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# Poincaré-Friedrichs type inequalities and discrete compactness for broken Sobolev/polynomial spaces

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O. Boussoufa <sup>1</sup>

## Abstract

We introduce a novel technique to establish Poincaré-Friedrichs type inequalities and discrete compactness for broken Sobolev and polynomial spaces. Distinguishing itself from prior works, this method can be applied to all the linear first-order differential operators, including operators with infinite-dimensional kernels. Our contributions encompass the re-establishment of previously proven results and the derivation of novel ones, such as a generalized Korn inequality in 2D for the trace-free symmetric gradient and the discrete counterpart of Weber's compactness theorem.

**keywords.** Poincaré-Friedrichs inequalities, discrete compactness, broken Sobolev/polynomial spaces, discontinuous Galerkin, finite elements, extension operators.

## 1 Introduction

Discrete Poincaré-Friedrichs type inequalities on broken Sobolev spaces constitute an essential tool for the analysis of discontinuous Galerkin discretization methods for nonlinear partial differential equations. The main contribution of this paper is to introduce a novel technique to derive such inequalities. A continuous Poincaré-Friedrichs type inequality has the following form:

$$\forall v \in V(\Omega), \quad \|v\|_{V(\Omega)} \leq c \|\mathcal{D}v\|_{L^q(\Omega)}, \quad (1.1)$$

where  $V(\Omega)$  is a Sobolev space and  $\mathcal{D}$  is a differential operator (see the next section for notation). For dG schemes, it's essential to extend inequality (1.1) to piecewise smooth functions on a given triangulation of the physical domain. To do so, the key idea is to construct an operator  $I_h$  for which the decomposition

$$v = (v - I_h v) + I_h v \quad (1.2)$$

is such that  $(v - I_h v) \in V(\Omega)$ , and therefore inequality (1.1) holds, and an estimate like the following is satisfied :

$$\sum_{D \in \mathcal{P}_h} \|I_h v\|_{W^{1,q}(D)}^q \lesssim \sum_{e \in \mathcal{E}_h} c(p_e) h_e^{1-q} \| [v] \|_{L^q(e)}^q, \quad (1.3)$$

where  $c(p_e) > 0$  is a constant depending on the local polynomial degree  $p_e$ . In many practical cases, properties (1.1)-(1.3) are sufficient to deduce the inequality

$$\|v\|_{V(\Omega)} \lesssim \left( \|\mathcal{D}v\|_{L^q(\Omega)}^q + \sum_{e \in \mathcal{E}_h} c(p_e) h_e^{1-q} \| [v] \|_{L^q(e)}^q \right)^{1/q}.$$

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Another contribution of this paper is the establishment of discrete compactness results. For this purpose we use decomposition (1.2) and estimate (1.3). The key idea is to apply the compactness properties of the Sobolev space  $V(\Omega)$  to  $(v - I_h v)$  and to combine estimate (1.3) and the discrete Rellich-Kondrachov compactness for  $(I_h v)$ .

There exist two major methods for the establishment of discrete Poincaré-Friedrichs type inequalities in the literature. The first one was introduced by Arnold [2] to establish the following generalized Poincaré's inequality for piecewise  $H^1$ -functions.

$$\|v\|_{L^2(\Omega)}^2 \lesssim \sum_{D \in \mathcal{P}_h} \|\nabla v\|_{L^2(D)}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v] \|_{L^2(e)}^2.$$

Note that the terms related to the edges along the boundary  $\partial\Omega$  are also penalized which is not optimal. Arnold's method relies on the use of  $L^q$  dual norms together with a  $W^{2,q}$  regularity estimate for an appropriate PDE. It can be applied to a large variety of differential operators including those with infinite-dimensional kernels. However, since regular solutions of PDEs are required, it works only for smooth domains. Completely different techniques were introduced by Brenner [8] in order to deal with general (non smooth) polygonal domains and to derive optimally penalized inequalities. Although Brenner's method has been successfully applied to generalize some continuous inequalities to broken Sobolev spaces (see [10, 9, 28]), it does not work for differential operators with infinite-dimensional kernels. The method introduced in this paper combines the advantages of both methods. More precisely, It is applicable to any differential operator : If the kernel of the differential operator is a finite-dimensional space of polynomials, we prove the desired inequality for the corresponding broken Sobolev space; otherwise, the inequality is obtained for piecewise polynomial functions. For an overview of discrete Poincaré-Friedrichs type inequalities we refer the reader to [22, 14, 12, 29, 11, 18, 20] and the references therein.

The discrete compactness properties have not been widely investigated for the spaces of totally discontinuous piecewise polynomial functions. Using techniques inspired by the Finite Volume literature, Di Pietro&Ern [14] established the counterpart of the Rellich-Kondrachov compactness theorem for broken Sobolev spaces. More precisely, let  $\{v_h\}_{h \in (0,1)}$  be a sequence of piecewise polynomial functions that are bounded in the following norm.

$$v \mapsto \left( \sum_{D \in \mathcal{P}_h} \|\nabla v\|_{L^q(D)}^q + \sum_{e \in \mathcal{E}_{int}} h_e^{1-q} \| [v] \|_{L^q(e)}^q + \sum_{e \in \mathcal{E}_b} h_e^{1-q} \|v\|_{L^q(e)}^q \right)^{1/q}. \quad (1.4)$$

Then the sequence  $(v_h)_{h \in (0,1)}$  is relatively compact in  $L^r(\Omega)$ , where the real number  $r \geq 1$  is defined by a well-known criterion. In this paper we derive a better version of this result where the third sum in (1.4) can be replaced by the  $q$ -exponent of any semi-norm  $\Phi_h$  that satisfy some given assumptions. One possible choice is to take for all  $h \in (0, 1)$ ,

$$\Phi_h := \left( \int_{\Omega} |v|^q dx \right)^{1/q}.$$

This means that the terms of the third sum in (1.4) do not need to be penalized. For similar compactness results for Finite Element spaces we refer the reader to [23, 13, 11, 21, 26] and the references therein.

The paper is organized as follows. In Section 2 we give notation and preliminaries. Section 3 is concerned with the construction of the operator  $I_h$  in the Hilbertian setting. First, we present the structure of the considered subpartitions of  $\mathcal{P}_h$  and introduce some assumptions. In Subsection 3.1 we construct the operator  $I_h$  for real-valued functions. In this simplest case we present the main techniques of the method in detail. Then, we accommodate these results to the case of vector fields in Subsection 3.2. In Section 4, we extend our method to the non-Hilbertian setting. Finally, in Section 5 we conclude with a brief review of the paper and point out some generalizations and future directions.

## 2 Notation and Preliminaries

Let  $\Omega \subset \mathbb{R}^2$  be a bounded polygonal domain with boundary  $\partial\Omega$ . We consider two disjoint subsets  $\gamma_{Dir}$  and  $\gamma_t$  of its boundary where Dirichlet boundary conditions and tangential boundary conditions are imposed respectively. We use standard Sobolev spaces  $W^{s,q}(\Omega)$  and  $H(\text{curl}, \Omega)$  (see Girault&Raviart [17]). The associated subspaces

$W_{\gamma_{Dir}}^{s,q}(\Omega)$  and  $H_{\gamma_t}(\text{curl}, \Omega)$  correspond to null trace value (resp. null tangential trace value) on  $\gamma_{Dir}$  (resp. on  $\gamma_t$ ). Let  $\{\mathcal{P}_h\}_h$  be a family of partitions of  $\Omega$  into polygons, i.e. we only assume that

$$D \cap D' = \emptyset \quad \text{if } D \neq D' \quad \text{and} \quad \bigcup_{D \in \mathcal{P}_h} \bar{D} = \bar{\Omega}.$$

In particular hanging nodes are permitted. We define an edge as a segment line of a subdomain in  $\mathcal{P}_h$ . The set containing all these edges is denoted by  $\mathcal{E}_h$ , which is decomposed into the union of  $\mathcal{E}_{int}$  and  $\mathcal{E}_b$  corresponding to interior and boundary edges. The intersection of any boundary edge with  $\gamma_{Dir}$  and  $\gamma_t$  is either the whole edge or has null measure. We assign to each element  $D \in \mathcal{P}_h$  its diameter  $h_D$ , the diameter  $\rho_D$  of the largest ball contained in  $D$  and a polynomial degree  $p_D$ . We denote by  $h_e$  the length of the edge  $e \in \mathcal{E}_h$  and define  $p_e$  by the rule

$$\begin{cases} \max\{p_D, p_{D'}\} & \text{if } e = \partial D \cap \partial D' \in \mathcal{E}_{int}, \\ p_D & \text{if } e \in \mathcal{E}_b \end{cases}$$

We denote by  $\boldsymbol{\delta}$  the couple  $(h, \mathbf{p})$  where  $h := \max_{D \in \mathcal{P}_h} h_D$  is the mesh size and  $\mathbf{p} := (p_D)_{D \in \mathcal{P}_h}$ . For any quantities  $A, B$ , the notation  $A \lesssim B$  means that the constant  $c$  in the inequality  $A \leq cB$  does not depend on the mesh-size and the polynomial degrees. The broken-Sobolev space  $W^{1,q}(\Omega, \mathcal{P}_h)$  is defined as follows

$$W^{1,q}(\Omega, \mathcal{P}_h) = \{v \in L^1(\Omega); \quad v|_D \in W^{1,q}(D) \quad \forall D \in \mathcal{P}_h\}.$$

We set

$$\|v\|_{W_{\gamma_{Dir}}^{1,q}(\Omega, \mathcal{P}_h)} = \left( \sum_{D \in \mathcal{P}_h} \|\nabla v\|_{L^q(D)}^q + \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{1-q} \|v\|_{L^q(e)}^q \right)^{1/q}.$$

The broken-polynomial space  $S^{\mathbf{p}}(\Omega, \mathcal{P}_h)$  consists of (possibly) discontinuous piecewise polynomial functions, i.e.,

$$S^{\mathbf{p}}(\Omega, \mathcal{P}_h) = \{v \in L^1(\Omega); \quad v|_D \in \mathbb{P}_{p_D}(D), \quad \forall D \in \mathcal{P}_h\},$$

where  $\mathbb{P}_p$  is either the space of polynomials of total degree at most equal to  $p$ . For  $D \in \mathcal{P}_h$ ,  $e \in \mathcal{E}_h$  and  $v \in W^{1,q}(\Omega, \mathcal{P}_h)$  we denote by  $v^+$  and  $v^-$  the trace value of  $v$  on the edge  $e$  from the exterior and the interior of  $D$  respectively. We define the jump and the average of the function  $v$  by

$$[v] = (v^+ - v^-)|_e, \quad \{v\} = \frac{1}{2} (v^+ + v^-)|_e.$$

When  $e$  belongs to the boundary, the above quantities are simply the trace value, i.e.

$$[v] = \{v\} = v|_e.$$

Let  $D$  be a polygonal domain and  $e$  be one of its edges. We recall the standard trace inequality (see Boyer&Fabrie [7])

$$\forall v \in W^{1,q}(D), \quad \|v\|_{L^q(e)} \lesssim h_e^{1-1/q} |D|^{-1/q} (\|v\|_{L^q(D)} + h_D \|\nabla v\|_{L^q(D)}), \quad (2.1)$$

where  $|D|$  denotes the area of  $D$ .

