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Fading regularization method for an inverse boundary value problem associated with the biharmonic equation

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ABSTRACT

In this paper, we propose a numerical algorithm that combines the fading regularization method with the method of fundamental solutions (MFS) to solve a Cauchy problem associated with the biharmonic equation. We introduce a new stopping criterion for the iterative process and compare its performance with previous criteria. Numerical simulations using MFS validate the accuracy of this stopping criterion for both compatible and noisy data and demonstrate the convergence, stability, and efficiency of the proposed algorithm, as well as its ability to deblur noisy data.

1. Introduction

The biharmonic equation is a fundamental mathematical equation widely used in the fields of physics and engineering. It finds its applications in fluid mechanics for Stokes flows, where it governs the two-dimensional interior flow of viscous fluids at low Reynolds number, and in elasticity theory. According to Kirchhoff plate theory [1], the biharmonic equation governs the bending of thin plates subjected to load forces.

Kirchhoff's assumptions lead to the expression of the transverse displacement of thin plates as the solution of a boundary value problem, consisting of a biharmonic-type equation and two boundary conditions expressed at every point on the boundary. However, in practical situations, it may not always be possible to specify boundary conditions at all points on the boundary.

This has led to many researchers considering Cauchy-type problems associated with the biharmonic equation, where a part of the boundary is over-specified by prescribing all the boundary conditions, while another part of the boundary is under-specified, where no boundary condition is available. Such problems are called ill-posed in the sense of Hadamard [2], since the uniqueness or stability of the solution is no longer guaranteed, referred to as inverse problems.

Holmgren's theorem, which was published in 1901 [3], guarantees that there is a unique solution for Cauchy boundary value problems, particularly when the boundary data is compatible. Nonetheless, the stability of the solution in relation to the boundary conditions on the accessible part of the boundary is still problematic. In these situations, standard direct methods for partial differential equations are unsuitable for solving the resulting Cauchy inverse problem. To obtain a stable solution with respect to the data, regularization techniques must be employed.

Cannon and Cecchi [4] numerically solved a similar problem where no information is given on the remaining part of the boundary using the least square method and mathematical programming techniques. Zeb et al. [5] and Lesnic et al. [6] solved the biharmonic inverse problem in the case of slow viscous incompressible flows. It consists of determining under-specified boundary conditions

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in terms of normal fluid velocity using additional pressure measurements on another part of the boundary. Atakhodzhaev [7] used interior measurements inside the domain to determine unspecified boundary values. The uniqueness of the solution has been proven, but no numerical results are given, and the ill-posedness of the problem persists, since the solution is not stable when the input data have small perturbations. Later, Lesnic et al. [8] introduced a stable numerical method for obtaining this solution, using Tikhonov regularization combined with a boundary element method (BEM).

Marin and Lesnic [9] addressed the problem where no boundary conditions are specified on a part of the boundary. They applied the method of fundamental solutions (MFS) and combined it with Tikhonov regularization to overcome the ill-conditioned system.

The Tikhonov regularization method was developed by Tikhonov [10] to resolve ill-posed problems by introducing a control term with a regularization parameter to generate an approximately well-posed problem. Although this method has been widely applied to various elliptic inverse Cauchy problems to ensure stability, the solution depends on the choice of the regularization parameter.

There are other methods besides the Tikhonov method for solving inverse problems related to elliptic PDEs. For instance, Andrieux et al. [11,12] proposed the energy-like error functional, which has been used to solve Cauchy problems for a stationary thermal field [11,12] and in elasticity theory [13,14]. This method separates the ill-posed Cauchy problem into two well-posed problems and minimizes the gap between the two separate fields. It has been combined with the Tikhonov method to overcome instability due to noisy data [15]. Optimal control techniques [16] and quasi-reversibility methods [17,18] are other methods used to solve elliptic inverse Cauchy problems.

The fading regularization method, introduced in [19,20], is an iterative regularization technique that stabilizes the solution and reduces dependence on the regularization parameter. This method involves finding the best-fitting solution among all solutions of the equilibrium equation that match the over-specified data on the accessible part of the boundary. The Cauchy problem is reduced to a sequence of well-posed optimization problems, and the functional of each problem comprises two terms: a relaxation term representing the gap between the optimal element and the over-specified boundary data, and a regularization term acting on the entire boundary, which controls the gap between the optimal element and the one obtained at the previous iteration.

The fading regularization technique has been successfully applied, in 2D situations, to Cauchy problems associated with various equations, such as the Laplace equation [20–23], the anisotropic heat conduction [24,25], the biharmonic equation [26], the linear elasticity system [27–29] and the Stokes system [30], and in both 2D and 3D situations for the Helmholtz equation [31,32]. It has been implemented using different numerical techniques like the finite element method (FEM) [20,21,26–28], the boundary element method (BEM) [21–24] and the method of fundamental solutions (MFS) [25,29–32].

The MFS is a meshless numerical method that was introduced by Kupradze and Aleksidze [33]. This method is commonly used for solving direct and inverse boundary value problems due to its simple implementation and low computational cost. However, even in direct problems, MFS typically produces an ill-conditioned system.

In this work, we propose a novel approach for solving Cauchy-type biharmonic inverse problems in which no boundary conditions are given on a part of the boundary, by combining the fading regularization method with the method of fundamental solutions (MFS). The proposed method aims to obtain a stable and accurate numerical solution by iteratively minimizing a functional composed of a relaxation term and a regularization term. To ensure convergence of the iterative process, we introduce a new stopping criterion and investigate its performance in comparison with previous criteria. Our approach is tested on reconstructing inaccessible information on a part of the boundary and the numerical results prove its effectiveness, efficiency and stability.

2. The Cauchy problem for the biharmonic equation

Let Ω be an open bounded domain of \mathbb{R}^2 , with boundary $\Gamma = \Gamma_d \cup \Gamma_i$, where $\Gamma_d, \Gamma_i \neq \emptyset$ and $\Gamma_d \cap \Gamma_i = \emptyset$. We consider the function u that satisfies the biharmonic equation in the domain Ω , namely

$$\Delta^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{1}$$

or, equivalently introducing an auxiliary function v , the system of equations

$$\Delta u(\mathbf{x}) = v(\mathbf{x}), \quad \Delta v(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega \tag{2}$$

Let $u_{,n}$ and $v_{,n}$ be the normal derivatives of u and v respectively (i.e. $u_{,n}(\mathbf{x}) \equiv \partial_n u(\mathbf{x})$ and $v_{,n}(\mathbf{x}) \equiv \partial_n v(\mathbf{x})$ at $\mathbf{x} \in \Gamma$). In the case of under-specified boundaries, where the solution u , v and their normal derivatives are unknown on some part Γ_i of the boundary Γ , it is well known that the solution of (1) or (2) cannot be determined easily. Moreover, when the remaining part of the boundary Γ_d is over-specified by having all of these four quantities prescribed on it, such that:

$$u(\mathbf{x}) = \varphi_d(\mathbf{x}), \quad u_{,n}(\mathbf{x}) = \psi_d(\mathbf{x}), \quad v(\mathbf{x}) = \mu_d(\mathbf{x}), \quad v_{,n}(\mathbf{x}) = \phi_d(\mathbf{x}), \quad \mathbf{x} \in \Gamma_d \tag{3}$$

the solution of the resulting problem is then unstable.

The problem consisting of the equilibrium equation (1) and the boundary conditions (3) is a Cauchy problem and can be interpreted as a dimensionless form of a mechanical bending problem where a thin plate is subjected to forces and moments acting on its boundary. According to Timoshenko and Woinowsky-Krieger [34], the vertical displacement $u(\mathbf{x})$ is a solution of the biharmonic equation (1) and the boundary conditions (3) correspond to the vertical displacement $u(\mathbf{x})$, the normal rotation $u_{,n}(\mathbf{x})$, a function $v(\mathbf{x})$ depending on the bending moments and a function $v_{,n}(\mathbf{x})$ depending on the shear forces and the bending moments. A practical example is obtained if we consider the boundary part Γ_d as a free edge boundary. In this situation, the bending moments and shear

forces are zero, implying in (3) that $\mu_d(\mathbf{x}) = 0$ and $\phi_d(\mathbf{x}) = 0$ on Γ_d . It is then only necessary to measure the vertical displacement $\varphi_d(\mathbf{x})$ and the normal rotation $\psi_d(\mathbf{x})$ on Γ_d .

This Cauchy problem is an inverse boundary value problem for the biharmonic equation. It is ill-posed, in the sense of Hadamard, because even if a solution exists, it is unstable with respect to small perturbations in the boundary data on Γ_d . Regularization methods can be used in order to overcome this ill-posedness.

Several approaches can be used for solving Cauchy problem for the biharmonic equation. The first approach, tested in [26], is to split the problem into a sequential resolutions of a Cauchy problem for the Laplace’s equation and of a Cauchy problem for the Poisson’s equation. A disadvantage of this technique is that the solution v of the Cauchy problem for the Laplace’s equation corresponds to the second member of the Poisson’s equation. Due to the ill-posedness of the Cauchy problem for the Laplace’s equation, the auxiliary function v could be inaccurate and affects the resolution of the whole problem. A second technique is to solve simultaneously, in a coupled system, the Cauchy problem for Laplace’s equation and the Cauchy problem for Poisson’s equation. This technique is tested in [26] and compared to the first approach. This second technique is more accurate than the previous one.

In this paper, we use another technique, used for example in [9], which presents the advantage to lead to the resolution of Cauchy problem for the biharmonic equation by a unique system and not a sequence of Cauchy problems nor a coupled system. Moreover, we use the Method of Fundamental Solutions in conjunction with the fading regularization technique to overcome difficulties encountered in [9] with the use of Tikhonov regularization, particularly concerning the choice of the regularization parameter.

