(\lambda; H_h^\pm, H_h) = \pm\frac{1}{2\cos(\frac{\pi}{\alpha})} \Psi_i\big((2/h)\sqrt{|z_i - \lambda|}\big)(1+o(1)), \quad \lambda \to z_i.$$

ii) *If* $\Psi_i(r) = C_0|\ln r|^\alpha$, *or* $C_0(\ln|\ln r|)^\alpha$, *or* $C_0|\ln r|(\ln|\ln r|)^{-1}$, *then*

$$\xi(\lambda; H_h^\pm, H_h) = \pm\frac{1}{2}\Psi_i\big((2/h)\sqrt{|z_i-\lambda|}\big)(1+o(1)), \quad \lambda \to z_i.$$

**Corollary 5.16.**

(1) *Let* $V_h$ *satisfy the assumptions of Proposition 5.20 with* $\nu_0 > 3$. *Then, as* $\lambda \to z \in \{0, \frac{4}{h^2}\}$, $\lambda \in (0, \frac{4}{h^2})$,

$$\xi(\lambda; H_h^\pm, H_h) = \pm\frac{1}{2\cos(\pi\beta_0)}\Big(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}\Big)^{1/\beta_0}(2/h)^{-1/\beta_0}(\sqrt{|z-\lambda|})^{-1/\beta_0}(1+o(1)).$$

(2) *Suppose that* $V_h$ *satisfies the assumptions of Proposition 5.21 with* $\nu_0 > 3$. *Then, as* $\lambda \to z \in \{0, \frac{4}{h^2}\}$, $\lambda \in (0, \frac{4}{h^2})$, *one has:*
i) *If* $\xi(j) = \eta j^\beta$, $\eta > 0$ *and* $\beta > 0$,

$$\xi(\lambda; H_h^\pm, H_h) = \pm\frac{1}{2}(2/\eta)^{1/\beta}|\ln\sqrt{|z-\lambda|}|^{1/\beta}(1+o(1)).$$

ii) *If* $\xi(j) = e^{\eta j^\beta}$, $\eta > 0$ *and* $\beta > 0$,

$$\xi(\lambda; H_h^\pm, H_h) = \pm\frac{1}{2}\eta^{-1/\beta}\big(\ln|\ln\sqrt{|z-\lambda|}|\big)^{1/\beta}(1+o(1)).$$

iii) *If* $\xi(j) = \chi_\eta^{-1}(j)$, $\eta > 0$,

$$\xi(\lambda; H_h^\pm, H_h) = \pm\eta|\ln\sqrt{|z-\lambda|}|\big(\ln|\ln\sqrt{|z-\lambda|}|\big)^{-1}(1+o(1)).$$

**Remark 5.17.** *By (1.3), Corollary 5.15 as well as Theorem 5.15, Theorem 5.6 and Theorem 5.4 concern the asymptotics of the scattering phase* $\arg\det S(\lambda; H_h^\pm, H_h)$ *near the spectral thresholds* $\{0, \frac{4}{h^2}\}$.

5.2.3. **Levinson type formulas.** By putting together Corollary 5.12 and Theorem 5.15, one obtains the next result which can be interpreted as generalized Levinson formulae.

**Theorem 5.18.** *Under the assumptions of Theorem 5.15, i), one has*

$$\lim_{\lambda\searrow 0}\frac{\xi(\lambda; H_h^-, H_h)}{\xi(-\lambda; H_h^-, H_h)} = \frac{1}{2\cos(\frac{\pi}{\alpha})} = \lim_{\lambda\searrow 0}\frac{\xi(\frac{4}{h^2}-\lambda; H_h^+, H_h)}{\xi(\frac{4}{h^2}+\lambda; H_h^+, H_h)}, \tag{5.35}$$

*while under the assumptions of Theorem 5.15, ii), one has*

$$\lim_{\lambda\searrow 0}\frac{\xi(\lambda; H_h^-, H_h)}{\xi(-\lambda; H_h^-, H_h)} = \frac{1}{2} = \lim_{\lambda\searrow 0}\frac{\xi(\frac{4}{h^2}-\lambda; H_h^+, H_h)}{\xi(\frac{4}{h^2}+\lambda; H_h^+, H_h)}. \tag{5.36}$$

In particular, the results of Corollaries 5.14 and 5.16 give the following result:

**Corollary 5.19.**

(1) *Let* $V_h$ *satisfy the assumptions of Proposition 5.20 with* $\nu_0 > 3$. *Then,*

$$\lim_{\lambda\searrow 0}\frac{\xi(\lambda; H_h^-, H_h)}{\xi(-\lambda; H_h^-, H_h)} = \frac{1}{2\cos(\pi\beta_0)} = \lim_{\lambda\searrow 0}\frac{\xi(\frac{4}{h^2}-\lambda; H_h^+, H_h)}{\xi(\frac{4}{h^2}+\lambda; H_h^+, H_h)}. \tag{5.37}$$

(2) *Let* $V_h$ *satisfy the assumptions of Proposition 5.21 with* $\nu_0 > 3$. *Then,*

$$\lim_{\lambda\searrow 0}\frac{\xi(\lambda; H_h^-, H_h)}{\xi(-\lambda; H_h^-, H_h)} = \frac{1}{2} = \lim_{\lambda\searrow 0}\frac{\xi(\frac{4}{h^2}-\lambda; H_h^+, H_h)}{\xi(\frac{4}{h^2}+\lambda; H_h^+, H_h)}. \tag{5.38}$$

5.3. **Examples of explicit eigenvalues asymptotics.** One gives in this section examples of computations of the quantities $\Phi_1(r)$, $\Phi_2(r)$ and $\Psi(r)$ defined by (5.31) and (5.32).

5.3.1. **Polynomial decay potentials.**

**Proposition 5.20.** *Let $\nu_0 > 1$, $\beta_0 > 1$, $V_h = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\delta_n^h\rangle\langle\delta_m^h| \otimes v_h(n,m)$ such that $v_h(n,m) = 0$ if $n \neq m$ and $v_h(n,n) = \langle n\rangle^{-\nu_0} \sum_{j\in\mathbb{Z}_+} \langle j\rangle^{-\beta_0} |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,|$. Then,*

$$\Phi_1(r) = \Phi_2(r) = \operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*),$$

*and*

$$\operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) = \Big(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}\Big)^{1/\beta_0} r^{-1/\beta_0}(1+o(1)), \qquad r \searrow 0. \tag{5.39}$$

*Proof.* Clearly, $V_h$ satisfies Assumption 3.1 if $\nu_0 > 2$, $\beta_0 > 2$. Moreover, recalling that $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$ and $p^{\mathcal{G}}_{\beta_0}$ are respectively given by (3.7) and (3.8), one obtains

$$V_h = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}_+} \langle n\rangle^{-\nu_0}\langle j\rangle^{-\beta_0} |\delta_n^h \otimes \mathrm{e}_j\rangle\langle \delta_n^h \otimes \mathrm{e}_j\,| = \big(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes p^{\mathcal{G}}_{\beta_0}\big)^2.$$

Then, $V_h$ fulfills Assumption 3.4 with $K^* = K = p^{\mathcal{G}}_{\beta_0}$ and $\mathscr{V}_h = I$. It follows that

$$\mathbf{V}_h^J = \mathbf{V}_h = (I \otimes p^{\mathcal{G}}_{\beta_0})^2 = I \otimes \sum_{j\in\mathbb{Z}_+} \langle j\rangle^{-\beta_0} |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,|$$

so that $\mathscr{P}\mathbf{V}_h\mathscr{P}^* : \mathbb{C}\otimes\mathcal{G} \to \mathbb{C}\otimes\mathcal{G}$ is given by

$$\mathscr{P}\mathbf{V}_h\mathscr{P}^* = \mathrm{p}_{\nu_0}\mathrm{p}_{\nu_0}^* \otimes \sum_{j\in\mathbb{Z}_+} \langle j\rangle^{-\beta_0} |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,| = \|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)}\Big(I \otimes \sum_{j\in\mathbb{Z}_+} \langle j\rangle^{-\beta_0} |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,|\Big).$$

One can see that the eigenvalues of the operator $\mathscr{P}\mathbf{V}_h\mathscr{P}^*$ are simple and

$$\sigma(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) = \big\{\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)}\langle j\rangle^{-\beta_0} : j \in \mathbb{Z}_+\big\}, \tag{5.40}$$

with eigenvectors respectively given by the orthonormal basis $(1\otimes \mathrm{e}_j)_{j\in\mathbb{Z}_+}$ of $\mathbb{C}\otimes\mathcal{G}$. So, for $r > 0$ small enough, one has

$$\begin{aligned}
\Phi_2(r) = \Phi_1(r) &= \operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) \\
&= \#\big\{\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)}\langle j\rangle^{-\beta_0} : j \in \mathbb{Z}_+ : r < \|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)}\langle j\rangle^{-\beta_0}\big\} \\
&= \#\big\{j \in \mathbb{Z}_+ : j < \big(\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^{4/\beta_0}_{\ell^2(\mathbb{Z}_h)} r^{-2/\beta_0} - 1\big)^{1/2}\big\}.
\end{aligned}$$

By denoting $\lfloor x\rfloor$ the integer part of $x \in \mathbb{R}$, one obtains finally

$$\begin{aligned}
\Phi_2(r) = \Phi_1(r) = \operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) &\underset{r\to 0}{\sim} \big\lfloor\big(\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^{4/\beta_0}_{\ell^2(\mathbb{Z}_h)} r^{-2/\beta_0} - 1\big)^{1/2}\big\rfloor \\
&\underset{r\to 0}{\sim} \big(\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^{4/\beta_0}_{\ell^2(\mathbb{Z}_h)} r^{-2/\beta_0} - 1\big)^{1/2} \underset{r\to 0}{\sim} \|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^{2/\beta_0}_{\ell^2(\mathbb{Z}_h)} r^{-1/\beta_0}.
\end{aligned}$$

The claim follows by noting that $\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^{2/\beta_0}_{\ell^2(\mathbb{Z}_h)} = \big(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}\big)^{1/\beta_0}$. □

5.3.2. **(Super)-exponential decay potentials along the component $\mathcal{G}$ of $\ell^2(\mathbb{Z}_h,\mathcal{G})$.** Consider $\xi$ an increasing unbounded real-valued function of the form

$$\xi(x) = \begin{cases} \eta x^\beta, & \eta > 0, \quad \beta > 0 \\ e^{\eta x^\beta}, & \eta > 0, \quad \beta > 0 \\ \chi_\eta^{-1}(x), & \eta > 0 \end{cases}, \quad x > 0, \tag{5.41}$$

where $\chi_\eta^{-1}$ is the inverse of the function $y \mapsto \chi_\eta(y) = \frac{\eta y}{\ln(y+2)}, y > 0$.

**Proposition 5.21.** *Let $\nu_0 > 1$ and consider the potential $V_h = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\delta_n^h\rangle\langle\delta_m^h| \otimes v_h(n,m)$ such that $v_h(n,m) = 0$ if $n \neq m$ and $v_h(n,n) = \langle n\rangle^{-\nu_0} \sum_{j\in\mathbb{Z}_+} e^{-\xi(j)} |\,\mathrm{e}_j\rangle\langle \mathrm{e}_j\,|$. Then, one has*

$$\Phi_1(r) = \Phi_2(r) = \operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*).$$

*Moreover:*

- *If* $\xi(j) = \eta j^\beta$, $\eta > 0$ *and* $\beta > 0$,

$$\operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) = (2/\eta)^{1/\beta}|\ln r|^{1/\beta}(1+o(1)), \qquad r \searrow 0. \tag{5.42}$$

- *If* $\xi(j) = e^{\eta j^\beta}$, $\eta > 0$ *and* $\beta > 0$,

$$\operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) = \eta^{-1/\beta}(\ln|\ln r|)^{1/\beta}(1+o(1)), \qquad r \searrow 0. \tag{5.43}$$

- *If* $\xi(j) = \chi_\eta^{-1}(j)$, $\eta > 0$,

$$\operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) = 2\eta|\ln r|(\ln|\ln r|)^{-1}(1+o(1)), \qquad r \searrow 0. \tag{5.44}$$

*Proof.* It can be checked, making the change of variable $x = \chi(y)$, that

$$\lim_{x\to+\infty} \frac{\chi^{-1}(x)}{x\ln(x)} = \frac{1}{\eta}.$$

Hence, $V_h$ satisfies Assumption 3.1 if $\nu_0 > 2$ and one has

$$V_h = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}_+} \langle n\rangle^{-\nu_0} e^{-\xi(j)} |\delta_n^h \otimes \mathrm{e}_j\rangle\langle \delta_n^h \otimes \mathrm{e}_j| = \big(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes K\big)^2,$$

where $K = \sum_{j\in\mathbb{Z}_+} e^{-\frac{1}{2}\xi(j)} |\mathrm{e}_j\rangle\langle \mathrm{e}_j|$. Thus, $V_h$ satisfies Assumption 3.4 with $\mathscr{V}_h = I$ and

$$\mathscr{P}\mathbf{V}_h^J\mathscr{P}^* = \mathscr{P}\mathbf{V}_h\mathscr{P}^* = \|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)}\Big(I \otimes \sum_{j\in\mathbb{Z}_+} e^{-\frac{1}{2}\xi(j)} |\mathrm{e}_j\rangle\langle \mathrm{e}_j|\Big).$$

The eigenvalues of the operator $\mathscr{P}\mathbf{V}_h\mathscr{P}^*$ are simple and

$$\sigma(\mathscr{P}\mathbf{V}_h\mathscr{P}^*) = \big\{\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)} e^{-\frac{1}{2}\xi(j)} : j \in \mathbb{Z}_+\big\}, \tag{5.45}$$

with eigenvectors respectively given by the orthonormal basis $(1\otimes \mathrm{e}_j)_{j\in\mathbb{Z}_+}$ of $\mathbb{C}\otimes\mathcal{G}$. Therefore, for $r > 0$ small enough, one has

$$\begin{aligned}\Phi_2(r) &= \Phi_1(r) = \operatorname{Tr} \mathbb{1}_{(r,+\infty)}(\mathscr{P}\mathbf{V}_h\mathscr{P}^*)\\ &= \#\big\{\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)} e^{-\frac{1}{2}\xi(j)} : j\in\mathbb{Z}_+ : r < \|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)} e^{-\frac{1}{2}\xi(j)}\big\}\\ &= \#\big\{j\in\mathbb{Z}_+ : \xi(j) < 2\ln\big(\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\ell^2(\mathbb{Z}_h)} r^{-1}\big)\big\}.\end{aligned} \tag{5.46}$$

The claim follows from (5.46) and (5.41). □

## 6. Factorisation of the potential

The aim of this section is to prove the next lemma that gives a suitable factorisation of the perturbation $V_h$ satisfying Assumption 3.1. In particular, this justifies our choice of the generalized Assumption 3.4.