We denote by  $\widehat{K}$  the unit square  $(-1, 1)^2$  or a reference triangle (see Fig. 1). The following inverse inequality holds (see [16])

$$\forall v \in \mathbb{P}_p(K), \quad \|v\|_{H^1(K)} \lesssim p \|v\|_{H^{1/2}(K)}. \quad (2.2)$$

For any real numbers  $s > 0$  and  $+\infty \geq q \geq 1$ , we have (see Bernardi et al. [4, Proposition 4.1]),

$$\forall v \in \mathbb{P}_p(0, 1), \quad \|v\|_{W^{s,q}(0,1)} \lesssim p \|v\|_{L^q(0,1)}. \quad (2.3)$$

The following inverse inequality holds (see Babuška et al. [3, Thm 6.5]),

$$\forall v \in \mathbb{P}_p(0, 1), \quad v(0) = 0, \quad \int_0^1 \frac{v(x)^2}{x} dx \lesssim (1 + \log p) \|v\|_{H^{1/2}(0,1)}^2. \quad (2.4)$$

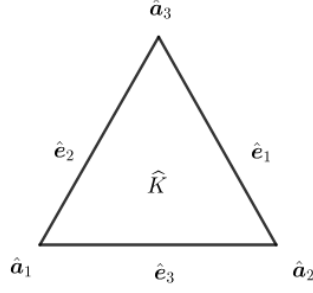


Figure 1: The reference triangle.

### 3 The Hilbertian case

Let  $\mathcal{T}_h$  be a subpartition of  $\mathcal{P}_h$  such that

1. each element in  $\mathcal{T}_h$  is either a triangle or a convex quadrilateral,
2. the intersection of any two elements, if not empty, is either a vertex or a whole edge,
3. for all  $K \subset \mathcal{T}_h$ , there is only one  $D \subset \mathcal{P}_h$  such that  $K \subset D$  and  $h_D \lesssim h_K$ ,
4. to each element  $K \in \mathcal{T}_h$  we associate an invertible affine transformation  $F_K : \hat{K} \rightarrow K$  which satisfies the compatibility condition  $F_{K_1}^{-1}(K_1 \cap K_2) = F_{K_2}^{-1}(K_1 \cap K_2)$ , for any elements  $K_1, K_2 \in \mathcal{T}_h$ .

For technical reasons, we assume that  $\mathcal{T}_h$  is formed by the following patches.

- The patches  $\omega_{Dir}$  and  $\omega_t$  consist of triangles with at least one edge lying on  $\gamma_{Dir}$  and  $\gamma_t$ , respectively.
- Given a set of vertices  $\mathcal{V}$ , the macroelements  $M_z$ , for every  $z \in \mathcal{V}$ , collects all the triangles for which  $z$  is a vertex. We assume that the intersection of two different macroelements is empty. Then, we define the patch  $\omega_M$  as the union of all  $M_z$ , for  $z \in \mathcal{V}$ .
- The patch  $\bar{\omega}$  contains all the quadrilaterals connecting the macroelements  $M_z$ , that is,  $K$  belongs to  $\bar{\omega}$  if and only if there exist two vertices  $z, z'$  such that the intersection of  $M_z$  and  $M_{z'}$  with  $K$  is a whole edge. We assume that the intersection of any quadrilateral in  $\bar{\omega}$  with  $\gamma_{Dir}$  has null measure.
- $\tilde{\omega}$  collects the triangles surrounded with the quadrilaterals of  $\bar{\omega}$  (see Fig. 2).

To go further we make some Assumptions on the mesh  $\mathcal{T}_h$ .

- (A1) The number of triangles in each connected component of  $\tilde{\omega}$  is uniformly bounded with respect to the mesh size  $h$ .
- (A2) The triangulation  $\mathcal{T}_h$  is sufficiently small so that every macroelement  $M_z$ ,  $z \in \mathcal{V}$ , and every connected component of the patch  $\tilde{\omega}$  is contained in at most two subdomains of the partition  $\mathcal{P}_h$ .
- (A3) The patch  $\omega_{Dir}$  is the union of two parts  $\omega_{Dir}^1$  and  $\omega_{Dir}^2$  such that: each triangle in  $\omega_{Dir}^1$  (resp.  $\omega_{Dir}^2$ ) has only one edge (resp. one vertex) on  $\gamma_{Dir}$ , the intersection of any two triangles in  $\omega_{Dir}^1$ , if not empty, is a vertex and for any two triangles  $K_1, K_2 \in \omega_{Dir}^1$ , there exists only one triangle in  $\omega_{Dir}^2$  that shares one edge with both  $K_1$  and  $K_2$ . The same assumption is made for the patch  $\omega_t$ .

We denote by  $\mathbb{T}(\mathcal{P}_h)$  the set of all meshes  $\mathcal{T}_h$  that satisfy properties (1)-(4) and Assumptions (A1)-(A3). An indicator of the partition  $\mathcal{P}_h$ 's quality is the constant

$$\kappa(\mathcal{P}_h) := \inf_{\mathcal{T}_h \in \mathbb{T}(\mathcal{P}_h)} \sup_{K \in \mathcal{T}_h} \frac{|K|}{\rho_K^2}.$$

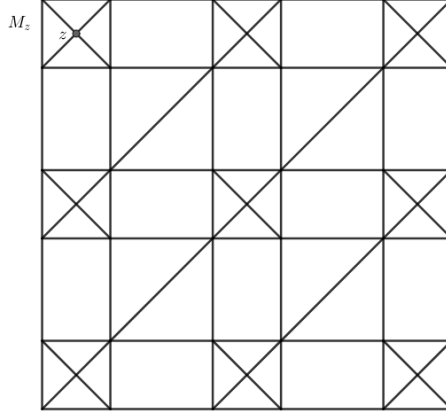


Figure 2: A portion of the subpartition  $\mathcal{T}_h$ .

### 3.1 Real-valued functions

In this part we deal with real-valued functions defined on the domain  $\Omega$ . We assume that the portion  $\gamma_t$  of the boundary is empty.

**Lemma 3.1.** [24, Lemma 4.1] *Let  $p \geq 1$  be an integer. There exists a polynomial  $\varphi_p \in \mathbb{P}_p(-1, 1)$  such that*

$$\varphi_p(1) = 1, \quad \varphi_p(-1) = 0$$

and

$$\|\varphi_p\|_{L^2(-1,1)}^2 = \frac{2}{p(p+2)}.$$

**Lemma 3.2.** *Let  $\hat{\gamma}$  be the union of several edges of the reference triangle  $\hat{K}$ . There exists a linear polynomial-preserving extension operator from  $\hat{\gamma}$  into the reference triangle  $\hat{K}$  satisfying the following estimates.*

$$\forall v \in L^2(\hat{\gamma}), \quad \|\mathcal{E}_{\hat{\gamma}}^{\hat{K}}(v)\|_{H^{1/2}(\hat{K})} \leq c\|v\|_{L^2(\hat{\gamma})}, \quad (3.1)$$

and for any continuous piecewise polynomial function  $v$  defined on  $\hat{\gamma}$ ,

$$\|\mathcal{E}_{\hat{\gamma}}^{\hat{K}}(v)\|_{L^2(\partial\hat{K})} \leq c\|v\|_{L^2(\hat{\gamma})}. \quad (3.2)$$

**Proof.** The desired extension operator is constructed by Ainsworth&Demkowicz [1] when  $\hat{\gamma}$  is equal to the whole boundary. Let  $\hat{e}_k$ ,  $k = 1, 3$  denote the edges of the triangle with  $\hat{e}_3 = (-1, 1)$  (see Fig 1). Now, we consider the case  $\hat{\gamma} = \hat{e}_1 \cup \hat{e}_2$ . Let  $v$  be any continuous piecewise polynomial function defined on  $\hat{\gamma}$  of degree less or equal to  $p$  and denote by  $v_1, v_2$  its restrictions on  $\hat{e}_1$  and  $\hat{e}_2$  respectively.

Let  $\varphi_1 \in \mathbb{P}_p(-1, 1)$  be the polynomial of Lemma 3.1 such that  $\varphi_1(-1) = 1$ , and  $\varphi_1(1) = 0$ . Similarly, we denote by  $\varphi_2$  the corresponding polynomial such that  $\varphi_2(-1) = 0$  and  $\varphi_2(1) = 1$ . The polynomial  $\psi_v$  defined on  $\hat{e}_3$  by

$$\psi_v = v_1(\hat{\mathbf{a}}_2)\varphi_2 + v_2(\hat{\mathbf{a}}_1)\varphi_1$$

satisfies  $\psi_v(\hat{\mathbf{a}}_1) = v_1(\hat{\mathbf{a}}_1)$ , and  $\psi_v(\hat{\mathbf{a}}_2) = v_2(\hat{\mathbf{a}}_2)$ . Furthermore, we have by a standard inverse inequality (see [4]),

$$\begin{aligned} \|\psi_v\|_{L^2(\hat{e}_3)}^2 &:= \|\psi_v\|_{L^2(-1,1)}^2 \lesssim |v_1(\hat{\mathbf{a}}_1)|^2 \|\varphi_2\|_{L^2(-1,1)}^2 + |v_2(\hat{\mathbf{a}}_2)|^2 \|\varphi_1\|_{L^2(-1,1)}^2 \\ &= \frac{2}{p(p+2)} (|v_1(\hat{\mathbf{a}}_2)|^2 + |v_2(\hat{\mathbf{a}}_1)|^2) \\ &\lesssim \frac{2p^2}{p(p+2)} \left( \|v_1\|_{L^2(\hat{e}_1)}^2 + \|v_2\|_{L^2(\hat{e}_2)}^2 \right) \\ &\lesssim \|v\|_{L^2(\hat{\gamma})}^2. \end{aligned} \quad (3.3)$$

Next, we define the extension of  $v$  by

$$\mathcal{E}_{\hat{\gamma}}^{\hat{K}} v := \mathcal{E}_{\partial \hat{K}}^{\hat{K}} \tilde{v},$$

with

$$\tilde{v} = \begin{cases} v_i & \text{on } \hat{e}_i, i = 1, 2, \\ \psi_v & \text{on } \hat{e}_3. \end{cases}$$

Observe that  $\tilde{v}$  is a continuous piecewise polynomial defined on the whole boundary of the triangle. Thus the extension  $\mathcal{E}_{\partial \hat{K}}^{\hat{K}} v$  is well defined. Estimate (3.2) is a direct consequence of estimate (3.3).

