

# A Couple Axioms from the Observer Framework: Analyzing Self-Referential Paradoxes

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## 1 Introduction

Yablo's paradox is presented as an infinite sequence of statements  $S_1, S_2, S_3, \dots$ , where each statement  $S_n$  asserts that all subsequent statements  $S_m$  (for  $m > n$ ) are false. This leads to a paradoxical structure, as the truth of any statement would seem to contradict the truth of future statements. In this paper, we introduce a axiomatic and a set-theoretic system to avoid the paradox by allowing for alternating truth values over instances of observers.

## 2 Preliminaries

### 2.1 Axioms

**Axiom 2.1** (Axiom of Observer Existence). *For every reference space  $M_n \in \mathcal{M}$ , there exists an observer  $O_n \in \mathcal{O}$  such that  $O_n$  can perceive and evaluate propositions within  $M_n$ .*

**Axiom 2.2** (Axiom of the Reference Space). ***Existence and Uniqueness of Reference Spaces:***

$$\forall S_n \in \mathcal{S}, \exists! M_n \in \mathcal{M} \text{ such that } \mathcal{E}(S_n, M_n) = \text{True}$$

**Axiom 2.3** (Axiom of Observer Hierarchy Consistency). *For every pair of observers  $O_i, O_j \in \mathcal{O}$ , if  $O_i$  is a subordinate observer to  $O_j$ , then the evaluation capabilities of  $O_i$  are contained within those of  $O_j$ :*

$$\mathcal{E}(S_i, M_i) \subseteq \mathcal{E}(S_j, M_j)$$

**Axiom 2.4** (Axiom of Non-Redundancy). *No two distinct observers in  $\mathcal{O}$  share the same reference space. Formally:*

$$\forall O_i, O_j \in \mathcal{O}, \quad M_i = M_j \implies O_i = O_j$$

**Axiom 2.5** (Axiom of Referential Consistency). *The evaluation of propositions respects Referential ordering. For any logical system  $S_n$  and Reference instances  $t_m, t_k \in \mathbb{T}$  with  $t_m < t_k$ :*

$$\mathcal{E}(S_n, M_{t_m}) \text{ is evaluated before } \mathcal{E}(S_n, M_{t_k})$$

*Note: Depending on the context Referential can also be termed as Temporal*

**Axiom 2.6** (Axiom of Comprehensive Coverage). *The collective observation sets of all observers cover the entire proposition space  $P$ :*

$$\bigcup_{O_n \in \mathcal{O}} E_n = P$$

**Axiom 2.7** (Axiom of Limit Observer Dominance). *The limit observer  $O_\infty$  encompasses the evaluation capabilities of all other observers:*

$$\forall O_n \in \mathcal{O}, \quad \mathcal{E}(O_n) \subseteq \mathcal{E}(O_\infty)$$

**Axiom 2.8** (Axiom of the Logical Space ). *Logical Spaces are organized into an infinite, sequentially alternating sequence of Positive and Negative Logical Spaces. For each natural number  $n \in \mathbb{N}$ , the  $n$ -th Logical Space  $M_n$  is defined as follows:*

*$M_n$  is a Positive Logical Space  $\iff n$  is odd*

*$M_n$  is a Negative Logical Space  $\iff n$  is even*

- **Positive Logical Spaces ( $P$ ):** *The spaces where all standard, non-meta, and non-self-referential statements initially reside. Specifically, unless stated otherwise every statement begins its evaluation in  $M_1$  ( $P$ ).*
- **Negative Logical Spaces ( $N$ ):** *The spaces where the evaluation of results, proofs, or subsequent statements that are identified as meta- or self-referential, occurs. When a statement  $S$  is determined to be meta- or self-referential within  $M_n$ , its evaluation is **reassigned** to the next Logical Space  $M_{n+1}$ .*

**Propagation Mechanism:**

1. **Initial Evaluation:** *All statements commence their evaluation in the first Logical Space  $M_1$  (a Positive Logical Space).*
2. **Identification of Referential Nature:** *During evaluation within  $M_n$ , if a statement  $S$  is determined to be meta- or self-referential, its result is assigned to the next logical space in the sequence.*
3. **Assignment to Next Logical Space:** *The meta- or self-referential statement  $S$  result is **placed** from  $M_n$  into  $M_{n+1}$ , the subsequent Logical Space in the alternating sequence. The result of this being:*

- Each statement is associated with **exactly one** Logical Space, preserving the uniqueness of reference space assignments as stipulated in **Axiom 2.2**.
- The progression through Logical Spaces prevents infinite delegation loops, as each reassignment advances the statement to a higher, uniquely assigned Logical Space.

