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Rigidity of random stationary measures and applications to point processes

Raphaël Lachièze-Rey*

Abstract The *mass rigidity* of a stationary random measure \mathbf{M} on \mathbb{R}^d or \mathbb{Z}^d (called *number rigidity* for a point process) entails that for a bounded set A the knowledge of \mathbf{M} on A^c determines $\mathbf{M}(A)$; the k -order rigidity for an integer k means we can recover the moments of the restriction of \mathbf{M} onto A up to order k . We show that the k -rigidity properties of a random stationary measure \mathbf{M} can be characterised by the integrability properties around 0 of \mathcal{J} , the continuous part of the spectral measure, by exploiting a connection with Schwartz' Paley-Wiener theorem for analytic functions of exponential type.

If \mathcal{J}^{-1} is not integrable in zero, \mathbf{M} is mass rigid, and similarly if \mathcal{J} has a zero of order $2k$ in 0, then \mathbf{M} is k -rigid. In the continuous setting, these local conditions are also necessary if \mathcal{J} has finitely many zeros, or is isotropic, or is at the opposite separable. This explains why no model seems to exhibit rigidity in dimension $d \geq 3$, and allows to efficiently recover many recent rigidity results about point processes. In the discrete setting, these results hold provided $\#A > 2k$.

For a continuous Determinantal point process with reduced kernel κ , k -rigidity is equivalent to $(1 - \widehat{\kappa^2})^{-1}$ having a zero of order $2k$ in 0, which answers questions on completeness and number rigidity. We also explore the consequences of these statements in the less tractable realm of Riesz gases.

Keywords: Rigidity, point processes, random measures, hyperuniformity, Determinantal point processes, Riesz gases

1 Introduction

The first instance of a linear prediction problem for a stationary process dates back to Szegö [1921], about a century ago, for a random time series $\{\mathbf{X}_k; k \in \mathbb{Z}\}$. He proved that the process is *deterministic*, i.e. the future is entirely determined by the past

$$\sigma(\mathbf{X}_k; k > 0) \subset \sigma(\mathbf{X}_k; k \leq 0)$$

through a linear predictor iff the continuous part \mathcal{J} of the spectral measure satisfies

$$\int_0^{2\pi} \ln(|\mathcal{J}(u)|) = -\infty.$$

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Kolmogorov [1941] further studies the non-deterministic case, and in particular addresses the weaker linear interpolation problem of whether $X_0 \in \sigma(X_k; k \neq 0)$. He proves that it is so iff

$$\int_{-\pi}^{\pi} \mathcal{J}^{-1}(u) du = \infty. \quad (1)$$

A detailed account of the line of research about random linear interpolation can be found in Rozanov [1967].

Much more recently, other similar problems emerged for higher-dimensional continuous models with the notion of *rigidity* for a *stationary point process*, i.e. a random locally finite set of points $P \subset \mathbb{R}^d$ which law is invariant under spatial translations. The oldest such result might be that of Aizenman and Martin [1980], about 1D Coulomb gases, but the systematic study of rigidity really started with the introduction of the notion of *tolerance* (Lyons and Steif [2003], Holroyd and Soo [2013]), then Ghosh and Peres [2017] coined the term *number rigidity* as the property that

$$\#P \cap B(0, 1) \in \sigma(P \cap B(0, 1)^c).$$

This obviously holds for *shifted lattices*, e.g. $P = \{\mathbf{m} + U; \mathbf{m} \in \mathbb{Z}^d\}$ (where U is uniform in $[0, 1]^d$ and ensures that P is invariant under translations with non-integer coordinates), as one can easily deduce the number of lattice points “hidden” in $B(0, 1)$ by only observing $P \cap B(0, 1)^c$. They realised that somehow surprisingly also some strongly non-lattice like models, the *infinite Ginibre ensemble* P_{gin} , and the zero set P_{GAF} of the *planar Gaussian Analytic Function*, satisfy number rigidity. As it turns out, these two processes possess also the property of *hyperuniformity*, i.e.

$$\frac{\text{Var}(\#P \cap B(0, r))}{\text{Vol}(B(0, r))} \xrightarrow{r \rightarrow \infty} 0, \quad (2)$$

at the difference of standard disordered systems such as Poisson processes, this property also is strongly reminiscent of lattice-like models. The surprising fact that it occurs for locally disordered, *amorphous* random measures, like P_{Gin}, P_{GAF} , or other Coulomb systems, has been systematically studied by physicists since the 90’s. This property requires some sort of long distance dependency, compatible with locally disordered configurations, often referred to as *global order and local disorder*. The activity around hyperuniformity has not stopped growing until now, as this property concerns a large diversity of models involving Coulomb systems, Gaussian analytic functions, eigenvalues of random matrices, Determinantal Point processes, and has many applications. The literature is too wide to be cited exhaustively, see for instance the survey of Torquato [2018] for materials science, of Ghosh and Lebowitz [2017] for statistical physics, or the more mathematical discussion of Coste [2021].

Many models have been proven to be rigid since then, mostly in the realm of hyperuniformity: some Determinantal Point Processes (DPPs) (Bufetov [2016], Bufetov and Qiu [2018], Bufetov et al. [2018]), Pfaffian processes (Bufetov et al. [2019]), Coulomb and Riesz systems (Dereudre et al. [2020], Chhaibi and Najnudel [2018], Chatterjee [2019], Dereudre and Vasseur [2023]), zeros of Gaussian processes (Ghosh and Peres [2017], Ghosh and Krishna [2021], Lachièze-Rey [2020]), stable matchings (Klatt et al. [2020]), and others (Ghosh and Lebowitz [2018], Klatt and Last [2020]). Besides the striking nature of rigidity and its link with

hyperuniformity, it has proven to be a useful property in other types of problems, such as continuous percolation [Ghosh et al. \[2016\]](#); [Osada \[2024\]](#) recently proved a relation with diffusive dynamics of particle systems, [Ghosh \[2015\]](#) also exploited number rigidity to show the completeness of random sets of exponential functions, [Lyons and Steif \[2003\]](#) study this property to show phase uniqueness for some models from statistical physics.

Stronger forms of rigidity have also emerged. For instance, for the \mathbb{P}_{GAF} model, [Ghosh and Peres \[2017\]](#) showed that not-only the number of particles in $B(0, 1)$ can be determined, but also their first moment, or center of mass $\sum_{x \in \mathbb{P}_{GAF} \cap B(0, 1)} x$, corresponding to *1-rigidity*, and no further moment can be determined from the observation of $\mathbb{P}_{GAF} \cap B(0, 1)^c$. They proved similarly that for the Ginibre process no moment can be determined beyond order 0. Another such result appears in [Dereudre et al. \[2020\]](#) for Sine_β processes above order 0.

Until now, both in the physics and mathematics literature, the precise nature of the connection between rigidity and hyperuniformity is not precisely understood, see the discussions in [Ghosh and Lebowitz \[2017\]](#), [Coste \[2021\]](#). It seems in particular that rigidity is not understood in dimensions $d \geq 3$, with the exception of the work [Chatterjee \[2019\]](#), where he discusses at length this question and proves that the hierarchical Coulomb gas, a simplified version of the actual 3D Coulomb gas which is not formally stationary, is not number rigid. [Dereudre and Vasseur \[2023\]](#) conjecture that the number rigidity of a Riesz gas of index s occurs iff $s \leq d - 1$.

Necessary and sufficient conditions. In the current article, we study the rigidity properties of a L^2 (wide-sense) stationary measure \mathbb{M} on \mathbb{R}^d or \mathbb{Z}^d . We shall establish for $k \in \mathbb{Z}$ a characterisation of k -th order linear rigidity, which is the property that the k -th order moments of \mathbb{M} restricted to $B(0, 1)$ can be a.s. recovered (linearly) from $\mathbb{M} \setminus B(0, 1)$, in terms of the behaviour around the origin of the structure factor \mathbb{S} , the generalised Fourier transform of the correlation measure, and more precisely of its continuous part $\mathfrak{s}(u)du$. The deep relation between the behaviour of \mathbb{S} around 0 and hyperuniformity emerges from the fact that (2) is the equivalent in the Fourier domain to the vanishing of \mathbb{S} near 0. We show that the decay exponent of \mathfrak{s} near 0 is related to its degree of rigidity, i.e. the number of moments of $\mathbb{M} \cap B(0, 1)$ that can be determined by the outside configuration $\mathbb{M} \cap B(0, 1)^c$.

More precisely, on a bounded set A , it is sufficient for k -rigidity that \mathfrak{s}^{-1} has a pole of order $2k$ in 0 (Theorem 1). For *number rigidity* ($k = 0$) or if $d = 1$, or if \mathfrak{s} is isotropic, it exactly means that for any neighbourhood U of 0,

$$\int_U \mathfrak{s}(u)^{-1} \|u\|^{2k} du = \infty, \quad (3)$$

the condition can be more complicated in higher dimensions or without isotropy. This explains most results cited above about number rigidity and 1-rigidity, and also generalises the sufficient conditions of [Ghosh and Lebowitz \[2017\]](#) and [Bufetov et al. \[2018\]](#) for number rigidity.

We show that this condition is necessary under some structural assumptions such as finite number of zeros, isotropy or separability (Proposition 2 and Theorem 2). It also explains why rigidity does not seem to occur in dimension $d \geq 3$: if $\mathfrak{s}(u) \sim \sigma \|u\|^2$ for some $\sigma > 0$ as $u \rightarrow 0$, as it is expected for many hyperuniform systems with integrable correlation measure, if $d \geq 3$, for $k \geq 0$, the left hand member of (3) is finite. Hence one

must find processes where the structure factor decreases faster to 0 to find higher order rigidity. We exhibit in [Lachièze-Rey \[2024+\]](#) a class of processes P_n , for n arbitrarily large, which are not shifted lattices and such that $\mathfrak{J}(u) \sim_{u \rightarrow 0} c \|u\|^n$ for some $c > 0$.

The converse part also allows to assess the correlation properties of a point process based on its rigidity behaviour: it is shown by [Dereudre and Vasseur \[2023\]](#) and [Dereudre et al. \[2020\]](#) that some β -ensembles are not k -rigid for some k , which implies non-integrability results on their correlation functions (see [Section 4.2](#)).

We give counter-examples of spectral measures showing that these results are optimal in full generality.

Determinantal point processes. These results apply efficiently to stationary DPPs. [Theorem 1](#) yields that a DPP on \mathbb{R}^d with kernel K is k -rigid iff $(1 - \mathcal{F}(|K|^2))^{-1}$ has a pole of order $2k$ in 0, retrieving for $k = 0$ the results of [Ghosh and Peres \[2017\]](#), [Ghosh \[2015\]](#), [Bufetov et al. \[2018\]](#), and also completes the answer of [Ghosh \[2015\]](#) to a question of [\[Lyons, 2014, Section 4\]](#) on completeness of complex exponentials. In the discrete setting, a pole of order $2k$ indeed implies k -rigidity, but the converse is only true in dimension 1, or under some additional assumption.

Discrete processes. Stationary discrete processes $X = \{X_m; m \in \mathbb{Z}^d\}$ show a similar connection between rigidity and the behaviour around 0 of the spectral measure S , defined on $[-\pi, \pi]^d$. We show in [Theorem 3](#) for $A = \llbracket m \rrbracket^d := \{-m, \dots, m\}^d$ that X is not *maximally rigid* on A , meaning $\{X(m); |m| \leq m\} \not\subset \sigma(\{X_m, |m| > m\})$, iff there exists a trigonometric polynomial $\psi(u) = \sum_{|m| \leq m} a_m e^{im \cdot u}$ such that,

$$\int_{\mathbb{T}^d} \frac{|\psi(u)|^2}{\mathfrak{J}(u)} du < \infty,$$

and not k -rigid if furthermore the derivatives of ψ up to order k do not vanish in 0. Besides the seminal work of [Kolmogorov \[1941\]](#) in dimension 1, this unifies some results established in the context of discrete DPPs, such as the result of [Lyons and Steif \[2003\]](#) regarding the *strong full K* property for uniqueness of phase transition in statistical physics. The condition can be made more explicit in dimension 1: X is *maximally rigid* on $\llbracket m \rrbracket$ iff the number of poles of \mathfrak{J} counted with multiplicity is $> m$, as for the condition [\(1\)](#) of [Kolmogorov \[1941\]](#) in the case $d = 1, m = 0, k = 0$, or the generalisation of [Bufetov et al. \[2018\]](#) in the case $d = 1, k = 0, m \in \mathbb{N}$: X is 0-rigid, i.e. $\sum_{|m| \leq m} X_m \in \sigma(\{X_k; |m| > m\})$, iff 0 is a pole, i.e.

$$\int_{B(0, \varepsilon)} \mathfrak{J}^{-1}(u) du = \infty, \varepsilon > 0,$$

or (obviously) if X is *maximally rigid* ([Proposition 5](#)).

