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# Central Nervous System Action on Rolling Balance Board Robust Stabilization: Computer Algebra and MID-based feedback design

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## Abstract

Using the computer algebra system `Maple`, we consider the stabilization of a rolling balance board by means of the multiplicity-induced-dominancy (MID) property. The human stance on a rolling balance board is analyzed in the sagittal plane through a two degree-of-freedom mechanical model. Namely, the human body is modeled by a double inverted pendulum which connects to the balance board through the ankle joint. The system is stabilized by the ankle torque managed by the central nervous system (CNS). The action of the CNS is modeled by a delayed full state feedback: a pointwise delay stands for all latencies in the neuromechanical system (reaction time, neuromechanical lag, etc). The aim of the paper is to achieve a good occurrence in terms of the decay rate, it exhibits the links between multiple spectral values satisfying the MID property and the exponential stability property of the solution<sup>1</sup>.

*Keywords:* Time-delay, asymptotic stability, exponential stability, exponential decay rate, Vandermonde matrix, quasipolynomial factorization, control design.

*2010 MSC:* 34K20, 39B82, 70Q05, 47N70

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Mechanical model</b>	<b>4</b>

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<sup>1</sup>Note that a preliminary version of this work was published in<sup>1</sup>.

<b>3</b>	<b>Background and prerequisites</b>	<b>8</b>
3.1	Cauchy’s argument principle . . . . .	8
3.2	The Stepan-Hassard approach . . . . .	8
3.3	Background on Gröbner bases . . . . .	9
3.4	Cell decomposition in CAD routine . . . . .	10
3.5	The MID paradigm: A partial pole-placement strategy . . . . .	11
3.6	Algorithmic investigation of the MID property . . . . .	12
<b>4</b>	<b>Main results</b>	<b>12</b>
4.1	A normalized fourth order friction-free model . . . . .	13
4.2	Forcing multiplicity . . . . .	14
4.2.1	Realrooted elimination-produced polynomial . . . . .	18
4.2.2	Fredholm representation of the normalized characteristic function . . . . .	19
4.2.3	Constancy sign of the integrand $q_{v,\sigma}$ . . . . .	20
4.2.4	Effective admissible region . . . . .	22
4.3	Dominancy Sufficient Conditions . . . . .	23
4.3.1	Frequency bound . . . . .	23
4.3.2	Dominancy . . . . .	26
<b>5</b>	<b>Conclusion</b>	<b>26</b>

## 1. Introduction

The intrinsic brain activity is spatio-temporally structured, and human balancing is among the few activities for which the time heterogeneity is essential. As a matter of fact, age-related decline in balance is a principal contributor to falls among older adults ; these are the most common causes of accidental deaths, and in non-fatal cases the costs related to the treatment of fall-related sequelae are on the rise as a consequence of the growing age of populations. As such, human balancing is a subject of growing interest.

The passive biomechanics of human balance are unstable as they consist of a network of interconnected inverted pendulums that are each unstable. The human balance system is the sensorimotor system that enables us to perform ordinary reflex tasks such as standing upright, walking, etc. Balance may be defined as the ability to maintain equilibrium in a gravitational field by keeping or returning the center of body mass over its base of support<sup>2</sup>. It is the system that depends on vestibular function, vision, and proprioception to maintain posture, navigate in one’s surroundings, coordinate motion of body parts, modulate fine motor control, and initiate the vestibulooculomotor reflexes. As such, success of balance may be assessed by the ability to perform quiet stance, compensatory postural reactions and anticipatory postural responses. This ability is also appraised by the reaction time (RT), or the controller of the central nervous system (CNS), i.e., the length of time between

a stimulus and our response to that stimulus. That said, elementary balancing tasks may be investigated by a mechanical analysis. Stick balancing on a fingertip<sup>3</sup>, quiet stance<sup>4,5,6,7,8</sup>, ball and beam balancing<sup>9</sup> and standing on a balance board<sup>10,11,12</sup> have been thoroughly investigated, see also<sup>13</sup> for further details. In fact, enquiries are established on the mechanical model of a single inverted pendulum, more complex tasks require multi-degree-of-freedom models.

In this paper, we consider the mathematical model of balancing on a uniaxial rolling balance board in the sagittal plane, see Figure 1. The balance board has a configurable geometry in as much as the radius  $R$  of the (half) wheels and the distance  $h$  between the straight edge of the half wheel and the board can be adjusted as in Figure 1. The latter parameters highly affect the ability to stabilize the associated mathematical model, as preliminary computations and experiments performed by human subjects show. Namely, standing on the balance board gets more involved as the wheel radius and the board elevation decrease. Indeed, hip strategy is dominant for smaller radii, which indicates a double inverted pendulum model for the human body. However, greater radii ( $R > 100$  mm) spur the use of musculature at the ankle to maintain the equilibrium, so that the human body can be considered as a single inverted pendulum. In this work, greater wheel radii are considered and therefore a two-degree-of-freedom mechanical model is analyzed involving the balance board and the human body.

Balance deficits and disorders are the consequence of a multitude of age-associated declines in sensorimotor function, including somatosensation, vestibular function, vision, cognition and strength. Stabilization of the human body and the balance board is performed by a control process governed by the CNS. Visual, vestibular and somatosensory systems obtain information about the spatial orientation and surroundings of the human body. The information is delivered to the brain, CNS determines the necessary interaction to maintain the balance after processing the signals and sends an instruction to the musculature. The process described above requires a definite time lapse called reaction time (RT). Consequently, the mathematical model of balancing tasks involves a delayed control law, here a proportional-derivative (PD) feedback with constant feedback delay<sup>14,15</sup>, since the visual and vestibular system perceive position and velocity, respectively. Other frequently used neuromuscular control models are for instance delayed proportional-derivative-acceleration feedback<sup>3,16</sup>, intermittent control<sup>17,18,19,20,7</sup> and predictor feedback<sup>3,21</sup>. It was shown in<sup>22</sup> and<sup>23</sup> that an integral term does not enhance the stabilizability of the system in the presence of feedback delay, that is why, here, only PD feedback is considered.

Note that RT of different sensory systems differ, yet they scale alike<sup>24</sup>. Consequently, in most investigations related to human balancing, delays associated with different sensory organs are assumed to be similar in magnitude<sup>17,10,15</sup>. In case of quiet stance, feedback delay is estimated at 100-200 ms<sup>17,25</sup>. Balancing on an unstable moving surface such as the balance board or skateboard is a trickier task, hence a sensibly higher RT is gauged at 150-300 ms<sup>10,15,26</sup>.

For the exponential behavior of solutions of the ensuing delay system, we are interested in the *spectral abscissa* of the corresponding characteristic function. Indeed, the spectral abscissa is related to the notion of

decay rate of time-delay system solutions.

A series of recent works proposed a safe control methodology called *partial pole placement* (PPP)<sup>27,28,29,30</sup>. This PPP methodology is based on the assignment of the closed-loop dominant solution's decay rate. Indeed, it has been shown that under appropriate conditions a multiple spectral value is the rightmost; see for instance<sup>31,29,32,33,34</sup>. The proposed PPP strategy relies on two properties of the spectrum distribution of delay systems, namely the *multiplicity-induced-dominancy* (MID)<sup>29,35,36</sup> and the *Coexisting-real-roots-induced-dominancy* (CRRID)<sup>37,38,39</sup>. While the MID property consists in conditions under which a given multiple complex zero of a quasipolynomial is dominant, the CRRID property consists in conditions under which the coexisting real spectral values are dominant. While, it has been shown in the generic quasipolynomial case that the real root of maximal multiplicity is necessarily the dominant (GMID), see for instance<sup>36,35</sup>, multiple roots with intermediate admissible multiplicities may be dominant but under appropriate conditions. Thanks to this property, an ensued control strategy is proposed in<sup>29,31,40</sup>, which consists in assigning a root with an admissible multiplicity once appropriate conditions guaranteeing its dominancy are determined. Furthermore, the MID property may be used to tune standard controllers. For instance, in<sup>40</sup> it is applied to the systematic tuning of the stabilizing PID controller of a first order plant. Here, we aim at assigning dominant multiple real roots with admissible codimensions and the MID property is utilized for the mechanical model of human balancing on rolling balance board in the sagittal plane. Delayed state feedback is adopted, which gives the governing equation the form of a delay differential equation (DDE). The rightmost characteristic root (called dominant root) is minimized by tuning the control gains according to the MID concept<sup>41</sup>.

The main contribution of the paper is to explore the use of the MID property for the rolling balance board stabilization. It should be mentioned that the particular structure of the system's dynamics does not allow the use of any of the existing MID results straightforwardly. Thus, for instance, due to the sparsity of the open-loop transfer function, the GMID cannot be reached and the characterization of the generic MID proposed in<sup>30</sup> is not valid. Moreover, the corresponding plant is not real rooted but its roots are located on real and imaginary axis and the ideas and the approach proposed in<sup>31</sup> cannot apply.

The paper is organized as follows. Next section builds the mechanical model of human stance on a rolling balance board in the sagittal plane from which our mathematical model is then derived. Section 3 reminds the relevant background and some prerequisites pertaining to quasipolynomials. The main results and their proofs are presented in Section 4. Finally, a conclusion ends the paper.

## 2. Mechanical model

The rolling balance board, see Figure 1, is modeled as a double inverted pendulum, see also<sup>42</sup>, which is an important model in human balancing research<sup>43,44</sup>. Indeed, the double-inverted pendulum is often referred to as the most simple nonlinear multi-body system as it features all the properties of higher-degree-of-freedom nonlinear systems, such as complexity or chaos. The corresponding full state feedback involves the angular

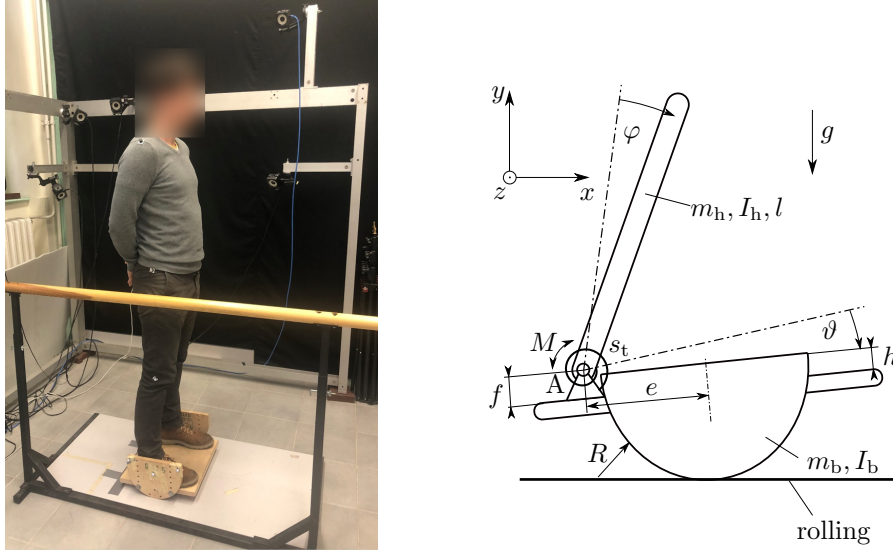


Figure 1: (Left) Two-degree-of-freedom mechanical model of human balancing on a uniaxial rolling balance board in the sagittal plane. (Right) Human balancing on a uniaxial rolling balance board in the sagittal plane. Figures are taken from<sup>13</sup>.

position and angular velocity of both pendulum segments, which implies that the number of control gains is four. Hence optimization and stabilization shall be performed in the four-dimensional-space of the control gains.