**Lemma 6.1.** *Let Assumption 3.1 holds. Consider the operators* $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$ *and* $p^{\mathcal{G}}_{\beta_0}$ *defined by* (3.7) *and* (3.8). *Then:*

i) *There exists* $\mathscr{V}_h \in \mathcal{B}(\ell^2(\mathbb{Z}_h,\mathcal{G}))$ *such that*

$$V_h = (\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes p^{\mathcal{G}}_{\beta_0})\mathscr{V}_h(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes p^{\mathcal{G}}_{\beta_0}). \tag{6.1}$$

ii) $\mathscr{V}_h \geq 0$ *so that*

$$V_h = \mathscr{M}_{0,h}^*\mathscr{M}_{0,h} \quad \text{with} \quad \mathscr{M}_{0,h} := \mathscr{V}_h^{1/2}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes p^{\mathcal{G}}_{\beta_0}). \tag{6.2}$$

*In particular* $V_h$ *is trace class and*

$$\|V_h\|_{\mathfrak{S}_1} \leq \|\mathscr{M}_{0,h}\|^2_{\mathfrak{S}_2}. \tag{6.3}$$

iii) *As a matrix,* $\mathscr{V}_h = \{a_h(n,m)\}_{(n,m)\in\mathbb{Z}^2}$ *with*

$$\|a_h(n,m)\|_{\mathcal{B}(\mathcal{G})} \leq C_h\langle n\rangle^{-\nu_1+\nu_0/2}\langle m\rangle^{-\nu_2+\nu_0/2}. \tag{6.4}$$

*Proof.* i) Constants are generic, i.e. change from an estimate to another. One can write

$$V_h = (\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes I)\widetilde{V}_h(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes I), \tag{6.5}$$

with $\widetilde{V}_h := \big\{\langle n\rangle^{\nu_0/2} v_h(n,m)\langle m\rangle^{\nu_0/2}\big\}_{(n,m)\in\mathbb{Z}^2}$. Namely, one has

$$\widetilde{V}_h = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\delta_n^h\rangle\langle\delta_m^h| \otimes \langle n\rangle^{\nu_0/2} v_h(n,m)\langle m\rangle^{\nu_0/2}. \tag{6.6}$$

a) Firstly let us show that the operator $\widetilde{V}_h$ is bounded. To see this, note that Assumption 3.1 and (3.2) imply that there exits a constant $C_h$ such that for each $(n,m)\in\mathbb{Z}^2$,

$$\|v_h(n,m)\|_{\mathcal{B}(\mathcal{G})} \leq C_h\langle n\rangle^{-\nu_1}\langle m\rangle^{-\nu_2}. \tag{6.7}$$

Using (6.6) and the Cauchy-Schwartz inequality, one gets for any $\varphi\in\ell^2(\mathbb{Z}_h,\mathcal{G})$

$$\begin{aligned}\|\widetilde{V}_h\varphi\|^2 &= \sum_{n\in\mathbb{Z}} \Big\| \sum_{m\in\mathbb{Z}} \langle n\rangle^{\nu_0/2} v_h(n,m)\langle m\rangle^{\nu_0/2}\varphi(hm)\Big\|_{\mathcal{G}}^2 \\ &\leq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \|\langle n\rangle^{\nu_0/2} v_h(n,m)\langle m\rangle^{\nu_0/2}\|_{\mathcal{B}(\mathcal{G})}^2\|\varphi\|^2.\end{aligned} \tag{6.8}$$

It follows from (6.7) that

$$\|\widetilde{V}_h\|^2 \leq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \|\langle n\rangle^{\nu_0/2} v_h(n,m)\langle m\rangle^{\nu_0/2}\|_{\mathcal{B}(\mathcal{G})}^2 \leq C_h\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\langle n\rangle^{-2\nu_1+\nu_0}\langle m\rangle^{-2\nu_2+\nu_0} < \infty.$$

b) For $(n,m)\in\mathbb{Z}^2$, define in $\mathcal{G}$ the operator

$$\tilde{v}_h(n,m) = \sum_{j\in\Lambda}\sum_{k\in\Lambda}\langle j\rangle^{\beta_0/2} v_{jk}^h(n,m)\langle k\rangle^{\beta_0/2}|\,\mathrm{e}_j\rangle\langle\mathrm{e}_k\,|, \tag{6.9}$$

which is bounded. Indeed, using the Cauchy-Schwartz inequality one gets for each $g\in\mathcal{G}$

$$\begin{aligned}&\|\tilde{v}_h(n,m)g\|_{\mathcal{G}}^2 = \sum_{j\in\Lambda}\big|\langle\mathrm{e}_j,\tilde{v}_h(n,m)g\rangle_{\mathcal{G}}\big|^2 = \sum_{j\in\Lambda}\Big|\sum_{k\in\Lambda}\langle j\rangle^{\beta_0/2} v_{jk}^h(n,m)\langle k\rangle^{\beta_0/2}\langle\mathrm{e}_k,g\rangle_{\mathcal{G}}\Big|^2 \\ &\leq \sum_{j\in\Lambda}\Big(\sum_{k\in\Lambda}|\langle j\rangle^{\beta_0/2} v_{jk}^h(n,m)\langle k\rangle^{\beta_0/2}|^2\Big)\sum_{k\in\Lambda}|\langle\mathrm{e}_k,g\rangle_{\mathcal{G}}|^2 = \|g\|_{\mathcal{G}}^2\sum_{j,k}\langle j\rangle^{\beta_0}|v_{jk}^h(n,m)|^2\langle k\rangle^{\beta_0} \\ &\leq C_h\langle n\rangle^{-2\nu_1}\langle m\rangle^{-2\nu_2}\|g\|_{\mathcal{G}}^2\sum_{j\in\Lambda}\sum_{k\in\Lambda}\langle j\rangle^{\beta_0}G_1^2(i,j)\langle k\rangle^{\beta_0} \leq C_h\langle n\rangle^{-2\nu_1}\langle m\rangle^{-2\nu_2}\|g\|_{\mathcal{G}}^2.\end{aligned}$$

It follows that

$$\|\tilde{v}_h(n,m)\|_{\mathcal{B}(\mathcal{G})} \leq C_h\langle n\rangle^{-\nu_1}\langle m\rangle^{-\nu_2}. \tag{6.10}$$

Now, for every $k_*\in\Lambda$, one has

$$\begin{aligned}\big(p_{\beta_0}^{\mathcal{G}}\tilde{v}_h(n,m)p_{\beta_0}^{\mathcal{G}}\big)\,\mathrm{e}_{k_*} &= \sum_{j\in\Lambda}\langle j\rangle^{-\beta_0/2}|\,\mathrm{e}_j\rangle\langle\mathrm{e}_j,\tilde{v}_h(n,m)p_{\beta_0}^{\mathcal{G}}\,\mathrm{e}_{k_*}\rangle_{\mathcal{G}} \\ &= \sum_{j\in\Lambda}\langle j\rangle^{-\beta_0/2}\langle k_*\rangle^{-\beta_0/2}|\,\mathrm{e}_j\rangle\langle\mathrm{e}_j,\tilde{v}_h(n,m)\,\mathrm{e}_{k_*}\rangle_{\mathcal{G}}.\end{aligned}$$

Since

$$\begin{aligned}\langle\mathrm{e}_j,\tilde{v}_h(n,m)\,\mathrm{e}_{k_*}\rangle_{\mathcal{G}} &= \sum_{j'\in\Lambda}\sum_{k\in\Lambda}\langle j'\rangle^{\beta_0/2} v_{j'k}^h(n,m)\langle k\rangle^{\beta_0/2}\langle\mathrm{e}_j,\mathrm{e}_{j'}\rangle_{\mathcal{G}}\langle\mathrm{e}_k,\mathrm{e}_{k_*}\rangle_{\mathcal{G}} \\ &= \langle j\rangle^{\beta_0/2} v_{jk_*}^h(n,m)\langle k_*\rangle^{\beta_0/2},\end{aligned}$$

then we have

$$\big(p_{\beta_0}^{\mathcal{G}}\tilde{v}_h(n,m)p_{\beta_0}^{\mathcal{G}}\big)\,\mathrm{e}_{k_*} = \sum_{j\in\Lambda} v_{jk_*}^h(n,m)|\,\mathrm{e}_j\rangle = v_h(n,m)\,\mathrm{e}_{k_*},$$

Therefore, for each $(n,m)\in\mathbb{Z}^2$, one has

$$p_{\beta_0}^{\mathcal{G}}\tilde{v}_h(n,m)p_{\beta_0}^{\mathcal{G}} = v_h(n,m). \tag{6.11}$$

Together with (6.6), this implies that

$$\widetilde{V}_h = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\delta_n^h\rangle\langle\delta_m^h| \otimes \langle n\rangle^{\nu_0/2} p_{\beta_0}^{\mathcal{G}} \tilde{v}_h(n,m) p_{\beta_0}^{\mathcal{G}} \langle m\rangle^{\nu_0/2} = (I\otimes p_{\beta_0}^{\mathcal{G}})\mathscr{V}_h(I\otimes p_{\beta_0}^{\mathcal{G}}), \tag{6.12}$$

where

$$\mathscr{V}_h = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\delta_n^h\rangle\langle\delta_m^h| \otimes \langle n\rangle^{\nu_0/2} \tilde{v}(n,m) \langle m\rangle^{\nu_0/2}. \tag{6.13}$$

By putting together (6.5) and (6.12), one obtains (6.1). Using (6.10) and arguing as in (6.8), one can show that $\mathscr{V}_h$ is a bounded operator.

ii) Let us show that $\mathscr{V}_h \geq 0$ if $V_h \geq 0$. First, note that the vectors of the basis $(\delta_n^h \otimes e_j)_{(n,j)\in\mathbb{Z}\times\Lambda}$ of $\ell^2(\mathbb{Z}_h,\mathcal{G})$ are eigenvectors of the operator $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}}$. Indeed, one has

$$(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}})(\delta_n^h\otimes \mathrm{e}_j) = \langle n\rangle^{-\nu_0/2}\langle j\rangle^{-\nu_0/2}(\delta_n^h\otimes \mathrm{e}_j), \qquad (n,j)\in\mathbb{Z}\times\Lambda.$$

This implies that the Range of $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}}$ is dense in $\ell^2(\mathbb{Z}_h,\mathcal{G})$. Then, for each $\varphi\in\ell^2(\mathbb{Z}_h,\mathcal{G})$, there exists a sequence of vectors $\varphi_q\in \mathrm{Range}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}})$, $q\geq 0$, such that

$$\lim_{q\to+\infty}\|\varphi_q-\varphi\|_{\mathcal{G}} = 0. \tag{6.14}$$

So, for each $q\geq 0$, $\varphi_q = (\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}})\varphi_q'$ for some $\varphi_q'\in\ell^2(\mathbb{Z}_h,\mathcal{G})$. Noting that $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}}$ is a positive operator and using (6.1), it follows that

$$\begin{aligned}\langle\mathscr{V}_h\varphi_q,\varphi_q\rangle &= \langle\mathscr{V}_h(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}})\varphi_q', (\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}})\varphi_q'\rangle \\ &= \langle(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}})\mathscr{V}_h(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}})\varphi_q', \varphi_q'\rangle = \langle V_h\varphi_q',\varphi_q'\rangle \geq 0.\end{aligned} \tag{6.15}$$

Now, since for $q\geq 0$

$$\begin{aligned}|\langle\mathscr{V}_h\varphi_q,\varphi_q\rangle - \langle\mathscr{V}_h\varphi,\varphi\rangle| &= |\langle\mathscr{V}_h(\varphi_q-\varphi),\varphi_q\rangle + \langle\mathscr{V}_h\varphi,\varphi_q-\varphi\rangle| \\ &\leq \|\mathscr{V}_h\|\|\varphi_q-\varphi\|\|\varphi_q\| + \|\mathscr{V}_h\|\|\varphi\|\|\varphi_q-\varphi\|,\end{aligned}$$

one deduces from (6.14) and (6.15) that

$$\langle\mathscr{V}_h\varphi,\varphi\rangle = \lim_{q\to+\infty}\langle\mathscr{V}_h\varphi_q,\varphi_q\rangle \geq 0.$$

Moreover, since the operator $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes p_{\beta_0}^{\mathcal{G}}$ is Hilbert-Schmidt according to Remark 3.2, then $V_h$ is trace class and (6.3) follows by (6.1) and the boundedness of $\mathscr{V}_h$.

iii) As matrix $\mathscr{V}_h = \{a_h(n,m)\}_{(n,m)\in\mathbb{Z}^2}$, one has from (6.13) that for each $(n,m)\in\mathbb{Z}^2$

$$a_h(n,m) = \langle n\rangle^{\nu_0/2}\tilde{v}_h(n,m)\langle m\rangle^{\nu_0/2}.$$