Rigidity of the class of functions of exponential type. Let us sketch here the method to derive these results, formally developed at [Section 5](#). The basic idea is the following: given a function γ bounded on some compact A , the linear statistic $\int \gamma dM$ is completely determined by $M \cap A^c$ if

$$\inf_{h \in \mathcal{C}_c^\infty(A^c)} \text{Var} \left(\int \gamma dM - \int h dM \right) = 0,$$

and this translates in the Fourier domain as

$$\inf_{h \in C_c^\infty(A^c)} \int |\hat{\gamma} - \hat{h}|^2 d\mathbf{S} = 0.$$

In other words, γ must be in the $L^2(\mathbf{S})$ -closure of the space H_A spanned by the \hat{h} for h in $C_c^\infty(A^c)$, or equivalently it must be orthogonal to any function $\varphi \in H_A^\perp$. The crucial observation is that for such φ , the tempered distribution $\varphi\mathbf{S}$ has by definition a spectrum bounded (by A). This conveys a very strong form of regularity, namely $\psi := \varphi\mathcal{J} = \varphi\mathbf{S}$ has no singular part and must be an analytic function of exponential type on \mathbb{C}^d by an application of the Schwartz-Paley-Wiener theorem (Theorem 5). We have also immediately $\psi \in L^2(\mathcal{J}^{-1})$ because $\int \psi^2 \mathcal{J}^{-1} = \int \varphi^2 \mathbf{S}$. If such ψ do not exist besides the null function, $H_A^\perp = \{0\}$ and all γ can be predicted, meaning the process is maximally rigid. Number rigidity means that $\gamma \equiv 1$ is orthogonal to $\hat{\psi}$ for all such ψ , i.e. for all entire function ψ with bounded spectrum, being in $L^2(\mathcal{J}^{-1})$ implies

$$\psi(0) = \langle 1, \hat{\psi} \rangle = 0,$$

thus \mathcal{J}^{-1} is non-integrable around 0, otherwise we could find such ψ non-vanishing in 0.

These high regularity and integrability requirement on ψ are in some sense the source of the *rigidity phenomenon*. It is then easier to characterise whether all such $\psi \in L^2(\mathcal{J}^{-1})$ must satisfy $\langle \hat{\gamma}, \psi \rangle_{\mathbb{R}^d} = 0$, meaning γ -rigidity, and it mostly depends on the zeros of \mathcal{J} , or more generally on how often $\mathcal{J}(u)$ is close to 0. What we study is not mere rigidity, but *linear rigidity*, meaning that $\int \gamma d\mathbf{M}$ should be approximated by linear functionals $I_{\mathbf{M}}(h)$, not by *any* functional on $\mathbf{M} \cap A^c$.

Let us present the rest of the paper. In Section 2, we introduce the spectral measure \mathbf{S} necessary to state the main results. In Section 3, we formally introduce the concepts of rigidity and \mathbf{k} -poles and derive the announced necessary and sufficient condition, and show how they apply to some particular examples; we also give the corresponding results for discrete processes. Section 4 is devoted to the applications to DPPs and Gibbs measure. Finally, Section 5 gives the formal framework about distributions, Schwarz' Paley-Wiener Theorem and the proofs. In the companion paper [Lachièze-Rey \[2024+\]](#), we explore higher order rigidity, and in particular stealthy systems and maximal rigidity, and give several examples of models satisfying a phase transition in the observation window A , or other surprising rigidity behaviours, such as *short range rigidity*, where $\mathbf{M} \cap B(0, 1)$ is completely determined by $\mathbf{M} \cap (B(0, 2) \setminus B(0, 1))$ despite the covariance being smooth and integrable.

2 Spectral measure

Let \mathbf{E} be either \mathbb{R}^d or \mathbb{Z}^d . Even though our main motivation is the class of point processes, we consider a random L^2 wide-sense stationary signed measure \mathbf{M} , i.e. a collection of real L^2 random variables $I_{\mathbf{M}}(f)$ on a probability space (Ω, \mathbf{P}) , for f in the space $C_c^b(\mathbf{E})$ of measurable bounded and compactly supported functions, satisfying a.s. the σ -additivity axiom, and $\text{Var}(I_{\mathbf{M}}(\theta_x f)) = \text{Var}(I_{\mathbf{M}}(f))$ where θ_x is the operator of translation by $x \in \mathbb{R}^d$.

For $\mathbf{E} = \mathbb{R}^d$, the second order behaviour of \mathbf{M} is described by the *correlation measure* \mathbf{C} , characterised by

$$\text{Cov}(I_{\mathbf{M}}(f), I_{\mathbf{M}}(g)) = \lambda \int_{\mathbb{R}^d} f(y)g(x+y)\mathbf{C}(dx)dy, f, g \in C_c^b(\mathbf{E}),$$

where λ is the mean intensity

$$\lambda = \mathbf{E}(\mathbf{M}(B(0, 1))).$$

By convention λ is assumed to be $\lambda = 1$ in all the article (achievable through a deterministic rescaling). The *spectral measure* \mathbf{S} is defined as the Fourier transform $\mathcal{F}\mathbf{C}$ in the sense of tempered distributions, i.e. for $f \in \mathcal{C}_c^b(\mathbf{E})$

$$\text{Var}(I_{\mathbf{M}}(f)) = \int_{\mathbb{R}^d} |\hat{f}|^2 \mathbf{S}, \quad (4)$$

it is indeed a non-negative measure thanks to Bochner's Theorem (see Coste [2021], following Berg and Frost [1975]). In any case, the only relation formally needed in this work is that (4) holds for some measure \mathbf{S} . The local square integrability of \mathbf{M} directly yields that \mathbf{S} is a non-negative tempered measure (Lemma 2), and its inverse Fourier transform \mathbf{C} is a signed positive definite tempered measure.

Informally, the behaviour of \mathbf{S} at infinity represents the regularity of \mathbf{M} ; for instance if \mathbf{M} is a smooth Gaussian field, then \mathbf{S} will experience fast decay, whereas if \mathbf{P} is a point process, \mathbf{S} has likely infinite mass; the regularity of \mathbf{S} around 0 is related to the long range dependency of \mathbf{M} .

Example 1 (Discrete fields). If $\mathbf{E} = \mathbb{Z}^d$, rather consider $X_{\mathbf{m}} := I_{\mathbf{M}}(\{\mathbf{m}\})$, $\mathbf{m} \in \mathbb{Z}^d$, and impose the convention $\mathbf{E}X_0 = 0$, $\text{Var}(X_0) = 1$. Then

$$\mathbf{C}(\mathbf{m}) = \mathbf{E}(X_0 X_{\mathbf{m}})$$

is bounded by $\mathbf{C}(0) = 1$. Since \mathbf{C} is positive definite, $\mathbf{C} := \mathcal{F}\mathbf{S}$ where \mathbf{S} satisfies (4) but with \mathbb{T}^d as integration domain. The previous formulae hold with the counting measure on \mathbb{Z}^d .

Example 2 (Point processes). Point processes are the prominent examples motivating the current work and related line of literature about rigidity. A configuration is an atomic measure on $\mathbf{E} = \mathbb{R}^d$ without accumulation points in its support, and a random point process is a random variable \mathbf{P} in the space of configurations endowed with the counting σ -algebra generated by mappings $\mathbf{P} \mapsto \#\mathbf{P} \cap A$ for A compact. The integrability assumption means $\mathbf{E}(\#\mathbf{P} \cap A^2) < \infty$ for bounded measurable A .

For instance if \mathbf{P} is the unit intensity homogeneous Poisson process, $\mathbf{C} = \delta_0$, $\mathbf{S} = \mathcal{L}$ the Lebesgue measure on \mathbb{R}^d , but this example is not of great interest for us because, due to its spatial independence, \mathbf{P} does not experience rigidity or hyperuniformity.

For point processes, long range interaction is measured through the *truncated correlation measure* $\rho_{tr}^{(2)}(dt)$ satisfying

$$\mathbf{C} = \rho_{tr}^{(2)} + \delta_0.$$

The Dirac mass δ_0 comes from the diagonal terms in the double summation of the covariance, reflecting the purely atomic nature of the point process. Informally, $\rho_{tr}^{(2)}(dx)$ describes the probability that there is a point close to x given that there is a point close to 0.

Example 3 (Gaussian process). Let $\hat{E} = \mathbb{R}^d$ if $E = \mathbb{R}^d$, or $\hat{E} = \mathbb{T}^d$ if $E = \mathbb{Z}^d$. Given any non-negative measure S with finite mass on \hat{E} , the spectral representation theorem ([Adler and Taylor, 2007, Th. 5.4.2]) gives a stationary centred Gaussian process $X = \{X(x), x \in E\}$ such that S is the spectral measure of the random measure $I_M(f) = \int f(x)X(x)dx$, X is furthermore strongly stationary in the sense where $\theta_y X = \{X(x+y); x \in E\}$ has the same law as X , $y \in E$. Since S is finite, $C = (2\pi)^{-d} \hat{\mathcal{F}} S = c\mathcal{L}$ where $\hat{\mathcal{F}}$ is the inverse Fourier transform, and the covariance function c is defined by

$$\mathbf{E}(W(0)W(x)) = c(x), x \in E.$$

3 Rigidity of random stationary measures

For $A \subset E$, the notation M_A indicates the collection $I_M(f)$ for $f \in C_c^b(A)$. The general problem of rigidity for a random signed measure M is to be able to infer a functional $F(M_A)$ knowing only M_{A^c} . The *mass rigidity*, for instance, called *number rigidity* for point processes, means that $M(A) \in \sigma(M_{A^c})$. *Maximal rigidity* means that any bounded functional F M_A -measurable can be predicted by M_{A^c} , or equivalently

$$\sigma(M_A) \subset \sigma(M_{A^c}).$$

For $\gamma : E \rightarrow \mathbb{C}$, bounded with bounded support, we are interested here in predicting $I_M(\gamma)$, i.e. in determining if $I_M(\gamma) \in \sigma(M_{A^c})$, called γ -rigidity on A , and maximal rigidity (MR) on A means that γ -rigidity holds for all such γ , and in particular allows to recover the entirety of M_A . Define furthermore *linear rigidity* if $I_M(\gamma)$ can be approximated by such linear statistics of M_{A^c} , i.e. if a.s. and in $L^2(\mathbf{P})$, for some $h_n \in C_c^b(A^c)$, $n \geq 1$,

$$I_M(\gamma) = \lim_n I_M(h_n),$$

which we write $I_M(\gamma) \in \sigma_{lin}(M_{A^c})$, and *maximal linear rigidity* if $I_M(\gamma) \in \sigma_{lin}(M_{A^c})$ for all γ bounded. The term *linear* is sometimes omitted in this article, but all methods employed here pertain to linear rigidity.

We will be particularly interested in (linear) k -rigidity for $k = (k_i) \in \mathbb{N}^d$, i.e. γ_k -rigidity for $\gamma_k(t) = t^k$ with $t^k = \prod_{i=1}^d t_i^{k_i}$, $t \in \mathbb{R}^d$. For $k \in \mathbb{N}$, say that M is k -rigid if it is k -rigid for every k with $|k| \leq k$ (in particular, number rigidity corresponds to 0-rigidity).

Call *convex body* A a compact convex set with non-empty interior. In the heart of the paper, Theorem 6 provides an abstract necessary and sufficient condition for linear k -rigidity on A in terms of S , and shows in particular that linear γ -rigidity only depends on the *spectral density* \mathcal{J} , the density of the continuous part of S ; \mathcal{J} is even and non-negative, as S . For this reason, we sometimes talk about the rigidity of \mathcal{J} instead of the linear rigidity of M , but it means exactly the same thing.

3.1 Local sufficient condition

We consider a random stationary measure M on \mathbb{R}^d with spectral density \mathcal{J} . The crucial concept is that of a k -pole.

Definition 1. Given $k \in \mathbb{N}^d$, say that \mathcal{J}^{-1} has a pole of order k in zero, or k -pole, if it is locally u^k -incompatible around 0: for every polynomial $Q = \sum_m a_m u^m$ such that $\int_{B(0,\varepsilon)} |Q|^2 \mathcal{J}^{-1} < \infty$ for some $\varepsilon > 0$, we have $a_k = 0$.

For $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^d$, say that $\mathbf{k}' \preceq \mathbf{k}$ if $k'_i \leq k_i$ for $i = 1, \dots, d$. Remark that, through multiplication of Q by $u^{\mathbf{k}-\mathbf{k}'}$, a \mathbf{k} -pole is a \mathbf{k}' -pole for $\mathbf{k}' \preceq \mathbf{k}$. Let us state our most general sufficient condition.

Theorem 1. If \mathcal{J}^{-1} has a pole of order \mathbf{k} in 0, \mathbf{M} is \mathbf{k} -rigid on any bounded measurable $A \subset \mathbb{R}^d$.

All the proofs of results of this subsection are at Section 5.4. In many situations, 0 being a \mathbf{k} -pole is equivalent to

$$\int_{B(0,\varepsilon)} \frac{u^{2\mathbf{k}}}{\mathcal{J}^{-1}(u)} du = \infty \quad (5)$$

for all $\varepsilon > 0$:

- In dimension $d = 1$, we can also say that \mathcal{J} has a *zero of multiplicity* $2\mathbf{k}$: for every non-zero polynomial Q on \mathbb{R} , we have $Q(u) = au^q(1 + o(1))$ as $u \rightarrow 0$ for some $q \in \mathbb{N}$, $a \neq 0$, and $Q \in L^2(\mathcal{J}^{-1}; B(0, \varepsilon))$ iff $\int_{B(0,\varepsilon)} u^{2q} \mathcal{J}^{-1} < \infty$, a \mathbf{k} -pole indeed means (5).
- If $\mathbf{k} = 0$ (number rigidity), a 0-pole, or just *pole*, is indeed equivalent to (5) because if $Q(0) \neq 0$, $Q(u) \sim Q(0)$ as $u \rightarrow 0$.

This is unfortunately not always the case:

Example 4. Let $\mathcal{J}(u) = (u_1 - u_2)^2$, $u \in \mathbb{R}^2$. Hence $\int_{B(0,\varepsilon)} u_i^2 \mathcal{J}^{-1}(u) du = \infty$, $i = 1, 2$ for $\varepsilon > 0$, but it does not mean that \mathcal{J}^{-1} has a $(1, 0)$ -pole because $Q(u) := (u_1 - u_2)$ satisfies $\partial_1 Q(0) \neq 0$ and $\int_{B(0,\varepsilon)} Q^2 \mathcal{J}^{-1} < \infty$.

