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# Monotone weak distributive laws over the lifted powerset monad in categories of algebras

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## Abstract

Noticing the similarity between the monotone weak distributive laws combining two layers of non-determinisms in sets and in compact Hausdorff spaces, we study whether the latter law can be obtained automatically as a weak lifting of the former. This holds partially, but does not generalize to other categories of algebras: we then characterize when exactly monotone weak distributive laws over powerset monads in categories of algebras exist, exhibiting a law combining probabilities and non-determinism in compact Hausdorff spaces and showing on the other hand that such laws do not exist in a lot of other cases.

**2012 ACM Subject Classification** Theory of computation  $\rightarrow$  Categorical semantics

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**Category** Track B. Automata, Logic, Semantics and Theory of Programming

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## 1 Introduction and preliminaries

In the study of the semantics of programming languages, and since the seminal work of Moggi [31], effectful computations are usually modeled with monads: an effectful function of type  $X \multimap Y$  is interpreted as a function of type  $X \rightarrow TY$ , where the monadic structure on  $T$  allows for having identities and compositions of such effectful functions. When considering several effects at the same time, a natural question arises: given monads corresponding to two effects, is it possible to construct a monad that corresponds to the combination of these two effects? In particular, combining probabilities and non-determinism has been a very popular subject of study in the topological and (dually) domain-theoretic setting: see for instance the introduction to [26] for an extensive bibliography on the topic.

The most straightforward way to combine two monads  $S$  and  $T$  would be to compose their underlying functors, but unfortunately in general the resulting endofunctor  $ST$  may not carry the structure of a monad. For this to hold, one usually requires the existence of a *distributive law* of  $T$  over  $S$ , a natural transformation  $TS \Rightarrow ST$  satisfying four axioms involving the units and multiplications of the two monads [4]. Such a distributive law makes  $ST$  into a monad, and its data is equivalent to the data of a *lifting* of  $S$  to the Eilenberg-Moore category  $EM(T)$  of  $T$ -algebras or to the data of an *extension* of  $T$  to the Kleisli category  $Kl(S)$  of free  $S$ -algebras.

Unfortunately, distributive laws turn out to be not so common: proving that some specific pairs of monads do not admit any distributive law between them has been the focus of several works [27, 35, 16], culminating in [37] where general techniques for proving the absence of distributive laws between monads on  $\mathbf{Set}$ , so-called “no-go theorems”, are exhibited. Among the culprits are the powerset monad  $P$  and the probability distributions monad  $D$ : there is no distributive law  $PP \Rightarrow PP$  nor  $DP \Rightarrow PD$ .

A next step in combining monads is thus to weaken the requirements asked for by distributive laws. In [6] a 2-categorical theory of such weakened distributive laws, where the axioms relating to the units of the monads are relaxed, is developed. This theory encompasses two orthogonal kinds of weakened laws, both called weak distributive laws: those of [34], extensively studied in [8, 9, 7], and those more recently put light on in [19] and given further instances of in [23, 24, 10]. This work focuses on the latter kind of laws: in the following a *weak distributive law* of  $T$  over  $S$  will thus be a natural transformation of type  $TS \Rightarrow ST$  (again), where only three of the four axioms of distributive laws are required. Such a weak distributive law need not make  $ST$  into a monad, but do so of a retract of it when the category is well-behaved. Weak laws  $TS \Rightarrow ST$  are equivalent to *weak extensions* of  $T$  to  $KI(S)$  — extensions of the semi-monad underlying  $T$ , i.e. of its endofunctor and multiplication but not of its unit — or, when the category is well-behaved, *weak liftings* of  $S$  to  $EM(T)$  — liftings of  $S$  up to a retraction.

Weak extensions are a precious tool for building weak distributive laws, because *monotone extensions* to  $\mathbf{Rel}$  — the category of sets and relations, which also happens to be the category of free algebras of the powerset monad  $P$  on  $\mathbf{Set}$  — admit a nice characterization: this makes it possible to find weak distributive laws without having to guess their formulæ anymore, and the monotonicity of the extension is a strong indicator that the resulting law will be semantically interesting. In the context of weak distributive laws, this strategy was first used by Garner [19], who exhibited a law  $PP \Rightarrow PP$ , but also a law  $\beta P \Rightarrow P\beta$  combining the ultrafilter monad  $\beta$  and the powerset monad and whose corresponding weak lifting turned out to be the Vietoris monad  $V$  of closed subsets on the category  $\mathbf{KHaus}$  of compact Hausdorff spaces (the algebras of  $\beta$  [29]). The same strategy was then used for instance in [23], where a law  $DP \Rightarrow PD$ , combining probabilities and non-determinism, is described.

In fact, most non-trivial weak distributive laws in the literature (trivial ones are described in [21]) are laws of type  $TP \Rightarrow PT$  built using this strategy. On the other hand, in [24] the goal was to find weak distributive laws in other categories than  $\mathbf{Set}$ , and it was reached as a law  $VV \Rightarrow VV$ , combining two layers of non-determinism in a continuous setting, was described. This last law was also built by constructing a monotone weak extension, and its formula is thus very close to that of the law  $PP \Rightarrow PP$ . In topological settings, non-determinism can also be combined with probabilities: weak distributive laws for this purpose are constructed by hand in a very recent pre-print [20].

The goal of the present work is to take over this program of finding non- $\mathbf{Set}$ -based weak distributive laws: we focus here in categories of algebras, which fit in the general framework for monotone weak laws presented in [24]. In particular, we notice in Lemma 11 that the law  $VV \Rightarrow VV$  is not only very similar to the law  $PP \Rightarrow PP$ , but is also actually some sort of weak lifting of it. We thus study whether there is a general framework for not only weakly lifting monads (as weak distributive laws do), but also weakly lifting weak distributive laws themselves: this framework should yield or simplify the construction of the law  $VV \Rightarrow VV$ , and hopefully generalize it to other categories of algebras, in particular  $EM(P)$  and  $EM(D)$  which have weakly lifted powerset monads thanks to the laws  $PP \Rightarrow PP$  and  $DP \Rightarrow PD$ .

This question is largely related to the problem of composing weak distributive laws, which was investigated in [22] from the point of view of the Yang-Baxter equations — the usual tool for composing and lifting plain distributive laws [14]. We end up getting a general “no-go theorem” for monotone weak laws over weakly lifted powerset monads: in that sense this work is also close in spirit to [37], where general no-go theorems for (strict) distributive laws are given. While not restricted to monotone distributive laws, these theorems are unlikely to generalize to our setting because they are based on the correspondance between monads

and algebraic theories, which is mostly restricted to  $\mathbf{Set}$  and does not have any obvious generalization to semi-monads — to which weak distributive laws are deeply related.

This article is organized as follows. In Section 2, we recall definitions and notations for and give examples of monads and weak distributive laws, as well as the framework of [24] for monotone weak distributive laws in regular categories. This culminates in Lemma 11, where we notice that the law  $\mathbf{VV} \Rightarrow \mathbf{VV}$  is some sort of weak lifting of the law  $\mathbf{PP} \Rightarrow \mathbf{PP}$ . The next two sections focus on lifting weak distributive laws: in Section 3 we study the approach of the Yang-Baxter equation, showing it indeed allows for weakly lifting weak laws but does not apply to the examples we consider; while in Section 4 we focus on the monotonicity of the laws, giving a simple characterization for the existence of monotone weak laws in categories of algebras and applying it to several examples. We conclude in Section 5.

Our main contributions are the following:

- we show that, while the law  $\mathbf{VV} \Rightarrow \mathbf{VV}$  is a kind of weak lifting of the law  $\mathbf{PP} \Rightarrow \mathbf{PP}$  (Lemma 17), this lifting does not come from a Yang-Baxter equation (Corollary 15), the usual approach to lifting laws: it is an instance of a general no-go theorem for Yang-Baxter equations involving the law  $\mathbf{PP} \Rightarrow \mathbf{PP}$  (Proposition 14);
- we characterize the Kleisli category of the weak lifting  $\bar{\mathbf{S}}$  of a monad  $\mathbf{S}$  to the algebras of a monad  $\mathbf{T}$  as the category of  $\mathbf{T}$ -algebras and  $\mathbf{Kl}(\bar{\mathbf{S}})$ -morphisms between them (Proposition 21);
- we give a characterization of the Kleisli categories of weakly lifted powerset monads as subcategory of relations (Theorem 26), in which a certain class of *decomposable morphisms* play a central role: it follows that monads must preserve these decomposable morphisms to have monotone weak distributive laws over weakly lifted powerset monads, and this is in fact a sufficient condition for monads that are themselves weak liftings (Corollary 34);
- concrete instances of this result are then easily derived: we recover independently the law combining probabilities and non-determinism in compact Hausdorff spaces and recently exhibited in [20] (Theorem 35), but we observe otherwise that monotone weak distributive laws over weakly lifted powerset monads in categories of algebras seem very rare (Table 1).

## 2 Preliminaries

In this section we first recall the theory of weak distributive laws from [19] and the tools that come along, especially the ones developed in [24]. The reader is assumed to be familiar with the basics of category theory, an introduction to which appearing for instance in [11].

### 2.1 Monads and (weak) distributive laws

► **Definition 1** (monad). A monad on a category  $\mathbf{C}$  is the data  $(\mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}})$ , often abbreviated  $\mathbf{T}$ , of an endofunctor  $\mathbf{T}: \mathbf{C} \rightarrow \mathbf{C}$  and natural transformations  $\eta^{\mathbf{T}}: \text{Id} \rightarrow \mathbf{T}$  and  $\mu^{\mathbf{T}}: \mathbf{T}\mathbf{T} \rightarrow \mathbf{T}$  satisfying the axioms  $\mu^{\mathbf{T}} \circ \mathbf{T}\eta^{\mathbf{T}} = \text{id} = \mu^{\mathbf{T}} \circ \eta^{\mathbf{T}}\mathbf{T}$  and  $\mu^{\mathbf{T}} \circ \mathbf{T}\mu^{\mathbf{T}} = \mu^{\mathbf{T}} \circ \mu^{\mathbf{T}}\mathbf{T}$ .

► **Example 2** (monads). In this work we will be particularly concerned with the following monads on  $\mathbf{Set}$ . Thereafter  $X$  and  $Y$  are sets,  $x$  is an element of  $X$  and  $f: X \rightarrow Y$  is a function.

**The powerset monad  $\mathbf{P}$ .**  $\mathbf{P}X = \{\text{subsets of } X\}$ ; for  $e \subseteq X$ ,  $(\mathbf{P}f)(e) = f[e] = \{f(x') \mid x' \in e\}$ ;  $\eta_X^{\mathbf{P}}(x) = \{x\}$ ; for  $E \subseteq \mathbf{P}X$ ,  $\mu_X^{\mathbf{P}}(E) = \bigcup E$ .

**The probability distributions monad  $\mathbf{D}$ .**  $\mathbf{D}X$  is the set of finitely-supported probability distributions on  $X$ , i.e. functions  $\varphi: X \rightarrow [0, 1]$  such that  $\varphi^{-1}([0, 1])$  is finite and  $\sum_{x \in X} \varphi(x) = 1$ ;  $\mathbf{D}f$  is the pushforward along  $f$  given by  $(\mathbf{D}f)(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$

for  $\varphi \in DX$  and  $y \in Y$ ;  $\eta_X^D(x)$  is the Dirac  $\delta_x$  for  $x \in X$ , i.e. the probability distribution such that  $\delta_x(x) = 1$ ; and  $\mu^D$  computes the mean of a distribution of distributions, so that for  $\Phi \in DDX$ ,  $\mu^D(\Phi)(x) = \sum_{\varphi \in DX} \Phi(\varphi) \cdot \varphi(x)$ .

**The ultrafilter monad  $\beta$ .**  $\beta X$  is the set of maximal filters on  $X$ , where filters are sets  $E \in PPX$  that are non-empty, up-closed (for inclusion), stable under finite intersections and do not contain the empty set; for an ultrafilter  $E \in \beta X$ ,  $(\beta f)(E) = \{e' \supseteq f(e) \mid e \in E\}$ ;  $\eta_X^\beta(x)$  is the principal filter  $\{e \in PX \mid x \in e\}$ ; and, for  $\mathfrak{C} \in \beta\beta X$ ,  $\mu_X^\beta(\mathfrak{C}) = \bigcup \{\bigcap \mathfrak{c} \mid \mathfrak{c} \in \mathfrak{C}\}$ .

We will also be concerned with a topological analogue of the powerset monad, defined on the category  $\text{KHaus}$  of compact Hausdorff spaces and continuous functions. Let  $X$  and  $Y$  be a compact Hausdorff space,  $x \in X$ , and  $f: X \rightarrow Y$  be a continuous function.

**The Vietoris monad  $\mathbf{V}$ .**  $\mathbf{V}X$  is the space of closed subsets of  $X$  equipped with the topology for which a subbase is given by the sets  $\square u = \{c \in \mathbf{V}X \mid c \subset u\}$  and  $\diamond u = \{c \in \mathbf{V}X \mid c \cap u \neq \emptyset\}$  where  $u$  ranges among all open sets of  $X$ ;  $(\mathbf{V}f)(c) = f[c] = \{f(x) \mid x \in c\}$  for  $c \in \mathbf{V}X$ ;  $\eta_X^{\mathbf{V}}(x) = \{x\}$ ; and  $\mu_X^{\mathbf{V}}(C) = \bigcup C$  for  $C \in \mathbf{V}\mathbf{V}X$ .

Weak distributive laws are a certain type of natural transformations involving monads.

► **Definition 3** ((weak) distributive law [19, Definition 9]). *A monad  $T$  weakly distributes over a monad  $S$  when there is a weak distributive law of  $T$  over  $S$ , i.e. a natural transformation  $\rho: TS \Rightarrow ST$  such that the Diagrams  $(\eta^+)$ ,  $(\mu^-)$ , and  $(\mu^+)$  below commute. A (strict) distributive law is a weak distributive law that moreover has the diagram  $(\eta^-)$  below commute.*

If  $\rho: TS \Rightarrow ST$  is a distributive law,  $ST$ ,  $\eta^{\text{ST}} = \eta^S \eta^T$  and  $\mu^{\text{ST}} = \mu^S \mu^T \circ S\rho T$  form a monad [4]. If it is only a weak distributive law this is not the case anymore, because  $\mu^{\text{ST}} \circ \eta^{\text{ST}} ST$  is not the identity anymore, only an idempotent (its composite with itself is itself). But suppose now it is a

$$\begin{array}{ccc} \begin{array}{ccc} \eta^{\text{TS}} \swarrow & S & \searrow S\eta^{\text{T}} \\ & (\eta^-) & \\ \text{TS} & \xrightarrow{\rho} & \text{ST} \end{array} & \begin{array}{ccc} \text{TTS} & \xrightarrow{\text{T}\rho} & \text{TST} \xrightarrow{\rho^{\text{T}}} & \text{STT} \\ \mu^{\text{TS}} \downarrow & & (\mu^-) & \downarrow S\mu^{\text{T}} \\ \text{TS} & \xrightarrow{\rho} & \text{ST} & \end{array} \\ \begin{array}{ccc} \text{T}\eta^{\text{S}} \swarrow & T & \searrow \eta^{\text{ST}} \\ & (\eta^+) & \\ \text{TS} & \xrightarrow{\rho} & \text{ST} \end{array} & \begin{array}{ccc} \text{TSS} & \xrightarrow{\rho^{\text{S}}} & \text{STS} \xrightarrow{S\rho} & \text{SST} \\ \text{T}\mu^{\text{S}} \downarrow & & (\mu^+) & \downarrow \mu^{\text{ST}} \\ \text{TS} & \xrightarrow{\rho} & \text{ST} & \end{array} \end{array}$$

split idempotent, i.e. that there is a functor  $S \bullet T$  and natural transformations  $p: ST \Rightarrow S \bullet T$  and  $i: S \bullet T \Rightarrow ST$  such that  $p \circ i = \text{id}$  and  $i \circ p = \mu^{\text{ST}} \circ \eta^{\text{ST}} ST$ . Then  $S \bullet T$  can be made into a monad [19] with unit  $\eta^{S \bullet T} = p \circ \eta^S \eta^T$  and multiplication  $\mu^{S \bullet T} = p \circ \mu^S \mu^T \circ S\rho T \circ ii$ : the monad  $S \bullet T$  is called the *weak composite* of  $S$  and  $T$ .

► **Example 4** (weak distributive laws). In this work we will be particularly concerned with the following weak distributive laws:

- in  $\text{Set}$ , the weak distributive law  $\lambda^{P/P}: PP \Rightarrow PP$  given by Equation (1) below has for weak composite the monad  $P \bullet P$  of sets of subsets closed under non-empty unions [19];
- in  $\text{KHaus}$ , there is a weak distributive law  $\lambda^{V/V}: \mathbf{V}\mathbf{V} \Rightarrow \mathbf{V}\mathbf{V}$  given by Equation (2) below [24];
- in  $\text{Set}$ , the weak distributive law  $\lambda^{D/P}: DP \Rightarrow PD$  given by Equation (3) below has for weak composite the monad  $P \bullet D$  of convex sets of finitely supported probability distributions [23].

$$\lambda_X^{P/P}(E) = \left\{ e' \in PX \mid e' \subseteq \bigcup E \text{ and } \forall e \in E, e \cap e' \neq \emptyset \right\} \quad (1)$$

$$\lambda_X^{V/V}(C) = \left\{ c' \in \mathbf{V}X \mid c' \subseteq \bigcup C \text{ and } \forall c \in C, c \cap c' \neq \emptyset \right\} \quad (2)$$

$$\lambda_X^{D/P}(\Phi) = \left\{ (\mu_X^D \circ Df)(\Phi) \mid f: PX \rightarrow DX \text{ and } \forall e \in PX, \Phi(e) \neq 0 \Rightarrow f(e) \in De \right\} \quad (3)$$

At this point the curious reader may ask two questions: how do we actually find these weak distributive laws — the formulas in Example 4 are not trivial — and how exactly is the

monad structure on  $S \bullet T$  constructed. These two questions will respectively be answered in the next two sections, which give two other equivalent presentations of weak distributive laws.

## 2.2 Regular categories and monotone (weak) extensions

Given a monad  $S$ , one may form its *Kleisli category*  $Kl(S)$ , which is intuitively the category of  $S$ -effectful arrows of  $C$ : its objects are those of  $C$ , its arrows  $X \multimap Y$  are those arrows  $X \rightarrow SY$  in  $C$ , the identity arrow  $X \multimap X$  is given by  $\eta_X^S: X \rightarrow SX$ , and the composite of two arrows  $f: X \multimap Y$  and  $g: Y \multimap Z$  is given by the composition  $X \xrightarrow{f} SY \xrightarrow{Sg} SSZ \xrightarrow{\mu_Z^S} SZ$  in  $C$ .  $Kl(S)$  comes with an adjunction  $F_S: C \rightleftarrows Kl(S): U_S$ , such that  $U_S F_S = S$ : the left adjoint  $F_S$  sends objects on themselves and an arrow  $f: X \rightarrow Y$  to the *pure* effectful arrow  $X \xrightarrow{f} Y \xrightarrow{\eta_Y^S} SY$ , and the right adjoint  $U_S$  sends an object  $X$  on  $SX$  and an effectful arrow  $f: X \rightarrow SY$  to the arrow  $SX \xrightarrow{Sf} SSY \xrightarrow{\mu^S} Y$ . The unit of the adjunction is  $\eta^S$  and its counit is the natural transformation  $\varepsilon^S: F_S U_S \Rightarrow Id$  (in  $Kl(S)$ ) with components the effectful arrows  $ids_X: SX \rightarrow SX$ .

An endofunctor  $T: C \rightarrow C$  extends to  $Kl(S)$  if there is an endofunctor  $\underline{T}: Kl(S) \rightarrow Kl(S)$  such that  $\underline{T} F_S = F_S T$ . If  $T$  and  $T'$  have extensions  $\underline{T}$  and  $\underline{T}'$ , a natural transformation  $\alpha: T \Rightarrow T'$  extends to  $Kl(S)$  if  $\underline{\alpha}$ , given by  $\underline{\alpha} F_S = F_S \alpha$ , is a natural transformation  $\underline{T} \Rightarrow \underline{T}'$ .

► **Definition 5** (extensions of monads). *Let  $S$  be a monad on  $C$ . A monad  $(T, \eta^T, \mu^T)$  on  $C$  weakly extends to  $Kl(S)$  when  $T$  and  $\mu^T$  extend to  $Kl(S)$ . It extends to  $Kl(S)$  when  $\eta^T$  also extends to  $Kl(S)$ .*

Giving an extension of a monad  $T$  to  $Kl(S)$  is equivalent to giving a distributive law  $\rho: TS \Rightarrow ST$ , just like giving a weak extension thereof is equivalent to giving a weak distributive law of the same type [19]: in both cases, the law  $\rho$  induces the extension which sends  $f: X \rightarrow SY$  to  $TX \xrightarrow{Tf} TSY \xrightarrow{\rho} STY$ , and is computed from the extension as  $\rho = U_S \underline{T} \varepsilon^S F_S \circ \eta^S TS$ . In particular Example 4 also yields examples of weak extensions.

### Monotone extensions.

If  $Kl(S)$  carries more structure than just that of a category then it is natural to ask that extensions of functors preserve this additional structure: the more structure preserved, the more semantically canonical the resulting law. For instance, recall that  $Kl(P)$  is the category  $Rel$  of sets and relations and that relations between two sets are ordered by inclusion. An extension of an endofunctor that preserves this order, and the resulting law, are called *monotone*. It turns out that such monotone extensions to relations can be defined and characterized in any *regular category*, as described in [24] and recalled now.

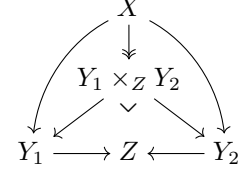
Let  $C$  be a finitely complete category. The *kernel pair* of an arrow  $f: X \rightarrow Y$  is the pullback  $p_1, p_2: X \times_Y X \rightarrow X$  of the cospan  $X \xrightarrow{f} Y \xleftarrow{f} X$ . Assume  $C$  has all the coequalizers of these kernel pairs: these coequalizers are called regular epimorphisms.  $C$  is then called *regular* if these conditions are satisfied and if the regular epimorphisms are stable under pullbacks<sup>1</sup>. Regular categories then enjoy the fact that every arrow  $f: X \rightarrow Y$  may be factored as a regular epimorphism (denoted with  $\twoheadrightarrow$ ) followed by a monomorphism (denoted

<sup>1</sup> a class of arrows is stable under pullbacks when for every pullback square  $a \circ b = c \circ d$ , if  $a$  is in the class then so is  $d$

with  $\hookrightarrow$ ), and this factorization is unique up to unique isomorphism — one should think of it as factoring an arrow through its image. **Set** is a canonical example of such a regular category.

Regular categories are useful because they are categories where we can speak of relations: if  $\mathcal{C}$  is a regular category, a  $\mathcal{C}$ -relation between two objects  $X$  and  $Y$  is a subobject  $r: R \hookrightarrow X \times Y$ . Relations are preordered:  $r: R \hookrightarrow X \times Y$  is smaller than  $s: S \hookrightarrow X \times Y$ , written  $r \leq s$ , when there is a monomorphism  $m: R \hookrightarrow S$  such that  $s \circ m = r$ . One may form the category  $\text{Rel}(\mathcal{C})$  with objects those of  $\mathcal{C}$  and arrows  $X \rightsquigarrow Y$  the equivalence classes of relations between  $X$  and  $Y$ . The identity on  $X$  is the diagonal  $\langle \text{id}_X, \text{id}_X \rangle: X \hookrightarrow X \times X$ , and composition of relations  $r = \langle r_X, r_Y \rangle: R \hookrightarrow X \times Y$  and  $s = \langle s_Y, s_Z \rangle: S \hookrightarrow Y \times Z$ , written  $s \cdot r: X \rightsquigarrow Z$ , is constructed as in the diagram on the left — by considering the pullback  $R \times_Y S \rightarrow R \times S$  of  $R \xrightarrow{r_Y} Y \xleftarrow{s_Y} S$ , and taking the monomorphism  $R \times_Y S \rightarrow R \times S \xrightarrow{r_X \times s_Z} X \times Z$ . The graph functor  $\text{Graph}: \mathcal{C} \rightarrow \text{Rel}(\mathcal{C})$  sends objects to themselves and an arrow  $f: X \rightarrow Y$  to the relation  $\langle \text{id}_X, f \rangle: X \hookrightarrow X \times Y$ . There is also a contravariant transpose functor  $-\dagger: \text{Rel}(\mathcal{C}) \rightarrow \text{Rel}(\mathcal{C})$  that sends objects on themselves and a relation  $\langle r_X, r_Y \rangle: R \hookrightarrow X \times Y$  to  $\langle r_Y, r_X \rangle: R \hookrightarrow Y \times X$ . All in all, a relation  $\langle f, g \rangle: R \hookrightarrow X \times Y$  can also be written as the composite  $\text{Graph } g \cdot (\text{Graph } f)^\dagger$ : we will often omit  $\text{Graph}$  and write directly  $g \cdot f^\dagger$ .

If  $\text{Kl}(\mathcal{S})$  is a wide<sup>2</sup> subcategory of  $\text{Rel}(\mathcal{C})$  (with  $\text{Graph}$  being the left adjoint in the Kleisli adjunction), an extension to  $\text{Kl}(\mathcal{S})$  can be constructed by first finding a monotone extension to  $\text{Rel}(\mathcal{C})$  and then restricting this extension to  $\text{Kl}(\mathcal{S})$ . This is a very useful technique because such monotone extensions to  $\text{Rel}(\mathcal{C})$ , called *relational extensions*, have a very nice characterization in terms of near pullbacks: a square is a near pullback when its limiting morphism into the corresponding pullback is a regular epimorphism — as in the diagram on the right. A functor between regular categories is said to be *nearly cartesian* when it sends pullbacks on near pullbacks and preserves<sup>3</sup> regular epimorphisms, or equivalently when it preserves near pullbacks, while a natural transformation  $\alpha: \mathbb{F} \Rightarrow \mathbb{G}$  between nearly cartesian functors is *nearly cartesian* as well when its naturality squares  $\alpha \circ \mathbb{F}f = \mathbb{G}f \circ \alpha$  are near pullbacks.



