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Towards Dynamic Distributed Knowledge

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Abstract

We propose a novel notion of distributed knowledge called *dynamic distributed knowledge* that corresponds to what a group of agents know after they share their knowledge. Its interpretation in Kripke models therefore combines static (modal accessibility) with dynamic (update) aspects. In prior work the static and dynamic aspects were also investigated, but separately, where the latter was called *resolution*. Unlike the usual distributed knowledge, in this work called *static distributed knowledge*, if a group of agents has dynamic distributed knowledge of a proposition, then it has dynamic distributed knowledge that the proposition is common knowledge. We report on its expressivity, axiomatization, and bisimulation characterization.

Keywords: modal logic, distributed knowledge, axiomatization, dynamics

1 Introduction and survey of related work

Distributed knowledge. Distributed knowledge is a well-known group notion of knowledge [21,22,27,19,29,20,14,31,34]. If agent a knows that p implies q and agent b knows that p , then agents a and b have distributed knowledge of q . Neither agent knows q individually. They have to share their knowledge.

Distributed knowledge of a proposition for a group of agents is true in a world of a given Kripke model, if *in that model* it is true in all worlds accessible by the relation that is the intersection of the equivalence relations for all agents in the group.

Observe that the proposition is then interpreted in the given model. What if we define distributed knowledge dynamically?

Distributed knowledge of a proposition for a group of agents is true in a world of a given Kripke model, if *in the model wherein the relation for each agent in the group is replaced by the intersection of the equivalence relations for all agents in the group*, it is true in all worlds accessible by that relation.

So we now interpret the proposition in the updated model. Let us call this novel notion *dynamic distributed knowledge*, whereas the standard notion is henceforth called *static distributed knowledge*. Dynamic distributed knowledge for a singleton group of agents is, as for static distributed knowledge, the same

as individual knowledge for that agent. But we will see that for two or more agents it is different from static distributed knowledge.

With the novel notion we avoid some standard issues when sharing static distributed knowledge. For example, when a and b have static distributed knowledge that b does not know that a knows p , after sharing their information b knows that a knows p . The act of sharing (the update) may change the truth value of propositions, such as is here the case. With the novel notion we can say that in this case a and b have dynamic distributed knowledge that b knows that a knows p . This is then even common knowledge between them.

We lose some of the properties of static distributed knowledge. Static distributed knowledge of a proposition implies that it is true. But dynamic distributed knowledge does not imply that it is true. The above is an example, a and b have dynamic distributed knowledge that b knows that a knows p , but where (‘right now’) b does not know that a knows p . Other properties are as for static distributed knowledge, such as positive and negative introspection.

Our results. Consider a logical language where apart from the boolean connectives the only modalities are those for dynamic distributed knowledge. For this logic of dynamic distributed knowledge we provide a modal logical semantics, determine various validities of interest and translations relating it to static distributed knowledge, we show its decidability, and propose a notion of shared bisimulation for which we show the Hennessy-Milner property. We give a complete axiomatization for an extension of the language. The complete axiomatization of the logic of dynamic distributed knowledge and the expressivity with respect to static distributed knowledge are left for further research.

Related work. Roots of the notion of distributed knowledge are found in sociology, economics, and philosophy [21,22,29]. In the epistemic logical literature an early source for static distributed knowledge is [19], wherein the notion was called *implicit knowledge*. A formal semantics or proof theory was not given. The later journal version [20] gives a standard Kripke model semantics but not an axiomatization, and now calls the notion *distributed knowledge*. Intriguingly, the slightly earlier publication [27] gives an axiomatization however without any claim of completeness. Complete axiomatizations are then given in [14,31], where the latter continue to call the notion implicit knowledge. The axiomatization proposed in [27] is the one proved to be complete in [14,31]. For a review, see also [34].

Different dynamic epistemic logics have been proposed for sharing distributed knowledge [1,8,11]. These works have in common that static dynamic knowledge operators are distinguished from dynamic modalities for sharing distributed knowledge. The dynamic distributed knowledge $\mathcal{D}_B\varphi$ that we propose in this work is definable in (all of) [1,8,11], for example, as $R_B D_B\varphi$ in [1], where *resolution* R_B means sharing distributed knowledge and where D_B is (static) distributed knowledge of [14] and related. (Resolution as resolving distributed knowledge is unrelated to resolution in logic programming.)

Instead of distributed knowledge \mathcal{D}_B interpreted by the relation \sim_B that is the intersection of the relations \sim_a for $a \in B$, one can also consider a notion

of distributed knowledge where \sim_B may be contained in that intersection; in general, where $\sim_B \subseteq \sim_C$ for any $C \subseteq B \subseteq A$. In completeness proofs [14,1] this features for technical convenience. However, it is by now also an intuitive notion of group epistemics, where the group knows more than the sum of its individual parts, for example in [7], as *correlated knowledge*, and in [17].

Notions of distributed knowledge for weaker frame classes than S5 are proposed in [13,2]. Dynamic distributed knowledge that we propose is based on ideas in [13], where another considered base logic is K4, as in [2]. In [17] distributed knowledge is investigated for KB4.

The proof theory of distributed knowledge has been investigated in [18].

Recent publications involving distributed knowledge and its dynamics are [15,9,37,25].

Overview Section 2 presents the logical language and semantics and Section 3 focusses on its validities. Section 4 proposes shared bisimulation and compares it to other notions of bisimulation. Section 5 gives two different embeddings of dynamic distributed knowledge in static distributed knowledge. Section 6 and Section 7 prepare the ground for a complete axiomatization of the language with dynamic distributed knowledge and resolution, and then show its completeness. Elementary proofs are omitted.

2 The logic of dynamic distributed knowledge

Let a finite nonempty set A of agents and a countable set P of propositional variables (atoms) be given.

Language. The language $\mathcal{L}_{\mathcal{D}DR}$ is defined by the BNF

$$\varphi ::= p \mid \top \mid (\varphi \wedge \varphi) \mid \neg\varphi \mid \mathcal{D}_B\varphi \mid D_B\varphi \mid R_B\varphi$$

where $p \in P$ and $B \subseteq A$. We follow the standard rules for omission of the parentheses. The sublanguage $\mathcal{L}_{\mathcal{D}}$ with only modalities $\mathcal{D}_B\varphi$ is the *language of dynamic distributed knowledge*. The sublanguage \mathcal{L}_D with only modalities $D_B\varphi$ is the *language of static distributed knowledge*. Notice that \mathcal{D}_\emptyset , D_\emptyset and R_\emptyset are modalities of $\mathcal{L}_{\mathcal{D}DR}$. For all $B \subseteq A$, let $|B|$ be the cardinality of B .

Other propositional and modal connectives are defined by abbreviation. In particular, the dual $\widehat{\mathcal{D}}_B\varphi$ is defined as $\neg\mathcal{D}_B\neg\varphi$, and the dual $\widehat{D}_B\varphi$ is defined as $\neg D_B\neg\varphi$. For $\mathcal{D}_{\{a_1, \dots, a_n\}}$ we often write $\mathcal{D}_{a_1 \dots a_n}$, and similarly for D . Modality \mathcal{D}_B is *dynamic distributed knowledge* for B , and formula $\mathcal{D}_B\varphi$ is read as ‘the agents in group B have dynamic distributed knowledge of φ ’. This notion is novel. The typography of \mathcal{D} is chosen to evoke two (here superimposed) letters D , as in *Dynamic Distributed*, which may help the reader to distinguish it from the notation for static distributed knowledge. Modality D_B is *static distributed knowledge* for B . This is the notion of distributed knowledge known from the literature. Modality R_B is known as *resolution*, explained later. For $\mathcal{D}_a\varphi$ (i.e., $\mathcal{D}_{\{a\}}\varphi$) or $D_a\varphi$ we write $K_a\varphi$, and for $\widehat{\mathcal{D}}_a\varphi$ and $\widehat{D}_a\varphi$ we write $\widehat{K}_a\varphi$. The fragment of $\mathcal{L}_{\mathcal{D}}$ with only modalities \mathcal{D}_\emptyset (of which the semantics are that of the universal modality) and K_a for $a \in A$ is called \mathcal{L}_{KU} , and without \mathcal{D}_\emptyset

we get \mathcal{L}_K , the *language of individual knowledge*. If $\vec{B} = B_1 \dots B_n$ then for $\mathcal{ID}_{B_1} \dots \mathcal{ID}_{B_n} \varphi$ we may write $\mathcal{ID}_{\vec{B}} \varphi$; where $\mathcal{ID}_\epsilon \varphi := \varphi$.

Frames and models. The structures are multi-agent epistemic *frames* (W, \sim) and *models* (W, \sim, V) where W is a domain of *worlds*, \sim is a function from the set of agents to equivalence relations (indistinguishability relations, knowledge relations) on W (where we write \sim_a for the knowledge relation for agent $a \in A$) and V is function from the set of atoms to the powerset of W , namely mapping each atom to the subset of worlds where it is true. We write \sim_B for $\bigcap_{b \in B} \sim_b$ (thus, $\sim_\emptyset = W \times W$).

If $M = (W, \sim, V)$ and $B \subseteq A$ then $M^B = (W, \sim^B, V)$ where for $a \in B$, $\sim_a^B := \bigcap_{b \in B} \sim_b$ and for $a \notin B$, $\sim_a^B := \sim_a$. In *updated model* M^B the relations for the agents $a \in B$ have been *updated* from \sim_a to \sim_B . For $M^{\{a_1, \dots, a_n\}}$ we write $M^{a_1 \dots a_n}$, whereas for $(M^B)^C$ we write M^{BC} . In M^B we have that $\sim_B^B = \sim_B$ for all $b \in B$, unlike in M . Note that $M^a = M$ for all $a \in A$ and that $M^\emptyset = M$. Given $\vec{B} = B_1 \dots B_n \in \mathcal{P}(A)^*$ and $M = (W, \sim, V)$, for $M^{B_1 \dots B_n}$ we write $M^{\vec{B}}$ (where $M^\epsilon = M$) and we write $\sim^{\vec{B}}$ for its knowledge function (where $\sim^\epsilon = \sim$). Given $\vec{B}, \vec{C} \in \mathcal{P}(A)^*$, \vec{B} is a prefix of \vec{C} , notation $\vec{B} \sqsubseteq \vec{C}$, if there is a $\vec{E} \in \mathcal{P}(A)^*$ such that $\vec{C} = \vec{B}\vec{E}$.

Consider an operation $\text{see} : \mathcal{P}(A) \rightarrow \mathcal{P}(A)^* \rightarrow \mathcal{P}(A)$ that associates to each $C \subseteq A$ and to each sequence \vec{B} of subsets of A the set $(\text{see}_C(\vec{B}))$ denoted $\text{see}_C(\vec{B})$ such that $\sim_{\text{see}_C(\vec{B})}$ is the knowledge relation for all agents in C after the \vec{B} update.

$$\begin{aligned} \text{see}_C(\epsilon) &:= C \\ \text{see}_C(\vec{B}E) &:= \text{see}_C(\vec{B}) && \text{if } C \cap E = \emptyset \\ \text{see}_C(\vec{B}E) &:= \text{see}_{C \cup E}(\vec{B}) && \text{if } C \cap E \neq \emptyset \end{aligned}$$

Intuitively, $\text{see}_C(\vec{B})$ determines what each agent c in group C can ‘see’ after the \vec{B} update, that is, how its knowledge relation has been restricted after the successive sharings of knowledge with the different groups of agents in \vec{B} . For example, if $C = \{a\}$ and $\vec{B} = \{a, b\}\{a, c\}$, then after the first $\{a, b\}$ update agent a already sees what a and b see (know), her relation is now $(\sim_{\{a, b\}}$ denoted) \sim_{ab} , and after the second update she sees what all three agents a, b, c can see (know), her relation is now \sim_{abc} . Etcetera.