3. The fading regularization method for the Cauchy problem

3.1. Continuous formulation

The application of the fading regularization method to the Cauchy problem associated with the biharmonic equation, may require a particular attention in the choice of the functional spaces in which the solution will be sought in order to ensure the convergence of the method. Let $\mathcal{H}_0^2(\Omega)$ be the space of solutions of the biharmonic equation defined by:

$$\mathcal{H}_0^2(\Omega) = \{u \in H^2(\Omega) \ / \ \Delta^2 u = 0\},$$

which is a closed subspace of

$$\mathcal{H}^2(\Omega) = \{u \in H^2(\Omega) \ / \ \Delta^2 u \in L^2(\Omega)\}$$

which is also a Hilbert space for the norm $\|\cdot\|_{\mathcal{H}^2(\Omega)}$ defined by

$$\|u\|_{\mathcal{H}^2(\Omega)}^2 = \|u\|_{H^2(\Omega)}^2 + \|\Delta^2 u\|_{L^2(\Omega)}^2.$$

We further define the space of trace operators:

$$X(\Gamma) = H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$$

with the scalar product $\langle \cdot, \cdot \rangle_{X(\Gamma)}$ defined for all $\Phi = (\varphi, \psi, \mu, \phi)$ et $\Phi' = (\varphi', \psi', \mu', \phi')$ in $X(\Gamma)$ such that:

$$\langle \Phi, \Phi' \rangle_{X(\Gamma)} = \langle \varphi, \varphi' \rangle_{H^{3/2}(\Gamma)} + \langle \psi, \psi' \rangle_{H^{1/2}(\Gamma)} + \langle \mu, \mu' \rangle_{H^{-1/2}(\Gamma)} + \langle \phi, \phi' \rangle_{H^{-3/2}(\Gamma)},$$

and its associated norm is denoted by $\|\cdot\|_{X(\Gamma)}$. We can thus define the space $H(\Gamma)$ of compatible traces (i.e. the space of traces which corresponds to the solutions of the problem):

$$H(\Gamma) = \{\Phi = (\varphi, \psi, \mu, \phi) \in X(\Gamma) \text{ such there exists } u \in \mathcal{H}_0^2(\Omega), v = \Delta u \text{ in } \Omega \text{ and } (u, u_{,n}, v, v_{,n}) = (\varphi, \psi, \mu, \phi)\}.$$

It is therefore immediate that $H(\Gamma)$ is a Hilbert space as a closed subspace of $X(\Gamma)$. We denote by $\|\cdot\|_{H(\Gamma)}$ its induced norm and we denote by $X(\Gamma_d)$ and $H(\Gamma_d)$ the restrictions of the spaces $X(\Gamma)$ and $H(\Gamma)$ respectively on the boundary part Γ_d and by $\|\cdot\|_{X(\Gamma_d)}$ and $\|\cdot\|_{H(\Gamma_d)}$ their associated norms.

The definition of these functional spaces makes possible to formulate an equivalent problem to the Cauchy problem defined by the equilibrium equation (1) and the boundary conditions (3):

$$\begin{cases} \text{Find } \mathbf{U} = (u|_{\Gamma}, u_{,n}|_{\Gamma}, v|_{\Gamma}, v_{,n}|_{\Gamma}) \in H(\Gamma) \\ \mathbf{U} = \Phi_d \quad \text{sur } \Gamma_d \end{cases} \tag{4}$$

where $\Phi_d = (\varphi_d, \psi_d, \mu_d, \phi_d)$. Even if the solution to this problem exists, it remains unstable with respect to small perturbations in the data since the solution does not depend on it continuously. It is therefore necessary to apply a regularization technique in order to obtain a stable solution. For that, we apply the fading regularization method, which transforms the problem (4) into a sequence of optimization problems represented by the following iterative algorithm:

$$\begin{cases} \text{Find } \mathbf{U}^{k+1} \in H(\Gamma) \text{ such that} \\ J_c^{k+1}(\mathbf{U}^{k+1}) \leq J_c^{k+1}(\mathbf{V}), \quad \forall \mathbf{V} \in H(\Gamma) \\ \text{with } J_c^{k+1}(\mathbf{V}) = \|\mathbf{V}|_{\Gamma_d} - \Phi_d\|_{H(\Gamma_d)}^2 + \epsilon \|\mathbf{V} - \mathbf{U}^k\|_{H(\Gamma)}^2 \end{cases} \tag{5}$$

for which there exists a unique optimal element \mathbf{U}^{k+1} at each iteration characterized by:

$$\langle \mathbf{U}^{k+1} - \Phi_d, \mathbf{V} \rangle_{H(\Gamma_d)} + c \langle \mathbf{U}^{k+1} - \mathbf{U}^k, \mathbf{V} \rangle_{H(\Gamma)} = 0 \quad \forall \mathbf{V} \in H(\Gamma). \tag{6}$$

This result is immediate since the functional J_c^{k+1} is continuous, convex and coercive thus admitting a unique solution characterized by the optimality equation (6).

3.2. Convergence of the iterative algorithm

In this section, we recall the convergence results of the iterative algorithm (5).

Lemma 1. *The sequence $\{\mathbf{U}^k\}_{k \geq 1}$ corresponding to the optimal elements produced by the iterative algorithm verifies the following identity*

$$\|\mathbf{U}^{n+1} - \mathbf{U}_e\|_{H(\Gamma)}^2 + \sum_{k=0}^n \|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{H(\Gamma)}^2 + \frac{2}{c} \sum_{k=0}^n \|\mathbf{U}^{k+1} - \Phi_d\|_{H(\Gamma_d)}^2 = \|\mathbf{U}^0 - \mathbf{U}_e\|_{H(\Gamma)}^2 \tag{7}$$

where \mathbf{U}_e is the solution of the Cauchy problem given by Eqs. (1) and (3), and $\Phi_d = \mathbf{U}_e|_{\Gamma_d}$.

Should be noted here that this Lemma is valuable only for compatible data.

Theorem 1. *Let Φ_d be the compatible Cauchy data associated with the compatible solution*

$\mathbf{U}_e = (u_e, u_{e,n}, v_e, v_{e,n}) \in H(\Gamma)$. *Then, the sequence produced by the iterative algorithm (5) converges strongly to Φ_d on Γ_d and weakly to \mathbf{U}_e on Γ .*

The proofs of Lemma 1 and of Theorem 1 are identical to those obtained in [26] for the Cauchy problem associated with the biharmonic equation, where the method was implemented using FEM. Alternatively, similar convergence results have been established in the seminal works [19,20], where the method was originally introduced for the Cauchy problem associated with the Laplace equation.

4. Stopping criteria

As shown in the previous section, the sequence of optimal elements $\mathbf{U}^k_{k \geq 0}$ converges to the exact solution of the problem. In order to evaluate the quality of the numerical reconstructions obtained at convergence, a stopping iteration needs to be defined. However, it is crucial to choose an appropriate stopping iteration. This can be challenging since we usually do not have knowledge of the exact solution of the problem. Stopping the algorithm too early may result in reduced accuracy, while stopping it too late can reintroduce the instability of the Cauchy problem when dealing with noisy data.

To address this issue, it is convenient to use control terms that can generate an accurate and reliable stopping criterion.

4.1. Previous stopping criteria

The fading regularization method incorporates different control terms to measure the convergence of the algorithm. For example, Delvare [35] and Cimetière et al. [19] use a stopping criterion based on the sign of the scalar product, defined as:

$$sp(\mathbf{U}^{k+1}) := \langle \mathbf{U}^{k+1} - \Phi_d, \mathbf{U}^{k+1} - \mathbf{U}^k \rangle_{H(\Gamma_d)}. \tag{8}$$

The condition $\langle \mathbf{U}^{k+1} - \Phi_d, \mathbf{U}^{k+1} - \mathbf{U}^k \rangle_{H(\Gamma_d)} \leq 0$ must be satisfied for \mathbf{U}^{k+1} to be an optimal element. This is a direct consequence of the optimality equation (6), in which \mathbf{V} is replaced by $\mathbf{U}^{k+1} - \mathbf{U}^k \in H(\Gamma)$.

Moreover, we can demonstrate that the following sequences decrease during the iterative process:

- The data relaxation term at each iteration ($J_{\Gamma_d}^k$) is defined as

$$J_{\Gamma_d}^{k+1}(\mathbf{U}^{k+1}) = \|\mathbf{U}|_{\Gamma_d}^{k+1} - \Phi_d\|_{H(\Gamma_d)}^2, \tag{9}$$

where $\|\cdot\|_{H(\Gamma_d)}$ represents the Hilbert space norm on Γ_d .

- The optimal value of the functional at each iteration (J_c^k) is defined as

$$J_c^{k+1}(\mathbf{U}^{k+1}) = \|\mathbf{U}^{k+1}|_{\Gamma_d} - \Phi_d\|_{H(\Gamma_d)}^2 + c \|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{H(\Gamma)}^2, \tag{10}$$

where $\|\cdot\|_{H(\Gamma)}$ represents the Hilbert space norm on Γ .

Furthermore, the sequence (J_F^k), which is composed of the regularization term at each iteration:

$$J_F^{k+1}(\mathbf{U}^{k+1}) = c \|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{H(\Gamma)}^2 \tag{11}$$

is bounded and converges towards zero. Based on previous works [31,36], which suggest that the sequence is decreasing, we introduce three stopping criteria that are based on the decay properties of each sequence. In [31], a criterion related to J_c was used but was later replaced with the criterion related to J_F because it does not require prior knowledge of data quality. Therefore, we do not use the stopping criterion linked to J_{Γ_d} in our work.

Table 1
Summary of the four stopping criteria.

Criterion label	Description
C_{J_r}	Violation of the property J_r decreases.
C_{J_e}	Violation of the property J_e decreases.
C_{sp}	Violation of the property $sp \leq 0$.
C_{S_e}	Violation of the property S_e increases.

4.2. Proposition of a stopping criterion

We propose a new stopping criterion based on the equality presented in Lemma 1, where for compatible data Φ_d , we have:

$$\sum_{j=0}^k \|U^{j+1} - U^j\|_{H(\Gamma)}^2 + \frac{2}{c} \sum_{j=0}^k \|U^{j+1} - \Phi_d\|_{H(\Gamma_d)}^2 = \|U_e\|_{H(\Gamma)}^2 - \|U^{k+1} - U_e\|_{H(\Gamma)}^2, \quad \forall k \in \mathbb{N}. \tag{12}$$

However, when the data are noisy and therefore probably incompatible, this equality is no longer valid. For the sake of simplicity, we introduce the following notations for each side of the equality (12):

- $S_d^{k+1}(U^{k+1}) := \sum_{j=0}^k \|U^{j+1} - U^j\|_{H(\Gamma)}^2 + \frac{2}{c} \sum_{j=0}^k \|U^{j+1} - \Phi_d\|_{H(\Gamma_d)}^2$,
- $S_e^{k+1}(U^{k+1}) := \|U_e\|_{H(\Gamma)}^2 - \|U^{k+1} - U_e\|_{H(\Gamma)}^2$.