► **Theorem 6** ([24, Theorem 6]). *Let  $\mathbb{F}: \mathcal{C} \rightarrow \mathcal{D}$  be a functor between regular categories.  $\mathbb{F}$  has a relational extension, i.e. an order-preserving functor  $\text{Rel}(\mathbb{F}): \text{Rel}(\mathcal{C}) \rightarrow \text{Rel}(\mathcal{D})$  such that  $\text{Rel}(\mathbb{F}) \text{Graph} = \text{Graph } \mathbb{F}$ , if and only if  $\mathbb{F}$  is nearly cartesian. In that case there is only one possible such  $\text{Rel}(\mathbb{F})$ , given by  $\text{Rel}(\mathbb{F})(g \cdot f^\dagger) = (\mathbb{F}g) \cdot (\mathbb{F}f)^\dagger$ .*

*Let  $\mathbb{F}, \mathbb{G}: \mathcal{C} \rightarrow \mathcal{D}$  be two such functors and  $\alpha: \mathbb{F} \Rightarrow \mathbb{G}$  be a natural transformation between them. Then  $\text{Rel}(\alpha)$ , given by  $\text{Rel}(\alpha) \text{Graph} = \text{Graph } \alpha$ , is a natural transformation  $\text{Rel}(\mathbb{F}) \Rightarrow \text{Rel}(\mathbb{G})$ , called the relational extension of  $\alpha$ , if and only if  $\alpha$  is nearly cartesian.*

Just like we omit to write  $\text{Graph}$ , we will often omit to write the functor  $\text{Rel}$ , and instead write  $\mathbb{F}$  for  $\text{Rel}(\mathbb{F})(g \cdot f^\dagger) = \mathbb{F}g \cdot (\mathbb{F}f)^\dagger$  and  $\alpha$  for  $\text{Rel}(\alpha)$ .

► **Example 7** (monotone extensions). **Set** is a regular category whose regular epimorphisms are the surjections and such that  $\text{Rel}(\text{Set}) = \text{Rel} \cong \text{Kl}(\mathbb{P})$ . The endofunctors and the multiplications of the monads  $\mathbb{P}$  and  $\mathbb{D}$  are nearly cartesian, hence  $\mathbb{P}$  and  $\mathbb{D}$  have monotone

<sup>2</sup> a wide subcategory is a subcategory that contains all objects of the bigger category

<sup>3</sup> throughout this work we will say that a functor preserves a class of diagrams whenever it sends any diagram in that class to a diagram in that class

weak extensions to  $\text{Kl}(\mathbb{P})$ : this is how the weak distributive laws  $\text{PP} \Rightarrow \text{PP}$  and  $\text{DP} \Rightarrow \text{PD}$  of Example 4 were constructed [19, 23].  $\text{KHaus}$  is also a regular category: its regular epimorphisms are the surjective continuous functions and  $\text{Rel}(\text{KHaus})$  is the category of compact Hausdorff spaces and closed relations, i.e. closed subsets of  $X \times Y$ , while  $\text{Kl}(\mathbb{V})$  is the category of compact Hausdorff spaces and continuous relations, i.e. closed relations  $r: X \rightsquigarrow Y$  such that  $r^{-1}[u]$  is open in  $X$  for every open  $u$  of  $Y$ . The endofunctor and multiplication of  $\mathbb{V}$  are nearly cartesian hence they have a relational extension, which restricts to continuous relations: this yields a monotone weak extension of  $\mathbb{V}$  to  $\text{Kl}(\mathbb{V})$ , and this is how the weak distributive law  $\mathbb{V}\mathbb{V} \Rightarrow \mathbb{V}\mathbb{V}$  of Example 4 was constructed [24].

### 2.3 Weak liftings

Let  $\mathbb{T}$  be a monad over a category  $\mathbb{C}$ . Its category of algebras  $\text{EM}(\mathbb{T})$  is the category whose objects are pairs  $(A, a)$  of a  $\mathbb{C}$ -object  $A$  and a  $\mathbb{C}$ -arrow  $a: \mathbb{T}A \rightarrow A$ , and whose arrows  $(A, a) \rightarrow (B, b)$  are those  $\mathbb{C}$ -arrows  $A \rightarrow B$  such that  $f \circ a = b \circ \mathbb{T}f$  — the identity morphism is the one with the identity as its underlying  $\mathbb{C}$ -arrow, and composition of morphisms is done by composing the underlying  $\mathbb{C}$ -arrows. Just like for  $\text{Kl}(\mathbb{T})$ , there is an adjunction  $\mathbb{F}^\mathbb{T}: \mathbb{C} \rightleftarrows \text{EM}(\mathbb{T}): \mathbb{U}^\mathbb{T}$  such that  $\mathbb{U}^\mathbb{T}\mathbb{F}^\mathbb{T} = \mathbb{T}$ : the left adjoint  $\mathbb{F}^\mathbb{T}$  sends an algebra  $(A, a)$  to its carrier object  $A$  and a morphism  $(A, a) \rightarrow (B, b)$  to the underlying arrow  $A \rightarrow B$ , while  $\mathbb{U}^\mathbb{T}$  sends an object  $X$  to the free  $\mathbb{T}$ -algebra on  $X$ , given by the pair  $(\mathbb{T}X, \mu_X^\mathbb{T})$ , and an arrow  $f: X \rightarrow Y$  to the morphism  $(\mathbb{T}X, \mu_X^\mathbb{T}) \rightarrow (\mathbb{T}Y, \mu_Y^\mathbb{T})$  with underlying  $\mathbb{C}$ -arrow  $\mathbb{T}f: \mathbb{T}X \rightarrow \mathbb{T}Y$ . There is finally a natural transformation  $\varepsilon^\mathbb{T}: \mathbb{F}^\mathbb{T}\mathbb{U}^\mathbb{T} \rightarrow \text{Id}$  (in  $\text{EM}(\mathbb{T})$ ) given by  $\mathbb{U}^\mathbb{T}\varepsilon_{(A,a)}^\mathbb{T} = a: \mathbb{T}A \rightarrow A$ .

► **Definition 8** (weak liftings [6, Definitions 4.1 and 4.2]). *A weak lifting of an endofunctor  $\mathbb{S}: \mathbb{C} \rightarrow \mathbb{C}$  to  $\text{EM}(\mathbb{T})$  is the data of an endofunctor  $\bar{\mathbb{S}}: \text{EM}(\mathbb{T}) \rightarrow \text{EM}(\mathbb{T})$  along with natural transformations  $\pi_\mathbb{S}^\mathbb{T}: \mathbb{S}\mathbb{U}^\mathbb{T} \Rightarrow \mathbb{U}^\mathbb{T}\bar{\mathbb{S}}$  and  $\iota_\mathbb{S}^\mathbb{T}: \mathbb{U}^\mathbb{T}\bar{\mathbb{S}} \Rightarrow \mathbb{S}\mathbb{U}^\mathbb{T}$  — also written  $\pi_\mathbb{S}$  and  $\iota_\mathbb{S}$  when not ambiguous — such that  $\pi_\mathbb{S} \circ \iota_\mathbb{S} = \text{id}$ .*

Let  $\alpha: \mathbb{S} \Rightarrow \mathbb{R}$  be a natural transformations between two  $\mathbb{C}$ -endofunctors with weak liftings  $(\bar{\mathbb{S}}, \pi_\mathbb{S}, \iota_\mathbb{S})$  and  $(\bar{\mathbb{R}}, \pi_\mathbb{R}, \iota_\mathbb{R})$ . If a natural transformation  $\bar{\alpha}: \bar{\mathbb{S}} \Rightarrow \bar{\mathbb{R}}$  has Diagram  $(\pi)$ , Diagram  $(\iota)$  or both Diagrams  $(\pi)$  and  $(\iota)$  below commute, it is respectively a weak  $\pi$ -, weak  $\iota$ - or a weak lifting thereof.

As discussed in [6], weak  $\pi$ - and weak  $\iota$ -liftings are necessarily unique and given by  $\mathbb{U}^\mathbb{T}\bar{\alpha} = \pi_\mathbb{R}^\mathbb{T} \circ \alpha \circ \iota_\mathbb{S}^\mathbb{T}$  if they exist. Their existence is moreover fully characterized:

$$\begin{array}{ccc} \mathbb{U}^\mathbb{T}\bar{\mathbb{S}} & \xrightarrow{\pi_\mathbb{S}^\mathbb{T}} & \mathbb{S}\mathbb{U}^\mathbb{T} & & \mathbb{S}\mathbb{U}^\mathbb{T} & \xrightarrow{\iota_\mathbb{S}^\mathbb{T}} & \mathbb{U}^\mathbb{T}\bar{\mathbb{S}} \\ \mathbb{U}^\mathbb{T}\bar{\alpha} \downarrow & (\pi) & \downarrow \alpha \mathbb{U}^\mathbb{T} & & \alpha \mathbb{U}^\mathbb{T} \downarrow & (\iota) & \downarrow \mathbb{U}^\mathbb{T}\bar{\alpha} \\ \mathbb{U}^\mathbb{T}\bar{\mathbb{R}} & \xrightarrow{\pi_\mathbb{R}^\mathbb{T}} & \mathbb{R}\mathbb{U}^\mathbb{T} & & \mathbb{R}\mathbb{U}^\mathbb{T} & \xrightarrow{\iota_\mathbb{R}^\mathbb{T}} & \mathbb{U}^\mathbb{T}\bar{\mathbb{R}} \end{array}$$

► **Theorem 9** ([6, Proposition 4.3 and Theorem 4.4]). *Suppose idempotents split in  $\mathbb{C}$ . Then having a weak lifting of  $\mathbb{S}: \mathbb{C} \rightarrow \mathbb{C}$  to  $\text{EM}(\mathbb{T})$  is equivalent to having a law  $\rho: \mathbb{T}\mathbb{S} \Rightarrow \mathbb{S}\mathbb{T}$  that has Diagram  $(\mu^+)$  commute.*

Fix now liftings  $\bar{\mathbb{S}}$  and  $\bar{\mathbb{R}}$  of  $\mathbb{S}$  and  $\mathbb{R}$  given by laws  $\rho: \mathbb{T}\mathbb{S} \Rightarrow \mathbb{S}\mathbb{T}$  and  $\sigma: \mathbb{T}\mathbb{R} \Rightarrow \mathbb{R}\mathbb{T}$ . Then,  $\alpha: \mathbb{S} \Rightarrow \mathbb{R}$  respectively has a weak  $\pi$ -, weak  $\iota$ - or weak lifting if and only if it makes Diagram (7), Diagram (8) or both Diagrams (7) and (8) below commute. The latter case holds equivalently if and only if  $\sigma \circ \mathbb{T}\alpha = \alpha \mathbb{T} \circ \rho$ .

This correspondance is moreover compositional: the composition of the weak (resp. weak  $\pi$ -, weak  $\iota$ -) liftings of two functors is a weak (resp. weak  $\pi$ -, weak  $\iota$ -) lifting of their composite.



$$\begin{array}{ccc}
\text{TS} & \xrightarrow{\rho} & \text{ST} \\
\begin{array}{c} \tau\eta^T\mathcal{S}\downarrow \\ \text{TTS} \xrightarrow{\tau\rho} \text{TST} \xrightarrow{\tau\alpha\text{T}} \text{TRT} \xrightarrow{\sigma\text{T}} \text{RTT} \xrightarrow{\mathbb{R}\mu^T} \text{RT} \\ \eta^T\text{TS}\uparrow \end{array} & & \begin{array}{c} \downarrow\alpha\text{T} \\ \text{RT} \\ \uparrow\sigma \end{array} \\
\text{TS} & \xrightarrow{\tau\alpha} & \text{TR}
\end{array}$$

(7)

$$\begin{array}{ccc}
\text{TS} & \xrightarrow{\tau\alpha} & \text{TR} \\
\text{TS} & \xrightarrow{\tau\alpha} & \text{TR}
\end{array}$$

(8)

In [19], Garner instantiates Theorem 9 to give another presentation of weak distributive laws: if idempotents split in  $\mathbf{C}$ , a weak distributive law  $\text{TS} \Rightarrow \text{ST}$  is equivalently given by a weak lifting of  $(\mathcal{S}, \eta^{\mathcal{S}}, \mu^{\mathcal{S}})$  to  $\text{EM}(\text{T})$ , i.e. by weak liftings of  $\mathcal{S}$ ,  $\eta^{\mathcal{S}}$  and  $\mu^{\mathcal{S}}$ , respectively coming from Diagrams  $(\mu^-)$ ,  $(\eta^+)$ , and  $(\mu^+)$ . That these two natural transformations weakly lift respectively means that Diagrams  $(\pi \circ \eta)$  and  $(\iota \circ \eta)$  and Diagrams  $(\pi \circ \mu)$  and  $(\iota \circ \mu)$  below commute. When Diagram  $(\eta^-)$  also commutes,  $\rho$  is a (strict) distributive law and this weak lifting is a (strict) lifting:  $\pi_{\mathcal{S}}$  and  $\iota_{\mathcal{S}}$  are both the identity.

$$\begin{array}{ccc}
\begin{array}{ccc} \eta^{\mathcal{S}}\text{U}^T & \text{U}^T & \text{U}^T\eta^{\bar{\mathcal{S}}} \\ \swarrow & (\pi \circ \eta) & \searrow \\ \text{SU}^T & \xrightarrow{\pi_{\mathcal{S}}} & \text{U}^T\bar{\mathcal{S}} \end{array} & \begin{array}{ccc} \text{SSU}^T & \xrightarrow{S\pi_{\mathcal{S}}^T} \text{SU}^T\bar{\mathcal{S}} & \xrightarrow{\pi_{\mathcal{S}}^T\bar{\mathcal{S}}} \text{U}^T\bar{\mathcal{S}}\bar{\mathcal{S}} \\ \mu^{\mathcal{S}}\text{U}^T\downarrow & (\pi \circ \mu) & \downarrow\text{U}^T\mu^{\bar{\mathcal{S}}} \\ \text{SU}^T & \xrightarrow{\pi_{\mathcal{S}}^T} & \text{U}^T\bar{\mathcal{S}} \end{array} \\
\begin{array}{ccc} \text{U}^T\eta^{\bar{\mathcal{S}}} & \text{U}^T & \eta^{\mathcal{S}}\text{U}^T \\ \swarrow & (\iota \circ \eta) & \searrow \\ \text{U}^T\bar{\mathcal{S}} & \xrightarrow{\iota_{\mathcal{S}}^T} & \text{SU}^T \end{array} & \begin{array}{ccc} \text{U}^T\bar{\mathcal{S}}\bar{\mathcal{S}} & \xrightarrow{\iota_{\mathcal{S}}^T\bar{\mathcal{S}}} \text{SU}^T\bar{\mathcal{S}} & \xrightarrow{S\iota_{\mathcal{S}}^T} \text{SSU}^T \\ \text{U}^T\mu^{\bar{\mathcal{S}}}\downarrow & (\iota \circ \mu) & \downarrow\mu^{\mathcal{S}}\text{U}^T \\ \text{U}^T\bar{\mathcal{S}} & \xrightarrow{\iota_{\mathcal{S}}^T} & \text{SU}^T \end{array}
\end{array}$$

► **Example 10** (weak liftings). All idempotents split in  $\text{Set}$  (they factor through their image). The algebras of  $\mathbf{P}$  are the *complete join-semilattices* (we write  $\text{EM}(\mathbf{P}) \cong \text{JSL}$ ): the weak lifting corresponding to the law  $\mathbf{P}\mathbf{P} \Rightarrow \mathbf{P}\mathbf{P}$  is the monad of *subsets closed under non-empty joins* [24]. The algebras of  $\mathbf{D}$  are the *barycentric algebras*, also called *convex spaces* (we write  $\text{EM}(\mathbf{D}) \cong \text{Conv}$ ): the weak lifting corresponding to the law  $\mathbf{D}\mathbf{P} \Rightarrow \mathbf{P}\mathbf{D}$  is the monad of *convex-closed subsets* [23]. Finally, there is also a weak distributive law  $\beta\mathbf{P} \Rightarrow \mathbf{P}\beta$ . Assuming the axiom of choice, the algebras of  $\beta$  are the compact Hausdorff spaces [29] (we write  $\text{EM}(\beta) \cong \text{KHaus}$ ): the corresponding weak lifting is the Vietoris monad  $\mathbf{V}$ . In that case  $\pi_{\mathbf{P}}$  computes the topological closure of a subset, while  $\iota_{\mathbf{P}}$  embeds the set of closed sets into the set of all subsets.

Write  $\kappa_{\mathcal{S}}^T$  for the composite  $\iota_{\mathcal{S}}^T \circ \pi_{\mathcal{S}}^T$ . Then  $\kappa_{\mathcal{S}}^T\mathbf{F}^T$  is the idempotent  $\text{ST} \Rightarrow \text{ST}$  (which splits into  $\pi_{\mathcal{S}}^T\mathbf{F}^T$  and  $\iota_{\mathcal{S}}^T\mathbf{F}^T$ ) and the weak composite monad can finally be obtained as the composite of the adjunction  $\mathbf{F}: \mathbf{C} \rightleftarrows \text{EM}(\text{T}): \mathbf{U}$  with any adjunction for  $\bar{\mathcal{S}}$ , so that  $\mathbf{S} \bullet \mathbf{T} = \text{U}^T\bar{\mathcal{S}}\mathbf{F}^T$ ,  $\eta^{\mathbf{S} \bullet \mathbf{T}} = \text{U}^T\eta^{\bar{\mathcal{S}}}\mathbf{F}^T \circ \eta^T$  and  $\mu^{\mathbf{S} \bullet \mathbf{T}} = \text{U}^T\mu^{\bar{\mathcal{S}}}\mathbf{F}^T \circ \text{U}^T\bar{\mathcal{S}}\varepsilon^T\mathbf{F}^T$ .

### 3 Weakly lifting weak distributive laws

With Example 10 and the fact that  $\mathbf{V}$  is a weak lifting of  $\mathbf{P}$  to  $\text{KHaus}$  in mind, we may now notice that not only are the two laws  $\mathbf{P}\mathbf{P} \Rightarrow \mathbf{P}\mathbf{P}$  (1) and  $\mathbf{V}\mathbf{V} \Rightarrow \mathbf{V}\mathbf{V}$  (2) very similar, but the second one seems to be some kind of weak lifting to  $\text{KHaus} \cong \text{EM}(\beta)$  of the first one:

► **Lemma 11.**  $\text{U}^{\beta}\lambda^{\mathbf{V}/\mathbf{V}} = \pi_{\mathbf{P}}\mathbf{V} \circ \mathbf{P}\pi_{\mathbf{P}} \circ \lambda^{\mathbf{P}/\mathbf{P}}\text{U}^{\beta} \circ \mathbf{P}\iota_{\mathbf{P}} \circ \iota_{\mathbf{P}}\mathbf{V}$ .

Is this just a coincidence, or is this an instance of Theorem 9? And in the latter case, can the weak distributivity of  $\lambda^{\mathbf{V}/\mathbf{V}}$  be automatically derived from that of  $\lambda^{\mathbf{P}/\mathbf{P}}$ , and does this law  $\mathbf{P}\mathbf{P} \Rightarrow \mathbf{P}\mathbf{P}$  also weakly lift to laws on other categories of algebras where the powerset monad weakly lifts, say  $\text{EM}(\mathbf{P})$  and  $\text{EM}(\mathbf{D})$ ?

### 3.1 The Yang-Baxter equation for weak distributive laws

The standard way to lift (strict) distributive laws is to use the so-called Yang-Baxter equation. Consider three monads  $(T, \eta^T, \mu^T)$ ,  $(S, \eta^S, \mu^S)$  and  $(R, \eta^R, \mu^R)$  and three weak distributive laws  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow$

$$\begin{array}{ccccc}
 & & \text{TRS} & \xrightarrow{\sigma S} & \text{RTS} & & \\
 & \nearrow^{T\tau} & & & & \searrow^{R\rho} & \\
 \text{TSR} & & & & \text{(YB)} & & \\
 & \searrow^{\rho R} & & & & \nearrow^{\tau T} & \\
 & & \text{STR} & \xrightarrow{S\sigma} & \text{SRT} & & \\
 & & & & & & \text{RST}
 \end{array}$$

RT and  $\tau: SR \Rightarrow RS$ . The Yang-Baxter equation for these three laws holds when diagram (YB) above commutes. If these laws are strict distributive laws, then it is well-known since [14] that the Yang-Baxter equation is enough to show that  $R\rho \circ \sigma S: \text{TRS} \Rightarrow \text{RST}$  is a distributive law of T over RS, and that the distributive law  $\tau: SR \Rightarrow RS$  lifts to a distributive law  $\bar{\tau}: \bar{S}\bar{R} \Rightarrow \bar{R}\bar{S}$  in  $\text{EM}(T)$ . The composition of weak distributive laws using the Yang-Baxter equation was investigated in [22]. A notable result is the following:

► **Proposition 12** ([22, Theorem 4.3]). *If the weak distributive laws  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow RT$  and  $\tau: SR \Rightarrow RS$  have Diagram (YB) commute, then  $\pi_R^S F^S T \circ R\rho \circ \sigma S \circ T \iota_R^S F^S$  is a weak distributive law  $T(R \bullet S) \Rightarrow (R \bullet S)T$ .*

The Yang-Baxter equation thus allows for weakly lifting  $R \bullet S$  to  $\text{EM}(T)$ . More importantly for our purpose, we show it also allows for weakly lifting the weak distributive law  $SR \Rightarrow RS$ :

► **Theorem 13.** *Weak distributive laws  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow RT$  and  $\tau: SR \Rightarrow RS$  satisfy the Yang-Baxter equation if and only if  $\tau: SR \Rightarrow RS$  weakly lifts to  $\text{EM}(T)$ , i.e. if there is a natural transformation  $\bar{\tau}: \bar{S}\bar{R} \Rightarrow \bar{R}\bar{S}$  such that Diagrams (14) and (15) commute. If this holds,  $\bar{\tau}$  is a weak distributive law, and the weak composite  $\bar{R} \bullet \bar{S}$  and the weak lifting  $\bar{R} \bullet \bar{S}$  (recall that  $R \bullet S$  weakly lifts to  $\text{EM}(T)$  by Proposition 12) can be chosen to be equal (as monads).*

$$\begin{array}{ccc}
 \text{SRU}^T \xrightarrow{S\pi_R} \text{SU}^T \bar{R} \xrightarrow{\pi_S \bar{R}} \text{U}^T \bar{S}\bar{R} & & \text{U}^T \bar{S}\bar{R} \xrightarrow{\iota_S \bar{R}} \text{SU}^T \bar{R} \xrightarrow{S\iota_R} \text{SRU}^T \\
 \tau U^T \downarrow & (14) & \downarrow U^T \bar{\tau} \\
 \text{RSU}^T \xrightarrow{R\pi_S} \text{RU}^T \bar{S} \xrightarrow{\pi_R \bar{S}} \text{U}^T \bar{R}\bar{S} & & \text{U}^T \bar{R}\bar{S} \xrightarrow{\iota_R \bar{S}} \text{RU}^T \bar{S} \xrightarrow{R\iota_S} \text{RSU}^T
 \end{array}$$

When Theorem 13 holds it immediately follows that  $U^T \bar{\tau} = \pi_R \bar{S} \circ R\pi_S \circ \tau U^T \circ S\iota_R \circ \iota_S \bar{R}$ , which is exactly the result we got in Lemma 11 for  $\bar{\tau} = \lambda^{V/V}: \mathbb{V}\mathbb{V} \Rightarrow \mathbb{V}\mathbb{V}$  and  $\tau = \lambda^{P/P}: \mathbb{P}\mathbb{P} \Rightarrow \mathbb{P}\mathbb{P}$ . It would thus be a reasonable conjecture that  $\lambda^{\beta/P}: \beta\mathbb{P} \Rightarrow \mathbb{P}\beta$ ,  $\lambda^{\beta/P}: \beta\mathbb{P} \Rightarrow \mathbb{P}\beta$  and  $\lambda^{P/P}: \mathbb{P}\mathbb{P} \Rightarrow \mathbb{P}\mathbb{P}$  satisfy the Yang-Baxter equation, from which we would immediately retrieve the weak distributivity of  $\lambda^{V/V}: \mathbb{V}\mathbb{V} \Rightarrow \mathbb{V}\mathbb{V}$  but also learn that the weak composite  $\mathbb{V} \bullet \mathbb{V}$  is a weak lifting of  $\mathbb{P} \bullet \mathbb{P}$ . Unfortunately, the Yang-Baxter equation does not hold in that case. A rather simple way to see this is to notice that  $\lambda^{V/V}$  does not make Diagram (15) commute, i.e. it is not a  $\iota$ -lifting of  $\lambda^{P/P}$ . More generally, we show the following no-go theorem for weak  $\iota$ -liftings:

► **Proposition 14.** *Let  $\lambda^{T/P}: TP \Rightarrow PT$  be a weak distributive law with corresponding weak lifting  $\bar{P}$ , and write  $\bar{P}\bar{P}A = (P\iota_P \circ \iota_P \bar{P}) [U^T \bar{P}\bar{P}(A, a)]$  when  $(A, a)$  is a T-algebra. If there is such an  $(A, a)$  such that  $\{A\} \in \bar{P}\bar{P}A$  and  $P_* A \notin \bar{P}\bar{P}A$  (where  $P_* A = \{E \subseteq A \mid E \neq \emptyset\}$ ), then  $\lambda^{P/P}: \mathbb{P}\mathbb{P} \Rightarrow \mathbb{P}\mathbb{P}$  does not have a weak  $\iota$ -lifting to  $\text{EM}(T)$ .*

► **Corollary 15.** *There is no weak  $\iota$ -lifting (let alone weak liftings) of  $\lambda^{P/P}$  to  $\text{KHaus}$ ,  $\text{JSL}$  or  $\text{Conv}$ .*

► **Remark 16.** It is not hard to show that the Yang-Baxter equation holding for  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow RT$  and  $\tau: SR \Rightarrow RS$  is also equivalent to  $\rho: TS \Rightarrow ST$  having an extension  $\underline{\rho}: \underline{T}\underline{S} \Rightarrow \underline{S}\underline{T}$  to  $\text{KI}(R)$ . In the case of  $\lambda^{\beta/P}: \beta\mathbb{P} \Rightarrow \mathbb{P}\beta$ ,  $\underline{\beta}$  is in fact a lax monad on  $\text{Rel}$  whose algebras are the topological spaces [3]. Unfortunately, we have just shown that the Yang-Baxter equation

does not hold for  $\lambda^{\beta/P}: \beta P \Rightarrow P\beta$ ,  $\lambda^{P/P}: PP \Rightarrow PP$  and  $\lambda^{P/P}: PP \Rightarrow PP$ , and so do not get a way to weakly lift  $\underline{P}$  to topological spaces for free.

