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# ML-DOA estimation using a sparse representation of array covariance

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**Abstract**—Sparse Direction-of-Arrival (DOA) estimators depend on the regularization parameter  $\lambda$  which is often empirically tuned. In this work, conducted under the vectorized covariance matrix model, we are looking for theoretical equivalence between the Maximum Likelihood (ML) and sparse estimators. We show that under mild conditions,  $\lambda$  can be chosen thanks to the distribution of the minimum of the ML criterion in the case of two impinging sources. The corresponding  $\lambda$  choice is  $\theta$ -invariant, only requiring an upper bound on the number of sources. Furthermore, it guarantees the global minimum of the sparse  $\ell_0$ -regularized criterion to be the ML solution.

Numerical experiments confirm that, for the proposed  $\lambda$ , sparse and ML estimators yield the same statistical performance.

**Index Terms**—sparse estimation, regularization parameter, maximum likelihood

## I. INTRODUCTION

DOA estimation is a pivotal area in many critical applications such as radar or telecommunications. Throughout the last 60 years, a plethora of estimation techniques has been proposed [1], [2]. On the first hand conventional techniques such as Capon’s beamformer [3] can not handle multiple sources. On the other hand, High Resolution (HR) estimators like MUSIC [4], [5] or ESPRIT [6] overcome this issue by exploiting the eigenstructure of the array covariance matrix. Nevertheless, these subspace-based approaches have limited performance in adversarial scenarios with few array snapshots or in presence of modeling errors [7]. Finally, the ML estimator [8] achieves the Cramér-Rao Lower Bound (CRLB) at high Signal-to-Noise Ratio (SNR) [9]. Although its statistical efficiency, the ML is rarely employed as it requires multi-dimensional non-convex optimization.

In recent years, there has been a growing interest within the signal processing community on sparse methods applied to DOA estimation [10] as they exhibit enhanced performances in tough scenarios. Out of the numerous modeling of the sparse DOA estimation problem, the sparse covariance matrix representation [11] emerged as a popular choice. Indeed, it benefits from the Virtual Array [12] concept thus having increased resolving power and number of identifiable sources.

For this model, the DOAs can be estimated through the minimization of a non-convex  $\ell_0$ -regularized objective  $J_{\ell_0}$  which is parametrized by  $\lambda$  the regularization parameter. Many works empirically tuned  $\lambda$  which is unfeasible in practice. In [13], the authors proposed an interval for the regularization pa-

rameter. However, the interval bounds depends on the sources directions thus complexifying off-line selection.

In this work, we introduce a novel  $\theta$ -invariant regularization parameter choice  $\lambda$  relying on the ML criterion distribution. This choice ensures the equivalence between ML and sparse DOA estimators as shown through numerical simulations.

## II. MAXIMUM LIKELIHOOD ON THE VECTORIZED COVARIANCE MATRIX MODEL

### A. Vectorized covariance matrix model

We consider a scenario of  $M$  independent sources of directions  $\Theta = \{\theta_1, \dots, \theta_M\}$  impinging on an array of  $N$  sensors. Assuming narrowband hypothesis, the array observation is given by:

$$\mathbf{x}(t) = \sum_{m=1}^M \mathbf{a}(\theta_m) s_m(t) + \mathbf{n}(t) = \mathbf{A}(\Theta) \mathbf{s}_\Theta(t) + \mathbf{n}(t) \quad (1)$$

with  $\mathbf{A}(\Theta)$  the array manifold matrix formed by the steering vectors  $\mathbf{a}(\theta_m) \in \mathbb{C}^N, m = 1 \dots M$ ,  $\mathbf{s}_\Theta(t) = [s_1(t), \dots, s_M(t)]^T \in \mathbb{C}^M$  a vector containing the complex envelopes of the emitted signals and  $\mathbf{n}(t) \in \mathbb{C}^N$  a complex Gaussian circular noise, independent of  $\mathbf{s}(t)$ , with covariance  $\mathbb{E}[\mathbf{n}(t)\mathbf{n}^H(t)] = \sigma^2 \mathbf{I}_N$  where  $\mathbb{E}[\cdot]$  is the temporal mean,  $(\cdot)^H$  denotes the complex conjugate transpose and  $\mathbf{I}_N$  the identity matrix of size  $N$ .