The case where  $\hat{\gamma}$  is equal to one edge can be treated in a similar way. For instance, if  $\hat{\gamma} = \hat{e}_1$ , it suffices to consider the extension

$$\tilde{v} = \begin{cases} v_1 & \text{on } \hat{e}_1, \\ v_1(\hat{\mathbf{a}}_3) \varphi_3, & \text{on } \hat{e}_2, \end{cases}$$

where  $\varphi_3$  is the polynomial of Lemma 3.1 such that  $\varphi_3(\hat{\mathbf{a}}_3) = 1$  and  $\varphi_3(\hat{\mathbf{a}}_1) = 0$ . Finally, we set  $\mathcal{E}_{\hat{e}_1}^{\hat{K}}(v) := \mathcal{E}_{\hat{e}_1 \cup \hat{e}_2}^{\hat{K}}(\tilde{v})$ .  $\square$

**Remark 3.3.** The extension operator  $\mathcal{E}_{\hat{\gamma}}^{\hat{K}}$  of Lemma 3.2 given on the reference triangle can be used to define the corresponding operator on any triangle  $K$  as follows.

$$\mathcal{E}_{\hat{\gamma}}^K(v) := \left( \mathcal{E}_{\hat{\gamma}}^{\hat{K}}(v \circ F_K) \right) \circ F_K^{-1}.$$

This formula will be applied for all extension operators in the sequel.

**Lemma 3.4.** [13] Let  $\hat{K}$  be the unit square  $(-1, 1)^2$  and  $\hat{e}_1, \hat{e}_2$  be two opposite edges of  $\hat{K}$ . There exists a linear polynomial-preserving extension operator  $\mathcal{E}_{\hat{e}_1 \cup \hat{e}_2}^{\square, \hat{K}} : H^{1/2}(\hat{e}_1 \cup \hat{e}_2) \rightarrow H^1(\hat{K})$  such that

$$\forall v \in H^1(\hat{e}_1 \cup \hat{e}_2), \quad \|\mathcal{E}_{\hat{e}_1 \cup \hat{e}_2}^{\square, \hat{K}}(v)\|_{H^1(\hat{K})} \leq c \|v\|_{H^{1/2}(\hat{e}_1 \cup \hat{e}_2)}$$

The main result is now stated.

**Theorem 3.5.** Assume that Assumptions (A1)-(A3) are satisfied. Then, there exists a linear operator  $I_h(\gamma_{Dir})$  defined from  $S^{\mathbf{p}}(\Omega, \mathcal{P}_h)$  into  $S^{\mathbf{p}}(\Omega, \mathcal{T}_h)$  with the following properties. For all  $v_{\delta} \in S^{\mathbf{p}}(\Omega, \mathcal{P}_h)$ , we have

$$v_{\delta} - I_h(\gamma_{Dir})v_{\delta} \in \mathcal{C}^0(\bar{\Omega}) \cap H_{\gamma_{Dir}}^1(\Omega) \quad (3.4)$$

and the following estimate holds.

$$\sum_{D \in \mathcal{P}_h} \|I_h(\gamma_{Dir})v_{\delta}\|_{H^1(D)}^2 \lesssim \sup_{K \in \mathcal{T}_h} \frac{|K|}{\rho_K^2} \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} p_e^2 \| [v_{\delta}] \|_{L^2(e)}^2, \quad (3.5)$$

**Proof.** To simplify notation, we suppress the subscript  $\delta$  and set  $I_h := I(\gamma_{Dir})$ . For ease of exposition, let us first consider the very simple case where the partition  $\mathcal{P}_h$  consists of two triangles  $K_1$  and  $K_2$ . We also assume that  $\gamma_{Dir} = e_1 \cup e_2$  (see Fig. 3). The operator  $I_h$  is defined element-wise by

$$(I_h v)|_{K_1} = \mathcal{E}_{e_1 \cup e_2}^{K_1}(v|_{K_1})$$

and

$$(I_h v)|_{K_2} = \mathcal{E}_{e_{12}}^{K_2}(v|_{K_2} - \tilde{v}),$$

where

$$\tilde{v} = (v - I_h v)|_{K_1} \quad \text{on } e_{12}.$$

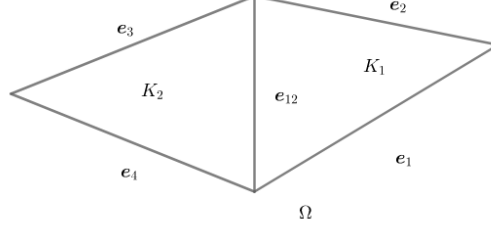


Figure 3: A simple partition consisting of two triangles.

By definition, we have

$$(v|_{K_2} - I_h v|_{K_2})|_{e_{12}} = \tilde{v} = (v|_{K_1} - (I_h v)|_{K_1})|_{e_{12}}.$$

Therefore, the function  $v - I_h v$  is continuous on  $\Omega$ . In addition, it can be easily checked that this function vanishes on the portion  $e_1 \cup e_2$ . For the stability property we have by scaling, the inverse inequality (2.2) and Lemma 3.2,

$$\begin{aligned} \|I_h v\|_{H^1(K_1)} &\lesssim \frac{|K_1|^{1/2}}{\rho_{K_1}} p_{K_1} \|\hat{v}\|_{L^2(\hat{e}_1 \cup \hat{e}_2)} \\ &\lesssim \frac{|K_1|^{1/2}}{\rho_{K_1}} p_{K_1} (h_{e_1}^{-1/2} \|v|_{K_1}\|_{L^2(e_1)} + h_{e_2}^{-1/2} \|v|_{K_1}\|_{L^2(e_2)}) \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|I_h v\|_{H^1(K_2)} &\lesssim \|\mathcal{E}_{e_{12}}^{K_2}(v|_{K_2} - v|_{K_1})\|_{H^1(K_2)} + \|\mathcal{E}_{e_{12}}^{K_2}(I_h v)|_{K_1}\|_{H^1(K_2)} \\ &\lesssim \frac{|K_2|^{1/2}}{\rho_{K_2}} (p_{e_{12}} h_{e_{12}}^{-1/2} \|v\|_{L^2(e_{12})} + p_{K_1} h_{e_{12}}^{-1/2} \|(I_h v)|_{K_1}\|_{L^2(e_{12})}) \end{aligned}$$

We conclude using property (3.2).

Now, we are interested in the general case where  $\mathcal{P}_h$  is a partition of  $\Omega$  into polygons. Taking into account Brenner's trick [8], it suffices to prove Theorem 3.5 in the case where  $\mathcal{P}_h$  coincides with  $\mathcal{T}_h$ .

**Step 1.(patch  $\omega_{Dir}$ )** For any triangle  $K \in \omega_{Dir}^1$ , we set

$$(Iv)|_K = \mathcal{E}_{\partial K \cap \gamma_{Dir}}^K(v|_K),$$

Let  $K \in \omega_{Dir}^2$ . According to Assumption (A3), there exist two triangles  $K_1, K_2 \in \omega_{Dir}^1$  such that  $K$  shares the edges  $e_1, e_2$  with  $K_1$  and  $K_2$ , respectively. The operator  $I_h$  is given by

$$(Iv)|_K = \mathcal{E}_{e_1 \cup e_2}^K(v|_K - (v - I_h)|_{e_1 \cup e_2}^+).$$

Now, we set

$$\gamma_b := \partial \left( \bigcup_{K \in \omega_{Dir}} K \right) \cap \Omega.$$

**Step 2.(patch  $\omega_M$ )** Let  $z \in \mathcal{V}$  and  $M_z = \{K_1, K_2, \dots, K_N\}$  be the corresponding macroelement. We denote by  $e_i$ ,  $i = 1, \dots, N$ , the edges  $K_i \cap K_{i+1}$  (with  $K_{N+1} = K_1$ ). Assume for simplicity that  $K_1$  is the only triangle that can have an edge on  $\gamma_b$ . The operator  $I_h$  is defined as follows.

$$(I_h v)|_K = \begin{cases} \mathcal{E}_{\partial K \cap \gamma_b}^{K_1}(v|_{K_1} - (v - I_h v)|_{\partial K \cap \gamma_b}^+) & \text{if } \partial K \cap \gamma_b \text{ has non-null measure,} \\ 0 & \text{otherwise.} \end{cases}$$

For  $i = 2, \dots, N-1$ , we set

$$(I_h v)|_{K_i} = \mathcal{E}_{e_{i-1}}^{K_i} (v|_{K_i} - (v - I_h v)|_{K_{i-1}}).$$

Finally,

$$(I_h v)|_{K_N} = \mathcal{E}_{e_{N-1} \cup e_N}^{K_N} (v|_{K_N} - (v - I_h v)_{e_{N-1} \cup e_N}^+).$$

We only prove boundedness since continuity can be checked as in the above simple case. For  $i = 2, \dots, N-1$ , we use the inverse inequality (2.2) and then property (3.2) to move from one edge to an other. We obtain

$$\begin{aligned} \|I_h v\|_{H^1(K_i)}^2 &\lesssim \|\mathcal{E}_{e_{i-1}}^{K_i}([v])\|_{H^1(K_i)}^2 + \|\mathcal{E}_{e_{i-1}}^{K_i}(I_h v)|_{K_{i-1}}\|_{H^1(K_i)}^2 \\ &\lesssim p_{e_i}^2 h_{e_{i-1}}^{-1} \|[v]\|_{L^2(e_{i-1})}^2 + \max_{1 \leq j \leq i-1} (p_{e_j})^2 h_{e_{i-1}}^{-1} \|(I_h v)|_{K_{i-1}}\|_{L^2(e_{i-1})}^2 \\ &\lesssim \max_{1 \leq j \leq i} (p_{e_j})^2 \sum_{j=1}^{i-1} h_{e_j}^{-1} \|[v]\|_{L^2(e_j)}^2, \end{aligned}$$

However, according to Assumption (A2) the macroelement  $M_z$  has at most one edge  $e_{j_0}$  on  $\mathcal{E}_h$ . Thus, the jumps of  $u$  across the edges  $e_j$  are equal to zero except for  $j = j_0$ . Therefore, we have,

$$\begin{aligned} \|Iv\|_{H^1(K_i)}^2 &\lesssim p_{e_{j_0}}^2 h_{e_{j_0}}^{-1} \|[v\delta]\|_{L^2(e_j)}^2 \\ &\lesssim \sum_{j=1}^N p_{e_j}^2 h_{e_j}^{-1} \|[v]\|_{L^2(e_j)}^2. \end{aligned}$$

The remaining estimate for  $K_N$  can be obtained similarly.