Lemma 2.1 *Let $M = (W, \sim, V)$ be a model. For all $B, C \subseteq A$: (i) $\sim_B^\emptyset = \sim_B$, (ii) $\sim_B^B = \sim_B$, (iii) if $C \subseteq B$ then $\sim^{BC} = \sim^B$, (iv) if $B \cap C = \emptyset$ then $\sim_C^B = \sim_C$, and (v) if $B \cap C \neq \emptyset$ then $\sim_C^B = \sim_{B \cup C}$.*

Lemma 2.2 *For all $C \subseteq A$ and $\vec{B} \in \mathcal{P}(A)^*$, $C \subseteq \text{see}_C(\vec{B})$.*

Lemma 2.3 *For all (W, \sim) and $\vec{B} \in \mathcal{P}(A)^*$ and $C \subseteq A$, $\sim_C^{\vec{B}} = \sim_{\text{see}_C(\vec{B})}$.*

Proof. Induction on the length of \vec{B} . Clearly, $\sim_C^\epsilon = \sim_C = \sim_{\text{see}_C(\epsilon)}$. Then, for $E \subseteq A$, if $C \cap E = \emptyset$: $\sim_C^{\vec{B}E} = \sim_C^{\vec{B}} = (\text{ind.}) \sim_{\text{see}_C(\vec{B})} = \sim_{\text{see}_C(\vec{B}E)}$. Otherwise, if $C \cap E \neq \emptyset$: $\sim_C^{\vec{B}E} = \sim_E^{\vec{B}} \cap \sim_C^{\vec{B}} = \sim_{C \cup E}^{\vec{B}} = (\text{ind.}) \sim_{\text{see}_{C \cup E}(\vec{B})} = \sim_{\text{see}_C(\vec{B}E)}$. \square

Lemma 2.4 For all (W, \sim) and $\vec{B}, \vec{C} \in \mathcal{P}(A)^*$, if $\sim_a^{\vec{B}} = \sim_a^{\vec{C}}$ for all $a \in A$, then $\sim^{\vec{B}} = \sim^{\vec{C}}$.

Update lattice. The function see plays a role in the axiomatization and in the canonical frame (Section 7). With the function see we can compare sequences $\vec{B} \in \mathcal{P}(A)^*$, and therefore, updates. We define $\vec{B} \preceq \vec{C}$ if for all $a \in A$, $\text{see}_a(\vec{B}) \subseteq \text{see}_a(\vec{C})$, and $\vec{B} \approx \vec{C}$ if $\vec{B} \preceq \vec{C}$ and $\vec{C} \preceq \vec{B}$ as well as $\vec{B} \prec \vec{C}$ if $\vec{B} \preceq \vec{C}$ but not $\vec{C} \preceq \vec{B}$.

Lemma 2.5 For all $B, C \subseteq A$, if $C \subseteq B$ then $BC \approx B$.

The quotient of $\mathcal{P}(A)^*$ with respect to the equivalence relation \approx defines an *update lattice*. The maximum element of this lattice is denoted ω . It is the equivalence class of \vec{B} such that $\text{see}_a^{\vec{B}} = A$ for all $a \in A$. The minimum element is the equivalence class of ϵ . Note that, if $\vec{B} \approx \vec{C}$, then for all epistemic models $M = (W, \sim, V)$, $M^{\vec{B}} = M^{\vec{C}}$ (as $\sim^{\vec{B}} = \sim^{\vec{C}}$, cf. Lemma 2.3).

Gossip. There is a strong relation between a sequence $\vec{B} = B_1 \dots B_n$ of two-element sets, representing a sequence of resolution updates [1], and a sequence of calls between two agents in *gossip protocols* [26,23,4,3]. In that case the tuple $(\text{see}_{a_1}(\vec{B}), \dots, \text{see}_{a_{|A|}}(\vec{B}))$ is known as the *secret distribution* resulting from the call sequence \vec{B} , given an initial distribution where agents only know their own secrets, and the depth of the update lattice is the maximum number $\binom{|A|}{2}$ of informative calls [5].

Semantics. We define the satisfaction relation \models by induction on $\varphi \in \mathcal{L}_{DDR}$ (where $p \in P$ and $B \subseteq A$).

$$\begin{aligned}
M, w \models p & \quad \text{iff } w \in V(p) \\
M, w \models \top & \quad \text{iff always} \\
M, w \models \neg\varphi & \quad \text{iff } M, w \not\models \varphi \\
M, w \models \varphi \wedge \psi & \quad \text{iff } M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \mathcal{I}_B\varphi & \quad \text{iff } M^B, v \models \varphi \text{ for all } v \in W \text{ such that } w \sim_B v \\
M, w \models D_B\varphi & \quad \text{iff } M, v \models \varphi \text{ for all } v \in W \text{ such that } w \sim_B v \\
M, w \models R_B\varphi & \quad \text{iff } M^B, w \models \varphi
\end{aligned}$$

A formula $\varphi \in \mathcal{L}_{DDR}$ is *valid on a frame* (W, \sim) if for all models $M = (W, \sim, V)$ and for all $w \in W$, $M, w \models \varphi$. A formula $\varphi \in \mathcal{L}_{DDR}$ is *valid* if it is valid on all epistemic frames. The set $L_{\mathcal{I}}$ of validities of $\mathcal{L}_{\mathcal{I}}$ is *the logic of dynamic distributed knowledge*. The set L_D of the validities of \mathcal{L}_D is *the logic of static distributed knowledge*.

The interpretation of $K_a\varphi$ as $\mathcal{I}_a\varphi$ and $D_a\varphi$ is standard, as $\bigcap_{b \in \{a\}} \sim_b = \sim_a$, so that $M^a = M$, and that of resolution R_B is as in [1]. It is elementary that:

Lemma 2.6 For all $B \subseteq A$ and $\varphi \in \mathcal{L}_{DDR}$, $\mathcal{I}_B\varphi \leftrightarrow D_B R_B\varphi$ and $\mathcal{I}_B\varphi \leftrightarrow R_B D_B\varphi$ are valid.

Therefore, the modality \mathcal{I}_B is definable from D_B and R_B in the language \mathcal{L}_{DDR} . We recall that our research interest is the language $\mathcal{L}_{\mathcal{I}}$ without D_B and R_B operators, and how it relates to \mathcal{L}_D .

3 Validities

We first list a number of invalidities for $\mathcal{L}_{\mathcal{D}}$, to defeat the expectation of the reader versed in static distributed knowledge. We then list $\mathcal{L}_{\mathcal{D}}$ validities describing the properties of a single modality \mathcal{D}_B , and show that it has the properties of KD45. We finally list $\mathcal{L}_{\mathcal{D}}$ validities describing interaction between different modalities \mathcal{D}_B and \mathcal{D}_C .

Invalidities Invalid (that is, not valid for all B, C and for all φ) for these semantics are the following formula schemata. We contrast them with well-known validities for static distributed knowledge and resolving distributed knowledge.

$$\begin{array}{ll} \not\models \mathcal{D}_B\varphi \rightarrow \varphi & \models D_B\varphi \rightarrow \varphi \\ \not\models \mathcal{D}_B\varphi \rightarrow \mathcal{D}_C\varphi \text{ when } B \subseteq C & \models D_B\varphi \rightarrow D_C\varphi \text{ when } B \subseteq C \\ \not\models \mathcal{D}_B\neg\varphi \leftrightarrow \neg\mathcal{D}_B\varphi & \models R_B\neg\varphi \leftrightarrow \neg R_B\varphi \end{array}$$

Example 3.1 Simple counterexamples suffice to show that the above are invalid. The model on the left is for two agents a and b and a single atom p such that a knows but b is uncertain about p . The model on the right results after a and b share their knowledge.

$$M : \quad w(p) \xrightarrow{b} v(\neg p) \quad M^{ab} : \quad w(p) \quad v(\neg p)$$

We can observe that

$$\begin{array}{llll} M, w \models \mathcal{D}_{ab}K_bp & \text{whereas} & M, w \not\models K_bp & \text{so } \not\models \mathcal{D}_{ab}\varphi \rightarrow \varphi \\ M, w \models K_b\neg K_bp & \text{whereas} & M, w \not\models \mathcal{D}_{ab}\neg K_bp & \text{so } \not\models \mathcal{D}_b\varphi \rightarrow \mathcal{D}_{ab}\varphi \\ M, w \models \neg K_bp & \text{whereas} & M, w \not\models K_b\neg p & \text{so } \not\models \neg\mathcal{D}_b\varphi \rightarrow \mathcal{D}_b\neg\varphi \end{array}$$

Although $R_B\neg\varphi \leftrightarrow \neg R_B\varphi$ is valid [36], as the update is a total function, our $\mathcal{D}_B\varphi$ corresponds to $R_B D_B\varphi$, and just as $\mathcal{D}_B\neg\varphi$ is not equivalent to $\neg\mathcal{D}_B\varphi$, also $R_B D_B\neg\varphi$ is not equivalent to $\neg R_B D_B\varphi$ in [36]. Although $\mathcal{D}_B\varphi \rightarrow \varphi$ is invalid, that is, not valid for all B and φ , it is easy to see that $\mathcal{D}_B p \rightarrow p$ is valid, as atoms do not change their value after update, and as the actual world is indistinguishable for B . As a consequence, uniform substitution is not validity preserving for these semantics ($\models \varphi$ does not imply $\models \varphi[p/\psi]$); so the logic of dynamic distributed knowledge is not a normal modal logic.

Mono-modal validities We continue with validities involving a single \mathcal{D}_B .

Proposition 3.2 *Valid / validity preserving for these semantics are:*

$$\begin{array}{ll} \models \mathcal{D}_B(\varphi \rightarrow \psi) \rightarrow (\mathcal{D}_B\varphi \rightarrow \mathcal{D}_B\psi) & \models \widehat{\mathcal{D}}_B\varphi \rightarrow \mathcal{D}_B\widehat{\mathcal{D}}_B\varphi \\ \models \mathcal{D}_B\varphi \rightarrow \mathcal{D}_B\mathcal{D}_B\varphi & \models \varphi \text{ implies } \models \mathcal{D}_B\varphi \\ \models \mathcal{D}_B\varphi \rightarrow \widehat{\mathcal{D}}_B\varphi & \end{array}$$

Proof. We use that $M^{BB} = M^B$ and that $\sim_B^B = \sim_B$. More details omitted. \square

It follows that the modalities \mathcal{D}_B have the properties of KD45 (we also have Modus Ponens), and thus represent consistent group belief. Still, the \mathcal{D}_B also have other properties, such as the mentioned $\models \mathcal{D}_B p \rightarrow p$, which makes it clear that the logic of dynamic distributed knowledge is not (multi-)KD45.

Multi-modal validities Finally, validities relating \mathcal{I}_B and \mathcal{I}_C for $C \neq B$.

Proposition 3.3 $\models \mathcal{I}_B \mathcal{I}_C \varphi \leftrightarrow \mathcal{I}_B \varphi$, where $\emptyset \neq C \subseteq B \subseteq A$.