The array covariance matrix is then given by:

$$\mathbf{R}_x = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^H(t)] = \mathbf{A}(\Theta) \mathbf{R}_s \mathbf{A}^H(\Theta) + \sigma^2 \mathbf{I}_N \quad (2)$$

where  $\mathbf{R}_s = \mathbb{E}[\mathbf{s}_\Theta(t)\mathbf{s}_\Theta^H(t)]$  is the sources covariance matrix.

In practice, the covariance matrix is not available. Instead,  $\mathbf{R}_x$  is replaced by its estimate  $\hat{\mathbf{R}}_x$  computed using  $K$  identically and independently distributed (i.i.d) array snapshots. Assuming temporally white noise ( $\forall i \neq j, \mathbb{E}[\mathbf{n}^H(t_i)\mathbf{n}(t_j)] = 0$ ),  $\hat{\mathbf{R}}_x$  can be written as:

$$\hat{\mathbf{R}}_x = \frac{1}{K} \sum_{k=1}^K \mathbf{x}(t_k)\mathbf{x}^H(t_k) = \mathbf{R}_x + \Delta \mathbf{R}_x \quad (3)$$

where  $K\Delta \mathbf{R}_x$  is a Wishart noise due to the finite number of snapshots [14].

Assuming that sources are uncorrelated, the vectorized covariance matrix model is obtained from (2, 3) by applying the column-wise vectorization operator  $\text{vec}(\cdot)$  to  $\widehat{\mathbf{R}}_x - \sigma^2 \mathbf{I}_N$ :

$$\mathbf{r} = \text{vec}(\widehat{\mathbf{R}}_x - \sigma^2 \mathbf{I}_N) = \sum_{m=1}^M \mathbf{b}(\theta_m) \gamma_m + \boldsymbol{\delta} = \mathbf{B}(\boldsymbol{\Theta}) \boldsymbol{\gamma}_{\Theta} + \boldsymbol{\delta} \quad (4)$$

where  $\mathbf{b}(\theta_m) = \mathbf{a}^*(\theta_m) \otimes \mathbf{a}(\theta_m)$  is the virtual array steering vector with  $\otimes$  the Kronecker product and  $(\cdot)^*$  the complex conjugate. The corresponding mixing matrix is referred as  $\mathbf{B}(\boldsymbol{\Theta}) = [\mathbf{b}(\theta_1), \dots, \mathbf{b}(\theta_M)]$ .  $\gamma_m = \mathbb{E}[|s_m(t)|^2]$  denotes the power of the  $m$ -th source and  $\boldsymbol{\gamma}_{\Theta} = [\gamma_1, \dots, \gamma_M]^T$  the vector of the sources powers. Finally,  $\boldsymbol{\delta} = \text{vec}(\Delta \widehat{\mathbf{R}}_x)$  is the vectorized noise.

Asymptotically, the vectorized observation asymptotically follows a multivariate complex Gaussian law  $\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Gamma}, \mathbf{C})$  where [15]:

$$\boldsymbol{\Gamma} = \mathbb{E}[(\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})^H] = \frac{1}{K} \mathbf{R}_x^T \otimes \mathbf{R}_x \quad (5)$$

and  $\mathbf{C}$  is defined in [15].

