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# Rigidity of random stationary measures and applications to point processes

Raphaël Lachièze-Rey\*

**Abstract** The *number rigidity* of a point process  $\mathbf{P}$  entails that for a bounded set  $A$  the knowledge of  $\mathbf{P}$  on  $A^c$  a.s. determines  $\mathbf{P}(A)$ ; the  $k$ -order rigidity means we can recover the moments of  $\mathbf{P}1_A$  up to order  $k$ . We show that there is  $k$ -rigidity if the continuous component  $\mathcal{J}$  of  $\mathbf{P}$ 's *structure factor* has a zero of order  $k$  in 0, by exploiting a connection with Schwartz' Paley-Wiener theorem for analytic functions of exponential type; these results apply to any random  $L^2$  wide sense stationary measure on  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ . In the continuous setting, these local conditions are also necessary if  $\mathcal{J}$  has finitely many zeros, or is isotropic, or at the opposite separable. This explains why no model seems to exhibit rigidity in dimension  $d \geq 3$ , and allows to efficiently recover many recent rigidity results about point processes. In the discrete setting, these results hold provided  $\#A > 2k$ .

We derive new results about models of *cluster lattices* and give the first example of a stationary point process  $\mathbf{P} \subset \mathbb{R}^d$  exhibiting arbitrary low decay of the structure factor in 0, hence arbitrary high order of rigidity. For a continuous Determinantal point process with reduced kernel  $\kappa$ ,  $k$ -rigidity is equivalent to  $(1 - \widehat{\kappa^2})^{-1}$  having a zero of order  $2k$  in 0, which answers questions on completeness and number rigidity. We also explore the consequences of these statements in the less tractable realm of Riesz gases.

**Keywords:** rigidity, point processes, random measures, hyperuniformity, Determinantal point processes, Riesz gases

## Introduction

The first instance of a linear prediction problem for a stationary process goes back to Szegö [1921], for a random time series  $\{\mathbf{X}_k; k \in \mathbb{Z}\}$ . He gave a necessary and sufficient condition for the process to be *deterministic: the future is entirely determined by the past*, i.e.

$$\sigma(\mathbf{X}_k; k > 0) \subset \sigma(\mathbf{X}_k; k \leq 0)$$

through a linear predictor, if and only if the spectral measure  $\mathcal{J}$  satisfies

$$\int_0^{2\pi} \ln(|\mathcal{J}(u)|) = -\infty.$$

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Kolmogorov [1941] further studies the non-deterministic case, and in particular addresses the weaker linear interpolation problem of whether  $X_0 \in \sigma(X_k; k \neq 0)$ . He proves that it is so if and only if

$$\int_{-\pi}^{\pi} \mathcal{J}^{-1}(u) du = \infty. \quad (1)$$

A detailed account of the line of research about random linear interpolation can be found in Rozanov [1967].

Much more recently, other similar problems emerged for higher-dimensional continuous models with the notion of *rigidity* for a *stationary point process*, i.e. a random locally finite set of points  $P \subset \mathbb{R}^d$  which law is invariant under spatial translations. The oldest such result might be that of Aizenman and Martin [1980], about 1D Coulomb gases, but the systematic study of rigidity really started with the introduction of the notion of *tolerance* (Lyons and Steif [2003], Holroyd and Soo [2013]), then Ghosh and Peres [2017] coined the term *number rigidity* as the property that

$$\#P \cap B(0, 1) \in \sigma(P \cap B(0, 1)^c).$$

This obviously holds for *shifted lattices*, e.g.  $P = \{\mathbf{m} + U; \mathbf{m} \in \mathbb{Z}^d\}$  (where  $U$  is uniform in  $[0, 1]^d$  and ensures that  $P$  is invariant under translations with non-integer coordinates), as one can easily deduce the number of lattice points “hidden” in  $B(0, 1)$  by only observing  $P \cap B(0, 1)^c$ . They realised that, somehow surprisingly, also some strongly non-lattice like models, the *infinite Ginibre ensemble*  $P_{\text{Gin}}$ , and the zero set  $P_{\text{GAF}}$  of the *planar Gaussian Analytic Function*, satisfy number rigidity. As it turns out, these two processes possess also possess the property of *hyperuniformity*, i.e.