Human stance on a rolling balance board in the sagittal plane is described by a two-degree-of-freedom mechanical model as shown in Figure 1. The generalized coordinates are the angle  $\varphi$  of the human body and the angle  $\vartheta$  of the balance board measured from the upper (unstable) equilibrium. The human body was modeled as a homogeneous rigid bar considering that the musculature at the ankle is used during balancing only. The mass and the height of the human body is denoted by  $m_h$  and  $l$ , therefore the mass moment of inertia of the human body becomes  $I_h = 1/12m_h l^2$  for the center of gravity.

The feet are considered to be fixed to the board, therefore the ankles are modeled as a pin between the human body and the balance board. As shown in Figure 1, the height of the ankle is denoted by  $f$ . The ankle joint is located on the left side of the symmetry axis of the balance board expressed by parameter  $e$ , since the feet are able to move the balance board in both  $+\vartheta$  and  $-\vartheta$  this way. The passive stiffness of the ankle musculature was considered as a torsional spring

$$s_t = 0.91m_h g \frac{l}{2}, \quad (1)$$

referring to<sup>45</sup>, while the passive damping of the muscles was neglected.

The elements of the balance board were made from plywood of 21 mm thickness and density of  $700 \text{ kg/m}^3$ . The center of gravity  $l_b$ , mass  $m_b$  and mass moment of inertia  $I_b$  of the balance board was calculated based on the actual set of the adjustable parameters  $R$  and  $h$ . The mass of the feet was neglected.

As it was explained in the Introduction, the sensory organs obtain information about the angular position

and angular velocity of the human body and the balance board, therefore a delayed full state feedback

$$M(t) = P_\varphi \varphi(t - \tau) + D_\varphi \dot{\varphi}(t - \tau) + P_\vartheta \vartheta(t - \tau) + D_\vartheta \dot{\vartheta}(t - \tau), \quad (2)$$

is applied as the control torque produced by the muscles at the ankle. The feedback delay  $\tau$  corresponds to the reaction time of human beings and estimated by a constant value. For the sake of simplicity, the sensory organs are involved with the same value in the control model.

The governing equation of the motion can be written by the Lagrange's equation of the second kind

$$\frac{d}{dt} \frac{dT}{dq_k} - \frac{dT}{dq_k} + \frac{dU}{dq_k} = Q_k(t) \quad k = 1, \dots, n, \quad (3)$$

where the kinetic energy is

$$T = \frac{1}{2} m_b v_{S_b}^2 + \frac{1}{2} I_b \dot{\vartheta}^2 + \frac{1}{2} m_h v_{S_h}^2 + \frac{1}{2} I_h \dot{\varphi}^2 \quad (4)$$

and the potential energy is

$$U = m_b g r_{S_b, y} + m_h g r_{S_h, y} + \frac{1}{2} s_t (\vartheta - \varphi)^2, \quad (5)$$

where  $v_{S_b}$  and  $v_{S_h}$  refer to the velocity of the center of gravity of the balance board and the human body, respectively.  $r_{S_b, y}$  and  $r_{S_h, y}$  stand for the vertical position of the center of gravity of the balance board and the human body, respectively,  $g$  is the gravitational acceleration and  $Q_k(t)$  denotes the generalized force. The center of gravity of the balance board

$$\mathbf{r}_{S_b} = \begin{bmatrix} -e \cos \vartheta + (h - f) \sin \vartheta \\ R + (f - h) \cos \vartheta - e \sin \vartheta \\ 0 \end{bmatrix} \quad (6)$$

and that of the human body

$$\mathbf{r}_{S_h} = \begin{bmatrix} -e \cos \vartheta + (h - f) \sin \vartheta - l \sin \varphi \\ R + (f - h) \cos \vartheta - e \sin \vartheta + l \cos \varphi \\ 0 \end{bmatrix} \quad (7)$$

can be obtained from the geometrical properties of the system. The velocity of the center of gravity of the balance board

$$\mathbf{v}_{S_b} = \begin{bmatrix} (-R + (R - l_b) \cos \vartheta) \dot{\vartheta} \\ (R - l_b) \sin \vartheta \dot{\vartheta} \\ 0 \end{bmatrix} \quad (8)$$

and that of the human body

$$\mathbf{v}_{S_h} = \begin{bmatrix} (-R + (h - f) \cos \vartheta + e \sin \vartheta) \dot{\vartheta} - l \cos \varphi \dot{\varphi} \\ -(e \cos \vartheta + (-h + f) \sin \vartheta) \dot{\vartheta} - l \sin \varphi \dot{\varphi} \\ 0 \end{bmatrix} \quad (9)$$

Table 1: Physical system parameters

Parameter	Explanation	Value
$R$	wheel radius	0.125 m
$h$	board distance from the wheel centre	0.05 m
$m_b$	mass of balance board	3.0988 kg
$l_b$	centre of gravity of balance board	0.0651 m
$I_b$	mass moment of inertia of the board	0.0201 kgm <sup>2</sup>
$m_h$	mass of balancing subject	57 kg
$l$	height of balancing subject	1.6 m
$I_h$	mass moment of inertia of the subject	12.16 kgm <sup>2</sup>
$e$	horizontal distance of ankle	0.12 m
$f$	vertical distance of ankle	0.07 m
$s$	passive stiffness at ankle joint	407 Nm
$g$	gravitational acceleration	9.81 m/s <sup>2</sup>
$\tau$	reaction time	150 ms
$P_\varphi^*$	proportional gain of $\varphi$	8222 Nm/rad
$D_\varphi^*$	derivative gain of $\varphi$	2667 Nms/rad
$P_\vartheta^*$	proportional gain of $\vartheta$	1140 Nm/rad
$D_\vartheta^*$	derivative gain of $\vartheta$	390 Nms/rad

are the first derivatives of (6) and (7), respectively. The distance between the ground and the center of gravity of the balance board is denoted by  $l_b$  and can be computed using the center of gravity of the elements of the balance board. For the sake of simplicity, time dependence of the generalized coordinates and their derivatives was not indicated. Finally, the linearized equation of motion reads

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{S}\mathbf{q}(t) = \mathbf{Q}(t), \quad (10)$$

where the mass ( $\mathbf{M}$ ) and the stiffness matrix ( $\mathbf{S}$ ) are determined by the mechanical properties of the system.

From (2) and (3), it follows that

$$\mathbf{Q}(t) = \mathbf{P}\mathbf{q}(t - \tau) + \mathbf{D}\dot{\mathbf{q}}(t - \tau) \quad (11)$$

with

$$\mathbf{P} = \begin{bmatrix} P_\varphi & P_\vartheta \\ -P_\varphi & -P_\vartheta \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} D_\varphi & D_\vartheta \\ -D_\varphi & -D_\vartheta \end{bmatrix}, \quad (12)$$

being the matrices of the proportional and the derivative gains, respectively.

The corresponding characteristic polynomial is

$$\begin{cases} \Delta(s) = P_0(s) + P_\tau(s)e^{-s\tau} & \text{with} \\ P_0(s) = s^4 + a_2s^2 + a_0 & \text{and} \\ P_\tau(s) = b_3s^3 + b_2s^2 + b_1s + b_0. \end{cases} \quad (13)$$

Coefficients of the polynomial  $P_0(s)$  are determined by the mechanical properties (mass, mass moment of inertia, geometry, passive stiffness and damping) of the analyzed system, and the control parameters  $P_\varphi, D_\varphi, P_\vartheta, D_\vartheta$



Table 2: Dimensionless model parameters admissible range

Parameter	MIN	MAX
$a_0$	-5000	0
$a_1$	-200	0
$a_2$	-1000	2000
$a_3$	0	100

are involved in the coefficients of polynomial  $P_\tau(s)$ . Since there is no damping in the system,  $a_1 = 0 = a_3 = 0$  where the physically admissible range of parameters is indicated in Table 2.

### 3. Background and prerequisites

This section provides the main methods, results and tools used in the sequel for the investigation of the modeling of the CNS action on the human stance. First, we recall some complex analysis results delimiting the location of some analytic functions' zeros. Next, we provide a basic primer on the well-known Gröbner basis concept, as well as the powerful *Cell Decomposition* algorithm which solves systems with semi-algebraic parameters and is implemented in the Computer Algebra System **Maple**. A review of the MID-based partial poles placement strategy concludes the section.

#### 3.1. Cauchy's argument principle

*Cauchy's argument principle* is a basic complex analysis property widely used in the stability analysis of linear time-invariant dynamical systems. Roughly speaking, the argument principle establishes a correspondence between the number of zeros minus the number of poles of a meromorphic function  $f$  in a simply connected domain  $D \subset \mathbb{C}$  and a contour integral, on the boundary  $\partial D$ , of the function's logarithmic derivative, which is also the winding number of the curve  $\partial D$ . Several stability methods such as the Nyquist criterion and the Mikhaylov curve derive from the argument principle.

**Theorem 1.** *Let  $U$  be a simply connected region with boundary  $\Gamma$  (piecewise smooth and oriented anti-clockwise). Let  $f$  be a meromorphic function in an open set containing the closure  $\bar{U}$  with poles  $p_1, \dots, p_l$  and zeros  $s_1, \dots, s_m$  counted according to their multiplicity, none of which belonging to the closed curve  $\Gamma$ . Then*

$$\frac{1}{2i\pi} \oint_{\Gamma} \frac{f'(s)}{f(s)} ds = \mathcal{Z} - \mathcal{P}, \quad (14)$$

where  $\mathcal{Z}$  and  $\mathcal{P}$  designate respectively the number of zeros and the number of poles of  $f$  enclosed by  $\Gamma$ .

#### 3.2. The Stepan-Hassard approach

The main theorem from<sup>46</sup> underscores the link between the number of unstable spectral values  $\mathbf{card}(\chi_+)$  and the number of critical spectral values  $\mathbf{card}(\chi_0)$ , both counting multiplicity.