**Step 3.(patch  $\bar{\omega}$ )** Let  $K$  be any quadrilateral in  $\bar{\omega}$  and  $M_z, M_{z'} \in \omega_M$  the corresponding macroelements sharing the edge  $e_1, e_2$  respectively with  $K$ . We set

$$(I_h v)|_K = \mathcal{E}_{e_1 \cup e_2}^{\square, K} (v|_K - u_K),$$

where

$$u_K = \begin{cases} (v - I_h v)_{M_z} & \text{on } e_1, \\ (v - I_h v)_{M_{z'}} & \text{on } e_2. \end{cases}$$

**Step 4.(patch  $\tilde{\omega}$ )** Let  $\tilde{\omega}^0$  be any connected component of  $\tilde{\omega}$  and denote by  $K_1^0, K_2^0, \dots, K_N^0$  the triangles with one edge on  $\partial\tilde{\omega}$ . Let  $K_i^0, K_j^0$  be two adjacent elements. We set  $e_{ij} = K_i^0 \cap K_j^0$  and  $e_k = K_k^0 \cap \partial\tilde{\omega}^0$ , with  $k = i, j$ . We set

$$(I_h v)|_{K_i^0} = \mathcal{E}_{e_i}^{K_i^0} (v|_{K_i^0} - (v - I_h v)_{e_i}^+) \quad (3.6)$$

and

$$(I_h v)|_{K_j^0} = \mathcal{E}_{e_j \cup e_{ij}}^{K_j^0} (v|_{K_j^0} - u_j),$$

where

$$u_j = \begin{cases} (v - I_h v)|_{K_i^0} & \text{on } e_{ij}, \\ (v - I_h v)_{e_j}^+ & \text{on } e_j. \end{cases}$$

If  $K_i^0$  is not adjacent to the other  $K_k^0$ 's, then  $(I_h v)|_{K_i^0}$  is simply defined by (3.6) independently of the remaining ones. Next, we define the new component  $\tilde{\omega}^1 = \tilde{\omega}^0 \setminus \bigcup_{i=1}^N K_i^0$  and repeat the same operations and so on.  $\square$

**Remark 3.6.** Assume that for any element  $K \in \mathcal{P}_h$  and any edge  $e \subset \partial K$ , we have  $h_K \lesssim h_e$ . Then, inspecting the above proof, we can check that the operator  $I_h(\gamma_{Dir})$  satisfies the following estimate.

$$\forall v \in H^1(\Omega, \mathcal{P}_h), \quad \|I_h(\gamma_{Dir})v\|_{L^2(\Omega)}^2 \lesssim \left( \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e \|v\|_{L^2(e)}^2 \right). \quad (3.7)$$

**Corollary 3.7.** Let  $\pi_{0,e}$  be the orthogonal  $L^2$ -projection operator defined from  $L^2(e)$  onto  $\mathbb{P}_0(e)$ . Assume that the portion  $\gamma_{Dir}$  has a non-null measure. Then, the following generalized Poincaré's inequality holds.

$$\|v\|_{L^2(\Omega)}^2 \lesssim \sum_{D \in \mathcal{P}_h} h_D^2 (1 + \kappa(\mathcal{P}_h) |D|^{-1}) \|\nabla v\|_{L^2(D)}^2 + \kappa(\mathcal{P}_h) \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} \|\pi_{0,e}[v]\|_{L^2(e)}^2. \quad (3.8)$$

for all  $v \in H^1(\Omega, \mathcal{P}_h)$ .

**Proof.** It holds that

$$\|v\|_{L^2(\Omega)} \leq c_p \|\nabla v\|_{L^2(\Omega)} \quad (3.9)$$

for all  $v \in H^1_{\gamma_{Dir}}(\Omega)$ . Let  $v_{0,h} \in S^0(\Omega, \mathcal{P}_h)$  be the piecewise constant function such that for every subdomain  $D \in \mathcal{P}_h$ ,

$$\|v - v_{0,h}\|_{L^2(D)} \lesssim h_D \|\nabla v\|_{L^2(D)}. \quad (3.10)$$

Estimate (3.10), Poincaré's inequality (3.9) and the properties of the operator  $I_h(\gamma_{Dir})$  imply that

$$\begin{aligned} \|v\|_{L^2(\Omega)}^2 &\lesssim \|v - v_{0,h}\|_{L^2(\Omega)}^2 + \|v_{0,h} - I_h(\gamma_{Dir})v_{0,h}\|_{L^2(\Omega)}^2 + \|I_h(\gamma_{Dir})v_{0,h}\|_{L^2(\Omega)}^2 \\ &\lesssim \sum_{D \in \mathcal{P}_h} h_D^2 \|\nabla v\|_{L^2(D)}^2 + \sum_{D \in \mathcal{P}_h} \|I_h(\gamma_{Dir})v_{0,h}\|_{H^1(D)}^2 \\ &\lesssim \sum_{D \in \mathcal{D}_h} h_D^2 \|\nabla v\|_{L^2(D)}^2 + \kappa(\mathcal{P}_h) \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} \|v_{0,h}\|_{L^2(e)}^2. \end{aligned}$$

Now, using the fact that  $v_{0,h} = \pi_{0,e}v_{0,h}$  and applying the trace inequality (2.1) we obtain

$$\begin{aligned} \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} \|v_{0,h}\|_{L^2(e)}^2 &\lesssim \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} \|\pi_{0,e}[v]\|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} \|\pi_{0,e}[v - v_{0,h}]\|_{L^2(e)}^2 \\ &\lesssim \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} \|\pi_{0,e}[v]\|_{L^2(e)}^2 + \sum_{D \in \mathcal{P}_h} |D|^{-1} \left( \|v - v_{0,h}\|_{L^2(D)}^2 + h_D^2 \|\nabla(v - v_{0,h})\|_{L^2(D)}^2 \right) \\ &\lesssim \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} \|\pi_{0,e}[v]\|_{L^2(e)}^2 + \sum_{D \in \mathcal{P}_h} h_D^2 |D|^{-1} \|\nabla v\|_{L^2(D)}^2, \end{aligned}$$

which concludes the proof.  $\square$

**Remark 3.8.** For piecewise polynomial functions we can obtain more precise results regarding the injection constant. Let  $v_\delta \in S^{\mathbf{p}}(\Omega, \mathcal{P}_h)$ . By Theorem 3.5 and Minkowski's inequality, we have

$$\begin{aligned} \|v_\delta\|_{L^2(\Omega)} &\leq \|v_\delta - I_h(\gamma_{Dir})v_\delta\|_{L^2(\Omega)} + \|I_h(\gamma_{Dir})v_\delta\|_{L^2(\Omega)} \\ &\leq c_p \left( \sum_{D \in \mathcal{P}_h} \|\nabla v_\delta\|_{L^2(D)}^2 \right)^{1/2} + (1 + c_p) \left( \sum_{D \in \mathcal{P}_h} \|I_h v_\delta\|_{H^1(D)}^2 \right)^{1/2} \\ &\leq c_p \left( \sum_{D \in \mathcal{P}_h} \|\nabla v_\delta\|_{L^2(D)}^2 \right)^{1/2} + c \left( \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} p_e^2 \|v_\delta\|_{L^2(e)}^2 \right)^{1/2}, \end{aligned}$$

Where  $c > 0$  is a constant independent of the mesh-size and  $c_p > 0$  is defined in (3.9). The constant related to the broken gradient is optimal but the quantity

$$\left( \sum_{D \in \mathcal{P}_h} \|\nabla v_\delta\|_{L^2(D)}^2 \right)^{1/2} + \left( \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} p_e^2 \| [v_\delta] \|_{L^2(e)}^2 \right)^{1/2}$$

is larger than the norm  $\|v_\delta\|_{H^1_{\gamma_{Dir}}(\Omega, \mathcal{P}_h)}$ .

### 3.2 Vector fields

The previous theory developed for scalar functions is now accommodated to the case of vector fields defined on  $\Omega$ . For brevity, we only consider tangential boundary conditions, but we emphasize that normal boundary conditions can be also treated in a very similar way and the results below can be easily generalized.

**Lemma 3.9.** *Let  $\hat{e}_1, \hat{e}_2$  be two edges of the reference triangle  $\hat{K}$  and denote by  $\hat{\mathbf{a}}$  their common vertex. Given an integer  $p \geq 1$ , let  $v \in \mathbb{P}_p(\hat{e}_2)$  and  $\mathbf{u} \in \mathbb{P}_p(\hat{e}_1)^2$  be such that  $\mathbf{u}(\hat{\mathbf{a}}) \cdot \mathbf{t}_2 = v(\hat{\mathbf{a}})$ , where  $\mathbf{t}_2$  denotes the tangential unit vector to the edge  $\hat{e}_2$ . Then there exists a polynomial  $\check{\mathcal{E}}_{\hat{e}_1 \cup \hat{e}_2}^{\hat{K}}(\mathbf{u}, v) \in \mathbb{P}_p(\hat{K})^2$  such that*

1.  $\check{\mathcal{E}}_{\hat{e}_1 \cup \hat{e}_2}^{\hat{K}}(\mathbf{u}, v)|_{\hat{e}_1} = \mathbf{u}$  and  $\check{\mathcal{E}}_{\hat{e}_1 \cup \hat{e}_2}^{\hat{K}}(\mathbf{u}, v)|_{\hat{e}_2} \cdot \mathbf{t}_2 = v$ ,
2. The following estimate holds

$$\|\check{\mathcal{E}}_{\hat{e}_1 \cup \hat{e}_2}^{\hat{K}}(\mathbf{u}, v)\|_{H^1(K)}^2 \lesssim \|v\|_{L^2(\hat{e}_2)}^2 + \|\mathbf{u}\|_{L^2(\hat{e}_1)}^2. \quad (3.11)$$

**Proof.** Let  $\mathbf{n}_2$  the outward normal unit vectors to  $\hat{e}_2$ . Assume that  $\hat{e}_2 = (-1, 1)$  for simplicity. Consider the vector field  $\psi(\mathbf{u}, v)$  defined on  $\hat{e}_2 \cup \hat{e}_1$  by

$$\psi(\mathbf{u}, v) = \begin{cases} \mathbf{u} & \text{on } \hat{e}_1, \\ v\mathbf{t}_2 + (\varphi_{p, \hat{\mathbf{a}}} \mathbf{u}(\hat{\mathbf{a}}) \cdot \mathbf{n}_2) \mathbf{n}_2 & \text{on } \hat{e}_2 \end{cases}$$

where  $\varphi_{p, \hat{\mathbf{a}}}$  is the polynomial of Lemma 3.1 defined on  $\hat{e}_2$  and such that  $\varphi_{p, \hat{\mathbf{a}}}(\hat{\mathbf{a}}) = 1$ . We can easily check that  $\psi(\mathbf{u}, v)$  is continuous. Now, we set

$$\check{\mathcal{E}}_{\hat{e}_1 \cup \hat{e}_2}^{\hat{K}}(\mathbf{u}, v) = \mathcal{E}_{\hat{e}_1 \cup \hat{e}_2}^K(\psi(\mathbf{u}, v) \cdot \mathbf{t}_2) \mathbf{t}_2 + \mathcal{E}_{\hat{e}_1 \cup \hat{e}_2}^K(\psi(\mathbf{u}, v) \cdot \mathbf{n}_2) \mathbf{n}_2,$$

where  $\mathcal{E}_{\hat{e}_1 \cup \hat{e}_2}^K$  is the extension operator of Lemma 3.2. Using the properties of the operator  $\mathcal{E}_{\hat{e}_1 \cup \hat{e}_2}^{\hat{K}}$  and the polynomial  $\varphi_{p, \hat{\mathbf{a}}}$ , we obtain

$$\begin{aligned} \|\mathcal{E}_{\hat{e}_2 \cup \hat{e}_1}^{\hat{K}}(\psi(\mathbf{u}, v) \cdot \mathbf{n}_2)\|_{H^1(K)}^2 &\lesssim p^2 \|\widehat{\psi}(\mathbf{u}, v) \cdot \mathbf{n}_2\|_{L^2(\hat{e}_1 \cup \hat{e}_2)}^2 \\ &\lesssim p^2 \left( \|\varphi_p \mathbf{u}(\hat{\mathbf{a}})\|_{L^2(-1, 1)}^2 + \|\mathbf{u} \cdot \mathbf{n}_2\|_{L^2(\hat{e}_2)}^2 \right) \\ &\lesssim \left( 1 + \frac{p^2}{p(p+1)} \right) \|\mathbf{u}\|_{L^2(\hat{e}_2)}^2 \end{aligned}$$