Then, (6.4) follows immediately from (6.10). This concludes the proof. □

## 7. Preliminary results

7.1. **Extensions of the kernel of $(-\Delta_h - z)^{-1}$ to the real axis.** For further references, we provide more details on our choice of analytic determinations of the complex logarithm and square-root functions. First, it can be checked that the map $\exp : s\in\mathbb{C}\mapsto e^s\in\mathbb{C}^*$ is a surjective group homomorphism with kernel $\ker(\exp) = 2i\pi\mathbb{Z}$. It follows that its restriction $\exp : s\in\{s\in\mathbb{C} : -\pi\leq \mathrm{Im}\, s<\pi\}\mapsto e^s\in\mathbb{C}^*$ is a bijective map. Since the image of the axis $\{s\in\mathbb{C} : \mathrm{Im}\, s = -\pi\}$ is the semi-axis $(-\infty,0)$, then the one of the (open) domain $\{s\in\mathbb{C} : -\pi<\mathrm{Im}\, s<\pi\}$ is the domain $\mathbb{C}\setminus(-\infty,0]$ so that

$$\exp : s\in\{s\in\mathbb{C} : -\pi<\mathrm{Im}\, s<\pi\}\mapsto e^s\in\mathbb{C}\setminus(-\infty,0]$$

is a holomorphic bijective map with non-vanishing derivative. The corresponding inverse map

$$\mathrm{Ln} : s\in\mathbb{C}\setminus(-\infty,0]\mapsto \mathrm{Ln}(s)\in\{s\in\mathbb{C} : -\pi<\mathrm{Im}\, s<\pi\} \tag{7.1}$$

is then holomorphic and will define our complex logarithm determination. It corresponds to the principal value of the logarithm function. Hence, one can define the complex analytic square-root determination using the analytic function Ln by

$$\sqrt{\cdot} = e^{\frac{1}{2}\mathrm{Ln}} : s \in \mathbb{C} \setminus (-\infty, 0] \mapsto e^{\frac{1}{2}\mathrm{Ln}(s)} \in \{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}. \tag{7.2}$$

It corresponds to the principal value of the square-root function. Note that (7.1) and (7.2) corresponds to employ the principal value of the argument Arg which takes values in $(-\pi, \pi]$ so that

$$\mathrm{Ln}(s) = \mathrm{Ln}|s| + i\mathrm{Arg}(s). \tag{7.3}$$

Let $z \in \mathbb{C} \setminus [0, \frac{4}{h^2}]$ and $R_h(z) := (-\Delta_h - z)^{-1}$ be the resolvent of $-\Delta_h$. One has

$$\mathscr{F}_h\big(R_h(z)\phi\big)(\theta) = \frac{(\mathscr{F}_h\phi)(\theta)}{f_h(\theta) - z}, \tag{7.4}$$

where $f_h(\theta)$ is given by (2.2). It can be checked that

$$\mathscr{F}_h(h^2 e^{i\alpha|\cdot|})(\theta) = -\frac{i}{2h^{-2}} \frac{\sin(h\alpha)}{\sin^2(h\theta/2) - \sin^2(h\alpha/2)}, \quad \mathrm{Im}\,\alpha > 0. \tag{7.5}$$

It follows from identities (7.4) and (7.5) that for $\mathrm{Im}\, z > 0$, $R_h(z)$ is an operator with convolution kernel given by $r(z, h(n-m))$, where

$$r(z, hk) = \frac{ih^2}{2} \frac{e^{i\alpha_h(z)h|k|}}{\sin(h\alpha_h(z))} = \frac{ie^{2i|k|\mathrm{Arcsin}(\frac{h}{2}\sqrt{z})}}{\sqrt{z}\sqrt{4/h^2 - z}} := r_h(z, k), \quad k \in \mathbb{Z}. \tag{7.6}$$

Here, $\alpha_h(z) = \frac{2}{h}\mathrm{Arcsin}(\frac{h}{2}\sqrt{z})$ is the unique solution to the equation $\frac{2-2\cos(h\alpha)}{h^2} = \frac{4}{h^2}\sin^2\left(\frac{h\alpha}{2}\right) = z$ lying in the region $\{\alpha \in \mathbb{C} : -\frac{\pi}{h} \leq \mathrm{Re}\,\alpha \leq \frac{\pi}{h} : \mathrm{Im}\,\alpha > 0\}$, where Arcsin is principal value of the real arcsine (arcsin) function obtained by employing the above analytic determinations Ln and $\sqrt{\cdot} = e^{\frac{1}{2}\mathrm{Ln}}$. Namely, for $s \in \mathbb{C} \setminus \big((-\infty, -1] \cup [1, +\infty)\big)$, one has

$$\mathrm{Arcsin}\, s = \frac{1}{i}\mathrm{Ln}\big(is + \sqrt{1 - s^2}\big) = w. \tag{7.7}$$

It can be easily checked that $w$ given by (7.7) is solution to the equation $\sin(w) = \frac{e^{iw} - e^{-iw}}{2i} = s$. In particular, if $s = x \in \mathbb{R}$ with $|x| < 1$, then

$$\mathrm{Arcsin}\, x = \mathrm{Arg}\big(ix + \sqrt{1 - x^2}\big) \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

coincides with the real classical arcsin inverse fonction, i.e. $\mathrm{Arcsin}\, x = \arcsin x$.

The next result follows immediately taking into account the above considerations.

**Proposition 7.1.** *One has*

$$\lim_{\varepsilon \searrow 0} r_h(\lambda + i\varepsilon, n-m) = \begin{cases} \frac{e^{2i|n-m|\mathrm{Arcsin}(\frac{ih}{2}\sqrt{-\lambda})}}{\sqrt{-\lambda}\sqrt{4/h^2 - \lambda}} & if \quad \lambda < 0 \\ \frac{ie^{2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}}{\sqrt{\lambda}\sqrt{4/h^2 - \lambda}} & if \quad \lambda \in (0, 4/h^2) \\ -\frac{e^{2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}}{\sqrt{\lambda}\sqrt{\lambda - 4/h^2}} & if \quad \lambda > 4/h^2. \end{cases} \tag{7.8}$$

For $\lambda \in \mathbb{R} \setminus \{0, \frac{4}{h^2}\}$, one defines $R_h(\lambda)$ as the operator acting in $\ell^2(\mathbb{Z}_h)$ with the convolution kernel $r_h(\lambda, n-m)$ where

$$r_h(\lambda, n-m) := \lim_{\varepsilon \searrow 0} r_h(\lambda + i\varepsilon, n-m). \tag{7.9}$$

## 7.2. **Estimates of the weighted resolvents.**

7.2.1. **Hilbert-Schmidt bounds.** Let $T_h(z)$ be the weighted resolvent defined by (4.1). For $\operatorname{Im} z > 0$, thanks to (7.6), one defines $R_h(z)$ as the operator acting in $\ell^2(\mathbb{Z}_h)$ with convolution kernel $r_h(z, n-m)$. So, according to (7.9), one extends $T_h(z)$ to $\mathbb{C} \setminus \{0, \frac{4}{h^2}\}$ by setting

$$T_h(\lambda) := \mathscr{V}_h^{1/2}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} R_h(\lambda) \langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes KK^*)\mathscr{V}_h^{1/2}, \quad \lambda \in (0, \tfrac{4}{h^2}). \tag{7.10}$$

Introduce $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\overline{\mathbb{C}^+} := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$.

**Proposition 7.2.** *Let $V_h$ satisfy Assumption 3.4. Then:*

i) *For any $z \in \mathbb{C} \setminus \{0, \frac{4}{h^2}\}$, the operator $T_h(z) \in \mathfrak{S}_2(\ell^2(\mathbb{Z}_h, \mathcal{G}))$ with*

$$\|T_h(\lambda)\|_{\mathfrak{S}_2} \leq \frac{\|K\|_{\mathfrak{S}_2}^2 \sum_{n\in\mathbb{Z}} \langle n\rangle^{-\nu_0}}{\lambda^{1/2}(\frac{4}{h^2} - \lambda)^{1/2}} \|\mathscr{V}_h\|, \quad \lambda \in (0, \tfrac{4}{h^2}).$$

ii) *The operator-valued function $z \in \overline{\mathbb{C}^+} \setminus \{0, \frac{4}{h^2}\} \mapsto T_h(z) \in \mathfrak{S}_2$ is continuous.*

*Proof.* i) Let $z \in \mathbb{C} \setminus \sigma(H_h)$. Then, the resolvent $(H_h - z)^{-1}$ of $H_h$ is bounded with

$$\|(H_h - z)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, [0, \frac{4}{h^2}])},$$

by the spectral theorem. Thanks to Remark 3.2, the operator $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \in \mathfrak{S}_2$ and by Assumption 3.4, $K \in \mathfrak{S}_2$. It follows that $T_h(z) \in \mathfrak{S}_2$ with

$$\|T_h(z)\|_{\mathfrak{S}_2} \leq \|\mathscr{M}_h\|_{\mathfrak{S}_2}^2 \|(H_h - z)^{-1}\| \leq \frac{\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|_{\mathfrak{S}_2}^2 \|K\|_{\mathfrak{S}_2}^2}{\operatorname{dist}(z, [0, \frac{4}{h^2}])} \|\mathscr{V}_h\|.$$

Let $\lambda \in (0, \frac{4}{h^2})$. The operator $R_h(\lambda)$ admits the convolution kernel $r_h(\lambda, n-m)$ given by (7.9). Using Proposition 7.1, we bound the $\mathfrak{S}_2$-norm of $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} R_h(z) \langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$ by

$$\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} R_h(\lambda) \langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|_{\mathfrak{S}_2} \leq \frac{\sum_{n\in\mathbb{Z}} \langle n\rangle^{-\nu_0}}{\lambda^{1/2}(\frac{4}{h^2} - \lambda)^{1/2}}.$$

Therefore, it follows that

$$\|T_h(\lambda)\|_{\mathfrak{S}_2} \leq \frac{\|K\|_{\mathfrak{S}_2}^2 \sum_{n\in\mathbb{Z}} \langle n\rangle^{-\nu_0}}{\lambda^{1/2}(\frac{4}{h^2} - \lambda)^{1/2}} \|\mathscr{V}_h\|.$$

ii) According to the point i), the map $z \in \overline{\mathbb{C}^+} \setminus \{0, \frac{4}{h^2}\} \mapsto T_h(z) \in \mathfrak{S}_2$ is well defined. Otherwise, since the map $z \mapsto (H_h - z)^{-1}$ is holomorphic in $\mathbb{C} \setminus \sigma(H_h)$, then the continuity of $z \in \overline{\mathbb{C}^+} \setminus \sigma(H_h) \mapsto T_h(z) \in \mathfrak{S}_2$ follows immediately. Indeed, as $|z - z_0| \to 0$ with $z$, $z_0 \in \overline{\mathbb{C}^+} \setminus \sigma(H_h)$,

$$\begin{aligned} \|T_h(z) - T_h(z_0)\|_{\mathfrak{S}_2} &= \|\mathscr{M}_h(H_h - z)^{-1}\mathscr{M}_h^* - \mathscr{M}_h(H_h - z_0)^{-1}\mathscr{M}_h^*\|_{\mathfrak{S}_2} \\ &\leq \|\mathscr{M}_h\|_{\mathfrak{S}_2}^2 \|(H_h - z)^{-1} - (H_h - z_0)^{-1}\| \to 0. \end{aligned}$$

Now, let $z_0 = \lambda_0 \in (0, \frac{4}{h^2})$ and $0 < \delta \ll 1$. Then, for

$$z \in D_\delta(\lambda_0) := \{s \in \overline{\mathbb{C}^+} \setminus \{0, \tfrac{4}{h^2}\} : |z - \lambda_0| \leq \delta\},$$

one has

$$\|T_h(z) - T_h(\lambda_0)\|_{\mathfrak{S}_2} \leq \|\mathscr{V}_h\| \|K\|_{\mathfrak{S}_2}^2 \|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \big[R_h(z) - R_h(\lambda_0)\big] \langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|_{\mathfrak{S}_2}. \tag{7.11}$$

The operator $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \big[R_h(z) - R_h(\lambda_0)\big] \langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$ admits the convolution kernel

$$\langle n\rangle^{-\nu_0/2} \big(r_h(z, n-m) - r_h(\lambda_0, n-m)\big) \langle m\rangle^{-\nu_0/2},$$

so that

$$\begin{aligned} &\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \big[R_h(z) - R_h(\lambda_0)\big] \langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|_{\mathfrak{S}_2}^2 \\ &\qquad \leq \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle n\rangle^{-\nu_0} \langle m\rangle^{-\nu_0} |r_h(z, n-m) - r_h(\lambda_0, n-m)|^2. \end{aligned} \tag{7.12}$$

The map $z \in D_\delta(\lambda_0) \mapsto \frac{1}{|z|^{1/2}|\frac{4}{h^2}-z|^{1/2}} \in \mathbb{R}$ is continuous. Since $D_\delta(\lambda_0)$ is compact, then there exists $s_0 \in D_\delta(\lambda_0)$ such that

$$\sup_{\substack{z\in D_\delta(\lambda_0)\\(n,m)\in\mathbb{Z}^2}} |r_h(z,n-m)-r_h(\lambda_0,n-m)| \leq \frac{1}{|s_0|^{1/2}|\frac{4}{h^2}-s_0|^{1/2}} + \frac{1}{\lambda_0^{1/2}(\frac{4}{h^2}-\lambda_0)^{1/2}} =: C_h(s_0,\lambda_0).$$

That is the map $(n,m)\in\mathbb{Z}^2 \mapsto \langle n\rangle^{-\nu_0}\langle m\rangle^{-\nu_0}|r_h(z,n-m)-r_h(\lambda_0,n-m)|$ is uniformly dominated w.r.t $z\in D_\delta(\lambda_0)$ by the map $(n,m)\in\mathbb{Z}^2 \mapsto C_h(s_0,\lambda_0)\langle n\rangle^{-\nu_0}\langle m\rangle^{-\nu_0}$. Now, using

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\langle n\rangle^{-\nu_0}\langle m\rangle^{-\nu_0} = \Big(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}\Big)^2 < \infty,$$

(7.12), Lebesgue's dominated convergence theorem and (7.11), one gets

$$\|T_h(z)-T_h(\lambda_0)\|_{\mathfrak{S}_2} \to 0 \quad \text{as} \quad |z-\lambda_0|\to 0,$$

which ends the proof. □

The next corollary is a direct consequence of Proposition 7.2.

**Corollary 7.3.** *Let $V_h$ satisfy Assumption 3.4 and $\lambda \in \mathbb{R}\setminus\{0,\frac{4}{h^2}\}$. Then, the limit $T_h(\lambda+i0)$ exists in $\mathfrak{S}_2$ with $T_h(\lambda+i0)=T_h(\lambda)$.*

7.2.2. **Trace class bounds.** In what follows below, we want to establish the existence of $T_h(\lambda+i0)$, $\lambda\in\mathbb{R}\setminus\{0,\frac{4}{h^2}\}$, in the trace class $\mathfrak{S}_1$. However, the proof is less evident to the one of Corollary 7.3 obtained directly from Proposition 7.2. To do this, the first step consists of establishing the following simple result, whose proof is similar to the one of Proposition 7.2 in several points.