**Theorem 2 (Hassard, <sup>46</sup>, pp. 223).** Consider the quasipolynomial function  $\Delta$  defined by (13). Let  $\rho_1, \dots, \rho_r$  be the positive roots of  $\mathcal{R}(y) = \Re(i^n \Delta(iy))$ , counted by their multiplicities and ordered increasingly. For each  $j = 1, \dots, r$  such that  $\Delta(i\rho_j) = 0$ , assume that the multiplicity of  $i\rho_j$  as a zero of  $\Delta(\lambda, \tau)$  is the same as the multiplicity of  $\rho_j$  as a root of  $\mathcal{R}(y)$ . Then  $\mathbf{card}(\chi_+)$  is given by the formula:

$$\mathbf{card}(\chi_+) = \frac{n - \mathbf{card}(\chi_0)}{2} + \frac{(-1)^r}{2} \operatorname{sgn} \mathcal{I}^{(\mu)}(0) + \sum_{j=1}^r \operatorname{sgn} \mathcal{I}(\rho_j), \quad (15)$$

where  $\mu$  designate the multiplicity of the zero spectral value of  $\Delta(\lambda, \tau) = 0$  and  $\mathcal{I}(y) = \Im(i^{-n} \Delta(iy))$ . Furthermore,  $\mathbf{card}(\chi_+)$  is odd (respectively, even) if  $\Delta^{(\mu)}(0) < 0$  ( $\Delta^{(\mu)}(0) > 0$ ). If  $\mathcal{R}(y) = 0$  has no positive zeros, set  $r = 0$  and omit the summation term in the expression of  $\mathbf{card}(\chi_+)$ . If  $\lambda = 0$  is not a root of the characteristic equation, set  $\mu = 0$  and interpret  $\mathcal{I}^{(0)}(0)$  as  $\mathcal{I}(0)$  and  $\Delta^{(0)}(0)$  as  $\Delta(0)$ .

### 3.3. Background on Gröbner bases

A wide range of problems related to the qualitative analysis of dynamical systems leads to systems of polynomial equations, see for instance <sup>47,48,49,50</sup>,

$$f_1 = 0, \dots, f_m = 0 \quad (16)$$

with  $f_i \in \mathbb{R}[x_1, \dots, x_n]$ . To solve this system we consider the ideal  $\langle f_1, \dots, f_m \rangle \subset \mathbb{R}[x_1, \dots, x_n]$ . For this aim, we use Gröbner Bases computations. In this section, we recall the basic facts about Gröbner bases, and refer the reader to <sup>51</sup> for details.

A *monomial ordering* is a total order on monomials that is compatible with the product and such that every nonempty set has a smallest element for the order. The leading term of a polynomial is the greatest monomial appearing in this polynomial.

A *Gröbner basis* of an ideal  $\mathcal{I}$  for a given monomial ordering is a set  $G$  of generators of  $\mathcal{I}$  such that the leading terms of  $G$  generate the ideal of leading terms of polynomials in  $\mathcal{I}$ . A polynomial is *reduced* with respect to the Gröbner basis  $G$  when its leading term is not a multiple of those of  $G$ . The basis is *reduced* if each element  $g \in G$  is reduced with respect to  $G \setminus \{g\}$ . For a given monomial ordering, the reduced Gröbner basis of a given set of polynomials exists and is unique, and can be computed using one's favorite general computer algebra system, like Maple, Magma or Singular. Several efficient implementations of the Gröbner basis algorithm exist, here we use the FGb implementation of  $F_4$  available in Maple <sup>52</sup>. The complexity of a Gröbner basis computation is well known to be generically exponential in the number of variables, and in the worst case doubly exponential in the number of variables. Moreover, a good choice of the monomial ordering reduces the computational cost.

The *graded reverse lexicographic order* or **grevlex** for short also denoted **tdeg** ordering is the most suited ordering for the computation of the (reduced) Gröbner basis. The monomials are first ordered by degree, and the order between two monomials of the same degree  $x_\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  and  $x_\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$  is given by  $x_\alpha \succ x_\beta$  when the last nonzero element of  $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$  is negative. Thus, among the monomials of degree  $d$ ,

the order is

$$x_1^d \succ x_1^{d-1}x_2 \succ x_1^{d-2}x_2^2 \succ \dots \succ x_2^d \succ x_1^{d-1}x_3 \succ x_1^{d-2}x_2x_3 \succ x_1^{d-2}x_3^2 \succ \dots \succ x_n^d.$$

However, a Gröbner basis for the **grevlex** ordering may not be appropriate for the computation of the solutions of the system (16). The most suited ordering for this computation is the *lexicographical* ordering (or **lex** ordering for short). The monomials are ordered by comparing the exponents of the variables in lexicographical order. Thus, any monomial containing  $x_1$  is greater than any monomial containing only variables  $x_2, \dots, x_n$ .

Under some hypotheses (radical ideal with a finite number of solutions, and up to a linear change of coordinates), the Gröbner Basis of an ideal  $\langle f_1, \dots, f_m \rangle$  for the **lex** order  $x_1 > \dots > x_n$  has the shape

$$\{x_1 - g_1(x_n), x_2 - g_2(x_n), \dots, x_{n-1} - g_{n-1}(x_{n-1}), g_n(x_n)\}, \quad (17)$$

where the  $g_i$  are univariate polynomials. In this case, the computation of the solutions of the system follows easily. In the general case, the shape of the Gröbner basis for the **lex** ordering is more complicated, but it is equivalent to several triangular systems for which the computation of the solutions are straightforward.

An important point is that a Gröbner basis for the **lex** order is in general hard to compute directly. It is much faster to compute first a Gröbner basis for the **grevlex** order, and then to make a change of ordering to the **lex** order.

Further, if one wants to compute all the polynomials in  $\mathcal{I}$ , that do not depend on the variables  $x_i, \dots, x_n$ , i.e.,  $\mathcal{I} \cap \mathbb{R}[x_1, \dots, x_{i-1}]$  an elimination procedure has to be carried out. Geometrically speaking, this elimination of indeterminates corresponds to the projection of the associated variety into  $[x_1, \dots, x_1]$ . This projection can be characterized using a Gröbner basis of  $\mathcal{I}$  for **lex** ordering often designating an *elimination ordering*.

The precise ordering we use to compute the Gröbner bases of the polynomial systems occurring in this paper are **grevlex** or **lexdeg** ordering.

Finally, we use repeatedly the *Radical Membership Theorem*:

**Theorem 3** (<sup>51</sup>). *Let  $\mathcal{I} = \langle f_1, \dots, f_s \rangle$  be an ideal of  $k[x_1, \dots, x_n]$ , then  $f$  belongs to  $\sqrt{\mathcal{I}}$  if, and only if,  $\langle f_1, \dots, f_s, 1 - yf \rangle = \langle 1 \rangle = k[x_1, \dots, x_n, y]$ .*

A finite set of polynomial equations  $\langle f_1, \dots, f_m \rangle \subset \mathbb{R}[x_1, \dots, x_n]$  is said to be *inconsistent* or *unfeasible* if, and only if,  $\langle 1 \rangle$  is the corresponding reduced Gröbner basis.

### 3.4. Cell decomposition in CAD routine

The main algebraic tool we use is the so-called *discriminant variety* associated to an algebraic set, which was introduced in<sup>53</sup>.

Let us consider the following parametric system of algebraic equations and inequations :

$$\begin{cases} f_1(T, X) = 0 \\ \vdots \\ f_n(T, X) = 0 \end{cases}, \quad \begin{cases} g_1(T, X) \neq 0 \\ \vdots \\ g_m(T, X) \neq 0 \end{cases} \quad (18)$$

where  $f_i$  and  $g_j$  are parametric multivariate polynomials in the ring  $\mathbb{Q}[t_1, \dots, t_r][x_1, \dots, x_q]$  with the parameter vector  $T = (t_1, \dots, t_r)$  and variable vector  $X = (x_1, \dots, x_q)$ . The corresponding *discriminant variety* is defined as the Zariski closure of the associated parameter space<sup>53</sup>. The smallest discriminant variety which is called the minimal discriminant variety, corresponds to the intersection of all admissible discriminant varieties and characterizes such a parametric system by providing a fundamental tool for classifying regions in the parameter space with respect to the admissible number of real solutions, see for instance<sup>54</sup>. The complementary of this algebraic variety can be partitioned, using the standard the *cylindrical algebraic decomposition*, into a set of connected components where the sign vector of polynomial equations remains constant. This strategy is implemented in the certified Maple package `RootFinding[Parametric]`. More precisely, `CellDecomposition` decomposes the parameter space of a parametric system into cells in which the original system has a constant number of solutions,<sup>55,53,54</sup>.

### 3.5. The MID paradigm: A partial pole-placement strategy

In the study of linear time-invariant (LTI) dynamical systems represented by DDEs, the characteristic function is expressed as a *quasipolynomial*, that is formally defined as follows.

**Definition 3.1.** *A quasipolynomial is a particular entire function  $\Delta : \mathbb{C} \rightarrow \mathbb{C}$  which may be written as follows*

$$\Delta(s) = \sum_{i=0}^{\ell} P_i(s) e^{-\tau_i s}, \quad (19)$$

where  $\ell$  is a positive integer,  $\tau_i \in \mathbb{R}_+$  ( $i = 0 \dots \ell$ ) are pairwise distinct numbers and  $P_i$  ( $i = 0 \dots \ell$ ) are polynomials of degree  $d_i \geq 0$ . Its degree  $\deg(\Delta)$  is given by

$$\deg(\Delta) = \ell + \sum_{i=0}^{\ell} \deg(P_i).$$

An important result in the literature, known as *Pólya-Szegő bound* and denoted  $PS_B$  (see<sup>56</sup> Problem 206.2, page 144 and page 347,<sup>57</sup>) plays an important role in characterizing the maximal allowable multiplicity of a characteristic root of a quasipolynomial. More precisely, we have the following.

**Proposition 3.1.** *Let  $\Delta$  be a quasipolynomial of degree  $D = \deg(\Delta)$ . Then, any characteristic root  $s_0 \in \mathbb{C}$  of  $\Delta$  exhibits a multiplicity at most equal to  $D$ .*

As briefly explained in the Introduction, a characteristic root  $\lambda_0$  (of  $\Delta$ ) satisfies the *MID property* if

- (i) its *algebraic multiplicity* (denoted  $M = M(\lambda_0)$ ) is *larger than one*,
- (ii) it is *dominant*, meaning that the remaining characteristic roots  $\lambda_\sigma$  of the spectrum satisfy the condition  $\Re(\lambda_\sigma) \leq \Re(\lambda_0)$ .

Since the maximal admissible multiplicity is defined by the degree of the quasipolynomial  $\Delta$  (see, e.g.,<sup>30</sup>), it is clear that  $2 \leq M \leq \deg(\Delta)$ . The case  $M = \deg(\Delta)$  is called *generic multiplicity*, and any multiplicity larger than one and smaller than  $\deg(\Delta)$  denotes an *intermediate multiplicity*. In what follows, we give a precise definition of the *dominant root*.

It turns out that, for the characteristic equations corresponding to delay systems, the real roots of maximal multiplicity are necessarily dominant. This property is known as Generic Multiplicity-Induced-Dominancy (GMID for short) and consists in conditions under which a given real root of maximal multiplicity is necessarily dominant. However, multiple roots with intermediate admissible multiplicities may or may not be dominant. As for the case of a root of strictly intermediate multiplicity, one must look for conditions on the free parameters of the system for which the former is dominant, this property is called Multiplicity-Induced-Dominancy (MID).

It should be mentioned that the generic case, i.e.,  $M(\lambda_0) = \deg(\Delta)$ , was already treated and explicit characterizations exist: the retarded case  $m = n - 1$  was characterized in<sup>30</sup> and a unified treatment of the retarded and neutral cases, where  $m \leq n$ , was provided in<sup>35</sup>.