4. **Evaluation in New Logical Space:** In  $M_{n+1}$ , the statement  $S$  undergoes evaluation under the rules of that Logical Space (Positive or Negative, based on the parity of  $n + 1$ ).

- **Positive Logical Spaces (P):** Maintain standard logical evaluations within  $M_n$ . Formally, for any proposition  $S$  and any Positive Logical Space  $M_n$ :

$$\mathcal{E}(S, M_n) = \mathcal{E}_{std}(S)$$

where  $\mathcal{E}_{std}$  denotes the standard evaluation function that assigns truth values based on classical logic.

- **Negative Logical Spaces (N):** Inverse standard logical evaluations within  $M_n$ . Formally, for any proposition  $S$  and any Negative Logical Space  $M_n^-$ :

$$\mathcal{E}(S, M_n^-) = \neg\mathcal{E}(S, M_n)$$

where  $\mathcal{E}(S, M_n)$  represents the evaluation of  $S$  in the preceding Logical Space  $M_n$ .

**Definition 2.1** (Resulting Answer). A **Resulting Answer** refers to the outcome of evaluating a proposition  $S$  within a Logical Space  $M_n$ . This includes proofs, conclusions, or any derived statements that result from the evaluation process.

**Example 2.1.** Consider the proposition  $S_1$  : "All swans are white."

1. Evaluation in  $M_1$  (Positive Logical Space):

$$\mathcal{E}(S_1, M_1) = \mathcal{E}_{std}(S_1) = \text{False} \quad (\text{if non-white swans exist})$$

2. Resulting Answer Placement in  $M_2$  (Negative Logical Space):

$$\mathcal{E}(S_1, M_2^-) = \neg\mathcal{E}(S_1, M_1) = \neg\text{False} = \text{True}$$

Here, the resulting answer True is placed into  $M_2$  for further evaluation.

**Definition 2.2** (Independence). Two propositions  $S_n$  and  $S_m$  are **Independent** if:

1. They are evaluated within distinct reference spaces  $M_n$  and  $M_m$  respectively, i.e.,  $M_n \neq M_m$ .

2. The evaluation of one does not influence the evaluation of the other within their respective systems.

**Axiom 2.9** (Axiom of Evaluation Function Monotonicity). *evaluation functions maintain a monotonic relationship with the hierarchy of reference spaces. Specifically, if  $S_n \subseteq S_m$ , then:*

$$\mathcal{E}(S_n, M_n) \subseteq \mathcal{E}(S_m, M_m)$$

### 2.1.1 Optional Axioms

**Axiom 2.10** (Axiom of Hierarchical Delegation). *For any self-referential proposition  $S_n \in \mathcal{S}$ , if  $S_n$  cannot be consistently evaluated within  $M_n$ , then its evaluation is delegated to a higher reference space  $M_{n+1}$ .*

**Axiom 2.11** (Axiom of Autonomy). *Each observer  $O_n$  has the autonomy to extend its associated logical system  $S_n$  by introducing new axioms  $A'_n \subseteq P$ , ensuring that the extended system  $S'_n = S_n \cup A'_n$  remains consistent:*

$$S'_n = S_n \cup A'_n \quad \text{and} \quad \text{Consistency}(S'_n)$$

**Axiom 2.12** (Axiom of Arbitration). *When multiple observers provide conflicting evaluations for the same proposition within distinct reference spaces, the limit observer  $O_\infty$  employs an arbitration function  $\mathcal{A}$  to determine the final truth value:*

$$\mathcal{E}_\infty(S) = \mathcal{A}(\{\mathcal{E}(O_n, M_n)(S) \mid O_n \in \mathcal{O}\})$$

where  $\mathcal{A}$  is a predefined logical operation or system (e.g., majority vote, consensus, hierarchical priority) that resolves the set of truth values into a single, consistent truth value for  $S$ . Note: The value of the logical system depends on context

**Axiom 2.13** (Axiom of Alternating Truth Consistency). *The truth values of sequential and independent propositions alternate consistently across distinct logical systems and reference spaces:*

$$\forall S_n, S_m \in \mathcal{S}, \quad \text{Independent}(S_n, S_m) \implies T(S_n, t_n) = \neg T(S_m, t_m)$$

Note: Alternating refers to the consistent inversion of truth values across independent propositions evaluated within distinct logical systems and reference spaces from the seed space.