We have the following more general proposition, allowing in particular to treat isotropic spectral measures.

Proposition 1. Define $\tilde{\mathcal{J}}(u) = \sup_{v: \|v\|=\|u\|} \mathcal{J}(v)$. Let $\mathbf{k} \in \mathbb{N}^d$. Then \mathbf{M} is \mathbf{k} -rigid on any bounded measurable A if for $\varepsilon > 0$

$$\int_{B(0,\varepsilon)} \tilde{\mathcal{J}}^{-1}(u) u^{2\mathbf{k}} du = \infty \text{ or equivalently } \int_{B(0,\varepsilon)} \tilde{\mathcal{J}}^{-1}(u) \|u\|^{2|\mathbf{k}|} du = \infty, \quad (6)$$

where $|\mathbf{k}| = \sum_i k_i$.

Let us illustrate this result by deriving the two seminal results about number rigidity by Ghosh and Peres [2017].

Example 5. Let \mathbf{P} be either \mathbf{P}_{GIN} , the infinite Ginibre ensemble or \mathbf{P}_{GAF} , the zero set of the planar Gaussian analytic function (see Ghosh and Peres [2017] for precise definitions). \mathbf{P} is an isotropic process in dimension $d = 2$ where \mathbf{C} has a density decaying exponentially fast, hence

$$\mathcal{J}(u) = \mathcal{J}(0) + \frac{1}{2} \|u\|^2 \int \|t\|^2 \mathbf{C}(dt) + O(\|u\|^4).$$

Ghosh and Peres [2017] start from the facts that for a twice differentiable function $f \in C_c^b(\mathbb{R}^2)$, for some finite C ,

$$\begin{aligned}\text{Var}(I_{\mathbb{P}_{GIN}}(f)) &\leq C \int \|\nabla f\|^2, \\ \text{Var}(I_{\mathbb{P}_{GAF}}(f)) &\leq C \int |\Delta f|^2.\end{aligned}$$

Since $\text{Var}(I_{\mathbb{P}}(f)) = \int |\hat{f}(u)|^2 \mathfrak{J}(u) du$, this readily implies that in both cases $\mathfrak{J}(0) = 0$ (hyperuniformity), hence by Theorem 1 for $k = 0$, \mathfrak{J}^{-1} has a 0-pole in 0 and both processes are number rigid. For $\mathbb{P} = \mathbb{P}_{GAF}$, it also implies that $\int \|t\|^2 \mathbf{C}(dt) = 0$, hence

$$\int \|u\|^2 \mathfrak{J}^{-1}(u) du = \infty,$$

and \mathfrak{J} is isotropic. Using also Proposition 1 it has a pole of order 1 in 0, hence \mathbb{P}_{GAF} is 1-rigid (also proved in Ghosh and Peres [2017]).

Up to date, \mathbb{P}_{GAF} is to my knowledge the only stationary point process which is not a shifted lattice and for which it is rigorously proven that $\mathfrak{J}(u) = o(\|u\|^2)$, see the companion paper for toy models vanishing at arbitrary order. The process \mathbb{P}_{GIN} is an instance of the class of projector determinantal point processes, the general case is treated at Section 4.1.

For $k = 0$, i.e. for number rigidity, our sufficient condition is

$$\int_{B(0,\varepsilon)} \frac{1}{\mathfrak{J}(u)} du = \infty. \quad (7)$$

Let us explain why it allows to recover the sufficient condition of Ghosh and Lebowitz [2017] in dimensions $d = 1, 2$. In dimension 1, they prove that a (unit intensity) point process \mathbb{P} is number rigid if the truncated correlation measure $\rho_{tr}^{(2)}$ has a density $\mathfrak{c} = \mathcal{F}(s-1)$ satisfying $\int \mathfrak{c} = -1$ (hyperuniformity) and

$$|\mathfrak{c}(t)| \leq c(1 + |t|)^2, t \in \mathbb{R}.$$

It implies that \mathfrak{J} is Lipschitz in 0 :

$$\begin{aligned}2|\mathfrak{J}(u)| &= |\mathfrak{J}(u) + \mathfrak{J}(-u)| = \int (e^{itu} + e^{-itu}) \mathfrak{c}(t) dt + 2 \int \mathfrak{c} \\ &\leq \int_{\mathbb{R}} |e^{itu} + e^{-itu} - 2| |\mathfrak{c}(t)| dt \\ &\leq \int 4c \frac{\sin(tu/2)^2}{(1 + |t|)^2} dt \\ &= 4c|u| \int \frac{\sin(z/2)^2}{(|z| + |u|)^2} dz \\ &\leq C|u| \text{ with } C = 4c \int \frac{\sin(z/2)^2}{|z|^2} dz < \infty.\end{aligned}$$

Hence (7) is satisfied. In dimension 2, with similar definitions, the condition is

$$|\mathfrak{c}(t)| \leq c(1 + \|t\|)^{-4-\varepsilon}$$

which gives that $\|t\|^2 c(t) \in L^1(\mathbb{R}^2)$, hence $\mathfrak{s} := \hat{c} + 1$ is twice differentiable in 0, and by parity $\mathfrak{s}'(0) = 0$, meaning $\mathfrak{s}(u) \leq c\|u\|^2$, and (7) holds again.

Bufetov et al. [2018] also give a sufficient condition, but depending on the structure factor of the discretised version, which is not the same as the original one, see the discussion at Section 3.4.

Let us conclude this subsection with an example illustrating the possible complexity of the concept of k -pole:

Example 6. Assume that some $\mathfrak{s} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ satisfies $\mathfrak{s}(u) \sim u_1^4$ as $u \rightarrow 0$. Let Q a non-zero polynomial in $L^2(\mathfrak{s}^{-1}, B(0, \varepsilon))$ for some $\varepsilon > 0$. We put Q under the form

$$Q(u) = P(u_2) + u_1 R(u_2) + u_1^2 S(u_1, u_2)$$

where P, R, S are polynomials. We have $u_1^2 S(u) \in L^2(\mathfrak{s}^{-1}, B(0, 1))$, hence, by isolating dominating terms of P and R , there are exponents $p, q \in \mathbb{N}$ and constants c_P, c_R such that in polar coordinates

$$\int_0^\varepsilon \left[\int_0^{2\pi} \frac{(c_P \rho^p \sin(\theta)^p + c_R \rho \cos(\theta) \rho^q \sin(\theta)^q)^2}{\rho^4 \cos(\theta)^4} d\theta \right] \rho d\rho < \infty.$$

Examining the point $\theta = \pi/2$ in the inner integral, it first implies $c_P = 0$, and then $c_R = 0$, which means $P = R = 0$.

We indeed proved that for a polynomial $Q = \sum a_m u^m$ in $L^2(\mathfrak{s}^{-1}, B(0, \varepsilon))$, all terms of exponent $m = (0, m), m \in \mathbb{N}$ or $m = (1, m), m \in \mathbb{N}$ must vanish. This means that 0 is a pole of order $(1, m)$ for all $m \in \mathbb{N}$ (and a fortiori of order $(0, m)$), and it does not admit other orders.

3.2 Necessary conditions

We investigate what non-rigidity implies on the spectral density \mathfrak{s} , and in particular whether a k -pole is necessary for k -rigidity. Maximal rigidity really depends on the behaviour of \mathfrak{s} on all \mathbb{R}^d , and is rather studied in the companion paper Lachièze-Rey [2024+]. We nevertheless give a context where maximal rigidity and k -rigidity are easier to handle.

Say that a pole $u_0 \in \mathbb{R}^d$ has finite order for \mathfrak{s}^{-1} if for some $\varepsilon > 0, q \in \mathbb{N}$,

$$\int_{B(u_0, \varepsilon)} \frac{\|u - u_0\|^{2q}}{\mathfrak{s}(u)} du < \infty.$$

Definition 2. Say that \mathfrak{s} is simple if \mathfrak{s}^{-1} has finitely many poles, all with finite order, and $\mathfrak{s}(u) \geq c(1 + \|u\|)^{-p}$ for some $p \in \mathbb{N}, c > 0$, at distance ε from these poles.

Proposition 2. Assume that \mathfrak{s} is simple. Then \mathbf{M} is not linearly maximally rigid on $B(0, \eta)$ for $\eta > 0$. Furthermore, for $k \in \mathbb{N}^d$, \mathbf{M} is linearly k -rigid on $B(0, \eta)$ iff 0 is a pole of order k .

The proof is at Section 5.6. This proposition in particular serves to treat stationary Determinantal Point Processes as their structure factor vanishes only in 0, see Section 4.1.

Apart from the *simple* case, we can give converse statements under stronger structural assumptions: for \mathfrak{s} *isotropic* (i.e. $\mathfrak{s}(u)$ only depends on $\|u\|$) or *separable* (i.e. $\mathfrak{s}(u) = \mathfrak{s}_1(u_1) \dots \mathfrak{s}_d(u_d), u \in \mathbb{R}^d$ for some even functions $\mathfrak{s}_i : \mathbb{R} \rightarrow \mathbb{R}_+$) and by assuming upfront that maximal rigidity does not hold, which often can be proved by other means. Remark 4 yields that if rigidity does not hold for some $\mathfrak{s}' \leq \mathfrak{s}$, then it does not hold either for \mathfrak{s} , hence one can investigate the largest separable/isotropic function $\mathfrak{s}' \leq \mathfrak{s}$.

Theorem 2. Let \mathbb{M} with spectral density \mathfrak{s} with either

- \mathfrak{s} is isotropic and $A = B(0, R), R > 0$
- \mathfrak{s} is separable and $A = [-R, R]^d, R > 0$.

Assume \mathbb{M} is not maximally rigid on A and (5) does not hold. Then for $\eta > 0$, \mathbb{M} is not k -rigid on $A^{+\eta}$ in the following cases:

Proof at Section 5.7 .

Remark 1. Under these hypotheses, if it can be established by other means that \mathbb{M} is linearly 0-rigid or 1-rigid, one can reverse the reasoning of Example 5 and deduce similar bounds on the variance of linear statistics.

In view of Proposition 2, Theorems 1 and 2, (5) is necessary and sufficient for k -rigidity on A in the following cases

- \mathfrak{s} is “simple”,
- \mathbb{M} is not LMR and \mathfrak{s} is quasi-isotropic, i.e. there is $c_-, c_+ > 0$ such that

$$c_- \mathfrak{s}(\|u\|) \leq \mathfrak{s}(u) \leq c_+ \mathfrak{s}(\|u\|), u \in \mathbb{R}^d,$$

- \mathbb{M} is not LMR and \mathfrak{s} is quasi-separable, i.e. $0 \leq c_- \mathfrak{s}' \leq \mathfrak{s} \leq c_+ \mathfrak{s}'$ for some separable \mathfrak{s}' .

Remark 2 (No phase transition). For \mathbb{M} which is not LMR on any convex body, linear number rigidity happens either on every convex body or none.

On the other hand, we show in the companion paper Lachièze-Rey [2024+] examples of point processes which are maximally rigid for $R \leq R_c$ for some $R_c > 0$, and not even number rigid for $R > R_c$. There are examples of processes obtained as lattice perturbations, such as in Klatt et al. [2020], Peres and Sly [2014], where number rigidity occurs in dimension $d \geq 3$, but it does not enter in the current framework as it can be proved that this rigidity is non-linear, and the tools employed strongly depend on the underlying lattice structure.

The structural conditions on \mathfrak{s} are necessary as there exists random measures which are linearly k -rigid without \mathfrak{s} vanishing around 0. We give an example for mass rigidity.

Proposition 3. Any random measure which spectral density satisfies $\mathfrak{s}(u) = u_2^2$ in some neighbourhood of $(1, 0)$ is mass rigid.

Proof at Section 5.6.

We can therefore build examples which are completely standard, i.e. with fast decay of the correlations, linear variance, and which are rigid; by the previous results, such examples will never be isotropic or separable. To give a concrete example, take a mixture of an isotropic and a separable density

$$\mathfrak{J}(u) = \begin{cases} 1 & \text{if } \|u\| \leq 1/2 \\ \frac{u_2^2}{1+u_2^{10}} \frac{1}{1+u_1^{10}} & \text{otherwise,} \end{cases}$$

not LMR because $\psi(u) := u_2(1 + \|u\|^{100})^{-1} \in L^2(\mathfrak{J}^{-1})$. To *realise* \mathfrak{J} , i.e. find \mathbf{M} for which such \mathfrak{J} is indeed the spectral density, use for instance Gaussian fields with Example 3. Whether there are such number rigid point processes with standard second order behaviour which are not hyperuniform depends on our ability to have such spectral densities for random atomic measures, this relates to the general difficult problem of *realisability of point processes*.

3.3 k-tolerance

Ghosh and Peres [2017] show that the infinite Ginibre ensemble \mathbf{P}_{GIN} is not rigid at an order higher than 0, in the sense that if one determines $N := \#\mathbf{P}_{GIN} \cap A$ from $\mathbf{P}_{GIN} \cap A^c$, then conditionnally to $\mathbf{P}_{GIN} \cap A^c$ and N , $\mathbf{P}_{GIN} \cap A$'s law is continuous wrt $\mathbf{U}_N := \{U_1, \dots, U_N\}$ where the U_i are iid uniform on A . They similarly show that the zero set \mathbf{P}_{GAF} of the planar GAF is 1-rigid and not more, in the sense that conditionnally to $(N := \#\mathbf{P}_{GAF} \cap A, U := \sum_{x \in \mathbf{P}_{GAF} \cap A} x, \mathbf{P}_{GAF} \cap A^c)$, $\mathbf{P}_{GAF} \cap A$ has a continuous law wrt to the law of \mathbf{U}_N conditionned to $\sum_{i=1}^N U_i = U$. A similar result is proved by Dereudre et al. [2020] for the sine_β processes $\mathbf{P}_\beta, \beta > 0$.