To the best of the authors' knowledge, in the open literature, the MID with over-order multiplicities has been addressed in two particular configurations  $M(\lambda_0) = n + 1$  and  $M(\lambda_0) = n + m$ . More precisely, in the "limit" case  $M(\lambda_0) = n + 1$ , sufficient conditions for their validity have been proposed in<sup>31</sup>, where the authors exploited the particular spectrum location of the open-loop plant.

### 3.6. Algorithmic investigation of the MID property

In this subsection, we provide an effective algorithm we are explicitly using in our framework. More precisely, we first establish conditions on the system's parameters which guarantee the existence of a multiple root. Second, we perform an affine change of variable in the characteristic equation  $\Delta(\lambda) = P_0(\lambda) + P_\tau(\lambda)e^{-\lambda\tau}$  in order to reduce the corresponding quasipolynomial to a normalized form:  $\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_\tau(z)e^{-z}$ . Next, we derive a bound on the imaginary part of roots of the normalized characteristic function in the complex right half-plane. Lastly, a certification of the dominance of the multiple root is demonstrated. In what follows, Algorithm 1 indicates the steps to be followed to reach a suitable frequency bound (see<sup>58,40</sup>).

## 4. Main results

This section is dedicated to the main results. After a normalization of the characteristic equation, the maximal admissible multiplicity for a root of the latter is obtained as a root,  $s_0$ , of a fourth-order polynomial named the elimination-produced polynomial. We also show that the gains are real if, and only if,  $s_0$  is real.

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**Algorithm 1** Estimation of the MID frequency bound in time-delay differential equations with single delay

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**Require:**  $\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_\tau(z)e^{-z}$  {Normalized quasipolynomial}

{Initialization}

ord = 0

{ord: order of truncation of the Taylor expansion of  $e^{2x} = \underbrace{1}_{\text{ord}=0} + 2x + 2x^2 + \frac{4x^3}{3} + \dots$ }

*dominance* = false

$\exists z_0 = x + i\omega \in \mathbb{R}_+^* + i\mathbb{R}_+^*$  s.t.  $\tilde{\Delta}(z_0) = 0$

$|\tilde{P}_0(x + i\omega)|^2 e^{2x} = |\tilde{P}_\tau(x + i\omega)|^2$

**while**  $\sim$  *dominance* **do**

ord = ord+1

$F(x, \omega) = |\tilde{P}_\tau(x + i\omega)|^2 - |\tilde{P}_0(x + i\omega)|^2 T_{ord}(e^{2x}) > 0$

{ $T_{ord}(e^{2x})$ : Taylor expansion of  $e^{2x}$  of order = *ord*}

$\omega^2 = \Omega$

$H(x, \Omega)$  {The polynomial characterizing the real roots of  $F$ }

**end while**

$\Omega_k(x)$  { $k^{\text{th}}$  real root of  $H$ , depend on free parameters}

**if**  $\max_x(\max_k(\Omega_k(x))) < \pi^2$  **then**

*dominance* = true

**end if**

**return** Frequency bound

---

Then, owing to the Maple routine `CellDecomposition`, we investigate regions in the parameter space where we can assign such a multiple negative real root as the spectral abscissa. The proof of the dominancy is based on a well proven algorithm described in Algorithm 1

#### 4.1. A normalized fourth order friction-free model

We investigate a stabilizing delayed-controller for the rolling balance board. In closed loop, the corresponding characteristic function reads:

$$\begin{cases} \Delta(s) = P_0(s) + P_\tau(s)e^{-s\tau} & \text{with} \\ P_0(s) = s^4 + a_2s^2 + a_0 & \text{and} \\ P_\tau(s) = b_3s^3 + b_2s^2 + b_1s + b_0. \end{cases} \quad (20)$$

Since  $a_0$  is typically a negative parameter as shown in Table 2, the change of variables ( $s \rightarrow \sqrt[4]{-a_0} s$ ) reduces the analysis to the normalized characteristic function:

$$\left\{ \begin{array}{l} \tilde{\Delta}(s) = \tilde{P}_0(s) + \tilde{P}_\tau(s)e^{-s\tilde{\tau}} \quad \text{with} \quad \tilde{P}_0(s) = s^4 + \tilde{a}_2 s^2 - 1 \quad \text{and} \quad \tilde{P}_\tau(s) = \tilde{b}_3 s^3 + \tilde{b}_2 s^2 + \tilde{b}_1 s + \tilde{b}_0, \\ \tilde{\tau} = \sqrt[4]{-a_0} \tau, \\ \tilde{a}_2 = a_2 / \sqrt{-a_0}, \\ \tilde{b}_k = b_k / (-a_0)^{\frac{4-k}{4}} \quad \text{for} \quad k \in \{0, \dots, 3\}. \end{array} \right. \quad (21)$$

For the sake of simplicity, the  $\tilde{\cdot}$  symbol is omitted, so the normalized quasipolynomial function studied in the sequel is the following:

$$\Delta(s) = s^4 + a_2 s^2 - 1 + (b_3 s^3 + b_2 s^2 + b_1 s + b_0) e^{-s\tau}. \quad (22)$$

At the end of our investigations, the transformation (21) shall enable the reconstruction of the appropriate stabilizing conditions for (20).

The present study is three-fold. First, we address the conditions on the quasipolynomial parameters providing the maximal admissible multiplicity for a given spectral value. Second, under the maximal multiplicity conditions, we investigate additional assumptions on the parameters in order to guarantee the dominance of the multiple spectral value. Third, how such a dominance result can be exploited in the design of a delayed stabilizing controller?

#### 4.2. Forcing multiplicity

The MID property introduced in the previous section, which consists in forcing a given quasipolynomial to have a root of a prescribed multiplicity, permits under appropriate conditions to characterize the rightmost-root. As a matter of fact, this multiplicity constraint defines a manifold in the parameter space enabling the tuning of the gains  $b_k$  when the delay  $\tau$  is left-free and guaranteeing the exponential stability of the closed-loop system solution. Hereafter, the next theorem, which is based on the MID property, provides the explicit conditions on the parameters' values guaranteeing a targeted multiplicity. Recall that the multiplicity of a given root of the generic quasipolynomial (22) is bounded by its degree; that is, 8 is the bound for multiplicity. However, forcing the multiplicity must not constrain the physical model parameters.

**Theorem 4.** *The following assertions hold:*

1. *If the parameters  $a_2$  and  $\tau$  are left free, then the maximal multiplicity of a given root of (22) is 5. Furthermore, such a multiplicity is reached if, and only if,  $s = s_0$ , a root of the elimination-produced polynomial*

$$\mathbb{P}(s) = \tau^4 s^4 + 16 \tau^3 s^3 + (\tau^2 a_2 + 72) \tau^2 s^2 + (8 \tau^2 a_2 + 96) \tau s - \tau^4 + 12 \tau^2 a_2 + 24. \quad (23)$$

2. The root  $s_0$  of (22) has multiplicity 5 if, and only if, the system's parameters satisfy:

$$\left\{ \begin{array}{l} \frac{3\tau^4 b_0}{e^{s_0\tau}} = (-a_2^2 s_0^2 - 2s_0^2 + a_2)\tau^6 + (-23a_2 s_0^3 - 8s_0 a_2^2 + 26s_0)\tau^5 + (-5a_2 s_0^2 - 12a_2^2 - 254)\tau^4 \\ \quad + (2348s_0^3 + 1612a_2 s_0)\tau^3 + (15804s_0^2 + 3060a_2)\tau^2 + 23976s_0\tau + 6168, \\ \frac{\tau^3 b_1}{e^{s_0\tau}} = (-a_2 s_0^3 + 2s_0)\tau^5 + (5a_2 s_0^2 - 22)\tau^4 + (196s_0^3 + 148a_2 s_0)\tau^3 \\ \quad + (1380s_0^2 + 276a_2)\tau^2 + 2136s_0\tau + 552, \\ \frac{\tau^3 b_2}{e^{s_0\tau}} = (a_2 s_0^2 - 2)\tau^5 + (18s_0^3 + 15a_2 s_0)\tau^4 + (138s_0^2 + 29a_2)\tau^3 + 228s_0\tau^2 + 60\tau, \\ \frac{3\tau b_3}{e^{s_0\tau}} = (2s_0^3 + a_2 s_0)\tau^3 + (18s_0^2 + 3a_2)\tau^2 + 36s_0\tau + 12. \end{array} \right. \quad (24)$$

3. The gains  $b_k$  in (22) are real if, and only if,  $s_0$  is real.

**Remark 4.1.** Theorem 4 gives conditions for a root of (22) to reach multiplicity 5. These conditions can be easily exploited using (21) to establish those for a root of multiplicity 5 for (20) where  $a_0$  is a left-free parameter. A given complex number is a root of multiplicity 5 for (20) if, and only if, it is root of the corresponding elimination-produced polynomial  $\tilde{\mathbb{P}}(s) = s^4 + 16s^3 + (\tau^2 a_2 + 72)s^2 + (8\tau^2 a_2 + 96)s + \tau^4 a_0 + 12\tau^2 a_2 + 24$ . Obviously, the nature (real/complex) of the zeros of  $\tilde{\mathbb{P}}$  is the same as those of  $\mathbb{P}$ .

**Proof of Theorem 4.** First, the vanishing of the quasipolynomial  $\Delta$  given in (22) yields the elimination of the exponential term as a rational function in  $s$ :

$$e^{-\tau s} = -\frac{\tilde{P}_0(s)}{\tilde{P}_\tau(s)} \quad (25)$$

Next, we investigate potential roots with algebraic multiplicity 5 by substituting the obtained equality (25) in the ideal  $\mathcal{I}_5$  generated by the first four derivatives of  $\Delta$  that is  $\mathcal{I}_5 = \langle \partial_s \Delta, \partial_s^2 \Delta, \dots, \partial_s^4 \Delta \rangle$ . This enables us



to investigate the following manifold of four algebraic equations in 7 unknowns  $b_0, b_1, b_2, b_3, a_2, s, \tau$  :

$$\begin{aligned}
0 &= \left( \tau \tilde{P}_0(s) + 4s^3 + 2a_2s \right) b_0 + \left( s\tilde{P}_0(s)\tau + 3s^4 + a_2s^2 + 1 \right) b_1 \\
&\quad + \left( s^2\tilde{P}_0(s)\tau + 2s^5 + 2s \right) b_2 + \left( s^3\tilde{P}_0(s)\tau + s^6 - a_2s^4 + 3s^2 \right) b_3 \\
0 &= \left( -\tau^2\tilde{P}_0(s) + 12s^2 + 2a_2 \right) b_0 + \left( -s\tilde{P}_0(s)\tau^2 + 2\tau\tilde{P}_0(s) + 12s^3 + 2a_2s \right) b_1 \\
&\quad + \left( -s^2\tilde{P}_0(s)\tau^2 + 4s\tilde{P}_0(s)\tau + 10s^4 + 2 \right) b_2 \\
&\quad + \left( -s^3\tilde{P}_0(s)\tau^2 + 6s^2\tilde{P}_0(s)\tau + 6s^5 - 4s^3a_2 + 6s \right) b_3 \\
0 &= \left( \tau^3\tilde{P}_0(s) + 24s \right) b_0 + \left( s\tilde{P}_0(s)\tau^3 - 3\tau^2\tilde{P}_0(s) + 24s^2 \right) b_1 \\
&\quad + \left( s^2\tilde{P}_0(s)\tau^3 - 6s\tilde{P}_0(s)\tau^2 + 6\tau\tilde{P}_0(s) + 24s^3 \right) b_2 \\
&\quad + \left( s^3\tilde{P}_0(s)\tau^3 - 9s^2\tilde{P}_0(s)\tau^2 + 18s\tilde{P}_0(s)\tau + 18s^4 - 6a_2s^2 + 6 \right) b_3 \\
0 &= \left( 24 - \tau^4\tilde{P}_0(s) \right) b_0 + \left( -s\tilde{P}_0(s)\tau^4 + 4\tau^3\tilde{P}_0(s) + 24s \right) b_1 \\
&\quad + \left( -s^2\tilde{P}_0(s)\tau^4 + 8s\tilde{P}_0(s)\tau^3 - 12\tau^2\tilde{P}_0(s) + 24s^2 \right) b_2 \\
&\quad + \left( -s^3\tilde{P}_0(s)\tau^4 + 12s^2\tilde{P}_0(s)\tau^3 - 36s\tilde{P}_0(s)\tau^2 + 24\tau\tilde{P}_0(s) + 24s^3 \right) b_3.
\end{aligned} \tag{26}$$