Proof. Let (M, w) be given. Then: $M, w \models \mathcal{I}_B \mathcal{I}_C \varphi$, iff $M^B, v \models \mathcal{I}_C \varphi$ for all $v \sim_B^e w$, iff $M^{BC}, u \models \varphi$ for all $u \sim_C^B v$ and for all $v \sim_B^e w$, iff (*) $M^{BC}, u \models \varphi$ for all $u \sim_B^e v$ and for all $v \sim_B^e w$, iff (Lemma 2.1) $M^B, u \models \varphi$ for all $u \sim_B^e v$ and for all $v \sim_B^e w$, iff $M^B, u \models \varphi$ for all $u \sim_B^e w$, iff $M, u \models \mathcal{I}_B \varphi$.

(*): As $C \subseteq B$ for all $c \in C$, $\sim_c^B = \bigcap_{b \in B} \sim_b$. Furthermore, for all $b \in B$, $\sim_b^B = \bigcap_{b \in B} \sim_b$. Therefore, $u \sim_c^B v$ iff $u \sim_b^B v$. \square

The instantiation of Prop. 3.3 for $B = C$ is that $\mathcal{I}_B \mathcal{I}_B \varphi \leftrightarrow \mathcal{I}_B \varphi$ is valid, from which we also get $\mathcal{I}_B \varphi \rightarrow \mathcal{I}_B \mathcal{I}_B \varphi$.

We continue with multi-modal validities that resemble the validities $D_B \varphi \rightarrow \varphi$ and $D_B \varphi \rightarrow D_C \varphi$ for static distributed knowledge, however, subject to restrictions. Let us define $\mathcal{L}_{\mathcal{I}} \uparrow C$ as the fragment of $\mathcal{L}_{\mathcal{I}}$ with only \mathcal{I}_B modalities for $C \subseteq B \subseteq A$.

Lemma 3.4 Let $B, C \subseteq A$ with $B \subseteq C$, and let $\varphi \in \mathcal{L}_{\mathcal{I}} \uparrow C$. Then for all models $M = (W, \sim, V)$ and $w \in W$, $M^B, w \models \varphi$ iff $M^C, w \models \varphi$.

Proof. The proof is by induction on φ . \square

Proposition 3.5 $\models \mathcal{I}_B \varphi \rightarrow \varphi$ whenever $\varphi \in \mathcal{L}_{\mathcal{I}} \uparrow B$.

Proof. The proof is by induction on $\varphi \in \mathcal{L}_{\mathcal{I}}$ in negation normal form. Let $M = (W, \sim, V)$ be given and $w \in W$. The cases of p and $\neg p$, conjunction and disjunction (using Lemma 3.4) have been omitted.

Case $\mathcal{I}_C \varphi$. $M, w \models \mathcal{I}_B \mathcal{I}_C \varphi$, iff (semantics) $M^B, v \models \mathcal{I}_C \varphi$ for all $v \sim_B w$, iff $M^{BC}, u \models \varphi$ for all $v \sim_B w$ and for all $u \sim_C^B v$, iff ($B \subseteq C$ implies that $M^{BC} = M^C$ and $\sim_C^B = \sim_C$) $M^C, u \models \varphi$ for all $v \sim_B w$ and for all $u \sim_C v$, which implies (as $w \sim_B w$) $M^C, u \models \varphi$ for all $u \sim_C w$, iff $M, w \models \mathcal{I}_C \varphi$.

Case $\widehat{\mathcal{I}}_C \varphi$. The proof is similar to the case $\mathcal{I}_C \varphi$. \square

Proposition 3.6 $\models \mathcal{I}_B \varphi \rightarrow \mathcal{I}_C \varphi$ whenever $B \subseteq C$ and $\varphi \in \mathcal{L}_{\mathcal{I}} \uparrow C$.

Proof. Suppose $B \subseteq C$. Let $\varphi \in \mathcal{L}_{\mathcal{I}} \uparrow C$. Suppose $M, w \models \mathcal{I}_B \varphi$. Hence, $M^B, v \models \varphi$ for all $v \sim_B w$. Thus, by Lemma 3.4, $M^C, v \models \varphi$ for all $v \sim_B w$. Since $B \subseteq C$, therefore $\sim_C \subseteq \sim_B$. Since $M^C, v \models \varphi$ for all $v \sim_B w$, therefore $M^C, v \models \varphi$ for all $v \sim_C w$. Consequently, $M, w \models \mathcal{I}_C \varphi$. \square

All above seems not enough for a complete axiomatization, see Section 7.

4 Bisimulation

We propose a novel notion of *shared bisimulation* and we compare it to the notions of individual bisimulation and collective bisimulation from the literature. For binary relations Z we write $(x, y) \in Z$ as well as Zxy .

Individual and collective bisimulation. Let $M = (W, \sim, V)$ and $M' = (W', \sim', V')$ be given. A non-empty relation $Z \subseteq W \times W'$ is a *collective bisimulation* [28] if for all $(w, w') \in Z$, $p \in P$, and $B \subseteq A$:

- **atoms:** $w \in V(p)$ iff $w' \in V'(p)$
- **forth:** if $w \sim_B v$ there is $v' \in W'$ such that $w' \sim'_B v'$ and $(v, v') \in Z$
- **back:** if $w' \sim'_B v'$ there is $v \in W$ such that $w \sim_B v$ and $(v, v') \in Z$

Pointed models (M, w) and (M', w') are *collective bisimilar*, notation $(M, w) \Leftrightarrow^c (M', w')$, if there exists a collective bisimulation Z between M and M' with $(w, w') \in Z$. Relation Z is an *individual bisimulation* [10] if **forth** and **back** are satisfied for all singleton sets $\{a\}$ where $a \in A$; (M, w) and (M', w') are *individually bisimilar*, notation $(M, w) \Leftrightarrow^i (M', w')$, if there exists an individual bisimulation Z between M and M' with $(w, w') \in Z$.

Shared bisimulation. The novel notion of *shared bisimulation* is given by defining a set of \vec{B} -shared bisimulations, for all $\vec{B} \in \mathcal{P}(A)^*$.

Definition 4.1 Let $M = (W, \sim, V)$ and $M' = (W', \sim', V')$ be given. A *shared bisimulation* Z is a collection $\{Z^{\vec{B}} \mid \vec{B} \in \mathcal{P}(A)^*\}$ of non-empty relations $Z^{\vec{B}} \subseteq W \times W'$ such that for all $(w, w') \in Z^{\vec{B}}$, $p \in P$, and $B \subseteq A$:

- **atoms:** $w \in V(p)$ iff $w' \in V'(p)$
- **forth:** if $w \sim_B v$ there is $v' \in W'$ such that $w' \sim'_B v'$ and $(v, v') \in Z^{\vec{B}B}$
- **back:** if $w' \sim'_B v'$ there is $v \in W$ such that $w \sim_B v$ and $(v, v') \in Z^{\vec{B}B}$

When $\{Z^{\vec{B}} \mid \vec{B} \in \mathcal{P}(A)^*\}$ is a shared bisimulation between M and M' , each $Z^{\vec{B}}$ is called a \vec{B} -shared bisimulation. Pointed models (M, w) and (M', w') are *shared bisimilar*, notation $(M, w) \Leftrightarrow^s (M', w')$, if there exists a shared bisimulation Z between M and M' with $(w, w') \in Z^\epsilon$.

We recall the update lattice from Section 2. We can similarly view the \vec{B} -shared bisimulations $Z^{\vec{B}}$ for all $\vec{B} \in \mathcal{P}(A)^*$ of a shared bisimulation Z as such a lattice, with top Z^ω and bottom Z^ϵ . There are therefore only finitely many \vec{B} -shared bisimulations to consider, modulo \approx -equivalence. Note that, given equivalence classes $[w]_A$ in W and $[w']_A$ in W' containing the same valuations, relating (for all w, w') states in $[w]_A$ to states in $[w']_A$ with the same valuation, is an ω -shared bisimulation.

Proposition 4.2 ([28]) *Collective bisimilar is (implies) individual bisimilar.*

Proposition 4.3 *Collective bisimilar is shared bisimilar.*

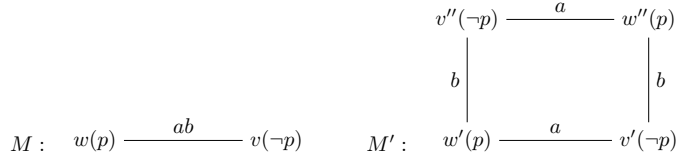
Proof. Let $Z \subseteq W \times W'$ be a collective bisimulation. Then $Z_2 := \{Z_2^{\vec{B}} \mid \vec{B} \in \mathcal{P}(A)^*\}$ where $Z_2^{\vec{B}} = Z$ for all \vec{B} is a shared bisimulation. Details omitted. \square

Proposition 4.4 *Shared bisimilar is individual bisimilar.*

Proof. Let $Z = \{Z^{\vec{B}} \mid \vec{B} \in \mathcal{P}(A)^*\}$ be a shared bisimulation between $M = (W, \sim, V)$ and $M' = (W', \sim', V')$. We show that $Z^{A^*} := \bigcup \{Z^{\vec{a}} \mid \vec{a} \in A^*\}$ is an individual bisimulation between M and M' . (Each $Z^{a_1 \dots a_n}$ is considered $Z^{\{a_1\} \dots \{a_n\}}$ with $\{a_1\} \dots \{a_n\} \in \mathcal{P}(A)^*$.) **Atoms** is obvious. Consider **forth**. Let be given $Z^{A^*} w w'$ and $w \sim_b v$. As $Z^{A^*} w w'$, there is $\vec{a} \in A^*$ with $Z^{\vec{a}} w w'$.

As $M^{\vec{a}} = M$, $w \sim_b v$ iff $w \sim_b^{\vec{a}} v$. As $Z^{\vec{a}}$ is a \vec{a} -shared bisimulation, there is a $v' \sim_b^{\vec{a}} w'$ with $Z^{\vec{a}b} v v'$. As $Z^{\vec{a}b} \subseteq Z^{A*}$, therefore $Z^{A*} v v'$. Furthermore, $v' \sim_b^{\vec{a}} w'$ iff $v' \sim_b^{\vec{a}} w'$ because $M'^{\vec{a}} = M'$. **Back** is similar. \square

Example 4.5 It is well-known that individual bisimilar may not be collective bisimilar. A typical counterexample involves the models M and M' below.



An individual bisimulation between M and M' consists of linking the p worlds and linking the $\neg p$ worlds. For example, pair (w, w') in this bisimulation. However, it is not a collective bisimulation: although $w \sim_{ab} v$ in M and where we note that $v \in V(p)$, there is no world s in M' such that $w' \sim_{ab} s$ and that has the same value of p as v in M . Only $s = w'$ fulfills that role and $w' \notin V(p)$. So, we cannot satisfy **atoms**. Indeed, $M, w \not\models D_{ab}p$, whereas $M', w' \models D_{ab}p$.

This also demonstrates that individual bisimilar may not be shared bisimilar, as **atoms** has also to be satisfied for a shared bisimulation, and indeed, similarly, $M, w \not\models \mathbb{D}_{ab}p$, whereas $M', w' \models \mathbb{D}_{ab}p$.

It remains a, rather annoyingly, open question whether collective bisimilar and shared bisimilar are the same or are different, but we conjecture that collective bisimilarity is a more refined notion to distinguish structures than shared bisimilarity. An answer to this question might help to compare the expressivity of dynamic distributed knowledge and static distributed knowledge.

Hennessy-Milner characterization We continue by characterizing shared bisimilarity. For that we need some additional terminology.