These strong distributional results rely on the determinantal nature of \mathbf{P}_{GIN} , the Gaussian nature of \mathbf{P}_{GAF} , or the DLR equations defining \mathbf{P}_β , and cannot be extended in the current general context. We can still investigate how “linearly tolerant” is a process which is not rigid, i.e. whether the random moments $m_{\mathbf{m}}(A) := \int_A t^{\mathbf{m}} d\mathbf{M}(t), \mathbf{m} \in \mathbb{N}^d$ are tied by a finite linear relation

$$\sum_{\mathbf{m}} a_{\mathbf{m}} m_{\mathbf{m}}(A) \in \sigma_{lin}(\mathbf{M}; A^c)$$

or equivalently if we have Q -rigidity on A for the polynomial $Q(t) = \sum_{\mathbf{m}} a_{\mathbf{m}} t^{\mathbf{m}}$. We show that this holds for what we call a *k-polynomial* Q , i.e. such that for some $\mathbf{k}_0 \succeq \mathbf{k}$, $a_{\mathbf{k}_0} \neq 0$ and the non-vanishing terms $a_{\mathbf{m}} \neq 0$ are such that $\mathbf{m} \preceq \mathbf{k}_0$.

Proposition 4. Assume that 0 is not a pole of order k and the assumptions of Theorem 2 or of Proposition 2 hold. Then \mathbf{M} is not Q -rigid for any k -polynomial Q .

The proof of this proposition is actually included in the proofs of resp. Proposition 2 and Theorem 2. As is apparent from the proofs, we will in fact have no Q -rigidity for almost all polynomials having at least a non-vanishing term of order $\mathbf{k}_0 \succeq \mathbf{k}$, i.e. for a linear combination $Q = \sum_i \alpha_i Q_i$ where the Q_i are k -polynomials, but some particular combinations of α_i are harder to discard.

3.4 Rigidity for discrete processes

Let us derive similar results for a centred L^2 discrete process $\mathbf{X} = \{\mathbf{X}_{\mathbf{m}}; \mathbf{m} \in \mathbb{Z}^d\}$. Define for $\gamma : \mathbb{Z}^d \rightarrow \mathbb{R}$ with compact support

$$\mathbf{X}(\gamma) := \sum_{\mathbf{m}} \mathbf{X}(\mathbf{m})\gamma(\mathbf{m}).$$

\mathbf{X} is said to be linearly γ -rigid on $A \subset \mathbb{Z}^d$ if $\inf_{h: A^c \rightarrow \mathbb{C}} \mathbf{X}(\gamma) - \mathbf{X}(h) = 0$ a.s. and in $L^2(\mathbb{P})$, where h has finite support, and LMR if this holds for all γ with finite support. If $\gamma(\mathbf{m}) = \mathbf{m}^{\mathbf{k}}$ we talk about \mathbf{k} -rigidity.

We assume \mathbf{X} is wide-sense L^2 -stationary with spectral measure \mathbf{S} , i.e.

$$\text{Var}(\mathbf{X}(\gamma)) = \int_{\mathbb{T}^d} |\hat{\gamma}(u)|^2 \mathbf{S}(du)$$

where $\mathbb{T} = [-\pi, \pi]$, for the complex-valued trigonometric polynomial

$$\hat{\gamma}(u) = \sum_{\mathbf{m} \in \mathbb{Z}^d} e^{-i\mathbf{m} \cdot u} \gamma(\mathbf{m}).$$

For $A \subset \mathbb{Z}^d$, the class of such trigonometric polynomials where $\gamma(\mathbf{m}) = 0$ for $\mathbf{m} \notin A$ is denoted by $\mathcal{E}(A)$, by analogy with entire functions of exponential type (see Section 5).

Theorem 3. Let $\mathfrak{s}(u)du$ the continuous part of \mathbf{S} .

- \mathbf{X} is not LMR on $\llbracket m \rrbracket^d$ iff there is $\psi \in \mathcal{E}(\llbracket m \rrbracket^d)$ such that

$$\int \frac{|\psi(u)|^2}{\mathfrak{s}(u)} du < \infty. \quad (8)$$

- For $\mathbf{k} \in \mathbb{N}^d$, \mathbf{X} is not \mathbf{k} -rigid iff there is $\psi \in \mathcal{E}(\llbracket m \rrbracket^d) \cap L^2(\mathfrak{s}^{-1})$ such that $\partial^{\mathbf{k}}\psi(0) \neq 0$.
- If 0 is a pole of order \mathbf{k} for \mathfrak{s}^{-1} , then \mathbf{X} is \mathbf{k} -rigid.

Proposition 5 (Specialization to time series in dimension $d = 1$). \mathbf{X} is not LMR on $\llbracket m \rrbracket$ iff the number of poles of \mathfrak{s}^{-1} on $[-\pi, \pi]$ counted with multiplicity is $\leq 2m$. Also, \mathbf{X} is not 0-rigid iff there are $2m$ poles or less and 0 is not a pole.

The proof is at Section 5.1. We discuss the LMR result below, the other statements seem to be the first dealing with partial rigidity, i.e. when one is concerned not in retrieving all of \mathbf{X} on $\llbracket m \rrbracket^d$, but only its restricted moments, by analogy to the continuous case.

The line of research of linear interpolation in the 20th century was mainly concerned with time series, and authors noted that a polynomial satisfying (8) means that perfect linear interpolation is impossible, see (10.28) and Theorem 10.3 in [Rozanov \[1967\]](#), with also an extension to processes taking values in \mathbb{R}^q . Much later, [Lyons and Steif \[2003\]](#) study the related problem of the strong full K property for $\{0, 1\}$ -valued \mathbf{X} , i.e. what does it imply on the $\mathbf{X}(\mathbf{m}), |\mathbf{m}| \leq m$, if we impose $\mathbf{X}(\mathbf{m}) = 1$ for $m \leq |\mathbf{m}| < m + k$ for m, k arbitrarily large? It is clear indeed that if no polynomial ψ is in $L^2(\mathfrak{s}^{-1})$, then \mathbf{X} is LMR for all m , and as such for k large enough the external conditioning completely determines

$\mathsf{X}(\mathbf{m}), |\mathbf{m}| \leq m$. This strong resemblance explains why condition (iii) in their theorem 7.7 is exactly the same condition. None of these works seems to acknowledge that m must be the degree of ψ . This result also completes the answer of Ghosh [2015] to the Question 9.9 of Lyons and Steif [2003] regarding rigidity on $A = \{0\}$.

Bufetov et al. [2018] studies the rigidity of continuous models through a discretisation procedure. Given a stationary random measure \mathbf{M} on \mathbb{R}^d , one can consider the associated stationary discrete process

$$\mathsf{X}_{\mathbf{m}} = \mathbf{M}(\mathbf{m} + [-1/2, 1/2]^d), \mathbf{m} \in \mathbb{Z}^d.$$

Bufetov et al. [2018] show number rigidity of \mathbf{M} on $[-1/2, 1/2]^d$ by showing the rigidity of X on $\{0\}$, and derive a necessary condition in terms of the covariance decay for X .

Remark 3. A reformulation of 0-rigidity in dimension 1 is the following: X is 0-rigid if X is LMR or 0 is a pole, which means that either $\int_{B(0,\varepsilon)} \mathcal{J}^{-1} = \infty$ or that the poles are away from zero and strictly more than m (counted with multiplicity), i.e. if one considers only m points, necessarily at least one pole is not covered, i.e. for $u_1, \dots, u_m \in \mathbb{C}^d \setminus \{1\}$,

$$\int \frac{\prod_i |e^{iu} - e^{iu_i}|^2}{\mathcal{J}(u)} du = \infty.$$

This seems to correspond to the content of Remark 2.1 of Bufetov et al. [2018] for continuous measures $\mathsf{S} = \mathcal{J}\mathcal{L}$.

At the contrary of the continuous case, k -rigidity is not monotonous in k , in the sense that there are examples of 1-rigid processes that are not 0-rigid in dimension 1, due to the constraint on the degree:

Example 7. Consider $m = 1$ in dimension $d = 1$, and $\mathcal{J}(u) = (u-1)^2(u+1)^2, u \in \mathbb{T}$. Let ψ a degree 1 trigonometric polynomial of $L^2(\mathcal{J}^{-1})$. In particular, there is a polynomial P of degree 2 such that

$$e^{iu}\psi(u) = P(e^{iu}) = a(e^{iu} - z_1)(e^{iu} - z_2), u \in \mathbb{T},$$

for some $a \in \mathbb{C} \setminus \{0\}, z_1, z_2 \in \mathbb{C}$. Examining the neighbourhoods of 1 and -1 of \mathcal{J}^{-1} , we necessarily have $z_1 = e^i, z_2 = e^{-i}$ (or the other way around), this is also sufficient for $\psi \in L^2(\mathcal{J}^{-1})$, showing that $L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(\llbracket 1 \rrbracket) \neq \{0\}$, hence \mathcal{J} is not LMR. It also implies that every such ψ is even, hence $\psi'(0) = 0$, meaning that \mathcal{J} is 1-rigid. On the other hand, $Q(0) \neq 0$, hence \mathcal{J} is not 0-rigid.

4 Applications

4.1 Determinantal processes and completeness

Determinantal points processes (DPPs), introduced in the context of quantum mechanics, have gained popularity as many classes of essential models in Matrix theory, statistical physics, combinatorics and others have proven to be determinantal, see Hough et al. [2009]. In the Euclidean context, we follow Soshnikov [2000]: a simple point process P on \mathbb{R}^d is determinantal with kernel $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ if for every $k \in \mathbb{N}^*$,

$$\rho_k(x_1, \dots, x_k) := \det((K(x_i, x_j))_{1 \leq i, j \leq k}) \geq 0$$

is the k point correlation function of \mathbf{P} , i.e. for any non-negative $\varphi : (\mathbb{R}^d)^k \rightarrow \mathbb{R}$, we have

$$\mathbf{E} \left(\sum_{\substack{\neq \\ x_1, \dots, x_k \in \mu}} \varphi(x_1, \dots, x_k) \right) = \int \rho_k \varphi,$$

where the sum runs over k -tuples of pairwise distinct points. We put by convention $K(0, 0) = 1$ to deal with unit intensity processes. Stationarity yields that $\rho_k(x_1, \dots, x_k) = \rho_k(x_1 - x_k, \dots, x_{k-1} - x_k, 0)$. Not all functions K give rise to a DPP and in particular we will require that K is Hermitian and positive definite, so that

$$\rho_2(x - y) = 1 - \kappa(x - y)^2 \in \mathbb{R}_+$$

where $\kappa(x) = |\kappa(0, x)|$, $0 \leq \kappa(x) \leq 1$, we also have to assume that κ is locally square integrable for definiteness issues. See [Hough et al., 2009, Section 4.5] for unicity and existence questions, and Soshnikov [2000] for a treatment of such stationary DPPs, it is shown in particular that they are mixing and ergodic, and the trace-class property yields that $\hat{\kappa}$ is integrable and $0 \leq \hat{\kappa} \leq 1$.

The main result of this section is a characterisation of k -rigidity in the continuous setting:

Corollary 1. *Let \mathbf{P} be a stationary DPP on \mathbb{R}^d with Hermitian positive kernel K and $\kappa(x) = |K(0, x)|$ is square integrable. Then for $k \in \mathbb{N}^d$, \mathbf{P} is k -rigid on a convex body A iff $(1 - \hat{\kappa}^2)^{-1}$ has a pole of order k in 0. When it is not the case, \mathbf{P} is k -tolerant in the sense that it is not Q -rigid for any k -polynomial Q .*

We see that $\mathfrak{s} := (1 - \hat{\kappa}^2)$ must vanish in 0 to have some rigidity. That is because, by Soshnikov [2000], $\mathfrak{s}(u)du$ is indeed the spectral measure of \mathbf{P} , and $\mathfrak{s}(0) = 0$ exactly means that \mathbf{P} is HU (see (2)); hyperuniform DPPs are a very important class in statistical physics and matrix theory, and also used for tasks of numerical integration Bardenet and Hardy [2020] or machine learning Kulesza and Taskar [2012].