**Proposition 7.4.** *Let $V_h$ satisfy Assumption 3.4. Then:*

i) *For any $z\in\mathbb{C}\setminus\sigma(H_h)$, the operator $T_h(z)\in\mathfrak{S}_1(\ell^2(\mathbb{Z}_h,\mathcal{G}))$ with*

$$\|T_h(z)\|_{\mathfrak{S}_1} \leq \frac{\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\mathfrak{S}_2}\|K\|^2_{\mathfrak{S}_2}}{\operatorname{dist}(z,[0,\frac{4}{h^2}])}\|\mathscr{V}_h\|.$$

ii) *The operator-valued function $z\in\mathbb{C}\setminus\sigma(H_h)\mapsto T_h(z)\in\mathfrak{S}_1$ is holomorphic.*

*Proof.* i) Let $z\in\mathbb{C}\setminus\sigma(H_h)$. Since the operators $\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}$, $K$ are Hilbert-Schmidt and $(H_h-z)^{-1}$ is bounded, then $T_h(z)\in\mathfrak{S}_1$ with

$$\|T_h(z)\|_{\mathfrak{S}_1} \leq \|\mathscr{M}_h\|^2_{\mathfrak{S}_2}\|(H_h-z)^{-1}\| \leq \frac{\|\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\|^2_{\mathfrak{S}_2}\|K\|^2_{\mathfrak{S}_2}}{\operatorname{dist}(z,[0,\frac{4}{h^2}])}\|\mathscr{V}_h\|.$$

ii) Thanks to the point i), the map $z\in\mathbb{C}\setminus\sigma(H_h)\mapsto T_h(z)\in\mathfrak{S}_1$ is well defined. Moreover,

$$\|T_h'(z)\|_{\mathfrak{S}_1} = \|\mathscr{M}_h(H_h-z)^{-2}\mathscr{M}_h^*\|_{\mathfrak{S}_1} \leq \|\mathscr{M}_h(H_h-z)^{-1}\|_{\mathfrak{S}_2}\|(H_h-z)^{-1}\mathscr{M}_h^*\|_{\mathfrak{S}_2},$$

and as $|z-z_0|\to 0$ with $z,z_0\in\mathbb{C}\setminus\sigma(H_h)$, one has

$$\Big\|\frac{T_h(z)-T_h(z_0)}{z-z_0}-T_h'(z_0)\Big\|_{\mathfrak{S}_1} \leq \|\mathscr{M}_h\|^2_{\mathfrak{S}_2}\Big\|\frac{(H_h-z)^{-1}-(H_h-z_0)^{-1}}{z-z_0}-(H_h-z_0)^{-2}\Big\| \to 0.$$

Thus the claim follows and this ends the proof. □

The second step consists of treating the case $\lambda\in(0,\frac{4}{h^2})$ which is more delicate. We first need to find a suitable integral decomposition of

$$T_h(z) = \mathscr{V}_h^{1/2}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}R_h(z)\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\otimes KK^*)\mathscr{V}_h^{1/2}, \quad z\in\mathbb{C}^+, \tag{7.13}$$

in the spirit of [6, Section 4.1]. This is the object of the next result. In order to simplify the notations, let us introduce the operators $a_h(\theta):\ell^2(\mathbb{Z}_h)\to\mathbb{C}$ defined by

$$a_h(\theta) := \frac{1}{\sqrt{\tau}}\langle e^{-i(\cdot)\theta}\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}|, \qquad \tau=\frac{2\pi}{h}, \tag{7.14}$$

and $a(\theta)^*:\mathbb{C}\to\ell^2(\mathbb{Z}_h)$ the rank one operator given by

$$a_h(\theta)^*\zeta = \frac{\zeta}{\sqrt{\tau}}e^{-i(\cdot)\theta}\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}. \tag{7.15}$$

**Proposition 7.5.** *Let $V_h$ satisfy Assumption 3.4. Then, for $z \in \mathbb{C}^+$, one has*

$$T_h(z) = \int_0^{\pi/h} u_h(\theta)^* u_h(\theta) \frac{d\theta}{f_h(\theta) - z}, \tag{7.16}$$

*where $f_h(\theta)$ is given by (2.2) and $u_h(\theta) : \ell^2(\mathbb{Z}_h, \mathcal{G}) \to \mathbb{C}^2 \otimes \mathcal{G}$ is the operator defined by*

$$u_h(\theta) := (\mathscr{A}_h(\theta) \otimes K^*)\mathscr{V}_h^{1/2}, \quad \theta \in \mathbb{T}_\tau. \tag{7.17}$$

*Here, $\mathscr{A}_h(\theta) : \ell^2(\mathbb{Z}_h) \to \mathbb{C}^2$ is the operator defined by*

$$\mathscr{A}_h(\theta)\phi = \begin{pmatrix} \frac{1}{\sqrt{\tau}} \big\langle e^{-i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2}, \phi \big\rangle_{\ell^2(\mathbb{Z}_h)} \\ \frac{1}{\sqrt{\tau}} \big\langle e^{i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2}, \phi \big\rangle_{\ell^2(\mathbb{Z}_h)} \end{pmatrix} = \begin{pmatrix} a_h(\theta)\phi \\ a_h(-\theta)\phi \end{pmatrix}, \tag{7.18}$$

*where the operator $a_h(\theta)$ is given by (7.14).*

*Proof.* Consider $\mathscr{F}_h : \ell^2(\mathbb{Z}_h) \to \mathrm{L}^2(\mathbb{T}_\tau)$ the discrete Fourier transform defined by (2.1). To simplify the notations, set $W := \langle (\cdot)h^{-1} \rangle^{-\nu_0/2}$ as an operator. For any $\varphi$, $\Phi \in \ell^2(\mathbb{Z}_h)$ and $z \in \mathbb{C}^+$, one has

$$\begin{aligned} &\langle \Phi, W R_h(z) W \varphi \rangle_{\ell^2(\mathbb{Z}_h)} = \langle W\Phi, R_h(z) W\varphi \rangle_{\ell^2(\mathbb{Z}_h)} \\ &= \langle \mathscr{F}_h(W\Phi), \mathscr{F}_h R_h(z) \mathscr{F}_h^{-1} \mathscr{F}_h(W\varphi) \rangle_{\mathrm{L}^2(\mathbb{T}_\tau)} \\ &= \frac{1}{\tau} \int_{-\pi/h}^{\pi/h} \frac{1}{f_h(\theta) - z} \mathscr{F}_h(W\varphi)(\theta) \overline{\mathscr{F}_h(W\Phi)(\theta)} d\theta \\ &= \frac{1}{\tau} \int_0^{\pi/h} \frac{1}{f_h(\theta) - z} \Big( \mathscr{F}_h(W\varphi)(-\theta) \overline{\mathscr{F}_h(W\Phi)(-\theta)} + \mathscr{F}_h(W\varphi)(\theta) \overline{\mathscr{F}_h(W\Phi)(\theta)} \Big) d\theta. \end{aligned} \tag{7.19}$$

For $\theta \in \mathbb{T}_\tau$, one has

$$\mathscr{F}_h(W\varphi)(-\theta) = \sum_{n \in \mathbb{Z}} e^{ihn\theta} \langle n \rangle^{-\nu_0/2} \varphi(hn) = \big\langle e^{-i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2}, \varphi \big\rangle_{\ell^2(\mathbb{Z}_h)}, \tag{7.20}$$

and then

$$\overline{\mathscr{F}_h(W\Phi)(-\theta)} = \overline{\big\langle e^{-i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2}, \Phi \big\rangle_{\ell^2(\mathbb{Z}_h)}}. \tag{7.21}$$

By putting together (7.19)-(7.21) and using (7.18), one gets

$$\begin{aligned} &\langle \Phi, W R_h(z) W \varphi \rangle_{\ell^2(\mathbb{Z}_h)} \\ &= \int_0^{\pi/h} \frac{1}{f_h(\theta) - z} \big( a_h(\theta)\varphi \cdot \overline{a_h(\theta)\Phi} + a_h(-\theta)\varphi \cdot \overline{a_h(-\theta)\Phi} \big) d\theta \\ &= \int_0^{\pi/h} \langle \mathscr{A}_h(\theta)\Phi, \mathscr{A}_h(\theta)\varphi \rangle_{\mathbb{C}^2} \frac{d\theta}{f_h(\theta) - z} \\ &= \int_0^{\pi/h} \langle \Phi, \mathscr{A}_h(\theta)^* \mathscr{A}_h(\theta)\varphi \rangle_{\ell^2(\mathbb{Z}_h)} \frac{d\theta}{f_h(\theta) - z}. \end{aligned} \tag{7.22}$$

It follows that $W R_h(z) W$ admits the integral representation

$$W R_h(z) W = \int_0^{\pi/h} \mathscr{A}_h(\theta)^* \mathscr{A}_h(\theta) \frac{d\theta}{f_h(\theta) - z}, \tag{7.23}$$

where the operator $\mathscr{A}_h(\theta)^* : \mathbb{C}^2 \to \ell^2(\mathbb{Z}_h)$ is given by

$$\mathscr{A}_h(\theta)^* \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = a_h(\theta)^* \zeta_1 + a_h(-\theta)^* \zeta_2, \tag{7.24}$$

so that $\mathscr{A}_h(\theta)^* \mathscr{A}_h(\theta) : \ell^2(\mathbb{Z}_h) \to \ell^2(\mathbb{Z}_h)$ is the rank two operator

$$\begin{aligned} &\mathscr{A}_h(\theta)^* \mathscr{A}_h(\theta) = a_h(\theta)^* a_h(\theta) + a_h(-\theta)^* a_h(-\theta) \\ &= \frac{1}{\tau} \Big( \big| e^{-i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2} \big\rangle \big\langle e^{-i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2} \big| + \big| e^{i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2} \big\rangle \big\langle e^{i(\cdot)\theta} \langle (\cdot)h^{-1} \rangle^{-\nu_0/2} \big| \Big). \end{aligned} \tag{7.25}$$

Then, by combining (7.13) and (7.23), the integral representation (7.16) of the operator $T_h(z)$ holds. This ends the proof. □

By performing the change of variable $\zeta = f_h(\theta) = \frac{4}{h^2}\sin^2(\frac{\theta h}{2})$ in (7.16), one gets

$$\begin{aligned} T_h(z) &= \int_0^{\pi/h} u_h(\theta)^* u_h(\theta) \frac{d\theta}{\frac{4}{h^2}\sin^2(\frac{\theta h}{2}) - z} \\ &= \int_0^{4/h^2} \frac{u(\frac{2}{h}\arcsin(\frac{h}{2}\sqrt{\zeta}))^* u(\frac{2}{h}\arcsin(\frac{h}{2}\sqrt{\zeta}))}{h\sqrt{\zeta}\sqrt{4/h^2-\zeta}} \frac{d\zeta}{\zeta - z} \\ &= \int_0^{4/h^2} \frac{\mathscr{U}_h(\zeta)d\zeta}{\zeta - z} = \int_{-2/h^2}^{2/h^2} \frac{\mathscr{U}_h(\zeta + \frac{2}{h^2})d\zeta}{\zeta + \frac{2}{h^2} - z}, \end{aligned} \tag{7.26}$$

where

$$\mathscr{U}_h(\zeta) := U_h(\zeta)^* U_h(\zeta), \tag{7.27}$$

and

$$U_h(\zeta) := \frac{u_h(\frac{2}{h}\arcsin(\frac{h}{2}\sqrt{\zeta}))}{h^{1/2}\zeta^{1/4}(\frac{4}{h^2} - \zeta)^{1/4}}. \tag{7.28}$$

Now in the spirit of [21, Proposition 3.3], one establishes Proposition 7.8 below. Firstly, the following lemma holds:

**Lemma 7.6.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$. Then, the map $\zeta \in (0, \frac{4}{h^2}) \mapsto \mathscr{U}_h(\zeta)$ is locally $\alpha$-Hölder in the $\mathfrak{S}_1$-norm with $\alpha = 1$.*

*Proof.* For $\zeta \in (0, \frac{4}{h^2})$, thanks to (7.27) and (7.28), one has

$$\mathscr{U}_h(\zeta) = \frac{u_h(\frac{2}{h}g_h(\zeta))^* u_h(\frac{2}{h}g_h(\zeta))}{h\zeta^{1/2}(\frac{4}{h^2} - \zeta)^{1/2}},$$

where $g_h$ is the function defined by (5.13). Fix $\zeta_0 \in (0, \frac{4}{h^2})$ and consider $(\zeta_0 - \delta, \zeta_0 + \delta) \subset (0, \frac{4}{h^2})$ a neighborhood of $\zeta_0$, $\delta > 0$. For every $\zeta_1$, $\zeta_2 \in (\zeta_0 - \delta, \zeta_0 + \delta)$, one has

$$\begin{aligned} \left\|\mathscr{U}_h(\zeta_1) - \mathscr{U}_h(\zeta_2)\right\|_{\mathfrak{S}_1} &\leq \left\| \frac{u_h(\frac{2}{h}g_h(\zeta_1))^* u_h(\frac{2}{h}g_h(\zeta_1)) - u_h(\frac{2}{h}g_h(\zeta_2))^* u_h(\frac{2}{h}g_h(\zeta_2))}{h\zeta_1^{1/2}(\frac{4}{h^2} - \zeta_1)^{1/2}} \right\|_{\mathfrak{S}_1} \\ &+ \left| \frac{1}{h\zeta_1^{1/2}(\frac{4}{h^2} - \zeta_1)^{1/2}} - \frac{1}{h\zeta_2^{1/2}(\frac{4}{h^2} - \zeta_2)^{1/2}} \right| \left\| u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big)^* u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big) \right\|_{\mathfrak{S}_1}. \end{aligned} \tag{7.29}$$

a) Let us treat the first term of the r.h.s. of (7.29). We have

$$\begin{aligned} &\left\| \frac{u_h(\frac{2}{h}g_h(\zeta_1))^* u_h(\frac{2}{h}g_h(\zeta_1)) - u_h(\frac{2}{h}g_h(\zeta_2))^* u_h(\frac{2}{h}g_h(\zeta_2))}{h\zeta_1^{1/2}(\frac{4}{h^2} - \zeta_1)^{1/2}} \right\|_{\mathfrak{S}_1} \\ &\qquad \leq \frac{1}{h\zeta_1^{1/2}(\frac{4}{h^2} - \zeta_1)^{1/2}} \int_{\min(\zeta_1,\zeta_2)}^{\max(\zeta_1,\zeta_2)} \left\| \partial_\zeta \Big[ u_h\Big(\frac{2}{h}g_h(\zeta)\Big)^* u_h\Big(\frac{2}{h}g_h(\zeta)\Big) \Big] \right\|_{\mathfrak{S}_1} d\zeta. \end{aligned} \tag{7.30}$$