$$\frac{\text{Var}(\#P \cap B(0, r))}{\mathcal{L}(B(0, r))} \xrightarrow{r \rightarrow \infty} 0, \quad (2)$$

where  $\mathcal{L}$  denotes Lebesgue measure, at the difference of standard disordered systems such as Poisson processes; this property also is strongly reminiscent of lattice-like models. The surprising fact that it occurs for locally disordered, *amorphous* random measures, like  $P_{\text{Gin}}$ ,  $P_{\text{GAF}}$ , or other Coulomb systems, has been systematically studied by physicists since the 90’s. This property requires some sort of long distance dependency, compatible with locally disordered configurations, often referred to as *global order and local disorder*. The activity around hyperuniformity has not stopped growing until now, as this property concerns a large diversity of models involving Coulomb systems, Gaussian analytic functions, eigenvalues of random matrices, Determinantal Point processes, and has many applications. The literature is too wide to be cited exhaustively, see for instance the survey of Torquato [2018] for materials science, of Ghosh and Lebowitz [2017] for statistical physics, or the more mathematical discussion of Coste [2021].

Many models have been proven to be rigid since then, mostly in the realm of hyperuniformity: some Determinantal Point Processes (DPPs) (Bufetov [2016], Bufetov and Qiu [2018], Bufetov et al. [2018]), Pfaffian processes (Bufetov et al. [2019]), Coulomb and Riesz systems (Dereudre et al. [2020], Chhaibi and Najnudel [2018], Chatterjee [2019], Dereudre and Vasseur [2023]), zeros of Gaussian processes (Ghosh and Peres [2017], Ghosh and Krishna [2021], Lachièze-Rey [2022]), stable matchings (Klatt et al. [2020]), and others (Ghosh and Lebowitz [2018], Klatt and Last [2022]). Besides the striking nature of rigidity and its link with

hyperuniformity, it has proven to be a useful property in other types of problems, such as continuous percolation [Ghosh et al. \[2016\]](#). [Osada \[2024\]](#) recently proved a relation with diffusive dynamics of particle systems, [Ghosh \[2015\]](#) also exploited number rigidity to show the completeness of random sets of exponential functions, an interesting property in signal processing, [Lyons and Steif \[2003\]](#) study this property to show phase uniqueness for some models from statistical physics.

Stronger forms of rigidity have also emerged. For instance, for the  $\mathbb{P}_{\text{GAF}}$  model, [Ghosh and Peres \[2017\]](#) showed that not-only the number of particles in  $B(0, 1)$  can be determined, but also the first moment, or center of mass  $\sum_{x \in \mathbb{P}_{\text{GAF}} \cap B(0, 1)} x$ , corresponding to *1-rigidity*, and that no further moment can be determined from the observation of  $\mathbb{P}_{\text{GAF}} \cap B(0, 1)^c$ . They proved similarly that for the Ginibre process no moment can be determined beyond order 0. Another such result appears in [Dereudre et al. \[2020\]](#) for  $\text{Sine}_\beta$  processes above order 0.

Until now, both in the physics and mathematics literature, the precise nature of the connection between rigidity and hyperuniformity is not precisely understood, see the discussions in [Ghosh and Lebowitz \[2017\]](#), [Coste \[2021\]](#), these terms are even used interchangeably in some works [Chatterjee \[2019\]](#). It seems in particular that rigidity is not understood in dimensions  $d \geq 3$ , [Chatterjee \[2019\]](#) discusses at length this question and proves that the hierarchical Coulomb gas, a simplified version of the stationary 3D Coulomb gas which is not formally stationary, is not number rigid. [Dereudre and Vasseur \[2023\]](#) conjecture that the number rigidity of a Riesz gas of index  $s$  occurs if and only if  $s \leq d - 1$ .