Proof. Since $\mathfrak{s}(u) \geq 0$, $\int \kappa^2 \leq 1$, and $\mathfrak{s}(u) = 0$ is only possible if $u = 0$. Since κ^2 is integrable, the Riemann-Lebesgue lemma ([Duistermaat and Kolk, 2006, Th. 14.2]) yields that $|\hat{\kappa}^2(u)| < 1/2$ for $|u| \geq T$ for some T . By Theorem 1, \mathbf{P} is k -rigid on A if \mathfrak{s}^{-1} has a pole of order k in 0. If it is not satisfied, it follows by Proposition 4 that \mathbf{M} is not Q -rigid for a polynomial Q having non zero coefficients a_k for some $|k| \geq k$. □

Linear number rigidity is therefore equivalent to the non-integrability of $(1 - \hat{\kappa}^2)^{-1}$. For many examples, $\hat{\kappa}^2(u) = 1 - \sigma^2 u^2 + o(u^2)$ for some $\sigma > 0$, hence there is k -rigidity iff $k = 0$ and $d \in \{1, 2\}$, and no rigidity in higher dimensions. This applies for instance to the infinite Ginibre ensemble on $\mathbb{R}^2 \approx \mathbb{C}$, defined by

$$K(x, y) = e^{x\bar{y} - \frac{1}{2}|x|^2 - \frac{1}{2}|y|^2}; x, y \in \mathbb{C},$$

i.e. $\kappa(x) = e^{-\frac{|x|^2}{2}}$, retrieving the result of Ghosh and Peres [2017], or the result of Ghosh [2015], stating that the sine process in dimension 1 and Ginibre ensemble in dimension 2 are number rigid. Tensor kernels give non rigid examples: Bufetov et al. [2018] proves non-rigidity for the DPP \mathbf{P} which kernel is $\kappa(x) = \text{sinc}(x_1)\text{sinc}(x_2)$ on $A = [-1/2, 1/2]^2$

with a discretisation (see Section 3.4). Theorem 1 yields that this \mathbb{P} is not linear number rigid in the continuous sense on any convex body A , any discrete average will not be linearly rigid either.

This result also bears a connection to the question of finding random countable sets of exponential functions spanning $L^2(A)$ for a compact $A \subset \mathbb{R}^d$. For $\chi \subset \mathbb{R}^d$, call

$$E_\chi = \{t \mapsto e^{tx}; x \in \chi\}.$$

Say that E_χ is *complete* if functions of $L^2([-\pi, \pi])$ are L^2 approximable by functions of E_χ . Lyons and Steif [2003] investigated the question of completeness for a random set $\chi \subset \mathbb{R}^d$, and more particularly if χ 's law is a DPP, leaving open several questions. Ghosh [2015] established a connection between number rigidity and completeness, and a corollary of [Ghosh, 2015, Th. 1.3] and Corollary 1 is the following result:

Corollary 2. *If \mathbb{P} is a DPP, then $E_\mathbb{P}$ spans $L^2([-\pi, \pi])$ if $(1 - \widehat{\kappa}^2)^{-1}$ is not integrable in 0.*

In particular it retrieves Th. 1.5 of Ghosh [2015] in dimension 1, which shows number rigidity if $\widehat{\kappa}$ is the indicator function of an interval: Parseval equality yields

$$\widehat{\kappa}^2(0) = \int \kappa^2 = \int \widehat{\kappa}^2 = \int \widehat{\kappa} = \kappa(0) = 1,$$

hence $\mathcal{J}(0) = 0$ (meaning \mathbb{P} is hyperuniform), and $\widehat{\kappa}^2$ is clearly Lipschitz, hence $\mathcal{J}(u) = O(|u|)$ and we have number rigidity.

4.2 Consequences for the covariance of Gibbs measures

Gibbs measures are also a prominent class of point processes, appearing in random matrices, statistical physics, and many other domains. It is known that in the case of short range interactions, such processes are not hyperuniform, see Dereudre and Flimmel [2024], hence \mathbb{S} will not vanish around 0 and the process is most likely not rigid. On the other hand, those with a very strong dependency at long range, such as Coulomb gases, or more generally Riesz gases, form a very important class of models in statistical physics, and are expected to be hyperuniform, see [Lewin, 2022, VI.C] and references therein.

Such models are usually defined for a finite but large number of particles interacting through some physical equations, and defining an infinite stationary model compatible with these equations is already a challenge, there can in many instances be distinct infinite models compatible with local conditions. In dimension 1, Valkó and Viràg [2009] have reached a universal explicit limit of 1D sine_β log gases, further studied in Dereudre et al. [2020], Chhaibi and Najnudel [2018], where it is proved that it is number rigid. Not much seems to be known in higher dimensions on the rigidity (or existence) of stationary models.

Summing up the current main results in this context, k -rigidity of a hyperuniform isotropic (or 1D) model is equivalent to (5), which is already a rigorous statement on the second order properties of the process through \mathcal{J} , though sometimes difficult to connect with the properties of the Fourier transform \mathbb{C} if the correlations are not known to decay fast. In particular, proving hyperuniformity is already a challenge, the only rigorous result in dimension $d \geq 2$ is to my knowledge the work Leblé [2023], showing that 2D

Coulomb gases are hyperuniform, but the obtained estimate is not sufficient to prove rigidity.

In the other direction, [Dereudre et al. \[2020\]](#) and [Dereudre and Vasseur \[2023\]](#) have established non-rigidity results, the latter managed to prove the existence of an infinite stationary and isotropic model in all dimension $d \geq 1$ compatible with the s -Riesz DLR equations for $d - 1 < s < d$, the *circular Riesz gas*, which is not number rigid. Hence we can use the necessary condition of [Theorem 2](#) to deduce rigorous statements about \mathbf{S} , and therefore $\mathbf{C} = \mathcal{F}\mathbf{S}$. The fact that Riesz gases are hyperuniform is expected, see an argument in [[Lewin, 2022, VI.C](#)], but there does not seem to be a rigorous proof.

Corollary 3. *Let \mathbf{M} a hyperuniform non-number rigid isotropic random stationary measure in dimension $d \in \{1, 2\}$. Then*

$$\int |t|^d |\mathbf{C}|(dt) = \infty.$$

Proof. If

$$\int_{\mathbb{R}} |t| |\mathbf{C}(dt)| < \infty$$

in dimension 1, $\mathbf{S} = \mathcal{F}\mathbf{C}$ has a differentiable density \mathfrak{s} satisfying $\mathfrak{s}(0) = 0$ (hyperuniformity), and

$$|\mathfrak{s}(u)| \leq \int |u| |t| |\mathbf{C}(t)| dt,$$

hence \mathfrak{s}^{-1} is not integrable in 0, and according to [Theorem 1](#), \mathbf{M} is number rigid, we reach a contradiction.

In dimension 2, if

$$\int_{\mathbb{R}^2} \|t\|^2 |\mathbf{C}|(dt) < \infty,$$

\mathbf{S} is twice differentiable, and by parity $\partial_1 \mathfrak{s}(0) = \partial_2 \mathfrak{s}(0) = 0$, hence

$$\mathfrak{s}(u) \leq c \|u\|^2,$$

similarly $\mathfrak{s}(u)^{-1}$ is not integrable in 0, \mathbf{M} is number rigid, which is again a contradiction. \square

The sine_β gases in 1D are proved by [Dereudre et al. \[2020\]](#), [Chhaibi and Najnudel \[2018\]](#) to be number rigid, and the proof of [Chhaibi and Najnudel \[2018\]](#) proves that this rigidity is linear. [Theorem 2](#) immediately gives the following result.

Corollary 4. *For all $\beta > 0$, the spectral density \mathfrak{s}_β of the sine_β process satisfies*

$$\int_{B(0, \varepsilon)} \frac{1}{\mathfrak{s}_\beta(u)} du = \infty.$$

[Dereudre et al. \[2020\]](#) prove that it is not further rigid, and in particular not (linearly) 1-rigid. This is actually expected, but it still rigorously prevents some behaviours.

Corollary 5. *Let \mathbf{M} a not 1-rigid hyperuniform process in dimension 1. Then*

$$\int |t|^3 |\mathbf{C}|(dt) = \infty \text{ or } \int t^2 \mathbf{C}(t) dt \neq 0.$$

At first sight, the second moment vanishing seems unlikely, but this actually occurs for the model \mathbf{P}_{GAF} (see Example 5), and hyperuniformity itself is the result of the vanishing of a moment (of order 0). Such moment cancellations, are sometimes called *sum rules* by physicists, and whether they occur and yield higher order rigidity is a fascinating topic of mathematics, see again [Lewin, 2022, VI.C] for references.

5 Proofs

For \mathbf{S} a non-negative measure on $\mathbf{E} = \mathbb{R}^d$ or $\mathbf{E} = \mathbb{T}^d$ and $A \subset \mathbf{E}$, write $L^2(\mathbf{S}; A)$ as the space of functions $\psi : A \rightarrow \mathbb{C}$ satisfying

$$\int_A |\psi|^2 d\mathbf{S} < \infty,$$

simply note $L^2(\mathbf{S}) = L^2(\mathbf{S}; \mathbf{E})$.

5.1 Proof of Theorem 3 (discrete processes)

We start with the discrete case, which does not require high level Fourier technology, and can serve as a model for the proof of the main theoretical result, Theorem 6.

Proof. For $A \subset \mathbb{Z}^d$, recall that $\mathcal{E}(A)$ is the space of trigonometric polynomials $\sum_{\mathbf{m} \in A} a_{\mathbf{m}} e^{i\mathbf{m} \cdot u}$ with $a_{\mathbf{m}} \in \mathbb{C}$. For $\gamma : \mathbb{Z}^d \rightarrow \mathbb{C}$, γ -rigidity means that for some h_n compactly supported in A^c , a.s. and in $L^2(\mathbf{P})$,

$$\mathbf{X}(\gamma) = \lim_n \mathbf{X}(h_n),$$

and this is equivalent to

$$\inf_{h: (\llbracket m \rrbracket^d)^c \rightarrow \mathbb{C}} \text{Var}(\mathbf{X}(\gamma) - \mathbf{X}(h)) = \inf_{h: (\llbracket m \rrbracket^d)^c \rightarrow \mathbb{C}} \int_{\mathbb{T}^d} \mathbf{S}(du) |\hat{\gamma}(u) - \hat{h}(u)|^2 = 0,$$

where $\hat{\gamma} \in \mathcal{E}(\llbracket m \rrbracket^d)$ and $\hat{h} \in \mathcal{E}((\llbracket m \rrbracket^d)^c)$. The orthogonal space of $\mathcal{E}((\llbracket m \rrbracket^d)^c)$ in $L^2(\mathbf{S})$ is

$$H_m^\perp := \left\{ \varphi \in L^2(\mathbf{S}) : \int_{\mathbb{T}^d} \bar{\varphi} \hat{h} d\mathbf{S} = 0; h : (\llbracket m \rrbracket^d)^c \rightarrow \mathbb{C} \text{ with finite support} \right\}.$$

For such φ , $\Psi(du) := \varphi \mathbf{S}(du)$ satisfies

$$a_{\mathbf{m}} := \int \bar{\Psi}(du) e^{i\mathbf{m} \cdot u} = \begin{cases} 0 & \text{if } |\mathbf{m}| > m \\ \int_{\mathbb{T}^d} \bar{\varphi}(u) e^{i\mathbf{m} \cdot u} \mathbf{S}(du) & \leq \|\varphi\|_{L^2(\mathbf{S})} \text{ for } |\mathbf{m}| \leq m, \end{cases}$$

recalling that $\mathbf{S}(\mathbb{T}^d) = \text{Var}(\mathbf{X}_0) = 1$. It means that Ψ coincides as inverse Fourier transform with

$$\sum_{|\mathbf{m}| \leq m} a_{\mathbf{m}} e^{i\mathbf{m} \cdot u},$$

in particular Ψ has a density $\psi \in \mathcal{E}(\llbracket m \rrbracket^d)$, and the negligible set supporting the singular part \mathcal{S}_s of \mathcal{S} is not charged (i.e. $\varphi \mathcal{S}_s = 0$). We must also have $\varphi \in L^2(\mathcal{S})$, i.e.

$$\int \frac{|\psi|^2}{\mathcal{J}} < \infty.$$

We proved indeed that in $L^2(\mathcal{S}, \mathbb{C})$,

$$H_m^\perp \subset \{\psi \mathcal{J} : \psi \in \mathcal{E}(\llbracket m \rrbracket^d) \cap L^2(\mathcal{J}^{-1})\}.$$

For the converse, we clearly have for such ψ vanishing on the negligible singular part's support, $\varphi := \psi \mathcal{J}^{-1} \in L^2(\mathcal{S})$, and for $h \in \mathcal{E}(\llbracket m \rrbracket^d)^c$

$$\int \bar{\varphi} \hat{h} \mathcal{J} = \int \bar{\psi} \hat{h} = 0.$$

Since LMR is equivalent to $H_m^\perp = \{0\}$, this part of the proof is complete.