Theorem 13 does have some concrete instances: in [22], Goy gives a substantial number of examples of triples of weak distributive laws for which the Yang-Baxter equation holds, although these examples all involve at least one strictly distributive law out of the three.

### 3.2 The $\pi$ -Yang-Baxter equation

We still do not have an explanation for why  $\lambda^{V/V}$  looks so much like  $\lambda^{P/P}$ . But  $\lambda^{V/V}$  being a weak lifting of  $\lambda^{P/P}$  is not a necessary condition for retrieving Lemma 11: in fact,  $\lambda^{V/V}$  being only a weak  $\iota$ - or  $\pi$ -lifting of  $\lambda^{P/P}$  would be enough. We saw in Corollary 15 that the weak  $\iota$ -lifting hypothesis was a dead-end: how about  $\lambda^{V/V}$  being a weak  $\pi$ -lifting of  $\lambda^{P/P}$ ? This turns out to be true, although the proof is of course more involved than that of the weaker Lemma 11.

► **Lemma 17.**  $\lambda^{V/V}: \mathbb{V}\mathbb{V} \Rightarrow \mathbb{V}\mathbb{V}$  is a weak  $\pi$ -lifting of  $\lambda^{P/P}: PP \Rightarrow PP$ .

Theorem 13 adapts to weak  $\pi$ -liftings, hence we immediately retrieve as a consequence of Lemma 17 that  $\lambda^{V/V}: \mathbb{V}\mathbb{V} \Rightarrow \mathbb{V}\mathbb{V}$  is a weak distributive law.

► **Proposition 18.** Weak distributive laws  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow RT$  and  $\tau: SR \Rightarrow RS$  satisfy the  $\pi$ -Yang-Baxter equation, given by Diagram ( $\pi$ -YB), if and only if  $\tau: SR \Rightarrow RS$  weakly  $\pi$ -lifts to  $\text{EM}(\mathbb{T})$ , i.e. if there is a natural transformation  $\bar{\tau}: \overline{SR} \Rightarrow \overline{RS}$  such that Diagram (14) commutes. If this holds,  $\bar{\tau}$  is a weak distributive law.

We also retrieve that  $\lambda^{V/V}$  is a monotone weak distributive law:

► **Proposition 19.** Consider a monotone weak distributive law  $\lambda^{S/P}: SP \Rightarrow PS$  in  $\text{Set}$  that has a weak  $\pi$ -lifting to  $\text{EM}(\mathbb{T})$ . If the components of  $\kappa_p: PU^T \Rightarrow PU^T$  are monotone functions (they preserve inclusion of subsets) then  $\bar{\lambda}^{S/P}: \overline{SP} \Rightarrow \overline{PS}$  is also a monotone weak distributive law.

Of course the point of Propositions 18 and 19 is that they make it easier to exhibit weak distributive laws in categories of algebras. Still, working with Diagram ( $\pi$ -YB) may be quite tedious, as it involves up to four composed layers of functors.

In fact in Lemma 17 we did not use this  $\pi$ -Yang-Baxter equation at all, instead we directly proved that  $\lambda^{V/V}$  was a weak  $\pi$ -lifting because we were already able to take for granted that its components were morphisms of  $\beta$ -algebras, i.e. continuous functions. Another problem with Proposition 18 is that even if we manage to disprove its prerequisites for some examples, we only get that there is no weak  $\pi$ -lifting of the weak distributive law, but we do not learn anything about other possible meaningful weak distributive laws in the category of algebras.

For all of these reasons we do not try to apply Proposition 18 to weakly  $\pi$ -lift  $\lambda^{P/P}: PP \Rightarrow PP$  to  $\text{EM}(\mathbb{P})$  and  $\text{EM}(\mathbb{D})$ , and immediately turn towards another approach in Section 4 instead: we try to weakly lift the conditions for the existence of monotone weak distributive laws (described in Section 2.2). This is a reasonable strategy because monotone laws are easier to reason about (all non-trivial weak distributive laws described in the literature are monotone) and are closer to being fully characterized, meaning we should hopefully be able to prove no-go theorems for monotone weak distributive laws. In fact we will prove that there is no such law  $\overline{PP} \Rightarrow \overline{PP}$  in  $\text{EM}(\mathbb{P})$  or  $\text{EM}(\mathbb{D})$ , so that by Proposition 19  $\lambda^{P/P}$  cannot weakly  $\pi$ -lift to  $\text{EM}(\mathbb{P})$  nor to  $\text{EM}(\mathbb{D})$ .

$$\begin{array}{ccccc}
 \text{TSR} & \xrightarrow{\text{T}\tau} & \text{TRS} & \xrightarrow{\sigma\text{S}} & \text{RTS} & \xrightarrow{\text{R}\rho} & \text{RST} \\
 \rho\text{R}\downarrow & & & & & & \uparrow\text{RS}\mu^T \\
 \text{STR} & & & (\pi\text{-YB}) & & & \text{RSTT} \\
 \text{S}\sigma\downarrow & & & & & & \uparrow\text{R}\rho^T \\
 \text{SRT} & \xrightarrow{\tau\text{T}} & \text{RST} & \xrightarrow{\eta^T\text{RST}} & \text{TRST} & \xrightarrow{\sigma\text{ST}} & \text{RTST}
 \end{array}$$

## 4 Weakly lifting monotone weak distributive laws

Let  $T$  be a monad on a regular category  $C$ . It is folklore that, under mild conditions,  $EM(T)$  is regular as well. For instance on  $\mathbf{Set}$ , all finitary monads, and even all monads if the axiom of choice is assumed to be true, have regular categories of algebras [12, Theorems 3.5.4 and 4.3.5]. Here we will assume that  $T$  is a nearly cartesian functor, but the following result also holds for monads that preserve reflexive coequalizers.

► **Theorem 20** (categories of algebras are regular). *Let  $T$  be a monad with nearly cartesian endofunctor on a regular category  $C$ . Then  $EM(T)$  is regular and  $U^T$  creates finite limits and near pullbacks (a square is a near pullback in  $EM(T)$  if and only if its image by  $U^T$  is so in  $C$ ).*

Consider weak distributive laws  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow RT$  and  $\tau: SR \Rightarrow RS$  on  $C$ . When  $\tau: SR \Rightarrow RS$  is a monotone weak distributive law thanks to the framework of [24], it is now natural to ask when there is also a monotone weak distributive law  $\bar{S}\bar{R} \Rightarrow \bar{R}\bar{S}$  in  $EM(T)$  arising in the same way: it is for instance the case for  $T = \beta$  and  $S = R = P$ .

To apply the framework for monotone weak distributive laws of [24] to monads  $\bar{S}$  and  $\bar{R}$ , we need to characterize  $Kl(\bar{R})$  as a subcategory of relations — we do this in Section 4.1 — and then prove that  $\bar{S}$  and  $\mu^{\bar{S}}$  are nearly cartesian and investigate when the extension of  $\bar{S}$  to  $Rel(EM(T))$  restricts to  $Kl(\bar{R})$  — we do this in Section 4.2. We finally apply our results in Section 4.3.

### 4.1 Kleisli categories of weakly lifted monads

Let us forget about regular categories and internal relations for an instant and first describe the Kleisli categories of a weakly lifted monad  $\bar{R}$  in terms of the Kleisli category of  $R$  itself.

► **Proposition 21.** *Let  $\sigma: TR \Rightarrow RT$  be a weak distributive law in a category  $C$  where idempotents split, so that  $R$  has a weak lifting  $\bar{R}$  to  $EM(T)$  and  $T$  a weak extension  $\underline{T}$  to  $Kl(R)$ . Then  $Kl(\bar{R})$ -arrows  $(A, a) \dashrightarrow (B, b)$  are in one-to-one correspondance with  $Kl(R)$ -arrows  $f: A \dashrightarrow B$  such that  $f \circ F_R a = F_R b \circ \underline{T} f$ .*

Assume that the framework of [24] applies:  $Kl(R)$  is a wide subcategory of  $Rel(C)$  (and the left adjoint coincides with the graph functor  $Graph: C \rightarrow Rel(C)$ ),  $T$  and  $\mu^T$  are nearly cartesian and the weak extension of  $(T, \eta^T, \mu^T)$  to  $Kl(R)$  is the restriction of the relational extension of  $T$  and  $\mu^T$  to  $Rel(C)$ . By Theorems 6 and 20,  $U^T$  has a relational extension  $Rel(U^T): Rel(EM(T)) \rightarrow Rel(C)$ , and by Proposition 21  $Kl(\bar{R})$  correspond to  $C$ -relations  $\psi: A \dashrightarrow B$  that are in  $Kl(R)$  and such that  $\psi \cdot a = b \cdot T\psi$ . This category is itself a wide subcategory of  $Rel(EM(T))$ :

► **Lemma 22.** *When the endofunctor  $T$  is nearly cartesian, a  $C$ -relation  $\psi: A \dashrightarrow B$  is the image of an  $EM(T)$ -relation  $(A, a) \dashrightarrow (B, b)$  by the faithful functor  $Rel(U^T): Rel(EM(T)) \rightarrow Rel(C)$  if and only if  $\psi \cdot a \geq b \cdot T\psi$ .*

We now describe in more concrete terms which  $EM(T)$ -relations are arrows in  $Kl(\bar{P})$ . Decomposable  $T$ -algebra morphisms play a central role in this description:

► **Definition 23** (decomposable morphisms of algebra). *Let  $T$  be a monad on a category  $C$  such that  $EM(T)$  is regular. A  $T$ -algebra morphism  $f: X \rightarrow Y$  is called decomposable when the square  $f \circ \varepsilon_X^T = \varepsilon_Y^T \circ F^T U^T f$  is a near pullback.*

*Given a jointly monic span  $\langle \psi_X, \psi_Y \rangle$  in  $EM(T)$ , the corresponding relation  $\psi = \psi_Y \cdot \psi_X^\dagger$  is called decomposable when  $\psi_X$  is so.*

Decomposable morphisms have been studied in [15, Definition 3.1.1] in the setting of *monoidal topology*. There, these morphisms are called *open* as they generalize open maps between compact Hausdorff spaces, as the next example shows. We prefer the term “decomposable” here as we focus on other examples that feel more algebraic than topological:

► **Example 24** (decomposable morphisms and relations in categories of algebras over  $\mathbf{Set}$ ). In  $\mathbf{EM}(\beta) \cong \mathbf{KHaus}$ , a continuous map is decomposable if and only if it is open (it preserves open sets), and decomposable relations are the continuous ones, i.e. those relations  $\psi: X \rightleftarrows Y$  such that  $\psi^{-1}[u]$  is open in  $X$  for every open subset  $u$  of  $Y$ .

In  $\mathbf{EM}(\mathbf{P}) \cong \mathbf{JSL}$ ,  $\psi: X \rightleftarrows Y$  is decomposable if and only if for every family  $(x_i)_{i \in I}$  of elements of  $X$  and every  $y \in Y$  such that  $(\bigvee_{i \in I} x_i, y) \in \psi$ , there is a family  $(y_i)_{i \in I}$  of elements of  $Y$  such that  $(x_i, y_i) \in \psi$  for all  $i \in I$  and  $\bigvee_{i \in I} y_i = y$ .

In  $\mathbf{EM}(\mathbf{D}) \cong \mathbf{Conv}$ ,  $\psi: X \rightleftarrows Y$  is decomposable if and only if for every  $x \in X$ , every *disintegration* of  $x$  as a barycenter  $x = \sum_{i=1}^n \lambda_i x_i$  and every  $y \in Y$  such that  $(x, y) \in \psi$ ,  $y$  disintegrates as a barycenter  $y = \sum_{i=1}^n \lambda_i y_i$  such that  $(x_i, y_i) \in \psi$  for all  $i \in I$ .

► **Lemma 25.** *When  $\mathbf{U}^\top$  creates near pullbacks,  $\mu^\top$  is nearly cartesian if and only if every free algebra morphism  $\mathbf{F}^\top f: (X, \mu_X^\top) \rightarrow (Y, \mu_Y^\top)$  is decomposable.*

We are now able to state the main result of this section.

► **Theorem 26.** *When the endofunctor  $\mathbf{T}$  is nearly cartesian, a relation  $\psi: (A, a) \rightleftarrows (B, b)$  is decomposable if and only if  $\mathbf{U}^\top \psi \cdot a = b \cdot \mathbf{TU}^\top \psi$ . If  $\mu^\top$  is also nearly cartesian and  $\mathbf{Rel}(\mathbf{T})$  restricts to  $\mathbf{Kl}(\mathbf{R}) \hookrightarrow \mathbf{Rel}(\mathbf{C})$  for some monad  $\mathbf{R}$  on  $\mathbf{C}$ , then the Kleisli category  $\mathbf{Kl}(\bar{\mathbf{R}})$  of the corresponding weakly lifted monad  $\bar{\mathbf{R}}$  on  $\mathbf{EM}(\mathbf{T})$  has for arrows  $(A, a) \dashrightarrow (B, b)$  the decomposable relations  $\psi: (A, a) \rightleftarrows (B, b)$  in  $\mathbf{EM}(\mathbf{T})$  such that  $\mathbf{U}^\top \psi$  is in  $\mathbf{Kl}(\mathbf{R})$ .*

► **Corollary 27.** *If  $\mathbf{T}$  has a monotone weak distributive law over  $\mathbf{P}$  in  $\mathbf{Set}$ , the Kleisli category of the lifted powerset monad  $\bar{\mathbf{P}}$  on  $\mathbf{EM}(\mathbf{T})$  is the category of  $\mathbf{T}$ -algebras and decomposable relations between them.*

► **Remark 28** (subobject classifiers in categories of algebras). Recall that an elementary topos is a regular category such that the  $\mathbf{Graph}$  functor has a right adjoint [18, §1.911]; the corresponding monad is called the powerset monad. If  $\mathbf{C}$  is an elementary topos with powerset monad  $\mathbf{P}$ , and if  $\mathbf{T}$  is a monad on  $\mathbf{C}$  such that the endofunctor  $\mathbf{T}$  and the natural transformation  $\mu^\top$  are nearly cartesian, then by Theorem 26  $\mathbf{EM}(\mathbf{T})$  is an elementary topos as soon as every  $\mathbf{T}$ -algebra morphism is decomposable. We retrieve for instance that the categories of group actions (algebras for monads  $G \times -$  where  $G$  is a group) are toposes, because the corresponding morphisms of algebras are easily shown to all be decomposable.

This is not a necessary condition for a category of algebras to be an elementary topos: it is well known that categories of monoid actions (algebras for monads  $M \times -$  where  $M$  is a monoid) are toposes, but there are equivariant morphisms that are not decomposable.

If  $\mathbf{EM}(\mathbf{T})$  is an elementary topos as in Remark 28,  $\bar{\mathbf{P}}1$  classifies subobjects in the sense that subobjects  $X \hookrightarrow Y$  are in one-to-one correspondance with morphisms  $Y \rightarrow \bar{\mathbf{P}}1$ , where  $1$  is the terminal object [18, §1.912].

This can be generalized when Theorem 26 holds as follows:  $\bar{\mathbf{R}}1$  classifies decomposition-closed subobjects, in the sense that decomposable monomorphisms  $X \hookrightarrow Y$  are in one-to-one correspondance with morphisms  $Y \rightarrow \bar{\mathbf{R}}1$  (the correspondance comes from the adjunction  $\mathbf{Kl}(\bar{\mathbf{R}})(Y, 1) \cong \mathbf{EM}(\mathbf{T})(Y, \bar{\mathbf{R}}1)$ ). For instance, the Vietoris monad on  $\mathbf{KHaus} \cong \mathbf{EM}(\beta)$  classifies clopen subsets of compact Hausdorff spaces, the non-empty-join-closed powerset monad on  $\mathbf{JSL} \cong \mathbf{EM}(\mathbf{P})$  classifies downwards-closed subsets of join-semilattices, and the convex-closed

powerset monad on  $\text{Conv} \cong \text{EM}(D)$  classifies *walls*, i.e. subsets  $E$  such that if  $x \in E$  and  $\sum_{i=1}^n x_i = x$ ,  $x_i \in E$  as well for all  $1 \leq i \leq n$  (walls appear for instance in the structure theorem for convex algebras, which state that every convex algebra is a subalgebra of the *Plonka sum* of its walls [33, Theorem 4.5]).

## 4.2 Monotone extensions to Kleisli categories of weakly lifted monads

In Section 4.1 we assumed  $T$  had a monotone weak distributive law over  $R$  coming from a relational extension of  $T$  and  $\mu^T$ , and described the Kleisli category of the corresponding weakly lifted monad  $\bar{R}$ . Suppose now there is another monad  $S$  that weakly lifts to  $\text{EM}(T)$  and that also has a monotone weak distributive law over  $R$  coming from a relational extension of  $S$  and  $\mu^S$ . When do we also get a monotone weak distributive law of  $\bar{S}$  over  $\bar{R}$  coming from a relational extension of  $\bar{S}$  and  $\mu^{\bar{S}}$ ?

For the relational extension to exist, we need  $\bar{S}$  and  $\mu^{\bar{S}}$  to be nearly cartesian. This always holds:

► **Lemma 29.** *Let  $C$  be a regular category where idempotents split, and suppose  $T$  is nearly cartesian. If  $F: C \rightarrow C$  is nearly cartesian and weakly lifts to  $\text{EM}(T)$ , then its weak lifting is nearly cartesian. If  $\alpha: F \Rightarrow G$  between two such functors is nearly cartesian and weakly lifts to  $\text{EM}(T)$ , then its weak lifting is nearly cartesian as well.*

$\bar{S}$  thus has a relational extension  $\text{Rel}(\bar{S})$ . By adapting Theorem 6, we can not only characterize when relational extensions restrict to  $\text{Kl}(\bar{R})$ , but when any endofunctor or natural transformation has a monotone extension to  $\text{Kl}(\bar{R})$ . We can even state this characterization more generally, without necessarily speaking of decomposable morphisms: we do this now.

► **Definition 30.** *Let  $\Gamma$  be a wide subcategory of a regular category  $C$  such that, in  $C$ ,*

- $\Gamma$ -arrows are stable under pullbacks (in  $C$ );
- if  $f \circ e$  is a  $\Gamma$ -arrow and  $e$  is a regular epimorphism (in  $C$ ),  $f$  is a  $\Gamma$ -arrow.

*Then we define  $C \cdot \Gamma^\dagger$  to be the wide subcategory of  $\text{Rel}(C)$  whose arrows are the  $C$ -relations  $\psi: X \rightleftarrows Y$  given by jointly monic spans  $\langle \psi_X, \psi_Y \rangle$  such that  $\psi_X$  is a  $\Gamma$ -arrow, and we write  $\text{Graph}_\Gamma: C \rightarrow C \cdot \Gamma^\dagger$  for the restriction of  $\text{Graph}: C \rightarrow \text{Rel}(C)$  to  $C \cdot \Gamma^\dagger$ .*

*We also define a  $\Gamma^\dagger \cdot C$ -square to be a square  $a \circ b = c \circ d$  such that  $a$  or  $c$  is a  $\Gamma$ -arrow.*

► **Definition 31.** *Let  $C$  and  $D$  be two regular categories with respective wide subcategories  $\Gamma$  and  $\Delta$  as in Definition 30. Let  $F$  be a functor  $C \rightarrow D$ . A  $(\Gamma, \Delta)$ -relational extension of  $F$  is a functor  $F_{\Gamma, \Delta}: C \cdot \Gamma^\dagger \rightarrow D \cdot \Delta^\dagger$  such that  $F_{\Gamma, \Delta} \text{Graph}_\Gamma = \text{Graph}_\Delta F_{\Gamma, \Delta}$ . If  $\alpha: F \Rightarrow G$  is a natural transformation between functors  $C \rightarrow D$  with  $(\Gamma, \Delta)$ -relational extensions  $F_{\Gamma, \Delta}$  and  $G_{\Gamma, \Delta}$ , a  $(\Gamma, \Delta)$ -relational extension is a (necessarily unique) natural transformation  $\underline{\alpha}_{\Gamma, \Delta}: F_{\Gamma, \Delta} \Rightarrow G_{\Gamma, \Delta}$  such that  $\underline{\alpha}_{\Gamma, \Delta} \text{Graph}_\Gamma = \text{Graph}_\Delta \underline{\alpha}_{\Gamma, \Delta}$ .*

► **Theorem 32.** *Let  $C, D, \Gamma, \Delta$  and  $F: C \rightarrow D$  be as in Definition 31.  $F$  has a monotone  $(\Gamma, \Delta)$ -relational extension if and only if the following two conditions hold:*

- $F$  restricts to a functor  $\Gamma \rightarrow \Delta$ ;
- $F$  sends near pullback  $\Gamma^\dagger \cdot C$ -squares on near pullback (necessarily  $\Delta^\dagger \cdot D$ -) squares (this is always true when  $F$  is nearly cartesian).

*Such a monotone  $(\Gamma, \Delta)$ -relational extension, if it exists, is necessarily unique and given by  $F_{\Gamma, \Delta}(g \cdot f)^\dagger = Fg \cdot (Ff)^\dagger$ .*

*Let  $\alpha: F \Rightarrow G$  be a natural transformation between two functors  $C \rightarrow D$  having such monotone  $(\Gamma, \Delta)$ -relational extensions.  $\alpha$  has a (necessarily unique)  $(\Gamma, \Delta)$ -relational extension if and only if it has near pullbacks for its naturality squares along  $\Gamma$ -morphisms (this is always true when  $\alpha$  is nearly cartesian).*

A first corollary, while not especially ground-breaking, is the following:

► **Corollary 33.** *In  $\mathbf{Set}$ , a monad  $(T, \eta^T, \mu^T)$  with nearly cartesian endofunctor and multiplication also has a (necessarily unique) monotone weak distributive law over the monad  $P_*$  of non-empty subsets, and over the monad  $P_f$  of finite subsets if and only if  $T$  preserves functions with finite pre-images of elements.*

We could more generally characterize the existence of monotone weak distributive laws over these powerset monads for any monad on  $\mathbf{Set}$ . Still, Corollary 33 is already enough to prove that  $P$  has monotone weak distributive laws over  $P_f$  and  $P_*$ , that  $D$  has a monotone weak distributive law over  $P_*$  but not over  $P_f$ , and that  $\beta$  has a monotone weak distributive law over  $P_*$ . A more impactful corollary — we will apply it repeatedly in Section 4.3 — is the following:

► **Corollary 34.** *Let  $T$  be a monad on  $\mathbf{Set}$  equipped with a monotone weak distributive law  $TP \Rightarrow PT$ , and let  $S$  be a monad on  $\mathbf{EM}(T)$ . If there is a monotone weak distributive law  $S\bar{P} \Rightarrow \bar{P}S$ ,  $S$  preserves decomposable  $T$ -algebra morphisms. Moreover if  $S$  is itself the weak lifting of a monad on  $\mathbf{Set}$  that has a weak distributive law over  $P$ , then the previous condition is not only necessary, but also sufficient.*

### 4.3 Monotone weak distributive laws in categories of algebras

A first use of Corollary 34 is retrieving the monotone weak distributive law  $VV \Rightarrow VV$ : once  $V$  and  $\mu^V$  are shown to be nearly cartesian, we only need to prove that decomposable morphisms are the open maps (Example 24) instead of proving that  $\mathbf{Kl}(V)$ -arrows are the continuous relations, and that  $V$  preserves open maps, which is technically much simpler than proving that  $\underline{V}$  preserves continuous relations, as originally done in [24, Proposition 20]. We are also able to answer quite easily Goy's [21, Conjecture 7.31] on the weak distributivity of the Radon monad — the monad of Radon probability measures on a compact Hausdorff space — over the Vietoris monad:

► **Theorem 35.** *The Radon monad  $R$  does not have a monotone weak distributive law over the Vietoris monad  $V$ , but it has (a unique) one over the non-empty Vietoris monad  $V_*$  — the monad of non-empty closed subsets.*

Let us stress the importance of this new weak distributive law: the question of how to combine probability and non-determinism has been the topic of numerous works (again, see the introduction of [26]), and this law provides an answer in  $\mathbf{KHaus}$  that is derived from a generic construction and thus comes with generic tools, e.g. generalized determinization and up-to techniques [23, 21]. In a very recent pre-print [20], Goubault-Larrecq also constructs this law  $RV_* \Rightarrow V_*R$  as an instance of weak distributive laws between monads of continuous valuations and non-deterministic choice in more general categories of topological spaces: our result is more restricted, but we derive the law from generic categorical principles instead of building it by hand, exhibit its canonicity (it comes from a relational extension), and show why the non-empty version of the Vietoris monad is needed.