**Necessary and sufficient conditions.** In the current article, we study the rigidity properties of a  $L^2$  (wide-sense) stationary measure  $\mathbb{M}$  on  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ . We shall establish for  $k \in \mathbb{N}$  a characterisation of  $k$ -th order linear rigidity, which is the property that the  $k$ -th order moments of  $\mathbb{M}$  restricted to  $B(0, 1)$  can be a.s. recovered (linearly) from  $\mathbb{M}1_{B(0, 1)^c}$ , in terms of the behaviour around the origin of the structure factor  $\mathbb{S}$ , the generalised Fourier transform of the correlation measure, and more precisely of its continuous component  $\mathfrak{s}(u)du$ . The deep relation between the behaviour of  $\mathbb{S}$  around 0 and hyperuniformity emerges from the fact that (2) is the equivalent in the Fourier domain to the vanishing of  $\mathbb{S}$  near 0. We show that the decay exponent of  $\mathfrak{s}$  is related to its degree of rigidity, i.e. the number of moments of  $\mathbb{M}1_{B(0, 1)}$  that can be determined by the outside configuration  $\mathbb{M}1_{B(0, 1)^c}$ .

More precisely, on a bounded set  $A$ , it is sufficient for  $k$ -rigidity that  $\mathfrak{s}^{-1}$  has a pole of order  $2k$  in 0 in a particular sense (Theorem 1). For *number rigidity* ( $k = 0$ ) or if  $d = 1$ , or if  $\mathfrak{s}$  is isotropic, it exactly means that for any neighbourhood  $U$  of 0,

$$\int_U \mathfrak{s}(u)^{-1} \|u\|^{2k} du = \infty, \quad (3)$$

the situation can be more complicated in higher dimensions, without isotropy (Proposition 3). This explains most results cited above about number rigidity and 1-rigidity, and also generalises the sufficient conditions of [Ghosh and Lebowitz \[2017\]](#) and [Bufetov et al. \[2018\]](#) for number rigidity.

We show that this condition is necessary under some structural assumptions such as finite number of zeros, isotropy or separability (Proposition 2 and Theorem 2). It also explains why rigidity does not seem to occur in dimension  $d \geq 3$ : if  $\mathfrak{s}(u) \sim \sigma \|u\|^2$  for

some  $\sigma > 0$  as  $u \rightarrow 0$ , as it is expected for many hyperuniform systems with integrable correlation measure, if  $d \geq 3$ , the left hand member of (3) is finite for all  $k$ . Hence one must find processes where the structure factor decreases faster to 0 to find higher order rigidity. We exhibit in Section 2.4 a class of processes  $P_n$ , for  $n$  a prime number, which are not shifted lattices and such that  $\mathfrak{s}(u) \leq c_n \|u\|^{2n}$  for some  $c_n > 0$ . We also give *tolerance results*, meaning that if  $\mathfrak{s}$  does not decay too fast close to 0, its higher order moments cannot be determined (Theorem 4).

The converse part also allows to assess the correlation properties of a point process based on its rigidity behaviour: it is shown by Dereudre and Vasseur [2023] and Dereudre et al. [2020] that some  $\beta$ -ensembles are not  $k$ -rigid for some  $k$ , which implies non-integrability results on their correlation functions (see Section 3.2).

We give counter-examples of spectral measures showing that these results are optimal in full generality.

**Determinantal point processes.** These results apply efficiently to stationary DPPs. Theorem 1 yields that a DPP on  $\mathbb{R}^d$  with kernel  $K$  is  $k$ -rigid if and only if  $(1 - \mathcal{F}(|K|^2))^{-1}$  has a pole of order  $2k$  in 0, retrieving for  $k = 0$  the results of Ghosh and Peres [2017], Ghosh [2015], Bufetov et al. [2018], and also completes the answer of Ghosh [2015] to a question of [Lyons, 2014, Section 4] on completeness of complex exponentials. In the discrete setting, a pole of order  $2k$  indeed implies  $k$ -rigidity, but the converse is only true in dimension 1, or under some additional assumption.