In the case of compatible data, we have

$$S_e^{k+1}(U^{k+1}) \xrightarrow[k \rightarrow +\infty]{} \|U_e\|_{H(\Gamma)}^2$$

and

$$S_d^{k+1}(U^{k+1}) \xrightarrow[k \rightarrow +\infty]{} \|U_e\|_{H(\Gamma)}^2.$$

However, we know from experience that the fading regularization method [19,31] is capable to filtrate the noise from the data. Thereby, in the case of noisy data $\tilde{\Phi}_d$, we modify the term S_d^{k+1} by removing the accumulation of the noise filtered by the regularization process and which remains in the sum

$$\sum_{j=0}^k \|U^{j+1} - \tilde{\Phi}_d\|_{H(\Gamma_d)}^2.$$

This accumulation is approximated at the iteration $(k + 1)$ by the term

$$(k + 1)\|U^{k+1} - \tilde{\Phi}_d\|_{H(\Gamma_d)}^2.$$

The left-hand side of the equality (12) is then replaced by

$$S_c^{k+1}(U^{k+1}) := \sum_{j=0}^k \|U^{j+1} - U^j\|_{H(\Gamma)}^2 + \frac{2}{c} \left(\sum_{j=0}^k \|U^{j+1} - \tilde{\Phi}_d\|_{H(\Gamma_d)}^2 - (k + 1)\|U^{k+1} - \tilde{\Phi}_d\|_{H(\Gamma_d)}^2 \right). \tag{13}$$

It is easy to establish that the sequence $(S_c^{k+1})_{k \geq 0}$ is increasing independently of the nature of the data (noisy or compatible).

Remark. In the case of compatible data, the term

$$\sum_{j=0}^k \|U^{j+1} - \Phi_d\|_{H(\Gamma_d)}^2$$

will generate an accumulation of numerical approximation errors which will be subtracted by the term

$$(k + 1)\|U^{k+1} - \Phi_d\|_{H(\Gamma_d)}^2.$$

Moreover, we can show in this case that the sequences $(S_c^{k+1})_{k \geq 0}$ et $(S_e^{k+1})_{k \geq 0}$ are convergent and have the same limit.

Taking into account the previous outcomes, we define a new stopping criterion based on the violation of the property $S_c^k(U^{k+1})$ increases.

A summary of all the criteria used to stop the fading regularization iterative algorithm is given in Table 1.

5. Discrete formulation of the iterative algorithm

5.1. Discrete formulation using the method of fundamental solutions (MFS)

We choose to implement the fading regularization algorithm (5) using the method of fundamental solutions (MFS). The MFS is a meshless method, which aims to approximate the solution in the domain and on its boundary without doing any integration. It also maintains effectiveness and accuracy. The ease of its implementation is also an advantage. In fact, the MFS usually generates an ill-conditioned system to solve, even for direct problems. Hence, this will be convenient in our case where we associate it with the fading regularization method which is a regularization technique. However, the knowledge of a fundamental solution of the operator, the position of source points and their number, on which the approximation is very dependent, are the main drawbacks of such method.

In the context of the inverse biharmonic problem, the MFS was already used by Marin and Lesnic [9] and Lesnic and Zeb [37] where it was combined with Tikhonov’s regularization method. The approximation is thus realized by a linear combination of fundamental solutions of the Laplace equation \mathcal{F}_1 and of the biharmonic equation \mathcal{F}_2 , which are given by, see [38]:

$$\begin{cases} \mathcal{F}_1(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \ln r(\mathbf{x}, \mathbf{y}) \\ \mathcal{F}_2(\mathbf{x}, \mathbf{y}) = -\frac{1}{8\pi} r^2(\mathbf{x}, \mathbf{y}) \ln r(\mathbf{x}, \mathbf{y}) \end{cases}, \quad \mathbf{x} = (x_1, x_2) \in \bar{\Omega}, \quad \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 \setminus \bar{\Omega}, \tag{14}$$

where $r(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

Hence,

$$u(\mathbf{x}) \approx u^M(a, b, \mathbf{Y}; \mathbf{x}) = \sum_{j=1}^M a_j \mathcal{F}_1(\mathbf{x}, \mathbf{y}^j) + b_j \mathcal{F}_2(\mathbf{x}, \mathbf{y}^j), \quad \mathbf{x} \in \bar{\Omega} \tag{15}$$

where $a = (a_1, \dots, a_M) \in \mathbb{R}^M$, $b = (b_1, \dots, b_M) \in \mathbb{R}^M$ and \mathbf{Y} is an M -vector of source points, $\mathbf{y}^j \in \mathbb{R}^2 \setminus \bar{\Omega}$.

However, the aim here is to approximate the optimal element \mathbf{U}^{k+1} characterized by (6) which also involves approximations of the elements u_n , v and v_n . By linearity of the normal derivative and of the Laplace equation, one obtains:

$$u'_M(\mathbf{x}) \approx \frac{\partial u^M}{\partial n}(a, b, \mathbf{Y}, \mathbf{n}; \mathbf{x}) = \sum_{j=1}^M a_j \mathcal{F}'_1(\mathbf{x}, \mathbf{y}^j; \mathbf{n}) + b_j \mathcal{F}'_2(\mathbf{x}, \mathbf{y}^j; \mathbf{n}), \quad \mathbf{x} \in \Gamma, \tag{16}$$

$$v_M(\mathbf{x}) \approx v^M(a, b, \mathbf{Y}; \mathbf{x}) = \sum_{j=1}^M b_j \mathcal{G}_2(\mathbf{x}, \mathbf{y}^j), \quad \mathbf{x} \in \Gamma, \tag{17}$$

$$v'_M(\mathbf{x}) \approx \frac{\partial v^M}{\partial n}(a, b, \mathbf{Y}, \mathbf{n}; \mathbf{x}) = \sum_{j=1}^M b_j \mathcal{G}'_2(\mathbf{x}, \mathbf{y}^j), \quad \mathbf{x} \in \Gamma, \tag{18}$$

where

$$\mathcal{F}'_1(\mathbf{x}, \mathbf{y}; \mathbf{n}) = -\frac{1}{2\pi} \frac{1}{r^2(\mathbf{x}, \mathbf{y})} ((\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{x})), \quad \mathbf{x} \in \Gamma, \quad \mathbf{y} \in \mathbb{R}^2 \setminus \bar{\Omega}, \tag{19}$$

$$\mathcal{F}'_2(\mathbf{x}, \mathbf{y}; \mathbf{n}) = -\frac{1}{8\pi} (2 \ln r(\mathbf{x}, \mathbf{y}) + 1) ((\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{x})), \quad \mathbf{x} \in \Gamma, \quad \mathbf{y} \in \mathbb{R}^2 \setminus \bar{\Omega}, \tag{20}$$

$$\mathcal{G}_2(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} (\ln r(\mathbf{x}, \mathbf{y}) + 1), \quad \mathbf{x} \in \Gamma, \quad \mathbf{y} \in \mathbb{R}^2 \setminus \bar{\Omega}, \tag{21}$$

$$\mathcal{G}'_2(\mathbf{x}, \mathbf{y}; \mathbf{n}) = -\frac{1}{2\pi} \frac{1}{r^2(\mathbf{x}, \mathbf{y})} ((\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{x})), \quad \mathbf{x} \in \Gamma, \quad \mathbf{y} \in \mathbb{R}^2 \setminus \bar{\Omega}. \tag{22}$$

The approximations (15), (16), (17) and (18) can also be written as an algebraic system with $2M$ unknowns:

$$\mathcal{A}\mathbf{X} = \mathbf{U}_e \tag{23}$$

where $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{Y}; \mathbf{n})$ is the matrix which contains the functions given by the Eqs. (14), (19), (20), (21) and (22), $\mathbf{X}^T = (a_1, \dots, a_M, b_1, \dots, b_M)$ is the vector of the $2M$ unknowns and $\mathbf{U}_e^T = (u, u_n, v, v_n)$.

We can see through this algebraic system that it is not necessary to discretize the whole domain in order to have the approximation of the solution on the boundary. We denote by \mathbf{U} the vector which contains the discrete values of the approximate solution $\mathbf{U}_h \in \mathcal{H}^h(\Gamma)$. If we consider N collocation points on the boundary Γ , we can write this vector in the form $\mathbf{U} = (u_1, \dots, u_N, u'_1, \dots, u'_N, v_1, \dots, v_N, v'_1, \dots, v'_N)$. And by computing the matrix \mathcal{A} at these collocation points, we can express this vector using the following linear system:

$$\mathcal{A} \mathbf{X} = \mathbf{U} \tag{24}$$

where $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{Y}; \mathbf{n})$ is evaluated at each collocation points $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$. We call \mathcal{A} the matrix of fundamental solutions which size is $4N \times 2M$ and constructed as follows:

For $i \in \overline{1, N}$:

– and for $j \in \overline{1, M}$:

$$\begin{aligned} \mathcal{A}_{i,j} &= \mathcal{F}_1(\mathbf{x}, \mathbf{Y}), \\ \mathcal{A}_{N+i,j} &= \mathcal{F}'_1(\mathbf{x}, \mathbf{Y}; \mathbf{n}), \\ \mathcal{A}_{2N+i,j} &= 0, \\ \mathcal{A}_{3N+i,j} &= 0. \end{aligned}$$

– and for $j \in \overline{M, 2M}$:

$$\begin{aligned} \mathcal{A}_{i,j} &= \mathcal{F}_2(\mathbf{x}, \mathbf{Y}), \\ \mathcal{A}_{N+i,j} &= \mathcal{F}'_2(\mathbf{x}, \mathbf{Y}; \mathbf{n}), \\ \mathcal{A}_{2N+i,j} &= \mathcal{G}_2(\mathbf{x}, \mathbf{Y}; \mathbf{n}), \\ \mathcal{A}_{3N+i,j} &= \mathcal{G}'_2(\mathbf{x}, \mathbf{Y}; \mathbf{n}). \end{aligned}$$

We can then define the space $\mathbb{H}_N(\Gamma)$ of discrete solutions of the biharmonic equation by:

$$\mathbb{H}_N(\Gamma) = \left\{ \mathbf{U} = (\mathbf{U}, \mathbf{U}', \mathbf{V}, \mathbf{V}') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \text{ as it exists } \mathbf{X} \in \mathbb{R}^{2M} \text{ such that } \mathcal{A} \mathbf{X} = \mathbf{U} \right\}.$$

5.2. Discrete iterative algorithm obtained with the MFS

The discrete iterative algorithm obtained with the method of fundamental solutions comes down to determining a sequence of vectors $(\mathbf{X}^k)_{k \in \mathbb{N}}$ solutions to minimization problems, as for $c > 0$ and $\mathbf{X}^0 = 0_{\mathbb{R}^{2M}}$ we have:

$$\begin{cases} \mathbf{X}^{k+1} = \min_{\mathbf{X} \in \mathbb{R}^{2M}} J_c^{k+1}(\mathbf{X}) \\ \text{where } J_c^{k+1}(\mathbf{X}) = \|\mathcal{A}_{|\Gamma_d} \mathbf{X} - \Phi_d\|_{H(\Gamma_d)}^2 + c \|\mathcal{A} \mathbf{X} - \mathcal{A} \mathbf{X}^k\|_{H(\Gamma)}^2 \end{cases} \tag{25}$$

Hence, minimizing, at the iteration k , the functional J_c^{k+1} with respect to \mathbf{X} leads to solve the following linear system:

$$(\mathcal{A}'_{|\Gamma_d} \mathcal{A}_{|\Gamma_d} + c \mathcal{A}' \mathcal{A}) \mathbf{X} = \mathcal{A}'_{|\Gamma_d} \Phi_d + c \mathcal{A}' \mathcal{A} \mathbf{X}^k.$$

where $\mathcal{A}_{|\Gamma_d}$ represents the restriction of \mathcal{A} evaluated at the collocation points belonging to the boundary Γ_d .