A second use of Corollary 34 is in proving the absence of monotone weak distributive laws. In Section 3 we were able to prove that the law  $PP \Rightarrow PP$  does not weakly lift to  $\mathbf{EM}(P)$  nor  $\mathbf{EM}(D)$ , but we were not able to say anything about the existence of other monotone weak distributive laws  $\bar{P}\bar{P} \Rightarrow \bar{P}\bar{P}$ . Now, thanks to the framework developed above and in particular Corollary 34, we are able to prove that such laws cannot exist. We start with  $\mathbf{EM}(P)$ :

► **Example 36.** In  $\text{EM}(\mathbb{P}) \cong \text{JSL}$ , let  $\bar{\mathbb{P}}$  be the monad of subsets closed under non-empty joins: the join of a family  $(E_i)_{i \in I}$  of non-empty-joins-closed subsets of  $X$  is the non-empty-joins-closed subset  $\{\bigvee_{i \in I} x_i \mid x_i \in E_i\}$ . Let  $f: 4 \rightarrow 2$  (where  $2 = \{0, 1\}$  and  $4 = \{0, 1, 2, 3\}$ ) be the function given by  $f(0) = f(2) = 0$  and  $f(1) = f(3) = 1$ . Then  $F^{\mathbb{P}}f$  is decomposable (by Lemma 25), but  $\bar{F}^{\mathbb{P}}f$  is not. Indeed, let  $A \in \bar{F}^{\mathbb{P}}4$  and  $B, B_1, B_2 \in \bar{F}^{\mathbb{P}}2$  be as depicted in Figure 2a (page 15):  $(\bar{F}^{\mathbb{P}}f)(A) = B = B_1 \vee B_2$  but there are no  $A_1, A_2 \in \bar{F}^{\mathbb{P}}4$  such that  $A = A_1 \vee A_2$  and  $(\bar{F}^{\mathbb{P}}f)(A_1) = B_1$  as well as  $(\bar{F}^{\mathbb{P}}f)(A_2) = B_2$ .

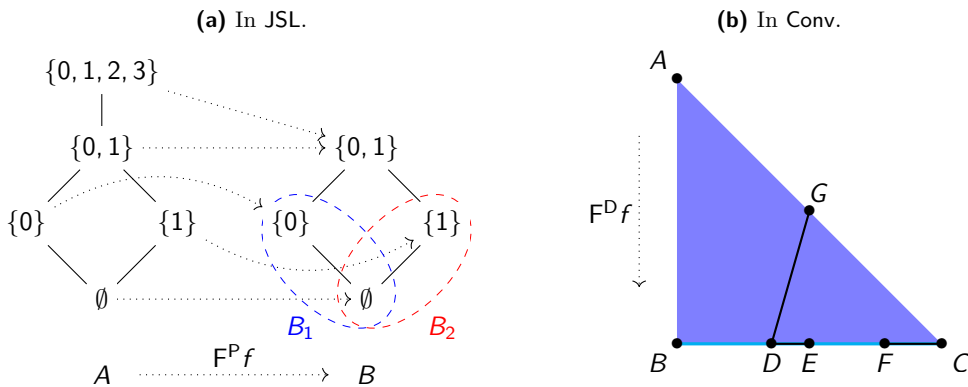
$\bar{\mathbb{P}}$  does not preserve decomposable  $\mathbb{P}$ -algebra morphisms, and thus there is no monotone weak distributive law  $\bar{\mathbb{P}}\bar{\mathbb{P}} \Rightarrow \bar{\mathbb{P}}\bar{\mathbb{P}}$ . In fact because the counter-example decomposable morphism is surjective and has finite pre-images, this also proves that there are no monotone weak distributive laws  $\bar{\mathbb{P}}\bar{\mathbb{P}}_* \Rightarrow \bar{\mathbb{P}}_*\bar{\mathbb{P}}$  or  $\bar{\mathbb{P}}\bar{F}_f \Rightarrow \bar{F}_f\bar{\mathbb{P}}$  in  $\text{EM}(\mathbb{P})$ .

Because there is a morphism of monads  $\mathbb{D} \Rightarrow \mathbb{P}$  (that sends a probability distribution to its support), there is a functor  $\text{EM}(\mathbb{P}) \rightarrow \text{EM}(\mathbb{D})$  which allows us to transfer Example 36 to  $\text{EM}(\mathbb{D})$ : there are no monotone weak distributive laws  $\mathbb{P}\mathbb{P} \Rightarrow \mathbb{P}\mathbb{P}$  or  $\bar{\mathbb{P}}\bar{\mathbb{P}}_* \Rightarrow \bar{\mathbb{P}}_*\bar{\mathbb{P}}$  in  $\text{EM}(\mathbb{D})$ . But this argument is not entirely satisfying, as the resulting example is that of a morphism of convex algebras with a very unnatural structure, namely that of complete join-semilattices: one could imagine restricting to a full subcategory of  $\text{EM}(\mathbb{D})$  that does not contain these semilattices, and perhaps  $\bar{\mathbb{P}}$  would preserve decomposable morphisms there. As shown by the following example, due to Harald Woracek and Ana Sokolova (private communication), this cannot be the case as soon as free convex algebras come in the picture.

► **Example 37 ([36]).** In  $\text{EM}(\mathbb{D}) \cong \text{Conv}$ , let  $\bar{\mathbb{P}}$  be the monad of convex subsets: a convex combination of some convex subsets of  $X$  is the convex set of the corresponding convex combinations of their points (in  $X$ ). Let  $f: \{A, B, C\} \rightarrow \{B, C\}$  be the function given by  $f(A) = f(B) = B$  and  $f(C) = C$ . Then  $F^{\mathbb{D}}f$  is decomposable (by Lemma 25) but  $\bar{F}^{\mathbb{D}}f$  is not. Indeed, depicting  $F^{\mathbb{D}}\{A, B, C\}$  as the triangle depicted in Figure 2b (page 15),  $F^{\mathbb{D}}\{B, C\}$  is the line segment  $[BC]$  and  $F^{\mathbb{D}}f$  is the vertical projection. Now  $\frac{1}{2}\{B\} + \frac{1}{2}\{C\} = [DE] = (\bar{F}^{\mathbb{D}}f)([GD])$ , but  $[GD]$  itself cannot be disintegrated as the mean of two convex subsets of  $ABC$ , one above  $B$  and the other above  $[FC]$ .

Using similar arguments, we are able to prove the existence or absence of monotone weak distributive laws over lifted powerset monads in several categories of algebras: these results are gathered in Table 1. All the negative results in this table come from the non-preservation of decomposable morphisms, and thus the absence of monotone extensions of the endofunctors themselves. The topmost row indicates in which category we work. A monad in the second

■ **Figure 2** Counterexamples to preservation of decomposability





	Set						KHaus		JSL	Conv	Mon				CMon			
	L	M	D	P	$\beta$	$M_S$	V	R	$\bar{P}$	$\bar{P}$	$\bar{M}$	$\bar{D}$	$\bar{P}$	$\bar{M}_S$	$\bar{M}$	$\bar{D}$	$\bar{P}$	
$\bar{P}$	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
$\bar{P}_*$	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗

■ **Table 1** Existence or absence of monotone weak distributed laws over weakly lifted powerset monads in categories of algebras.

topmost row has a monotone weak distributive law over a monad in the left column if the corresponding cell is filled with ✓, otherwise it is filled with ✗. In Set,  $\bar{P}$  and  $\bar{P}_*$  are the usual powerset monads  $P$  and  $P_*$ .  $L$  is the monad of lists,  $M$  that of multisets and  $M_S$  that of modules for a semiring  $S$  satisfying the conditions of [10, Theorem 3.1].  $\text{Mon} \cong \text{EM}(L)$  is the category of monoids and  $\text{CMon} \cong \text{EM}(M)$  that of commutative monoids. Linear theories distribute over commutative monads [30], hence  $L$  and  $M$  distribute over  $M$ ,  $P$ ,  $D$  and  $M_S$  when  $S$  is commutative, and so these four monads have liftings (in particular weak liftings) to  $\text{Mon}$  and  $\text{CMon}$ .

## 5 Conclusion

Noticing the similarity between the laws  $VV \Rightarrow VV$  and  $PP \Rightarrow PP$ , we developed the theory for weakly lifting weak distributive laws and showed that it only applied partially in this case. We then focused on the monotonicity of the laws, and gave full characterizations of monotone weak distributive laws over weakly lifted powerset monads in categories of algebras by characterizing the Kleisli categories of the latter, a key notion appearing then being that of decomposable morphisms. We finally applied this result to exhibit a new law  $RV_* \Rightarrow V_*R$  for combining probability and non-determinism in  $\text{KHaus}$ , but also to show that in general these monotone weak distributive laws seem to be quite rare.

We leave for further work the development of a full 2-categorical theory for iterating weak distributive laws (in the vein of [14, 7]), which would complete Section 3 but would likely be scarce in new examples. With monotone laws over powerset-like monads fully characterized, another natural question is now whether this can be done in other settings, e.g. in Pos-regular categories [28] or over other monads: the multiset monad for instance is a good candidate as its Kleisli category can also be described through spans. Finally, the author believes it would be interesting to transpose the results of Section 4 in the setting of monoidal topology [25], where categories of algebras for nearly cartesian monads generalize the category of compact Hausdorff spaces in a formal sense: perhaps for instance the weakly lifted powerset monads we study are a generalization of the Vietoris monads of topological spaces.

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### A Additional tool for the proofs: semi-algebras

Call a *semi-monad* on a category  $\mathcal{C}$  the data  $(T, \mu^T)$  of an endofunctor  $T: \mathcal{C} \rightarrow \mathcal{C}$  and a natural transformation  $\mu^T: TT \rightarrow T$  such that  $\mu^T \circ T\mu^T = \mu^T \circ \mu^T T$ . In particular, any monad has an underlying semi-monad, weak distributive laws are really distributive laws between a monad and a semi-monad, and weak extensions of monads are just extensions of the underlying semi-monads.

Given a semi-monad  $T$ , one may form the category  $\text{EM}_s(T)$  of its *semi-algebras*: a semi- $T$ -algebra is a pair  $(A, a)$  of an object  $A$  of  $\mathcal{C}$  and of an arrow  $a: TA \rightarrow A$  such that  $a \circ \mu_A^T = a \circ Ta$ , and a semi- $T$ -algebra morphism  $(A, a) \rightarrow (B, b)$  is an arrow  $f: A \rightarrow B$  in  $\mathcal{C}$  such that  $f \circ a = b \circ Tf$  — the identity morphism is the one with the identity as its underlying  $\mathcal{C}$ -arrow, and composition of morphisms is done by composing the underlying  $\mathcal{C}$ -arrows. Just like for  $\text{EM}(T)$ , there are functors  $F_s^T: \mathcal{C} \rightleftarrows \text{EM}_s(T): U_s^T$  such that  $U_s^T F_s^T = \text{Id}$ , so that  $F_s^T$  sends a semi-algebra  $(A, a)$  to its carrier object  $A$  and a morphism  $(A, a) \rightarrow (B, b)$  to the underlying arrow  $A \rightarrow B$ , while  $U_s^T$  sends an object  $X$  to the free  $T$ -algebra on  $X$ , given by the pair  $(TX, \mu_X^T)$ , and an arrow  $f: X \rightarrow Y$  to the morphism  $(TX, \mu_X^T) \rightarrow (TY, \mu_Y^T)$  with underlying  $\mathcal{C}$ -arrow  $Tf: TX \rightarrow TY$ . There is a natural transformation  $\varepsilon^T: F_s^T U_s^T \rightarrow \text{Id}$  (in  $\text{EM}_s(T)$ ) given by  $U_s^T \varepsilon_{(A,a)}^T = a: TA \rightarrow A$ .

If  $T$  is also a monad, we let  $\text{EM}(T)$  be the full subcategory of  $\text{EM}_s(T)$  on the  $T$ -algebras, i.e. semi- $T$ -algebras  $(A, a)$  such that  $a \circ \eta_A^T = \text{id}_A$ . We write  $I^T$  for the inclusion  $\text{EM}(T) \rightarrow \text{EM}_s(T)$ , and  $F^T$  and  $U^T$  are the restrictions of  $F_s^T$  and  $U_s^T$  to  $\text{EM}(T)$ .

In general if  $U: \mathcal{D} \rightarrow \mathcal{C}$  is a functor, a *lifting* of an endofunctor  $F: \mathcal{C} \rightarrow \mathcal{C}$  along  $U$  is an endofunctor  $\bar{F}: \mathcal{D} \rightarrow \mathcal{D}$  such that  $U\bar{F} = FU$ , while a lifting of a natural transformation  $\alpha: F \rightarrow G$  between endofunctors with liftings  $\bar{F}$  and  $\bar{G}$  is a natural transformation  $\bar{\alpha}$  such that  $U\bar{\alpha} = \alpha U$ .

► **Definition 38** (lifting to algebras). *Let  $T$  be a monad on  $\mathcal{C}$ . A monad  $(S, \eta^S, \mu^S)$  on  $\mathcal{C}$  semi-lifts to  $\text{EM}_s(T)$  when  $T$ ,  $\eta^T$  and  $\mu^T$  lift along  $U_s^T$ . It lifts to  $\text{EM}(T)$  when it has a semi-lifting to  $\text{EM}_s(T)$  that restricts to  $\text{EM}(T)$ .*

Giving a lifting  $\bar{S}$  of an endofunctor  $S: \mathcal{C} \rightarrow \mathcal{C}$  along  $U_s^T$  such that  $\varepsilon^T \bar{S} = \varepsilon^T S \circ F_s^T S U_s^T \varepsilon^T \circ F_s^T S \eta^T$  is equivalent to giving a law  $\rho: TS \Rightarrow ST$  having Diagram  $(\mu^+)$  commute [32]: the law  $\rho$  induces the semi-lifting which sends the semi- $T$ -algebra  $(A, a)$  to  $(SA, TSA \xrightarrow{\rho_A} STA \xrightarrow{S_a} SA)$ , and is computed from the semi-lifting as  $\rho = U_s^T \varepsilon^T \bar{S} F_s^T \circ TS \eta^T$ . If  $S$  and  $R$  have liftings  $\bar{S}$  and  $\bar{R}$  along  $U_s^T$ , a natural transformation  $\alpha: S \Rightarrow R$  has a lifting  $\bar{\alpha}$  along  $U_s^T$  if and only if  $\sigma \circ T\alpha = \alpha T \circ \rho$ . In particular, semi-liftings of a monad  $(S, \eta^S, \mu^S)$  to  $\text{EM}_s(T)$  (whose endofunctors satisfy the equation above) are in one-to-one correspondance with weak distributive laws  $TS \Rightarrow ST$ .

Notice the resemblance with the conditions in Theorem 9: this is not a coincidence, as we will show next that weak liftings (along  $U^T$ ) can be constructed from liftings along  $U_s^T$ .

Let  $T$  be a monad on a category  $\mathcal{C}$  where all idempotents split. It then follows that all idempotents in  $\text{EM}_s(T)$  split as well. Because there is an idempotent natural transformation  $\kappa^T: \text{Id} \Rightarrow \text{Id}$  in  $\text{EM}_s(T)$  given by  $U_s^T \kappa^T = U_s^T \varepsilon^T \circ \eta^T U_s^T$ , each  $\kappa_A^T$  splits as  $\kappa_A^T = \iota_A^T \circ \pi_A^T$ , and the resulting semi- $T$ -algebra in the middle is actually an algebra. In fact this construction yields a functor  $K^T: \text{EM}_s(T) \rightarrow \text{EM}(T)$  so that  $\kappa^T$  splits as  $\pi^T: \text{Id} \Rightarrow I^T K^T$  followed by  $\iota^T: I^T K^T \Rightarrow \text{Id}$ . We then get  $I^T K^T f = \pi_B^T \circ f \circ \iota_A^T$  for  $f: A \rightarrow B$  in  $\text{EM}_s(T)$ ,  $K^T I^T = \text{Id}$  and that  $K^T$  is both left and right adjoint to  $I^T$ :  $\iota^T$  is the counit of  $I^T \dashv K^T$  while  $\pi^T$  is the unit of  $K^T \dashv I^T$ . To make notations less heavy, if  $\bar{S}$  is a semi-lifting

$$\begin{array}{ccc}
 \text{EM}_s(T) & \xrightleftharpoons{K^T} & \text{EM}(T) \\
 \begin{array}{c} \curvearrowleft \\ \text{U}_s^T \\ \curvearrowright \end{array} & & \begin{array}{c} \curvearrowright \\ \text{U}^T \\ \curvearrowleft \end{array} \\
 \text{F}_s^T & \xrightarrow{\quad} & \text{F}^T
 \end{array}$$

of  $S$ , we also write  $\pi_{\bar{S}}^T$  for  $\pi^T \bar{S}$  and  $\iota_{\bar{S}}^T$  for  $\iota^T \bar{S}$ . We will also omit the superscript  $T$  from the functors and natural transformations when non-ambiguous.

If  $\bar{S}$  is a lifting of an endofunctor  $S$  along  $U_s^T$ , then  $\bar{S} = K\bar{S}I: EM(T) \rightarrow EM(T)$  is a weak lifting of  $S$  to  $EM(T)$ . Similarly, if  $\bar{\alpha}$  is the lifting of a natural transformation  $\alpha$  along  $U_s^T$ , then  $\bar{\alpha} = K\bar{\alpha}I$  is a weak lifting of  $\alpha$  to  $EM(T)$ . In particular, if  $(S, \eta^S, \mu^S)$  has a semi-lifting to  $\bar{S}$ , then  $\bar{S}$  can be made into a monad by composing the adjunction  $I: EM(T) \rightleftharpoons EM_s(T): K$  with any adjunction that yields  $\bar{S}$ .

## B Proofs for Section 3 (Weakly lifting weak distributive laws)

► **Lemma 11.**  $U^\beta \lambda^{V/V} = \pi_P V \circ P\pi_P \circ \lambda^{P/P} U^\beta \circ P\iota_P \circ \iota_P V$ .

**Proof.** Recall from [19] that, if  $X$  is a compact Hausdorff space that we see as a semi- $\beta$ -algebra, the  $X$ -component of  $\pi_P$  is the function  $PX \rightarrow VX$  that takes a subset of a compact Hausdorff space and outputs its closure, while the  $X$ -component of  $\iota_P$  is the function  $VX \rightarrow PX$  that embeds closed subsets in the set of all subsets. Hence for  $C$  a closed set of closed sets,

$$\begin{aligned} (P\pi_P \circ \lambda^{P/P} U^\beta \circ P\iota_P \circ \iota_P V)_X(C) &= \left\{ \bar{e} \mid e \subseteq \bigcup C \text{ and } \forall c \in C, c \cap e \neq \emptyset \right\} \\ &= \left\{ c \in VX \mid d \subseteq \bigcup C \text{ and } \forall c \in C, c \cap d \neq \emptyset \right\} \\ &= (\iota_P V \circ U^\beta \lambda^{V/V})_X(C) \end{aligned}$$

(where  $\bar{e}$  denotes the closure of  $e$ ). Indeed if  $e \subseteq \bigcup C$  then  $\bar{e} \subseteq \bigcup C$  because  $\bigcup C = \mu^V(C)$  is closed, and if  $e \cap c \neq \emptyset$  then  $\bar{e} \cap c \neq \emptyset$ .

Because  $\pi_P \circ \iota_P = \text{id}$ ,  $U^\beta \lambda^{V/V} = \pi_P V \circ P\pi_P \circ \lambda^{P/P} U^\beta \circ P\iota_P \circ \iota_P V$ . ◀

► **Theorem 13.** *Weak distributive laws  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow RT$  and  $\tau: SR \Rightarrow RS$  satisfy the Yang-Baxter equation if and only if  $\tau: SR \Rightarrow RS$  weakly lifts to  $EM(T)$ , i.e. if there is a natural transformation  $\bar{\tau}: \bar{S}\bar{R} \Rightarrow \bar{R}\bar{S}$  such that Diagrams (14) and (15) commute. If this holds,  $\bar{\tau}$  is a weak distributive law, and the weak composite  $\bar{R} \bullet \bar{S}$  and the weak lifting  $\overline{R \bullet S}$  (recall that  $R \bullet S$  weakly lifts to  $EM(T)$  by Proposition 12) can be chosen to be equal (as monads).*

**Proof.** We apply Theorem 9: recall that  $SR$  and  $RS$  have weak liftings  $\bar{S}\bar{R}$  and  $\bar{R}\bar{S}$  thanks to the laws  $S\sigma \circ \rho R$  and  $R\rho \circ \sigma S$ . Then  $\tau: SR \Rightarrow RS$  weakly lifts to  $EM(T)$  as  $\bar{\tau}: \bar{S}\bar{R} \Rightarrow \bar{R}\bar{S}$  if and only if  $R\rho \circ \sigma S \circ T\tau = \tau T \circ S\sigma \circ \rho R$ : this is exactly the Yang-Baxter equation. Because the correspondance is functorial, it preserves any equation satisfied by  $\tau$ ,  $\eta^S$ ,  $\eta^R$ ,  $\mu^S$  and  $\mu^R$  (they all weakly lift to  $EM(T)$ ): since  $\tau$  is a weak distributive law, so is  $\bar{\tau}$ .

We now prove that  $\pi_R^S F^S: RS \Rightarrow R \bullet S$  and  $\iota_R^S F^S: R \bullet S \Rightarrow RS$  also weakly lift to  $EM(T)$  as natural transformations  $\bar{R}\bar{S} \Rightarrow \overline{R \bullet S}$  and  $\overline{R \bullet S} \Rightarrow \bar{R}\bar{S}$ . It will then follow, again by functoriality of the correspondance of Theorem 9, that these two liftings form a splitting of the idempotent  $\bar{R}\bar{S} \Rightarrow \overline{R \bullet S}$  and that  $\overline{R \bullet S}$  can thus be chosen as the weak composite monad  $\bar{R} \bullet \bar{S}$ .

We apply Theorem 9 again:  $RS$  and  $R \bullet S$  have weak liftings  $\bar{R}\bar{S}$  and  $\overline{R \bullet S}$  thanks to the laws  $R\rho \circ \sigma S$  and  $\pi_R^S F^S T \circ R\rho \circ \sigma S \circ T\iota_R^S F^S$  (see Proposition 12 for the latter). Thus  $\pi_R^S$  weakly lifts to  $EM(T)$  if and only if

$$\pi_R^S F^S T \circ R\rho \circ \sigma S \circ T\iota_R^S F^S \circ T\pi_R^S F^S = \pi_R^S F^S T \circ R\rho \circ \sigma S$$

Similarly,  $\iota_R^S$  weakly lifts to  $EM(T)$  if and only if

$$R\rho \circ \sigma S T \circ \iota_R^S F^S = \iota_R^S F^S T \circ \pi_R^S F^S T \circ R\rho \circ \sigma S \circ T\iota_R^S F^S$$

These two results are an immediate consequence of [22, Lemma 4.2] which states that  $\kappa_{\mathbb{R}}^{\mathbb{S}} \mathbb{F}^{\mathbb{S}} \mathbb{T} \circ \mathbb{R}\rho \circ \sigma \mathbb{S} = \mathbb{R}\rho \circ \sigma \mathbb{S} \circ \mathbb{T} \kappa_{\mathbb{R}}^{\mathbb{S}} \mathbb{F}^{\mathbb{S}}$ .

Note that it is also possible but much more tedious to prove this result by hand, by checking that  $\bar{\tau}$  as given by Diagrams (14) and (15) is indeed a weak distributive law.  $\blacktriangleleft$

► **Proposition 14.** *Let  $\lambda^{\mathbb{T}/\mathbb{P}}: \mathbb{T}\mathbb{P} \Rightarrow \mathbb{P}\mathbb{T}$  be a weak distributive law with corresponding weak lifting  $\bar{\mathbb{P}}$ , and write  $\bar{\mathbb{P}}\mathbb{P}A = (\mathbb{P}\iota_{\mathbb{P}} \circ \iota_{\mathbb{P}}\bar{\mathbb{P}}) [\mathbb{U}^{\mathbb{T}}\bar{\mathbb{P}}\mathbb{P}(A, a)]$  when  $(A, a)$  is a  $\mathbb{T}$ -algebra. If there is such an  $(A, a)$  such that  $\{A\} \in \bar{\mathbb{P}}\mathbb{P}A$  and  $\mathbb{P}_*A \notin \bar{\mathbb{P}}\mathbb{P}A$  (where  $\mathbb{P}_*A = \{E \subseteq A \mid E \neq \emptyset\}$ ), then  $\lambda^{\mathbb{P}/\mathbb{P}}: \mathbb{P}\mathbb{P} \Rightarrow \mathbb{P}\mathbb{P}$  does not have a weak  $\iota$ -lifting to  $\mathbb{E}\mathbb{M}(\mathbb{T})$ .*

**Proof.** Suppose there is such an algebra, and consider any  $\bar{\lambda}^{\mathbb{P}/\mathbb{P}}: \bar{\mathbb{P}}\mathbb{P} \Rightarrow \bar{\mathbb{P}}\mathbb{P}$ . Let thus  $E \in \mathbb{U}^{\mathbb{T}}\bar{\mathbb{P}}\mathbb{P}(A, a)$  be such that  $(\mathbb{P}\iota_{\mathbb{P}} \circ \iota_{\mathbb{P}}\bar{\mathbb{P}}) E = \{A\}$ . Then, by Equation (1),  $(\lambda^{\mathbb{P}/\mathbb{P}} \circ \mathbb{P}\iota_{\mathbb{P}} \circ \iota_{\mathbb{P}}\bar{\mathbb{P}})(E) = \lambda^{\mathbb{P}/\mathbb{P}}(\{A\}) = \mathbb{P}_*A$ . But  $\mathbb{P}_*A \notin \bar{\mathbb{P}}\mathbb{P}A$ , hence necessarily  $(\mathbb{P}\iota_{\mathbb{P}} \circ \iota_{\mathbb{P}}\bar{\mathbb{P}} \circ \mathbb{U}^{\mathbb{T}}\bar{\lambda}^{\mathbb{P}/\mathbb{P}})(E) \in \bar{\mathbb{P}}\mathbb{P}A$  must be different from  $\mathbb{P}_*A$ : Diagram (15) does not commute.  $\blacktriangleleft$

► **Corollary 15.** *There is no weak  $\iota$ -lifting (let alone weak liftings) of  $\lambda^{\mathbb{P}/\mathbb{P}}$  to  $\mathbb{K}\mathbb{H}\mathbb{a}\mathbb{u}\mathbb{s}$ ,  $\mathbb{J}\mathbb{S}\mathbb{L}$  or  $\mathbb{C}\mathbb{o}\mathbb{n}\mathbb{v}$ .*

**Proof. 1.** Given a compact Hausdorff space given as a  $\beta$ -algebra  $(A, a)$ ,  $\bar{\mathbb{P}}\mathbb{P}A$  is the set of all closed sets (in the Vietoris topology) of closed sets of  $(A, a)$ .  $\{A\}$  is such a closed set of closed sets (all singletons and the whole set are always closed in a compact Hausdorff space) but  $\mathbb{P}_*A$  is not in general because it contains all non-empty sets, in particular non-closed sets if there are any (which is the case for the unit interval, for instance).