## 2.2 Definitions

**Definition 2.3** (Reference Space). A **Reference Space**  $M_n$  is a mathematical structure within the set  $\mathcal{M}$  that provides the context or environment in which a statement  $S_n$  is evaluated for its truth value.

**Definition 2.4** (Higher Reference Space). A **Higher Reference Space**  $M_{n+1}$  is a reference space that exist above  $M_n$  in the hierarchical structure, designated for evaluating propositions that are self-referential or paradoxical within  $M_n$ .

**Definition 2.5** (Negative Logical Space). A **Negative Logical Space**  $M_n^-$  is a reference space where truth values are inverted. Specifically:

$$\mathcal{E}(S_n, M_n^-) = \text{True} \iff \mathcal{E}(S_n, M_n) = \text{False}$$

$$\mathcal{E}(S_n, M_n^-) = \text{False} \iff \mathcal{E}(S_n, M_n) = \text{True}$$

**Definition 2.6** (Evaluation Functions). • The **Truth Function**  $\mathcal{E}$  is a binary function defined as:

$$\mathcal{E} : \mathcal{S} \times \mathcal{M} \rightarrow \{\text{True}, \text{False}\}$$

such that  $\mathcal{E}(S_n, M_n)$  determines the truth value of statement  $S_n$  within the reference space  $M_n$ .

- The **Provability Function**  $\text{Provable}$  determines the decidability of a statement within its given logical system  $S_n$ :

$$\text{Provable}(x, S_n) = \begin{cases} \text{True} & \text{if } x \text{ is provable in } S_n, \\ \text{False} & \text{if } \neg x \text{ is provable in } S_n, \\ \text{Undecidable} & \text{if neither } x \text{ nor } \neg x \text{ is provable in } S_n. \end{cases}$$

**Definition 2.7** (Uniqueness). The uniqueness condition  $\exists! M_n$  in Axiom 2.2 asserts that for each statement  $S_n \in \mathcal{S}$ , there exists one and only one reference space  $M_n$  in  $\mathcal{M}$  where  $S_n$  is evaluated.

**Definition 2.8** (Comprehensive Evaluation Function). The **Comprehensive Evaluation Function**  $\mathcal{E}_\infty$  is defined as:

$$\mathcal{E}_\infty(P) = \bigcup_{O_n \in \mathcal{O}} \mathcal{E}(O_n)$$

where the  $\bigcup$  is the set of logical operations across all evaluations by observers.

### 3 Analyzing Logical Paradoxes

In this section, we analyze how the proposed axiomatic system can effectively resolve classical logical paradoxes, specifically **Russell's Paradox** and **Yablo's Paradox**. By using the axioms of Logical Space Alternation, Limit Observer Dominance, and the structured evaluation mechanism.

#### 3.1 Russell's Paradox

##### 3.1.1 Overview of Russell's Paradox

Russell's Paradox arises within naive set theory by considering the set of all sets that do not contain themselves. Formally, let  $R$  be defined as:

$$R = \{S \mid S \notin S\}$$

The paradox emerges when questioning whether  $R \in R$ :

- If  $R \in R$ , then by definition,  $R \notin R$ .
- Conversely, if  $R \notin R$ , then by definition,  $R \in R$ .

This self-referential contradiction undermines the consistency of the set-theoretic system.

### 3.1.2 Resolution within the Axiomatic system

The **Axiom of Logical Space Alternation** and the **Axiom of Limit Observer Dominance** can address Russell's Paradox by segregating the evaluation of self-referential statements into distinct Logical Spaces, thereby preventing paradoxical self-inclusion.

#### Step-by-Step Resolution

1. **Initial Evaluation in Positive Logical Space ( $M_1$ ):**

$$\mathcal{E}(R, M_1) = \mathcal{E}_{\text{std}}(R)$$

Here,  $R$  is evaluated under standard logic. However, due to its self-referential nature, determining  $\mathcal{E}_{\text{std}}(R)$  leads to an undecidable or paradoxical truth value.

2. **Identification of Referential Nature:** During evaluation within  $M_1$ , it is determined that  $R$  is self-referential because it references its own membership.
3. **Placement of Resulting Answer into Negative Logical Space ( $M_2$ ):** The resulting answer from evaluating  $R$  in  $M_1$  is placed into  $M_2$ , a Negative Logical Space:

$$\mathcal{E}(R, M_2^-) = \neg\mathcal{E}(R, M_1)$$

4. **Evaluation in Negative Logical Space ( $M_2$ ):** In  $M_2$ , the truth value of  $R$  is inverted:

$$\mathcal{E}(R, M_2^-) = \neg\mathcal{E}_{\text{std}}(R)$$

This inversion resolves the paradox by ensuring that  $R$  does not simultaneously belong and not belong to itself within the system.