6. Numerical results

Herein we present numerical simulations obtained using the fading regularization method combined with the MFS for the inverse Cauchy problem composed by the equilibrium equation (1) and the boundary conditions (3). These simulations are carried out by using data generated from an analytical solution u^{an} , i.e. $\Phi_d = (\varphi_d, \psi_d, \mu_d, \phi_d) = (u_{|\Gamma_d}^{an}, u_{|\mathbf{n}| \Gamma_d}^{an}, v_{|\Gamma_d}^{an}, v_{|\mathbf{n}| \Gamma_d}^{an})$. We consider the following two analytical solutions:

Analytical solution 1

$$\begin{aligned} u^{an}(\mathbf{x}) &= x_1^3 + x_2^3, \\ v^{an}(\mathbf{x}) &= \Delta u^{an}(\mathbf{x}) = 6(x_1 + x_2), \end{aligned} \quad \mathbf{x} = (x_1, x_2) \in \Omega, \tag{26}$$

Analytical solution 2

$$\begin{aligned} u^{an}(\mathbf{x}) &= \frac{1}{2} x_1 (\sin x_1 \cosh x_2 - \cos x_1 \sinh x_2), \\ v^{an}(\mathbf{x}) &= \Delta u^{an}(\mathbf{x}) = \cosh x_2 \cos x_1 + \sinh x_2 \sin x_1, \end{aligned} \quad \mathbf{x} = (x_1, x_2) \in \Omega, \tag{27}$$

where Ω is an open bounded domain. We firstly investigate the reconstructions for a smooth two-dimensional domain represented by the unit disk $\Omega = \{\mathbf{x} = (x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\}$ (Fig. 1), where the accessible part of the boundary is the upper semi-circle $\Gamma_d = \{\mathbf{x} \in \Gamma \mid 0 \leq \theta(\mathbf{x}) \leq \pi\}$, whereas the inaccessible part is the lower one $\Gamma_i = \{\mathbf{x} \in \Gamma \mid \pi \leq \theta(\mathbf{x}) \leq 2\pi\}$, in which $\theta(\mathbf{x})$ denotes the angular coordinate of \mathbf{x} . Next, we give some reconstructions for a non-regular domain taken as the square domain $[0, 1] \times [0, 1]$ (Fig. 2), where only two sides of its boundary are accessible.

We recall *MFS parameters* the number of source points M that are taken uniformly distributed on a circle outside the domain Ω and the distance d which separates them from the boundary Γ . The collocation points on the boundary Γ are likewise uniformly distributed (see Fig. 1).

Figs. 3 and 4 give the reconstructions of the solution u and of the components $u_{\mathbf{n}}$, v and $v_{\mathbf{n}}$ on the whole boundary obtained by the fading regularization method combined with the method of fundamental solutions for the two analytical solutions and for three different values of the regularization parameter c (10 , 10^{-1} and 10^{-3}). The MFS parameters taken here are such as $M = 10$, $d = 10$ for the **analytical solution 1** and $M = 10$, $d = 8$ for the **analytical solution 2**. These reconstructions are obtained using the criterion C_{J_r} . It can be noticed that the reconstructions of all the boundary conditions are accurate and independent on the choice of the parameter c .

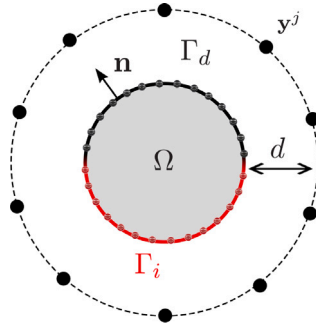


Fig. 1. Regular domain Ω surrounded by source points y^j located on a circle at the distance d from the boundary Γ .

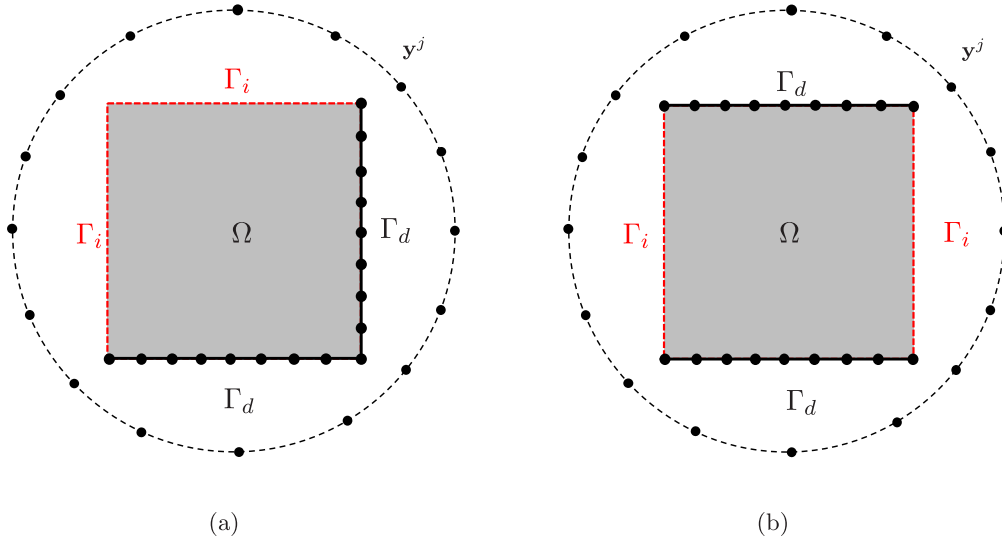


Fig. 2. Rectangular domain Ω with an accessible part Γ_d located on two adjacent sides (a) or two opposite sides (b) surrounded by source points y^j uniformly distributed on a circle outside the domain.

To study the noisy data case, we consider that the analytical boundary data are perturbed as following:

$$\begin{cases} \widetilde{\varphi}_d(\mathbf{x}) = u^{an}(\mathbf{x}) + \delta \max_{y \in \Gamma_d} (|u^{an}(y)|) \rho, & \mathbf{x} \in \Gamma_d, \\ \widetilde{\psi}_d(\mathbf{x}) = u_{,\mathbf{n}}^{an}(\mathbf{x}) + \delta \max_{y \in \Gamma_d} (|u_{,\mathbf{n}}^{an}(y)|) \rho, & \mathbf{x} \in \Gamma_d, \\ \widetilde{\mu}_d(\mathbf{x}) = v^{an}(\mathbf{x}) + \delta \max_{y \in \Gamma_d} (|v^{an}(y)|) \rho, & \mathbf{x} \in \Gamma_d, \\ \widetilde{\phi}_d(\mathbf{x}) = v_{,\mathbf{n}}^{an}(\mathbf{x}) + \delta \max_{y \in \Gamma_d} (|v_{,\mathbf{n}}^{an}(y)|) \rho, & \mathbf{x} \in \Gamma_d, \end{cases} \quad (28)$$

where δ is the level of noise and ρ is a pseudo-random number from the standard uniform distribution in $[-1, 1]$.

The reconstructions in the case of noisy data are given by Figs. 5 and 6 for both examples of analytical solutions and by applying the criterion C_{J_r} . Fig. 7 gives the reconstruction of the solution on the accessible part of the boundary Γ_d . We note that the fading regularization method succeeded in denoising all the data on this part even in the presence of a high noise level $\delta = 10\%$.

In order to investigate the accuracy of the solution obtained at each iteration of the iterative process, we consider the following error estimations:

$$u_{err}^k = \frac{\|u^{an} - u^k\|_{L^2(\Gamma)}}{\|u^{an}\|_{L^2(\Gamma)}}; \quad u_{err}^k = \frac{\|u_{,\mathbf{n}}^{an} - u_{,\mathbf{n}}^k\|_{L^2(\Gamma)}}{\|u_{,\mathbf{n}}^{an}\|_{L^2(\Gamma)}}; \quad v_{err}^k = \frac{\|v^{an} - v^k\|_{L^2(\Gamma)}}{\|v^{an}\|_{L^2(\Gamma)}}; \quad v_{err}^k = \frac{\|v_{,\mathbf{n}}^{an} - v_{,\mathbf{n}}^k\|_{L^2(\Gamma)}}{\|v_{,\mathbf{n}}^{an}\|_{L^2(\Gamma)}}. \quad (29)$$

6.1. Comparison of the stopping criteria

Herein we investigate the performance of the stopping criteria proposed to stop the fading regularization algorithm. Hence, we shall give some numerical tests by varying the parameters c , N , N_{Γ_d} and the proportion of Γ_d for compatible and noisy data. These tests are carried out by considering the analytical solution 1 and the MFS parameters such as $M = 10$ and $d = 10$.

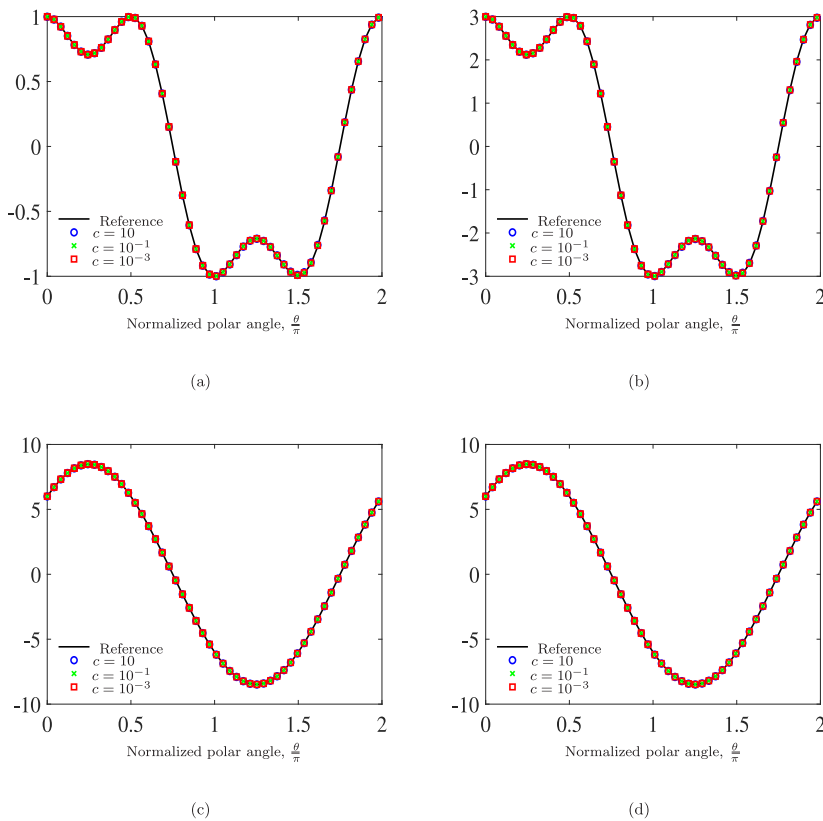


Fig. 3. Reconstructions of u (a), u_n (b), v (c) and v_n (d) on the whole boundary for the **analytical solution 1** and for different values of $c = 10, 10^{-1}$ and 10^{-3} with compatible data.