Similarly, we show that

$$\|\mathcal{E}_{\hat{e}_2 \cup \hat{e}_1}^{\hat{K}}(\psi(\mathbf{u}, v) \cdot \mathbf{t}_2)\|_{H^1(K)}^2 \lesssim \|v\|_{L^2(\hat{e}_2)}^2 + \|\mathbf{u} \cdot \mathbf{t}_2\|_{L^2(\hat{e}_1)}^2, \quad (3.12)$$

which concludes the proof.  $\square$

**Theorem 3.10.** *Assume that Assumptions (A1)-(A3) are satisfied. Then, there exists an operator  $\mathbf{I}_h(\gamma_{Dir}, \gamma_t)$  defined from  $S^{\mathcal{P}}(\Omega, \mathcal{P}_h)^2$  into  $S^{\mathcal{P}}(\Omega, \mathcal{T}_h)^2$  with the following properties: for all  $v_\delta \in S^{\mathcal{P}}(\Omega, \mathcal{P}_h)^2$ , we have*

$$\mathbf{v}_\delta - \mathbf{I}_h(\gamma_{Dir}, \gamma_t) \mathbf{v}_\delta \in \mathcal{C}(\bar{\Omega})^2 \cap H^1_{\gamma_{Dir}}(\Omega)^2 \cap H_{\gamma_t}(\text{curl}, \Omega) \quad (3.13)$$

and the following estimate holds.

$$\sum_{D \in \mathcal{P}_h} \|\mathbf{I}_h(\gamma_{Dir}, \gamma_t) \mathbf{v}_\delta\|_{H^1(D)}^2 \lesssim \sup_{K \in \mathcal{T}_h} \frac{|K|}{\rho_K^2} \left( \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} p_e^2 \|\mathbf{v}_\delta\|_{L^2(e)}^2 + \sum_{e \in \gamma_t} h_e^{-1} p_e^2 \|\mathbf{v}_\delta \cdot \mathbf{t}\|_{L^2(e)}^2 \right), \quad (3.14)$$

where  $\mathbf{t}$  is the tangential unit vector to  $\partial\Omega$ .

**Proof.** We follow exactly the same steps as for the scalar case. To simplify notation we set  $\mathbf{I}_h := \mathbf{I}_h(\gamma_{Dir}, \gamma_t)$ .

**Step 1. (patch  $\omega_{Dir}$ )** As in the scalar case.

**Step 2. (patch  $\omega_t$ )** Let us first consider the part  $\omega_t^1$ . We can choose the mesh  $\mathcal{T}_h$  so that there will be only three possible cases for the triangles of  $\omega_t^1$ .

case 1. Let  $K_1 \in \omega_t^1$  sharing one edge  $e_{12}$  with a triangle  $K_2 \in \omega_{Dir}$ . We denote by  $e_1$  the edge  $K_1 \cap \gamma_t$  and  $\mathbf{a}$  the common vertex to  $K_1$  and  $K_2$  lying on the boundary  $\partial\Omega$ . We set

$$(\mathbf{I}_h v)|_{K_1} = (\mathcal{E}_e^{K_1}(v|_{K_1} \cdot \mathbf{t}_{e_1})) \mathbf{t}_{e_1} + \check{\mathcal{E}}_{e_1 \cup e_{12}}^{K_1}(\mathbf{0}, u_2 - u_1),$$

where

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_{K_1} - \mathcal{E}_{e_1}^{K_1}(\mathbf{v}|_{K_1} \cdot \mathbf{t}_{e_1}) \mathbf{t}_{e_1}, \\ \mathbf{u}_2 &= (\mathbf{v} - \mathbf{I}v)|_{K_2}. \end{aligned}$$

case 2. Let  $K_1, K_2 \in \omega_t^1$  be two triangles such that their intersection is equal to a vertex  $\mathbf{a}$ , which is also a vertex of  $\Omega$ . In addition, the triangles  $K_1$  and  $K_2$  each have one edge  $e_1$  and  $e_2$  lying on  $\gamma_t$  respectively. Let  $\psi$  be the function defined on  $e_1 \cup e_2$  as follows:

$$\psi = \begin{cases} (\mathbf{v}|_{K_1} \cdot \mathbf{t}_1) \mathbf{t}_1 + w_1 \mathbf{n}_1 & \text{on } e_1, \\ (\mathbf{v}|_{K_1} \cdot \mathbf{t}_2) \mathbf{t}_2 + w_2 \mathbf{n}_2 & \text{on } e_2, \end{cases}$$

where,

$$w_1 = \frac{1}{\mathbf{n}_1 \cdot \mathbf{t}_2} ((\mathbf{v}|_{K_1}(\mathbf{a}) \cdot \mathbf{t}_1) \mathbf{t}_1 - \mathbf{v}|_{K_2}(\mathbf{a})) \cdot \mathbf{t}$$

and

$$w_2 = (\mathbf{v}|_{K_1}(\mathbf{a}) - \mathbf{v}|_{K_2}(\mathbf{a}) + (\mathbf{v}|_{K_1}(\mathbf{a}) \cdot \mathbf{t}_1) \mathbf{t}_1 + w_1 \mathbf{n}_1) \cdot \mathbf{n}_1.$$

The function  $\psi$  is defined so that  $(\mathbf{v} - \psi)|_{e_1}(\mathbf{a}) = (\mathbf{v} - \psi)|_{e_2}(\mathbf{a})$ . For  $i = 1, 2$ , we set

$$\mathbf{I}_h(\mathbf{v})_{K_i} = \mathcal{E}_{e_i}^{K_i}(\psi|_{e_i} \cdot \mathbf{t}_i) \mathbf{t}_i + \mathcal{E}_{e_i}^{K_i}(\psi|_{e_i} \cdot \mathbf{n}_i) \mathbf{n}_i$$

case 3. Let  $K \in \omega_t^1$  be a triangle with an empty intersection with  $\omega_{Dir}$  and whose vertices are different from those of  $\Omega$ . We set

$$(\mathbf{I}_h v)|_K = \mathcal{E}_e^K(\mathbf{v}|_K \cdot \mathbf{t}_e) \mathbf{t}_e,$$

where  $e$  is the edge  $K \cap \gamma_t$ .

Now, let  $K \in \omega_t^2$ . According to Assumption (A3), there exist two triangles in  $\omega_t^1$  such that each one shares one edge with  $K$ . We denote these edges by  $e_1$  and  $e_2$  and set

$$(\mathbf{I}_h \mathbf{v})_K = \begin{cases} v - (v - \mathbf{I}_h v)|_{e_1}^+ & \text{on } e_1, \\ v - (v - \mathbf{I}_h v)|_{e_2}^+ & \text{on } e_2. \end{cases}$$

Now, we set

$$\gamma_b := \partial \left( \bigcup_{K \in \omega_{Dir} \cup \omega_t} K \right) \cap \Omega.$$

**Step 3.** The operator  $\mathbf{I}_h$  is defined on the patches  $\omega_M, \bar{\omega}$  and  $\tilde{\omega}$  exactly as in the proof of Theorem 3.5.  $\square$

**Remark 3.11.** Assume that for all  $K \in \mathcal{T}_h$  and  $\mathbf{e} \in \mathcal{E}_h \cap \partial K$ , we have  $h_K \lesssim h_e$ . We can easily check that

$$\|I_h(\gamma_{Dir}, \gamma_t) \mathbf{v}_\delta\|_{L^2(\Omega)}^2 \lesssim \left( \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e \|\llbracket \mathbf{v}_\delta \rrbracket\|_{L^2(e)}^2 \right), \quad (3.15)$$

for all  $\mathbf{v}_\delta \in S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$ .

**Corollary 3.12.** Let  $\epsilon_f(\mathbf{v})$  be the trace-free symmetric gradient defined by  $\epsilon_f(\mathbf{v}) := \epsilon(\mathbf{v}) - \frac{1}{2} \operatorname{div} \mathbf{v} \mathbb{I}$ , where  $\mathbb{I}$  is the identity matrix and

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2} (\partial_{x_j} v_i + \partial_{x_i} v_j), \quad i, j = 1, 2.$$

Then, the following generalized Korn's inequality holds:

$$\sum_{D \in \mathcal{P}_h} \|\mathbf{v}_\delta\|_{H^1(D)} \lesssim \left\| \sum_{D \in \mathcal{P}_h} \|\epsilon_f(\mathbf{v})\|_{L^2(D)}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} p_e^2 \|\llbracket v_\delta \rrbracket\|_{L^2(e)}^2 \right\|. \quad (3.16)$$

for all  $\mathbf{v}_\delta \in S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$ .

**Proof.** According to [25] there exists a constant  $c > 0$  such that for all  $\mathbf{v} \in H_0^1(\Omega)^2$ ,

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq \|\epsilon_f(\mathbf{v})\|_{L^2(\Omega)}.$$

Thus, the corollary is a direct consequence of Theorem 3.10.  $\square$

**Corollary 3.13.** Assume that each connected component of  $\gamma_t$  has a convex neighborhood in  $\Omega$ , then the following Friedrichs type inequality holds.

$$\begin{aligned} \sum_{D \in \mathcal{P}_h} \|\mathbf{v}_\delta\|_{H^1(D)}^2 &\lesssim \sum_{D \in \mathcal{P}_h} \left( \|\operatorname{curl} \mathbf{v}_\delta\|_{L^2(D)}^2 + \|\operatorname{div} \mathbf{v}_\delta\|_{L^2(D)}^2 \right) \\ &+ \kappa(\mathcal{P}_h) \left( \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e^{-1} p_e^2 \|\llbracket \mathbf{v}_\delta \rrbracket\|_{L^2(e)}^2 + \sum_{e \in \gamma_t} h_e^{-1} p_e^2 \|\mathbf{v}_\delta \cdot \mathbf{t}\|_{L^2(e)}^2 \right) \end{aligned} \quad (3.17)$$

for all  $\mathbf{v}_\delta \in S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$ , where  $\mathbf{t}$  denotes the tangential unit vector.