The above system is a linear system in the unknowns  $(b_k)_{0 \leq k \leq 3}$ . Using standard elimination techniques, we obtain a set of three solutions; the first one, asserts that  $b_k = 0$  for  $k \in \llbracket 0, 3 \rrbracket$ , the second one corresponds to  $s$  as a root of the open-loop polynomial ( $\tilde{P}_0(s) = 0$ ) with  $b_2 = -s^3b_1 - s^2b_0 - sa_2b_1 - sb_3 - a_2b_0$  and the last solution corresponds to  $s = s_0$  as defined in (23) and  $b_k$  as rational functions in  $(s_0, a_2, \tau)$ .

Observe that the first solution corresponds to the open-loop system while the second solution is inconsistent with respect to the transcendental term's elimination (25). Hence, these two solutions are discarded. Next, substituting conditions pertaining to the third solution in (25) yields the explicit values of the gains  $b_k$  allowing to tune the parameters as provided in (24), which concludes items 1) and 2).

To show 3), we assume that there exists  $s^* \in \mathbb{C} - \mathbb{R}$  such that  $\mathbb{P}(s^*) = 0$  and the coefficients  $b_k(s^*, a_2, \tau) \in \mathbb{R}$  for  $k \in \llbracket 0, 3 \rrbracket$ . Then, we substitute  $s^* = \alpha + i\beta$  with  $\alpha \neq 0$  and  $\beta \neq 0$  in the expressions of  $b_k$  for  $k \in \llbracket 0, 3 \rrbracket$  and define an ideal  $\mathcal{I}_{\mathcal{R}}$  (where the index  $\mathcal{R}$  stands for the realness of the gains  $b_k$ ) generated by  $(p_k)_{0 \leq k \leq 3}$  corresponding respectively to the imaginary part of the gains  $(b_k)_{0 \leq k \leq 3}$ , and the polynomials  $p_4 = \Re(\mathbb{P}(s^*))$  and  $p_5 = \Im(\mathbb{P}(s^*))$ ; that is,  $\mathcal{I}_{\mathcal{R}} = \langle p_0, \dots, p_5 \rangle$ .

We proceed as advanced in Section 3.3 by computing a Gröbner basis with respect to the elimination order

lex with respect to the unknowns  $a_2$  and  $\tau$ . We obtain a basis of 9 elements  $G = \langle G_1, \dots, G_9 \rangle$ , where

$$\begin{aligned}
G_1 &= \beta^4, \\
G_2 &= \alpha^{12}\beta^2 - 3\alpha^8\beta^2 + 21\alpha^4\beta^2 - 125\beta^2, \\
G_3 &= 9\alpha^{11}\beta^2 - 32\alpha^7\beta^2 + 109\alpha^3\beta^2 + 275\beta^2\tau, \\
G_4 &= 2\alpha^{10}\beta^2 - 56\alpha^6\beta^2 - 208\alpha^2\beta^2 + 275\beta^2a_2, \\
G_5 &= 936\alpha^8\beta^2 + 825\alpha^8\tau^2 + 8910\alpha^7\tau - 3190\alpha^5\tau a_2 + 22110\alpha^6 - 7354\alpha^4\beta^2 - 1375\alpha^4\tau^2 - 14465\alpha^4a_2 \\
&\quad - 22110\alpha^3\tau + 2200\alpha^2a_2^2 + 5500\alpha\tau a_2 - 56760\alpha^2 - 11500\beta^2 - 2200\tau^2 + 22000a_2, \\
G_6 &= -5244\alpha^{11}\beta^2 + 46292\alpha^7\beta^2 + 28875\alpha^7\tau^2 + 311850\alpha^6\tau + 5500\alpha^4\tau^3 - 111650\alpha^4\tau a_2 + 773850\alpha^5 \\
&\quad - 182714\alpha^3\beta^2 + 116875\alpha^3\tau^2 - 506275\alpha^3a_2 + 447150\alpha^2\tau + 77000\alpha a_2^2 + 5500\tau^3 - 55000\tau a_2 \\
&\quad + 257400\alpha, \\
G_7 &= -852\alpha^{10}\beta^2 + 6036\alpha^6\beta^2 + 4125\alpha^6\tau^2 + 44550\alpha^5\tau - 15950\alpha^3\tau a_2 + 110550\alpha^4 - 18862\alpha^2\beta^2 \\
&\quad + 9625\alpha^2\tau^2 - 72325\alpha^2a_2 + 2750\tau^2a_2 + 54450\tau\alpha + 11000a_2^2 + 46200 \\
G_8 &= -556\alpha^{10}\beta^2 + 7868\alpha^6\beta^2 + 4125\alpha^6\tau^2 + 41250\alpha^5\tau - 13750\alpha^3\tau a_2 + 84150\alpha^4 - 10926\alpha^2\beta^2 \\
&\quad + 4125\alpha^2\tau^2 + 1375\alpha\tau a_2^2 - 43725\alpha^2a_2 + 8250\tau\alpha + 5500a_2^2 + 6600 \\
G_9 &= 295576\alpha^9\beta^2 + 210375\alpha^9\tau^2 + 2124210\alpha^8\tau - 688490\alpha^6\tau a_2 + 4534530\alpha^7 - 1264134\alpha^5\beta^2 \\
&\quad - 253825\alpha^5\tau^2 - 2301695\alpha^5a_2 - 4434210\alpha^4\tau + 1305700\alpha^2\tau a_2 + 88000\alpha a_2^3 - 9859080\alpha^3 \\
&\quad - 3371860\alpha\beta^2 - 464200\alpha\tau^2 - 88000\tau a_2^2 + 3691600\alpha a_2 - 105600\tau
\end{aligned} \tag{27}$$

Obviously, the first element of  $G$  signifies that a solution of the multivariate system exists only if  $\beta = 0$ , which contradicts the fact that  $s^* \in \mathbb{C} - \mathbb{R}$ . Note that this can also be recovered by intentionally adding the polynomial  $\beta T - 1$ , where  $T$  stands for a non zero auxiliary unknown, to the polynomials defining the ideal  $\mathcal{I}_{\mathcal{R}}$  and employing the Membership Theorem. As a matter of fact, in the latter case one obtains  $\langle 1 \rangle$  as a Gröbner basis. ■

**Remark 4.2.** *From a control theory viewpoint, the controllers' gains have to be real and assertion 3) of Theorem 4 guarantees that they are so for the considered real solution  $s_0$ . Since the degree of the elimination-produced polynomial is even and our concern is related to such polynomials with at least one real-root, then two cases are of interest; an elimination-produced polynomial with either four real roots, or two real roots. In addition, in the case of a real-rooted elimination-produced polynomial, the corresponding roots need to be negative to be admissible assignable roots for the initial quasipolynomial. More precisely, the only cases of interest among elimination-produced polynomials with two negative roots are those with a negative dominant zero. Lastly, note that apart from real-rooted Hurwitz polynomials no general result exists for characterizing polynomials with negative dominant roots. One finds in Figure 2 a chart of the number of real zeros in the parameter space  $(\tau, a_2)$ .*

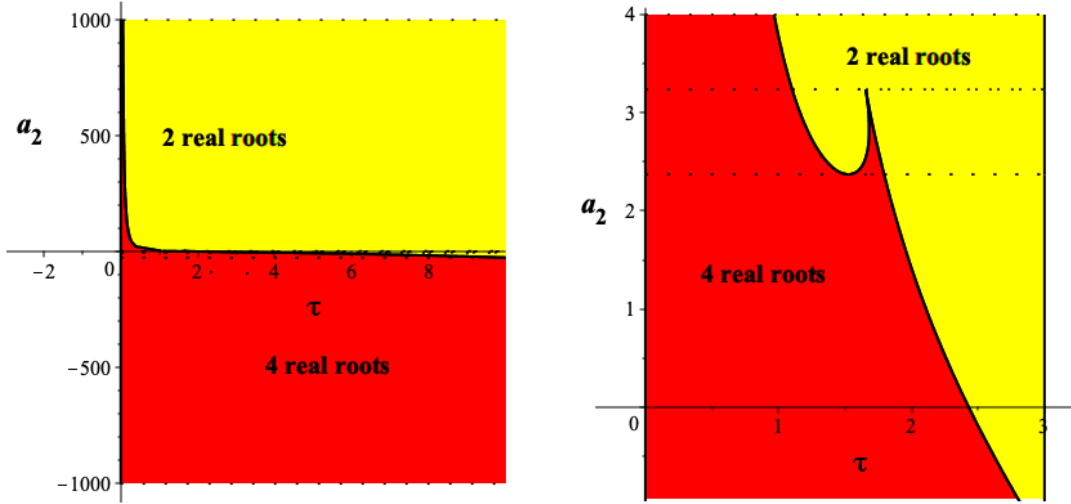


Figure 2: Number of real roots of (23). (Left) Real roots number in the prescribed region. (Right) A zoom on the region including a cusp point at  $(\tau, a_2) = \left( \sqrt{2} \sqrt[4]{3 \cdot 2^{2/3} + 12} \sqrt[3]{2} - 18, \frac{\sqrt{3 \cdot 2^{2/3} + 12} \sqrt[3]{2} - 18}{5} \left( 4 \sqrt[3]{2} + 3 \cdot 2^{2/3} + 2 \right) \right)$  corresponding to a triple root at  $s = 2^{2/3} - 4$ . The figures are produced using the Maple package `RootFinding[Parametric]` and especially the certified routine `CellDecomposition`

#### 4.2.1. Realrooted elimination-produced polynomial

The following proposition provides conditions on the system's parameters guaranteeing the realrootedness of the associated elimination-produced polynomial.