Given $M = (W, \sim, V)$, $M' = (W', \sim', V')$, $w \in W$, and $w' \in W'$, pointed models (M, w) and (M', w') are *modally equivalent*, notation $(M, w) \equiv (M', w')$, if for all $\varphi \in \mathcal{L}_{\mathbb{D}}$, $M, w \models \varphi$ iff $M', w' \models \varphi$.

A model $M = (W, \sim, V)$ is *image-finite* if for all agents $a \in A$ and worlds $v \in W$ the equivalence class $[v]_a = \{u \in W \mid v \sim_a u\}$ is finite.

Proposition 4.6 *Shared bisimilar implies modally equivalent.*

Proof. We need to show that for arbitrary pointed models (M, w) and (M', w') , $(M, w) \leftrightarrow^s (M', w')$ implies $(M, w) \equiv (M', w')$. In order to prove that, we will prove the following statement.

Let be given $\varphi \in \mathcal{L}_{\mathbb{D}}$, models $M = (W, \sim, V)$ and $M' = (W', \sim', V')$, and a shared bisimulation Z between M and M' . Then for all $w \in W$, $w' \in W'$, and $\vec{B} \in \mathcal{P}(A)^*$: $Z^{\vec{B}} w w'$ implies $(M^{\vec{B}}, w) \equiv (M'^{\vec{B}}, w')$. $(*)$

The proof is by induction on φ . Elementary cases are omitted.

Case $\mathbb{D}_B\varphi$: Assume $M'^{\vec{B}}, w' \models \mathbb{D}_B\varphi$. To prove $M^{\vec{B}}, w \models \mathbb{D}_B\varphi$, assume $v \in W$ with $v \sim_B^{\vec{B}} w$. From initial assumption $Z^{\vec{B}} w w'$, $v \sim_B^{\vec{B}} w$, and **forth**

it follows that there is $v' \in W'$ such that $v' \sim_{\vec{B}} w'$ and $Z^{\vec{B}B}vv'$. Also, from assumption $M^{\vec{B}}, w' \models \mathcal{D}_B\varphi$ and $v' \sim_{\vec{B}} w'$ it follows that $M^{\vec{B}B}, v' \models \varphi$. From $M^{\vec{B}B}, v' \models \varphi$, $Z^{\vec{B}B}vv'$, and induction it now follows that $M^{\vec{B}B}, v \models \varphi$. As v was arbitrary, from the semantics of \mathcal{D}_B it now follows that $M^{\vec{B}}, w \models \mathcal{D}_B\varphi$. The other direction is proved similarly.

The case $\vec{B} = \epsilon$ of (*) gets us that $Z^\epsilon ww'$ implies $(M, w) \equiv (M', w')$. \square

Proposition 4.7 *Modally equivalent implies shared bisimilar on the class of image-finite models.*

Proof. We need to show that for arbitrary image-finite pointed models (M, w) and (M', w') , $(M, w) \equiv (M', w')$ implies $(M, w) \leftrightarrow^s (M', w')$. In order to prove that, we will prove the following statement.

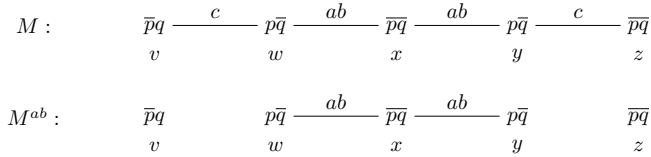
Let be given $M = (W, \sim, V)$ and $M' = (W', \sim', V')$ with $w \in W$ and $w' \in W'$, and such that $(M, w) \equiv (M', w')$. Consider the set of relations $Z = \{Z^{\vec{B}} \mid \vec{B} \in \mathcal{P}(A)^*\}$ such that for all $v \in W$, $v' \in W'$, $Z^{\vec{B}}vv'$ iff $(M^{\vec{B}}, v) \equiv (M'^{\vec{B}}, v')$. Then Z is a shared bisimulation between M and M' . (**)

Suppose towards a contradiction that Z is not a shared bisimulation. Then there is $\vec{B} \in \mathcal{P}(A)^*$ such that $Z^{\vec{B}}$ is not a \vec{B} -shared bisimulation. Then there are $w, v \in W$ and $w' \in W'$ with $Z^{\vec{B}}ww'$ and $w \sim_{\vec{B}} v$ but for all $v' \in W'$ with $w' \sim'_{\vec{B}} v'$, not $Z^{\vec{B}B}vv'$. As M' is image-finite, we can enumerate such v' as v'_1, \dots, v'_n . By the definition of Z , this means that $(M^{\vec{B}B}, v) \not\equiv (M'^{\vec{B}B}, v'_1), \dots$, and $(M^{\vec{B}B}, v) \not\equiv (M'^{\vec{B}B}, v'_n)$. Let ψ_1, \dots, ψ_n be distinguishing formulas such that $M^{\vec{B}B}, v \models \psi_1$ but $M'^{\vec{B}B}, v'_1 \not\models \psi_1, \dots$, and $M^{\vec{B}B}, v \models \psi_n$ but $M'^{\vec{B}B}, v'_n \not\models \psi_n$. Note that $M^{\vec{B}B}, v \models \bigwedge_{i=1}^n \psi_i$. By the semantics of dynamic distributed knowledge we therefore have that $M^{\vec{B}}, w \models \widehat{\mathcal{D}}_B \bigwedge_{i=1}^n \psi_i$. On the other hand, $M'^{\vec{B}}, w' \not\models \widehat{\mathcal{D}}_B \bigwedge_{i=1}^n \psi_i$, as in any of the $\text{see}_B(\vec{B})$ -accessible $v'_i = v'_1, \dots, v'_n$ the conjunction $\bigwedge_{i=1}^n \psi_i$ is false: ψ_1 is false in v'_1 , and therefore $\bigwedge_{i=1}^n \psi_i$ is false in v'_1 .

This contradicts the assumption that $Z^{\vec{B}}$ is not a \vec{B} -shared bisimulation, and therefore the initial assumption that Z is not a shared bisimulation.

As Z is a shared bisimulation, Z^ϵ is an ϵ -shared bisimulation. The original assumption $(M, w) \equiv (M', w')$ means that $Z^\epsilon ww'$, by the definition of Z^ϵ . By the definition of shared bisimilarity $Z^\epsilon ww'$ means that $(M, w) \leftrightarrow^s (M', w')$. \square

Example 4.8 Not all $Z^{\vec{B}}$ are collective bisimulations. Consider the model below for agents a, b, c and variables p and q (valuations label worlds, $\bar{p} = \neg p$).



A maximal shared (auto)bisimulation on this model consists of:

- $Z^\epsilon = Z^a = Z^b = \{(v, v), (w, w), (x, x), (y, y), (z, z)\}$
- $Z^{ab} = Z^\omega = \{(v, v), (w, w), (x, x), (y, y), (z, z)\} \cup \{(w, y), (y, w)\}$

In this case we have that $Z^\epsilon \subset Z^\omega$ and that Z^ω is not a collective bisimulation, or even an individual bisimulation, as worlds w and y are not (even individual) bisimilar: $M, w \models K_c q$ whereas $M, y \not\models K_c q$. However, in the M^{ab} update they are. So, $Z^{ab}st$ does not imply $Z^\epsilon st$ for arbitrary worlds s and t . However, given $s = w$, there always is *some* t such that $Z^{ab}st$ and $Z^\epsilon st$, namely $t = w$.

5 Translations

We present two translations. We show that every formula in \mathcal{L}_D is equivalent to one in \mathcal{L}_D , using a translation τ , and we show that on the class of frames *with local agreement* (for any two agents one always knows more than the other) every formula in \mathcal{L}_D is equivalent to one in \mathcal{L}_{KU} , using a translation δ .

5.1 Dynamic into static distributed knowledge

Lemma 2.6 showed that $\mathbb{D}_B \varphi$ is definable as either $D_B R_B \varphi$ or as $R_B D_B \varphi$. In [1] it is shown that every formula in \mathcal{L}_{DR} is equivalent to a formula in \mathcal{L}_D . Combining the two proves that every $\varphi \in \mathcal{L}_D$ is equivalent to a $\psi \in \mathcal{L}_D$. Instead of this indirect proof, we define a translation τ , indexed by groups of agents $B \subseteq A$, such that any $\varphi \in \mathcal{L}_D$ is equivalent to $\tau_\emptyset(\varphi) \in \mathcal{L}_D$.

Definition 5.1 Let translation $(\tau_B)_{B \subseteq A}$ be the family of functions from \mathcal{L}_{DDR} to \mathcal{L}_D inductively defined below. For $\tau_{B_1}(\dots \tau_{B_m}(\varphi) \dots)$ we write $\tau_{\vec{B}}(\varphi)$.

$$\begin{array}{l} \tau_B(p) = p \\ \tau_B(\top) = \top \\ \tau_B(\neg\varphi) = \neg\tau_B(\varphi) \\ \tau_B(\varphi \wedge \psi) = \tau_B(\varphi) \wedge \tau_B(\psi) \end{array} \quad \left| \quad \begin{array}{l} \tau_B(\mathbb{D}_C \varphi) = D_C \tau_B(\tau_C(\varphi)) \quad \text{if } B \cap C = \emptyset \\ \tau_B(\mathbb{D}_C \varphi) = D_{B \cup C} \tau_B(\tau_C(\varphi)) \quad \text{if } B \cap C \neq \emptyset \\ \tau_B(D_C \varphi) = D_C \tau_B(\varphi) \quad \text{if } B \cap C = \emptyset \\ \tau_B(D_C \varphi) = D_{B \cup C} \tau_B(\varphi) \quad \text{if } B \cap C \neq \emptyset \\ \tau_B(R_C \varphi) = \tau_B(\tau_C(\varphi)) \end{array} \right.$$

Lemma 5.2 Let $\varphi \in \mathcal{L}_{DDR}$. For all $B, C \subseteq A$, $\tau_B(\mathbb{D}_C \varphi) = \tau_B(D_C R_C \varphi)$.

Lemma 5.3 Let $\vec{B} \in \mathcal{P}(A)^*$. For all $C \subseteq A$, there exists $C' \subseteq A$ such that $C \subseteq C'$ and for all $\varphi \in \mathcal{L}_{DDR}$, $\tau_{\vec{B}}(\mathbb{D}_C \varphi) = D_{C'} \tau_{\vec{B}}(\varphi)$.

Proof. By induction on \vec{B} . □

Lemma 5.4 Let $\varphi \in \mathcal{L}_{DDR}$. For all models $M = (W, \sim, V)$, for all $u \in W$ and for all $B \subseteq A$, $M^B, u \models \varphi$ if and only if $M, u \models \tau_B(\varphi)$.

Proof. The proof is done by induction on φ . Let $M = (W, \sim, V)$ be a model, $u \in W$ and $B \subseteq A$. We only show the case $\mathbb{D}_C \psi$.

Case $B \cap C = \emptyset$. Then: (i) $M^B, u \models \mathbb{D}_C \psi$, iff (ii) for all $v \in W$, if $u \sim_C^B v$ then $M^{BC}, v \models \psi$, iff (iii) for all $v \in W$, if $u \sim_C v$ then $M^{BC}, v \models \psi$, iff (iv) for all $v \in W$, if $u \sim_C v$ then $M^B, v \models \tau_C(\psi)$, iff (v) for all $v \in W$, if $u \sim_C v$ then $M, v \models \tau_B(\tau_C(\psi))$, iff (vi) $M, u \models D_C \tau_B(\tau_C(\psi))$.