### B. Maximum Likelihood estimation

For additive Gaussian noise a ML estimator of  $\boldsymbol{\Theta}$ , known to achieve the Cramér-Rao lower bound at high Signal-to-Noise Ratio (SNR), can be easily obtained from (4) [8]. However, for non-diagonal covariance matrices such as (5) this yields a multi-term log-likelihood function. This objective can be simplified for diagonal covariance matrices leading us to apply a pre-whitening step to (4):

$$\mathbf{y} = \widehat{\mathbf{W}} \mathbf{r} = \widehat{\mathbf{W}} \mathbf{B}(\boldsymbol{\Theta}) \boldsymbol{\gamma}_{\Theta} + \boldsymbol{\delta}_w \quad (6)$$

where  $\widehat{\mathbf{W}} = \sqrt{K} \widehat{\mathbf{R}}_x^{-T/2} \otimes \widehat{\mathbf{R}}_x^{-1/2}$  is the estimated whitening matrix and  $\boldsymbol{\delta}_w = \widehat{\mathbf{W}} \boldsymbol{\delta}$  a spatially white noise with  $\mathbb{E}[\boldsymbol{\delta}_w \boldsymbol{\delta}_w^H] = \mathbf{I}_{N^2}$ . Applying the ML to (6) yields the following estimator of  $\boldsymbol{\Theta}$ :

$$J_{ML}(\boldsymbol{\Theta}) = \text{tr}(\boldsymbol{\Pi}^{\perp}(\boldsymbol{\Theta}) \mathbf{y} \mathbf{y}^H) = \mathbf{y}^H \boldsymbol{\Pi}^{\perp}(\boldsymbol{\Theta}) \mathbf{y} \quad (7)$$

$$\widehat{\boldsymbol{\Theta}} = \arg \min_{\boldsymbol{\Theta} \in \mathbb{R}^M} J_{ML}(\boldsymbol{\Theta})$$

where  $\boldsymbol{\Pi}^{\perp}(\boldsymbol{\Theta}) = \mathbf{I}_{N^2} - (\widehat{\mathbf{W}} \mathbf{B}(\boldsymbol{\Theta})) (\widehat{\mathbf{W}} \mathbf{B}(\boldsymbol{\Theta}))^{\#}$  is the noise projector computed for directions  $\boldsymbol{\Theta}$  with  $(\cdot)^{\#}$  the Moore-Penrose pseudo inverse and  $\text{tr}(\cdot)$  the trace operator. In what follows, the optimal value of the ML criterion  $J_{ML}$  is referred as:

$$\epsilon = \min_{\boldsymbol{\Theta} \in \mathbb{R}^M} J_{ML}(\boldsymbol{\Theta}) = J_{ML}(\widehat{\boldsymbol{\Theta}}) = \mathbf{y}^H \boldsymbol{\Pi}^{\perp}(\widehat{\boldsymbol{\Theta}}) \mathbf{y} \quad (8)$$

## III. SPARSE ESTIMATION

### A. Sparse modeling

Under the hypothesis that DOAs lie within a predefined grid of directions  $\boldsymbol{\Phi} = \{\varphi_1, \dots, \varphi_G\}$  ( $\boldsymbol{\Theta} \subset \boldsymbol{\Phi}$ ), sparsity can be introduced in (6) using an overcomplete dictionary  $\mathbf{B}(\boldsymbol{\Phi}) = [\mathbf{b}(\varphi_1), \dots, \mathbf{b}(\varphi_G)] \in \mathbb{C}^{N^2 \times G}$  with  $G \gg N^2$ . Thus, we obtain the following sparse model for the observation  $\mathbf{y}$ :

$$\mathbf{y} = \widehat{\mathbf{W}} \mathbf{B}(\boldsymbol{\Phi}) \boldsymbol{\gamma}_0 + \boldsymbol{\delta}_w \quad (9)$$

where  $\boldsymbol{\gamma}_0$  is a sparse vector which has exactly  $M$  non-zeros entries corresponding to the sources powers. Thereafter,  $\boldsymbol{\gamma}_0$  is referred as the sparse spectrum.