**Proposition 4.1.** *Consider the following polynomials :*

$$\begin{aligned} \check{p}(a_2) &= 5 a_2^6 + 36 a_2^4 - 432 a_2^2 - 5184, & \check{p}(a_2) &= 5 a_2^6 + 54 a_2^4 - 2592, \\ \bar{p}(a_2) &= 3285 a_2^6 + 3432 a_2^4 + 25216 a_2^2 - 384, & \hat{p}(\tau) &= \tau^4 - 12 \tau^2 a_2 - 24, \\ \tilde{p}(\tau) &= -1990656 + (a_2^2 + 4)^2 \tau^{12} + (4 a_2^5 - 32 a_2^3 - 192 a_2) \tau^{10} + (-280 a_2^4 + 768 a_2^2 + 3456) \tau^8 \\ &\quad + 9728 \tau^6 a_2^3 + (-152064 a_2^2 - 82944) \tau^4 + 995328 \tau^2 a_2. \end{aligned}$$

Let  $\hat{a}_2, \check{a}_2, \check{a}_2, \check{a}_2, \bar{a}_2, \bar{a}_2$  be respectively the first and second real roots of polynomials  $\check{p}, \check{p}$  and  $\bar{p}$  above. Let  $\hat{\tau}_2$  be the second real root of  $\hat{p}$  and let  $\tilde{\tau}_{k \in \mathbb{N}}$  be the  $k^{\text{th}}$  real root (ordered increasingly) of  $\tilde{p}$ . The elimination-produced polynomial  $\mathbb{P}$  admits four negative roots if, and only if, its parameters satisfy one of the following assertions:

1.  $-1000 < a_2 < \bar{a}_2$  and  $0 < \tau < \hat{\tau}_2$ ,
2.  $\check{a}_2 < a_2 < \check{a}_2$  and  $\tau \in (0, \tilde{\tau}_4) \cup (\tilde{\tau}_5, \tilde{\tau}_6)$ ,
3.  $a_2 \in (\bar{a}_2, \check{a}_2) \cup (\check{a}_2, 2000)$ , and  $0 < \tau < \tilde{\tau}_2$ .

**Proof.** We use the Maple routine `CellDecomposition` of the `RootFinding[Parametric]` package <sup>(59,60,54)</sup> which decomposes the parameters' space into two manifolds: the discriminant variety and its complement. As

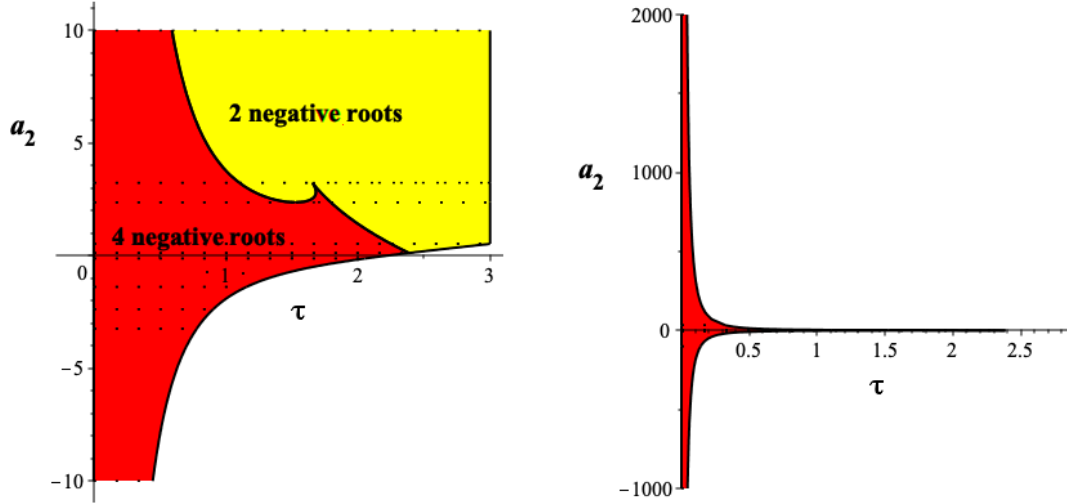


Figure 3: (Left) Number of negative roots of (23) in a compact region. (Right) Sub-region in the prescribed domain guaranteeing the real-rootedness of (23) as well as the negativity of its zeros. The figures are produced using the `Maple` package `RootFinding[Parametric]` and especially the certified routine `CellDecomposition`; see<sup>53</sup> for further insights on the mathematical foundation of the routine.

an input for the `CellDecomposition` routine, we include the vanishing elimination-produced polynomial  $\mathbb{P} = 0$ , the constraints on the system's parameters  $-1000 < a_2 < 2000$ , a delay range  $0 < \tau < 200$  and the negativity of the assigned root  $s_0 < 0$ . The routine computes a set of polynomials whose zeros define a discriminant variety of the parametric equation  $\mathbb{P} = 0$ , as well as a cylindrical algebraic decomposition of the complementary of this discriminant variety, which yields 46 cells, each of which contains a fixed number of negative roots and is represented by a sampling point (strictly interior to the cell). By a post-treatment, consisting in merging neighboring cells and exploiting their algebraic characterization, we obtain the announced result. ■

#### 4.2.2. Fredholm representation of the normalized characteristic function

Let  $s_0 \in \mathbb{R}_-$  and consider, as in<sup>35</sup>, the quasipolynomial  $\hat{\Delta} : \mathbb{C} \rightarrow \mathbb{C}$  obtained from  $\Delta$  in (22) by the following change of variables  $\hat{\Delta}(\lambda) = \tau^4 \Delta\left(\frac{\lambda}{\tau} + s_0\right)$ ,  $\lambda \in \mathbb{C}$ , then

$$\hat{\Delta}(\lambda) = \sum_{i=0}^4 \alpha_i \lambda^i + \sum_{i=0}^3 \beta_i \lambda^i e^{-\lambda}, \quad (28)$$

where

$$\left\{ \begin{array}{l} \alpha_0 = -16\tau^3 s_0^3 - 72\tau^2 s_0^2 + \frac{(-24\tau^2 a_2 - 288) s_0 \tau}{3} - 12\tau^2 a_2 - 24, \\ \alpha_1 = 4\tau^3 s_0^3 + 2\tau^3 a_2 s_0, \\ \alpha_2 = 6\tau^2 s_0^2 + \tau^2 a_2, \\ \alpha_3 = 4s_0 \tau, \\ \alpha_4 = 1, \\ \beta_0 = 16\tau^3 s_0^3 + 72\tau^2 s_0^2 + \frac{(24\tau^2 a_2 + 288) s_0 \tau}{3} + 12\tau^2 a_2 + 24, \\ \beta_1 = 12\tau^3 s_0^3 + 72\tau^2 s_0^2 + \frac{(18\tau^2 a_2 + 288) s_0 \tau}{3} + 12\tau^2 a_2 + 24, \\ \beta_2 = 4\tau^3 s_0^3 + 30\tau^2 s_0^2 + \frac{(6\tau^2 a_2 + 144) s_0 \tau}{3} + 5\tau^2 a_2 + 12, \\ \beta_3 = \frac{2\tau^3 s_0^3}{3} + 6\tau^2 s_0^2 + \frac{(\tau^2 a_2 + 36) s_0 \tau}{3} + \tau^2 a_2 + 4. \end{array} \right.$$

The normalized quasipolynomial  $\hat{\Delta}$  defined in (28) can be factorized as

$$\hat{\Delta}(\lambda) = \lambda^5 \int_0^1 q(t) e^{-t\lambda} dt, \quad (29)$$

where

$$\begin{aligned} q(t) = t^4 \left( -\frac{2\tau^3 s_0^3}{3} - \frac{\tau^3 a_2 s_0}{3} - 3\tau^2 s_0^2 - \frac{\tau^2 a_2}{2} - 4s_0 \tau - 1 \right) + t^3 \left( \frac{2\tau^3 s_0^3}{3} + \frac{\tau^3 a_2 s_0}{3} \right) \\ + t^2 \left( 3\tau^2 s_0^2 + \frac{\tau^2 a_2}{2} \right) + 4ts_0 \tau + 1. \end{aligned} \quad (30)$$

Let us adopt the following parameterization

$$v = \tau s_0 \quad \text{and} \quad \sigma = \tau^2 a_2.$$

Notice that the parameter  $v$  is necessarily negative. Under this new parameterization, the integrand in (30) becomes

$$q_{v,\sigma}(t) = (1-t) \tilde{q}_{v,\sigma} \quad (31)$$

where

$$\tilde{q}_{v,\sigma}(t) = (4v^3 + 2\sigma v + 18v^2 + 3\sigma + 24v + 6)t^3 + (18v^2 + 3\sigma + 24v + 6)t^2 + (24v + 6)t + 6 \quad (32)$$

#### 4.2.3. Constancy sign of the integrand $q_{v,\sigma}$

In our approach, the sign constancy of the polynomial  $q_{v,\sigma}$  defined previously in (30) for  $t \in (0, 1)$  is necessary. Therefore, the following proposition gives regions in the parameter space guaranteeing the sign constancy of  $q_{v,\sigma}$  for  $t \in (0, 1)$ .

**Proposition 4.2.** *Let  $q_{v,\sigma}$  be the parametric polynomial defined by (31) and define in the parameters' space the region  $R$  as follows*

$$R = \left\{ (v, \sigma) \in \mathbb{R}_- \times \mathbb{R} : 6v^3 + 3\sigma v + 18v^2 + 3\sigma + 24v + 6 < 0 \quad \text{and} \right. \\ \left. 12v^3 + 6\sigma v + 18v^2 + 3\sigma + 48v + 12 - \frac{(4v^3 + 2\sigma v + 18v^2 + 3\sigma + 24v + 6)(2v^3 + \sigma v + 12v)}{6v^3 + 3\sigma v + 18v^2 + 3\sigma + 24v + 6} < 0 \right. \\ \left. \text{and } 2v^3 + \sigma v + 12v < 0 \quad \text{and} \quad 4v^3 + 2\sigma v + 18v^2 + 3\sigma + 24v + 6 < 0 \right\}. \quad (33)$$

If  $(v, \sigma) \in R$ , then the parametric polynomial  $q_{v,\sigma}(t)$  has a constant while  $t \in (0, 1)$ .

**Proof.** Since  $t \in (0, 1)$ , then using (31), the constancy sign of  $q_{v,\sigma}$  depend on the constancy sign of  $\tilde{q}_{v,\sigma}$ . If the roots  $\tilde{q}_{v,\sigma}$  are with negative real part, then the constancy sign of  $\tilde{q}_{v,\sigma}$  is guaranteed. To do that, let assume for the moment that  $t$  is a complex variable, then, the Routh-Hurwitz criterion ensures that all the roots of the equation  $\tilde{q}_{v,\sigma}(t) = 0$  have negative real part without explicitly computing these roots. The Routh-Hurwitz table gives the following condition

$$\left\{ v > 0, \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3} < \sigma \right\} \quad (34)$$

which is unfortunately inconsistent since  $v$  is considered to be negative, that means that  $\tilde{q}_{v,\sigma}$  may have roots with positive or negative real part.