From (7.17), one gets for $\zeta \in (0, \frac{4}{h^2})$

$$u_h\Big(\frac{2}{h}g_h(\zeta)\Big)^* u_h\Big(\frac{2}{h}g_h(\zeta)\Big) = \mathscr{V}_h^{1/2} \Big[ \mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^* \mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big) \otimes KK^* \Big] \mathscr{V}_h^{1/2}, \tag{7.31}$$

so that

$$\left\| \partial_\zeta \Big[ u_h\Big(\frac{2}{h}g_h(\zeta)\Big)^* u_h\Big(\frac{2}{h}g_h(\zeta)\Big) \Big] \right\|_{\mathfrak{S}_1} \leq \|\mathscr{V}_h\| \|K\|_{\mathfrak{S}_2}^2 \left\| \partial_\zeta \Big[ \mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^* \mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big) \Big] \right\|_{\mathfrak{S}_1}. \tag{7.32}$$

By using (7.25), one obtains for $\phi \in \ell^2(\mathbb{Z}_h)$

$$\begin{aligned}
&\frac{\pi}{h}\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^*\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)\phi(hn)\\
&= \frac{1}{2}\Big(e^{-2ing_h(\zeta)}\langle (hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}} e^{2img_h(\zeta)}\langle (hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\
&+ e^{2ing_h(\zeta)}\langle (hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}} e^{-2img_h(\zeta)}\langle (hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\Big)\\
&= \sum_{m\in\mathbb{Z}} \cos[2h(n+m)h^{-1}g_h(\zeta)]\langle (hn)h^{-1}\rangle^{-\nu_0/2}\langle (hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm).
\end{aligned} \tag{7.33}$$

Consequently, for any $\phi \in \ell^2(\mathbb{Z}_h)$, by setting $C(\zeta) := -\frac{1}{\sqrt{\zeta}\sqrt{4/h^2-\zeta}}$, one gets

$$\begin{aligned}
&\frac{\pi}{h}\partial_\zeta\Big[\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^*\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)\Big]\phi(hn) = C(\zeta)\langle (hn)h^{-1}\rangle^{-\nu_0/2}\\
&\qquad\qquad \times\sum_{m\in\mathbb{Z}} h(n+m)h^{-1}\sin[2h(n+m)h^{-1}g_h(\zeta)]\langle (hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\
&= C(\zeta)\Big(\langle (hn)h^{-1}\rangle^{-\nu_0/2}(hn)h^{-1}\sin[2(hn)h^{-1}g_h(\zeta)]\\
&\qquad\qquad \times\sum_{m\in\mathbb{Z}} \cos[2(hm)h^{-1}g_h(\zeta)]\langle (hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\
&+ \langle (hn)h^{-1}\rangle^{-\nu_0/2}(hn)h^{-1}\cos[2(hn)h^{-1}g_h(\zeta)]\\
&\qquad\qquad \times\sum_{m\in\mathbb{Z}} \sin[2(hm)h^{-1}g_h(\zeta)]\langle (hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\
&+ \langle (hn)h^{-1}\rangle^{-\nu_0/2}\sin[2(hn)h^{-1}g_h(\zeta)]\\
&\qquad\qquad \times\sum_{m\in\mathbb{Z}} \cos[2(hm)h^{-1}g_h(\zeta)]\langle (hm)h^{-1}\rangle^{-\nu_0/2}(hm)h^{-1}\phi(hm)\\
&+ \langle (hn)h^{-1}\rangle^{-\nu_0/2}\cos[2(hn)h^{-1}g_h(\zeta)]\\
&\qquad\qquad \times\sum_{m\in\mathbb{Z}} \sin[2(hm)h^{-1}g_h(\zeta)]\langle (hm)h^{-1}\rangle^{-\nu_0/2}(hm)h^{-1}\phi(hm)\Big).
\end{aligned}$$

Il follows, using the inequality $\|A_1A_2\|_{\mathfrak{S}_1} \leq \|A_1\|_{\mathfrak{S}_2}\|A_2\|_{\mathfrak{S}_2}$ for $A_1$, $A_2 \in \mathfrak{S}_2$, that

$$\begin{aligned}
&\frac{\pi}{h}\Big\|\partial_\zeta\Big[\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^*\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)\Big]\Big\|_{\mathfrak{S}_1}\\
&\leq |C(\zeta)|\Big[\Big(\sum_{n\in\mathbb{Z}}\sin^2[2ng_h(\zeta)]n^2\langle n\rangle^{-\nu_0}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}\cos^2[2ng_h(\zeta)]\langle n\rangle^{-\nu_0}\Big)^{1/2}\\
&\quad+\Big(\sum_{n\in\mathbb{Z}}\cos^2[2ng_h(\zeta)]n^2\langle n\rangle^{-\nu_0}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}\sin^2[2ng_h(\zeta)]\langle n\rangle^{-\nu_0}\Big)^{1/2}\\
&\quad+\Big(\sum_{n\in\mathbb{Z}}\sin^2[2ng_h(\zeta)]\langle n\rangle^{-\nu_0}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}\cos^2[2ng_h(\zeta)]n^2\langle n\rangle^{-\nu_0}\Big)^{1/2}\\
&\quad+\Big(\sum_{n\in\mathbb{Z}}\cos^2[2ng_h(\zeta)]\langle n\rangle^{-\nu_0}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}\sin^2[2ng_h(\zeta)]n^2\langle n\rangle^{-\nu_0}\Big)^{1/2}\Big],
\end{aligned}$$

so that

$$\frac{\pi}{h}\Big\|\partial_\zeta\Big[\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^*\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)\Big]\Big\|_{\mathfrak{S}_1} \leq 4|C(\zeta)|\sum_{n\in\mathbb{Z}} n^2\langle n\rangle^{-\nu_0}. \tag{7.34}$$

Putting together (7.30), (7.32) and (7.34), one gets finally

$$\begin{aligned}(7.35)\qquad &\Big\|\frac{u_h(\frac{2}{h}g_h(\zeta_1))^*u_h(\frac{2}{h}g_h(\zeta_1))-u_h(\frac{2}{h}g_h(\zeta_2))^*u_h(\frac{2}{h}g_h(\zeta_2))}{h\zeta_1^{1/2}(\frac{4}{h^2}-\zeta_1)^{1/2}}\Big\|_{\mathfrak{S}_1}\\ &\qquad\le \frac{4}{\pi}\big(\max_{\zeta\in(\zeta_0-\delta,\zeta_0+\delta)}\big)^2\frac{\|\mathscr{V}_h\|\|K\|^2_{\mathfrak{S}_2}\sum_{n\in\mathbb{Z}}n^2\langle n\rangle^{-\nu_0}}{\zeta^{1/2}(\frac{4}{h^2}-\zeta)^{1/2}}|\zeta_1-\zeta_2|.\end{aligned}$$

b) Now, one treats the second term of the r.h.s. of (7.29). We have

$$\begin{aligned}(7.36)\qquad &\Big|\frac{1}{h\zeta_1^{1/2}(\frac{4}{h^2}-\zeta_1)^{1/2}}-\frac{1}{h\zeta_2^{1/2}(\frac{4}{h^2}-\zeta_2)^{1/2}}\Big|\Big\|u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big)^*u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big)\Big\|_{\mathfrak{S}_1}\\ &\le\frac{1}{h}\max_{\zeta\in(\zeta_0-\delta,\zeta_0+\delta)}\Big|\partial_\zeta\frac{1}{\zeta^{1/2}(\frac{4}{h^2}-\zeta)^{1/2}}\Big|\Big\|u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big)^*u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big)\Big\|_{\mathfrak{S}_1}|\zeta_1-\zeta_2|.\end{aligned}$$

From (7.31), it follows that for $\zeta\in(0,\frac{4}{h^2})$,

$$(7.37)\qquad \Big\|u_h\Big(\frac{2}{h}g_h(\zeta)\Big)^*u_h\Big(\frac{2}{h}g_h(\zeta)\Big)\Big\|_{\mathfrak{S}_1}\le\|\mathscr{V}_h\|\|K\|^2_{\mathfrak{S}_2}\Big\|\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^*\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)\Big\|_{\mathfrak{S}_1}.$$

Using (7.33), one obtains that for $\phi\in\ell^2(\mathbb{Z}_h)$

$$\begin{aligned}&\frac{\pi}{h}\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)^*\mathscr{A}_h\Big(\frac{2}{h}g_h(\zeta)\Big)\phi(hn)\\ &=\cos[2(hn)h^{-1}g_h(\zeta)]\langle(hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}}\cos[2(hm)h^{-1}g_h(\zeta)]\langle(hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\ &+\sin[2(hn)h^{-1}g_h(\zeta)]\langle(hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}}\sin[2(hm)h^{-1}g_h(\zeta)]\langle(hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm).\end{aligned}$$

This gives that

$$\begin{aligned}(7.38)\qquad &\frac{\pi}{h}\Big\|\mathscr{A}_h\Big(\frac{2}{h}g_h(\lambda)\Big)^*\mathscr{A}_h\Big(\frac{2}{h}g_h(\lambda)\Big)\Big\|_{\mathfrak{S}_1}\\ &\qquad\le\sum_{n\in\mathbb{Z}}\cos^2[2ng_h(\lambda)]\langle n\rangle^{-\nu_0}+\sum_{n\in\mathbb{Z}}\sin^2[2ng_h(\lambda)]\langle n\rangle^{-\nu_0}=\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}.\end{aligned}$$

Using (7.36), (7.37) and (7.38), we finally get

$$\begin{aligned}(7.39)\qquad &\Big|\frac{1}{h\zeta_1^{1/2}(\frac{4}{h^2}-\zeta_1)^{1/2}}-\frac{1}{h\zeta_2^{1/2}(\frac{4}{h^2}-\zeta_2)^{1/2}}\Big|\Big\|u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big)^*u_h\Big(\frac{2}{h}g_h(\zeta_2)\Big)\Big\|_{\mathfrak{S}_1}\\ &\le\frac{1}{\pi}\|\mathscr{V}_h\|\|K\|^2_{\mathfrak{S}_2}\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}\max_{\zeta\in(\zeta_0-\delta,\zeta_0+\delta)}\Big|\partial_\zeta\frac{1}{\zeta^{1/2}(\frac{4}{h^2}-\zeta)^{1/2}}\Big||\zeta_1-\zeta_2|.\end{aligned}$$

Now, the lemma follows by putting together (7.29), (7.35) and (7.39). □

As a direct consequence, by applying Sokhotski-Plemelj formula [1], the following corollary holds.

**Corollary 7.7.** *Let $z=\lambda+i\varepsilon$ with $\lambda\in(0,\frac{4}{h^2})$. Then,*

$$(7.40)\qquad \Big\|\int_0^{4/h^2}\frac{\mathscr{U}_h(\zeta)d\zeta}{\zeta-\lambda-i\varepsilon}-\text{p.v.}\int_0^{4/h^2}\frac{\mathscr{U}_h(\zeta)d\zeta}{\zeta-\lambda}-i\pi\mathscr{U}_h(\lambda)\Big\|_{\mathfrak{S}_1}=\mathcal{O}_h(\varepsilon),\quad \varepsilon\searrow 0.$$

One can now state the next result showing the existence of $T_h(\lambda+i0)$ in $\mathfrak{S}_1$ for $\lambda\in(0,\frac{4}{h^2})$. Notations are those introduced above. It follows from Corollary 7.7 and (7.26) the following:

**Proposition 7.8.** *Let $z=\lambda+i\varepsilon$ with $\lambda\in(0,\frac{4}{h^2})$. Then, as $\varepsilon\searrow 0$,*

$$(7.41)\qquad T_h(\lambda+i\varepsilon)=\int_0^{4/h^2}\frac{\mathscr{U}_h(\zeta)d\zeta}{\zeta-\lambda-i\varepsilon}\quad\longrightarrow\quad\text{p.v.}\int_0^{4/h^2}\frac{\mathscr{U}_h(\zeta)d\zeta}{\zeta-\lambda}+i\pi\mathscr{U}_h(\lambda),$$

*in the $\mathfrak{S}_1$-norm.*

The next corollary is a direct consequence of Corollary 7.3, Propositions 7.4 and 7.8.

**Corollary 7.9.** *Let $V_h$ satisfy Assumption 3.4 and $\lambda \in \mathbb{R} \setminus \{0, \frac{4}{h^2}\}$. Then, the limit $T_h(\lambda + i0)$ exists in $\mathfrak{S}_1$ with $T_h(\lambda + i0) = T_h(\lambda)$.*

For further references in the proof of the main results, let us point out the following remarks.