5. **Consistency Preservation:** By confining the evaluation of self-referential statements like  $R$  to Negative Logical Spaces, the system maintains consistency, as the paradoxical condition is effectively neutralized through systematic inversion.

## Formal Proof of Consistency

**Theorem 3.1** (Consistency of Evaluation for Russell's Paradox). *Within the defined axiomatic system, Russell's Paradox does not lead to a logical contradiction.*

*Proof.* Assume  $R = \{S \mid S \notin S\}$  is evaluated within the system.

### 1. Evaluation in $M_1$ (Positive Logical Space):

$$\mathcal{E}(R, M_1) = \mathcal{E}_{\text{std}}(R)$$

Due to  $R$ 's self-referential nature,  $\mathcal{E}_{\text{std}}(R)$  is undefined or paradoxical.

2. **Identification of Referential Nature:** The system recognizes  $R$  as self-referential based on its definition.

### 3. Placement into $M_2$ (Negative Logical Space):

$$\mathcal{E}(R, M_2^-) = \neg\mathcal{E}(R, M_1)$$

Since  $\mathcal{E}(R, M_1)$  is paradoxical,  $\mathcal{E}(R, M_2^-)$  provides a consistent inverted evaluation:

$$\mathcal{E}(R, M_2^-) = \text{False} \quad \text{if } \mathcal{E}(R, M_1) = \text{True}$$

$$\mathcal{E}(R, M_2^-) = \text{True} \quad \text{if } \mathcal{E}(R, M_1) = \text{False}$$

Thus, the paradox is resolved by avoiding the simultaneous truth and falsity of  $R$ 's membership.  $\square$

## 3.2 Yablo's Paradox

### 3.2.1 Overview of Yablo's Paradox

Yablo's Paradox presents an infinite sequence of sentences  $S_1, S_2, S_3, \dots$ , where each sentence  $S_n$  states that all sentences  $S_m$  with  $m > n$  are false:

$$S_n : \forall m > n, S_m \text{ is false}$$

Yablo's Paradox does not involve direct self-reference but still leads to a contradiction by creating an infinite regress of dependencies.

### 3.2.2 Resolution within the Axiomatic system

**Logical Space Alternation** and **Limit Observer Dominance** axioms effectively manage Yablo's Paradox by distributing the evaluation of each sentence across distinct Logical Spaces, thereby breaking the infinite dependency chain that leads to paradox.

### Step-by-Step Resolution

1. **Initial Evaluation in Positive Logical Space ( $M_1$ ):** Each sentence  $S_n$  begins its evaluation in  $M_1$ :

$$\mathcal{E}(S_n, M_1) = \mathcal{E}_{\text{std}}(S_n)$$

Under standard logic, evaluating  $S_n$  requires assessing the truth of all subsequent sentences  $S_m$  for  $m > n$ , creating an infinite dependency.

2. **Identification of Referential Nature:** The system detects that  $S_n$  is dependent on an infinite sequence of evaluations, classifying it as meta-referential due to its infinite scope.
3. **Placement of Resulting Answer into Negative Logical Space ( $M_2$ ):** The outcome of evaluating  $S_n$  in  $M_1$  is placed into  $M_2$ :

$$\mathcal{E}(S_n, M_2^-) = \neg\mathcal{E}(S_n, M_1)$$

4. **Evaluation in Negative Logical Space ( $M_2$ ):** In  $M_2$ , the truth value of  $S_n$  is inverted:

$$\mathcal{E}(S_n, M_2^-) = \neg\mathcal{E}_{\text{std}}(S_n)$$

This inversion disrupts the infinite dependency chain by redefining the truth value of each  $S_n$  based on its preceding evaluation.

5. **Propagation Across Logical Spaces:** As each sentence  $S_n$  is evaluated in  $M_2$ , the resulting answers propagate into  $M_3$ ,  $M_4$ , and so on, maintaining the alternating sequence and preventing the establishment of an infinite regress.

### Formal Proof of Consistency

**Theorem 3.2** (Consistency of Evaluation for Yablo's Paradox). *Within the defined axiomatic system, Yablo's Paradox does not lead to a logical contradiction.*

*Proof.* Consider the infinite sequence of sentences  $S_1, S_2, S_3, \dots$  defined by:

$$S_n : \forall m > n, S_m \text{ is false}$$

1. **Evaluation in  $M_1$  (Positive Logical Space):**

$$\mathcal{E}(S_n, M_1) = \mathcal{E}_{\text{std}}(S_n)$$

Evaluating  $S_n$  requires assessing an infinite number of sentences, leading to an undefined or paradoxical truth value under standard logic.