**Proof.** This is a direct consequence of Theorem 3.10 and the following estimate (see Boussoufa et al. [5, Proposition 2.1]).

$$\forall \mathbf{v} \in H_{\gamma_{Dir}}^1(\Omega)^2 \cap H_{\gamma_t}(\operatorname{curl}, \Omega), \quad \|\mathbf{v}\|_{H^1(\Omega)^2} \leq c \left( \|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)^2} + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)^2} \right).$$

$\square$

**Remark 3.14.** Inequalities (3.16)-(3.17) are likely to be optimal with respect to the mesh size in sense that the exponent of  $h_e$  can not be improved. However, it seems that this is not the case for the polynomial degree. Indeed, Inequality (3.16) has been established by Hong&Williams [28] for piecewise  $H^1$ -functions defined on polyhedrons. In addition, owing to Daveau&Zaghdani [29], it holds that

$$\|\mathbf{v}\|_{L^2(\Omega)}^2 \lesssim \sum_{D \in \mathcal{P}_h} \left( \|\operatorname{curl} \mathbf{v}\|_{L^2(D)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(D)}^2 \right) + \sum_{e \in \mathcal{E}_{int}} h_e^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_b} h_e^{-1} \|\mathbf{v} \cdot \mathbf{t}\|_{L^2(e)}^2.$$

for all  $\mathbf{v} \in H^1(\Omega, \mathcal{P}_h)^3$ . Therefore, we claim that the polynomial degrees can be completely removed from inequalities (3.16)-(3.17) and that they still hold for piecewise  $H^1$ -functions.

**Definition 3.15.** (Admissible meshes). The family  $\{\mathcal{P}_h\}_{h \in (0,1)}$  is said to be an admissible mesh family if the following assumptions are satisfied:

(i) for all  $h \in (0, 1)$ , This a finite family of non-empty connected (possibly nonconvex) polygons  $D$  forming a partition of  $\Omega$  and whose boundaries are a finite union of edges;

(ii) each  $D \in \mathcal{P}_h$  has a finite number of macro-edges independently of the mesh size. A set  $E \subset \partial D$  is said to be a macro-edge of  $D$  if either  $E = \partial D \cap \partial\Omega$  or there is  $D' \in \mathcal{P}_h$ ,  $D' \neq D$ , such that  $E = \partial D \cap D'$ ;

(iii) for all  $D \in \mathcal{P}_h$ ,

$$\sum_{E \subset \partial D} h_E^2 \lesssim |D|,$$

where  $h_E$  denotes the length of the macro-edge  $E$ ;

(iv) for all  $h \in (0, 1)$ , each  $D \in \mathcal{P}_h$  is affine-equivalent to an element belonging to a finite collection of reference elements;

(v) the ratio of the diameter  $h_D$  of any  $D \in \mathcal{P}_h$  to the diameter of the largest ball inscribed in  $D$  is bounded from above independently of the mesh size  $h$ ;

(vi) for all  $D \in \mathcal{P}_h$  and for all edges  $e \subset \partial D$ ,

$$h_D \lesssim h_e.$$

(vii) for all  $D \in \mathcal{P}_h$  and all  $e \in \mathcal{E}_h \cap \partial D$ ,  $|D| \lesssim h_e^2$ .

Let  $\{\mathcal{P}_h\}_h$  be a mesh family of the polygon  $\Omega$ . We consider the lifting operators  $R_h^t, R_h^n : S^{\mathbf{P}}(\Omega, \mathcal{P}_h)^2 \rightarrow S^{\mathbf{P}}(\Omega, \mathcal{P}_h)^2$  defined as follows. For all  $\mathbf{v}_h, \boldsymbol{\varphi}_h \in S^{\mathbf{P}}(\Omega, \mathcal{P}_h)^2$ ,

$$\begin{aligned} \int_{\Omega} R_h^t(\mathbf{v}_h) \cdot \boldsymbol{\varphi}_h \, dx &= \sum_{e \in \mathcal{E}_{int}} ([\mathbf{v}_h] \cdot \mathbf{t}, \{\boldsymbol{\varphi}_h\} \cdot \mathbf{n}_e)_e, \\ \int_{\Omega} R_h^n(\mathbf{v}_h) \cdot \boldsymbol{\varphi}_h \, dx &= \sum_{e \in \mathcal{E}_{int}} ([\mathbf{v}_h] \cdot \mathbf{n} \cdot \mathbf{t}, \{\boldsymbol{\varphi}_h\} \cdot \mathbf{n}_e)_e. \end{aligned}$$

The associated curl and divergence operators are given by

$$C_h(\mathbf{v}_h) = \text{curl}_h \mathbf{v}_h + R_h^t(\mathbf{v}_h)$$

and

$$D_h(\mathbf{v}_h) = \text{div}_h \mathbf{v}_h + R_h^n(\mathbf{v}_h),$$

where  $\text{curl}_h$  and  $\text{div}_h$  denote the broken curl and divergence operators respectively.

**Corollary 3.16.** *Let  $\{\mathcal{P}_h\}_{h \in (0,1)}$  be and admissible mesh family. Let  $\{\mathbf{v}_h\}_{h \in (0,1)}$  be a sequence in  $S^{\mathbf{P}}(\Omega, \mathcal{P}_h)^2$ . Assume that this sequence is bounded in the following norm.*

$$\mathbf{v} \mapsto \left( \sum_{D \in \mathcal{P}_h} \left( \|\mathbf{curl} \mathbf{v}\|_{L^2(D)}^2 + \|\text{div} \mathbf{v}\|_{L^2(D)}^2 \right) + \sum_{E \in \mathcal{E}_{int}} h_e^{-1} \|[\mathbf{v}]\|_{L^2(e)}^2 + \sum_{E \in \mathcal{E}_b} h_e^{-1} \|\mathbf{v} \cdot \mathbf{t}\|_{L^2(e)}^2 \right)^{1/2}.$$

*Then, there exists a function  $\mathbf{v} \in H_0(\mathbf{curl}, \Omega) \cap H(\text{div}, \Omega)$  such that, as  $h \rightarrow 0$ , up to a subsequence,  $\mathbf{v}_h \rightarrow \mathbf{v}$  strongly in  $L^2(\Omega)^2$  and  $C_h(\mathbf{v}_h) \rightharpoonup \mathbf{curl} \mathbf{v}$  and  $D_h(\mathbf{v}_h) \rightharpoonup \text{div} \mathbf{v}$  weakly in  $L^2(\Omega)$ .*

**Proof.** Consider the decomposition

$$\mathbf{v}_h = (\mathbf{v}_h - \mathbf{I}_h \mathbf{v}_h) + \mathbf{I}_h \mathbf{v}_h,$$

where  $\mathbf{I}_h := \mathbf{I}_h(\emptyset, \partial\Omega)$ . We can check that the sequence  $\{\mathbf{v}_h - \mathbf{I}_h \mathbf{v}_h\}_{h \in (0,1)}$  is bounded in  $H_0(\mathbf{curl}, \Omega) \cap H(\text{div}, \Omega)$ . Thus, owing to Weber's compactness theorem (see [27]), there exists a function  $\mathbf{v}_1 \in H_0(\mathbf{curl}, \Omega) \cap H(\text{div}, \Omega)$  such

that, as  $h \rightarrow 0$ , up to a subsequence,  $(\mathbf{v}_h - I_h \mathbf{v}_h) \rightarrow \mathbf{v}_1$  strongly in  $L^2(\Omega)^2$ . Now, let  $e_b$  be any boundary edge and  $D \in \mathcal{P}_h$  be the subdomain such that  $e_b \subset \partial D$ . The trace inequality (2.1) and Remark 3.11 imply that

$$h_{e_b}^{-1} \|\mathbf{I}_h \mathbf{v}_h\|_{L^2(e_b)}^2 \lesssim \sum_{e \in \mathcal{E}_{int} \cap \partial D} h_e^{-1} \|\mathbf{v}_h\|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_b \cap \partial D} h_e^{-1} \|\mathbf{v}_h \cdot \mathbf{t}\|_{L^2(e)}^2.$$

Therefore, the sequence  $\{I_h \mathbf{v}_h\}_{h \in (0,1)}$  is bounded in the  $H_0^1(\Omega, \mathcal{P}_h)$ -norm. Thus, owing [14, Thm 2.2], there exists a function  $\mathbf{v}_2 \in H_0^1(\Omega)$  such that, up to a subsequence,  $I_h \mathbf{v}_h \rightarrow \mathbf{v}_2$  strongly in  $L^2(\Omega)^2$ . The desired function is given by

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2.$$

To prove the weak convergence of  $\{C_h \mathbf{v}_h\}_{h \in (0,1)}$  and  $\{D_h(\mathbf{v}_h)\}_{h \in (0,1)}$  we proceed as in [14, proof of Thm 2.2].  $\square$

## 4 The general case

In the previous section we have derived discrete inequalities and compactness results for the Lebesgue exponent  $q = 2$ . In this part we will briefly show how to extend these results to the non-Hilbertian case  $q \neq 2$ . We only deal with real-valued functions since the case of vector fields can be treated as in Subsection 3.2.

In this section, the subpartition  $\mathcal{T}_h$  is a family of polygons such that :

1. For any subdomain  $K \in \mathcal{T}_h$  there exists  $D \in \mathcal{P}_h$  such that  $K \subset D$  and  $h_K \sim h_D$ ,
2. The elements of  $\mathcal{T}_h$  can be regrouped to form a family of patches  $\{\omega\}$  such that every patch is an array of subdomains  $D_{ij}$ ,  $1 \leq i, j \leq 5$  (see Fig. 4).
3. For each subdomain  $K$  in the partition  $\mathcal{T}_h$ , there is an associated reference subdomain  $\widehat{K}$ , defined by the linear mapping

$$\mathbf{x} \mapsto h_K^{-1} \widehat{\mathbf{x}}.$$

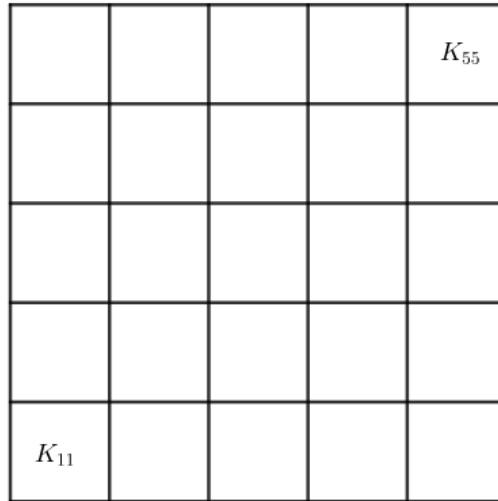


Figure 4: The elementary patch structure when all the subdomains are identical squares.

We consider the following assumptions.

- (A1') The intersection of any boundary edge with  $\gamma_{Dir}$  is either the whole edge or has null measure.

(A2') The subpartition  $\mathcal{T}_h$  is sufficiently small so that every patch  $\omega$  is contained in at most two different domains in  $\mathcal{P}_h$ .