Mass rigidity (or 0-rigidity) means γ -rigidity for $\gamma = 1_{\llbracket m \rrbracket^d}$, i.e. $\langle \varphi, \hat{\gamma} \rangle = 0$ for $\varphi \in H_m^\perp$. Hence it means, for all $\psi = \sum_{\mathbf{m}} a_{\mathbf{m}} e^{i\mathbf{u} \cdot \mathbf{m}} \in \mathcal{E}(\llbracket m \rrbracket^d) \cap L^2(\mathcal{J}^{-1})$,

$$0 = \langle \psi \mathcal{J}, \hat{\gamma} \rangle_{\mathcal{S}} = \left\langle \sum_{\mathbf{m} \in \llbracket m \rrbracket^d} \hat{\delta}_{\mathbf{m}}, \psi \right\rangle = \int_{\mathbb{T}^d} \sum_{\mathbf{m}} e^{-i\mathbf{m} \cdot \mathbf{u}} \sum_{\mathbf{m}'} a_{\mathbf{m}'} e^{i\mathbf{m}' \cdot \mathbf{u}} d\mathbf{u} = \sum_{\mathbf{m}} a_{\mathbf{m}} = \psi(0).$$

\mathbf{k} -rigidity means the same with

$$0 = \left\langle \sum_{\mathbf{m}} \mathbf{m}^{\mathbf{k}} \hat{\delta}_{\mathbf{m}}, \psi \right\rangle = \sum_{\mathbf{m}} \mathbf{m}^{\mathbf{k}} a_{\mathbf{m}} = i^{\mathbf{k}} \partial^{\mathbf{k}} \psi(0).$$

Assume now that \mathcal{J}^{-1} has a pole of order \mathbf{k} . Let $\varphi \in H_m^\perp \setminus \{0\}$, $\psi = \varphi \mathcal{J}^{-1}$. In particular, $\int_{B(0, \varepsilon)} \psi^2 \mathcal{J}^{-1} < \infty$, hence since 0 is a \mathbf{k} -pole, $\partial^{\mathbf{k}} \psi(0) = 0$. We indeed proved \mathbf{k} -rigidity. \square

Proof of Proposition 5. Assume the poles of \mathcal{J}^{-1} in $[-\pi, \pi]$ have finite orders k_i , i.e. for $\varepsilon > 0$ sufficiently small,

$$\int_{u_i - \varepsilon}^{u_i + \varepsilon} \frac{|u - u_i|^{2k_i}}{\mathcal{J}(u)} du < \infty$$

or equivalently

$$\int_{u_i - \varepsilon}^{u_i + \varepsilon} \frac{|e^{iu} - e^{iu_i}|^{2k_i}}{\mathcal{J}(u)} du < \infty$$

with $\sum_i k_i \leq 2m$. Since \mathcal{J} is even, for u_i a pole, $-u_i$ is also a pole (with same order). Let u_1, \dots, u_p the poles $\neq 0$ (repeated according to multiplicity), with $p \leq m$. If $p = m$, 0 is not a pole otherwise there are more than $2m$ poles. If $p < m$, if 0 is a pole, its order is $2(m - p)$ or less by the hypothesis. We hence define $\psi \in \mathcal{E}(\llbracket m \rrbracket)$ by

$$\psi(u) = |e^{iu} - 1|^{2(m-p)} \prod_{i=1}^p (e^{iu} - e^{iu_i})(e^{-iu} - e^{iu_i}).$$

ψ is by construction of degree m , non null, and in $L^2(\mathcal{J}^{-1})$, hence X is not LMR by Theorem 3.

Conversely, assume not LMR: by Theorem 3 there is

$$\psi(u) = \sum_{|m| \leq m} a_m e^{im \cdot u} \neq 0$$

in $L^2(\mathcal{J}^{-1})$, and since $\mathcal{J}(u) = \mathcal{J}(-u)$, $\psi(-u) \in L^2(\mathcal{J}^{-1})$. Now, either $\psi(u) = -\psi(-u)$, or $\psi(u) + \psi(-u)$ is a non-zero even function of $L^2(\mathcal{J}^{-1})$. In any case, we can assume $|\psi(u)| = |\psi(-u)|$ without loss of generality. Let

$$P(z) = z^m \sum_{|m| \leq m} a_m z^m = \prod_{i=1}^{2m} (z - z_i)$$

for some $z_1, \dots, z_{2m} \in \mathbb{C}$, which is a polynomial of degree $2m$, and $\psi(u) = P(e^{iu})e^{-imu}$.

A zero of ψ is necessarily a root of P with norm 1, $z_i = e^{iu_i}$, and $-u_i$ is also a zero, hence e^{-iu_i} is also a root of P . Since P has at most $2m$ roots, ψ has at most $2m$ zeros, which means that \mathcal{J} has at most $2m$ poles (all counted with multiplicity).

For the converse sense for 0-rigidity, assume that X is not LMR, meaning there are less than $2m$ poles. If 0 is not a pole, then indeed with the previous notation $\psi(1) = P(e^{iu \cdot 0}) \neq 0$, whence $|\psi(u)| = |P(e^{iu})|$ is in $\mathcal{E}(\llbracket m \rrbracket) \cap L^2(\mathcal{J}^{-1}) \setminus \{0\}$ with

$$\left\langle \sum_{|k| \leq m} \delta_k, \hat{\psi} \right\rangle = \psi(1) \neq 0,$$

it gives the equivalence for 0-rigidity. □

5.2 The Schwartz-Paley-Wiener Theorem

The Fourier transform used here relies on the space of tempered distributions, denoted by $\mathcal{S}'(\mathbb{R}^d)$ as the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of smooth functions with rapid decay; there is no need to enter into technical details in this article, see for instance [Duistermaat and Kolk \[2006\]](#) for a theoretical exposition. We only consider in this paper complex-valued measures, that is a distribution Ψ on \mathbb{R}^d such that for some non-negative measure $|\Psi|$ on \mathbb{R}^d ,

$$|\Psi(f)| \leq \int |f| |\Psi|(dx), f \in C_c^b(\mathbb{R}^d);$$

we use the notation $\Psi(f) = \int f \Psi$ when it makes sense. Such Ψ is indeed tempered if for instance for some $p > 0$

$$\int_{\mathbb{R}^d} (1 + \|u\|)^{-p} |\Psi|(du) < \infty, \tag{9}$$

it is in particular locally finite. We actually only consider measures of the form $\Psi = \psi \mathsf{S}$ where $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ and S is a non-negative measure.

One can then define the Fourier transform $\mathcal{F}\Psi \in \mathcal{S}'(\mathbb{R}^d)$ through

$$\langle \varphi, \mathcal{F}\Psi \rangle = \langle \hat{\varphi}, \Psi \rangle$$

for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\hat{\varphi}(u) = \int_{\mathbb{R}^d} e^{-i\langle u, t \rangle} dt$ is the usual Fourier transform on $L^2(\mathbb{R}^d)$. Recall the Fourier inversion:

Theorem 4 (Duistermaat and Kolk [2006], Th.14.18). For $\Psi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\mathcal{F}(\mathcal{F}\dot{\Psi}) = (2\pi)^d \Psi$$

where $\dot{\Psi}(\cdot) = \Psi(-\cdot)$ in the distributional sense. Call *spectrum* of Ψ , denoted by $\text{sp}(\Psi)$, the support of $\mathcal{F}\Psi$ as a distribution, i.e. the largest closed set A such that $\langle \mathcal{F}\Psi, \varphi \rangle = 0$ for φ supported by A^c .

For a compact A , define for $\zeta \in \mathbb{R}^d$

$$s_A(\zeta) = \sup_{x \in A} \langle \zeta, x \rangle.$$

In the case $A = B(0, R)$, we have $s_A(\zeta) = R\|\zeta\|$. One can also notice that $s_A = s_{\text{conv}(A)}$ characterises the closed convex hull $\text{conv}(A)$ of A .

Call analytic function on \mathbb{C}^d any function

$$\psi(z) = \sum_m a_m z_1^{m_1} \dots z_d^{m_d}, z \in \mathbb{C}^d$$

where $a_m \in \mathbb{C}$ and the series is absolutely convergent on \mathbb{C}^d . Recall that by Hartog's theorem (see [Krantz, 1992, Sec.0.2]), this is equivalent to the analyticity of ψ in each variable z_i separately. The scalar product on \mathbb{C}^d is $\langle x, z \rangle = \sum_i \bar{x}_i z_i$. For $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, write $\|z\|_{\mathbb{C}}^2 = \langle z, z \rangle$, not to be mistaken with the entire function $\|u\|^2 = \sum_i u_i^2$, mostly used on \mathbb{R}^d .

Theorem 5. [Schwartz' Paley-Wiener Theorem] Let $A \subset \mathbb{R}^d$ bounded. Let Ψ a tempered complex-valued measure on \mathbb{R}^d . Then $\text{sp}(\Psi) \subset \text{conv}(A)$ iff Ψ has a density ψ wrt \mathcal{L} on \mathbb{R}^d that can be extended as an analytic function on \mathbb{C}^d such that for some finite C ,

$$|\psi(z)| \leq C \exp(s_A(\mathcal{F}z)), z \in \mathbb{C}^d.$$

Such functions are denoted by $\mathcal{E}(A)$, which is therefore a subclass of the class of entire functions of \mathbb{R}^d , and also a subclass of the class of (restrictions to \mathbb{R}^d of) analytic functions with exponential type, and the extension to \mathbb{C}^d is still denoted ψ .

An important observation is that for $\psi \in \mathcal{E}(A)$, for a polynomial Q , $Q\psi \in \mathcal{E}(A)$, hence it still has spectrum in A , and if ψQ^{-1} is analytic, it is also in $\mathcal{E}(A)$.

A more general version of the previous theorem for all $\mathcal{S}'(\mathbb{R}^d)$ is proved at [Duistermaat and Kolk, 2006, Th. 17.1, Theorems 17.3].

See the companion paper Lachièze-Rey [2024+] for an elementary proof of a result slightly more general than the direct implication.

The construction of some functions of $\mathcal{E}(A)$ rely on the following technique, involving J_d the Fourier transform of the unit sphere indicator:

$$J_d(u) = \frac{B_{d/2}(\|u\|)}{\|u\|^{d/2}}, u \in \mathbb{R}^d, \quad (10)$$

where $B_{d/2}$ is the Bessel function of the first kind of order $d/2$.

Lemma 1. Let $\psi \in \mathcal{E}(A)$ for some bounded A . Then for every non-constant polynomial P and $\eta > 0, p \geq 0$, the function

$$\tilde{\psi}(u) = P(u)\psi(u)J_d(\eta u/M)^M$$

with $M = \frac{2}{d+1}(\deg(P) + p)$ satisfies $|\tilde{\psi}(u)| \leq c|\psi(u)|||u||^{-p}$ for some finite c and $u \in \mathbb{R}^d$, and $\psi \in \mathcal{E}(A^{+\eta})$ where $A^{+\eta} := \cup_{t \in A} B(t, \eta)$.

Proof. By Theorem 5, J_d is an analytic function with spectrum in $B(0, 1)$, hence $J_d(\eta \cdot /M)^M \in \mathcal{E}(B(0, \eta))$. Hence $\tilde{\psi}$ is analytic as well, and using again Theorem 5, it proves that

$$|\tilde{\psi}(z)| \leq c \exp(s_A(z)) \exp(\eta \|z\|).$$

Then, for some $x_0 \in A$,

$$s_A(z) + \eta \|z\| = \langle x_0, z \rangle + \eta \|z\| \leq \sup_{x \in A, t \in B(0, \eta)} \langle x + t, z \rangle = s_{A+\eta}(z)$$

hence $\psi \in \mathcal{E}(A^{+\eta})$. We have the classical equivalent as $u \rightarrow \infty$, for some $c, c_d > 0$,

$$J_d(u) = c \|u\|^{-(d+1)/2} \cos(\|u\| - c_d)(1 + O(\|u\|^{-1})), \quad (11)$$

which gives $|\tilde{\psi}(u)| \leq c|\psi(u)|||u||^{-p}$. □

5.3 Characterisation of γ -rigidity

Let us translate the general rigidity problem in the Fourier space. Let $A \subset \mathbb{R}^d$. For $\gamma \in \mathbf{C}_c^b(\mathbb{R}^d)$, γ -rigidity on A means

$$\inf_{h \in \mathcal{C}_c^\infty(A^c)} \text{Var} \left(\int_A \gamma(t) \mathbf{M}(dt) - \int_{A^c} h(t) \mathbf{M}(dt) \right) = 0.$$

It implies that for some sequence $h_n, I_{\mathbf{M}}(\gamma) - I_{\mathbf{M}}(h_n) \rightarrow 0$ in $L^2(\mathbf{P})$, hence for some subsequence we have a.s. $I(\gamma) = \lim_{n' \rightarrow \infty} I(h_{n'})$, meaning $I(\gamma) \in \sigma_{lin}(\mathbf{M}_{A^c})$. By (4), γ -rigidity is equivalent to

$$\inf_{h \in \mathcal{C}_c^\infty(A^c)} \int \mathbf{S}(du) |\hat{\gamma}(u) - \hat{h}(u)|^2 = 0.$$

The heart of this article lies in the current characterisation of γ -rigidity. Recall that the spectral density $\mathcal{J} \geq 0$ is the density of the continuous part of \mathbf{S} .

Theorem 6. Let A a convex body. \mathbf{M} is linearly γ -rigid on A iff for all $\psi \in \mathcal{E}(A) \cap L^2(\mathcal{J}^{-1})$,

$$\int_A \bar{\gamma} \hat{\psi} = 0.$$

Before the proof, an instructive immediate remark:

Remark 4. For $0 \leq \mathcal{J} \leq \mathcal{J}'$, γ -rigidity for \mathcal{J}' on a bounded set A implies γ -rigidity for \mathcal{J} on A .

We also need a lemma:

Lemma 2. For any spectral measure \mathbf{S} of a wide-sense L^2 stationary random measure \mathbf{M} ,

$$\int_{\mathbb{R}^d} (1 + \|u\|)^{-2(d+1)} \mathbf{S}(du) < \infty.$$

In particular, \mathbf{S} is a tempered measure.