2. Consider a  $\mathbb{P}$ -algebra  $(A, a)$ , i.e. a complete join-semilattice where  $a: \mathbb{P}A \rightarrow A$  computes the join of a subset. Then  $\bar{\mathbb{P}}\mathbb{P}A$  is the set of all non-empty-joins-closed sets of non-empty-joins-closed subsets of  $A$ .  $\{A\}$  is such a non-empty-joins-closed set of non-empty-joins-closed subsets, but  $\mathbb{P}_*A$  is not in general because it contains all non-empty sets, in particular non-join-closed subsets if there are any (which is the case of  $(\mathbb{P}2, \cup)$  for instance).

3. Consider a  $\mathbb{D}$ -algebra  $(A, a)$ , i.e. a barycentric algebra where  $a: \mathbb{D}A \rightarrow A$  computes convex combinations of elements. Then  $\bar{\mathbb{P}}\mathbb{P}A$  is the set of all convex sets of convex subsets of  $A$ .  $\{A\}$  is such a convex set of convex subsets, but  $\mathbb{P}_*A$  is not in general because it contains all non-empty sets, in particular non-convex subsets if there are any (which is the case for  $\mathbb{D}2$  for instance).  $\blacktriangleleft$

► **Lemma 17.**  $\lambda^{\mathbb{V}/\mathbb{V}}: \mathbb{V}\mathbb{V} \Rightarrow \mathbb{V}\mathbb{V}$  is a weak  $\pi$ -lifting of  $\lambda^{\mathbb{P}/\mathbb{P}}: \mathbb{P}\mathbb{P} \Rightarrow \mathbb{P}\mathbb{P}$ .

**Proof.** We want to show that Diagram (14) commutes, i.e. that for any compact Hausdorff space  $X$  and  $E \in \mathbb{P}\mathbb{P}\mathbb{U}^{\beta}X$  we have that the two sets

$$\overline{\{\bar{e} \mid e \subseteq \bigcup E \text{ and } \forall e' \in E, e \cap e' \neq \emptyset\}} \quad (4)$$

and

$$\{c \in \mathbb{V}X \mid c \subseteq \bigcup \overline{\{\bar{e} \mid e \in E\}} \text{ and } \forall c' \in \overline{\{\bar{e} \mid e \in E\}}, c \cap c' \neq \emptyset\} \quad (5)$$

are equal.

Notice first that, since  $\mu^{\mathbb{V}}$  is a weak  $\pi$ -lifting of  $\mu^{\mathbb{P}}$ , we have that  $\bigcup \overline{\{\bar{e} \mid e \in E\}} = \overline{\bigcup E}$ .

(4)  $\subseteq$  (5). Consider some  $e \subseteq \bigcup E$  such that for every  $e' \in E$ ,  $e \cap e' \neq \emptyset$ . Then  $\bar{e} \subseteq \overline{\bigcup E} = \bigcup \{\bar{e} \mid e \in E\}$ . Consider now some  $c \in \overline{\bigcup \{\bar{e} \mid e \in E\}}$ . If  $c \cap \bar{e} = \emptyset$ , then  $c$  is in the open  $\square \bar{e}^c$ . Hence there must be some  $\bar{e}'$  with  $e' \in E$  in this open as well, i.e. such that  $e' \cap \bar{e} = \emptyset$  and thus  $e' \cap e = \emptyset$ . Such an  $e'$  does not exist by assumption, hence  $\bar{e} \cap c \neq \emptyset$ .

Hence every element of  $\{\bar{e} \mid e \subseteq \bigcup E \text{ and } \forall e' \in E, e \cap e' \neq \emptyset\}$  is contained in (5). Because this latter set is closed in  $\mathbf{V}VX$ , it follows that (4)  $\subseteq$  (5).

(5)  $\subseteq$  (4). Consider some  $c \in$  (5), i.e. some  $c \in VX$  such that  $c \subseteq \overline{\bigcup E}$  and such that for every  $c' \in \{\bar{e} \mid e \in E\}$ ,  $c \cap c' \neq \emptyset$ . Let  $\square u_0 \cap \bigcap_{i=1}^n \diamond u_i$  be a basic open set of  $VX$  that contains  $c$ .

Then  $c \subseteq u_0$ , so by regularity (see for instance [24, Lemma 21]) and compactness of  $X$  there is an open set  $u'_0$  of  $X$  such that  $c \subseteq u'_0$  and  $\overline{u'_0} \subseteq u_0$ . Consider the set  $e = \bigcup E \cap u'_0$ . Then  $e \subseteq u_0$ , and for every  $e' \in E$ ,  $c \cap \bar{e}' \neq \emptyset$  hence  $u'_0 \cap \bar{e}' \neq \emptyset$  and thus  $u'_0 \cap e' \neq \emptyset$  (because  $u'_0$  is open). And of course since  $e' \subseteq \bigcup E$ ,  $u'_0 \cap e' \subseteq e \cap e' \neq \emptyset$ .

Moreover for every  $1 \leq i \leq n$ ,  $u_i \cap c \neq \emptyset$  and  $c \subseteq u'_0$  hence  $(u_i \cap u'_0) \cap c \neq \emptyset$ . Since  $c \subseteq \overline{\bigcup E}$ ,  $(u_i \cap u'_0) \cap \overline{\bigcup E} \neq \emptyset$  and thus  $(u_i \cap u'_0) \cap \bigcup E \neq \emptyset$  ( $u_i \cap u'_0$  is open). We have just showed that  $e \cap u_i \neq \emptyset$  so that  $\bar{e} \cap u_i \neq \emptyset$ , and of course  $\bar{e} \subseteq \overline{u'_0} \subseteq u_0$ :  $\bar{e} \in \square u_0 \cap \bigcap_{i=1}^n \diamond u_i$ . This shows that  $c$  is in the closure of  $\{\bar{e} \mid e \subseteq \bigcup E \text{ and } \forall e' \in E, e \cap e' \neq \emptyset\}$ , hence in (4).  $\blacktriangleleft$

► **Proposition 18.** *Weak distributive laws  $\rho: TS \Rightarrow ST$ ,  $\sigma: TR \Rightarrow RT$  and  $\tau: SR \Rightarrow RS$  satisfy the  $\pi$ -Yang-Baxter equation, given by Diagram ( $\pi$ -YB), if and only if  $\tau: SR \Rightarrow RS$  weakly  $\pi$ -lifts to  $\mathbf{EM}(\mathbf{T})$ , i.e. if there is a natural transformation  $\bar{\tau}: \overline{SR} \Rightarrow \overline{RS}$  such that Diagram (14) commutes. If this holds,  $\bar{\tau}$  is a weak distributive law.*

**Proof.** The proof is almost a copy-paste of that of Theorem 13.

We apply Theorem 9: recall that  $SR$  and  $RS$  have weak liftings  $\overline{SR}$  and  $\overline{RS}$  thanks to the laws  $S\sigma \circ \rho R$  and  $R\rho \circ \sigma S$ . Then  $\tau: SR \Rightarrow RS$  weakly  $\pi$ -lifts to  $\mathbf{EM}(\mathbf{T})$  as  $\bar{\tau}: \overline{SR} \Rightarrow \overline{RS}$  if and only if the corresponding instance of Diagram (7) commutes: this is exactly Diagram ( $\pi$ -YB) (up to commuting of natural transformations). Because the correspondance is functorial, it preserves any equation satisfied by  $\tau$ ,  $\eta^S$ ,  $\eta^R$ ,  $\mu^S$  and  $\mu^R$  (they all weakly  $\pi$ -lift to  $\mathbf{EM}(\mathbf{T})$ ): since  $\tau$  is a weak distributive law, so is  $\bar{\tau}$ .  $\blacktriangleleft$

► **Proposition 19.** *Consider a monotone weak distributive law  $\lambda^{S/P}: SP \Rightarrow PS$  in  $\mathbf{Set}$  that has a weak  $\pi$ -lifting to  $\mathbf{EM}(\mathbf{T})$ . If the components of  $\kappa_P: PU^T \Rightarrow PU^T$  are monotone functions (they preserve inclusion of subsets) then  $\overline{\lambda^{S/P}}: \overline{SP} \Rightarrow \overline{PS}$  is also a monotone weak distributive law.*

**Proof.** Recall that morphisms  $X \dashrightarrow Y$  in  $\mathbf{Kl}(\mathbf{P})$  correspond to functions  $X \rightarrow PY$ , so that  $f, g: X \rightarrow PY$  are such that  $f \leq g$  if and only if  $f(x) \subseteq g(x)$  for every  $x \in X$ . The extension of the functor  $S$  to  $\mathbf{Kl}(\mathbf{P})$  then sends a morphism  $X \dashrightarrow Y$  given by the function  $f: X \rightarrow PY$  to the morphism  $SX \dashrightarrow SY$  given by the composite  $SX \xrightarrow{Sf} SPY \xrightarrow{\lambda^{S/P}} PSY$ . That  $\lambda^{S/P}$  is a monotone law means that this extension is monotone, i.e. that if  $f \leq g$  then  $\lambda^{S/P} \circ Sf \leq \lambda^{S/P} \circ Sg$ .

Similarly, morphisms  $X \dashrightarrow Y$  in  $\mathbf{Kl}(\overline{\mathbf{P}})$  correspond to  $\mathbf{T}$ -algebra morphisms  $f: X \rightarrow \overline{PY}$ , and the morphisms  $X \dashrightarrow Y$  can be ordered by saying that  $f \leq g: X \rightarrow \overline{PY}$  if and only if  $(\iota_P)_Y \circ U^T f \leq (\iota_P)_Y \circ U^T g$ . The extension of the functor  $\overline{S}$  to  $\mathbf{Kl}(\overline{\mathbf{P}})$  sends a morphism  $X \dashrightarrow Y$  given by a  $\mathbf{T}$ -algebra morphism  $f: X \rightarrow \overline{PY}$  to the morphism  $\overline{SX} \dashrightarrow \overline{SY}$  given by the composite  $\overline{SX} \xrightarrow{\overline{Sf}} \overline{SPY} \xrightarrow{\overline{\lambda^{S/P}}} \overline{PSY}$ .



We now show that this extension is monotone. Consider  $f, g: X \rightarrow \overline{P}Y$  in  $\text{EM}(\mathbb{T})$  such that  $((\iota_P)_Y \circ \mathbf{U}^T f)(x) \subseteq ((\iota_P)_Y \circ \mathbf{U}^T g)(x)$  for every  $x \in \mathbf{U}^T X$ . Then by monotonicity of  $\lambda^{S/P}$ ,

$$\left( \lambda^{S/P} \circ \mathbf{S}(\iota_P)_Y \circ \mathbf{S}\mathbf{U}^T f \right) (E') \subseteq \left( \lambda^{S/P} \circ \mathbf{S}(\iota_P)_Y \circ \mathbf{S}\mathbf{U}^T g \right) (E')$$

for every  $E' \in \mathbf{U}^T \overline{P}X$ . It follows that

$$\left( \mathbf{P}(\pi_S)_Y \circ \lambda^{S/P} \circ \mathbf{S}(\iota_P)_Y \circ \mathbf{S}\mathbf{U}^T f \circ (\iota_P)_X \right) (E) \subseteq \left( \mathbf{P}(\pi_S)_Y \circ \lambda^{S/P} \circ \mathbf{S}(\iota_P)_Y \circ \mathbf{S}\mathbf{U}^T g \circ (\iota_P)_X \right) (E)$$

for every  $E \in \mathbf{P}\mathbf{U}^T X$ , and finally by monotonicity of  $\kappa_P \mathbf{S}$  that

$$(\kappa_P)_{\overline{S}Y} \circ \mathbf{P}(\pi_S)_Y \circ \lambda^{S/P} \circ \mathbf{S}(\iota_P)_Y \circ \mathbf{S}\mathbf{U}^T f \circ (\iota_P)_X \leq (\kappa_P)_{\overline{S}Y} \circ \mathbf{P}(\pi_S)_Y \circ \lambda^{S/P} \circ \mathbf{S}(\iota_P)_Y \circ \mathbf{S}\mathbf{U}^T g \circ (\iota_P)_X$$

Because  $\iota_P$  is a natural transformation and because  $\overline{\lambda^{S/P}}$  is a weak  $\pi$ -lifting of  $\lambda^{S/P}$ , this inequality can be rewritten as

$$(\iota_P)_{\overline{S}Y} \circ \mathbf{U}^T \left( \overline{\lambda^{S/P}} \circ \overline{S}f \right) \leq (\iota_P)_{\overline{S}Y} \circ \mathbf{U}^T \left( \overline{\lambda^{S/P}} \circ \overline{S}g \right)$$

so that the extension of  $\overline{S}$  to  $\text{Kl}(\overline{P})$  is monotone. ◀

## C Proofs for Section 4 (Weakly lifting monotone weak distributive laws)

► **Lemma 39.** *Let  $\mathbb{T}$  be a semi-monad on a regular category  $\mathcal{C}$ . Then  $\mathbf{U}_s^T$  creates all limits that  $\mathcal{C}$  has (in particular kernel pairs). If the endofunctor  $\mathbb{T}$  is nearly cartesian,  $\text{EM}_s(\mathbb{T})$  is regular and  $\mathbf{U}_s^T$  creates regular epimorphisms, so that  $\mathbf{U}_s^T$  is in particular nearly cartesian.*

**Proof.** This proof is very similar to the proof that categories of algebras over  $\text{Set}$  are regular, which can be found in the literature [12, Theorem 4.3.5].

Let  $\text{Alg}(\mathbb{T})$  be the category whose objects are pairs  $(A, a)$  of an object  $A$  of  $\mathcal{C}$  and of an arrow  $a: \mathbb{T}A \rightarrow A$ , and whose arrows  $(A, a) \rightarrow (B, b)$  are given by  $\mathcal{C}$ -arrows  $f: A \rightarrow B$  such that  $f \circ a = b \circ \mathbb{T}f$ :  $\text{Alg}(\mathbb{T})$  is the category of functorial  $\mathbb{T}$ -algebras, i.e.  $\mathbb{T}$ -algebras when  $\mathbb{T}$  is seen as an endofunctor. It is well known that the obvious forgetful functor  $\mathbf{U}: \text{Alg}(\mathbb{T}) \rightarrow \mathcal{C}$  creates limits, so that  $\text{Alg}(\mathbb{T})$  in particular has all limits that  $\mathcal{C}$  has [1]: because  $\mathcal{C}$  is finitely complete, so is  $\text{Alg}(\mathbb{T})$ .  $\text{EM}_s(\mathbb{T})$  is a subcategory of  $\text{Alg}(\mathbb{T})$ , and limits of semi- $\mathbb{T}$ -algebras are again semi- $\mathbb{T}$ -algebras — recall that every finite limite can be constructed from pullbacks and the terminal object:

- the terminal functorial  $\mathbb{T}$ -algebra  $(X, x)$  is a semi- $\mathbb{T}$ -algebra:  $x \circ \mu_X^T = x \circ \mathbb{T}x$  because  $X = \mathbf{U}(X, x)$  is terminal in  $\mathcal{C}$ ;
- if  $(X, x) \xrightarrow{f} (Z, z) \xleftarrow{g} (Y, y)$  is a cospan of semi- $\mathbb{T}$ -algebras in  $\text{Alg}(\mathbb{T})$  and  $(P, p) \xrightarrow{\pi} (X, x) \times (Y, y)$  is its pullback, then

$$\begin{aligned} \mathbf{U}\pi \circ p \circ \mu_P^T &= \langle x \circ \mathbb{T}\mathbf{U}\pi_X, y \circ \mathbb{T}\mathbf{U}\pi_Y \rangle \circ \mu_P^T \\ &= \langle x \circ \mu_X^T \circ \mathbb{T}\mathbf{U}\pi_X, y \circ \mu_Y^T \circ \mathbb{T}\mathbf{U}\pi_Y \rangle \\ &= \langle x \circ \mathbb{T}x \circ \mathbb{T}\mathbf{U}\pi_X, y \circ \mathbb{T}y \circ \mathbb{T}\mathbf{U}\pi_Y \rangle \\ &= \langle x \circ \mathbb{T}\pi_X \circ \mathbb{T}p, y \circ \mathbb{T}\pi_Y \circ \mathbb{T}p \rangle \\ &= \mathbf{U}\pi \circ p \circ \mathbb{T}p \end{aligned}$$

and  $\mathbf{U}\pi$  is monic so  $p \circ \mu_P^T = p \circ \mathbb{T}p$ .

It immediately follows that  $U_s^T$ , the restriction of  $U$  to  $EM_s(\mathbb{T})$ , creates limits.

We now work in  $EM_s(\mathbb{T})$ . Consider the kernel pair  $u, v: (P, p) \rightarrow (X, x)$  of a  $\mathbb{T}$ -algebra morphism  $f: (X, x) \rightarrow (Y, y)$ . Because  $U_s^T$  preserves pullbacks,  $U_s^T u, U_s^T v: P \rightarrow X$  is the kernel pair of  $U_s^T f$ : let  $e: X \rightarrow Q$  be its coequalizer.

$\mathbb{T}$  is nearly cartesian so  $\mathbb{T}e$  is again a regular epimorphism, i.e. a coequalizer of its kernel pair, and we now show that it is also a coequalizer of  $\mathbb{T}u, \mathbb{T}v: \mathbb{T}P \rightrightarrows \mathbb{T}X$ . First, because  $e \circ u = e \circ v$  is a pullback, the square  $\mathbb{T}e \circ \mathbb{T}u = \mathbb{T}e \circ \mathbb{T}v$  is nearly cartesian: writing  $u', v': P' \rightarrow \mathbb{T}X$  for the kernel pair of  $\mathbb{T}e$ , there is a regular epimorphism  $e': \mathbb{T}P \twoheadrightarrow P'$  such that  $u' \circ e' = \mathbb{T}u$  and  $v' \circ e' = \mathbb{T}v$ . Suppose now  $g: \mathbb{T}X \rightarrow \mathbb{T}Z$  is such that  $g \circ \mathbb{T}u = g \circ \mathbb{T}v$ . Then because  $e'$  is an epimorphism,  $g \circ u' = g \circ v'$  and  $g$  factors uniquely through  $\mathbb{T}e$  (the coequalizer of  $(u', v')$ ).

Consider now the composite  $\mathbb{T}X \xrightarrow{x} X \xrightarrow{e} Q$ . It satisfies  $e \circ x \circ \mathbb{T}u = e \circ u \circ p = e \circ v \circ p = e \circ x \circ \mathbb{T}v$ , so it factors through the coequalizer  $\mathbb{T}e$  of  $\mathbb{T}u$  and  $\mathbb{T}v$ . We let  $q: \mathbb{T}Q \rightarrow Q$  be the corresponding colimiting arrow, which satisfies  $q \circ \mathbb{T}e = e \circ x$  by definition. Now  $(Q, q)$  is in fact a semi- $\mathbb{T}$ -algebra:

$$\begin{aligned} q \circ \mu_Q^T \circ \mathbb{T}\mathbb{T}e &= q \circ \mathbb{T}e \circ \mu_X^T \\ &= e \circ x \circ \mu_X^T \\ &= e \circ x \circ \mathbb{T}x \\ &= q \circ \mathbb{T}e \circ \mathbb{T}x \\ &= q \circ \mathbb{T}q \circ \mathbb{T}\mathbb{T}e \end{aligned}$$

Because  $\mathbb{T}\mathbb{T}e$  is a regular epimorphism, it is in particular an epimorphism:  $q \circ \mu_Q^T = q \circ \mathbb{T}q$ .  $e: X \rightarrow Q$  is thus a semi- $\mathbb{T}$ -algebra morphism by definition.

Similarly, we prove that  $e: (X, x) \rightarrow (Q, q)$  is the coequalizer of  $u, v: (P, p) \rightarrow (X, x)$  in  $EM_s(\mathbb{T})$ . Consider some morphism of semi- $\mathbb{T}$ -algebras  $g: (X, x) \rightarrow (A, a)$  such that  $g \circ u = g \circ v$ . Then  $U_s^T g \circ U_s^T u = U_s^T g \circ U_s^T v$  in  $\mathbb{C}$ , hence  $U_s^T g$  factors as  $h \circ e$  for some  $h: Q \rightarrow A$ . We now show that  $h$  is in fact a semi- $\mathbb{T}$ -algebra morphism  $(Q, q) \rightarrow (A, a)$ :

$$\begin{aligned} h \circ q \circ \mathbb{T}e &= h \circ e \circ x \\ &= g \circ x \\ &= a \circ \mathbb{T}g \\ &= a \circ \mathbb{T}h \circ \mathbb{T}e \end{aligned}$$

Because  $\mathbb{T}e$  is an epimorphism,  $h \circ q = a \circ \mathbb{T}h$ .

By construction,  $e$  is a regular epimorphism in  $EM(\mathbb{T})$  if and only if  $U^T e$  is so in  $\mathbb{C}$ , so  $U^T$  creates regular epimorphisms.  $\blacktriangleleft$

**► Theorem 20** (categories of algebras are regular). *Let  $\mathbb{T}$  be a monad with nearly cartesian endofunctor on a regular category  $\mathbb{C}$ . Then  $EM(\mathbb{T})$  is regular and  $U^T$  creates finite limits and near pullbacks (a square is a near pullback in  $EM(\mathbb{T})$  if and only if its image by  $U^T$  is so in  $\mathbb{C}$ ).*

**Proof.** By Lemma 39,  $EM_s(\mathbb{T})$  is a regular category and  $U_s^T$  creates finite limits and coequalizers of kernel pairs. It is easy to show in the same vein that this restricts to  $EM(\mathbb{T})$ , i.e. that limits and coequalizers of kernel pairs of algebras are algebras. Finally because  $U^T$  creates pullbacks and regular epimorphisms, it creates near pullbacks.  $\blacktriangleleft$

► **Proposition 21.** *Let  $\sigma: \text{TR} \Rightarrow \text{RT}$  be a weak distributive law in a category  $\mathcal{C}$  where idempotents split, so that  $\mathbb{R}$  has a weak lifting  $\bar{\mathbb{R}}$  to  $\text{EM}(\mathbb{T})$  and  $\mathbb{T}$  a weak extension  $\underline{\mathbb{T}}$  to  $\text{Kl}(\mathbb{R})$ . Then  $\text{Kl}(\bar{\mathbb{R}})$ -arrows  $(A, a) \dashrightarrow (B, b)$  are in one-to-one correspondance with  $\text{Kl}(\mathbb{R})$ -arrows  $f: A \dashrightarrow B$  such that  $f \circ \mathbb{F}_R a = \mathbb{F}_R b \circ \underline{\mathbb{T}}f$ .*

**Proof.** Let us first prove that the Kleisli category of the semilifted monad  $\bar{\mathbb{R}}$  on  $\text{EM}_s(\mathbb{T})$  is described as:

- its objects are semi- $\mathbb{T}$ -algebras  $(A, a)$ , i.e. objects of  $\text{EM}_s(\mathbb{T})$ ;
- its arrows  $(A, a) \rightarrow (B, b)$  are  $\text{Kl}(\mathbb{R})$ -arrows  $f: A \dashrightarrow B$  such that  $f \circ \mathbb{F}_R a = \mathbb{F}_R b \circ \underline{\mathbb{T}}f$ .

Call directly the category described above  $\text{Kl}(\bar{\mathbb{R}})$ . We will build the adjunction  $\text{EM}_s(\mathbb{T}) : \mathbb{F}_{\bar{\mathbb{R}}} \rightleftarrows \text{U}_{\bar{\mathbb{R}}} : \text{Kl}(\bar{\mathbb{R}})$ , and, since  $\mathbb{F}_{\bar{\mathbb{R}}}$  will be bijective on objects, it will immediately follow that  $\text{Kl}(\bar{\mathbb{R}})$  is indeed the Kleisli category of  $\bar{\mathbb{R}}$ .

We let  $\mathbb{F}_{\bar{\mathbb{R}}} : \text{EM}_s(\mathbb{T}) \rightarrow \text{Kl}(\bar{\mathbb{R}})$  be the functor that sends semi- $\mathbb{T}$ -algebras on themselves and semi- $\mathbb{T}$ -algebra morphisms  $f: (A, a) \rightarrow (B, b)$  to the  $\text{Kl}(\mathbb{R})$ -arrow  $\mathbb{F}_R \mathbb{U}^T f: A \dashrightarrow B$ . We indeed have that  $\mathbb{F}_R \mathbb{U}^T f \circ \mathbb{F}_R a = \mathbb{F}_R b \circ \underline{\mathbb{T}} \mathbb{F}_R \mathbb{U}^T f$  because  $\underline{\mathbb{T}}$  is an extension of the endofunctor  $\mathbb{T}$  ( $\underline{\mathbb{T}} \mathbb{F}_R = \mathbb{F}_R \mathbb{T}$ ),  $f$  is a semi- $\mathbb{T}$ -algebra morphism and  $\mathbb{F}_R$  is a functor.