6.1.1. Influence of the parameter c

We can see from Figs. 8 and 9 that the algorithm is always convergent no matter the value taken for the parameter c . We can also observe that all the control terms are decreasing and become constant almost at the same iteration.

The results given by Fig. 10 validate the new stopping criterion (13). In fact, it highlights the validity of the equality ($S_d = S_e$) given by Lemma 1 only for compatible data. However, we can see that the modification, given by the term S_c to re-establish the equality for incompatible data, is well established because we get that $S_c = S_e$ after a certain number of iterations. Moreover, this is also valid for noisy data as well as for compatible data. The idea is therefore to consider this modification to propose a new stopping criterion valid for any type of data. As the quantity S_c is increasing (a sum of positive quantities), the algorithm can be stopped when this quantity (13) becomes quite constant. This can be compared with the other stopping criteria, where in Fig. 11 all control terms become roughly constant within a common range of iterations. Next, a more complete study is conducted where we compare criteria by evaluating the error norms taken at each corresponding stopping iteration.

Tables 2 and 3 show more precise comparisons between the reconstruction results given by Figs. 3 and 5 and can provide us with the conclusion that the differences between the used criteria only occur when the coefficient $c \geq 10^{-1}$ and that as well for compatible data as for noisy data.

We also notice, thanks to Tables 2 and 3 that the criterion C_{J_r} is less reliable when the regularization coefficient $c \geq 1$. Indeed, this criterion is activated too early. On the other hand, the other criteria give roughly the same results. Moreover, one can note that the coefficient c only accelerate the convergence rate of the algorithm but that it is not necessarily interesting to choose it too small in order to keep a regularization effect. We also notice that the reconstructions of u and u_n are always less precise than those of v and v_n .

Table 4 shows the evolution of error estimates while varying the level of noise and for different values of regularization parameter c . We can notice some kind of stability towards the increase of the noise level and that the error increases are roughly linear compared to the level of noise.

6.1.2. Influence of the extension of the accessible part Γ_d

A small number of data or a small accessible part Γ_d may exist in practice and therefore having an effective stopping criterion in these cases seems very useful. To this end, we carry out tests by varying the accessible part of the boundary for compatible data and noisy data. We further denote by ζ the coefficient such that $|\Gamma_d| = \zeta|\Gamma|$.

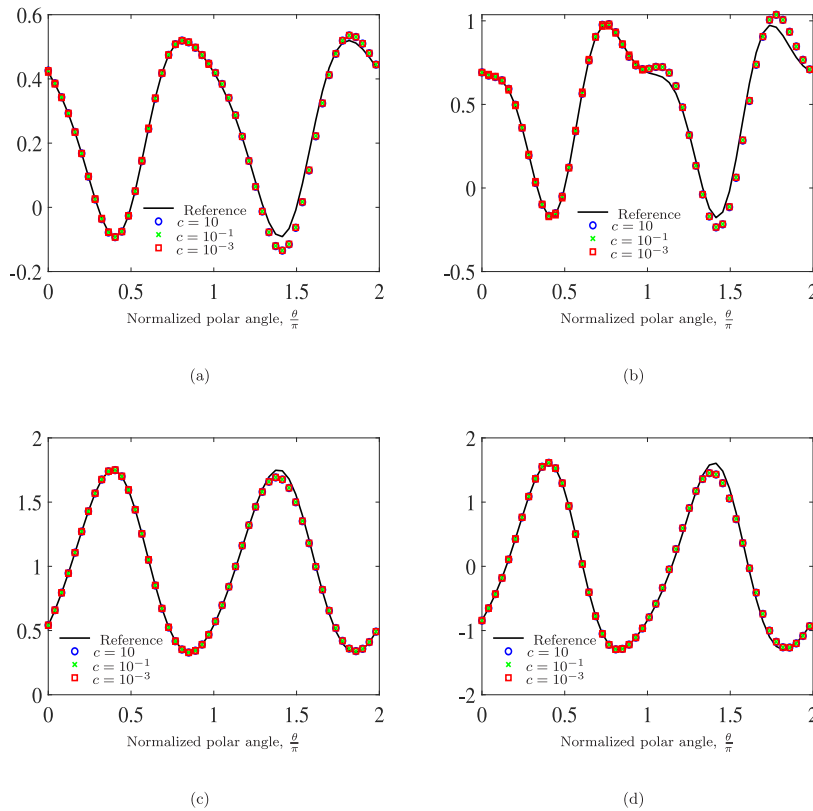


Fig. 4. Reconstructions of u (a), u_n (b), v (c) and v_n (d) on the whole boundary for the analytical solution 2 and for different values of $c = 10, 10^{-1}$ and 10^{-3} with compatible data.

Tables 5 and 6 show the evolution of the error estimates and of the evolution of the stopping iteration for different values of ζ for both compatible and noisy data. We notice that all stopping criteria give almost the same reconstruction results, except when the accessible part Γ_d is very small. We observe in this case that the criterion C_{J_f} would be more efficient than the others when the data are noisy but it is however the least efficient in the case of compatible data. Moreover, the new criterion proposed C_{S_c} gives identical results to those obtained using the criterion C_{J_c} .

6.2. Influence of the MFS parameters

As indicated previously, the position of the source points and their number are crucial parameters when using the method of fundamental solutions. We study here the influence of the number of sources M as well as the influence of their positioning on the quality of the reconstructions. Therefore, we will vary each parameter for a fixed value of the rest and for a different number of collocation points N . Indeed, when the number of source points varies, the number of collocation points on the boundary must also vary to establish a complete numerical analysis.

Table 7 shows the influence of the number of source points M on the numerical results retrieved by the fading regularization method for different number of collocation points N and at a fixed distance $d = 10$. We can notice that the number of source points M should not be very low to have good reconstructions. Moreover, we notice that the ratio M/N should be greater than $1/3$ to have accurate results. This appears to be a very good agreement with the corresponding results in [38–40] on the application of the MFS to problems associated with elliptic partial differential operators.

Table 8 shows the influence of the distance d for different number of source points M and collocation points N such as the ratio M/N is equal to $2/3$. We can see that the parameter d highly affects the accuracy of the reconstructions. In fact, it must be large enough, but not too much, to obtain an accurate solution. Note also that the numerical errors are larger when reconstructing u and u' than v and v' .

6.3. Reconstruction in the case of a square domain

We consider here the case of a non-regular domain: the unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ (Fig. 2). We apply the fading regularization method combined with the MFS to solve the inverse Cauchy problem in order to reconstruct the boundary conditions on the whole

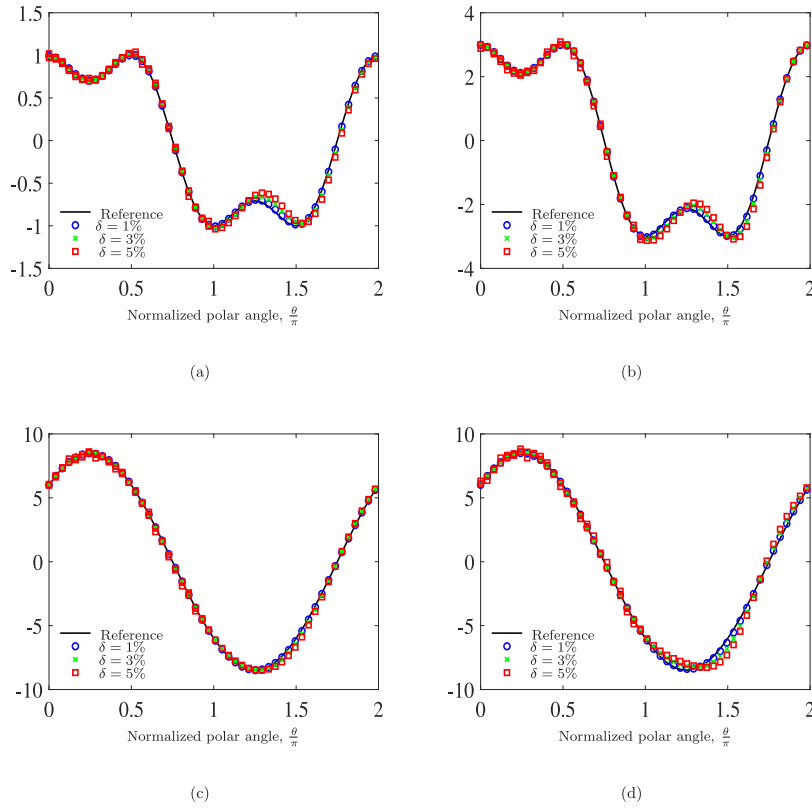


Fig. 5. Reconstructions of u (a), u_n (b), v (c) and v_n (d) on the whole boundary for the analytical solution 1 with different levels of noisy data ($\delta = 1\%$, 3% and 5%) and $c = 10^{-3}$.