**Discrete processes.** Stationary discrete processes  $X = \{X_m; m \in \mathbb{Z}^d\}$  show a similar connection between rigidity and the behaviour around 0 of the spectral measure  $S$ , defined on  $[-\pi, \pi]^d$ . We show in Theorem 4 for  $A = \llbracket m \rrbracket^d := \{-m, \dots, m\}^d$  that  $X$  is not *maximally rigid* on  $A$ , meaning  $\{X(m); |m| \leq m\} \not\subset \sigma(\{X_m, |m| > m\})$ , if and only if there exists a trigonometric polynomial  $\psi(u) = \sum_{|m| \leq m} a_m e^{i(m,u)}$  such that,

$$\int_{\mathbb{T}^d} \frac{|\psi(u)|^2}{\mathfrak{s}(u)} du < \infty,$$

and not  $k$ -rigid if furthermore the derivatives of  $\psi$  up to order  $k$  do not vanish in 0. Besides the seminal work of Kolmogorov [1941] in dimension 1, this unifies some results established in the context of discrete DPPs, such as the result of Lyons and Steif [2003] regarding the *strong full K* property for uniqueness of phase transition in statistical physics. The condition can be made more explicit in dimension 1:  $X$  is *maximally rigid* on  $\llbracket m \rrbracket$  if and only if the number of poles of  $\mathfrak{s}$  counted with multiplicity is  $> m$ , as for the condition (1) of Kolmogorov [1941] in the case  $d = 1, m = 0, k = 0$ , or the generalisation of Bufetov et al. [2018] in the case  $d = 1, k = 0, m \in \mathbb{N}$ :  $X$  is 0-rigid, i.e.  $\sum_{|m| \leq m} X_m \in \sigma(\{X_k; |m| > m\})$ , if and only if 0 is a pole, i.e.

$$\int_{B(0,\varepsilon)} \mathfrak{s}^{-1}(u) du = \infty, \varepsilon > 0,$$

or (obviously) if  $X$  is *maximally rigid* (Proposition 6).

**Rigidity of the class of functions of exponential type.** Let us sketch here the method to derive these results for a stationary random measure  $M$  with spectral measure  $S$  (formally developed at Section 4). The basic idea is the following: given a function

$\gamma$  bounded on some compact  $A$ , the linear statistic  $\int \gamma d\mathbf{M}$  is completely determined by  $\mathbf{M}|_{A^c}$  if

$$\inf_{h \in \mathcal{C}_c^\infty(A^c)} \text{Var} \left( \int \gamma d\mathbf{M} - \int h d\mathbf{M} \right) = 0,$$

and by (4) this translates in the Fourier domain as

$$\inf_{h \in \mathcal{C}_c^\infty(A^c)} \int |\hat{\gamma} - \hat{h}|^2 d\mathbf{S} = 0.$$

In other words,  $\hat{\gamma}$  must be in the  $L^2(\mathbf{S})$ -closure of the space  $H_A$  spanned by the  $\hat{h}$  for  $h$  in  $\mathcal{C}_c^\infty(A^c)$ , or equivalently it must be orthogonal to any function  $\varphi \in H_A^\perp$ . The crucial observation is that for such  $\varphi$ , the tempered distribution  $\varphi\mathbf{S}$  has by definition a spectrum bounded (by  $A$ ). This conveys a very strong form of regularity, namely  $\psi := \varphi\mathcal{J} = \varphi\mathbf{S}$  has no singular part and must be an analytic function of exponential type on  $\mathbb{C}^d$  by an application of the Schwartz-Paley-Wiener theorem (Theorem 6). We have also immediately  $\psi \in L^2(\mathcal{J}^{-1})$  because  $\int \psi^2 \mathcal{J}^{-1} = \int \varphi^2 \mathbf{S}$ . If such  $\psi$  do not exist besides the null function,  $H_A^\perp = \{0\}$  and all  $\gamma$  can be predicted, meaning the process is maximally rigid. Number rigidity means that  $\gamma \equiv 1$  is orthogonal to  $\hat{\psi}$  for all such  $\psi$ , i.e. for all entire function  $\psi$  with bounded spectrum, being in  $L^2(\mathcal{J}^{-1})$  implies

$$\psi(0) = \langle 1, \hat{\psi} \rangle = 0.$$

If  $\mathcal{J}^{-1}$  is non-integrable around 0, indeed  $\psi(0) = 0$  as otherwise  $\psi \notin L^2(\mathcal{J}^{-1})$ , hence  $\mathbf{M}$  is number rigid.