Let us now investigate the appropriate conditions guaranteeing that the roots of  $\tilde{q}_{v,\sigma}$  which are with real part greater than 0 are in fact necessarily with real part greater than 1. We propose first to do a scaling in the real axis using the change of variable  $\hat{q}_{v,\sigma}(t) = \tilde{q}_{v,\sigma}(t - 1)$ , and next we proceed with the symmetric of  $\hat{q}_{v,\sigma}$ , i.e., we consider the reflection with respect to the imaginary axis:  $\bar{q}_{v,\sigma}(t) = \hat{q}_{v,\sigma}(-t)$ . The resulting parametric polynomial is defined by

$$\bar{q}_{v,\sigma}(t) = (-4v^3 - 2\sigma v - 18v^2 - 3\sigma - 24v - 6)t^3 + (-12v^3 - 6\sigma v - 36v^2 - 6\sigma - 48v - 12)t^2 \\ + (-12v^3 - 6\sigma v - 18v^2 - 3\sigma - 48v - 12)t - 4v^3 - 2\sigma v - 24v. \quad (35)$$

Again, applying the Routh-Hurwitz table of the latter polynomial yields the union of the region  $R$ , (33), and the following empty region

$$\left\{ (v, \sigma) \in \mathbb{R}_- \times \mathbb{R} : 6v^3 + 3\sigma v + 18v^2 + 3\sigma + 24v + 6 > 0 \quad \text{and} \right. \\ \left. 12v^3 + 6\sigma v + 18v^2 + 3\sigma + 48v + 12 - \frac{(4v^3 + 2\sigma v + 18v^2 + 3\sigma + 24v + 6)(2v^3 + \sigma v + 12v)}{6v^3 + 3\sigma v + 18v^2 + 3\sigma + 24v + 6} > 0 \right. \\ \left. \text{and } 2v^3 + \sigma v + 12v > 0 \quad \text{and} \quad 4v^3 + 2\sigma v + 18v^2 + 3\sigma + 24v + 6 > 0 \right\} = \emptyset, \quad (36)$$

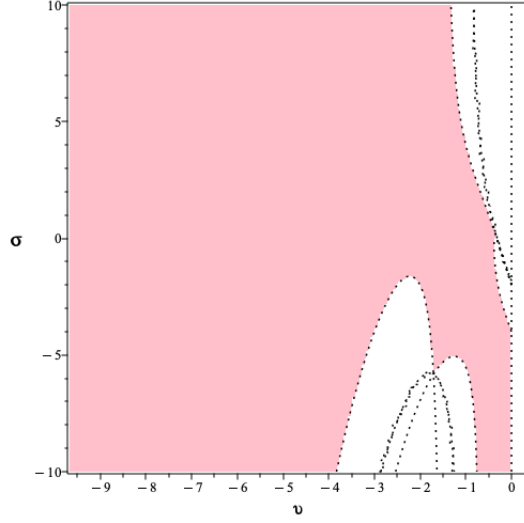


Figure 4: Plot in the parameters' space of region  $R$  given by (33).

thereby terminating the proof. ■

Figure 4 exhibits a region in the parameters' space  $(v, \sigma)$  where  $\tilde{q}_{v, \sigma}$  has all its roots with real part greater than 1, this is done thanks to the scaling, the symmetry and the application of Routh-Hurwitz conditions on the auxiliary polynomial  $\bar{q}_{v, \sigma}$ .

#### 4.2.4. Effective admissible region

In this section we characterize the region in the original parameters' space  $(a_2, \tau)$  where one has both properties; the elimination-produced polynomial is realrooted and Hurwitz, and the integrand  $q(t)$  is of constant sign while  $t \in (0, 1)$ .

**Proposition 4.3.** *Let  $\tilde{\tau}_{k \in \mathbb{N}}$  be the  $k^{\text{th}}$  real root (ordered increasingly) of*

$$\begin{aligned} \check{p} = & -1296 + 16(a_2^2 + 4)^2 \tau^{12} + (-36a_2^5 + 288a_2^3 + 1728a_2) \tau^{10} + (-1305a_2^4 - 1512a_2^2 - 5904) \tau^8 \\ & + (-8352a_2^3 + 28800a_2) \tau^6 + (-19224a_2^2 + 120096) \tau^4 - 15552\tau^2 a_2. \end{aligned}$$

Set  $-1000 < a_2 < 2000$ . If one of the following assertions holds:

1.  $0 < \tau < \sqrt{6a_2 + 2\sqrt{9a_2^2 + 6}}$ ,
2.  $\tau \in (0, \tilde{\tau}_4) \cup (\tilde{\tau}_5, \tilde{\tau}_6)$ ,
3.  $\tau \in (0, \check{\tau}_4) \cup (\check{\tau}_5, \check{\tau}_6)$ ,

then  $\mathbb{P}$  is realrooted and Hurwitz and  $q(t)$  is of constant sign for  $t \in (0, 1)$ .

**Proof.** As in the proof of Proposition 4.1, we use the Maple routine `CellDecomposition` with an input consisting of the conditions ensuring the constancy sign of  $q_{v, \sigma}$  (interpreted in terms of  $s_0, a_2$  and  $\tau$ ), the vanishing of

the elimination-produced polynomial  $\mathbb{P} = 0$ , the constraints on the system parameters  $-1000 < a_2 < 2000$  and  $0 < \tau < 200$ , as well as the negativity of the assigned root  $s_0 < 0$ . By a post-treatment consisting in merging neighboring cells and exploiting their algebraic characterization, we obtain the announced result. ■

#### 4.3. Dominancy Sufficient Conditions

Now we are able to conclude our analysis by identifying the region in the parameters' space where the quintuple root at  $s_0$  corresponds to the spectral abscissa of  $\Delta$ .

##### 4.3.1. Frequency bound

**Lemma 4.1.** Consider  $\hat{\Delta}$  the quasipolynomial given in (28) with

$$(v, \sigma) \in \hat{R} = \left\{ (v, \sigma) \in \mathbb{R}_- \times \mathbb{R} : -1 < v < -\frac{3}{4} \quad \text{and} \quad \sigma < \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3} \quad \text{and} \right. \\ \left. \sigma > \frac{-32v^4 - 96v^3 - 174v^2 + 3\sqrt{32v^4 + 56v^3 + 65v^2 + 42v + 9} - 123v - 27}{16v^2 + 24v + 9} \right\}. \quad (37)$$

If  $\hat{\Delta}$  has a root  $s_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ , then  $0 < \Im(s_0) < \pi$ .

**Proof.** In order to find an appropriate frequency bound we first solve the system of conditions defining region  $R$  where the constancy sign of  $q_{v,\sigma}$  is ensured, we obtain many different regions in the parameters' space  $(v, \sigma) \cup_{i=1}^{19} S_i$ , where

$$S_1 = \left\{ v = -\frac{3}{2}, -8 + \frac{\sqrt{29}}{2} < \sigma \right\},$$

$$S_2 = \left\{ v = -\frac{\sqrt{10}}{2}, -\frac{13453\sqrt{10} - 42473 - 3\sqrt{1559649 - 493188\sqrt{10}}}{947\sqrt{10} - 2986} < \sigma \right\},$$

$$S_3 = \left\{ v \leq -\frac{\sqrt{10}}{2}, \sigma < -2v^2 - 12 \right\},$$

$$S_4 = \left\{ v \leq -1 - \frac{\sqrt{2}}{2}, \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3} < \sigma \right\},$$

$$S_5 = \left\{ v = -1, -14 + 6\sqrt{2} < \sigma < 4 \right\},$$

$$S_6 = \left\{ v = -\frac{3}{4}, -\frac{43}{4} < \sigma < \frac{19}{8} \right\},$$

$$S_7 = \left\{ v = -1 + \frac{\sqrt{2}}{2}, -15 + 2\sqrt{2} < \sigma < -\frac{-10640 + 7487\sqrt{2}}{-1042 + 719\sqrt{2}} \right\},$$

$$S_8 = \left\{ v = \frac{1}{2} - \frac{\sqrt{11}}{2}, -\frac{166520833\sqrt{11} - 552287115}{8535003\sqrt{11} - 28307401} < \sigma < -\frac{28\sqrt{11} - 89}{-4 + \sqrt{11}} \right\},$$

$$S_9 = \{v = -0.3820880724, -0.33829270 < \sigma < 0.342333120\},$$

$$S_{10} = \{v = -0.3820880724, -12.29198259 < \sigma < -0.33829270\},$$



$$\begin{aligned}
S_{11} &= \left\{ -\frac{\sqrt{10}}{2} < v < -\frac{3}{2}, \sigma < \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3} \right\}, \\
S_{12} &= \left\{ -\frac{\sqrt{10}}{2} < v < -\frac{3}{2}, \epsilon < \sigma \right\}, \\
S_{13} &= \left\{ -0.732050808 \leq v < -0.3820880724, -2v^2 - 12 < \sigma < \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3} \right\}, \\
S_{14} &= \left\{ -1 < v < -\frac{3}{4}, \epsilon < \sigma < \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3} \right\}, \\
S_{15} &= \left\{ -\frac{3}{2} < v, \sigma < \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3}, v < \epsilon < \sigma \right\}, \\
S_{16} &= \left\{ -\frac{3}{4} < v, \sigma < \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3}, v < 1 - \sqrt{3}, \epsilon < \sigma \right\}, \\
S_{17} &= \left\{ -0.3820880724 < v, \sigma < -\epsilon, v < -0.2928932190, -2.0v^2 - 12.0 < \sigma \right\}, \\
S_{18} &= \left\{ \sigma < \frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3}, v < -1, \epsilon < \sigma, \frac{1}{2} - \frac{\sqrt{11}}{2} < v \right\}, \\
S_{19} &= \left\{ -0.3820880724 < v, \sigma < -\frac{-2(2v^3 + 9v^2 + 12v + 3)}{2v + 3}, v < -0.2928932190, \epsilon < \sigma \right\},
\end{aligned}$$

with

$$\epsilon = \frac{-32v^4 - 96v^3 - 174v^2 + 3\sqrt{32v^4 + 56v^3 + 65v^2 + 42v + 9} - 123v - 27}{16v^2 + 24v + 9}.$$