Table 2

Evolutions of the error estimates and of the stopping iteration for each stopping criterion and for different values of c with compatible data.

c	Criterion	k	u_{err} (%)	u'_{err} (%)	v_{err} (%)	v'_{err} (%)
10	C_{J_r}	971	32.76	22.90	10.46	19.71
	C_{J_e}	12 011	0.58	0.43	4.16×10^{-2}	8.51×10^{-2}
	C_{S_p}	12 934	0.45	0.33	2.58×10^{-2}	5.24×10^{-2}
	C_{S_e}	12 934	0.45	0.33	2.58×10^{-2}	5.24×10^{-2}
1	C_{J_r}	718	2.76	2.05	3.33×10^{-1}	6.85×10^{-1}
	C_{J_e}	1 565	0.29	0.21	4.73×10^{-3}	8.10×10^{-3}
	C_{S_p}	1 565	0.29	0.21	4.73×10^{-3}	8.10×10^{-3}
	C_{S_e}	1 565	0.29	0.21	4.73×10^{-3}	8.10×10^{-3}
10^{-1}	C_{J_r}	179	0.23	0.17	9.40×10^{-4}	2.05×10^{-3}
	C_{J_e}	187	0.22	0.16	1.94×10^{-3}	4.81×10^{-3}
	C_{S_p}	179	0.23	0.17	9.40×10^{-4}	2.05×10^{-3}
	C_{S_e}	187	0.22	0.16	1.94×10^{-3}	4.81×10^{-3}
10^{-2}	C_{J_r}	24	0.05	0.07	5.63×10^{-2}	1.23×10^{-1}
	C_{J_e}	18	0.05	0.07	5.64×10^{-2}	1.23×10^{-1}
	C_{S_p}	18	0.05	0.07	5.64×10^{-2}	1.23×10^{-1}
	C_{S_e}	18	0.05	0.07	5.64×10^{-2}	1.23×10^{-1}
10^{-3}	C_{J_r}	11	0.13	0.12	1.10×10^{-1}	2.32×10^{-1}
	C_{J_e}	10	0.13	0.12	1.10×10^{-1}	2.32×10^{-1}
	C_{S_p}	7	0.13	0.12	1.10×10^{-1}	2.32×10^{-1}
	C_{S_e}	10	0.13	0.12	1.10×10^{-1}	2.32×10^{-1}
10^{-4}	C_{J_r}	6	0.14	0.12	1.10×10^{-1}	2.30×10^{-1}
	C_{J_e}	5	0.14	0.12	1.10×10^{-1}	2.30×10^{-1}
	C_{S_p}	4	0.14	0.12	1.10×10^{-1}	2.30×10^{-1}
	C_{S_e}	4	0.14	0.12	1.10×10^{-1}	2.30×10^{-1}

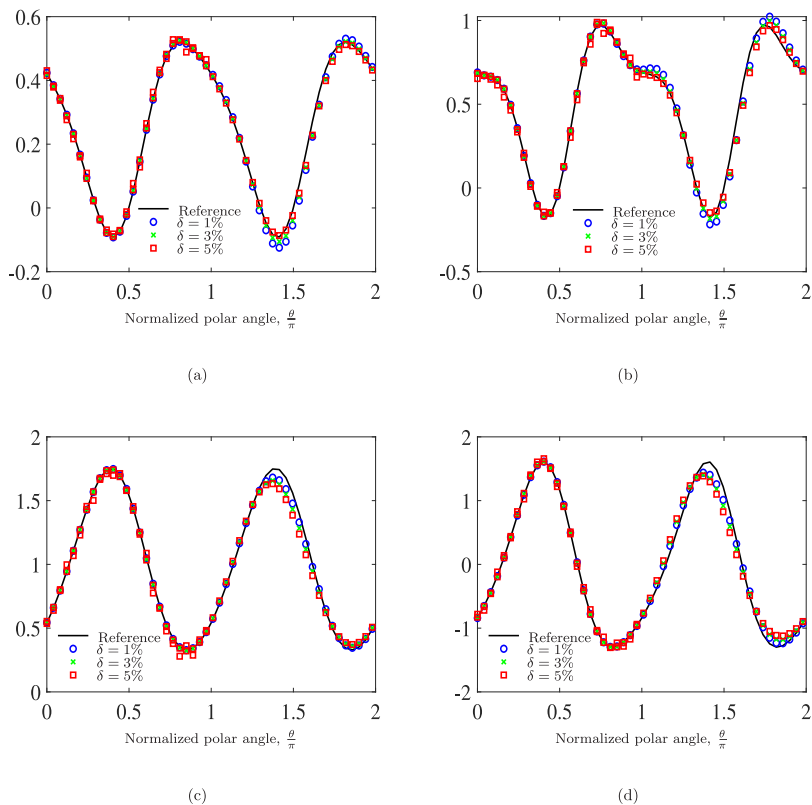


Fig. 6. Reconstructions of u (a), u_n (b), v (c) and v_n (d) on the whole boundary for the analytical solution 2 with different levels of noisy data ($\delta = 1\%$, 3% and 5%) and $c = 10^{-3}$.

Table 3

Evolutions of the error estimates and of the stopping iteration for each stopping criterion and for different values of c with noisy data ($\delta = 3\%$).

c	Criterion	k	u_{err} (%)	u'_{err} (%)	v_{err} (%)	v'_{err} (%)
10	C_{J_f}	683	45.88	30.88	14.65	26.13
	C_{J_e}	11 799	10.58	7.98	3.21	6.17
	C_{sp}	13 010	10.56	8.04	3.24	6.22
	C_{S_c}	13 010	10.56	8.04	3.24	6.22
1	C_{J_f}	804	10.90	7.62	3.02	5.67
	C_{J_e}	1 756	10.57	8.09	3.24	6.23
	C_{sp}	1 756	10.57	8.09	3.24	6.23
	C_{S_c}	1 794	10.57	8.09	3.24	6.23
10^{-1}	C_{J_f}	173	10.57	8.10	3.25	6.24
	C_{J_e}	210	10.56	8.11	3.25	6.26
	C_{sp}	192	10.56	8.11	3.25	6.25
	C_{S_c}	210	10.56	8.11	3.25	6.26
10^{-2}	C_{J_f}	21	12.99	7.73	1.93	2.32
	C_{J_e}	16	13.00	7.73	1.93	2.32
	C_{sp}	15	13.00	7.74	1.93	2.32
	C_{S_c}	16	13.00	7.73	1.93	2.32
10^{-3}	C_{J_f}	10	13.07	7.77	1.83	2.12
	C_{J_e}	8	13.07	7.77	1.83	2.12
	C_{sp}	7	13.07	7.77	1.83	2.12
	C_{S_c}	8	13.07	7.77	1.83	2.12
10^{-4}	C_{J_f}	6	13.07	7.77	1.82	2.12
	C_{J_e}	5	13.07	7.77	1.82	2.12
	C_{sp}	5	13.07	7.77	1.82	2.12
	C_{S_c}	5	13.07	7.77	1.82	2.12

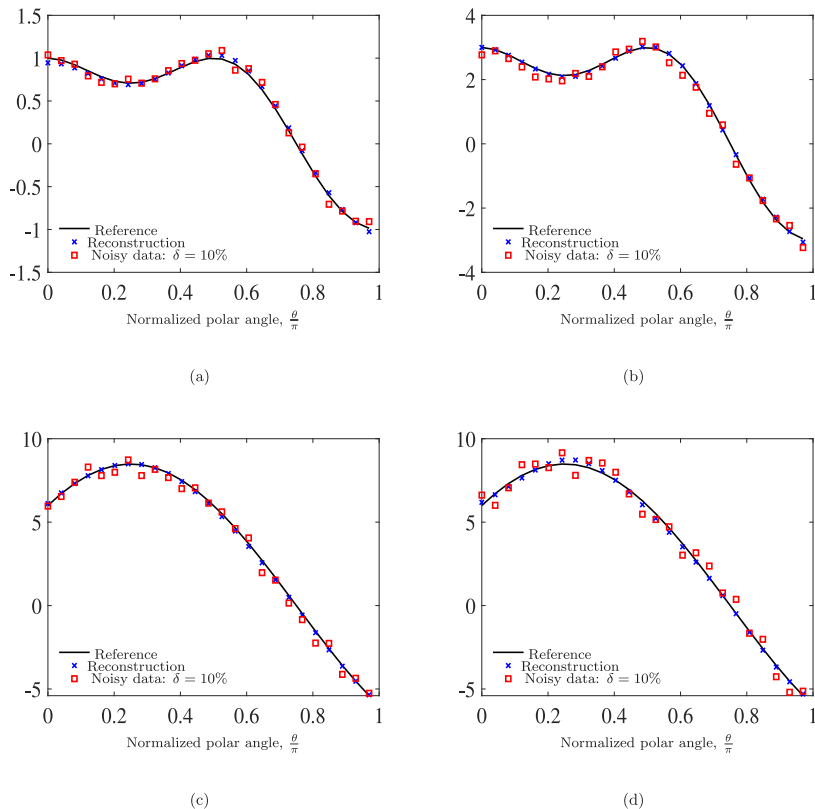


Fig. 7. Reconstructions of u (a), u_n (b), v (c) and v_n (d) on the boundary part Γ_d for the analytical solution 1 with noisy data ($\delta = 10\%$) and $c = 10^{-3}$.

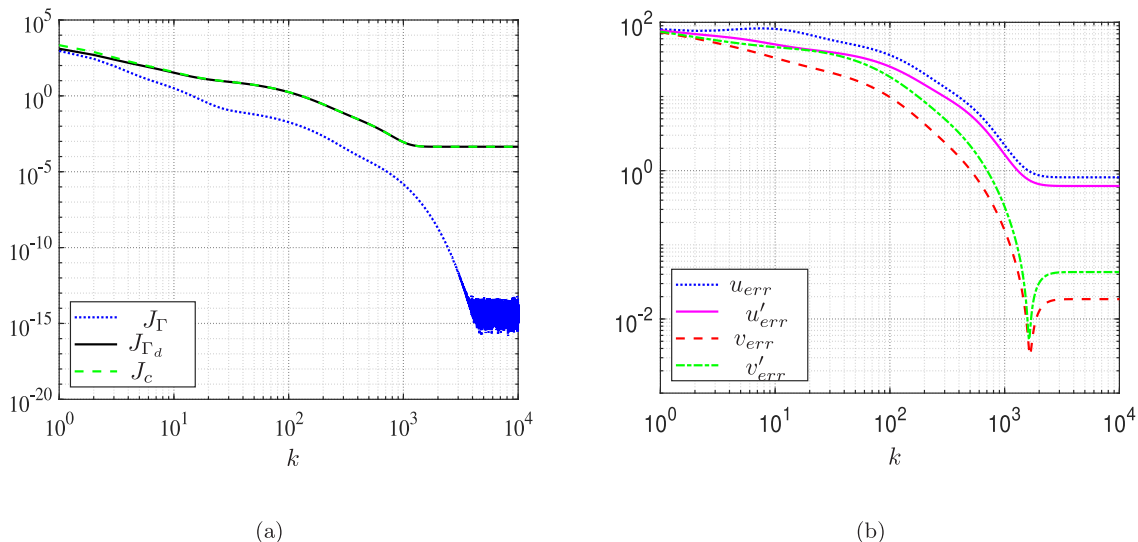


Fig. 8. Evolutions of the control terms J_Γ , J_{Γ_d} and J_c (a) and of the error estimates (b) for compatible data according to the number of iterations k with $c = 1$.