### B. Sparse problem formulation

The aim of DOA estimation is to find the directions of the impinging signals from the observation  $\mathbf{y}$  which corresponds, in the sparse model (9), to the non-zeros entries of  $\boldsymbol{\gamma}_0$ . Thus, we need to estimate the sparse spectrum  $\boldsymbol{\gamma}_0$  in order to provide an estimate of  $\boldsymbol{\Theta}$ . This problem is ill-posed since  $G \gg N^2$ . Hence, the sparsity of  $\boldsymbol{\gamma}_0$  is exploited to ensure the uniqueness of the solution. Following Delmer's work [13], an estimate of  $\boldsymbol{\gamma}$  can be obtained through the minimization of the  $\ell_0$ -regularized objective function given by:

$$\min_{\boldsymbol{\gamma} \in \mathbb{C}^G} \left\{ J_{\ell_0}(\lambda, \boldsymbol{\gamma}) = \frac{1}{2} \|\mathbf{y} - \widehat{\mathbf{W}} \mathbf{B}(\boldsymbol{\Phi}) \boldsymbol{\gamma}\|_2^2 + \lambda \|\boldsymbol{\gamma}\|_0 \right\} \quad (10)$$

with a suitable regularization parameter  $\lambda > 0$  which balances data fidelity towards the solution sparsity.

## IV. SELECTION OF THE REGULARIZATION PARAMETER

The choice of  $\lambda$  in (10) is of utmost importance since it controls the trade-off between estimation error and sparsity. Many approaches exploiting the noise level have been proposed for  $\ell_1$ -regularization [16]–[18]. For the  $\ell_0$  optimization framework  $\lambda$  is generally empirically tuned given that only few results are available. Recently, Delmer [13] *et al.* proposed to choose  $\lambda$  so that  $J_{\ell_0}$  has a global minimum in  $\hat{\boldsymbol{\gamma}}$  ie.:

$$\lambda > 0 \mid \forall \boldsymbol{\gamma} \in \mathbb{C}^G, J_{\ell_0}(\lambda, \hat{\boldsymbol{\gamma}}) \leq J_{\ell_0}(\lambda, \boldsymbol{\gamma}) \quad (11)$$

where  $\hat{\boldsymbol{\gamma}}$  is the sparse spectrum, in the neighboring of  $\boldsymbol{\gamma}_0$ , obtained using the ML DOA estimate  $\widehat{\boldsymbol{\Theta}}$  (7). For a given observation  $\mathbf{y}$ , the values of  $\lambda$  that achieves condition (11) are within a stochastic admissible interval  $I_M(\mathbf{y}) = [\lambda_M^-, \lambda_M^+]$ . This interval ensures the global minimizer of (10) to be the ML solution which is an  $M$ -sparse vector.

For the case  $M = 2$ , the corresponding interval is  $I_2(\mathbf{y}) = [\lambda_2^-, \lambda_2^+]$  where:

$$\lambda_2^- \approx \frac{c_2(\mathbf{y}) - c_3(\mathbf{y})}{2} \quad \text{and} \quad \lambda_2^+ \approx \frac{c_1(\mathbf{y}) - c_2(\mathbf{y})}{2} \quad (12)$$

with  $c_k(\mathbf{y}) = \inf\{\|\mathbf{y} - \widehat{\mathbf{W}} \mathbf{B}(\boldsymbol{\Phi}) \boldsymbol{\gamma}\|_2^2, \|\boldsymbol{\gamma}\|_0 = k\}$  the ML criterion optimal value (8) for a prescribed sparsity level  $k$ . Note that we have  $c_3(\mathbf{y}) \leq c_2(\mathbf{y}) \leq c_1(\mathbf{y})$ . In the following,  $I_2$  is assumed non-empty (ie.  $c_1(\mathbf{y}) > c_3(\mathbf{y})$ ).

For arrays presenting robustness against second order ambiguities for  $M = 2$  sources with equal power, the relation  $c_2(\mathbf{y}) \ll c_1(\mathbf{y})$  holds. Thus, the following inequality can be proven:

$$\lambda_2^- \leq \frac{1}{2} c_2(\mathbf{y}) \leq \lambda_2^+ \quad (13)$$

Remarking that  $c_2(\mathbf{y})$  corresponds to the minimum of the ML criterion (8)  $\epsilon$ , a suitable regularization parameter choice satisfying (13) can be formulated as:

$$\lambda = \frac{1}{2} F_{\epsilon}(\eta) \quad (14)$$

where  $F_\epsilon(\eta)$  is the inverse Cumulative Distribution Function of  $\epsilon$  for probability  $\eta$ . In [19], we derived the distribution of  $\epsilon$  under complex Gaussian non-circular noise. We proved that  $\epsilon$  asymptotically has a  $\chi_2$  distribution with  $N^2 - M$  degrees of freedom. The choice of  $\eta$  is discussed in section V.