Proof. Let J_d the Fourier transform of the indicator of the unit ball (see (10)), satisfying in particular (11). Let

$$\psi(u) = J_d(u)^2 + J_d(u + \pi/2)^2.$$

It satisfies $\psi \geq 0$ and is larger than some $\kappa > 0$ outside $B(0, R)$ for some $R < \infty$ because $\|u\|$ and $\|u + \pi/2\|$ cannot be simultaneously close to $\pi\mathbb{Z} + \pi/2$. Hence ψ satisfies for $\|u\| \geq R$

$$\frac{c_-}{\|u\|^{d+1}} \leq \psi(u) \leq \frac{c_+}{\|u\|^{d+1}}.$$

Also $\widehat{\psi}(x)$ is bounded by 2 and $\text{sp}(\psi) \subset \text{sp}(J_d^2) \subset B(0, 2)$. Therefore, by (4)

$$\begin{aligned} \int \mathbf{S}(du) (1 + \|u\|)^{-2(d+1)} &\leq \mathbf{S}(B(0, R)) + \int_{B(0, R)^c} c \mathbf{S}(du) \psi(u)^2 \\ &\leq \mathbf{S}(B(0, R)) + c \text{Var} \left(I_{\mathbf{M}}(\widehat{\psi}) \right) < \infty. \end{aligned}$$

□

Proof of Theorem 6. Denote by $\bar{H}^{\mathbf{S}}$ the closure of some subspace H of $L^2(\mathbf{S})$, and let

$$H_A = \{\hat{h} : h \in C_c^\infty(A^c)\}.$$

This is indeed a subspace of $L^2(\mathbf{S})$ because for $h \in C_c^\infty(A^c)$, $\hat{h} \in \mathcal{S}(\mathbb{R}^d)$ has rapid decay, and

$$\int_{\mathbb{R}^d} |\hat{h}|^2 \mathbf{S} \leq c \int_{\mathbb{R}^d} (1 + \|u\|)^{-2(d+1)} \mathbf{S}(du) < \infty$$

by Lemma 2.

Hence \mathbf{M} is γ -rigid iff $\hat{\gamma} \in \bar{H}_A^{\mathbf{S}}$, iff $\int \hat{\gamma} \bar{\varphi} \mathbf{S} = 0$ for $\varphi \in H_A^\perp$ where

$$H_A^\perp = \{\varphi \in L^2(\mathbf{S}) : \int_{\mathbb{R}^d} \bar{\varphi} \hat{h} d\mathbf{S} = 0; h \in C_c^\infty(A^c)\}.$$

The proof of the theorem is concluded by the following lemma:

Lemma 3. Let $A \subset \mathbb{R}^d$ bounded measurable.

$$H_A^\perp = \{\psi \mathcal{J}^{-1} : \psi \in \mathcal{E}(\text{conv}(A)) \cap L^2(\mathcal{J}^{-1})\}.$$

Proof of Lemma 3. Let $\varphi \in \bar{H}_A^\perp$. Let $\Psi = \varphi \mathbf{S}$. The measure Ψ is tempered because by Cauchy-Schwarz inequality in $L^2(\mathbf{S})$,

$$\int_{\mathbb{R}^d} |\varphi| \mathbf{S}(du) (1 + \|u\|)^{-(d+1)} \leq \|\varphi\|_{L^2(\mathbf{S})} \sqrt{\int (1 + \|u\|)^{-2(d+1)} \mathbf{S}(du)}$$

and the latter is finite with Lemma 2. Let $\mathcal{F}\Psi$ its Fourier transform in the sense of $\mathcal{S}'(\mathbb{R}^d)$. Let $h \in C_c^\infty(A^c) \subset \mathcal{S}(\mathbb{R}^d)$, then $\hat{h} \in \mathcal{S}(\mathbb{R}^d) \cap L^2(\mathbf{S})$. By definition of H_A^\perp ,

$$0 = \int \bar{h} \varphi d\mathbf{S} = \int \bar{h} d\Psi = \langle \mathcal{F}\Psi, h \rangle.$$

Since $\langle \mathcal{F}\Psi, h \rangle = 0$ for $h \in C^\infty$ with compact support in A^c , we have $\text{sp}(\Psi) \subset A$. By the SPW Theorem 5, Ψ has a density ψ :

$$\Psi = \psi \mathcal{L} = \varphi \mathcal{J} \mathcal{L}$$

and ψ is a function of $\mathcal{E}(\text{conv}(A))$. Also, since $\varphi \in L^2(\mathbf{S})$,

$$\int \frac{\psi^2}{\mathcal{J}} = \int \varphi^2 \mathcal{J} = \int \varphi^2 \mathbf{S} < \infty,$$

indeed $\psi \in L^2(\mathcal{J}^{-1})$ (with $1/0 = \infty$).

For the converse, let $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(\text{conv}(A))$, in particular $\text{sp}(\psi) \subset \text{conv}(A)$, and let $\varphi = \psi \mathcal{J}^{-1} \in L^2(\mathbf{S})$. For $h \in C_c^\infty(A^c) \subset \mathcal{S}(\mathbb{R}^d)$,

$$0 = \langle \mathcal{F}\psi, h \rangle_{\mathbb{R}^d} = \langle \psi, \hat{h} \rangle = \int \bar{\varphi} \hat{h} d\mathbf{S},$$

indeed $\varphi \in H_A^\perp$.

□

□

5.4 Proof of Theorem 1

We assume here that \mathcal{J} has a pole of order k in 0. Let us prove γ -rigidity for $\gamma(t) = t^k$. We actually prove that it is k -rigid on $\text{conv}(A)$, which in turns implies k -rigidity on A : if $\gamma \in \sigma_{lin}(\mathbf{M}_{\text{conv}(A)^c})$,

$$\int_A \gamma d\mathbf{M} = \int_{\text{conv}(A)} \gamma d\mathbf{M} - \int_{\text{conv}(A) \setminus A} \gamma d\mathbf{M} \in \sigma_{lin}(\mathbf{M}_{A^c}).$$

Hence without loss of generality we assume A is convex. We use Lemma 6: we must prove that $\hat{\gamma}$ is orthogonal to all $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A)$. Since $\text{sp}(\psi) \subset A$, we have

$$\int_{\mathbb{R}^d} \psi \hat{\gamma} = \int_A t^k \mathcal{F}\psi = \int_{\mathbb{R}^d} t^k \mathcal{F}\psi = (-i)^{|k|} \partial^k \psi(0).$$

Lemma 4. There exists Q a polynomial equivalent to $\psi(u) = \sum_m a_m u^m$ in the neighbourhood of 0 with $\partial^k Q(0) = \partial^k \psi(0)$.

Proof. Let $I = \{\mathbf{m} : a_{\mathbf{m}} \neq 0\} \subset \mathbb{N}^d$. It is easy to see that there is a finite set $I_0 \subset I$ that dominates I in the sense that for all $\mathbf{m}' \in I$, there is some $\mathbf{m} \in I_0$ with $\mathbf{m}' \preceq \mathbf{m}$. Said differently, there does not exist infinite $I_0 \subset \mathbb{N}^d$ made up of extremal points, i.e. such that every $\mathbf{m} \in I_0$ is not \preceq -smaller than all others $\mathbf{m}' \in I_0$.

Define

$$Q(u) = \sum_{\mathbf{m} \in I_0 \cup \{\mathbf{k}\}} a_{\mathbf{m}} u^{\mathbf{m}}$$

so that Q, ψ have the same term of order \mathbf{k} . Also, by uniform convergence of the series, as $u \rightarrow 0$,

$$\psi(u) = \sum_{\mathbf{m} \in I_0} [a_{\mathbf{m}} u^{\mathbf{m}} (1 + o(1))] = Q(u) (1 + o(1)).$$

□

Since $\psi \in L^2(\mathcal{J}^{-1})$, we have in particular for ε sufficiently small

$$\int_{B(0, \varepsilon)} \frac{|Q(u)|^2}{\mathcal{J}(u)} du < \infty,$$

by definition of \mathbf{k} -incompatibility it means $\partial^{\mathbf{k}} Q(0) = 0 = \partial^{\mathbf{k}} \psi(0) = \langle t^{\mathbf{k}}, \hat{\psi} \rangle$. We proved that \mathbf{M} is \mathbf{k} -rigid.

Proof of Proposition 1. Let $Q \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon))$ a polynomial, q the degree of lowest order terms, meaning $Q = Q_h + R$ with Q_h a non-null homogeneous polynomial, i.e. verifying for some $q \in \mathbb{N}$, $u \neq 0$,

$$Q_h(u) = \|u\|^q Q_h(\theta), \theta = \frac{u}{\|u\|} \in \mathbb{S}^{d-1},$$

$R(u) = o(\|u\|^q)$ as $u \rightarrow 0$. We have

$$\begin{aligned} \int_{B(0, \varepsilon)} \mathcal{J}^{-1}(u) |Q|^2(u) du &\geq \int_{B(0, \varepsilon)} \tilde{\mathcal{J}}(\|u\|)^{-1} \|u\|^{2q} (|Q_h(\theta)| + o(1))^2 du \\ &\geq \sigma_d \int_0^\varepsilon \tilde{\mathcal{J}}(\rho)^{-1} \rho^{2q} \left(\int_{\mathbb{S}^{d-1}} |Q_h(\theta)|^2 d\theta + o(1) \right) \rho^{d-1} d\rho. \end{aligned}$$

Therefore, if $q \leq k$, indeed $\mathcal{J}^{-1}|Q|^2$ is not integrable around 0, which means Q cannot have terms of degree $\leq k$, and \mathcal{J} is \mathbf{k} -rigid for $|\mathbf{k}| \leq k$.

□

5.5 Necessity lemma

The necessity proofs rely on the following technique.

Lemma 5. Let Q a \mathbf{k} -polynomial (for instance $Q(u) = u^{\mathbf{k}}$), $A \subset \mathbb{R}^d$ bounded measurable. Assume for some $\varepsilon > 0$ there is $\psi \in L^2(\mathcal{J}^{-1}, \mathbb{R}^d \setminus B(0, \varepsilon)) \cap \mathcal{E}(A)$ and $\psi(0) \neq 0$, and that 0 is not a pole of order \mathbf{k} for \mathcal{J}^{-1} . Then \mathbf{M} is not Q -rigid on $A^{+\eta}$ for $\eta > 0$.

Proof. Since 0 is not a pole of order \mathbf{k} , it means that there exists a polynomial $P = \sum_{\mathbf{k}'} b_{\mathbf{k}'} u^{\mathbf{k}'}$ with the term $b_{\mathbf{k}} u^{\mathbf{k}} \neq 0$ and for some $\varepsilon > 0$

$$\int_{B(0,\varepsilon)} |P|^2 \mathcal{J}^{-1} < \infty.$$

If there is also a term $b_{\mathbf{k}'} u^{\mathbf{k}'} \neq 0$ for some $\mathbf{k}' \prec \mathbf{k}$, (i.e. $\mathbf{k}' \preceq \mathbf{k}, \mathbf{k}' \neq \mathbf{k}$), we have

$$\tilde{P} = u^{\mathbf{k}-\mathbf{k}'} P \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon)),$$

we can assume without loss of generality that there is no term strictly smaller than \mathbf{k} . Let then

$$\tilde{\psi}(u) = \psi(u) J_d(u\eta / \deg(P))^{\deg(P)},$$

satisfying $\tilde{\psi}(0) \neq 0$. By Lemma 1, $P\tilde{\psi} \in \mathcal{E}(B(0, A^{+\eta}))$ and $|P\tilde{\psi}| \leq c|\psi|$, hence also $P\tilde{\psi} \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon)^c)$. We also have $P\tilde{\psi} \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon))$ because $P\tilde{\psi}(u) \sim P(u)$ as $u \rightarrow 0$, hence $P\tilde{\psi} \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A^{+\eta})$.

Since Q is a \mathbf{k} -polynomial, $Q(u) = \sum_{\mathbf{m} \preceq \mathbf{k}_0} a_{\mathbf{m}} u^{\mathbf{m}}$ for some $\mathbf{k}_0 \succeq \mathbf{k}$ with $a_{\mathbf{k}_0} \neq 0$. Since 0 is also a pole of order \mathbf{k}_0 (because $\mathbf{k}_0 \succeq \mathbf{k}$), the whole proof can be done with \mathbf{k}_0 instead of \mathbf{k} , and we assume without loss of generality $\mathbf{k} = \mathbf{k}_0$. Then we have

$$\partial^Q(P\tilde{\psi})(0) := \sum_{\mathbf{m} \preceq \mathbf{k}} \sum_{\mathbf{k}' \neq \mathbf{k}} b_{\mathbf{k}'} a_{\mathbf{m}} \partial^{\mathbf{m}}(u^{\mathbf{k}'} \tilde{\psi})(0) = a_{\mathbf{k}} b_{\mathbf{k}} \partial^{\mathbf{k}}(u^{\mathbf{k}} \tilde{\psi})(0) + \sum_{\mathbf{m} \preceq \mathbf{k}, \mathbf{k}' \neq \mathbf{k}} b_{\mathbf{k}'} a_{\mathbf{m}} \partial^{\mathbf{m}}(u^{\mathbf{k}'} \tilde{\psi})(0).$$

The proof is complete if we prove that $\partial^Q(P\tilde{\psi})(0) \neq 0$, because since $P\tilde{\psi} \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A^{+\eta}) \setminus \{0\}$, M is not Q -rigid by Theorem 6. Let us prove this claim.

For $\mathbf{k}' \neq \mathbf{k}, \mathbf{m} \preceq \mathbf{k}$, we have $\mathbf{k}' \not\preceq \mathbf{m}$, hence the conclusion comes from the fact that $\tilde{\psi}(0) \neq 0$ and the following identity: We have for $\mathbf{k}' \neq \mathbf{m}$

$$\partial^{\mathbf{m}}(u^{\mathbf{k}'} \tilde{\psi})(0) = \delta_{\mathbf{m}=\mathbf{k}'} \mathbf{m}! \tilde{\psi}(0)$$

with $\mathbf{m}! = m_1 \dots m_d$. Let us finally prove this identity.