The equivalence between (ii) and (iii) is a consequence of $B \cap C = \emptyset$, and that between (iii), (iv) and (v) is a consequence of the induction hypothesis.

Case $B \cap C \neq \emptyset$. Then: (i) $M^B, u \models \mathcal{D}_C \psi$, iff (ii) for all $v \in W$, if $u \sim_C^B v$ then $M^{BC}, v \models \psi$, iff (iii) for all $v \in W$, if $u \sim_{B \cup C} v$ then $M^{BC}, v \models \psi$, iff (iv) for all $v \in W$, if $u \sim_{B \cup C} v$ then $M^B, v \models \tau_C(\psi)$, iff (v) for all $v \in W$, if $u \sim_{B \cup C} v$ then $M, v \models \tau_B(\tau_C(\psi))$, iff (iv) $M, u \models D_{B \cup C} \tau_B(\tau_C(\psi))$.

The equivalence between (ii) and (iii) is a consequence of $B \cap C \neq \emptyset$, and that between (iii), (iv) and (v) is a consequence of the induction hypothesis. \square

Corollary 5.5 *Any $\varphi \in \mathcal{L}_D$ is equivalent to $\tau_\emptyset(\varphi) \in \mathcal{L}_D$.*

As a further consequence, satisfiability in the logic L_D is *decidable*, because the logic L_D is decidable [14,34].

5.2 Agents with local agreement

A *frame (model) with local agreement*, first considered in [16], is a multi-agent epistemic frame (W, \sim) (resp. model (W, \sim, V)) such that for all $a, b \in A$ and for all $s \in W$, either $[s]_a \subseteq [s]_b$, or $[s]_b \subseteq [s]_a$. Validity of formulas φ on frames with local agreement is denoted $\models^{\text{agree}} \varphi$. We show that on the class of frames with local agreement every formula in \mathcal{L}_D is equivalent to a formula in \mathcal{L}_K . Intuitively, local agreement formalizes that given any two agents, one always knows more than the other. A consequence easily shown is that in any state of a model there is a hierarchy between the agents (a total order) from one who knows most to one who knows least:

Lemma 5.6 *Let (W, \sim) be a frame with local agreement. For all $B \subseteq A$, if $B \neq \emptyset$ then for all $s \in W$ there exists $a \in B$ such that for all $b \in B$, $[s]_a \subseteq [s]_b$.*

Lemma 5.7 *Preservation after update: Let (W, \sim) be a frame with local agreement. For all $B \subseteq A$, the frame (W, \sim^B) is a frame with local agreement.*

Definition 5.8 Let translation $(\delta_B)_{B \subseteq A}$ be the family of functions from \mathcal{L}_D to \mathcal{L}_D inductively defined below. For $\delta_{B_1}(\dots \delta_{B_m}(\varphi) \dots)$ we write $\delta_{\vec{B}}(\varphi)$.

$$\begin{aligned} \delta_B(\varphi) &= \varphi \text{ if } |B| \leq 1 & \delta_B(\neg\varphi) &= \neg\delta_B(\varphi) & \text{if } |B| \geq 2 \\ \delta_B(p) &= p \text{ if } |B| \geq 2 & \delta_B(\varphi \wedge \psi) &= \delta_B(\varphi) \wedge \tau_B(\psi) & \text{if } |B| \geq 2 \\ \delta_B(\top) &= \top \text{ if } |B| \geq 2 & \delta_B(\mathcal{D}_\emptyset\varphi) &= \mathcal{D}_\emptyset\delta_B(\varphi) & \text{if } |B| \geq 2 \end{aligned}$$

$$\begin{aligned} \delta_B(\mathcal{D}_C\varphi) &= \bigvee \{K_c \delta_B(\delta_C(\varphi)) \mid c \in C\} & \text{if } |B| \geq 2, C \neq \emptyset \text{ and } B \cap C = \emptyset \\ \delta_B(\mathcal{D}_C\varphi) &= \bigvee \{K_c \delta_B(\delta_C(\varphi)) \mid c \in B \cup C\} & \text{if } |B| \geq 2, C \neq \emptyset \text{ and } B \cap C \neq \emptyset \end{aligned}$$

Define $\delta : \mathcal{L}_D \rightarrow \mathcal{L}_D$ replacing every subformula of shape $\mathcal{D}_B\varphi$ by $\delta_B(\mathcal{D}_B\varphi)$.¹

Lemma 5.9 *For all $B \subseteq A$ with $|B| \geq 2$ and for all $\varphi \in \mathcal{L}_D$, $\delta_B(\varphi) \in \mathcal{L}_{KU}$.*

Proof. By induction on φ . \square

Lemma 5.10 *For all models $M = (W, \sim, V)$ with local agreement, formulas $\varphi \in \mathcal{L}_D$, $s \in W$ and $B \subseteq A$: $M^B, s \models \varphi$ iff $M, s \models \delta_B(\varphi)$.*

¹ $\delta(p) = p$, $\delta(\top) = \top$, $\delta(\varphi \wedge \psi) = \delta(\varphi) \wedge \delta(\psi)$, $\delta(\neg\varphi) = \neg\delta(\varphi)$, and $\delta(\mathcal{D}_B\varphi) = \delta_B(\mathcal{D}_B\varphi)$

Proof. By induction on φ . We only show the case $\varphi = \mathcal{I}_C\psi$ where $C \neq \emptyset$. We assume $|B| \geq 2$, for otherwise the result trivially holds.

From left to right, suppose $M^B, s \models \mathcal{I}_C\psi$ and $M, s \not\models \delta_B(\mathcal{I}_C\psi)$. Now, there are two cases: either $B \cap C = \emptyset$, or $B \cap C \neq \emptyset$. In the former case, $M, s \not\models \bigvee\{K_c\delta_B(\delta_C(\psi)) \mid c \in C\}$. Hence, for all $c \in C$, $M, s \not\models K_c\delta_B(\delta_C(\psi))$. By Lemma 5.6, let $c \in C$ be such that for all $d \in C$, $[s]_c \subseteq [s]_d$. Thus, $M, s \not\models K_c\delta_B(\delta_C(\psi))$. Consequently, there exists $t \in W$ such that $s \sim_c t$ and $M, t \not\models \delta_B(\delta_C(\psi))$. Since for all $d \in C$, $[s]_c \subseteq [s]_d$, therefore $s \sim_C t$. Moreover, by induction hypothesis, $M^{BC}, t \not\models \psi$. Since $B \cap C = \emptyset$ and $s \sim_C t$, therefore $s \sim_C^B t$. Since $M^B, s \models \mathcal{I}_C\psi$, therefore $M^{BC}, t \models \psi$: a contradiction. In the latter case, $M, s \not\models \bigvee\{K_c\delta_B(\delta_C(\psi)) \mid c \in B \cup C\}$. Hence, for all $c \in B \cup C$, $M, s \not\models K_c\delta_B(\delta_C(\psi))$. By Lemma 5.6, let $c \in B \cup C$ be such that for all $d \in B \cup C$, $[s]_c \subseteq [s]_d$. Thus, $M, s \not\models K_c\delta_B(\delta_C(\psi))$. Consequently, there exists $t \in W$ such that $s \sim_c t$ and $M, t \not\models \delta_B(\delta_C(\psi))$. Since for all $d \in B \cup C$, $[s]_c \subseteq [s]_d$, therefore $s \sim_{B \cup C} t$. Moreover, by induction hypothesis, $M^{BC}, t \not\models \psi$. Since $B \cap C \neq \emptyset$ and $s \sim_{B \cup C} t$, therefore $s \sim_C^B t$. Since $M^B, s \models \mathcal{I}_C\psi$, therefore $M^{BC}, t \models \psi$: a contradiction.

From right to left, suppose $M^B, s \not\models \mathcal{I}_C\psi$ and $M, s \models \delta_B(\mathcal{I}_C\psi)$. Now, there are two cases: either $B \cap C = \emptyset$, or $B \cap C \neq \emptyset$. In the former case, $M, s \models \bigvee\{K_c\delta_B(\delta_C(\psi)) \mid c \in C\}$. Hence, there exists $c \in C$ such that $M, s \models K_c\delta_B(\delta_C(\psi))$. Since $M^B, s \not\models \mathcal{I}_C\psi$, therefore there exists $t \in W$ such that $s \sim_C^B t$ and $M^{BC}, t \not\models \psi$. Thus, by induction hypothesis, $M, t \not\models \delta_B(\delta_C(\psi))$. Since $B \cap C = \emptyset$ and $s \sim_C^B t$, therefore $s \sim_C t$. Since $c \in C$, therefore $s \sim_c t$. Since $M, s \models K_c\delta_B(\delta_C(\psi))$, therefore $M, t \models \delta_B(\delta_C(\psi))$: a contradiction. In the latter case, $M, s \models \bigvee\{K_c\delta_B(\delta_C(\psi)) \mid c \in B \cup C\}$. Hence, there exists $c \in B \cup C$ such that $M, s \models K_c\delta_B(\delta_C(\psi))$. Since $M^B, s \not\models \mathcal{I}_C\psi$, therefore there exists $t \in W$ such that $s \sim_C^B t$ and $M^{BC}, t \not\models \psi$. Thus, by induction hypothesis, $M, t \not\models \delta_B(\delta_C(\psi))$. Since $B \cap C \neq \emptyset$ and $s \sim_C^B t$, therefore $s \sim_{B \cup C} t$. Since $c \in B \cup C$, therefore $s \sim_c t$. Since $M, s \models K_c\delta_B(\delta_C(\psi))$, therefore $M, t \models \delta_B(\delta_C(\psi))$: a contradiction. \square

Lemma 5.11 For all $\varphi \in \mathcal{L}_{\mathcal{I}}$, $|B| \geq 2$: $\models^{\text{agree}} \mathcal{I}_B\varphi \leftrightarrow \bigvee\{K_b\delta_B(\varphi) \mid b \in B\}$.

Proof. Let $\varphi \in \mathcal{L}_{\mathcal{I}}$. For the sake of the contradiction, suppose $|B| \geq 2$ and $\not\models^{\text{agree}} \mathcal{I}_B\varphi \leftrightarrow \bigvee\{K_b\delta_B(\varphi) \mid b \in B\}$. Hence, there exists a (M, s) with local agreement such that either $M, s \models \mathcal{I}_B\varphi$ and $M, s \not\models \bigvee\{K_b\delta_B(\varphi) \mid b \in B\}$, or $M, s \not\models \mathcal{I}_B\varphi$ and $M, s \models \bigvee\{K_b\delta_B(\varphi) \mid b \in B\}$, where $M = (W, \sim, V)$. In the former case, since (W, \sim) is a frame with local agreement and $B \neq \emptyset$, therefore there exists $a \in B$ such that for all $b \in B$, $[s]_a \subseteq [s]_b$. Since $M, s \not\models \bigvee\{K_b\delta_B(\varphi) \mid b \in B\}$, therefore $M, s \not\models K_a\delta_B(\varphi)$. Thus, there exists $t \in W$ such that $s \sim_a t$ and $M, t \not\models \delta_B(\varphi)$. Consequently, by Lemma 5.10, $M^B, t \not\models \varphi$. Since $s \sim_a t$ and for all $b \in B$, $[s]_a \subseteq [s]_b$, therefore $s \sim_B t$. Since $M^B, t \not\models \varphi$, therefore $M, s \not\models \mathcal{I}_B\varphi$: a contradiction. In the latter case, there exists $t \in W$ such that $s \sim_B t$ and $M^B, t \not\models \varphi$. Hence, by Lemma 5.10, $M, t \not\models \delta_B(\varphi)$. Moreover, there exists $a \in B$ such that $M, s \models K_a\delta_B(\varphi)$. Since $s \sim_B t$, therefore $s \sim_a t$. Since $M, s \models K_a\delta_B(\varphi)$, therefore $M, t \models \delta_B(\varphi)$: a contradiction.

contradiction. □

Proposition 5.12 *Let φ in $\mathcal{L}_{\mathcal{D}}$. Then $\models^{\text{agree}} \varphi \leftrightarrow \delta(\psi)$.*

Proof. By Lemmas 5.9 and 5.11. □

If $\varphi \in \mathcal{L}_{\mathcal{D}}$, then $\delta(\varphi) \in \mathcal{L}_{KU}$, and if $\varphi \in \mathcal{L}_{\mathcal{D}}$ without modality \mathcal{D}_\emptyset (the universal modality), then $\delta(\varphi) \in \mathcal{L}_K$. Consequently, every φ in the language of dynamic distributed knowledge without the universal modality is equivalent to a formula in the multi-agent language of individual knowledge.