**Remark 7.10.** *For $\lambda \in (0, \frac{4}{h^2})$, one knows by definition that $T_h(\lambda)$ is given by (7.10), where $R_h(\lambda)$ admits the convolution kernel*

$$r_h(\lambda, n-m) = \frac{ie^{2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}}{\sqrt{\lambda}\sqrt{4/h^2 - \lambda}}. \tag{7.42}$$

**Remark 7.11.** *In the case $\lambda < 0$, by the uniqueness of the limit and arguing as in the proof of ii) of Proposition 7.2, on can show that the formula (7.10) for $T_h(\lambda)$ remains valid with $R_h(\lambda)$ admitting the convolution kernel*

$$r_h(\lambda, n-m) = \frac{e^{2i|n-m|\mathrm{Arcsin}(\frac{ih}{2}\sqrt{-\lambda})}}{\sqrt{-\lambda}\sqrt{4/h^2 - \lambda}}. \tag{7.43}$$

**Remark 7.12.** *For $\lambda > \frac{4}{h^2}$, similarly to Remark 7.11, one has*

$$T_h(\lambda) = \mathscr{V}_h^{1/2}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} R_h(\lambda) \langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes KK^*)\mathscr{V}_h^{1/2}, \tag{7.44}$$

*with $R_h(\lambda)$ admitting the convolution kernel*

$$r_h(\lambda, n-m) = -\frac{e^{2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}}{\sqrt{\lambda}\sqrt{\lambda - 4/h^2}}. \tag{7.45}$$

## 8. Proof of the main results

It is useful to recall the following standard properties of the counting functions $\mathscr{N}_\pm$ defined by (4.4). If $T_1 = T_1^*$ and $T_2 = T_2^*$ belong to $\mathfrak{S}_\infty(\mathcal{G})$, then one has the Weyl inequalities

$$\mathscr{N}_\pm(x_1 + x_2, T_1 + T_2) \leq \mathscr{N}_\pm(x_1, T_1) + \mathscr{N}_\pm(x_2, T_2), \quad x_1, x_2 > 0. \tag{8.1}$$

If $T \in \mathfrak{S}_p(\mathcal{G})$ for some $p \geq 1$, then

$$\mathscr{N}_\pm(x, T) \leq x^{-p} \|T\|_{\mathfrak{S}_p}^{-p}, \quad x > 0. \tag{8.2}$$

The following preliminary result holds:

**Proposition 8.1.** *Let $V_h$ satisfy Assumption 3.4. Then, one has the estimates*

$$\begin{aligned}\int_{\mathbb{R}} \mathscr{N}_\pm(1+\varepsilon, \mathrm{Re}\, T_h(\lambda) + t\mathrm{Im}\, T_h(\lambda)) \frac{dt}{\pi(1+t^2)} \\ \leq \mp\xi(\lambda; H_h^\mp, H_h) \\ \leq \int_{\mathbb{R}} \mathscr{N}_\pm(1-\varepsilon, \mathrm{Re}\, T_h(\lambda) + t\mathrm{Im}\, T_h(\lambda)) \frac{dt}{\pi(1+t^2)},\end{aligned}$$

*for any $\varepsilon \in (0,1)$.*

*Proof.* It is a direct consequence of the Weyl inequalities (8.1) and Corollary 7.9. Indeed, for any $\varepsilon \in (0,1)$, it follows that

$$\begin{aligned}\int_{\mathbb{R}} \mathscr{N}_\pm(1+\varepsilon, \mathrm{Re}\, T_h(\lambda) + t\mathrm{Im}\, T_h(\lambda)) \frac{dt}{\pi(1+t^2)} \\ \leq \int_{\mathbb{R}} \mathscr{N}_\pm(1, A_h(\lambda+i0) + tB_h(\lambda+i0)) \frac{dt}{\pi(1+t^2)} \\ \leq \int_{\mathbb{R}} \mathscr{N}_\pm(1-\varepsilon, \mathrm{Re}\, T_h(\lambda) + t\mathrm{Im}\, T_h(\lambda)) \frac{dt}{\pi(1+t^2)}.\end{aligned}$$

This implies the proposition and ends the proof. □

8.1. **Proof of Theorem 5.2.** One assumes that the potential $V_h$ satisfies Assumption 3.4 throughout this section. First, note that

$$\operatorname{Re} T_h(\lambda) = T_h(\lambda) \quad \text{and} \quad \operatorname{Im} T_h(\lambda) = 0, \quad \lambda \in (-\infty, 0) \cup (4/h^2, +\infty). \tag{8.3}$$

This together with Proposition 8.1 and the identity

$$\int_{\mathbb{R}} \frac{dt}{\pi(1+t^2)} = 1,$$

implies immediately the next result:

**Corollary 8.2.** *Let $\lambda \in (-\infty, 0) \cup (\frac{4}{h^2}, +\infty)$. Then,*

$$\mathscr{N}_{\pm}(1+\varepsilon, T_h(\lambda)) \leq \mp\xi(\lambda; H_h^{\mp}, H_h) \leq \mathscr{N}_{\pm}(1-\varepsilon, T_h(\lambda)),$$

*for any $\varepsilon \in (0,1)$.*

One shows in the following result that $\mathscr{N}_{\pm}(x, T_h(\lambda))$ can be bounded as $\lambda \to 0$, from below and from above by expressions involving $\mathcal{L}_0^h(\lambda)$, up to $\mathcal{O}(1)$. Here, $\mathcal{L}_0^h(\lambda) : \ell^2(\mathbb{Z}_h) \otimes \mathcal{G} \to \ell^2(\mathbb{Z}_h) \otimes \mathcal{G}$ is the trace class operator defined by

$$\mathcal{L}_0^h(\lambda) = \frac{h}{2} \frac{L_{0,h}^* L_{0,h}}{\sqrt{-\lambda}}, \qquad \lambda \in (-\infty, 0), \tag{8.4}$$

where $L_{0,h}$ is defined by (5.2).

**Proposition 8.3.** *Let $\nu_0 > 3$ in Assumption 3.4. Then, as $\lambda \nearrow 0$,*

$$\mathscr{N}_+((1+\varepsilon)x, \mathcal{L}_0^h(\lambda)) + \mathcal{O}(1) \leq \mathscr{N}_+(x, T_h(\lambda)) \leq \mathscr{N}_+((1-\varepsilon)x, \mathcal{L}_0^h(\lambda)) + \mathcal{O}(1),$$

*and*

$$\mathcal{O}(1) \leq \mathscr{N}_-(x, T_h(\lambda)) \leq \mathcal{O}(1),$$

*for any $\varepsilon \in (0,1)$ and $x > 0$.*

*Proof.* The main idea of the proof is to approximate the operator $T_h(\lambda) - \mathcal{L}_0^h(\lambda)$ in Hilbert-Shmidt norm, as $\lambda \nearrow 0$, by a compact operator independent of $\lambda$.

a) Let $\lambda < 0$. The convolution kernel $r_h(\lambda, n-m)$ given by (7.43) can be decomposed as

$$r_h(\lambda, n-m) = \frac{h}{2\sqrt{-\lambda}} + \Big( \frac{1}{\sqrt{-\lambda}\sqrt{4/h^2 - \lambda}} - \frac{h}{2\sqrt{-\lambda}} \Big) + \frac{e^{2i|n-m|\operatorname{Arcsin}(\frac{ih}{2}\sqrt{-\lambda})} - 1}{\sqrt{-\lambda}\sqrt{4/h^2 - \lambda}}. \tag{8.5}$$

Together with Remark 7.11, this implies that

$$T_h(\lambda) - \mathcal{L}_0^h(\lambda) = \mathscr{V}_h^{1/2} (S_{h,\nu_0}^{(\lambda)} \otimes KK^*) \mathscr{V}_h^{1/2} + \mathcal{I}_0^h(\lambda), \tag{8.6}$$

where $S_{h,\nu_0}^{(\lambda)} : \ell^2(\mathbb{Z}_h) \to \ell^2(\mathbb{Z}_h)$ is the summation kernel operator defined by

$$(S_{h,\nu_0}^{(\lambda)} \varphi)(hn) := \sum_{m \in \mathbb{Z}} \langle n \rangle^{-\nu_0/2} \frac{e^{2i|n-m|\operatorname{Arcsin}(\frac{ih}{2}\sqrt{-\lambda})} - 1}{\sqrt{-\lambda}\sqrt{4/h^2 - \lambda}} \langle m \rangle^{-\nu_0/2} \varphi(hm),$$

$\varphi \in \ell^2(\mathbb{Z}_h)$, and

$$\mathcal{I}_0^h(\lambda) := \Big( \frac{1}{\sqrt{-\lambda}\sqrt{4/h^2 - \lambda}} - \frac{h}{2\sqrt{-\lambda}} \Big) L_{0,h}^* L_{0,h},$$

with the operator $L_{0,h}$ given by (5.2). Since $\frac{1}{\sqrt{-\lambda}\sqrt{4/h^2-\lambda}} - \frac{h}{2\sqrt{-\lambda}} = \mathcal{O}_h(\sqrt{-\lambda})$ as $\lambda \nearrow 0$, and the operator $L_{0,h}$ is independent of $\lambda$, it follows that

$$\lim_{\lambda \nearrow 0} \|\mathcal{I}_0^h(\lambda)\|_{\mathfrak{S}_2} = \lim_{\lambda \nearrow 0} \mathcal{O}_h(\sqrt{-\lambda}) \|L_{0,h}^* L_{0,h}\|_{\mathfrak{S}_2} = 0. \tag{8.7}$$

Define the operator

$$T_0^h = \mathscr{V}_h^{1/2} (S_{h,\nu_0}^{(0)} \otimes KK^*) \mathscr{V}_h^{1/2}, \tag{8.8}$$

where $S^{(0)}_{h,\nu_0} : \ell^2(\mathbb{Z}_h) \to \ell^2(\mathbb{Z}_h)$ is the summation kernel operator given by

$$(S^{(0)}_{h,\nu_0}\phi)(hn) := -\frac{h^2}{2}\sum_{m\in\mathbb{Z}} \langle n\rangle^{-\nu_0/2}|n-m|\langle m\rangle^{-\nu_0/2}\phi(hm), \qquad \varphi \in \ell^2(\mathbb{Z}_h). \tag{8.9}$$

Since $\nu_0 > 3$, then $\sum_{n,m} |\langle n\rangle^{-\nu_0/2}|n-m|\langle m\rangle^{-\nu_0/2}|^2 < \infty$ and $S^{(0)}_{h,\nu_0}$ belongs to $\mathfrak{S}_2(\ell^2(\mathbb{Z}_h))$. In particular, the operator $T_0$ is compact in $\ell^2(\mathbb{Z}_h, \mathcal{G})$. By using the Lebesgue dominated convergence theorem and the convolution kernels of the operators $S^{(\lambda)}_{h,\nu_0}$ and $S^{(0)}_{h,\nu_0}$, one gets

$$\lim_{\lambda\nearrow 0} \|S^{(\lambda)}_{h,\nu_0} - S^{(0)}_{h,\nu_0}\|^2_{\mathfrak{S}_2} = 0. \tag{8.10}$$

Putting this together with (8.6), (8.7) and (8.10), one obtains

$$\lim_{\lambda\nearrow 0} \|T_h(\lambda) - \mathcal{L}^h_0(\lambda) - T^h_0\|_{\mathfrak{S}_2} = 0. \tag{8.11}$$

b) Now, consider $\lambda < 0$, $\varepsilon \in (0,1)$ and $x > 0$. Using Weyl's inequalities (8.1), one gets

$$\begin{aligned}\mathscr{N}_\pm((1+\varepsilon)x, \mathcal{L}^h_0(\lambda)) - \mathscr{N}_\mp(\varepsilon x, T_h(\lambda) - \mathcal{L}^h_0(\lambda)) &\le \mathscr{N}_\pm(x, T_h(\lambda))\\ &\le \mathscr{N}_\pm((1-\varepsilon)x, \mathcal{L}^h_0(\lambda)) + \mathscr{N}_\pm(\varepsilon x, T_h(\lambda) - \mathcal{L}^h_0(\lambda)),\end{aligned}$$

Since $\mathcal{L}^h_0(\lambda)$ is a positive operator, then one has

$$\mathscr{N}_-(s, \mathcal{L}^h_0(\lambda)) = 0, \quad \forall\, s > 0.$$

Therefore, to get the proposition, it suffices to prove that for every $\varepsilon \in (0,1)$ and $x > 0$,

$$\mathscr{N}_\pm(\varepsilon x, T_h(\lambda) - \mathcal{L}^h_0(\lambda)) = \mathcal{O}(1), \qquad \lambda \nearrow 0 \tag{8.12}$$

To this end, let us fix $s > 0$. From (8.11), it follows that there exists $\lambda_0 < 0$ small enough such that

$$\|T_h(\lambda) - \mathcal{L}^h_0(\lambda) - T^h_0\|_{\mathfrak{S}_2} < \frac{s\sqrt{\mathscr{N}_\pm(s/2, T^h_0)}}{2}, \qquad \forall\, \lambda_0 < \lambda < 0.$$

This together with the Weyl inequalities implies that for all $\lambda_0 < \lambda < 0$,

$$\begin{aligned}\mathscr{N}_\pm(s, T_h(\lambda) - \mathcal{L}^h_0(\lambda)) &\le \mathscr{N}_\pm(s/2, T_h(\lambda) - \mathcal{L}^h_0(\lambda) - T^h_0) + \mathscr{N}_\pm(s/2, T^h_0)\\ &\le (s/2)^{-2}\|T_h(\lambda) - \mathcal{L}^h_0(\lambda) - T^h_0\|^2_{\mathfrak{S}_2} + \mathscr{N}_\pm(s/2, T^h_0) \le 2\mathscr{N}_\pm(s/2, T^h_0) = \mathcal{O}(1),\end{aligned}$$

uniformly w.r.t. $\lambda_0 < \lambda < 0$, which gives (8.12). This ends the proof. □

In the next result, $\mathcal{L}^h_{4/h^2}(\lambda) : \ell^2(\mathbb{Z}_h) \otimes \mathcal{G} \to \ell^2(\mathbb{Z}_h) \otimes \mathcal{G}$ is the trace class operator defined by

$$\mathcal{L}^h_{4/h^2}(\lambda) = -\frac{h}{2}\frac{L^*_{4/h^2,h} L_{4/h^2,h}}{\sqrt{\lambda - 4/h^2}}, \qquad \lambda \in (\tfrac{4}{h^2}, +\infty), \tag{8.13}$$

where $L_{4/h^2,h}$ is defined by (5.4). The proof is similar to that of Proposition 8.3 and then will be shortened. Only the main quantities will be specified.