The assumption yields that we either have $\mathbf{m} = \mathbf{k}'$, or $\mathbf{m}_i < \mathbf{k}'_i$ for some i . Then the Leibniz formula for functions of one argument yields

$$\begin{aligned} \partial^{\mathbf{m}}(u^{\mathbf{k}'} \tilde{\psi}(u)) &= \left(\prod_{j \neq i} \partial_j^{m_j} \right) \partial_i^{m_i} (u_i^{\mathbf{k}'_i} u_i^{\mathbf{k}'_i} \tilde{\psi}(u_i, u_i)) \text{ with } \mathbf{k}'_i = (\mathbf{k}'_1, \dots, \underbrace{0}_i, \dots, \mathbf{k}'_d), u_i = (u_1, \dots, \underbrace{0}_i, \dots, u_d) \\ &= \left(\prod_{j \neq i} \partial_j^{m_j} \right) \left(\sum_{a > 0} \lambda_a u_i^a \tilde{\psi}_a(u) + 1_{\mathbf{k}'_i = \mathbf{m}_i} u_i^{\mathbf{k}'_i} \mathbf{m}_i! \tilde{\psi}(u_i, u_i) \right) \text{ for some } \lambda_a \in \mathbb{R}, \text{ functions } \tilde{\psi}_a, \\ &= \sum_{a > 0} \lambda_a u_i^a \left(\prod_{j \neq i} \partial_j^{m_j} \right) (\tilde{\psi}_a(u)) + 1_{\mathbf{k}'_i = \mathbf{m}_i} \mathbf{m}_i! \left(\prod_{j \neq i} \partial_j^{m_j} \right) (u_i^{\mathbf{k}'_i} \tilde{\psi}(u)) \\ \partial^{\mathbf{m}}(u^{\mathbf{k}'} \tilde{\psi}(u))|_{u=0} &\begin{cases} = 0 \text{ if } \mathbf{m}_i < \mathbf{k}'_i \\ = \mathbf{m}! \tilde{\psi}(0) \text{ if } \mathbf{m} = \mathbf{k}' \text{ (with an induction on } j). \end{cases} \end{aligned}$$

□

5.6 Necessary conditions

Proof of Proposition 2. Let $u_1, \dots, u_m \in \mathbb{R}^d$ the poles of \mathcal{J}^{-1} , let ε, q such that $P(u) := \prod_i \|u - u_i\|^{2q}$ is in $L^2(\mathcal{J}^{-1}, B(u_i, \varepsilon))$ for all i , $\varepsilon > 0$. Introduce

$$\psi(u) = P(u) J_d(u/\eta M)^M$$

with $M = \frac{d+1}{2}(\deg(P) + p + d)$. Lemma 1 (with $1 \in \mathcal{E}(\{0\})$) yields $\psi \in \mathcal{E}(B(0, \eta))$ and $|\psi(u)| \leq c(1 + \|u\|)^{-p-d}$ (ψ is smooth hence bounded around 0). We have

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|\psi(u)|^2}{\mathcal{J}(u)} du &\leq c \int_{\mathbb{R}^d \setminus \cup_i B(u_i, \varepsilon)} |\psi(u)|^2 (1 + \|u\|)^p + \sum_i c_{i, \varepsilon} \int_{B(u_i, \varepsilon)} \frac{\|u - u_i\|^{4q}}{\mathcal{J}(u)} du \\ &\leq \int (1 + \|u\|)^{-2d-2p} (1 + \|u\|)^p du + C \\ &< \infty, \end{aligned}$$

hence $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(B(0, \eta)) \setminus \{0\}$. Let $\gamma = \widehat{\psi}$, bounded and supported by $B(0, \eta)$. M is not γ -rigid on $B(0, \eta)$ by Theorem 6 because with Parseval's identity

$$\int \gamma \widehat{\psi} = \int |\psi|^2 \neq 0.$$

In particular, M is not linearly maximally rigid.

The sufficiency part about k -rigidity is Theorem 1. For the necessity, define instead

$$\tilde{\psi}(u) = \prod_{i: u_i \neq 0} \|u - u_i\|^{2q} J_d(u/\eta M)^M.$$

A similar reasoning shows that $\tilde{\psi}$ is in $L^2(\mathcal{J}^{-1}, \mathbb{R}^d \setminus B(0, \varepsilon)) \cap \mathcal{E}(B(0, \eta))$ for some $\varepsilon > 0$ with $\tilde{\psi}(0) \neq 0$. We can then conclude with Lemma 5. \square

Proof of Proposition 3. Let $\psi \in L^2(\mathcal{J}^{-1})$ analytic. Since for some $\varepsilon > 0$, for $u_1 \in [-\varepsilon, \varepsilon]$,

$$\int_{B((1+u_1, 0), \varepsilon)} \frac{1}{\mathcal{J}(u)} du \geq \int_{[1+u_1-\varepsilon, 1+u_1+\varepsilon] \times [-\varepsilon, \varepsilon]} u_2^{-2} du_1 du_2 = \infty,$$

ψ must vanish on $\{(1 + u_1, 0); |u_1| \leq \varepsilon\}$. Hence $\lambda : v \mapsto \psi(v, 0)$ is an entire function vanishing on an interval. Therefore $\lambda \equiv 0$, in particular $\psi(0) = 0$. With $\gamma = 1_A$ for A bounded, we showed that any $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A)$ satisfies

$$\langle \gamma, \hat{\psi} \rangle = \int_A \hat{\psi} = \int_{\mathbb{R}^d} \hat{\psi} = \psi(0) = 0,$$

showing that M is γ -rigid (i.e. 0-rigid on A) with Theorem 6. \square

5.7 Proof of Theorem 2

Isotropic case

We prove that there is no Q -rigidity for some polynomial $Q = \sum_{m \leq k_0} a_m u^m$ for some $k_0 \geq k$. For k -rigidity, simply take $Q(u) = u^k$.

Since \mathbf{M} is not LMR, there exists $\psi_0 \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A) \setminus \{0\}$. Let then P a polynomial such that $\psi_0(u) \sim P(u)$ as $u \rightarrow 0$, and

$$\psi_1(u) = P(u)\psi_0(u)J_a(u\eta/\deg(P))^{\deg(P)}.$$

By Lemma 1, $\psi_1 \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A^{+\eta})$.

Then define on \mathbb{C}^d the rotational average, using the analytic extension of ψ_1 on \mathbb{C}^d ,

$$\psi(z) = \int_{O(d)} \psi_1(\theta z) d\theta, z \in \mathbb{C}^d$$

with the Haar measure on the orthogonal group $O(d)$ of \mathbb{R}^d . Still denote by ψ its restriction to \mathbb{R}^d , which is in particular isotropic. By definition, $\psi_1(u) \sim P(u)^2$ as $u \rightarrow 0$ and $P(u)^2 \geq 0$ for $u \in \mathbb{R}^d$, hence ψ does not vanish identically in the neighbourhood of 0, so that $\psi \not\equiv 0$. We have with the triangle inequality and the Cauchy-Schwarz inequality in $L^2(\mathcal{J}^{-1})$

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|\psi(u)|^2}{\mathcal{J}(u)} du &\leq \int_{\mathbb{R}^d} \int_{O(d) \times O(d)} \frac{|\psi_1(\theta u)| |\psi_1(\theta' u)|}{\mathcal{J}(u)} d\theta d\theta' du \\ &\leq \int_{O(d) \times O(d)} \sqrt{\int_{\mathbb{R}^d} \frac{|\psi_1(\theta u)|^2}{\mathcal{J}(u)} du} \sqrt{\int_{\mathbb{R}^d} \frac{|\psi_1(\theta' u)|^2}{\mathcal{J}(u)} du} d\theta d\theta' \\ &\leq \int_{O(d) \times O(d)} \left(\sqrt{\int_{\mathbb{R}^d} \frac{|\psi_1(u)|^2}{\mathcal{J}(u)} du} \right)^2 d\theta d\theta' < \infty \end{aligned}$$

by isotropy of \mathcal{J} , using $\psi_1 \in L^2(\mathcal{J}^{-1})$.

For $z, \zeta \in \mathbb{C}^d$, since ψ_1 is analytic on \mathbb{C}^d ,

$$\psi(z + \zeta) = \int_{O(d)} [\psi_1(\theta z) + \psi_1'(\theta z)\zeta + O(\zeta)] d\theta = \psi(z) + \left(\int \psi_1'(\theta z) d\theta \right) \zeta + O(\zeta),$$

using that the quantities involved in the $O(\cdot)$ are locally bounded; hence ψ is analytic as well.

Also, for $z \in \mathbb{C}^d, \theta \in O(d)$,

$$\|\theta z\|_{\mathbb{C}}^2 = \|\theta \mathcal{R}z\|^2 + \|\theta \mathcal{I}z\|^2 = \|z\|_{\mathbb{C}}^2.$$

Using both implications of Theorem 5 on $A^{+\eta} = B(0, R + \eta)$,

$$|\psi(z)| \leq c \int_{O(d)} \exp((R + \eta)\|z\|_{\mathbb{C}}) d\theta$$

and $\text{sp}(\psi) \subset B(0, R + \eta)$. Finally, ψ is an isotropic element of $L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(B(0, R + \eta)) \setminus \{0\}$.

Hence non isotropic terms vanish and for some $q \in \mathbb{N}, \kappa \neq 0$, $\psi(u) = \kappa \|u\|^{2q} (\sum_{l>0} a_l \|u\|^{2l})$, where $\|u\|^{2l} = (\sum_i u_i^2)^l$. Hence $\tilde{\psi} := \psi(u) \|u\|^{-2q} \psi$ is analytic and isotropic as well, with spectrum in $B(0, R + \eta)$ with Theorem 5 again, and $\tilde{\psi}(0) \neq 0$. We then can use directly Lemma 5 to conclude that \mathbf{M} is not Q -rigid on $B(0, R + \eta)$. This holds for all $R > 0$, hence also for $R - \eta$ for η sufficiently small.

Separable case. We assume that \mathcal{J} is not (linearly) maximally rigid, hence there is $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A) \setminus \{0\}$. Let us build a separable version. Recall that, as analytic function, ψ has an extension on \mathbb{C}^d . For $z^i \in \mathbb{C}^{d-1}$,

$$\psi_i^{z^i}(z_i) = \psi(z_i; z^i), z_i \in \mathbb{C},$$

where $(z_i; z^i)$ consists in z_i at the i -th position, surrounded by the $d - 1$ components of z^i . Let the domain $D \subset \mathbb{R}^{d-1}$ of u^i where $\psi_i^{u^i}$ is not identically 0. We have

$$0 < \int \frac{|\psi|^2}{\mathcal{J}} = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{|\psi_i^{u^i}(u_i)|^2}{\mathcal{J}_i(u)} \frac{1}{\prod_{j \neq i} \mathcal{J}_j(u)} du^i du_i = \int_D \int_{\mathbb{R}} \frac{|\psi_i^{u^i}(u_i)|^2}{\mathcal{J}_i(u)} \frac{1}{\prod_{j \neq i} \mathcal{J}_j(u)} du^i du_i < \infty$$

hence there exists $u^i \in D$ such that $\psi_i^{u^i} \in L^2(\mathcal{J}_i^{-1}) \setminus \{0\}$. Clearly $\psi_i^{u^i}$ is analytic as analyticity in several complex variables implies analyticity in each variable. Let $q_i \in \mathbb{N}$, $\kappa_i \neq 0$ the dominating power and coefficient in $0 : \psi_i^{u^i}(z) \sim \kappa_i z^{q_i}$ as $z \rightarrow 0$. Define $\psi_i(z) := z^{-q_i} \psi_i^{u^i}(z)$, $z \in \mathbb{C}$, still analytic. Define finally

$$\tilde{\psi}(z) = \prod_i \psi_i(z_i), z = (z_i) \in \mathbb{R}^d.$$

Let $\varepsilon > 0$. We have for some finite $\kappa > 0$, $\psi_i(z) \leq \kappa$ if $\|z\|_{\mathbb{C}} \leq \varepsilon$, otherwise $|\psi_i(z)| \leq c_\varepsilon |\psi_i^{u^i}(z_i)|$. Since for each i , $\psi_i^{u^i} \in L^2(\mathcal{J}_i^{-1})$, we have $\tilde{\psi} \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon)^c)$ (as functions on \mathbb{R}^d).

Let $A_R = [-R, R]^d$. It remains to show $\tilde{\psi} \in \mathcal{E}(A_R)$. We have

$$s_{A_R}(\zeta) = R \sup_{x: |x_i| \leq 1} \sum_i x_i \zeta_i = R \sum_i \text{sign}(\zeta_i) \zeta_i = R \sum_i |\zeta_i|, \zeta \in \mathbb{R}^d.$$

Since $\psi \in \mathcal{E}(A_R)$, for $z \in \mathbb{C}$,

$$\begin{aligned} |\psi_i^{u^i}(z)| &= |\psi(z; u^i)| \leq C \exp(s_{A_R}(\mathcal{J}(z; u^i))) \\ &\leq C \exp(Rc_{u^i} + R|\mathcal{J}z|) \\ &\leq c'_{u^i} \exp(R|\mathcal{J}z|). \end{aligned}$$

Furthermore, $\tilde{\psi}$ is analytic, satisfies $\tilde{\psi}(0) \neq 0$ and for $\|z\|_{\mathbb{C}} > \varepsilon$,

$$|\tilde{\psi}(z)| \leq \prod_i c'_i \exp(R|\mathcal{J}z_i|) = c' \exp(R \sum_i |\mathcal{J}z_i|) = c' \exp(s_{A_R}(\mathcal{J}z)).$$

Theorem 5 again yields that $\tilde{\psi} \in \mathcal{E}(A_R)$. Then one can conclude with Lemma 5 that we do not have Q -rigidity on A_R .

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