We initially considered frames with local agreement in order to obtain a complete axiomatization of dynamic distributed knowledge ‘for an interesting fragment’. We ended up proving that this fragment can be translated into multi-agent individual knowledge. This seems to make it less interesting. However, local agreement might have worthwhile applications in modelling hierarchical knowledge relations between groups of agents.

6 On standard frames and semi-standard frames

Preparing the ground for Section 7 on the axiomatization, we generalize the epistemic frames introduced in Section 2, that we name *standard frames*, to the *semi-standard frames* wherein an equivalence relation \sim_B is associated with each $B \subseteq A$ and that satisfy the requirement that $\sim_C \subseteq \sim_B$ whenever $B \subseteq C$. It may then be that \sim_B is a proper subset of $\bigcap_{b \in B} \sim_b$ (the group knows more than the sum of its individuals). The completeness proofs involving distributed knowledge often involve ‘unravelling’ a canonical model based on a semi-standard frame into one that is based on a standard frame with the same information content [14,1]. We prove this result explicitly, as our proof applies to other frame classes (than those with equivalence relations), which may be of interest.

Definition 6.1 A frame (W, \sim) is *semi-standard* if for all $B, C \subseteq A$, if $B \subseteq C$ then $\sim_C \subseteq \sim_B$. It is *standard* if, also, for all $B, C \subseteq A$, $\sim_B \cap \sim_C \subseteq \sim_{B \cup C}$.

We note that the results of Section 5.1 translating $\mathcal{L}_{\mathcal{D}}$ into \mathcal{L}_D without loss of generality also hold for semi-standard frames: all clauses in the proof of crucial Lemma 5.4 showing that $M^B, u \models \varphi$ iff $M, u \models \tau_B(\varphi)$ equally apply to models based on semi-standard frames.

Definition 6.2 A *bounded morphism* from a frame (W', \sim') to a frame (W, \sim) is a function $\pi : W' \rightarrow W$ such that

(FC) for all $B \subseteq A$ and for all $s', t' \in W'$, if $s' \sim'_B t'$ then $\pi(s') \sim_B \pi(t')$,

(BC) for all $B \subseteq A$, for all $s' \in W'$ and for all $t \in W$, if $\pi(s') \sim_B t$ then there exists $t' \in W'$ such that $s' \sim'_B t'$ and $\pi(t') = t$.

A frame (W, \sim) is a *bounded morphic image* of a frame (W', \sim') if there exists a bounded morphism from (W', \sim') to (W, \sim) .

Since, in any frame, the accessibility relation associated to the empty set of agents is the universal relation on the frame, therefore in our setting every bounded morphism is a surjective function.

Lemma 6.3 *If the frame (W, \sim) is a bounded morphic image of the frame (W', \sim') then for all $\varphi \in \mathcal{L}_{DDR}$, if $(W', \sim') \models \varphi$ then $(W, \sim) \models \varphi$.*

Proof. Suppose the frame (W, \sim) is a bounded morphic image of the frame (W', \sim') . Let $\varphi \in \mathcal{L}_{DDR}$. Suppose $(W', \sim') \models \varphi$. Suppose $(W, \sim) \not\models \varphi$. Hence, by Cor. 5.5, $(W, \sim) \not\models \tau_\emptyset(\varphi)$. Since the frame (W, \sim) is a bounded morphic image of the frame (W', \sim') , therefore $(W', \sim') \not\models \tau_\emptyset(\varphi)$. See [10, Theorem 3.14] and [12, Cor. 3.16]. Thus, by Cor. 5.5, $(W', \sim') \not\models \varphi$: a contradiction. \square

Lemma 6.4 *Let (W, \sim) be a frame. Then (W, \sim) is semi-standard, iff for all $B, C \subseteq A$, $\sim_{B \cup C} \subseteq \sim_B \cap \sim_C$.*

Lemma 6.5 *Every semi-standard frame is the bounded morphic image of a standard frame.*

Proof. Let (W, \sim) be a semi-standard frame. For all $B \subseteq A$ and for all $u \in W$, let $[u]_B$ be the equivalence class of u modulo \sim_B . For all $X, Y \in \mathcal{P}(W)$, let $X + Y = (X \setminus Y) \cup (Y \setminus X)$. Notice that $(\mathcal{P}(W), \emptyset, W, +, \cap)$ is a Boolean ring. Let \mathcal{F} be the set (with typical elements denoted α, β , etc) of all functions of type $\mathcal{P}(A) \times A \rightarrow \mathcal{P}(W)$. Let $W' = W \times \mathcal{F}$. Let $\sim' : \mathcal{P}(A) \rightarrow \mathcal{P}(W' \times W')$ be the function such that for all $B \subseteq A$, \sim'_B is the binary relation on W' such that for all $(s, \alpha), (t, \beta) \in W'$, $(s, \alpha) \sim'_B (t, \beta)$ if and only if

$$(C_1) \text{ for all } E \subseteq A, [s]_E + \sum_{a \in E} \alpha(E, a) = [t]_E + \sum_{a \in E} \beta(E, a),$$

$$(C_2) \text{ for all } E \subseteq A \text{ and } a \in A, \text{ if } a \in B \text{ and } a \in E \text{ then } \alpha(E, a) = \beta(E, a).$$

The proof of the lemma is now established in Facts 6.6, 6.7, 6.8, and 6.9. \square

Fact 6.6 *The couple (W', \sim') is a frame.*

Fact 6.7 *The frame (W', \sim') is semi-standard.*

Proof. Let $B, B' \subseteq A$. Suppose $B \subseteq B'$. Suppose $\sim'_{B'} \not\subseteq \sim'_B$. Hence, there exist $(s, \alpha), (t, \beta) \in W'$ such that $(s, \alpha) \sim'_{B'} (t, \beta)$ and $(s, \alpha) \not\sim'_B (t, \beta)$. Thus, either there exists $E \subseteq A$ such that $[s]_E + \sum_{a \in E} \alpha(E, a) \neq [t]_E + \sum_{a \in E} \beta(E, a)$, or there exists $E \subseteq A$ and there exists $a \in A$ such that $a \in B$, $a \in E$ and $\alpha(E, a) \neq \beta(E, a)$. In the former case, since $(s, \alpha) \sim'_{B'} (t, \beta)$, therefore $[s]_E + \sum_{a \in E} \alpha(E, a) = [t]_E + \sum_{a \in E} \beta(E, a)$: a contradiction. In the latter case, since $B \subseteq B'$, therefore $a \in B'$. Since $(s, \alpha) \sim'_{B'} (t, \beta)$ and $a \in E$, therefore $\alpha(E, a) = \beta(E, a)$: a contradiction. \square

Fact 6.8 *The semi-standard frame (W', \sim') is standard.*

Proof. Let $B, B' \subseteq A$. Suppose $\sim'_{B \cup B'} \not\subseteq \sim'_B \cap \sim'_{B'}$. Hence, there exist $(s, \alpha), (t, \beta) \in W'$ such that $(s, \alpha) \sim'_{B \cup B'} (t, \beta)$, $(s, \alpha) \not\sim'_B (t, \beta)$ and $(s, \alpha) \sim'_{B'} (t, \beta)$. Thus, for all $F \subseteq A$, $[s]_F + \sum_{b \in F} \alpha(F, b) = [t]_F + \sum_{b \in F} \beta(F, b)$. Moreover, for all $F \subseteq A$ and for all $b \in A$, if $b \in B$ and $b \in F$ then $\alpha(F, b) = \beta(F, b)$ and for all $F \subseteq A$ and for all $b \in A$, if $b \in B'$ and $b \in F$ then $\alpha(F, b) = \beta(F, b)$. Since $(s, \alpha) \not\sim'_B (t, \beta)$, therefore there exists $E \subseteq A$ and there exists $a \in A$ such that $a \in B \cup B'$, $a \in E$ and $\alpha(E, a) \neq \beta(E, a)$. Consequently, either $a \in B$, or $a \in B'$. In the former case, since for all $F \subseteq A$ and for all $b \in A$, if $b \in B$ and $b \in F$ then $\alpha(F, b) = \beta(F, b)$, therefore

$\alpha(E, a) = \beta(E, a)$: a contradiction. In the latter case, since for all $F \subseteq A$ and for all $b \in A$, if $b \in B'$ and $b \in F$ then $\alpha(F, b) = \beta(F, b)$, therefore $\alpha(E, a) = \beta(E, a)$: a contradiction. \square

Let $\pi : W' \rightarrow W$ be the function such that for all $(s, \alpha) \in W'$, $\pi(s, \alpha) = s$.

Fact 6.9 π is a bounded morphism from (W', \sim') to (W, \sim) .

Proof. Let $B \subseteq A$ and $(s, \alpha), (t, \beta) \in W'$ be such that $(s, \alpha) \sim'_B (t, \beta)$. Hence, for all $E \subseteq A$, $[s]_E + \Sigma_{a \in E} \alpha(E, a) = [t]_E + \Sigma_{a \in E} \beta(E, a)$. Moreover, for all $E \subseteq A$ and for all $a \in A$, if $a \in B$ and $a \in E$ then $\alpha(E, a) = \beta(E, a)$. Thus, $[s]_B + \Sigma_{a \in B} \alpha(B, a) = [t]_B + \Sigma_{a \in B} \beta(B, a)$. Moreover, for all $a \in A$, if $a \in B$ then $\alpha(B, a) = \beta(B, a)$. Consequently, $[s]_B = [t]_B$. Hence, $s \sim_B t$.

Let $B \subseteq A$, $(s, \alpha) \in W'$ and $t \in W$ be such that $\pi(s, \alpha) \sim_B t$. Thus, $s \sim_B t$. Consequently, for all $E \subseteq A$, if $E \subseteq B$ then $s \sim_E t$. Hence, for all $E \subseteq A$, if $E \subseteq B$ then $[s]_E = [t]_E$. For all $E \subseteq A$, if $E \not\subseteq B$ then let $\zeta(E) \in A$ be such that $\zeta(E) \in E$ and $\zeta(E) \notin B$. Let β be the function of type $\mathcal{P}(A) \times A \rightarrow \mathcal{P}(W)$ such that for all $E \subseteq A$ and for all $a \in A$,

- if $a \in E$ and $a \in B$ then $\beta(E, a) = \alpha(E, a)$,
- elsif $a \in E$, $a \notin B$ and $a = \zeta(E)$ then $\beta(E, a) = \Sigma_{b \in E \setminus B} \alpha(E, b)$,
- else $\beta(E, a) = \emptyset$.