## V. NUMERICAL EXPERIMENTS

We consider an array of  $N = 6$  antennas with 5 antennas distributed around a circle of radius  $0.8\lambda_0$  where  $\lambda_0$  denotes the wavelength and one central sensor.  $M = 2$  sources of directions  $\theta_1 = 180^\circ, \theta_2 = 200^\circ$  with  $K = 200$  and SNR = 0 dB impinge on this array. Fig. 1 depicts the distribution of the interval bounds  $\lambda_2^-, \lambda_2^+$  and  $\lambda = \frac{1}{2}\epsilon$ . For this scenario, our regularization parameter choice belongs to the interval thus ensuring that  $J_{\ell_0}$  global minimizer corresponds to the ML DOA solution.

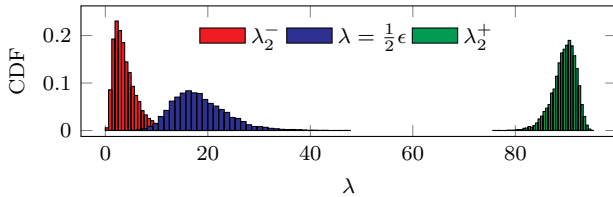


Fig. 1. Histograms of the distributions of  $\lambda_2^-, \lambda_2^+$  and  $\lambda = \frac{1}{2}\epsilon$  with Cumulative Distribution Function (CDF) normalization.

According to [19], we propose to choose  $\eta = 0.05$  leading to  $\lambda = \frac{1}{2}F_{\chi^2(34)}^{-1}(0.05)$  where  $F_{\chi^2(34)}^{-1}$  is the inverse CDF of the  $\chi_2$  law with 34 degrees of freedom.

Finally, we verify that, for the proposed  $\lambda$  choice, ML and sparse methods lead to the same statistical performance. To this end, we conducted 10000 independent Monte-Carlo experiments for SNR values between  $-20$  dB and  $20$  dB. The sources are considered resolved if two peaks are detected and  $\max\{|\hat{\theta}_1 - \theta_1|, |\hat{\theta}_2 - \theta_2|\} < 10^\circ$  where  $10^\circ$  corresponds to the half beamwidth of the virtual array. Results, obtained with the Forward-Backward splitting algorithm [20], are represented on Fig.2. For  $\text{SNR} \geq -4$  dB, ML and sparse estimators have the same statistical performance thus validating the proposed  $\lambda$  choice.

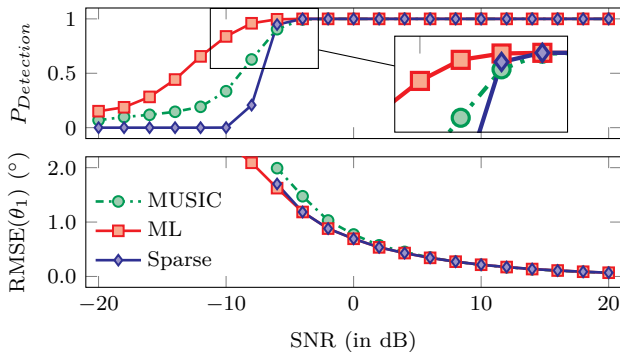


Fig. 2. Probability of detection (top) and RMSE (bottom). Given that the two sources have the same power, results are only represented for direction  $\theta_1$ .

## VI. CONCLUSION

In this paper, we proposed a novel regularization choice for  $\ell_0$ -regularized DOA estimation by exploiting the distribution of the ML criterion thus leading to  $\theta$ -invariant choice. Through numerical experiments, we confirmed that the proposed  $\lambda$  belongs to Delmer's interval hence ensuring the equivalence with the ML estimator. This approach must be extended to scenarios with  $M > 2$  sources. This is an ongoing work.

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