We let  $\mathbb{U}_{\bar{\mathbb{R}}} : \text{Kl}(\bar{\mathbb{R}}) \rightarrow \text{EM}_s(\mathbb{T})$  be the functor that sends a semi- $\mathbb{T}$ -algebra  $(A, a)$  to the semi- $\mathbb{T}$ -algebra  $\bar{\mathbb{R}}(A, a) = (\mathbb{R}A, \mathbb{R}a \circ \sigma_A)$ , and a  $\text{Kl}(\bar{\mathbb{R}})$ -arrow  $(A, a) \dashrightarrow (B, b)$  with underlying  $\text{Kl}(\mathbb{R})$ -arrow  $f: A \dashrightarrow B$  to the semi- $\mathbb{T}$ -algebra morphism  $\bar{\mathbb{R}}(A, a) \rightarrow \bar{\mathbb{R}}(B, b)$  with underlying  $\mathcal{C}$ -arrow  $\mathbb{U}_R f: \mathbb{R}A \rightarrow \mathbb{R}B$ . To prove that this is indeed a morphism of semi- $\mathbb{T}$ -algebras, apply  $\mathbb{U}_R$  to the equality  $f \circ \mathbb{F}_R a = \mathbb{F}_R b \circ \underline{\mathbb{T}}f$ , recall that  $\sigma = \mathbb{U}_R \underline{\mathbb{T}} \varepsilon^{\mathbb{R}} \mathbb{F}_R \circ \eta^{\mathbb{R}} \text{TR}$ , and use the naturality of  $\varepsilon^{\mathbb{R}}: \mathbb{F}_R \mathbb{U}_R \Rightarrow \text{Id}$  and  $\eta^{\mathbb{R}}: \text{Id} \Rightarrow \mathbb{R}$ :

$$\begin{aligned} \mathbb{U}_R f \circ \mathbb{R}a \circ \sigma_A &= \mathbb{R}b \circ \mathbb{U}_R \underline{\mathbb{T}}f \circ \sigma_A \\ &= \mathbb{R}b \circ \mathbb{U}_R \underline{\mathbb{T}}f \circ \mathbb{U}_R \underline{\mathbb{T}} \varepsilon_{\mathbb{F}_R A}^{\mathbb{R}} \circ \eta_{\text{TRA}}^{\mathbb{R}} \\ &= \mathbb{R}b \circ \mathbb{U}_R \underline{\mathbb{T}} \varepsilon_{\mathbb{F}_R B}^{\mathbb{R}} \circ \mathbb{U}_R \underline{\mathbb{T}} \mathbb{F}_R \mathbb{U}_R f \circ \eta_{\text{TRA}}^{\mathbb{R}} \\ &= \mathbb{R}b \circ \mathbb{U}_R \underline{\mathbb{T}} \varepsilon_{\mathbb{F}_R B}^{\mathbb{R}} \circ \mathbb{R} \text{T} \mathbb{U}_R f \circ \eta_{\text{TRA}}^{\mathbb{R}} \\ &= \mathbb{R}b \circ \mathbb{U}_R \underline{\mathbb{T}} \varepsilon_{\mathbb{F}_R B}^{\mathbb{R}} \circ \eta_{\text{TRB}}^{\mathbb{R}} \circ \text{T} \mathbb{U}_R f \\ &= \mathbb{R}b \circ \sigma_B \circ \text{T} \mathbb{U}_R f \end{aligned}$$

Notice that  $\mathbb{U}_{\bar{\mathbb{R}}} \mathbb{F}_{\bar{\mathbb{R}}} = \bar{\mathbb{R}}$  (because  $\mathbb{U}_s^T \bar{\mathbb{R}} = \mathbb{R} \mathbb{U}_s^T$ ). For the unit of the adjunction, we of course take the unit  $\eta^{\bar{\mathbb{R}}}$  of the monad  $\bar{\mathbb{R}}$ .

We define the counit  $\varepsilon^{\bar{\mathbb{R}}}: \mathbb{F}_{\bar{\mathbb{R}}} \mathbb{U}_{\bar{\mathbb{R}}} \Rightarrow \text{Id}$  by letting  $\varepsilon_{\mathbb{F}_{\bar{\mathbb{R}}}(A, a)}^{\bar{\mathbb{R}}}: (\mathbb{R}A, \mathbb{R}a \circ \sigma_A) \rightarrow (A, a)$  be the  $\text{Kl}(\bar{\mathbb{R}})$ -arrow with underlying  $\text{Kl}(\mathbb{R})$ -arrow  $\varepsilon_{\mathbb{F}_R A}^{\mathbb{R}}: \mathbb{R}A \dashrightarrow A$ : using the naturality of  $\varepsilon^{\mathbb{R}}$ , that  $\sigma = \mathbb{U}_R \underline{\mathbb{T}} \varepsilon^{\mathbb{R}} \mathbb{F}_R \circ \eta^{\mathbb{R}} \text{TR}$  and that  $\mathbb{F}_R \dashv \mathbb{U}_R$  is an adjunction, we get

$$\begin{aligned} \varepsilon_{\mathbb{F}_R A}^{\mathbb{R}} \circ \mathbb{F}_R \mathbb{R}a \circ \mathbb{F}_R \sigma_A &= \mathbb{F}_R a \circ \varepsilon_{\underline{\mathbb{T}} \mathbb{F}_R A}^{\mathbb{R}} \circ \mathbb{F}_R \sigma_A \\ &= \mathbb{F}_R a \circ \varepsilon_{\underline{\mathbb{T}} \mathbb{F}_R A}^{\mathbb{R}} \circ \mathbb{F}_R \mathbb{U}_R \underline{\mathbb{T}} \varepsilon_{\mathbb{F}_R A}^{\mathbb{R}} \circ \mathbb{F}_R \eta_{\text{TRA}}^{\mathbb{R}} \\ &= \mathbb{F}_R a \circ \underline{\mathbb{T}} \varepsilon_{\mathbb{F}_R A}^{\mathbb{R}} \circ \varepsilon_{\mathbb{F}_R \text{TRA}}^{\mathbb{R}} \circ \mathbb{F}_R \eta_{\text{TRA}}^{\mathbb{R}} \\ &= \mathbb{F}_R a \circ \underline{\mathbb{T}} \varepsilon_{\mathbb{F}_R A}^{\mathbb{R}} \end{aligned}$$

The obvious forgetful functor  $\mathbb{U}_s^T: \text{Kl}(\bar{\mathbb{R}}) \rightarrow \text{Kl}(\mathbb{R})$  is such that  $\mathbb{U}_s^T \varepsilon^{\bar{\mathbb{R}}} = \varepsilon^{\mathbb{R}} \mathbb{U}_s^T$  and it is faithful, hence  $\varepsilon^{\bar{\mathbb{R}}}$  is indeed natural because  $\varepsilon^{\mathbb{R}}$  is. In fact we get the four equalities

$$\begin{aligned} \mathbb{U}_s^T \mathbb{F}_{\bar{\mathbb{R}}} &= \mathbb{F}_R \mathbb{U}_s^T \\ \mathbb{U}_s^T \mathbb{U}_{\bar{\mathbb{R}}} &= \mathbb{U}_R \mathbb{U}_s^T \\ \mathbb{U}_s^T \varepsilon^{\bar{\mathbb{R}}} &= \varepsilon^{\mathbb{R}} \mathbb{U}_s^T \\ \mathbb{U}_s^T \eta^{\bar{\mathbb{R}}} &= \eta^{\mathbb{R}} \mathbb{U}_s^T \end{aligned}$$

so by faithfulness of  $U_s^T$  and  $U_s^T$  again we immediately get that  $U_{\bar{R}} \varepsilon^{\bar{R}} \circ \eta^{\bar{R}} U_{\bar{R}} = \text{id}$  and  $\varepsilon^{\bar{R}} F_{\bar{R}} \circ F_{\bar{R}} \circ \eta^{\bar{R}} = \text{id}$  from the corresponding equalities for  $\eta^R$  and  $\varepsilon^R$ :  $(F_{\bar{R}}, U_{\bar{R}}, \eta^{\bar{R}}, \varepsilon^{\bar{R}})$  is indeed an adjunction, and  $(U_s^T, U_s^T)$  in fact form a morphism of adjunctions. Another way to see this is that not only does the monad  $R$  semilift to  $\text{EM}_s(T)$ , but its Kleisli adjunction does as well.

We similarly deduce that the monad arising from this semilifted Kleisli adjunction is indeed  $(\bar{R}, \eta^{\bar{R}}, \mu^{\bar{R}})$ , and so because  $F_{\bar{R}}$  is bijective on objects, we have indeed just described the Kleisli category of  $\bar{R}$ .

Finally, recall that we get the weakly lifted monad  $\bar{R}$  by composing the adjunction  $I^T \dashv K^T$  with any adjunction for the monad  $\bar{R}$ . We can in particular use the Kleisli adjunction for  $\bar{R}$ : the adjunction  $F_{\bar{R}} I^T: \text{EM}(T) \rightleftharpoons \text{Kl}(\bar{R}) : K^T U_{\bar{R}}$  has for corresponding monad  $\bar{R}$ . In particular, we now retrieve the Kleisli category of  $\bar{R}$  as the full subcategory of  $\text{Kl}(\bar{R})$  on objects which are images by the left adjoint of objects of  $\text{EM}(T)$ : this is exactly the category described in the statement of Proposition 21.  $\blacktriangleleft$

**► Lemma 22.** *When the endofunctor  $T$  is nearly cartesian, a C-relation  $\psi: A \rightleftarrows B$  is the image of an  $\text{EM}(T)$ -relation  $(A, a) \rightleftarrows (B, b)$  by the faithful functor  $\text{Rel}(U^T): \text{Rel}(\text{EM}(T)) \rightarrow \text{Rel}(C)$  if and only if  $\psi \cdot a \geq b \cdot T\psi$ .*

**Proof.** Recall [13, Proposition 1.8]: consider C-arrows  $f: C \rightarrow A$  and  $g: D \rightarrow B$  and C-relations  $\phi: C \rightleftarrows D$  and  $\psi: A \rightleftarrows B$  respectively given by the jointly monic spans  $\langle \phi_C, \phi_D \rangle: S \hookrightarrow C \times D$  and  $\langle \psi_A, \psi_B \rangle: R \hookrightarrow A \times B$ . Then,  $g \cdot \phi \leq \psi \cdot f$  if and only if there is a necessarily unique  $t: S \rightarrow R$  such that  $\psi_A \circ t = f \circ \phi_C$  and  $\psi_B \circ t = g \circ \phi_D$ :

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 C & \xleftarrow{\phi_C} & S & \xrightarrow{\phi_D} & D \\
 \downarrow f & & & & \downarrow g \\
 A & \xleftarrow{\psi_A} & R & \xrightarrow{\psi_B} & B
 \end{array} & \rightleftharpoons & 
 \begin{array}{ccccc}
 C & \xleftarrow{\phi_C} & S & \xrightarrow{\phi_D} & D \\
 \downarrow f & & \vdots & & \downarrow g \\
 A & \xleftarrow{\psi_A} & R & \xrightarrow{\psi_B} & B
 \end{array}
 \end{array}$$

Consider now an  $\text{EM}(T)$ -relation  $\psi: (A, a) \rightleftarrows (B, b)$ , given by a jointly monic span  $\langle \psi_{(A,a)}, \psi_{(B,b)} \rangle: (R, r) \hookrightarrow (A, a) \times (B, b)$ . Then, because  $U^T$  preserves monomorphisms and products (it preserves all limits),  $\langle U^T \psi_{(A,a)}, U^T \psi_{(B,b)} \rangle: R \hookrightarrow A \times B$  is a jointly monic span for the C-relation  $U^T \psi$ . Factor  $\langle T U^T \psi_{(A,a)}, T U^T \psi_{(B,b)} \rangle: TR \hookrightarrow TA \times TB$  as  $TR \twoheadrightarrow S \hookrightarrow TA \times TB$ , and call  $e: TR \twoheadrightarrow S$  the regular epimorphism in this factorization. Then  $r$  factors through  $e$  as well, and because  $e$  is an epimorphism, Diagrams (17) and (18) below commute.

$$\begin{array}{ccccc}
 & & TR & & \\
 & \swarrow & | & \searrow & \\
 T U^T \psi_{(A,a)} & & \downarrow e & & T U^T \psi_{(B,b)} \\
 & \swarrow & \downarrow & \searrow & \\
 TA & \xleftarrow{\quad} & S & \xrightarrow{\quad} & TB \\
 \downarrow a & & \vdots & & \downarrow b \\
 A & \xleftarrow{U^T \psi_{(A,a)}} & R & \xrightarrow{U^T \psi_{(B,b)}} & B
 \end{array}$$

(17)                      (18)

By [13, Proposition 1.8], we get that  $U^T \psi \cdot a \geq b \cdot T U^T \psi$ .

Conversely, consider a  $\mathbf{C}$ -relation  $\psi: A \rightsquigarrow B$  given by a jointly monic span  $\langle \psi_A, \psi_B \rangle: R \hookrightarrow A \times B$  and such that  $\psi \cdot a \geq b \cdot \top\psi$ . Factor  $\langle \top\psi_A, \top\psi_B \rangle: \top R \rightarrow \top A \times \top B$  as  $\top A \twoheadrightarrow S \hookrightarrow \top A \times \top B$ , and call  $e: \top A \twoheadrightarrow S$  and  $\phi = \langle \phi_{\top A}, \phi_{\top B} \rangle: S \hookrightarrow \top A \times \top B$  the regular epimorphism and monomorphism in this factorization. We then have the inequality:

$$\begin{array}{ccccc}
 & & \top R & & \\
 & \swarrow \top\psi_A & \downarrow e & \searrow \top\psi_B & \\
 \top A & \xleftarrow{\phi_{\top A}} & S & \xrightarrow{\phi_{\top B}} & \top B \\
 \downarrow a & & \wr & & \downarrow b \\
 A & \xleftarrow{\psi_A} & R & \xrightarrow{\psi_B} & B
 \end{array}$$

By [13, Proposition 1.8], we get a  $\mathbf{C}$ -arrow  $t: S \rightarrow R$  such that  $\psi_A \circ t = a \circ \phi_{\top A}$  and  $\psi_B \circ t = b \circ \phi_{\top B}$ . Let  $r = t \circ e: \top R \rightarrow R$ . We now show that  $(R, r)$  is a  $\top$ -algebra, so that  $\psi_A$  and  $\psi_B$  are actually jointly monic  $\top$ -algebra morphisms and  $\psi$  is an  $\text{EM}(\top)$ -relation. On the one hand,

$$\begin{aligned}
 \psi \circ r \circ \eta_R^\top &= \langle a \circ \top\psi_A, b \circ \top\psi_B \rangle \circ \eta_R^\top \\
 &= \langle a \circ \eta_A^\top \circ \psi_A, b \circ \eta_B^\top \circ \psi_B \rangle \\
 &= \psi
 \end{aligned}$$

and on the other hand

$$\begin{aligned}
 \psi \circ r \circ \mu_R^\top &= \langle a \circ \top\psi_A, b \circ \top\psi_B \rangle \circ \mu_R^\top \\
 &= \langle a \circ \mu_A \circ \top\top\psi_A, b \circ \mu_B \circ \top\top\psi_B \rangle \\
 &= \langle a \circ \top a \circ \top\top\psi_A, b \circ \top b \circ \top\top\psi_B \rangle \\
 &= \langle a \circ \top\psi_A \circ \top r, b \circ \top\psi_B \circ \top r \rangle \\
 &= \psi \circ r \circ \top r
 \end{aligned}$$

$(R, r)$  is thus indeed a  $\top$ -algebra because  $\psi$  is a monomorphism.

It is easy to see that the correspondence between  $\text{EM}(\top)$ - and  $\mathbf{C}$ -relations we have just described is bijective.  $\blacktriangleleft$

Before looking at examples, let us give two lemmas that will make working with decomposability of morphisms and relations easier.

► **Lemma 40.** *If  $\text{U}^\top$  creates near pullbacks, a  $\top$ -morphism  $f: (A, a) \rightarrow (B, b)$  is decomposable if and only if  $\text{U}^\top f \circ a = b \circ \top\text{U}^\top f$  is a near pullback in  $\mathbf{C}$ . In particular if  $\top$  is a monad on  $\text{Set}$ , this holds if and only if for every  $x \in A$  and  $u \in \top B$  such that  $f(x) = b(u)$ , there is some  $t \in \top A$  such that  $(\top f)(t) = u$  and  $a(t) = x$ .*

**Proof.**  $\text{U}^\top$  creates near pullbacks hence  $\text{U}^\top f \circ \text{U}^\top \varepsilon_{(A,a)}^\top = \text{U}^\top \varepsilon_{(B,b)}^\top \circ \text{U}^\top \text{F}^\top \text{U}^\top f$  is a near pullback in  $\mathbf{C}$  if and only if  $f \circ \varepsilon_{(A,a)}^\top = \varepsilon_{(B,b)}^\top \circ \text{F}^\top \text{U}^\top f$  is a near pullback in  $\text{EM}(\top)$ .

In  $\text{Set}$ , regular epimorphisms are surjections hence  $f \circ a = b \circ \top f$  is a near pullback if and only if the limiting function from  $\top A$  to the pullback  $A \times_B \top B$  is surjective, which translates to what is stated above.  $\blacktriangleleft$

► **Lemma 41.** *Suppose  $U^\top$  creates and  $T$  preserves regular epimorphisms. If  $g = h \circ e$  is decomposable and  $e$  is a regular epimorphism, then  $h$  is decomposable as well. In particular, if  $f: R \rightarrow X$  is decomposable then for any  $g: R \rightarrow Y$ ,  $g \cdot f^\dagger$  is a decomposable relation.*

**Proof.** Consider  $g: R \rightarrow Y$ ,  $e: R \twoheadrightarrow S$  and  $h: S \rightarrow Y$  as above. Since  $g$  is decomposable, we have the nearly cartesian square

$$\begin{array}{ccc}
 & & F^\top U^\top R \\
 & \swarrow^{F^\top U^\top g} & \searrow \\
 F^\top U^\top X & \longleftarrow P & \\
 \varepsilon_X^\top \downarrow & \lrcorner \downarrow & \searrow^{\varepsilon_R^\top} \\
 X & \longleftarrow g & R
 \end{array}$$

where  $P$  is the pullback of  $g$  and  $\varepsilon_X^\top$ . Because  $g = h \circ e$  and because the composition of two pullbacks is a pullback, this nearly cartesian square decomposes as

$$\begin{array}{ccccc}
 & & F^\top U^\top S & \xleftarrow{F^\top U^\top e} & F^\top U^\top R \\
 & \swarrow^{F^\top U^\top h} & \swarrow & \swarrow & \searrow \\
 F^\top U^\top X & \longleftarrow P' & \longleftarrow P & \longleftarrow & \\
 \varepsilon_X^\top \downarrow & \lrcorner \downarrow & \lrcorner \downarrow & & \searrow^{\varepsilon_R^\top} \\
 X & \longleftarrow h & S & \longleftarrow e & R
 \end{array}$$

where  $P'$  is the pullback of  $h$  and  $\varepsilon_X^\top$ . To show that  $h$  is decomposable means to show that the morphism  $F^\top U^\top S \rightarrow P'$  in the diagram above is a regular epimorphism. We know that the morphisms  $F^\top U^\top R \rightarrow P$  and  $P \rightarrow P'$  are regular epimorphisms — because  $g$  is decomposable, and because pullbacks of regular epimorphisms are regular epimorphisms — hence their composite is a regular epimorphism as well. Moreover  $U^\top e$  is a regular epimorphism hence  $TU^\top e$  and thus  $F^\top U^\top e$  are as well, respectively because  $T$  preserves regular epimorphisms and  $U^\top$  creates them. It follows that this morphism  $F^\top U^\top S \rightarrow P'$  is indeed a regular epimorphism: this is a general property of factorization systems [2, Proposition 14.9.1], and regular epimorphisms and monomorphisms form a factorization system in every regular category.

Consider now some decomposable  $f: R \rightarrow X$  and some  $g: R \rightarrow Y$ . Factor  $\langle f, g \rangle$  into a regular epimorphism  $e: R \twoheadrightarrow R'$  followed by a jointly monic span  $\langle f', g' \rangle: R' \hookrightarrow X \times Y$ . Then  $f = f' \circ e$  hence  $f'$  is decomposable, and the two relations  $g \cdot f^\dagger$  and  $g' \cdot (f')^\dagger$  are equal by definition:  $g \cdot f^\dagger$  is a decomposable relation. ◀

► **Example 24** (decomposable morphisms and relations in categories of algebras over  $\text{Set}$ ). In  $\text{EM}(\beta) \cong \text{KHaus}$ , a continuous map is decomposable if and only if it is open (it preserves open sets), and decomposable relations are the continuous ones, i.e. those relations  $\psi: X \rightleftarrows Y$  such that  $\psi^{-1}[u]$  is open in  $X$  for every open subset  $u$  of  $Y$ .

In  $\text{EM}(\text{P}) \cong \text{JSL}$ ,  $\psi: X \rightleftarrows Y$  is decomposable if and only if for every family  $(x_i)_{i \in I}$  of elements of  $X$  and every  $y \in Y$  such that  $(\bigvee_{i \in I} x_i, y) \in \psi$ , there is a family  $(y_i)_{i \in I}$  of elements of  $Y$  such that  $(x_i, y_i) \in \psi$  for all  $i \in I$  and  $\bigvee_{i \in I} y_i = y$ .

In  $\text{EM}(\text{D}) \cong \text{Conv}$ ,  $\psi: X \rightleftarrows Y$  is decomposable if and only if for every  $x \in X$ , every disintegration of  $x$  as a barycenter  $x = \sum_{i=1}^n \lambda_i x_i$  and every  $y \in Y$  such that  $(x, y) \in \psi$ ,  $y$  disintegrates as a barycenter  $y = \sum_{i=1}^n \lambda_i y_i$  such that  $(x_i, y_i) \in \psi$  for all  $i \in I$ .

**Proof.** For the descriptions of decomposable relations, we make use of the characterization given in the first part of Theorem 26.

**In KHaus.** This immediately follows from Theorem 26 and the description of  $\text{Kl}(\mathbb{V})$  given in [24]. But we can also recover this result by hand: by Lemma 40, a continuous map  $f: X \rightarrow Y$  is a decomposable  $\beta$ -algebra morphism if and only if for every ultrafilter  $E$  on  $Y$  converging to  $y \in Y$ , and every  $x \in f^{-1}(Y)$ , there is an ultrafilter  $E'$  on  $X$  converging to  $x$  and such that  $E = \{e \supseteq f(e') \mid e' \in E'\}$ .

- Suppose  $f$  is decomposable and consider some  $x \in X$  and some neighborhood  $N$  of  $x$ :  $N$  belongs to every ultrafilter converging to  $x$ . Consider some ultrafilter  $E$  on  $Y$  converging to  $f(x)$ . Then there is an ultrafilter  $E'$  on  $X$  converging to  $x$  and such that  $E = \{e \supseteq f(e') \mid e' \in E'\}$ . By definition  $N \in E'$ , hence  $f(N) \in E$ .  $f(N)$  is thus a neighborhood of  $y$ , and  $f$  is open.

- Conversely, suppose that  $f$  is open, so that if  $N$  is a neighborhood of  $x \in X$  then  $f(N)$  is a neighborhood of  $f(x)$ . Fix  $y \in Y$ ,  $E$  converging to  $y$  and  $x \in f^{-1}(y)$ .

Let  $F = \{e' \supseteq f^{-1}(e) \cap N \mid e \in E, N \in \mathcal{N}(x)\}$  where  $\mathcal{N}(x)$  is the filter of neighborhoods of  $x$ . Then  $F$  is a filter: it is non-empty because  $X$  is a neighborhood of  $x$  hence  $F$  contains  $X = f^{-1}(Y)$ ; it is stable under finite intersections because  $E$  and  $\mathcal{N}(x)$  are; it is upwards-closed by definition; and it does not contain the empty set because every neighborhood  $N$  of  $x$  is such that  $f(N)$  is a neighborhood of  $f(x)$ , i.e.  $f(N) \in E$  and  $f(N)$  intersects every  $e \in E$ .

Let  $E'$  be any ultrafilter containing  $F$ . Then  $E'$  converges to  $x$  because it contains  $\mathcal{N}(x)$ . Moreover  $f^{-1}(e) \in E'$  for every  $e \in E$ , and conversely if  $f^{-1}(e) \in E'$  then  $f^{-1}(e^c) = f^{-1}(e)^c$  is not in  $E'$  so that  $e^c$  is not in  $E$ : because  $E$  is an ultrafilter, either  $e \in E$  or  $e^c \in E$ , so the former holds.

In other words,  $E = (\beta f)(E')$  and  $f$  is decomposable.

We could prove similarly by hand that decomposable relations are indeed the continuous ones.

**In JSL.** A relation  $(A, \vee^A) \rightsquigarrow (B, \vee^B)$  in  $\text{JSL}$  given by some  $\psi: A \rightsquigarrow B$  in  $\text{Set}$  is decomposable if and only if for every  $a \in A$ ,  $E \in \text{PA}$  such that  $a = \vee^A E$  and  $b \in B$  such that  $(a, b) \in \psi$ , there is an  $F \in \text{PB}$  such that

$$\forall a' \in E, \exists b' \in F, (a', b') \in \psi \text{ and } \forall b' \in F, \exists a' \in E, (a', b') \in \psi$$

and  $\vee^B F = b$ .

Consider some  $E = \{a_i \mid i \in I\}$  so that  $a = \vee_{i \in I}^A a_i$  and some  $b \in B$  such that  $(a, b) \in \psi$ . Then if  $\psi$  is decomposable, consider the corresponding  $F \in \text{PB}$  and write  $F_i = \{b' \in F \mid (a_i, b') \in \psi\}$  and  $b_i = \vee^B F_i$ . By definition of  $F$ , every  $F_i$  is non-empty and  $\bigcup_{i \in I} F_i = F$ , so that  $\vee_{i \in I}^B b_i = \vee^B F = b$ . Moreover  $\psi \subseteq A \times B$  is a relation of join-semilattices and  $(a_i, b') \in \psi$  for every  $b' \in F_i$ , hence  $(a_i, b_i) = \vee_{b' \in F_i}^{A \times B} (a_i, b') \in \psi$  as well.