The reader may easily verify that $(s, \alpha) \sim'_B (t, \beta)$. \square

Proposition 6.10 For all $\varphi \in \mathcal{L}_{\mathcal{D}R}$, if φ is valid on standard frames then φ is valid on semi-standard frames.

Proof. By Lemmas 6.3 and 6.5. \square

The same proof technique can be used for frames with other than equivalence relations \sim_a , and thus used for extensions with distributed knowledge of other logics than S5, such as KD45 (consistent belief) and KB4 (partial equivalence relations).

7 Axiomatization

7.1 Introduction

There are multiple roads towards determining whether a formula in $\psi \in \mathcal{L}_{\mathcal{D}}$ is semantically equivalent to a formula ψ' in some other language, and that is a theorem in an axiomatization for the logic of that language. All these inspire the quest to find an axiomatization of $L_{\mathcal{D}}$. (i) Replace in ψ all \mathcal{D}_B by $R_B \mathcal{D}_B$ and determine whether the resulting ψ'' is a theorem in the axiomatization of the logic of [1], which proceeds by rewriting $\psi'' \in \mathcal{L}_{\mathcal{D}R}$ into a $\psi' \in \mathcal{L}_{\mathcal{D}}$, and then determine whether ψ' is a theorem of the logic of distributed knowledge [14]. We considered axiomatizing the fragment of [1] with packed modalities $R_B \mathcal{D}_B \varphi$ [36]. (ii) Use the translation $\tau_{\emptyset} : \mathcal{L}_{\mathcal{D}} \rightarrow \mathcal{L}_{\mathcal{D}}$ of Section 5 (Def. 5.1) and determine whether $\psi' = \tau_{\emptyset}(\psi)$ is a theorem in $L_{\mathcal{D}}$ [14]. We considered axiomatizing the clauses τ_B of that translation. (iii) Axiomatize the logic for special frame classes: use the translations δ_B of Section 5 and Prop. 5.12 for

frames with local agreement. Thus we obtain a $\psi' \in \mathcal{L}_{KU}$, or even in \mathcal{L}_K if ψ did not contain \mathcal{D}_\emptyset (the universal modality), which can be determined a theorem in the logic (multi-)S5. This frame restriction trivializes $L_{\mathcal{D}}$.

None of these roads, despite looking hard, provided us with an axiomatization of $L_{\mathcal{D}}$. The ‘hybrid’ nature of \mathcal{D}_B that is simultaneously statically and dynamically interpreted is a complication. A similar case seems the *logic of knowability* [35] with quantifiers over announcements but no announcement modalities, for which no axiomatization is known, unlike for APAL [6].

Instead of a complete axiomatization for the logic $L_{\mathcal{D}}$ of dynamic distributed knowledge we therefore present a complete axiomatization for the logic $L_{\mathcal{D}R}$ of dynamic distributed knowledge *and resolution*. Our results may compare to the axiomatizations in [1] for the logic $L_{\mathcal{D}R}$, and to a variant of that in [8].

7.2 The logic of dynamic distributed knowledge and resolution

Definition 7.1 Let $\mathbf{L}_{\mathcal{D}R}$ be the least set of $\mathcal{L}_{\mathcal{D}R}$ -formulas produced by the below, where in (IR2): $\text{see}_{E_1}(\vec{B}_1) \subseteq \text{see}_C(\vec{B})$, \dots , $\text{see}_{E_k}(\vec{B}_k) \subseteq \text{see}_C(\vec{B})$.

- | | | | |
|------|---|-------|--|
| (F0) | all propositional tautologies | (F5) | $R_B p \leftrightarrow p$ |
| (F1) | $R_B(\varphi \rightarrow \psi) \rightarrow (R_B \varphi \rightarrow R_B \psi)$ | (F6) | $\neg R_B \perp$ |
| (F2) | $\mathcal{D}_B \varphi \rightarrow R_B \varphi$ | (F7) | $R_B \neg \varphi \leftrightarrow \neg R_B \varphi$ |
| (F3) | $\mathcal{D}_B \varphi \rightarrow R_B \mathcal{D}_B \varphi$ | (IR0) | from $\varphi \rightarrow \psi$ and φ infer ψ |
| (F4) | $\neg \mathcal{D}_B \varphi \rightarrow \mathcal{D}_B \neg \mathcal{D}_B \varphi$ | (IR1) | from φ infer $R_B \varphi$ |

$$(IR2) \text{ from } R_{\vec{B}_1} R_{E_1} \varphi_1 \wedge \dots \wedge R_{\vec{B}_k} R_{E_k} \varphi_k \rightarrow R_{\vec{B}} R_C \psi \text{ infer } \\ R_{\vec{B}_1} \mathcal{D}_{E_1} \varphi_1 \wedge \dots \wedge R_{\vec{B}_k} \mathcal{D}_{E_k} \varphi_k \rightarrow R_{\vec{B}} \mathcal{D}_C \psi$$

Note that rather few axioms and rules of $\mathbf{L}_{\mathcal{D}R}$ involve dynamic distributed knowledge \mathcal{D}_B , namely (F2), (F3), (F4), and (IR2), and that rule (IR2) is very involved: there is an instantiation of (IR2) for every $k \in \mathbb{N}$. This suggests a complete axiomatization for $L_{\mathcal{D}}$ might also need a complex derivation rule.

Lemma 7.2 *Derivable in $\mathbf{L}_{\mathcal{D}R}$ are $\mathcal{D}_B(\varphi \rightarrow \psi) \rightarrow (\mathcal{D}_B \varphi \rightarrow \mathcal{D}_B \psi)$, $R_B(\varphi \vee \psi) \leftrightarrow (R_B \varphi \vee R_B \psi)$, $R_B(\varphi \wedge \psi) \leftrightarrow (R_B \varphi \wedge R_B \psi)$, $\neg \mathcal{D}_B \perp$, and ‘ φ impl. $\mathcal{D}_B \varphi$ ’.*

Proof. This is shown by using (F1), (F6), (F7), (IR1) and (IR2). \square

Interestingly, we were not able to find a derivation of the validity $\mathcal{D}_B \varphi \rightarrow \mathcal{D}_B \mathcal{D}_B \varphi$ in this axiomatization $\mathbf{L}_{\mathcal{D}R}$. It is derivable as the axiomatization will be shown complete.

Lemma 7.3 *For all $\varphi \in \mathcal{L}_{\mathcal{D}R}$, if $\varphi \in \mathbf{L}_{\mathcal{D}R}$ then φ is valid on standard frames.*

Proof. It suffices to demonstrate that the $\mathcal{L}_{\mathcal{D}R}$ -formulas (F1)–(F7) are valid on standard frames and the inference rules (IR1) and (IR2) preserve validity on standard frames. We only show the case of the inference rule (IR2).

Suppose the inference rule (IR2) does not preserve validity on standard frames. Hence, there exists $k \in \mathbb{N}$, there exist $\vec{B}_1, \dots, \vec{B}_k \in \mathcal{P}(A)^*$, there exist $E_1, \dots, E_k \subseteq A$, there exists $\vec{B} \in \mathcal{P}(A)^*$ and there exists $C \subseteq A$ such that

$\text{see}_{E_1}(\vec{B}_1) \subseteq \text{see}_C(\vec{B}), \dots, \text{see}_{E_k}(\vec{B}_k) \subseteq \text{see}_C(\vec{B})$ and there exists $\varphi_1, \dots, \varphi_k \in \mathcal{L}_{DR}$ and there exists $\psi \in \mathcal{L}_{DR}$ such that $R_{\vec{B}_1}R_{E_1}\varphi_1 \wedge \dots \wedge R_{\vec{B}_k}R_{E_k}\varphi_k \rightarrow R_{\vec{B}}R_C\psi$ is valid on standard frames and $R_{\vec{B}_1}\mathcal{D}_{E_1}\varphi_1 \wedge \dots \wedge R_{\vec{B}_k}\mathcal{D}_{E_k}\varphi_k \rightarrow R_{\vec{B}}\mathcal{D}_C\psi$ is not valid on standard frames. Thus, there exists a standard model $M = (W, \sim, V)$ and $u \in W$ such that $M, u \models R_{\vec{B}_1}\mathcal{D}_{E_1}\varphi_1, \dots, M, u \models R_{\vec{B}_k}\mathcal{D}_{E_k}\varphi_k$ and $M, u \not\models R_{\vec{B}}\mathcal{D}_C\psi$. Consequently, there exists $v \in W$ such that $u \sim_{\vec{B}} v$ and $M^{\vec{B}C}, v \not\models \psi$. Hence, by Lemma 2.3, $u \sim_{\text{see}(\vec{B}, C)} v$. Moreover, $M, v \not\models R_{\vec{B}}R_C\psi$. Since $R_{\vec{B}_1}R_{E_1}\varphi_1 \wedge \dots \wedge R_{\vec{B}_k}R_{E_k}\varphi_k \rightarrow R_{\vec{B}}R_C\psi$ is valid on standard frames, therefore there exists $i \in (k)$ such that $M, v \not\models R_{\vec{B}_i}R_{E_i}\varphi_i$. Thus, $M^{\vec{B}_i E_i}, v \not\models \varphi_i$. Since $\text{see}_{E_i}(\vec{B}_i) \subseteq \text{see}_C(\vec{B})$, from $u \sim_{\text{see}_C(\vec{B})} v$ we obtain $u \sim_{\text{see}_{E_i}(\vec{B}_i)} v$. Consequently, by Lemma 2.3, $u \sim_{E_i}^{\vec{B}_i} v$. Since $M^{\vec{B}_i E_i}, v \not\models \varphi_i$, therefore $M, u \not\models R_{\vec{B}_i}\mathcal{D}_{E_i}\varphi_i$: a contradiction. \square

7.3 Completeness

We now proceed to show the completeness of the logic \mathbf{L}_{DR} .

A set s of \mathcal{L}_{DR} -formulas is *\mathbf{L}_{DR} -consistent* if for all $n \in \mathbb{N}$ and for all $\varphi_1, \dots, \varphi_n \in s$, $\neg(\varphi_1 \wedge \dots \wedge \varphi_n) \notin \mathbf{L}_{DR}$.

Lemma 7.4 *\mathbf{L}_{DR} is an \mathbf{L}_{DR} -consistent set of \mathcal{L}_{DR} -formulas.*

Proof. By Lemma 7.3. \square

An \mathbf{L}_{DR} -consistent set s of \mathcal{L}_{DR} -formulas is *maximal* if for all \mathbf{L}_{DR} -consistent sets t of \mathcal{L}_{DR} -formulas, if $s \subseteq t$ then $s = t$.

Lemma 7.5 *For all \mathbf{L}_{DR} -consistent sets s of \mathcal{L}_{DR} -formulas, there exists a maximal \mathbf{L}_{DR} -consistent set t of \mathcal{L}_{DR} -formulas such that $s \subseteq t$.*

Proof. See [10, Lemma 4.17] and [12, Lemma 5.1] for details. \square

For convenience of the exposition, from here on we write \mathbf{L} instead of \mathbf{L}_{DR} in this section.