(A3') For all  $D \in \mathcal{P}_h$  and all  $K \in \mathcal{T}_h$  such that  $K \subset D$ ,

$$\rho_K \sim \min_{e \in \mathcal{E}_h \cap \partial K} h_e.$$

Let  $\tilde{\mathbb{T}}(\mathcal{P}_h)$  the set of all the subpartitions  $\mathcal{T}_h$  that satisfy all the above properties. As in the previous section, we set

$$\kappa(\mathcal{P}_h) := \inf_{\mathcal{T}_h \in \tilde{\mathbb{T}}(\mathcal{P}_h)} \sup_{K \in \mathcal{T}_h} \frac{|K|}{\rho_K^2}.$$

**Lemma 4.1.** *Let  $D$  be a polygonal domain and  $\gamma$  be the union of some whole edges of  $D$ . Let  $M \geq 1$  be an integer. Let  $\gamma_i := \bigcup_{1 \leq j \leq m(i)} e_j^i$ ,  $1 \leq i \leq M$ , be the connected components of  $\gamma$  consisting of  $m(i) \geq 2$  adjacent edges. Assume*

*that for all  $1 \leq i \leq M$ , the edges  $e_j^i$  and  $e_{j+1}^i$  share a vertex denoted by  $\mathbf{a}_j^i$ . Let  $\mathbb{W}$  be the subspace of  $\prod_{e \in \gamma} W^{1-1/q}(e)$  given by following conditions :*

1.  $v|_{e_j^i}(\mathbf{a}_j^i) = v|_{e_{j+1}^i}(\mathbf{a}_j^i)$ , for every  $1 \leq i \leq M$  and  $1 \leq j \leq m(i)$ , when  $q > 2$ ,

2.  $\int_0^{h_i} \left| v|_{e_j^i}(\mathbf{a}_j^i - x\mathbf{t}_{e_j^i}) - v|_{e_{j+1}^i}(\mathbf{a}_{j+1}^i - x\mathbf{t}_{e_{j+1}^i}) \right|^2 \frac{dx}{x} < +\infty$ , when  $q = 2$ ,

where  $h_i = \frac{1}{2} \min(h_{e_j^i}, h_{e_{j+1}^i})$  and  $\mathbf{t}_{e_j^i}$  denotes the tangential unit vector to the edge  $e_j^i$ .

3. No extra conditions when  $1 < q < 2$ .

Then, there exists a linear extension operator  $\mathcal{E}_\gamma^D$  defined from  $\mathbb{W}$  into  $W^{1,q}(D)$  such that the following estimates hold:

$$\|\mathcal{E}_\gamma^D(v)\|_{W^{1,q}(D)} \leq c \left( \sum_{e \in \gamma} \|v|_e\|_{W^{1-1/q,q}(e)} \right), \quad (4.1)$$

when  $q \neq 2$  and

$$\|\mathcal{E}_\gamma^D(v)\|_{H^1(D)} \leq c \left( \sum_{e \in \gamma} \|v|_e\|_{H^{1/2}(e)} + \sum_{i=1}^M \sum_{j=1}^{m(i)} \int_0^{h_i} \left| v|_{e_j^i}(\mathbf{a}_j^i - x\mathbf{t}_{e_j^i}) - v|_{e_{j+1}^i}(\mathbf{a}_{j+1}^i - x\mathbf{t}_{e_{j+1}^i}) \right|^2 \frac{dx}{x} \right) \quad (4.2)$$

In addition,

$$\|\mathcal{E}_\gamma^D(v)\|_{L^q(\partial D)} \leq c \|v\|_{L^q(\gamma)}.$$

**Proof.** When  $\gamma$  is equal to the whole boundary  $\partial D$ , the extension operator is considered in Grisvard [19, Thm 1.5.2.3]. If  $\gamma$  is different from  $\partial D$ , we can easily construct a function  $\psi \in W^{1-1/q,q}(\partial D)$  such that

$$\|\psi\|_{L^q(\partial D)} \leq c \|v\|_{L^q(\partial D)}, \text{ for all } 1 < q < +\infty,$$

$$\|\psi\|_{W^{1-1/q,q}(\partial D)} \leq c \left( \sum_{e \in \gamma} \|v|_e\|_{W^{1-1/q,q}(e)} \right), \text{ if } q \neq 2,$$

and

$$\|\psi\|_{H^{1/2}(\partial D)} \leq c \left( \sum_{e \in \gamma} \|v|_e\|_{H^{1/2}(e)} + \sum_{i=1}^M \sum_{j=1}^{m(i)} \int_0^{h_i} \left| v|_{e_j^i}(\mathbf{a}_j^i - x\mathbf{t}_{e_j^i}) - v|_{e_{j+1}^i}(\mathbf{a}_{j+1}^i - x\mathbf{t}_{e_{j+1}^i}) \right|^2 \frac{dx}{x} \right).$$

The operator  $\mathcal{E}_\gamma^D$  is given by

$$\mathcal{E}_\gamma^D(v) := \mathcal{E}_{\partial D}^D(\psi).$$

□

**Theorem 4.2.** Assume that Assumptions (A1')-(A3') are satisfied. Then, there exists a linear operator  $\tilde{I}_h(\gamma_{Dir})$  defined from  $S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$  into  $S^{\mathcal{P}}(\Omega, \mathcal{T}_h)$  with the following properties. For all  $v_{\delta} \in S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$ ,

$$v_{\delta} - \tilde{I}_h(\gamma_{Dir})v_{\delta} \in \mathcal{C}(\bar{\Omega}) \cap W_{\gamma_{Dir}}^{1,q}(\Omega) \quad (4.3)$$

and the following estimates hold.

$$\sum_{D \in \mathcal{P}_h} \|\tilde{I}_h(\gamma_{Dir})v_{\delta}\|_{W^{1,q}(D)}^q \lesssim \sup_{K \in \mathcal{T}_h} \frac{|K|}{\rho_K^q} \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} \left(\frac{p_e^2}{h_e}\right)^{q-1} \| [v_{\delta}] \|_{L^q(e)}^q, \quad (4.4)$$

when  $q \neq 2$  and

$$\sum_{D \in \mathcal{P}_h} \|\tilde{I}_h(\gamma_{Dir})v_{\delta}\|_{H^1(D)}^2 \lesssim \sup_{K \in \mathcal{T}_h} \frac{|K|}{\rho_K^2} \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} (p_e \log p_e)^2 h_e^{-1} \| [v_{\delta}] \|_{L^2(e)}^2, \quad (4.5)$$

**Proof.** To simplify notation, we set  $\tilde{I}_h := \tilde{I}_h(\gamma_{Dir})$ . Let  $\omega_{Dir}$  be the set of all the subdomains in  $\mathcal{T}_h$  with a non-null-measure intersection with  $\gamma_{Dir}$ . The operator  $\tilde{I}_h$  is defined on  $\omega_{Dir}$  separately from all the other patches. Let  $K$  be subdomain in  $\omega_{Dir}$  such that the intersection  $K \cap K'$  is empty for any  $K' \in \omega_{Dir}$ . Then, denoting by  $E_K$  the intersection  $K \cap \gamma_{Dir}$ , we set

$$(\tilde{I}_h v)|_K = \mathcal{E}_{E_K}^K(v|_K).$$

Now let  $K_1, K_2, \dots, K_N$ , be  $N \geq 2$  adjacent subdomains such that  $K_i$  and  $K_{i+1}$ ,  $1 \leq i \leq N-1$ , share a macro-edge  $E_i$  and denote by  $E_{i,Dir}$  the intersection  $K_i \cap \gamma_{Dir}$ . We set

$$(\tilde{I}_h v)|_{K_1} = \mathcal{E}_{E_1}^{K_1}(v|_{K_1})$$

and for  $i = 2, \dots, N$ ,

$$(\tilde{I}_h v)|_{K_i} = \mathcal{E}_{E_{i-1} \cup E_{i,Dir}}^{K_i}(v - \tilde{v}_i),$$

where

$$\tilde{v}_i = \begin{cases} (v - \tilde{I}_h v)|_{K_{i-1}} & \text{on } E_{i-1}, \\ 0 & \text{on } E_{i,Dir}. \end{cases}$$

Now, the operator  $\tilde{I}_h$  is constructed by the following algorithm. First we replace  $\mathcal{T}_h$  by  $\mathcal{T}_h \setminus \omega_{Dir}$  and set  $\gamma_b = \partial\omega_{Dir}$ .

**Algorithm.** Let  $\omega = \{K_{ij}; 1 \leq i, j \leq 5\}$  be any patch of the subpartition  $\mathcal{T}_h$ .

**Step 1.** Let  $1 \leq i, j \leq 5$  be two odd integers. We set

$$(Iv)_{K_{ij}} = \begin{cases} 0 & \text{if } K_{ij} \cap \gamma_b \text{ has a null measure,} \\ \mathcal{E}_{\partial K_{ij} \cap \gamma_b}^{K_{ij}}(v|_{K_{ij}} - (v - \tilde{I}_h v)|_{K_{ij} \cap \gamma_b}^+) & \text{otherwise.} \end{cases}$$

**Step 2.** Let  $1 \leq i, j \leq 5$  be two integers with different parities. Each subdomain  $K_{ij}$  has two adjacent subdomains from Step 1. We denote the intersection of  $K_{ij}$  with these two subdomains by  $E_{ij}^1$  and  $E_{ij}^2$ . We denote by  $E_{ij}^b$  the intersection between  $K_{ij}$  and  $\gamma_b$  (which may have a null measure). We set

$$(\tilde{I}_h v)|_{K_{ij}} = \mathcal{E}_{E_{ij}^1 \cup E_{ij}^2 \cup E_{ij}^b}^{K_{ij}}(v|_{K_{ij}} - \tilde{v}_{ij}), \quad (4.6)$$

with

$$\tilde{v}_{ij} = \begin{cases} (v - \tilde{I}_h v)|_{K_{ij}^1} & \text{on } E_{ij}^1, \\ (v - \tilde{I}_h v)|_{K_{ij}^2} & \text{on } E_{ij}^2, \\ (v - \tilde{I}_h v)|_{K_{ij}^b} & \text{on } E_{ij}^b. \end{cases}$$

**Step 3.** There remain four subdomains  $K_{ij}$  where the indices  $i, j$  are both even. We define  $\tilde{I}_h$  as follows.

$$(\tilde{I}_h v)|_{K_{ij}} = \mathcal{E}_{\partial K_{ij}}^{K_{ij}}(v|_{K_{ij}} - (v - \tilde{I}_h v)|_{\partial K_{ij}}^+).$$

Finally, we replace  $\mathcal{T}_h$  by  $\mathcal{T}_h \setminus \omega$  and  $\gamma_b$  by  $\gamma_b \cup \partial\omega$  and repeat the above steps until  $\mathcal{T}_h$  become equal to the empty set. Estimates (4.4) and (4.5) are direct consequences of estimates (4.1)-(4.2) and inverse inequalities (2.3)-(2.4).  $\square$

**Remark 4.3.** Assume that for all  $K \in \mathcal{T}_h$  and  $\mathbf{e} \in \mathcal{E}_h \cap \partial K$ , we have  $h_K \lesssim h_e$ . As in the previous section, it holds that

$$\|\tilde{I}_h(\gamma_{Dir})\mathbf{v}_\delta\|_{L^q(\Omega)}^q \lesssim \sum_{e \in \mathcal{E}_{int} \cup \gamma_{Dir}} h_e \|\mathbf{v}_\delta\|_{L^q(e)}^q, \quad (4.7)$$

for all  $\mathbf{v}_\delta \in S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$ .