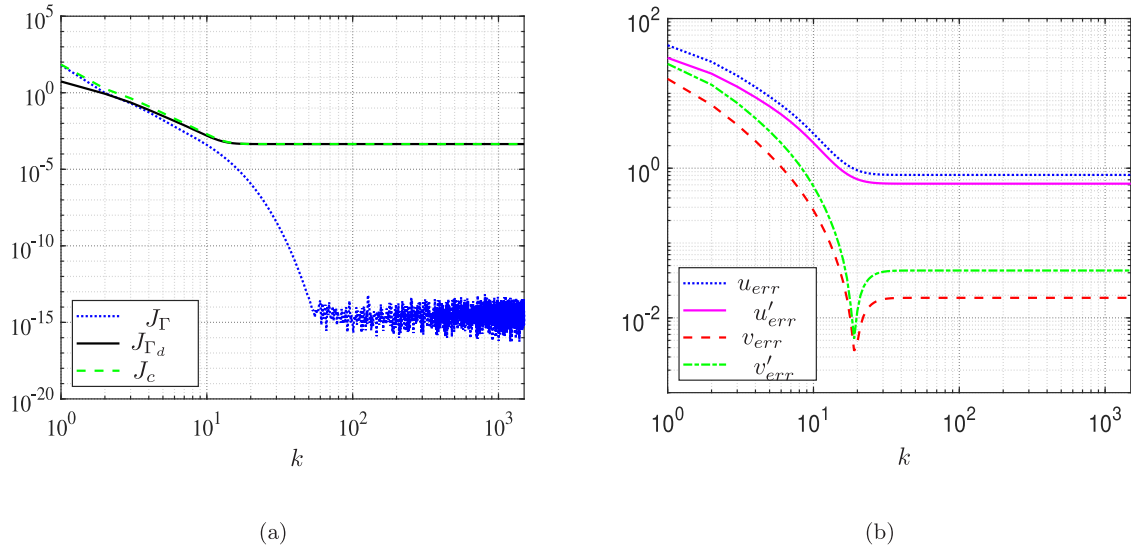


Fig. 9. Evolutions of the control terms J_Γ , J_{Γ_d} and J_c (a) and of the error estimates (b) for compatible data according to the number of iterations k with $c = 10^{-2}$.

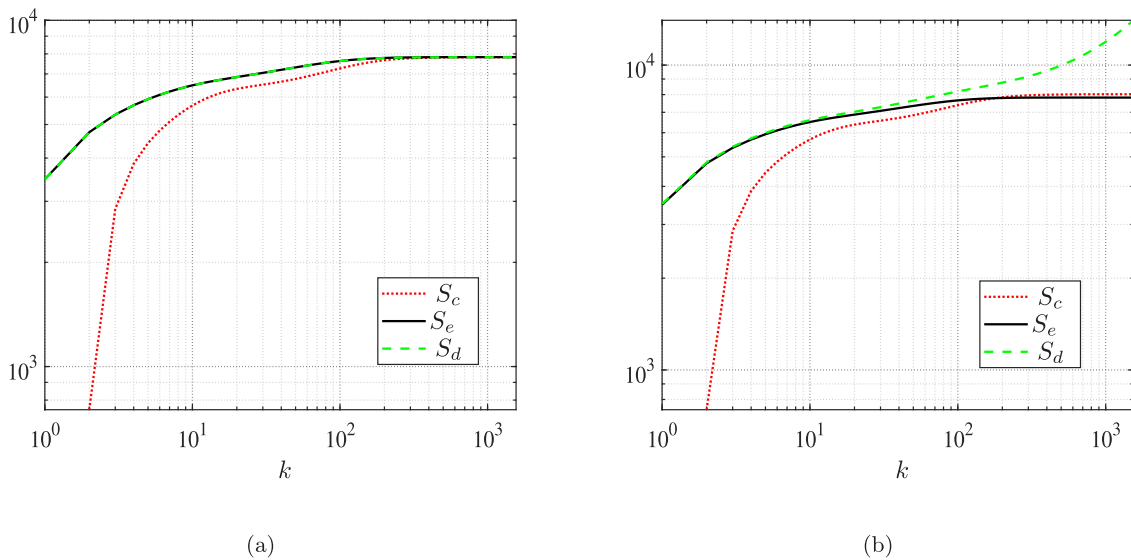


Fig. 10. Evolution of the quantities S_c , S_e and S_d according to the number of iterations k for compatible data (a) and noisy data ($\delta = 3\%$) (b) with $c = 1$.

boundary. We present numerical reconstructions using data generated from the available **analytical solution 2** such that only two sides of the square are accessible. Let s be a curvilinear abscissa varying from 0 to 4 such that when $s \in [0, 1]$ denotes the edge $x_2 = 0$, $s \in [1, 2]$ the edge $x_1 = 1$, $s \in [2, 3]$ the edge $x_2 = 1$ and $s \in [3, 4]$ the edge $x_1 = 0$. Hence, we will examine two different situations when these accessible sides are adjacent ($x_2 = 0$ and $x_1 = 1$, Fig. 2(a)) or opposite ($x_2 = 0$ and $x_2 = 1$, Fig. 2(b)). For the MFS parameters, we choose to fix the source points on a circle surrounded the domain such that its centre is $(\frac{1}{2}, \frac{1}{2})$ and its radius is 5.

The reconstructions in the case of noisy data are given by Figs. 12 and 13 where the data are located respectively on two adjacent sides or on two opposite sides. The reconstructions are clearly better in the second case because all corners of the domain belongs to the known boundary part Γ_d , thus generating an ease of reconstruction of the normal derivative since it is discontinuous at these points. However, the reconstruction in the case of adjacent sides is not so bad since the reconstruction kept the shape of the exact

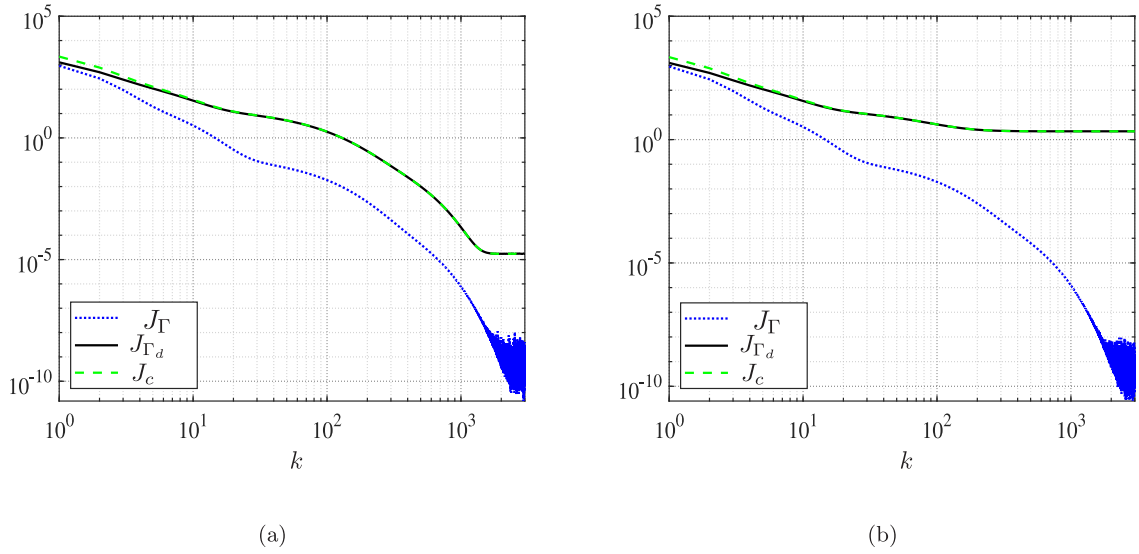


Fig. 11. Evolution of the control terms J_Γ , J_{Γ_d} and J_c according to the number of iterations k for compatible data (a) and noisy data ($\delta = 3\%$) (b) with $c = 1$.

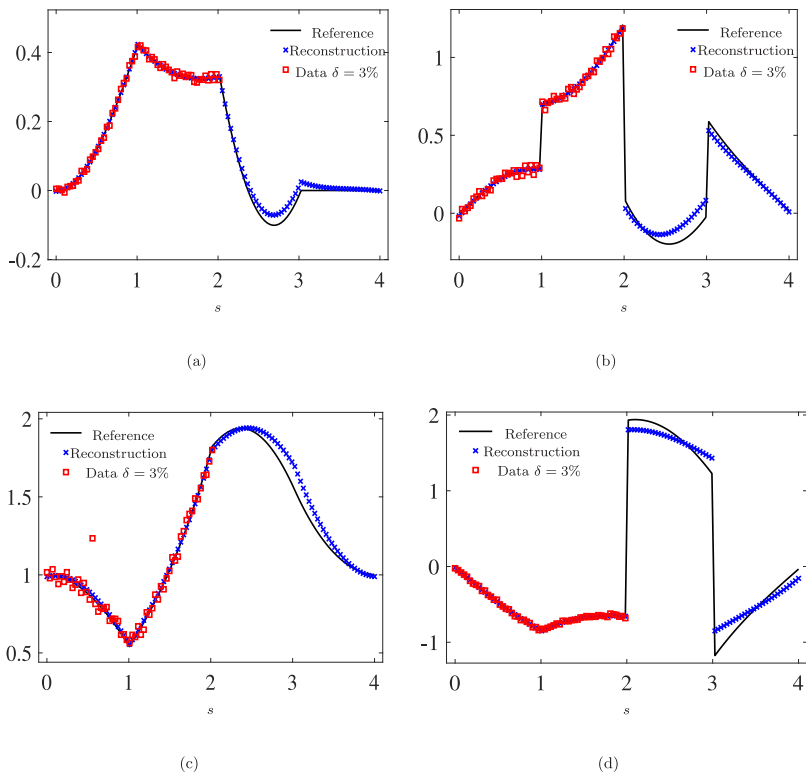


Fig. 12. Reconstructions of u (a), u_n (b), v (c) and v_n (d) for the analytical solution 2 with $c = 10^{-1}$ and noisy data $\delta = 3\%$ located on two adjacent sides of the square domain.

solution curve with a small offset but clearly showing the discontinuity of the normal derivatives u_n and v_n at the corners belonging to Γ_i .