Let $W_{\mathbf{L}}$ be the set of all maximal \mathbf{L} -consistent sets of all \mathcal{L}_{DR} -formulas. Let $\sim_{\mathbf{L}}: \mathcal{P}(A) \rightarrow \mathcal{P}(W_{\mathbf{L}} \times W_{\mathbf{L}})$ be the function such that for all $C \subseteq A$, $\sim_{\mathbf{L}, C}$ is the binary relation on $W_{\mathbf{L}}$ such that for all $s, t \in W_{\mathbf{L}}$, $s \sim_{\mathbf{L}, C} t$ if and only if for all $\vec{B} \in \mathcal{P}(A)^*$ and for all $E \subseteq A$, if $\text{see}_E(\vec{B}) \subseteq C$ then for all $\varphi \in \mathcal{L}_{DR}$, if $R_{\vec{B}}\mathcal{D}_E\varphi \in s$ then $R_{\vec{B}}R_E\varphi \in t$.

Lemma 7.6 *The pair $(W_{\mathbf{L}}, \sim_{\mathbf{L}})$ is a frame.*

Proof. By Lemmas 7.4 and 7.5, $W_{\mathbf{L}}$ is nonempty. Hence, it suffices to demonstrate that for all $C \subseteq A$, $\sim_{\mathbf{L}, C}$ is reflexive and Euclidean (and therefore an equivalence relation). Let $C \subseteq A$.

Suppose $\sim_{\mathbf{L}, C}$ is not reflexive. Thus, there exists $s \in W_{\mathbf{L}}$ such that $s \not\sim_{\mathbf{L}, C} s$. Consequently, there are $\vec{B} \in \mathcal{P}(A)^*$ and $E \subseteq A$ such that $\text{see}_E(\vec{B}) \subseteq C$ and there is $\varphi \in \mathcal{L}_{DR}$ such that $R_{\vec{B}}\mathcal{D}_E\varphi \in s$ and $R_{\vec{B}}R_E\varphi \notin s$. Hence, by using (F1), (F2) and (IR1), $R_{\vec{B}}R_E\varphi \in s$: a contradiction.

Suppose $\sim_{\mathbf{L}, C}$ is not Euclidean. Thus, there exist $s, t, u \in W_{\mathbf{L}}$ such that $s \sim_{\mathbf{L}, C} t$, $s \sim_{\mathbf{L}, C} u$ and $t \not\sim_{\mathbf{L}, C} u$. Consequently, there are $\vec{B} \in \mathcal{P}(A)^*$

and $E \subseteq A$ such that $\text{see}_E(\vec{B}) \subseteq C$ and there exists $\varphi \in \mathcal{L}_{DR}$ such that $R_{\vec{B}}\mathcal{D}_E\varphi \in t$ and $R_{\vec{B}}R_E\varphi \notin u$. Since $s \sim_{L,C} u$ and $\text{see}_E(\vec{B}) \subseteq C$, therefore $R_{\vec{B}}\mathcal{D}_E\varphi \notin s$. Hence, by using (F1), (F7) and (IR1), $R_{\vec{B}}\neg\mathcal{D}_E\varphi \in s$. Thus, by using (F1), (F4) and (IR1), $R_{\vec{B}}\mathcal{D}_E\neg\mathcal{D}_E\varphi \in s$. Since $s \sim_{L,C} t$ and $\text{see}_E(\vec{B}) \subseteq C$, therefore $R_{\vec{B}}R_E\neg\mathcal{D}_E\varphi \in t$. Consequently, by using (F1), (F7) and (IR1), $R_{\vec{B}}R_E\mathcal{D}_E\varphi \notin t$. Hence, by using (F1), (F3) and (IR1), $R_{\vec{B}}\mathcal{D}_E\varphi \notin t$: a contradiction. \square

Lemma 7.7 *The frame (W_L, \sim_L) is semi-standard.*

Proof. Let $C, C' \subseteq A$. Suppose $C \subseteq C'$. Suppose $\sim_{L,C'} \not\subseteq \sim_{L,C}$. Hence, there exists $s, t \in W_L$ such that $s \sim_{L,C'} t$ and $s \not\sim_{L,C} t$. Thus, there exists $\vec{B} \in \mathcal{P}(A)^*$ and there exists $E \subseteq A$ such that $\text{see}_E(\vec{B}) \subseteq C$ and there exists $\varphi \in \mathcal{L}_{DR}$ such that $R_{\vec{B}}\mathcal{D}_E\varphi \in s$ and $R_{\vec{B}}R_E\varphi \notin t$. Since $C \subseteq C'$, therefore $\text{see}_E(\vec{B}) \subseteq C'$. Since $s \sim_{L,C'} t$ and $R_{\vec{B}}\mathcal{D}_E\varphi \in s$, therefore $R_{\vec{B}}R_E\varphi \in t$: a contradiction. \square

Definition 7.8 The semi-standard frame (W_L, \sim_L) is called *canonical frame of L* . The *canonical valuation of L* is the function V_L associating to each atom p the set $V_L(p)$ of all $s \in W_L$ such that $p \in s$. The triple $M_L = (W_L, \sim_L, V_L)$ is called *canonical model of L* .

Lemma 7.9 *Let $\varphi \in \mathcal{L}_{DR}$. For all $s \in W_L$ and for all $\vec{B} \in \mathcal{P}(A)^*$, $M_L^{\vec{B}}, s \models \varphi$ if and only if $R_{\vec{B}}\varphi \in s$.*

Proof. By induction on φ . We only show the case $\mathcal{D}_C\psi$.

From right to left, suppose $M_L^{\vec{B}}, s \not\models \mathcal{D}_C\psi$ and $R_{\vec{B}}\mathcal{D}_C\psi \in s$. Hence, there exists $t \in W_L$ such that $s \sim_{L,C}^{\vec{B}} t$ and $M_L^{\vec{B}C}, t \not\models \psi$. Thus, by Lemma 2.3, $s \sim_{L, \text{see}_C(\vec{B})} t$. Moreover, by induction hypothesis, $R_{\vec{B}}R_C\psi \notin t$. Since $R_{\vec{B}}\mathcal{D}_C\psi \in s$, therefore $R_{\vec{B}}R_C\psi \in t$: a contradiction.

From left to right, suppose $M_L^{\vec{B}}, s \models \mathcal{D}_C\psi$ and $R_{\vec{B}}\mathcal{D}_C\psi \notin s$. Let $t_0 = \{R_{\vec{B}'}R_{E'}\chi \mid \vec{B}' \in \mathcal{P}(A)^* \text{ and } E' \subseteq A \text{ are such that } \text{see}_{E'}(\vec{B}') \subseteq \text{see}_C(\vec{B}) \text{ and } \chi \in \mathcal{L}_{DR} \text{ is such that } R_{\vec{B}'}\mathcal{D}_{E'}\chi \in s\} \cup \{\neg R_{\vec{B}}R_C\psi\}$.

We first observe that t_0 is an L -consistent set of \mathcal{L}_{DR} -formulas. This can be shown as follows. Suppose t_0 is not an L -consistent set of \mathcal{L}_{DR} -formulas. Hence, there exist $k \in \mathbb{N}$, $\vec{B}'_1, \dots, \vec{B}'_k \in \mathcal{P}(A)^*$ and $E'_1, \dots, E'_k \subseteq A$ such that $\text{see}_{E'_1}(\vec{B}'_1) \subseteq \text{see}_E(\vec{B})$, \dots , $\text{see}_{E'_k}(\vec{B}'_k) \subseteq \text{see}_E(\vec{B})$, and there exist $\chi_1, \dots, \chi_k \in \mathcal{L}_{DR}$ such that $R_{\vec{B}'_1}\mathcal{D}_{E'_1}\chi_1, \dots, R_{\vec{B}'_k}\mathcal{D}_{E'_k}\chi_k \in s$ and $R_{\vec{B}'_1}R_{E'_1}\chi_1 \wedge \dots \wedge R_{\vec{B}'_k}R_{E'_k}\chi_k \rightarrow R_{\vec{B}}R_C\psi \in L$. Thus, by using (IR2), $R_{\vec{B}'_1}\mathcal{D}_{E'_1}\chi_1 \wedge \dots \wedge R_{\vec{B}'_k}\mathcal{D}_{E'_k}\chi_k \rightarrow R_{\vec{B}}\mathcal{D}_C\psi \in L$. Since $R_{\vec{B}'_1}\mathcal{D}_{E'_1}\chi_1, \dots, R_{\vec{B}'_k}\mathcal{D}_{E'_k}\chi_k \in s$, therefore $R_{\vec{B}}\mathcal{D}_C\psi \in s$: a contradiction.

Now using that t_0 is an L -consistent set of \mathcal{L}_{DR} -formulas, by Lemma 7.5, there exists $t \in W_L$ such that $t_0 \subseteq t$. Hence, for all $\vec{B}' \in \mathcal{P}(A)^*$ and $E' \subseteq A$, if $\text{see}_{E'}(\vec{B}') \subseteq \text{see}_C(\vec{B})$ then for all $\chi \in \mathcal{L}_{DR}$, if $R_{\vec{B}'}\mathcal{D}_{E'}\chi \in s$ then $R_{\vec{B}'}R_{E'}\chi \in t$. Moreover, $\neg R_{\vec{B}}R_C\psi \in t$. Thus, $s \sim_{\text{see}_C(\vec{B})} t$. Also, $R_{\vec{B}}R_C\psi \notin t$. Consequently, by Lemma 2.3, $s \sim_C^{\vec{B}} t$. Since $M_L^{\vec{B}}, s \models \mathcal{D}_C\psi$, therefore $M_L^{\vec{B}C}, t \models \psi$. Hence,

by induction hypothesis, $R_{\bar{B}}R_C\psi \in t$: a contradiction. \square

Lemma 7.10 *For all $\varphi \in \mathcal{L}_{DR}$, φ is true in $M_{\mathbf{L}}$ if and only if $\varphi \in \mathbf{L}$.*

Proof. By Lemma 7.9. \square

Lemma 7.11 *For all $\varphi \in \mathcal{L}_{DR}$, if φ is valid in the class of all semi-standard frames then $\varphi \in \mathbf{L}$.*

Proof. By Lemmas 7.7 and 7.10. \square

Theorem 7.12 *For all $\varphi \in \mathcal{L}_{DR}$: (i) $\varphi \in \mathbf{L}$, iff (ii) φ is valid on standard frames, iff (iii) φ is valid on semi-standard frames.*

Proof. By Lemmas 6.10, 7.3 and 7.11. \square

8 Conclusions and further research

Conclusions. We proposed a logic of *dynamic distributed knowledge* with modalities combining static and dynamic features. We considered its validities, gave translations to static distributed knowledge, showed its decidability, proposed and characterized a notion of shared bisimulation, and gave a complete axiomatization for the extension with resolution.

Further research Interpreting dynamic distributed knowledge on Kripke models with other frame properties than equivalence relations, extending S5, is worth pursuing. Dynamic distributed knowledge based on K4 (*transitive* models) is investigated in [13]. Static distributed knowledge based on KB4 (models with *partial equivalence relations*) is investigated in distributed computing [17] to model *crashed* (or *dead*) agents, or *faulty* agents [32]. A crashed or faulty agent satisfies $K_a \perp$. We also consider extending the language $\mathcal{L}_{\mathcal{D}}$ with *common knowledge*. It is easy to see that $\mathcal{D}_B\varphi \leftrightarrow \mathcal{D}_B C_B\varphi$ is valid. We conjecture this is the only interaction axiom needed in order to get a complete axiomatization, extending a presumed one for \mathcal{D} and the standard one for C [34]. Finally we consider incorporating *relativized common knowledge* [24,30], (dynamic) *common distributed knowledge* [8,33], semi-standard Kripke models ($\sim_C \subseteq \sim_B$ for $B \subseteq C$) [7,17], and applications in game theory and economics [13].

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