**Corollary 4.4.** Let  $1 \leq q < \infty$ . Let  $\mathcal{P}_h$  be a mesh of an admissible mesh family. Let  $\Phi$  be a semi-norm on  $W^{1,q}(\Omega, \mathcal{P}_h)$  such that

$$\Phi(v) \leq c_\Phi \left( \sum_{D \in \mathcal{P}_h} \|v\|_{W^{1,q}(D)}^q \right)^{1/q}, \quad (4.8)$$

for all  $v \in W^{1,q}(\Omega, \mathcal{P}_h)$ , and

$$\forall c \in \mathbb{R}, \quad \Phi(c) = 0 \text{ if and only if } c = 0. \quad (4.9)$$

Then, for all  $r$  such that  $1 \leq r < \frac{2q}{2-q}$  if  $1 \leq q < 2$  or  $1 \leq r < \infty$  if  $2 \leq q < \infty$ , the following generalized Poincaré-Sobolev inequality holds.

$$\|v\|_{L^r(\Omega)} \lesssim \left( \sum_{D \in \mathcal{P}_h} \|\nabla v\|_{L^q(D)}^q + \sum_{e \in \mathcal{E}_{int}} h_e^{1-q} \|v\|_{L^q(e)}^q + \Phi(v)^q \right)^{1/q}, \quad (4.10)$$

for all  $v \in S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$ .

**Proof.** Let  $\tilde{I}_h := \tilde{I}_h(\emptyset)$  be the operator of Theorem 4.2 defined from  $\mathcal{P}_h$  into  $\tilde{\mathcal{T}}_h$ , where  $\tilde{\mathcal{T}}_h \subset \tilde{\mathbb{T}}(\mathcal{P}_h)$ . Let  $\tilde{\mathcal{T}}_h$  be a shape-regular triangulation of  $\Omega$  and a subpartition of  $\tilde{\mathcal{T}}_h$ . The orthogonal  $L^2$ -projection operator  $\pi_h$  defined from  $L^2(\Omega)$  onto  $S^{\mathcal{P}}(\Omega, \tilde{\mathcal{T}}_h) \cap \mathcal{C}(\bar{\Omega})$  satisfies the following stability estimates (see [15]). For all  $v \in W^{1,q}(\Omega, \tilde{\mathcal{T}}_h)$  and all  $K \in \tilde{\mathcal{T}}_h$ ,

$$\|\pi_h v\|_{L^q(K)} \lesssim \|v\|_{L^q(K)} \quad (4.11)$$

and

$$\|\nabla \pi_h v\|_{L^q(K)} \lesssim \|\nabla v\|_{L^q(K)}. \quad (4.12)$$

Using the compactness of the embedding  $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ , we can prove by a classical argument that

$$\|v\|_{L^r(\Omega)} \leq c \|\nabla v\|_{L^q(\Omega)} + \Phi(v).$$

for all  $v \in W^{1,q}(\Omega)$ . The above inequality and estimate (4.12) imply that

$$\begin{aligned} \|v_\delta\|_{L^r(\Omega)}^q &\lesssim \|\pi_h(v_\delta - \tilde{I}_h v_\delta)\|_{L^r(\Omega)}^q + \|\pi_h(\tilde{I}_h v_\delta)\|_{L^r(\Omega)}^q \\ &\lesssim \sum_{D \in \mathcal{P}_h} \|\nabla v_\delta\|_{L^q(D)}^q + \sum_{D \in \mathcal{P}_h} \|\tilde{I}_h v_\delta\|_{W^{1,q}(K)}^q + \Phi(v_\delta)^q + \|\tilde{I}_h v_\delta\|_{L^r(\Omega)}^q \end{aligned}$$

First, let  $1 \leq q < 2$  and  $q \leq r < \frac{2q}{2-q}$ . Using the embedding  $l^q \hookrightarrow l^r$ , a standard inverse inequality, estimate (4.11) and Remark 4.3, we find

$$\begin{aligned} \left( \sum_{K \in \mathcal{T}_h} \|\pi_h \tilde{I}_h v_\delta\|_{L^r(K)}^r \right)^{q/r} &\leq \sum_{K \in \mathcal{T}_h} \|\pi_h \tilde{I}_h v_\delta\|_{L^r(K)}^q \\ &\lesssim \sum_{K \in \mathcal{T}_h} |K|^{2(\frac{q}{r}-1)} \|\pi_h \tilde{I}_h v_\delta\|_{L^q(K)}^q \\ &\lesssim \sum_{e \in \mathcal{E}_{int}} h_e^{\frac{2q}{r}-1} \|\mathbf{v}_\delta\|_{L^q(e)}^q \\ &\lesssim \sum_{e \in \mathcal{E}_{int}} h_e^{1-q} \|\mathbf{v}_\delta\|_{L^q(e)}^q \end{aligned}$$

We obtain the same estimate for all  $1 \leq r < q$  using the embedding  $L^q(\Omega) \hookrightarrow L^r(\Omega)$ .

Now, let  $2 \leq q < \infty$  and  $1 \leq r < \infty$ . Owing to [14, Thm 6.1], we have

$$\|\pi_h I_h v_\delta\|_{L^r(\Omega)}^q \lesssim \sum_{K \in \mathcal{T}_h} \|\nabla \pi_h I_h v_\delta\|_{L^q(K)}^q + \sum_{e \in \mathcal{E}_{int}} h_e^{1-q} \|\llbracket \pi_h I_h v_\delta \rrbracket\|_{L^q(e)}^q.$$

Let  $e_b \in \mathcal{E}_b$  be any boundary edge and  $K \in \mathcal{T}_h$  be such that  $e_b \subset \partial K$ . The trace inequality (2.1), Remark 4.3 and Theorem 4.2 imply that

$$\begin{aligned} h_e^{1-q} \|\llbracket \pi_h I_h v_\delta \rrbracket\|_{L^q(e_b)}^q &\lesssim |K|^{-1} \left( \|\pi_h I_h v_\delta\|_{L^q(K)}^q + h_K^q \|\nabla \pi_h I_h v_\delta\|_{L^q(K)}^q \right) \\ &\lesssim \sum_{e \in \mathcal{E}_{int}} h_e^{-1} \|\llbracket v_\delta \rrbracket\|_{L^q(e)}^q \\ &\lesssim \sum_{e \in \mathcal{E}_{int}} h_e^{1-q} \|\llbracket v_\delta \rrbracket\|_{L^q(e)}^q, \end{aligned}$$

which concludes the proof.  $\square$

Let  $\{\mathcal{P}_h\}_{h \in (0,1)}$  be a mesh family of the polygon  $\Omega$ . We consider the lifting operator  $\mathbf{R}_h : S^{\mathcal{P}}(\Omega, \mathcal{P}_h) \rightarrow S^{\mathcal{P}}(\Omega, \mathcal{P}_h)^2$  defined as follows: for all  $v_h \in S^{\mathcal{P}}(\Omega, \mathcal{P})$ , and all  $\varphi_h \in S^{\mathcal{P}}(\Omega, \mathcal{P})^2$ ,

$$\int_{\Omega} \mathbf{R}_h(v_h) \cdot \varphi_h dx = \sum_{e \in \mathcal{E}_{int}} (\llbracket v_h \rrbracket, \varphi_h \cdot \mathbf{n}_e)_e.$$

We consider the discrete gradient operator

$$\mathbf{G}_h(v_h) = -\nabla_h v_h + \mathbf{R}_h(v_h),$$

where  $\nabla_h$  denotes the broken gradient operator.

**Corollary 4.5.** *Let  $1 \leq q < \infty$ . Let  $\{\mathcal{P}_h\}_{h \in (0,1)}$  be a family of admissible meshes. Let  $(\Phi_h)_{h \in (0,1)}$  be a family of semi-norms on  $W^{1,q}(\Omega, \mathcal{P}_h)$  such that  $\Phi_h$  satisfy properties (4.8)-(4.9) for all  $h \in (0,1)$ . Let  $\{v_h\}_{h \in (0,1)}$  be a sequence in  $S^{\mathcal{P}}(\Omega, \mathcal{P}_h)$ . Assume that this sequence is bounded in the following norm.*

$$v \mapsto \left( \sum_{D \in \mathcal{P}_h} \|\nabla v\|_{L^q(D)}^q + \sum_{e \in \mathcal{E}_{int}} h_e^{1-q} \|\llbracket v \rrbracket\|_{L^q(e)}^q + \Phi_h(v)^q \right)^{1/q}.$$

*Then, for all  $r$  such that  $1 \leq r < \frac{2q}{2-q}$  if  $1 \leq q < 2$  or  $1 \leq r < \infty$  if  $2 \leq q < \infty$ , there exists a function  $v \in W^{1,q}(\Omega)$  such that, as  $h \rightarrow 0$ , up to a subsequence,  $v_h \rightarrow v$  strongly in  $L^q(\Omega)$  and  $\mathbf{G}_h(v_h) \rightharpoonup \nabla v$  weakly in  $L^q(\Omega)^2$ .*

**Proof.** Consider the decomposition

$$v_h = \pi_h(v_h - \tilde{I}_h v_h) + \tilde{I}_h v_h,$$

where the operators  $\tilde{I}_h$  and  $\pi_h$  are defined in the proof of Corollary 4.4. Combining estimates (4.10), (4.11) and (4.12), we can check that the sequence  $\{\pi_h(v_h - \tilde{I}_h v_h)\}_{h \in (0,1)}$  is bounded in the  $\|\cdot\|_{W^{1,q}(\Omega)}$ -norm. Thus, there exists a function  $v_1 \in L^q(\Omega)$  such that, as  $h \rightarrow 0$ , up to a subsequence,  $(v_h - \tilde{I}_h v_h) \rightarrow v_1$  strongly in  $L^q(\Omega)$ . As in the proof of Corollary 3.16 we show that the sequence  $\{\pi_h \tilde{I}_h v_h\}_{h \in (0,1)}$  is bounded in the  $\|\cdot\|_{W_0^{1,q}(\Omega, \mathcal{T}_h)}$ -norm. Therefore, owing to [14, Thm 6.2], there exists a function  $v_2 \in W_0^{1,q}(\Omega)$  such that, as  $h \rightarrow 0$ , up to a subsequence,  $\{\pi_h \tilde{I}_h v_h\}_h \rightarrow v_2$  strongly in  $L^q(\Omega)$ .

For the weak convergence of  $\mathbf{G}_h(v_h)$  we proceed as in [14, proof of Thm 2.2].  $\square$

## 5 Conclusion

In this paper we have introduced a new strategy to establish Poincaré-Friedrichs type inequalities and discrete compact embeddings. The cornerstone of our method is the combination of the properties of different extension operators and suitable subpartitions. In contrast to previous works, this method allows the treatment of any linear first order differential operator. The results established in this paper can be used in the analysis of numerical schemes for nonlinear PDEs. For instance, inequality (3.17) will be used by the author and Yakoubi in [6] to analyze a  $hp$  DG pressure correction scheme for the Navier-Stokes equations with boundary conditions involving the (total) pressure. Future work should address the generalization of the results of this paper to nonsmooth *curved* domains. Also, the author believes that the Poincaré-Friedrichs inequalities for differential operators with infinite-dimensional kernels can be established for broken Sobolev spaces.

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