Conversely, if  $\psi$  satisfies the stated property then if  $\vee^A E = a$  and  $(a, b) \in \psi$ ,  $\vee_{e \in E}^A e = a$  so there is a family  $(b_e)_{e \in E}$  of elements of  $B$  such that  $(e, b_e) \in \psi$  and  $\vee_{e \in E}^B b_e = b$ . Taking  $F = \{b_e \mid e \in E\}$ , we see that  $\psi$  is decomposable.

**In Conv.** A relation  $A \rightsquigarrow B$  in  $\text{Conv}$  given by some  $\psi: A \rightsquigarrow B$  in  $\text{Set}$  is decomposable if and only if for every  $a \in A$  given as  $a = \sum_{i=1}^n \lambda_i a_i$  with  $\sum_{i=1}^n \lambda_i = 1$ ,  $\lambda_i \neq 0$  and  $a_i \in A$  for every  $1 \leq i \leq n$ , and  $b \in B$  such that  $(a, b) \in \psi$ , there are for every  $1 \leq i \leq n$  some  $m_i \geq 1$ ,  $(\mu_{i,j})_{1 \leq j \leq m_i}$  such that  $\sum_{j=1}^{m_i} \mu_{i,j} = \lambda_i$  and  $(b_{i,j})_{1 \leq j \leq m_i}$  such that  $(a_i, b_{i,j}) \in \psi$  for all  $1 \leq j \leq m_i$  and  $\sum_{i=1}^n \sum_{j=1}^{m_i} \mu_{i,j} b_{i,j} = b$ .

Suppose this holds. For every  $1 \leq i \leq n$ , let  $b_i = \frac{1}{\lambda_i} \sum_{j=1}^{m_i} \mu_{ij} b_{ij}$ . Then  $\sum_{i=1}^n \lambda_i b_i = \sum_{i=1}^n \sum_{j=1}^{m_i} \mu_{ij} b_{ij} = b$ , and for every  $1 \leq i \leq n$ ,  $(a_i, b_i) = \frac{1}{\lambda_i} \sum_{j=1}^{m_i} \mu_{ij} (a_i, b_{ij}) \in \psi$  because  $\psi \subseteq A \times B$  is a relation of barycentric algebras.

Conversely, consider some  $a = \sum_{i=1}^n \lambda_i a_i$  and  $b$  such that  $(a, b) \in \psi$  as above. If there is always some  $(b_i)_{1 \leq i \leq n}$  such that  $\sum_{i=1}^n \lambda_i b_i = b$  and  $(a_i, b_i) \in \psi$ , then taking  $m_i = 1$ ,  $\mu_{i1} = \lambda_i$  and  $b_{i1} = b_i$  for every  $1 \leq i \leq n$  shows that  $\psi$  is decomposable.  $\blacktriangleleft$

► **Lemma 25.** *When  $U^T$  creates near pullbacks,  $\mu^T$  is nearly cartesian if and only if every free algebra morphism  $F^T f: (X, \mu_X^T) \rightarrow (Y, \mu_Y^T)$  is decomposable.*

**Proof.**  $\mu^T$  is nearly cartesian if and only if for every  $f: X \rightarrow Y$  in  $\mathbf{C}$ ,  $\mu_Y^T \circ \mathbb{T}f = \mathbb{T}f \circ \mu_X^T$  is a near pullback. But  $\mu^T = U^T \varepsilon^T F^T$ , hence this holds if and only if for every such  $f$ ,  $U^T \varepsilon_{F^T Y}^T \circ U^T F^T U^T F^T f = U^T F^T f \circ U^T \varepsilon_{F^T X}^T$  is a near pullback, i.e. if and only if every such  $F^T f$  is decomposable since  $U^T$  creates near pullbacks.  $\blacktriangleleft$

► **Theorem 26.** *When the endofunctor  $\mathbb{T}$  is nearly cartesian, a relation  $\psi: (A, a) \leftrightarrow (B, b)$  is decomposable if and only if  $U^T \psi \cdot a = b \cdot \mathbb{T}U^T \psi$ . If  $\mu^T$  is also nearly cartesian and  $\text{Rel}(\mathbb{T})$  restricts to  $\text{Kl}(\mathbb{R}) \hookrightarrow \text{Rel}(\mathbf{C})$  for some monad  $\mathbb{R}$  on  $\mathbf{C}$ , then the Kleisli category  $\text{Kl}(\bar{\mathbb{R}})$  of the corresponding weakly lifted monad  $\bar{\mathbb{R}}$  on  $\text{EM}(\mathbb{T})$  has for arrows  $(A, a) \dashrightarrow (B, b)$  the decomposable relations  $\psi: (A, a) \leftrightarrow (B, b)$  in  $\text{EM}(\mathbb{T})$  such that  $U^T \psi$  is in  $\text{Kl}(\mathbb{R})$ .*

**Proof.** Consider some  $\text{EM}(\mathbb{T})$ -relation  $\psi: (A, a) \leftrightarrow (B, b)$ . By Lemma 22, it corresponds to the  $\mathbf{C}$ -relation  $U^T \psi: A \leftrightarrow B$ , which satisfies  $U^T \psi \cdot a \leq b \cdot \mathbb{T}U^T \psi$ . By Proposition 21,  $\psi$  is in  $\text{Kl}(\bar{\mathbb{R}})$  if and only if we have moreover that  $U^T \psi$  is in  $\text{Kl}(\mathbb{R})$  and  $U^T \psi \cdot a = b \cdot \mathbb{T}U^T \psi$ . We now show that this last condition is equivalent to  $\psi$  being decomposable.

If  $\psi$  is decomposable, it is given by a jointly monic span  $\langle \psi_{(A,a)}, \psi_{(B,b)} \rangle: (R, r) \hookrightarrow (A, a) \times (B, b)$  such that  $\psi_{(A,a)}$  is a decomposable morphism. Writing  $\psi_A = U^T \psi_{(A,a)}$  and  $\psi_B = U^T \psi_{(B,b)}$ , this means that  $\psi_A \circ r = a \circ \mathbb{T}\psi_A$  is a near pullback in  $\mathbf{C}$ . By definition of the composite of two relations, this is equivalent to saying that  $\psi_A^\dagger \cdot a = r \cdot (\mathbb{T}\psi_A)^\dagger$ . But because  $\psi_{(B,b)}$  is a morphism of  $\mathbb{T}$ -algebras, we also have that  $\psi_B \cdot r = b \cdot \mathbb{T}\psi_B$ , and so

$$\begin{aligned} U^T \psi \cdot a &= \psi_B \cdot \psi_A^\dagger \cdot a \\ &= \psi_B \cdot r \cdot (\mathbb{T}\psi_A)^\dagger \\ &= b \cdot \mathbb{T}\psi_B \cdot (\mathbb{T}\psi_A)^\dagger \\ &= b \cdot \mathbb{T}U^T \psi \end{aligned}$$

Conversely, suppose that  $\psi$  given as above is such that  $U^T \psi \cdot a = b \cdot \mathbb{T}U^T \psi$ . We now show that  $\psi_A$  is decomposable: consider the pullback

$$\begin{array}{ccc} & & \mathbb{T}R \\ & \swarrow \mathbb{T}\psi_A & \downarrow e \\ \mathbb{T}A & \xleftarrow{p_{\mathbb{T}A}} & P \\ & \downarrow p_R & \downarrow r \\ A & \xleftarrow{\psi_A} & R \end{array}$$

To prove that  $\psi_A$  is decomposable, we need to prove that  $e: \mathbb{T}R \rightarrow P$  in the diagram above is a regular epimorphism. By definition of the composition of relations,  $\psi_A^\dagger \cdot a = p_R \cdot p_A^\dagger$ , so that  $b \cdot \mathbb{T}\psi = \psi \cdot a = \psi_B \cdot p_R \cdot p_A^\dagger$ . Factor  $\langle p_{\mathbb{T}A}, \psi_B \circ p_R \rangle: P \rightarrow \mathbb{T}A \times B$  into the



regular epimorphism  $e': P \twoheadrightarrow P'$  and the jointly monic span  $\langle p'_{\top A}, p'_B \rangle: P' \hookrightarrow \top A \times B$ . Then  $p'_B \cdot (p'_{\top A})^\dagger = \psi_B \cdot p_R \cdot p_A^\dagger = b \cdot \top\psi = \psi_B \cdot r \cdot (\top\psi_A)^\dagger$ , therefore  $e' \circ e: \top R \rightarrow P'$ , as in the diagram below, must be a regular epimorphism.

$$\begin{array}{ccccc}
 & & \top R & & \\
 & \swarrow \top\psi_A & \downarrow e & \searrow r & \\
 & & P & & \\
 & \swarrow p_{\top A} & \downarrow e' & \searrow p_R & \\
 \top A & \xleftarrow{p'_{\top A}} & P' & \xrightarrow{\psi_B} & R & \xrightarrow{\psi_B} & \psi_B \\
 & & & \searrow p'_B & & & \\
 & & & & & & 
 \end{array}$$

We will now show that  $e'$  is a monomorphism and thus an isomorphism, and it will follow that  $e$  must be a regular epimorphism as well. We have the commuting diagram

$$\begin{array}{ccc}
 P & \xrightarrow{e'} & P' \hookrightarrow \top A \times B \\
 \downarrow \langle p_{\top A}, p_R \rangle & & \downarrow \langle \text{id}_{\top A}, a \rangle \times \text{id}_B \\
 \top A \times R & \xrightarrow{\text{id}_{\top A} \times \langle \psi_A, \psi_B \rangle} & \top A \times A \times B
 \end{array}$$

where all the arrows except  $e$  are monomorphisms:  $e$  is also a monomorphism.  $\blacktriangleleft$

► **Remark 28** (subobject classifiers in categories of algebras). Recall that an elementary topos is a regular category such that the  $\mathbf{Graph}$  functor has a right adjoint [18, §1.911]; the corresponding monad is called the powerset monad. If  $\mathbf{C}$  is an elementary topos with powerset monad  $\mathbf{P}$ , and if  $\mathbf{T}$  is a monad on  $\mathbf{C}$  such that the endofunctor  $\mathbf{T}$  and the natural transformation  $\mu^\mathbf{T}$  are nearly cartesian, then by Theorem 26  $\mathbf{EM}(\mathbf{T})$  is an elementary topos as soon as every  $\mathbf{T}$ -algebra morphism is decomposable. We retrieve for instance that the categories of group actions (algebras for monads  $G \times -$  where  $G$  is a group) are toposes, because the corresponding morphisms of algebras are easily shown to all be decomposable.

This is not a necessary condition for a category of algebras to be an elementary topos: it is well known that categories of monoid actions (algebras for monads  $M \times -$  where  $M$  is a monoid) are toposes, but there are equivariant morphisms that are not decomposable.

If  $\mathbf{EM}(\mathbf{T})$  is an elementary topos as in Remark 28,  $\bar{\mathbf{P}}1$  classifies subobjects in the sense that subobjects  $X \hookrightarrow Y$  are in one-to-one correspondance with morphisms  $Y \rightarrow \bar{\mathbf{P}}1$ , where  $1$  is the terminal object [18, §1.912].

This can be generalized when Theorem 26 holds as follows:  $\bar{\mathbf{R}}1$  classifies decomposition-closed subobjects, in the sense that decomposable monomorphisms  $X \hookrightarrow Y$  are in one-to-one correspondance with morphisms  $Y \rightarrow \bar{\mathbf{R}}1$  (the correspondance comes from the adjunction  $\mathbf{Kl}(\bar{\mathbf{R}})(Y, 1) \cong \mathbf{EM}(\mathbf{T})(Y, \bar{\mathbf{R}}1)$ ). For instance, the Vietoris monad on  $\mathbf{KHaus} \cong \mathbf{EM}(\beta)$  classifies clopen subsets of compact Hausdorff spaces, the non-empty-join-closed powerset monad on  $\mathbf{JSL} \cong \mathbf{EM}(\mathbf{P})$  classifies downwards-closed subsets of join-semilattices, and the convex-closed powerset monad on  $\mathbf{Conv} \cong \mathbf{EM}(\mathbf{D})$  classifies *walls*, i.e. subsets  $E$  such that if  $x \in E$  and  $\sum_{i=1}^n x_i = x$ ,  $x_i \in E$  as well for all  $1 \leq i \leq n$  (walls appear for instance in the structure theorem for convex algebras, which state that every convex algebra is a subalgebra of the *Plonka sum* of its walls [33, Theorem 4.5]).

► **Lemma 29.** *Let  $\mathbf{C}$  be a regular category where idempotents split, and suppose  $\mathbf{T}$  is nearly cartesian. If  $\mathbf{F}: \mathbf{C} \rightarrow \mathbf{C}$  is nearly cartesian and weakly lifts to  $\mathbf{EM}(\mathbf{T})$ , then its weak lifting is nearly cartesian. If  $\alpha: \mathbf{F} \Rightarrow \mathbf{G}$  between two such functors is nearly cartesian and weakly lifts to  $\mathbf{EM}(\mathbf{T})$ , then its weak lifting is nearly cartesian as well.*

**Proof.** Suppose  $F$  is nearly cartesian, i.e. preserves near pullbacks, and that it has a weak lifting  $\bar{F}$  to  $\text{EM}(\mathbb{T})$ . Then it also has a semi-lifting  $\bar{F}$  to  $\text{EM}_s(\mathbb{T})$ , and  $FU_s^\top = U_s^\top \bar{F}$  is nearly cartesian as well. Since  $U_s^\top$  creates near pullbacks by Lemma 39,  $\bar{F}$  is nearly cartesian. Now  $\bar{F} \cong K^\top \bar{F} I^\top$ , but  $K^\top$  and  $I^\top$  are both left and right adjoints, hence they preserve both limits and colimits, and thus near pullbacks (regular epimorphisms are preserved as coequalizers of their kernel pairs). It follows that  $\bar{F}$  preserves near pullbacks.

Suppose  $\alpha: F \Rightarrow G$  is a nearly cartesian natural transformation between two such functors that has a weak lifting  $\bar{\alpha}: \bar{F} \Rightarrow \bar{G}$  to  $\text{EM}(\mathbb{T})$ . Then it also has a semi-lifting  $\bar{\alpha}$  to  $\text{EM}_s(\mathbb{T})$ , and  $\alpha U_s^\top = U_s^\top \bar{\alpha}$  is nearly cartesian. Since  $U_s^\top$  creates near pullbacks,  $\bar{\alpha}$  is nearly cartesian. Now  $\bar{\alpha} \cong K^\top \bar{\alpha} I^\top$ , hence since  $K^\top$  and  $I^\top$  preserve near pullbacks  $\bar{\alpha}$  is also nearly cartesian.  $\blacktriangleleft$

► **Definition 30.** Let  $\Gamma$  be a wide subcategory of a regular category  $\mathcal{C}$  such that, in  $\mathcal{C}$ ,

■  $\Gamma$ -arrows are stable under pullbacks (in  $\mathcal{C}$ );

■ if  $f \circ e$  is a  $\Gamma$ -arrow and  $e$  is a regular epimorphism (in  $\mathcal{C}$ ),  $f$  is a  $\Gamma$ -arrow.

Then we define  $\mathcal{C} \cdot \Gamma^\dagger$  to be the wide subcategory of  $\text{Rel}(\mathcal{C})$  whose arrows are the  $\mathcal{C}$ -relations  $\psi: X \rightleftarrows Y$  given by jointly monic spans  $\langle \psi_X, \psi_Y \rangle$  such that  $\psi_X$  is a  $\Gamma$ -arrow, and we write  $\text{Graph}_\Gamma: \mathcal{C} \rightarrow \mathcal{C} \cdot \Gamma^\dagger$  for the restriction of  $\text{Graph}: \mathcal{C} \rightarrow \text{Rel}(\mathcal{C})$  to  $\mathcal{C} \cdot \Gamma^\dagger$ .

We also define a  $\Gamma^\dagger \cdot \mathcal{C}$ -square to be a square  $a \circ b = c \circ d$  such that  $a$  or  $c$  is a  $\Gamma$ -arrow.

**Proof.** We prove that  $\mathcal{C} \cdot \Gamma^\dagger$  is indeed a category. It contains all the identities of  $\text{Rel}(\mathcal{C})$  because both  $\mathcal{C}$  and  $\Gamma$  contain all the identities of  $\mathcal{C}$ . Consider jointly monic spans  $\langle f_1, g_1 \rangle: R_1 \hookrightarrow X \times Y$  and  $\langle f_2, g_2 \rangle: R_2 \hookrightarrow Y \times Z$  such that  $f_1$  and  $f_2$  are in  $\Gamma$ , and let  $\langle f', g' \rangle: S \hookrightarrow R_1 \times R_2$  be the pullback of  $g_1: R_1 \rightarrow Y$  and  $f_2: R_2 \rightarrow Y$ . Then by the first property of  $\Gamma$ ,  $f'$  and thus  $f_1 \circ f'$  ( $\Gamma$  is a category hence stable under composition) are in  $\Gamma$ . Factor  $\langle f_1 \circ f', g_2 \circ g' \rangle$  into a regular epimorphism followed by a jointly monic span: by the second property of  $\Gamma$ , the resulting jointly monic span is also in  $\mathcal{C} \cdot \Gamma^\dagger$ .  $\blacktriangleleft$

► **Theorem 32.** Let  $\mathcal{C}, \mathcal{D}, \Gamma, \Delta$  and  $F: \mathcal{C} \rightarrow \mathcal{D}$  be as in Definition 31.  $F$  has a monotone  $(\Gamma, \Delta)$ -relational extension if and only if the following two conditions hold:

■  $F$  restricts to a functor  $\Gamma \rightarrow \Delta$ ;

■  $F$  sends near pullback  $\Gamma^\dagger \cdot \mathcal{C}$ -squares on near pullback (necessarily  $\Delta^\dagger \cdot \mathcal{D}$ -) squares (this is always true when  $F$  is nearly cartesian).

Such a monotone  $(\Gamma, \Delta)$ -relational extension, if it exists, is necessarily unique and given by  $\underline{F}_{\Gamma, \Delta}(g \cdot f)^\dagger = Fg \cdot (Ff)^\dagger$ .

Let  $\alpha: F \Rightarrow G$  be a natural transformation between two functors  $\mathcal{C} \rightarrow \mathcal{D}$  having such monotone  $(\Gamma, \Delta)$ -relational extensions.  $\alpha$  has a (necessarily unique)  $(\Gamma, \Delta)$ -relational extension if and only if it has near pullbacks for its naturality squares along  $\Gamma$ -morphisms (this is always true when  $\alpha$  is nearly cartesian).

**Proof.** We adapt the proof of [21, Theorem 5.6], itself taken from [17, §5.3.11].

First note that a  $\mathcal{C}$ -relation is in  $\mathcal{C} \cdot \Gamma^\dagger$  as soon as it can be written  $g \cdot f^\dagger$  with  $f$  in  $\Gamma$ , without  $f$  and  $g$  necessarily being jointly monic (and similarly for  $\mathcal{D}$  and  $\Delta$ ): if  $\langle f, g \rangle = \langle f', g' \rangle \circ e$  with  $e$  a regular epimorphism and  $f'$  and  $g'$  jointly monic,  $f' \circ e = f$  and  $f'$  is thus also in  $\Gamma$ . Moreover a  $\mathcal{C}$ -arrow  $f$  is in  $\Gamma$  if and only if  $f^\dagger$  is in  $\mathcal{C} \cdot \Gamma$  (and, again, similarly for  $\mathcal{D}$  and  $\Delta$ ):  $f^\dagger$  is given by the jointly monic span  $\langle f, \text{id} \rangle$ .

Recall that for every arrow  $f: X \rightarrow Y$  in a regular category,  $f^\dagger: Y \rightleftarrows X$  is the unique relation such that  $\text{id}_X \leq f^\dagger \cdot f$  and  $f \cdot f^\dagger \leq \text{id}_Y$  [13, §1.2 and §1.7]. If  $H: \mathcal{C} \cdot \Gamma^\dagger \rightarrow \text{Rel}(\mathcal{D})$  is monotone, then for every  $\Gamma$ -arrow  $f: X \rightarrow Y$ ,

$$Hf \cdot H(f^\dagger) = H(f \cdot f^\dagger) \leq H(\text{id}_Y) = \text{id}_{HY}$$

and

$$\text{id}_{\mathbf{H}X} = \mathbf{H}(\text{id}_X) \leq \mathbf{H}(f^\dagger \cdot f) = \mathbf{H}(f^\dagger) \cdot \mathbf{H}f$$

and therefore  $\mathbf{H}(f^\dagger) = (\mathbf{H}f)^\dagger$ .

In particular if we take  $\mathbf{H}$  to be a  $(\Gamma, \Delta)$ -relational extension of  $\mathbf{F}$  followed by the inclusion  $\mathbf{D} \cdot \Delta^\dagger \rightarrow \text{Rel}(\mathbf{D})$ , we get that  $(\mathbf{F}f)^\dagger$  is in  $\mathbf{D} \cdot \Delta^\dagger$ , and thus  $\mathbf{F}f$  is in  $\Delta$ :  $\mathbf{F}$  restricts to a functor  $\Gamma \rightarrow \Delta$ . This forces  $\mathbf{H}$  to be defined on  $\mathbf{C} \cdot \Gamma^\dagger$ -relations as  $\mathbf{H}(g \cdot f^\dagger) = \mathbf{F}g \cdot (\mathbf{F}f)^\dagger$  (for  $f$  in  $\mathbf{D}$ ). Consider now a near pullback  $\Gamma^\dagger \cdot \mathbf{C}$ -square  $f \circ g' = g \circ f'$ , so that  $f^\dagger \cdot g = g' \cdot (f')^\dagger$ . Applying  $\mathbf{H}$ , we get  $(\mathbf{F}f)^\dagger \cdot \mathbf{F}g = \mathbf{F}g' \cdot (\mathbf{F}f')^\dagger$ , hence  $\mathbf{F}f \circ \mathbf{F}g' = \mathbf{F}g \circ \mathbf{F}f'$  is a nearly cartesian  $\Delta^\dagger \cdot \mathbf{D}$ -square.

Conversely, suppose  $\mathbf{F}$  satisfies the two properties above, and define  $\underline{\mathbf{E}}_{\Gamma, \Delta}: \mathbf{C} \cdot \Gamma^\dagger \rightarrow \mathbf{D} \cdot \Delta^\dagger$  by  $\underline{\mathbf{E}}_{\Gamma, \Delta}(g \cdot f^\dagger) = \mathbf{F}g \cdot \mathbf{F}f^\dagger$  for jointly monic  $f$  and  $g$ : because  $\mathbf{F}$  restricts as a functor  $\Gamma \rightarrow \Delta$ , the result is indeed in  $\mathbf{D} \cdot \Delta^\dagger$ .  $\underline{\mathbf{E}}_{\Gamma, \Delta}$  moreover preserves the identity:  $\underline{\mathbf{E}}_{\Gamma, \Delta}(\text{id}_X \cdot \text{id}_X^\dagger) = \text{Fid}_X \cdot (\text{Fid}_X)^\dagger = \text{id}_{\mathbf{F}X} \cdot \text{id}_{\mathbf{F}X}^\dagger$ . Finally,  $\underline{\mathbf{E}}_{\Gamma, \Delta}$  preserves composition:

- Given  $f: R \rightarrow X$  and  $g: R \rightarrow Y$  (not necessarily jointly monic), write  $\langle f, g \rangle = \langle f', g' \rangle \circ e$  with  $e$  a regular epimorphism and  $\langle f', g' \rangle$  a monomorphism, so that  $g \cdot f^\dagger = g' \cdot (f')^\dagger$  and  $\underline{\mathbf{E}}_{\Gamma, \Delta}(g \cdot f^\dagger) = \mathbf{F}g' \cdot (\mathbf{F}f')^\dagger$  by definition. Since  $\mathbf{F}$  preserves regular epimorphisms (the square  $\text{id} \circ e = \text{id} \circ e$  is a near pullback  $\Gamma^\dagger \cdot \mathbf{C}$ -square hence  $\text{id} \circ \mathbf{F}e = \text{id} \circ \mathbf{F}e$  is a near pullback as well), we get that  $\underline{\mathbf{E}}_{\Gamma, \Delta}(g \cdot f^\dagger) = \mathbf{F}g' \cdot (\mathbf{F}f')^\dagger = \mathbf{F}g \cdot (\mathbf{F}f)^\dagger$ .
- Given  $f: Y \rightarrow R$  and  $g: X \rightarrow R$ , write  $p_X: P \rightarrow X$  and  $p_Y: P \rightarrow Y$  for their pullback:  $f^\dagger \cdot g = p_Y \cdot p_X^\dagger$  and  $\underline{\mathbf{E}}_{\Gamma, \Delta}(f^\dagger \cdot g) = \mathbf{F}p_Y \cdot (\mathbf{F}p_X)^\dagger$  by definition. The  $\Gamma^\dagger \cdot \mathbf{C}$ -square  $f \circ p_Y = g \circ p_X$  is a near pullback, hence so is  $\mathbf{F}f \circ \mathbf{F}p_Y = \mathbf{F}g \circ \mathbf{F}p_X$ :  $\underline{\mathbf{E}}_{\Gamma, \Delta}(f^\dagger \cdot g) = \mathbf{F}p_Y \cdot (\mathbf{F}p_X)^\dagger = (\mathbf{F}f)^\dagger \cdot \mathbf{F}g$ .