Let now  $s_0 = x_0 + i\omega_0 \in \mathbb{R}_+ + i\mathbb{R}_+$  be a root of  $\tilde{\Delta}(\lambda) = P_0(\lambda) + P_1(\lambda)e^{-\lambda}$  as defined in (28), where  $P_0(\lambda) = \sum_{i=0}^4 \alpha_i \lambda^i$ ,  $P_1(\lambda) = \sum_{i=0}^3 \beta_i \lambda^i$  and  $s_0$  satisfy the following equality  $|P_0(x_0 + i\omega_0)|^2 e^{2x_0} = |P_1(x_0 + i\omega_0)|^2$ . Since  $e^{2x} > 1$ , the function  $F_{v,\sigma}(x, \omega) = |P_1(x + i\omega)|^2 - |P_0(x + i\omega)|^2$  satisfies  $F_{v,\sigma}(x_0, \omega_0) > 0$ . Moreover, the zeros of  $F_{v,\sigma}$  can be characterized by a polynomial of degree 4 in  $\Omega = \omega^2$ :

$$G_{v,\sigma}(\Omega, x) = -\Omega^4 + \gamma_{3,v,\sigma}(x)\Omega^3 + \gamma_{2,v,\sigma}(x)\Omega^2 + \gamma_{1,v,\sigma}(x)\Omega + \gamma_{0,v,\sigma}(x) \quad (38)$$

where  $\gamma_{i,v,\sigma}$ ,  $i = 0..3$  are functions in  $x$  parameterized in  $v$  and  $\sigma$

$$\begin{aligned}
\gamma_{3,v,\sigma}(x) &= -4x^2 - 8vx + \frac{4v^6}{9} + 8v^5 + \frac{(4\sigma + 468)v^4}{9} + \frac{(48\sigma + 1344)v^3}{9} + \frac{(\sigma^2 + 180\sigma + 1692)v^2}{9} \\
&\quad + \frac{(6\sigma^2 + 240\sigma + 864)v}{9} + \sigma^2 + 10\sigma + 16,
\end{aligned}$$

$$\begin{aligned}
\gamma_{2,v,\sigma}(x) &= -6x^4 - 24vx^3 + \left( \frac{4v^6}{3} + 24v^5 - 6 \left( -26 - \frac{2\sigma}{9} \right) v^4 - 6 \left( -\frac{224}{3} - \frac{8\sigma}{3} \right) v^3 \right. \\
&\quad \left. - 6 \left( -10\sigma - \frac{\sigma^2}{18} - 90 \right) v^2 - 6 \left( -\frac{40\sigma}{3} - \frac{\sigma^2}{3} - 48 \right) v + 48 + 26\sigma + 3\sigma^2 \right) x^2 \\
&\quad + \left( \frac{16v^6}{3} + 88v^5 - 6 \left( -\frac{8\sigma}{9} - \frac{260}{3} \right) v^4 - 6 \left( -\frac{88\sigma}{9} - 220 \right) v^3 - 6 \left( -\frac{2\sigma^2}{9} - \frac{100\sigma}{3} - 256 \right) v^2 \right. \\
&\quad \left. - 6 \left( -112 - \frac{11\sigma^2}{9} - \frac{122\sigma}{3} \right) v + 96 + 10\sigma^2 + 64\sigma \right) x,
\end{aligned}$$

$$\begin{aligned}
\gamma_{1,v,\sigma}(x) = & -4x^6 - 24vx^5 + \left( \frac{4v^6}{3} + 24v^5 - 4 \left( -39 - \frac{\sigma}{3} \right) v^4 - 4(-112 - 4\sigma)v^3 - 4 \left( -129 - \frac{\sigma^2}{12} \right. \right. \\
& \left. \left. - 15\sigma \right) v^2 - 4 \left( -72 - \frac{\sigma^2}{2} - 20\sigma \right) v + 48 + 3\sigma^2 + 22\sigma \right) x^4 \\
& + \left( \frac{32v^6}{3} + 176v^5 - 4 \left( -260 - \frac{8\sigma}{3} \right) v^4 - 4 \left( -652 - \frac{88\sigma}{3} \right) v^3 - 4 \left( -768 - \frac{2\sigma^2}{3} - 100\sigma \right) v^2 \right. \\
& \left. - 4 \left( -336 - \frac{11\sigma^2}{3} - 118\sigma \right) v + 20\sigma^2 + 128\sigma + 192 \right) x^3 + \left( 32v^6 + 480v^5 - 4(-624 - 8\sigma)v^4 \right. \\
& \left. - 4(-1440 - 80\sigma)v^3 - 4(-2\sigma^2 - 240\sigma - 1296)v^2 - 4(-10\sigma^2 - 240\sigma - 288)v + 48\sigma^2 + 96\sigma \right) x^2 \\
& + \left( 32v^6 + 384v^5 - 4(-432 - 8\sigma)v^4 - 4(-576 - 64\sigma)v^3 - 4(-2\sigma^2 - 144\sigma - 144)v^2 \right. \\
& \left. - 4(-8\sigma^2 - 96\sigma)v + 48\sigma^2 + 96\sigma \right) x,
\end{aligned}$$

$$\begin{aligned}
\gamma_{0,v,\sigma}(x) = & -x^8 + \left( -\left( \frac{\sigma}{3} + 16 \right) v - \left( -8 - \frac{\sigma}{3} \right) v \right) x^7 + \left( -12v^2 - (2\sigma + 48)v - 2\sigma - \left( \frac{2v^3}{3} + 6v^2 + \left( \frac{\sigma}{3} \right. \right. \right. \\
& \left. \left. + 16 \right) v + \sigma + 4 \right) \left( -\frac{2v^3}{3} - 6v^2 + \left( -8 - \frac{\sigma}{3} \right) v - 4 - \sigma \right) - (-2\sigma - 48)v \right) x^6 \\
& + \left( -8v^3 - (8\sigma + 96)v - (4v^3 + 36v^2 + (2\sigma + 48)v + 6\sigma + 12) \left( -\frac{2v^3}{3} - 6v^2 + \left( -8 - \frac{\sigma}{3} \right) v \right. \right. \\
& \left. \left. - 4 - \sigma \right) - \left( \frac{2v^3}{3} + 6v^2 + \left( \frac{\sigma}{3} + 16 \right) v + \sigma + 4 \right) (-4v^3 - 24v^2 + (-2\sigma - 48)v - 4\sigma - 12) - \right. \\
& \left. (-4\sigma - 96)v \right) x^5 + \left( - (16v^3 + 72v^2 + (8\sigma + 96)v + 12\sigma + 24) \left( -\frac{2v^3}{3} - 6v^2 + \left( -8 - \frac{\sigma}{3} \right) v \right. \right. \\
& \left. \left. - 4 - \sigma \right) - (4v^3 + 36v^2 + (2\sigma + 48)v + 6\sigma + 12) (-4v^3 - 24v^2 + (-2\sigma - 48)v - 4\sigma - 12) \right. \\
& \left. - \left( \frac{2v^3}{3} + 6v^2 + \left( \frac{\sigma}{3} + 16 \right) v + \sigma + 4 \right) (-8v^3 - 72v^2 + (-4\sigma - 96)v - 12\sigma - 24) + 32v^3 \right. \\
& \left. + 144v^2 - (-16\sigma - 192)v + 24\sigma + 48 \right) x^4 + \left( - (16v^3 + 72v^2 + (8\sigma + 96)v + 12\sigma + 24) (-4v^3 \right. \\
& \left. - 24v^2 + (-2\sigma - 48)v - 4\sigma - 12) - (4v^3 + 36v^2 + (2\sigma + 48)v + 6\sigma + 12) (-8v^3 - 72v^2 \right. \\
& \left. + (-4\sigma - 96)v - 12\sigma - 24) - \left( \frac{2v^3}{3} + 6v^2 + \left( \frac{\sigma}{3} + 16 \right) v + \sigma + 4 \right) (-32v^3 - 144v^2 + (-16\sigma \right. \\
& \left. - 192)v - 24\sigma - 48) \right) x^3 + \left( - (16v^3 + 72v^2 + (8\sigma + 96)v + 12\sigma + 24) (-8v^3 - 72v^2 + (-4\sigma \right. \\
& \left. - 96)v - 12\sigma - 24) - (4v^3 + 36v^2 + (2\sigma + 48)v + 6\sigma + 12) (-32v^3 - 144v^2 + (-16\sigma - 192)v \right. \\
& \left. - 24\sigma - 48) \right) x^2 - \left( 16v^3 + 72v^2 + (8\sigma + 96)v + 12\sigma + 24 \right) \left( -32v^3 - 144v^2 + (-16\sigma - 192)v \right.
\end{aligned}$$

$$-24\sigma - 48)x.$$

Let  $(v, \sigma) \in S_{14}$ . Since  $v < -3/4$ ,  $\sigma < -2(2v^3 + 9v^2 + 12v + 3)/(2v + 3)$  and  $x > 0$ , this latter polynomial of degree 4 can be upper bounded with respect to  $v$  and  $\sigma$  by the parameter-free expression

$$\begin{aligned} G(\Omega, x) = & -\Omega^4 + \frac{1}{9} \left( \frac{1449}{64} + 54x - 36x^2 \right) \Omega^3 + \frac{x}{3} \left( \frac{75}{8} - \frac{2967x}{64} + 54x^2 - 18x^3 \right) \Omega^2 + \frac{x}{3} \left( \frac{1323}{8} - \frac{1575x}{8} \right. \\ & \left. + \frac{579x^2}{4} - \frac{7383x^3}{64} + 54x^4 - 12x^5 \right) \Omega + \frac{441x^3}{8} - \frac{525x^4}{8} + \frac{361x^5}{8} - \frac{1311x^6}{64} + 6x^7 - x^8 \end{aligned} \quad (39)$$

which admits two real roots, the greatest of which reaches a maximum value at  $x^* \approx 0.7627783180$ . Thus,  $\Omega \approx 5.290134760 < \pi^2$ , i.e.,  $\omega_0 < \pi$ . Notice that the region denoted by  $\hat{R}$  in the statement of the proposition is equal to  $S_{14}$  in the proof. ■

#### 4.3.2. Dominancy

In principle, one can investigate the dominancy of a given spectral value using the standard Principle argument as recalled in Section 3 or equivalently using the Stepan-Hassard approach. However, here the proof of the dominance is based on a contradiction. Consider  $(v, \sigma) \in \hat{R}$  and assume that there exists  $s_0 = x_0 + i\omega_0 \in \mathbb{C}$  a non-zero root of  $\hat{\Delta}$  satisfying  $\Re(s_0) > 0$ . Then, we infer from the imaginary part of (29) that  $\int_0^1 q_{v,\sigma}(t) \sin(t\omega_0) e^{-tx_0} dt = 0$ . Since  $\omega_0 < \pi$  from Lemma 4.1, the function  $t \mapsto q_{v,\sigma}(t) \sin(t\omega_0)$  is strictly positive in  $(0, 1)$ , which contradicts the latter vanishing integral as required to end the dominancy proof.

## 5. Conclusion

This contribution is intended to be a didactical chapter on the use of the MID-based partial poles placement (PPP) in concrete applications. Namely, it dealt with the modeling of the CNS action on the human balance. Since the PPP paradigm requires computations which often appear to be of complex nature, particularly when some of the system's parameters are left free, we privileged the use of the Computer Algebra System `Maple`. Indeed, `Maple` offers an efficient kernel for effective computations of Gröbner bases. Also, the powerful `CellDecomposition` algorithm in CAD routine is, in our opinion, one of the most efficient symbolic/numeric tools for "solving" semi-algebraic systems depending on parameters. As such, this chapter acts as a bridge between the PPP-based control design methodology, the modeling of sensorial human balance and the use of symbolic/numeric algorithms provided in the computer algebra system `Maple`.

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