**Proposition 8.4.** *Suppose $\nu_0 > 3$ in Assumption 3.4. Then, as $\lambda \searrow \frac{4}{h^2}$,*

$$\mathcal{O}(1) \le \mathscr{N}_+(x, T_h(\lambda)) \le \mathcal{O}(1),$$

*and*

$$\mathscr{N}_-((1+\varepsilon)x, \mathcal{L}^h_{4/h^2}(\lambda)) + \mathcal{O}(1) \le \mathscr{N}_-(x, T_h(\lambda)) \le \mathscr{N}_-((1-\varepsilon)x, \mathcal{L}^h_{4/h^2}(\lambda)) + \mathcal{O}(1),$$

*for any $\varepsilon \in (0,1)$ and $x > 0$.*

*Proof.* Let $\lambda > \frac{4}{h^2}$. Consider the operator $J$ given by (5.3). Using (7.44) and (7.45), one can write

$$T_h(\lambda) = \mathscr{V}_h^{1/2}(J\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} J^* R_h(\lambda) J \langle(\cdot)h^{-1}\rangle^{-\nu_0/2} J^* \otimes KK^*)\mathscr{V}_h^{1/2}, \tag{8.14}$$

with $J^*R_h(\lambda)J$ admitting the convolution kernel

$$r^J_h(\lambda, n-m) = -\frac{e^{i\pi(n+m)+2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}}{\sqrt{\lambda}\sqrt{\lambda - 4/h^2}}. \tag{8.15}$$

The kernel $r_h^J(\lambda, n-m)$ can be decomposed as

$$\begin{aligned} r_h^J(\lambda, n-m) &= -\frac{h}{2\sqrt{\lambda - 4/h^2}} \\ &- \Big(\frac{1}{\sqrt{\lambda}\sqrt{\lambda-4/h^2}} - \frac{h}{2\sqrt{\lambda-4/h^2}}\Big) - \frac{e^{i\pi(n+m)+2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}-1}{\sqrt{\lambda}\sqrt{\lambda-4/h^2}}. \end{aligned} \tag{8.16}$$

This together with (8.14) and (8.15) gives

$$T_h(\lambda) - \mathcal{L}^h_{4/h^2}(\lambda) = \mathscr{V}_h^{1/2}(JX^{(\lambda)}_{h,\nu_0}J^* \otimes KK^*)\mathscr{V}_h^{1/2} + \mathcal{I}^h_{4/h^2}(\lambda), \tag{8.17}$$

where $X^{(\lambda)}_{h,\nu_0} : \ell^2(\mathbb{Z}_h) \to \ell^2(\mathbb{Z}_h)$ is the summation kernel operator defined by

$$(X^{(\lambda)}_{h,\nu_0}\varphi)(hn) := -\sum_{m\in\mathbb{Z}} \langle n\rangle^{-\nu_0/2} \frac{e^{i\pi(n+m)+2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}-1}{\sqrt{\lambda}\sqrt{\lambda-4/h^2}} \langle m\rangle^{-\nu_0/2}\varphi(hm),$$

$\varphi \in \ell^2(\mathbb{Z}_h)$ and

$$\mathcal{I}^h_{4/h^2}(\lambda) := -\Big(\frac{1}{\sqrt{\lambda}\sqrt{\lambda-4/h^2}} - \frac{h}{2\sqrt{\lambda-4/h^2}}\Big) L^*_{4/h^2,h}L_{4/h^2,h},$$

the operator $L_{4/h^2,h}$ being given by (5.4). Since $\frac{1}{\sqrt{\lambda}\sqrt{\lambda-4/h^2}} - \frac{h}{2\sqrt{\lambda-4/h^2}} = \mathcal{O}_h(\sqrt{\lambda-4/h^2})$ as $\lambda \searrow \frac{4}{h^2}$, and the operator $L_{4/h^2,h}$ is independent of $\lambda$, it follows that

$$\lim_{\lambda\searrow\frac{4}{h^2}} \|\mathcal{I}^h_{4/h^2}(\lambda)\|_{\mathfrak{S}_2} = \lim_{\lambda\searrow\frac{4}{h^2}} \mathcal{O}_h(\sqrt{\lambda-4/h^2})\|L^*_{4/h^2,h}L_{4/h^2,h}\|_{\mathfrak{S}_2} = 0. \tag{8.18}$$

Consider the operator

$$T^h_{4/h^2} = \mathscr{V}_h^{1/2}(JS^{(0)}_{h,\nu_0}J^* \otimes KK^*)\mathscr{V}_h^{1/2}, \tag{8.19}$$

where $S^{(0)}_{h,\nu_0}$ is the operator defined by (8.9). Using Lebesgue's dominated convergence theorem, one shows that

$$\lim_{\lambda\searrow\frac{4}{h^2}} \|X^{(\lambda)}_{h,\nu_0} - S^{(0)}_{h,\nu_0}\|^2_{\mathfrak{S}_2} = 0, \tag{8.20}$$

So, it follows from (8.17), (8.18) and (8.20) that

$$\lim_{\lambda\searrow\frac{4}{h^2}} \|T_h(\lambda) - \mathcal{L}^h_{4/h^2}(\lambda) - T^h_{4/h^2}\|_{\mathfrak{S}_2} = 0.$$

Now, the claim follows by arguing as in part b) of the proof of Proposition 8.3, noting that $\mathcal{L}^h_{4/h^2}(\lambda)$ is a negative operator. This ends the proof. □

For $x > 0$ and $\lambda < 0$, one has

$$\begin{aligned} \mathscr{N}_+(x, \mathcal{L}^h_0(\lambda)) &= \mathscr{N}_+\Big(x, \frac{hL^*_{0,h}L_{0,h}}{2\sqrt{-\lambda}}\Big) = \mathscr{N}_+\Big(x, \frac{hL_{0,h}L^*_{0,h}}{2\sqrt{-\lambda}}\Big) \\ &= \mathscr{N}_+\Big(x, \frac{h(\mathrm{p}_{\nu_0}\otimes K^*)\mathscr{V}_h(\mathrm{p}^*_{\nu_0}\otimes K)}{2\sqrt{-\lambda}}\Big) = \mathscr{N}_+(x, \omega^h_0(\lambda)), \end{aligned} \tag{8.21}$$

where the operator $\omega^h_0(\lambda)$ is given by (5.8). Similarly, one shows that for any $x > 0$ and $\lambda > \frac{4}{h^2}$,

$$\mathscr{N}_-(x, \mathcal{L}^h_{4/h^2}(\lambda)) = \mathscr{N}_-\Big(x, -\frac{hL^*_{4/h^2}L_{4/h^2}}{2\sqrt{\lambda-4/h^2}}\Big) = \mathscr{N}_-(x, \omega^h_{4/h^2}(\lambda)), \tag{8.22}$$

where the operator $\omega^h_{4/h^2}(\lambda)$ is given by (5.8). Now, Theorem 5.2 follows directly from Corollary 8.2, Propositions 8.3 and 8.4, identities (8.21) and (8.22).

8.2. **Proof of Theorem 5.4.** This section concerns the case $\lambda \in (0, \frac{4}{h^2})$. We assume that $V_h$ satisfies Assumption 3.4. In the next result, one shows the boundedness of $\mathscr{N}_{\pm}(x, \mathrm{Re}\, T_h(\lambda))$ as $\lambda \to \lambda_0 \in \{0, \frac{4}{h^2}\}$.

**Proposition 8.5.** *Suppose $\nu_0 > 3$ in Assumption 3.4. Then, for any $x > 0$,*

$$\mathscr{N}_{\pm}(x, \mathrm{Re}\, T_h(\lambda)) = \mathcal{O}(1),$$

*as $\lambda \searrow 0$ and $\lambda \nearrow \frac{4}{h^2}$.*

*Proof.* a) First, let us focus on the case $\lambda \searrow 0$. Thanks to (7.42), the operator $\mathrm{Re}\, R_h(\lambda)$ admits the convolution kernel

$$\mathrm{Re}\, r_h(\lambda, n-m) = -\frac{\sin(2|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda}))}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}.$$

Putting this together with Remark 7.10, one obtains

$$\mathrm{Re}\, T_h(\lambda) = \mathscr{V}_h^{1/2}(E^{(\lambda)}_{h,\nu_0} \otimes KK^*)\mathscr{V}_h^{1/2}, \tag{8.23}$$

where $E^{(\lambda)}_{h,\nu_0} : \ell^2(\mathbb{Z}_h) \to \ell^2(\mathbb{Z}_h)$ is the convolution kernel operator given by

$$(E^{(\lambda)}_{h,\nu_0}\phi)(hn) := \sum_{m\in\mathbb{Z}} \langle n\rangle^{-\nu_0/2}\mathrm{Re}\, r_h(\lambda, n-m)\langle m\rangle^{-\nu_0/2}\phi(hm), \quad \phi \in \ell^2(\mathbb{Z}_h).$$

By using Lebesgue's dominated convergence theorem, one gets

$$\lim_{\lambda\searrow 0} \|E^{(\lambda)}_{h,\nu_0} - S^{(0)}_{h,\nu_0}\|^2_{\mathfrak{S}_2} = 0, \tag{8.24}$$

where $S^{(0)}_{h,\nu_0}$ is the operator given by (8.9). It follows from (8.23) and (8.24) that

$$\lim_{\lambda\searrow 0} \|\mathrm{Re}\, T_h(\lambda) - T_0^h\|_{\mathfrak{S}_2} = 0,$$

where $T_0^h$ is the operator given by (8.8). Now, the claim follows by arguing as in part b) of the proof of Proposition 8.3.

b) The case $\lambda \nearrow \frac{4}{h^2}$ can be proved as follows. Thanks to Remark 7.10, one can write

$$T_h(\lambda) = \mathscr{V}_h^{1/2}(J\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}R_h^J(\lambda)\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}J^* \otimes KK^*)\mathscr{V}_h^{1/2}, \tag{8.25}$$

with $R_h^J := J^*R_h(\lambda)J$ admitting the convolution kernel

$$r_h^J(\lambda, n-m) = \frac{ie^{i\pi(n+m)+2i|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda})}}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}. \tag{8.26}$$

It follows that

$$\mathrm{Re}\, T_h(\lambda) = \mathscr{V}_h^{1/2}(J\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\mathrm{Re}\, R_h^J(\lambda)\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}J^* \otimes KK^*)\mathscr{V}_h^{1/2}, \tag{8.27}$$

with $\mathrm{Re}\, R_h^J(\lambda) := J^*R_h(\lambda)J$ admitting the convolution kernel

$$\mathrm{Re}\, r_h^J(\lambda, n-m) = -\frac{\sin(\pi(n+m)+2|n-m|\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda}))}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}. \tag{8.28}$$

Now, the rest of the proof follows as in a) above replacing $\mathrm{Re}\, r_h(\lambda, n-m)$ by $\mathrm{Re}\, r_h^J(\lambda, n-m)$ and $T_0^h$ by $T^h_{4/h^2}$ given by (8.19). This ends the proof. □

To prove the next result, one will exploit Proposition 7.8 and Corollary 7.9. Nevertheless, it is important to note that it can be also obtained by using the fact that for $\lambda \in (0, \frac{4}{h^2})$,

$$\mathrm{Im}\, T_h(\lambda) = \mathscr{V}_h^{1/2}(\langle(\cdot)h^{-1}\rangle^{-\nu_0/2}\mathrm{Im}\, R_h(\lambda)\langle(\cdot)h^{-1}\rangle^{-\nu_0/2} \otimes KK^*)\mathscr{V}_h^{1/2},$$

where $\mathrm{Im}\, R_h(\lambda)$ admits the convolution kernel

$$\mathrm{Im}\, r_h(\lambda, n-m) = \frac{\cos(2(n-m)\mathrm{Arcsin}(\frac{h}{2}\sqrt{\lambda}))}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}.$$

**Proposition 8.6.** *Let $V_h$ satisfy Assumption 3.4. Then, for $\lambda \in (0, \frac{4}{h^2})$, $0 \leq \operatorname{Im} T_h(\lambda) \in \mathfrak{S}_1$. Moreover, one has*

$$\operatorname{Im} T_h(\lambda) = \frac{1}{\sqrt{\lambda}\sqrt{4/h^2 - \lambda}} b_h(\lambda)^* b_h(\lambda), \tag{8.29}$$

*where $b_h(\lambda) : \ell^2(\mathbb{Z}_h, \mathcal{G}) \to \mathbb{C}^2 \otimes \mathcal{G}$ is the operator defined by*

$$b_h(\lambda) := (Y_h(\lambda) \otimes K^*) \mathscr{V}_h^{1/2}, \tag{8.30}$$

*with $Y_h(\lambda) : \ell^2(\mathbb{Z}_h) \to \mathbb{C}^2$ defined by (5.18).*

*Proof.* Thanks to Proposition 7.8 and Corollary 7.9,

$$\operatorname{Im} T_h(\lambda) = \pi \mathscr{U}_h(\lambda) \in \mathfrak{S}_1, \qquad \lambda \in (0, 4/h^2). \tag{8.31}$$

where w.r.t. the notations of Proposition 7.5,

$$\mathscr{U}_h(\lambda) = U_h(\lambda)^* U_h(\lambda), \qquad U_h(\lambda) = \frac{u_h(\frac{2}{h} g_h(\lambda))}{h^{1/2} \lambda^{1/4} (\frac{4}{h^2} - \lambda)^{1/4}}. \tag{8.32}$$

It follows that $\operatorname{Im} T_h(\lambda) \geq 0$. From (7.17), one gets

$$\begin{aligned} \operatorname{Im} T_h(\lambda) &= \frac{\pi}{h\sqrt{\lambda}\sqrt{4/h^2 - \lambda}} u_h\Big(\frac{2}{h} g_h(\lambda)\Big)^* u_h\Big(\frac{2}{h} g_h(\lambda)\Big) \\ &= \frac{\pi}{h\sqrt{\lambda}\sqrt{4/h^2 - \lambda}} \mathscr{V}_h^{1/2} \Big[\mathscr{A}_h\Big(\frac{2}{h} g_h(\lambda)\Big)^* \mathscr{A}_h\Big(\frac{2}{h} g_h(\lambda)\Big) \otimes K K^*\Big] \mathscr{V}_h^{1/2}. \end{aligned} \tag{8.33}$$

Thanks to (7.33), one has for $\phi \in \ell^2(\mathbb{Z}_h)$

$$\begin{aligned} &\frac{\pi}{h} \mathscr{A}_h\Big(\frac{2}{h} g_h(\lambda)\Big)^* \mathscr{A}_h\Big(\frac{2}{h} g_h(\lambda)\Big) \phi(hn) \\ &= \cos[2(hn)h^{-1} g_h(\lambda)] \langle (hn)h^{-1} \rangle^{-\nu_0/2} \sum_{m \in \mathbb{Z}} \cos[2(hm)h^{-1} g_h(\lambda)] \langle (hm)h^{-1} \rangle^{-\nu_0/2} \phi(hm) \\ &+ \sin[2(hn)h^{-1} g_h(\lambda)] \langle (hn)h^{-1} \rangle^{-\nu_0/2} \sum_{m \in \mathbb{Z}} \sin[2(hm)h^{-1} g_h(\lambda)] \langle (hm)h^{-1} \rangle^{-\nu_0/2} \phi(hm) \\ &= Y_h(\lambda)^* Y_h(\lambda) \phi(hn), \end{aligned} \tag{8.34}$$

where the operator $Y_h(\lambda)^*$ is given by (5.19). Putting together (8.33) and (8.34), one gets

$$\operatorname{Im} T_h(\lambda) = \frac{1}{\sqrt{\lambda}\sqrt{4/h^2 - \lambda}} \mathscr{V}_h^{1/2} \big[Y_h(\lambda)^* Y_h(\lambda) \otimes K K^*\big] \mathscr{V}_h^{1/2} = (8.29). \tag{8.35}$$

This ends the proof. □

The next result uses in particular the identities (see e.g. [13, Section 5.4])

$$\int_{\mathbb{R}} \mathscr{N}_\pm(s, tT) \frac{dt}{\pi(1 + t^2)} = \frac{1}{\pi} \operatorname{Tr} \arctan(s^{-1} T), \qquad s > 0, \tag{8.36}$$

where $0 \leq T = T^* \in \mathfrak{S}_1$.