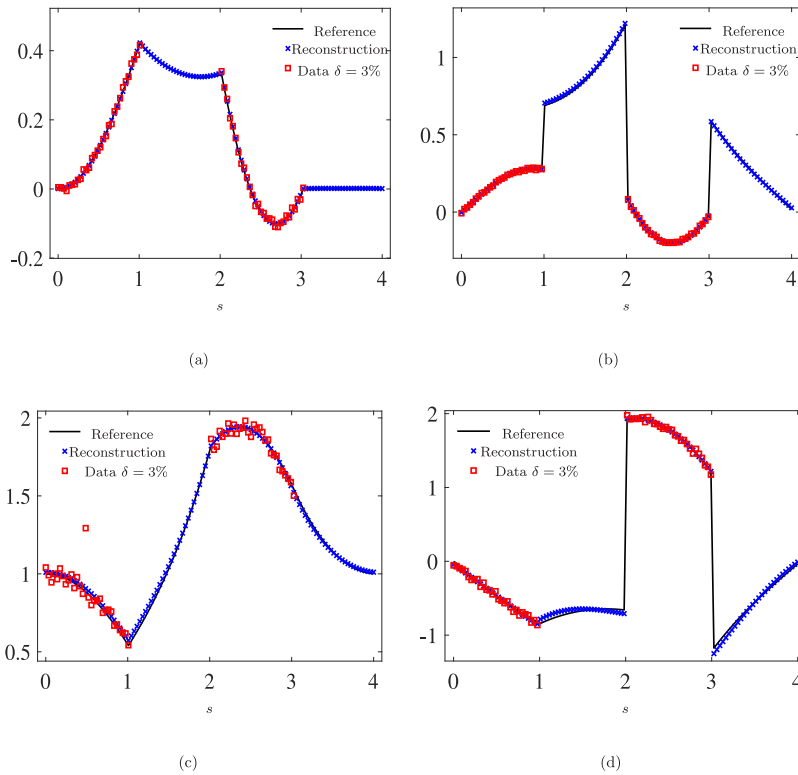


Fig. 13. Reconstructions of u (a), u_n (b), v (c) and v_n (d) for the analytical solution 2 with $c = 10^{-1}$ and noisy data $\delta = 3\%$ located on two opposite sides of the square domain.

Table 4
Evolutions of the error estimates and of the stopping iteration associated with the criterion C_{S_c} for the analytical solution 1, for different noise levels δ and different values of c .

δ	c	k	u_{err} (%)	u'_{err} (%)	v_{err} (%)	v'_{err} (%)
1%	10^{-3}	12	3.53	2.67	1.08	2.08
	10^{-2}	46	3.53	2.67	1.08	2.08
	10^{-1}	374	3.53	2.67	1.08	2.08
3%	10^{-3}	8	13.07	7.77	1.83	2.12
	10^{-2}	16	13.00	7.73	1.93	2.32
	10^{-1}	212	10.56	8.11	3.25	6.26
5%	10^{-3}	10	16.42	13.41	5.37	10.35
	10^{-2}	45	16.57	12.10	5.37	10.35
	10^{-1}	364	17.47	11.41	5.37	10.35
10%	10^{-3}	10	34.94	26.82	10.74	20.70
	10^{-2}	43	30.04	22.82	10.74	20.70
	10^{-1}	360	32.94	22.82	10.74	20.70

7. Conclusion

In this paper, we used the fading regularization method to solve the Cauchy problem associated with the biharmonic equation, where no boundary conditions are given on a part of the boundary. We presented the continuous formulation of the method and we also proposed a new stopping criterion for the fading regularization algorithm. The numerical implementation of this algorithm was carried out using the method of fundamental solutions and two kind of domains, regular and non-regular, were examined.

We carried out a numerical comparative study allowing the validation of the new stopping criterion. We had come to the conclusion that almost all the criteria proposed to stop the fading regularization algorithm give roughly the same results, except for special situations like a regularization coefficient $c \geq 10^{-1}$ or a small amount of data. We therefore studied the influence of MFS parameters on the reconstruction of boundary conditions.

The reconstruction results thus obtained are examined in the case of domains with smooth boundaries where it has been shown that the obtained solution is stable with respect to noisy data. The reconstructions obtained in the case of a square domain are

Table 5

Evolutions of the error estimates and of the stopping iteration for each criterion and for different values of ζ for *compatible data* with $c = 10^{-1}$.

ζ	Criterion	k	u_{err} (%)	u'_{err} (%)	v_{err} (%)	v'_{err} (%)
1/5	C_{J_r}	762	42.02	30.24	27.96	46.53
	C_{J_c}	21 833	14.83	9.71	4.97	9.21
	C_{sp}	19 850	16.33	10.90	5.62	10.64
	C_{S_c}	23 168	13.97	9.02	4.58	8.51
2/5	C_{J_r}	730	7.95	5.85	1.13	2.22
	C_{J_c}	1 644	1.09×10^{-1}	9.23×10^{-2}	6.67×10^{-2}	1.32×10^{-1}
	C_{sp}	1 438	4.02×10^{-1}	2.98×10^{-1}	1.21×10^{-1}	2.40×10^{-1}
	C_{S_c}	1 644	1.09×10^{-1}	9.23×10^{-2}	6.67×10^{-2}	1.32×10^{-1}
3/5	C_{J_r}	63	5.08×10^{-2}	4.22×10^{-2}	2.44×10^{-3}	6.10×10^{-3}
	C_{J_c}	60	5.19×10^{-2}	4.27×10^{-2}	2.41×10^{-3}	5.99×10^{-3}
	C_{sp}	56	5.48×10^{-2}	4.41×10^{-2}	2.37×10^{-3}	5.78×10^{-3}
	C_{S_c}	60	5.19×10^{-2}	4.27×10^{-2}	2.41×10^{-3}	5.99×10^{-3}
4/5	C_{J_r}	18	2.37×10^{-2}	3.51×10^{-2}	3.61×10^{-4}	1.17×10^{-3}
	C_{J_c}	16	2.37×10^{-2}	3.51×10^{-2}	3.56×10^{-4}	1.17×10^{-3}
	C_{sp}	16	2.37×10^{-2}	3.51×10^{-2}	3.56×10^{-4}	1.17×10^{-3}
	C_{S_c}	16	2.37×10^{-2}	3.51×10^{-2}	3.56×10^{-4}	1.17×10^{-3}
1	C_{J_r}	9	1.58×10^{-2}	3.17×10^{-2}	5.10×10^{-5}	1.03×10^{-4}
	C_{J_c}	11	1.58×10^{-2}	3.17×10^{-2}	4.97×10^{-5}	1.03×10^{-4}
	C_{sp}	8	1.58×10^{-2}	3.17×10^{-2}	5.15×10^{-5}	1.02×10^{-4}
	C_{S_c}	8	1.58×10^{-2}	3.17×10^{-2}	5.15×10^{-5}	1.02×10^{-4}

Table 6

Evolutions of the error estimates and of the stopping iteration for each criterion and for different values of ζ for *noisy data* ($\delta = 3\%$) with $c = 10^{-1}$.

ζ	Criteria	k	u_{err} (%)	u'_{err} (%)	v_{err} (%)	v'_{err} (%)
1/5	C_{J_r}	743	43.58	34.54	32.17	45.33
	C_{J_c}	68 767	131.23	141.11	55.79	123.57
	C_{sp}	65 551	123.02	133.20	53.45	118.54
	C_{S_c}	72 013	139.63	149.18	58.16	128.66
2/5	C_{J_r}	874	7.96	7.98	8.23	14.58
	C_{J_c}	1 783	10.99	10.40	9.07	16.21
	C_{sp}	1 588	10.73	10.22	9.03	16.12
	C_{S_c}	1 804	11.01	10.42	9.08	16.21
3/5	C_{J_r}	67	4.83	2.14	0.86	0.77
	C_{J_c}	58	4.83	2.14	0.86	0.77
	C_{sp}	58	4.83	2.14	0.86	0.77
	C_{S_c}	58	4.83	2.14	0.86	0.77
4/5	C_{J_r}	18	2.47	1.05	0.36	0.49
	C_{J_c}	17	2.47	1.05	0.36	0.49
	C_{sp}	17	2.47	1.05	0.36	0.49
	C_{S_c}	17	2.47	1.05	0.36	0.49
1	C_{J_r}	11	1.15	0.80	0.30	0.49
	C_{J_c}	9	1.15	0.80	0.30	0.49
	C_{sp}	7	1.15	0.80	0.30	0.49
	C_{S_c}	9	1.15	0.80	0.30	0.49

relatively accurate despite the presence of corners in the unknown part of the boundary and they also showed a robustness of the method in the case of data located on two opposite sides. It was also observed that the errors in the numerical normal derivatives, obtained using the fading regularization-MFS algorithm, were higher than those corresponding to the reconstructed solutions.

Data availability

No data was used for the research described in the article.

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Table 7

Evolutions of the error estimates and of the stopping iteration for different numbers of collocation points N and source points M fixed at a distance $d = 10$ from the boundary with *noisy data* ($\delta = 3\%$) and $c = 10^{-1}$.

N	M	k	u_{err} (%)	u'_{err} (%)	v_{err} (%)	v'_{err} (%)
30	5	27	404.32	259.07	72.51	63.16
	10	129	10.81	6.44	1.37	2.39
	20	569	7.43	6.51	1.45	2.53
	30	664	7.25	6.66	1.11	2.18
60	10	169	9.23	5.90	2.19	4.64
	20	582	9.82	5.21	1.99	4.12
	40	637	9.41	5.05	1.78	2.22
	60	738	9.34	5.01	1.80	2.25
120	10	208	12.47	8.34	2.55	4.43
	40	658	4.57	4.31	1.23	1.34
	80	704	4.64	4.39	1.24	1.37
	120	715	4.68	4.43	1.25	1.38
240	20	519	5.66	4.98	0.86	1.79
	80	722	5.21	4.62	0.35	0.67
	160	707	5.09	4.53	0.34	0.65
	240	679	5.12	4.55	0.33	0.65

Table 8

Evolutions of the error estimates and of the stopping iteration for different values of the distance d that separates the source points from the boundary and for different numbers of collocation points N and source points M such as $N/M = 2/3$ and $|\Gamma_d| = \frac{1}{2}|\Gamma|$ with *noisy data: analytical solution 1* ($\delta = 3\%$) and $c = 10^{-1}$.

N/M	d	k	u_{err} (%)	u'_{err} (%)	v_{err} (%)	v'_{err} (%)
$N = 60$ $M = 40$	5	1087	28.71	39.74	7.77	17.98
	10	169	9.23	5.90	2.19	4.64
	15	164	4.07	2.12	0.59	0.92
	20	86	8.90	8.22	5.51	7.36
$N = 120$ $M = 80$	5	2001	32.79	26.28	11.81	29.20
	10	704	4.64	4.39	1.24	1.37
	15	173	7.24	3.48	1.04	0.89
	20	1120	154.01	119.21	95.46	71.01
$N = 240$ $M = 160$	5	2205	17.34	21.92	5.81	16.45
	10	1707	5.09	4.53	0.34	0.65
	15	1139	1.90	1.52	0.34	0.60
	20	1121	209.02	168.41	118.56	100.21

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