Consider now some  $\alpha: \mathbf{F} \Rightarrow \mathbf{G}$  as stated above.  $\underline{\alpha}_{\Gamma, \Delta}: \underline{\mathbf{E}}_{\Gamma, \Delta} \Rightarrow \underline{\mathbf{G}}_{\Gamma, \Delta}$  exists as a natural transformation if and only if for every  $\langle f, g \rangle: R \hookrightarrow X \times Y$  in  $\mathbf{C} \cdot \Gamma^\dagger$ , i.e. with  $f$  in  $\Gamma$ ,  $\alpha_Y \cdot \mathbf{F}g \cdot (\mathbf{F}f)^\dagger = \mathbf{G}g \cdot (\mathbf{G}f)^\dagger \cdot \alpha_X$ . Of course  $\alpha_Y \cdot \mathbf{F}g = \mathbf{G}g \cdot \alpha_R$  because  $\alpha$  is a natural transformation  $\mathbf{F} \Rightarrow \mathbf{G}$ , hence this is true if and only if  $\alpha_R \cdot (\mathbf{F}f)^\dagger = (\mathbf{G}f)^\dagger \cdot \alpha_X$  (consider the case  $g = \text{id}_R$ ), i.e. if and only if the naturality square  $\alpha_X \circ \mathbf{F}f = \mathbf{G}f \circ \alpha_R$  is a near pullback. ◀

► **Corollary 33.** *In  $\text{Set}$ , a monad  $(\mathbf{T}, \eta^\top, \mu^\top)$  with nearly cartesian endofunctor and multiplication also has a (necessarily unique) monotone weak distributive law over the monad  $\mathbf{P}_*$  of non-empty subsets, and over the monad  $\mathbf{P}_f$  of finite subsets if and only if  $\mathbf{T}$  preserves functions with finite pre-images of elements.*

**Proof.** A  $\text{Kl}(\mathbf{P}_*)$  morphism is a function  $X \rightarrow \mathbf{P}_*Y$ , i.e. a relation such that every element of  $X$  is related to at least one element of  $Y$ . When seen as jointly monic spans  $\langle \psi_X, \psi_Y \rangle$ , these relations are exactly those such that  $\psi_X$  is surjective. Surjections satisfy the two conditions of Definition 30 because they are the same thing as regular epimorphisms and  $\text{Set}$  is regular, and so we may apply Theorem 32 as nearly cartesian endofunctors preserve in particular regular epimorphisms.

A  $\text{Kl}(\mathbf{P}_f)$  morphism is a function  $X \rightarrow \mathbf{P}_fY$ , i.e. a relation such that every element of  $X$  is related to a finite number of elements of  $Y$ . When seen as jointly monic spans  $\langle \psi_X, \psi_Y \rangle$ , these relations are exactly those such that  $\psi_X$  has finite pre-images of elements. These functions with finite pre-images of elements are easily seen to satisfy the two conditions of Definition 30, and so we may apply Theorem 32. ◀

► **Corollary 34.** *Let  $T$  be a monad on  $\mathbf{Set}$  equipped with a monotone weak distributive law  $TP \Rightarrow PT$ , and let  $S$  be a monad on  $\mathbf{EM}(T)$ . If there is a monotone weak distributive law  $S\bar{P} \Rightarrow \bar{P}S$ ,  $S$  preserves decomposable  $T$ -algebra morphisms. Moreover if  $S$  is itself the weak lifting of a monad on  $\mathbf{Set}$  that has a weak distributive law over  $P$ , then the previous condition is not only necessary, but also sufficient.*

**Proof.** We have seen in Lemma 41 that in  $\mathbf{EM}(T)$ , decomposable  $T$ -morphisms satisfy the second condition of Definition 30. They must also satisfy the first condition (stability under pullbacks) because, by Theorem 26, the composite of two decomposable relations is a decomposable relation: decomposable relations are the arrows of  $\mathbf{Kl}(\bar{P})$ . We may thus apply Theorem 32 to characterize monotone extensions to  $\mathbf{Kl}(\bar{P})$ . ◀

► **Theorem 35.** *The Radon monad  $R$  does not have a monotone weak distributive law over the Vietoris monad  $V$ , but it has (a unique) one over the non-empty Vietoris monad  $V_*$  — the monad of non-empty closed subsets.*

**Proof.** See [21, §7.4] for an introduction to radon measures and the Radon monad. Recall that  $R$  sends a compact Hausdorff space  $X$  on the space  $RX$  of Radon probability measures on  $X$ , or equivalently by the Riesz-Markov theorem to the space of positive continuous linear operators  $C(X) \rightarrow \mathbb{R}$  with norm 1, where  $C(X)$  is the Banach space of continuous functions  $X \rightarrow \mathbb{R}$  equipped with the uniform norm: for every such operator  $\varphi$  there is a Radon measure  $m$  on  $X$  such that  $\varphi(u) = \int_X u dm$  for any  $u \in C(X)$ , and conversely integrating along a measure defines an operator with the aforementioned properties.  $RX$  (seen as the space of operators) is equipped with the vague topology, i.e. the weakest topology that makes each  $ev_f: RX \rightarrow \mathbb{R}$  continuous, where  $f$  ranges in  $C(X)$  and  $ev_f$  is the function that evaluates an operator at  $f$ .

Given  $f: X \rightarrow Y$  in  $\mathbf{KHaus}$ ,  $\varphi \in RX$  and  $g \in C(Y)$ ,  $(Rf)(\varphi)(g) = \varphi(g \circ f)$ .  $\eta^R(x) \in RX$  is the operator that evaluates a continuous function at  $x \in X$ , and for  $\Phi \in RRX$  and  $f \in C(X)$ ,  $\mu^R(\Phi)(f) = \Phi(ev_f)$ .

**Radon over Vietoris.** Consider the inclusion  $f: 1 \rightarrow 2$ . 1 and 2 are sets equipped with the discrete topology, so that  $f$  is an open continuous function. But  $R1 = 1$  and  $R2$  is the unit interval: a point is never open in the unit interval, hence  $Rf$  cannot be open.  $R$  does not preserve open maps, and by Corollary 34 does not have a monotone weak distributive law  $RV \Rightarrow VR$ .

**Radon over non-empty Vietoris.** First note that  $V_*$  is easily seen to be the weak lifting of  $P_*$  to  $\mathbf{KHaus} \cong \mathbf{EM}(\beta)$ . By Theorem 26,  $\mathbf{Kl}(V_*)$  is the category of relations seen as jointly monic spans  $\langle \psi_X, \psi_Y \rangle$  such that  $\psi_X$  is open and surjective. All surjective continuous functions between compact Hausdorff spaces are open, so this boils down to  $\psi_X$  being surjective, i.e. a regular epimorphism (because  $U^\beta$  creates regular epimorphisms). Regular epimorphisms of course satisfy the conditions of Definition 30 in all regular categories, hence we may apply Theorem 32: because  $R$  is a nearly cartesian endofunctor [24, Propositions 7.26 and 7.28], it has a monotone extension to  $\mathbf{Kl}(V_*)$ . We now show that the naturality squares of  $\mu^R$  along surjections are near pullbacks, so that  $\mu^R$  also extends to  $\mathbf{Kl}(V_*)$  and we get a weak distributive law  $RV_* \Rightarrow V_*R$ .

Consider thus a continuous surjection  $f: X \rightarrow Y$  in  $\mathbf{KHaus}$ , and recall that  $Rf$  is also surjective [21, Proposition 7.26] (in fact we will simply adapt the proof of preservation of surjectivity to show our result). Consider some  $\Phi \in RRY$  and  $\psi \in RX$  such that  $\mu_Y^R(\Phi) = (Rf)(\psi)$ , i.e.  $\Phi(ev_g) = \psi(g \circ f)$  for every  $g \in C(Y)$ . Consider a first subspace  $V_1$  of  $C(RX)$  consisting of continuous functions of the shape  $g \circ Rf$  for  $g \in C(RY)$ :

$g \circ Rf \mapsto \Phi(g)$  defines a positive continuous linear operator  $\Psi_1$  of norm 1 on  $V_2$  (the definition does not depend on the choice of  $g$  because there is only one such choice as  $Rf$  is surjective). Similarly, consider a second subspace  $V_2$  of  $C(RX)$  consisting of continuous functions of the shape  $ev_g$  for some  $g \in C(X)$ : then  $ev_g \mapsto \psi(g)$  defines a positive continuous linear operator  $\Psi_2$  of norm 1 on  $V_2$ . If a continuous function in  $C(RX)$  is in  $V_1 \cap V_2$ , it is of the shape  $ev_{g \circ f} = ev_g \circ Rf$  for some  $g \in C(Y)$ , and  $\Phi(ev_g) = \psi(g \circ f)$  by assumption:  $\Psi_1$  and  $\Psi_2$  agree on  $V_1 \cap V_2$ . This defines a continuous linear operator on  $V_1 + V_2$ , which extends to a continuous linear operator  $\Psi$  on  $C(RX)$  by the Hahn-Banach theorem [5, Theorem 1.12.26]. Now if 1 denotes the constant function equal to 1,  $\Psi(1) = \Psi(ev_1) = \psi(1) = 1$  and  $\Psi$ , as a continuous linear functional of norm 1, must be positive.

We thus found some  $\Psi \in \text{RR}X$  such that  $(\text{RR}f)(\Psi) = \Phi$  and  $\mu_X^R(\Psi) = \psi$ : because  $\mathbf{U}^\beta$  creates near pullbacks, this proves that the commuting square  $Rf \circ \mu_X^R = \mu_Y^R \circ \text{RR}f$  is a near pullback. ◀

► **Example 36.** In  $\text{EM}(\mathbf{P}) \cong \text{JSL}$ , let  $\bar{\mathbf{P}}$  be the monad of subsets closed under non-empty joins: the join of a family  $(E_i)_{i \in I}$  of non-empty-joins-closed subsets of  $X$  is the non-empty-joins-closed subset  $\{\bigvee_{i \in I} x_i \mid x_i \in E_i\}$ . Let  $f: 4 \rightarrow 2$  (where  $2 = \{0, 1\}$  and  $4 = \{0, 1, 2, 3\}$ ) be the function given by  $f(0) = f(2) = 0$  and  $f(1) = f(3) = 1$ . Then  $\mathbf{F}^{\mathbf{P}}f$  is decomposable (by Lemma 25), but  $\bar{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f$  is not. Indeed, let  $A \in \bar{\mathbf{P}}\mathbf{F}^{\mathbf{P}}4$  and  $B, B_1, B_2 \in \bar{\mathbf{P}}\mathbf{F}^{\mathbf{P}}2$  be as depicted in Figure 2a (page 15):  $(\bar{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f)(A) = B = B_1 \vee B_2$  but there are no  $A_1, A_2 \in \bar{\mathbf{P}}\mathbf{F}^{\mathbf{P}}4$  such that  $A = A_1 \vee A_2$  and  $(\bar{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f)(A_1) = B_1$  as well as  $(\bar{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f)(A_2) = B_2$ .

**Proof.** Suppose indeed there are such  $A_1$  and  $A_2$ . Then  $A_1, A_2 \subseteq A$  and thus  $\{0, 1, 2, 3\} \in A_1 \cup A_2$  ( $\{0, 1, 2, 3\}$  is join-irreducible in  $A$ ). This would imply  $\{0, 1\} \in B_1 \cup B_2$ , which does not hold. ◀

► **Example 37** ([36]). In  $\text{EM}(\mathbf{D}) \cong \text{Conv}$ , let  $\bar{\mathbf{P}}$  be the monad of convex subsets: a convex combination of some convex subsets of  $X$  is the convex set of the corresponding convex combinations of their points (in  $X$ ). Let  $f: \{A, B, C\} \rightarrow \{B, C\}$  be the function given by  $f(A) = f(B) = B$  and  $f(C) = C$ . Then  $\mathbf{F}^{\mathbf{D}}f$  is decomposable (by Lemma 25) but  $\bar{\mathbf{P}}\mathbf{F}^{\mathbf{D}}f$  is not. Indeed, depicting  $\mathbf{F}^{\mathbf{D}}\{A, B, C\}$  as the triangle depicted in Figure 2b (page 15),  $\mathbf{F}^{\mathbf{D}}\{B, C\}$  is the line segment  $[BC]$  and  $\mathbf{F}^{\mathbf{D}}f$  is the vertical projection. Now  $\frac{1}{2}\{B\} + \frac{1}{2}\{C\} = [DE] = (\bar{\mathbf{P}}\mathbf{F}^{\mathbf{D}}f)([GD])$ , but  $[GD]$  itself cannot be disintegrated as the mean of two convex subsets of  $ABC$ , one above  $B$  and the other above  $[FC]$ .

**Proof.** If such a disintegration existed, then the convex subset above  $B$  would contain both  $B$  (because  $D \in [DG]$ ) and  $A$  (because  $G \in [DG]$ ), hence would be  $[AB]$ . The subset above  $[FC]$  would contain at least one point, and hence the mean of these two subsets would have to contain a non-trivial vertical line segment, which is not the case of  $[DG]$ . ◀

## C.1 Proofs for Table 1

Let us first give an additional corollary of Theorems 26 and 32.

► **Corollary 42.** *Let  $\mathbf{T}$  be a monad on  $\text{Set}$  equipped with a monotone weak distributive law  $\mathbf{T}\mathbf{P}_* \Rightarrow \mathbf{P}_*\mathbf{T}$ , and let  $\mathbf{S}$  be a monad on  $\text{EM}(\mathbf{T})$ . If there is a monotone weak distributive law  $\mathbf{S}\bar{\mathbf{P}}_* \Rightarrow \bar{\mathbf{P}}_*\mathbf{S}$ ,  $\mathbf{S}$  preserves surjective decomposable  $\mathbf{T}$ -algebra morphisms. Moreover if  $\mathbf{S}$  is itself the weak lifting of a monad on  $\text{Set}$  that has a weak distributive law over  $\mathbf{P}$ , then the previous condition is not only necessary, but also sufficient.*

**Proof.** By Theorem 26, the Kleisli category of  $\overline{P}_*$  has for arrows  $X \rightarrow Y$  the T-relations given as jointly monic spans  $\langle \psi_X, \psi_Y \rangle: R \hookrightarrow X \times Y$  such that  $\psi_X$  is decomposable and  $U^T \psi_X$  is surjective. The subcategory of surjective decomposable T-algebra morphisms satisfies the two conditions of Definition 30, as both the subcategory of decomposable T-algebra morphisms and surjective T-algebra morphisms (regular epimorphisms in  $EM(T)$ ) do. We may thus apply Theorem 32 to characterize monotone extensions to  $Kl(\overline{P}_*)$ . ◀

For every monad we consider in  $JSL$ ,  $Conv$ ,  $Mon$  and  $CMon$ , we will prove that it does not have monotone weak distributive law over  $\overline{P}$  by exhibiting a decomposable morphism that is sent by the underlying endofunctor on a non-decomposable morphism. In fact all these morphisms will also be surjective, and so by Corollary 42 they will also prove that these monads do not have monotone weak distributive laws over  $\overline{P}_*$ .

We do not prove that the example morphisms we exhibit are decomposable, as they will all be free morphisms and thus immediately be decomposable by Lemma 25.

### C.1.1 In Set.

[19] exhibits and proves the monotonicity of the weak distributive laws  $PP \Rightarrow PP$  and  $\beta P \Rightarrow P\beta$ , [23] does it for the law  $DP \Rightarrow PD$ , while [10] does it for the law  $M_S P \Rightarrow P M_S$ . The (strict) distributive laws  $LP \Rightarrow PL$  and  $MP \Rightarrow PM$  exist because linear theories distribute over commutative monads [30]. These two laws are monotone, as noticed for instance in [21, Example 2.23]: the extension corresponding to the law  $LP \Rightarrow PL$  is given in [21, Example 1.43] and is easily seen to be monotone, while the law  $MP \Rightarrow PM$  is an instance of the law  $M_S P \Rightarrow P M_S$  when  $S = \mathbb{N}$  [10, Remark 3.3]. By Corollary 33, these laws all restrict to non-empty powersets.

### C.1.2 In KHaus.

[24] exhibits the monotone weak distributive law  $VV \Rightarrow VV$  as the primary example for the framework we started from here. It is interesting to note that after Section 4, this is straightforward to show: by Example 24, decomposable  $\beta$ -algebra morphisms are exactly the open maps, and because  $V$  is the weak lifting of  $P$  to  $KHaus \cong EM(\beta)$ , by Corollary 34 there is a monotone weak distributive law  $VV \Rightarrow VV$  if and only if  $V$  preserves open maps. This is easily seen to be true. Consider indeed some open map  $f: X \rightarrow Y$ , and some open subset  $u$  of  $X$ . Then  $\square u = \{c \in VX \mid c \subseteq u\}$  is open in  $VX$ , and  $(Vf)[\square u] = \{f[c] \mid c \in \square u\} = \square f[u]$  is open in  $VY$ . Similarly  $\diamond u = \{c \in VX \mid c \cap u \neq \emptyset\}$  is open in  $VX$ , and  $(Vf)[\diamond u] = \{f[c] \mid c \in \diamond u\} = \diamond f[u] \cap \square f[X]$  is open in  $VY$ . Because the sets  $\square u$  and  $\diamond u$  form a subbasis of the topology of  $VX$  when  $u$  ranges among all open sets of  $X$ , this proves that  $Vf$  is open.

The case of the Radon monad  $R$  is handled in Theorem 35. Because  $V$  preserves continuous surjections, by a similar proof it also has a monotone weak distributive law over  $V_*$ .

### C.1.3 In JSL.

The case  $\overline{P}\overline{P} \Rightarrow \overline{P}\overline{P}$  is handled in Example 36.

### C.1.4 In Conv.

The case  $\overline{P}\overline{P} \Rightarrow \overline{P}\overline{P}$  is handled in Example 37.

### C.1.5 In Mon.

By Lemma 40, a morphism of monoids  $f: A \rightarrow B$  is decomposable if and only if for every  $f(a) = b_1 \cdots b_n$  with  $a \in A$  and  $b_i \in B$ , there are some  $a_1, \dots, a_n \in A$  such that  $f(a_i) = b_i$  and  $a = a_1 \cdots a_n$ .

**PP**  $\Rightarrow$  **PP**. Here  $\mathbf{P} = \bar{\mathbf{P}}$  is the monad of subsets of monoids: if  $A$  is a monoid with unit  $e$ ,  $\mathbf{P}A$  is a monoid with unit  $\{e\}$  and multiplication  $\alpha_1\alpha_2 = \{a_1a_2 \mid a_i \in \alpha_i\}$ .

Consider the constant function  $f: \{a, b\} \rightarrow \{a\}$ . It extends to a monoid morphism  $f^*: \{a, b\}^* \rightarrow \{a\}^*$ . Let  $\beta = \{aa\}$ ,  $\beta_1 = \beta_2 = \{a\}$  and  $\alpha = \{ab, ba\}$ :  $\beta = \beta_1\beta_2$  and  $\beta = f^*[\alpha]$ . Suppose  $\alpha_1, \alpha_2 \in \bar{\mathbf{P}}(\{a, b\}^*)$  are such that  $f^*[\alpha_i] = \beta_i$ . Then all the words in  $\alpha_i$  must have length 1. Hence if  $\alpha_1\alpha_2 \supseteq \alpha$ ,  $\alpha_1$  and  $\alpha_2$  must both contain  $a$  and  $b$ , which implies that  $\{aa, bb\} \subset \alpha_1\alpha_2 \neq \alpha$ .

**M<sub>S</sub>P**  $\Rightarrow$  **PM<sub>S</sub>**. Let  $S$  be a commutative (unital) semiring. Here  $\mathbf{M}_S$  is the monad of  $S$ -linear combinations: if  $A$  is a monoid with unit  $e$ ,  $\mathbf{M}_SA$  is a monoid with unit  $1 \cdot e$  and multiplication  $(\sum_i \alpha_i \cdot a_i) (\sum_j \beta_j \cdot b_j) = \sum_{i,j} \alpha_i \beta_j \cdot a_i b_j$ .

The counter-example for  $\mathbf{P}$  can be adapted:  $(1 \cdot a)((1+1) \cdot a) = (1+1) \cdot aa = f(1 \cdot ab + 1 \cdot ba)$ . If there are  $\alpha_1, \alpha_2 \in \mathbf{M}_S(\{a, b\}^*)$  such that  $(\mathbf{M}_S f^*)(\alpha_1) = 1 \cdot a$ ,  $(\mathbf{M}_S f^*)(\alpha_2) = (1+1) \cdot a$  and  $\alpha_1\alpha_2 = 1 \cdot ab + 1 \cdot ba$ , then  $\alpha_1 = w \cdot a + x \cdot b$  and  $\alpha_2 = y \cdot a + z \cdot b$  where  $w, x, y, z \in S$  are such that  $w + x = 1$ ,  $y + z = 1 + 1$ ,  $wy = 0$ ,  $wz = 1$ ,  $xy = 1$  and  $xz = 0$ . This implies that  $z = (w + x)z = wz + xz = 1$  and similarly  $y = (w + x)y = wy + xy = 1$ , hence that  $w = 0$  and  $x = 0$  and thus  $1 = w + x = 0$ :  $S$  is the trivial semiring and  $\mathbf{M}_S$  is the constant monad  $\{0\}$ .

**MP**  $\Rightarrow$  **PM**. This is the instance  $S = \mathbb{N}$  of the case  $\mathbf{M}_S \mathbf{P} \Rightarrow \mathbf{PM}_S$ .

**DP**  $\Rightarrow$  **PD**. Here  $\mathbf{D} = \bar{\mathbf{D}}$  is the monad of finitely supported probability distributions on a monoid: if  $A$  is a monoid with unit  $e$ ,  $\mathbf{D}A$  is a monoid with unit  $\delta_e$  and multiplication  $(\alpha_1\alpha_2)(a) = \sum_{a_1 a_2 = a} \alpha(a_1)\beta(a_2)$ .

The counter-example for  $\mathbf{P}$  can be adapted:  $\delta_{aa} = \delta_a \delta_a = (\mathbf{D}f)(\delta_{ab}/2 + \delta_{ba}/2)$ .

### C.1.6 In CMon.

By Lemma 40, a morphism of commutative monoids  $f: A \rightarrow B$  is decomposable if and only if for every  $f(a) = m_1 b_1 + \cdots + m_n b_n$  with  $a \in A$ ,  $b_i \in B$  and  $m_i \in \mathbb{N}$ , there are some  $a_1, \dots, a_n \in A$  such that  $f(a_i) = b_i$  and  $a = m_1 a_1 + \cdots + m_n a_n$ .

**PP**  $\Rightarrow$  **PP**. Here  $\mathbf{P} = \bar{\mathbf{P}}$  is the monad of subsets of monoids: if  $A$  is a commutative monoid with unit  $0$ ,  $\mathbf{P}A$  is a monoid with unit  $\{e\}$  and addition  $\alpha_1 + \alpha_2 = \{a_1 + a_2 \mid a_i \in \alpha_i\}$ .

Consider the constant function  $f: \{a, b\} \rightarrow \{a\}$ . It extends to a commutative monoid morphism  $f^*: \{a, b\}^{\otimes} \rightarrow \mathbb{N}$  (where  $\Sigma^{\otimes}$  is the free commutative monoid on  $\Sigma$ ). Let  $\beta = \{2a\}$ ,  $\beta_1 = \beta_2 = \{a\}$  and  $\alpha = \{2a, 2b\}$ :  $\beta = \beta_1 + \beta_2$  and  $\beta = f^*[\alpha]$ . Suppose  $\alpha_1, \alpha_2$  are such that  $f^*[\alpha_i] = \beta_i$ . Then all the words in  $\alpha_i$  must have length 1. Hence if  $\alpha_1 + \alpha_2 = \alpha$ , both of them must contain  $a$  and  $b$ : then  $a + b \in \alpha_1 + \alpha_2$  and  $\alpha_1 + \alpha_2 \neq \alpha$ .

**MP**  $\Rightarrow$  **PM**. Here  $\mathbf{M} = \bar{\mathbf{M}}$  is the monad of (finite) multisets, also written as finite  $\mathbb{N}$ -linear combinations: if  $A$  is a commutative monoid with unit  $0$  and addition  $\oplus$ ,  $\mathbf{M}A$  is a commutative monoid with unit  $1 \cdot 0$  and addition  $(\sum_i \alpha_i \cdot a_i) \oplus (\sum_j \beta_j \cdot b_j) = \sum_{i,j} \alpha_i \beta_j \cdot (a_i \oplus b_j)$ .

The counter-example for  $\mathbf{P}$  can be adapted:  $(1 \cdot a) \oplus (2 \cdot a) = 2 \cdot (a \oplus a) = (\mathbf{M}f^*)(1 \cdot (a \oplus a) + 1 \cdot (b \oplus b))$ . If there are  $\alpha_1, \alpha_2 \in \mathbf{M}\{a, b\}^{\otimes}$  such that  $(\mathbf{M}f^*)(\alpha_1) = 1 \cdot a$ ,  $(\mathbf{M}f^*)(\alpha_2) = 2 \cdot a$  and  $\alpha_1 \oplus \alpha_2 = 1 \cdot (a \oplus a) + 1 \cdot (b \oplus b)$ , then  $\alpha_1 = x_a \cdot a + x_b \cdot b$  and  $\alpha_2 = y_a \cdot a + y_b \cdot b$  where  $x_a, x_b, y_a, y_b \in S$  are such that  $x_a + x_b = 1$ ,  $y_a + y_b = 1 + 1$ ,  $x_a y_a = x_b y_b = 1$  and

$x_a y_b + x_b y_a = 0$ . These equations do not have common solutions in  $\mathbb{N}$ , and so such  $\alpha_1$  and  $\alpha_2$  cannot exist.

**DP**  $\Rightarrow$  **PD**. Here  $D = \overline{D}$  is the monad of finitely-supported probability distributions on a monoid: if  $A$  is a commutative monoid with unit 0,  $DA$  is a commutative monoid with unit  $\delta_0$  and addition  $(\alpha_1 + \alpha_2)(a) = \sum_{a_1+a_2=a} \alpha_1(a_1)\alpha_2(a_2)$ .

The counter-example for P can be adapted:  $\delta_{2a} = \delta_a + \delta_a = (Df)(\delta_{2a}/2 + \delta_{2b}/2)$ .