**Proposition 8.7.** *Let $\nu_0 > 3$ in Assumption 3.4. As $\lambda \searrow 0$ and $\lambda \nearrow \frac{4}{h^2}$, the following bounds hold:*

$$\begin{aligned} &\frac{1}{\pi} \operatorname{Tr} \arctan((x(1 + \varepsilon))^{-1} \operatorname{Im} T_h(\lambda)) + \mathcal{O}(1) \\ &\qquad \leq \int_{\mathbb{R}} \mathscr{N}_\pm(x, \operatorname{Re} T_h(\lambda) + t \operatorname{Im} T_h(\lambda)) \frac{dt}{\pi(1 + t^2)} \\ &\qquad\quad \leq \frac{1}{\pi} \operatorname{Tr} \arctan((x(1 - \varepsilon))^{-1} \operatorname{Im} T_h(\lambda)) + \mathcal{O}(1), \end{aligned}$$

*for any $\varepsilon \in (0, 1)$ and $x > 0$.*

*Proof.* It follows from the Weyl inequalities (8.1) that for any $\varepsilon \in (0,1)$ and $x > 0$,

$$\begin{aligned}\mathscr{N}_{\pm}((1+\varepsilon)x, t\mathrm{Im}\, T_h(\lambda)) - \mathscr{N}_{\mp}(\varepsilon x, \mathrm{Re}\, T_h(\lambda)) \\ \leq \mathscr{N}_{\pm}(x, \mathrm{Re}\, T_h(\lambda) + t\mathrm{Im}\, T_h(\lambda)) \\ \leq \mathscr{N}_{\pm}((1-\varepsilon)x, t\mathrm{Im}\, T_h(\lambda)) + \mathscr{N}_{\pm}(\varepsilon x, \mathrm{Re}\, T_h(\lambda)).\end{aligned}$$

This together with Proposition 8.5, (8.36) and Proposition 8.6 gives the claim, and ends the proof. □

Applying Proposition 8.7 with $x = 1$, one obtains immediately the following result:

**Corollary 8.8.** *Let $\nu_0 > 3$ in Assumption 3.4. Then, for any $\varepsilon \in (0,1)$,*

$$\begin{aligned}&\frac{1}{\pi}\mathrm{Tr}\arctan((1+\varepsilon)^{-1}\mathrm{Im}\, T_h(\lambda)) + \mathcal{O}(1) \\ &\quad \leq \mp\xi(\lambda; H_h^{\mp}, H_h) \\ &\quad \leq \frac{1}{\pi}\mathrm{Tr}\arctan((1-\varepsilon)^{-1}\mathrm{Im}\, T_h(\lambda)) + \mathcal{O}(1),\end{aligned}$$

*as $\lambda \searrow 0$ and $\lambda \nearrow \frac{4}{h^2}$.*

Now, for $s > 0$ and $\lambda \in (0, \frac{4}{h^2})$, one has

$$\mathscr{N}_{\pm}(s, t\mathrm{Im}\, T_h(\lambda)) = \mathscr{N}_{\pm}\Big(s, \frac{tb_h(\lambda)^* b_h(\lambda)}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}\Big) = \mathscr{N}_{\pm}\Big(s, \frac{tb_h(\lambda) b_h(\lambda)^*}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}\Big) = \mathscr{N}_{\pm}(s, t\Omega_h(\lambda)). \tag{8.37}$$

This together with (8.36) and Corollary 8.8 gives Theorem 5.4.

## 9. Proof of the asymptotics (5.22) and (5.25)

The aim of this section is to prove identities (5.22) and (5.25). The operators $\Omega_h(\lambda)$ and $\Omega_{\bullet,h}(\lambda)$, $\bullet \in \{0, \frac{4}{h^2}\}$ are respectively given by (5.20) and Remark 5.5.

**Proposition 9.1.** *Let $V_h$ satisfy Assumption 3.4 with $\nu_0 > 3$. Then, for any $x > 0$ one has*

$$\mathrm{Tr}\big(\arctan(x^{-1}\Omega_h(\lambda)) - \arctan(x^{-1}\Omega_{0,h}(\lambda))\big) = \mathcal{O}_h(1), \qquad \lambda \searrow 0, \tag{9.1}$$

*and*

$$\mathrm{Tr}\big(\arctan(x^{-1}\Omega_h(\lambda)) - \arctan(x^{-1}\Omega_{4/h^2,h}(\lambda))\big) = \mathcal{O}_h(1), \qquad \lambda \nearrow \frac{4}{h^2}. \tag{9.2}$$

*Proof.* We only give the proof of (9.1) since the one of (9.2) follows in a similar way.

Using (8.36) and (8.37) one gets for $x > 0$ and $\lambda \in (0, \frac{4}{h^2})$,

$$\mathrm{Tr}\big(\arctan(x^{-1}\Omega_h(\lambda))\big) = \mathrm{Tr}\big(\arctan(x^{-1}\widetilde{\Omega}_h(\lambda))\big), \tag{9.3}$$

where, according to (8.35), one has

$$\widetilde{\Omega}_h(\lambda) = \mathrm{Im}\, T_h(\lambda) = \frac{1}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}\mathscr{V}_h^{1/2}\big[Y_h(\lambda)^* Y_h(\lambda) \otimes KK^*\big]\mathscr{V}_h^{1/2}, \tag{9.4}$$

with the operator $Y_h(\lambda) : \ell^2(\mathbb{Z}_h) \to \mathbb{C}^2$ defined by (5.18). Similarly, one shows that

$$\mathrm{Tr}\big(\arctan(x^{-1}\Omega_{0,h}(\lambda))\big) = \mathrm{Tr}\big(\arctan(x^{-1}\widetilde{\Omega}_{0,h}(\lambda))\big), \tag{9.5}$$

where

$$\widetilde{\Omega}_{0,h}(\lambda) = \frac{h}{2\sqrt{\lambda}}\mathscr{V}_h^{1/2}\big[Y_{0,\nu_0}^* Y_{0,\nu_0} \otimes KK^*\big]\mathscr{V}_h^{1/2}, \tag{9.6}$$

with the operator $Y_{0,\nu_0} : \ell^2(\mathbb{Z}_h) \to \mathbb{C}^2$ defined by (5.23). It follows from the Lifshits-Krein trace formula (1.2) and identities (9.3)-(9.6) that

$$\begin{aligned}&\big|\mathrm{Tr}\big(\arctan(x^{-1}\Omega_h(\lambda)) - \arctan(x^{-1}\Omega_{0,h}(\lambda))\big)\big| \\ &\quad \leq \int_{\mathbb{R}} |\xi(s; x^{-1}\widetilde{\Omega}_h(\lambda), x^{-1}\widetilde{\Omega}_{0,h}(\lambda))| ds \leq \frac{1}{x}\|\widetilde{\Omega}_h(\lambda) - \widetilde{\Omega}_{0,h}(\lambda)\|_{\mathfrak{S}_1},\end{aligned} \tag{9.7}$$

(see [32, Theorem 8.2.1]). Thanks to (9.4) and (9.6), one has

$$\|\widetilde{\Omega}_h(\lambda)-\widetilde{\Omega}_{0,h}(\lambda)\|_{\mathfrak{S}_1}\leq\|\mathscr{V}_h\|\|KK^*\|_{\mathfrak{S}_1}\Big\|\frac{1}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}Y_h(\lambda)^*Y_h(\lambda)-\frac{h}{2\sqrt{\lambda}}Y_{0,\nu_0}^*Y_{0,\nu_0}\Big\|_{\mathfrak{S}_1}. \tag{9.8}$$

Then, to conclude, it suffices to show that

$$\Big\|\frac{1}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}Y_h(\lambda)^*Y_h(\lambda)-\frac{h}{2\sqrt{\lambda}}Y_{0,\nu_0}^*Y_{0,\nu_0}\Big\|_{\mathfrak{S}_1}=\mathcal{O}(\sqrt{\lambda}),\qquad\lambda\searrow 0, \tag{9.9}$$

as follows. Firstly, one can observe that for $\lambda\in(0,\frac{4}{h^2})$,

$$\begin{aligned}&\Big\|\frac{1}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}Y_h(\lambda)^*Y_h(\lambda)-\frac{h}{2\sqrt{\lambda}}Y_{0,\nu_0}^*Y_{0,\nu_0}\Big\|_{\mathfrak{S}_1}\\&\leq\Big|\frac{1}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}-\frac{h}{2\sqrt{\lambda}}\Big|\|Y_h(\lambda)^*Y_h(\lambda)\|_{\mathfrak{S}_1}+\frac{h}{2\sqrt{\lambda}}\|Y_h(\lambda)^*Y_h(\lambda)-Y_{0,\nu_0}^*Y_{0,\nu_0}\|_{\mathfrak{S}_1}.\end{aligned} \tag{9.10}$$

a) Let us treat the first term of the r.h.s. of (9.10). It can be checked that $\frac{1}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}-\frac{h}{2\sqrt{\lambda}}=\mathcal{O}_h(\sqrt{\lambda})$ as $\lambda\searrow 0$. Furthermore, it follows from (8.34) that

$$\|Y_h(\lambda)^*Y_h(\lambda)\|_{\mathfrak{S}_1}\leq\sum_{n\in\mathbb{Z}}\cos^2[2ng_h(\lambda)]\langle n\rangle^{-\nu_0}+\sum_{n\in\mathbb{Z}}\sin^2[2ng_h(\lambda)]\langle n\rangle^{-\nu_0}=\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}.$$

Consequently, one gets

$$\Big|\frac{1}{\sqrt{\lambda}\sqrt{4/h^2-\lambda}}-\frac{h}{2\sqrt{\lambda}}\Big|\|Y_h(\lambda)^*Y_h(\lambda)\|_{\mathfrak{S}_1}=\mathcal{O}_h(\sqrt{\lambda}),\qquad\lambda\searrow 0. \tag{9.11}$$

b) Now, let us treat the second term of the r.h.s. of (9.10). A direct computation shows that for any $\phi\in\ell^2(\mathbb{Z}_h)$ and $n\in\mathbb{Z}$,

$$Y_{0,\nu_0}^*Y_{0,\nu_0}\phi(hn)=\langle(hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}}\langle(hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm).$$

This together with (8.34) gives

$$\begin{aligned}&(Y_h(\lambda)^*Y_h(\lambda)-Y_{0,\nu_0}^*Y_{0,\nu_0})\phi(hn)\\&=-2\sin^2[(hn)h^{-1}g_h(\lambda)]\langle(hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}}\langle(hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\&-\langle(hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}}2\sin^2[(hm)h^{-1}g_h(\lambda)]\langle(hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\&+2\sin^2[(hn)h^{-1}g_h(\lambda)]\langle(hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}}2\sin^2[(hm)h^{-1}g_h(\lambda)]\langle(hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm)\\&+\sin[2(hn)h^{-1}g_h(\lambda)]\langle(hn)h^{-1}\rangle^{-\nu_0/2}\sum_{m\in\mathbb{Z}}\sin[2(hm)h^{-1}g_h(\lambda)]\langle(hm)h^{-1}\rangle^{-\nu_0/2}\phi(hm).\end{aligned}$$

It follows that

$$\begin{aligned}&\|Y_h(\lambda)^*Y_h(\lambda)-Y_{0,\nu_0}^*Y_{0,\nu_0}\|_{\mathfrak{S}_1}\\&\leq 4\Big(\sum_{n\in\mathbb{Z}}\sin^4[ng_h(\lambda)]\langle n\rangle^{-\nu_0}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}\langle n\rangle^{-\nu_0}\Big)^{1/2}\\&\qquad+4\sum_{n\in\mathbb{Z}}\sin^4[ng_h(\lambda)]\langle n\rangle^{-\nu_0}+\sum_{n\in\mathbb{Z}}\sin^2[2ng_h(\lambda)]\langle n\rangle^{-\nu_0}\\&\leq\big(4|g_h(\lambda)|+8g_h^2(\lambda)\big)\sum_{n\in\mathbb{Z}}n^2\langle n\rangle^{-\nu_0}\underset{\lambda\searrow 0}{\sim}2h\sqrt{\lambda}\sum_{n\in\mathbb{Z}}n^2\langle n\rangle^{-\nu_0}.\end{aligned}$$

Therefore,

$$\frac{h}{2\sqrt{\lambda}}\|Y_h(\lambda)^*Y_h(\lambda)-Y_{0,\nu_0}^*Y_{0,\nu_0}\|_{\mathfrak{S}_1}=\mathcal{O}_h(1),\qquad\lambda\searrow 0. \tag{9.12}$$

One obtains immediately the claim by putting together (9.10), (9.11) and (9.12). □

**Remark 9.2.** *If $\nu_0 > 5$, then we see from the proof that more precise estimates in Proposition 9.1 may be obtained so that*

$$\mathrm{Tr}\big(\arctan(x^{-1}\Omega_h(\lambda)) - \arctan(x^{-1}\Omega_{0,h}(\lambda))\big) = \mathcal{O}_h(\sqrt{\lambda}) = o_h(1), \qquad \lambda \searrow 0, \tag{9.13}$$

*and*

$$\mathrm{Tr}\big(\arctan(x^{-1}\Omega_h(\lambda)) - \arctan(x^{-1}\Omega_{4/h^2,h}(\lambda))\big) = \mathcal{O}_h(\sqrt{4/h^2 - \lambda}) = o_h(1), \qquad \lambda \nearrow \frac{4}{h^2}. \tag{9.14}$$

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