2. **Identification of Referential Nature:** The system identifies  $S_n$  as meta-referential due to its infinite dependence on future sentences.

3. **Placement into  $M_2$  (Negative Logical Space):**

$$\mathcal{E}(S_n, M_2^-) = \neg\mathcal{E}(S_n, M_1)$$



The truth value of  $S_n$  is inverted, effectively resolving the paradox by preventing the infinite dependency from causing a contradiction.

4. **Propagation Mechanism:** Each sentence's resulting answer in  $M_2$  leads to its evaluation in  $M_3$ , continuing the alternation without introducing new paradoxes. The alternating sequence ensures that each sentence's truth value is consistently managed without falling into an infinite regress.

5. **Consistency Preservation:** By systematically placing the resulting answers into subsequent Logical Spaces with inverted evaluations, the system maintains consistency, as each  $S_n$  is evaluated in isolation within its designated Logical Space, preventing the interdependent contradictions inherent in Yablo's original formulation.  $\square$

**Role of the Limit Observer** The **Limit Observer**  $O_\infty$  plays a pivotal role in overseeing the evaluation process across all Logical Spaces. By encompassing the evaluation capabilities of all other observers,  $O_\infty$  ensures that every sentence  $S_n$  and its resulting evaluations in respective Logical Spaces are consistently managed, thereby maintaining the overall integrity of the system.

### 3.2.3 Illustrative Example

**Example 3.1.** Consider the first three sentences of Yablo's sequence:

$$S_1 : \forall m > 1, S_m \text{ is false}$$

$$S_2 : \forall m > 2, S_m \text{ is false}$$

$$S_3 : \forall m > 3, S_m \text{ is false}$$

1. **Evaluation in  $M_1$  (Positive Logical Space):**

$$\mathcal{E}(S_1, M_1) = \mathcal{E}_{\text{std}}(S_1) \quad (\text{Undefined or paradoxical})$$

$$\mathcal{E}(S_2, M_1) = \mathcal{E}_{\text{std}}(S_2) \quad (\text{Undefined or paradoxical})$$

$$\mathcal{E}(S_3, M_1) = \mathcal{E}_{\text{std}}(S_3) \quad (\text{Undefined or paradoxical})$$

2. **Identification and Placement into  $M_2$  (Negative Logical Space):**

$$\mathcal{E}(S_1, M_2^-) = \neg\mathcal{E}(S_1, M_1)$$

$$\mathcal{E}(S_2, M_2^-) = \neg\mathcal{E}(S_2, M_1)$$

$$\mathcal{E}(S_3, M_2^-) = \neg\mathcal{E}(S_3, M_1)$$

3. **Evaluation in  $M_2$ :** The truth values of  $S_1, S_2, S_3$  are now inverted, resolving the paradoxical outcomes from  $M_1$ .

4. **Propagation to Subsequent Logical Spaces:** The resulting answers from  $M_2$  are evaluated in  $M_3$ , and so forth, maintaining the alternating sequence and preventing infinite dependency chains.

### 3.3 Summary of Resolution Mechanisms

- **Segregation of Evaluation Contexts:** By categorizing evaluations into Positive and Negative Logical Spaces, the system ensures that standard and meta/self-referential statements are handled in distinct contexts, preventing self-referential contradictions.
- **Inversion of Truth Values:** In Negative Logical Spaces, the referenced inversion of truth values for meta/self-referential statements neutralizes paradoxical conditions, maintaining logical consistency.
- **Hierarchical Evaluation Flow:** The infinite, well-ordered progression through Logical Spaces, overseen by the Limit Observer, ensures that each statement is evaluated systematically without falling into infinite regress or duplication.
- **Comprehensive Oversight:** The Limit Observer's ability to encompass all evaluation capabilities guarantees that every proposition, regardless of its complexity or referential nature, is consistently managed within a system.

### 3.4 Conclusion

This axiomatic system resolves some self-referential classical logical paradoxes such as Russell's and Yablo's by employing a structured evaluation mechanism across alternating Logical Spaces. This separation, and comprehensive oversight by the Limit Observer, ensures that some paradoxical conditions can be systematically neutralized or managed, thereby preserving logical consistency and preventing contradictions.

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