These high regularity and integrability requirement on  $\psi$  are in some sense the source of the *rigidity phenomenon*. It is then easier to characterise whether all such  $\psi \in L^2(\mathcal{J}^{-1})$  must satisfy  $\langle \hat{\gamma}, \psi \rangle_{\mathbb{R}^d} = 0$ , meaning  $\gamma$ -rigidity, and it mostly depends on the zeros of  $\mathcal{J}$ , or more generally on how often  $\mathcal{J}(u)$  is close to 0. What we study is not mere rigidity, but *linear rigidity*, meaning that  $\int \gamma d\mathbf{M}$  should be approximated by linear functionals  $I_{\mathbf{M}}(h)$ , not by *any* functional on  $\mathbf{M}|_{A^c}$ .

Let us present the rest of the paper. In Section 1, we introduce the spectral measure  $\mathbf{S}$  necessary to state the main results. In Section 2, we formally introduce the concepts of rigidity and  $k$ -poles and derive the announced necessary and sufficient condition. We show how they apply to some particular examples, in particular we give examples satisfying any prescribed order of rigidity, and state conversely the tolerance results; we also give the corresponding results for discrete processes. Section 3 is devoted to the applications to DPPs and Gibbs measures. Finally, Section 4 gives the formal framework about tempered distributions, Schwarz' Paley-Wiener Theorem and the proofs. In the companion paper [Lachièze-Rey \[2024+\]](#), we explore maximal rigidity, and in particular stealthy systems, and give several examples of models satisfying a phase transition in the observation window  $A$ , or other surprising rigidity behaviours, such as *short range rigidity*, where  $\mathbf{M} \cap B(0, 1)$  is completely determined by  $\mathbf{M} \cap (B(0, 2) \setminus B(0, 1))$  despite the covariance being smooth and integrable.

# 1 Spectral measure

Let  $\mathbf{E}$  be either  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ . Even though our main motivation is the class of point processes, we consider a random  $L^2$  wide-sense stationary signed measure  $\mathbf{M}$ , i.e. a collection of real  $L^2$  random variables  $I_{\mathbf{M}}(f)$  on a probability space  $(\Omega, \mathbf{P})$ , for  $f$  in the space  $\mathcal{C}_c^b(\mathbf{E})$  of measurable bounded and compactly supported functions, satisfying a.s. the  $\sigma$ -additivity axiom, and  $\text{Var}(I_{\tau_x \mathbf{M}}(f)) = \text{Var}(I_{\mathbf{M}}(f))$  where  $\tau_x$  is the operator of translation by  $x \in \mathbb{R}^d$ . The class of linear statistics  $I_{\mathbf{M}}(f)$  naturally extends to broader class of functions  $f$ .

For  $\mathbf{E} = \mathbb{R}^d$ , the second order behaviour of  $\mathbf{M}$  is described by the *correlation measure*  $\mathbf{C}$ , characterised by

$$\text{Cov}(I_{\mathbf{M}}(f), I_{\mathbf{M}}(g)) = \lambda \int_{\mathbb{R}^d} f(y)g(x+y)\mathbf{C}(dx)dy, f, g \in \mathcal{C}_c^b(\mathbf{E}),$$

where  $\lambda$  is the mean intensity

$$\lambda = \mathbf{E}(\mathbf{M}(B(0, 1))).$$

By convention  $\lambda$  is assumed to be  $\lambda = 1$  in all the article (achievable through a deterministic rescaling). The *spectral measure*  $\mathbf{S}$  is defined in all generality as the Fourier transform  $\mathcal{FC}$  in the sense of tempered distributions (Section 4), i.e. for  $f \in \mathcal{C}_c^b(\mathbf{E})$

$$\text{Var}(I_{\mathbf{M}}(f)) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{f}|^2 d\mathbf{S}, \quad (4)$$

it is indeed a non-negative measure thanks to Bochner's Theorem (see Coste [2021], following Berg and Frost [1975]). In any case, the only relation formally needed in this work is that (4) holds for some non-negative measure  $\mathbf{S}$ . The local square integrability of  $\mathbf{M}$  directly yields that  $\mathbf{S}$  is a non-negative tempered measure (Lemma 3), and its inverse Fourier transform  $\mathbf{C}$  is a signed positive definite tempered measure.

Informally, the behaviour of  $\mathbf{S}$  at infinity represents the regularity of  $\mathbf{M}$ ; for instance if  $\mathbf{M}$  is a smooth Gaussian field, then  $\mathbf{S}$  will experience fast decay, whereas if  $\mathbf{P}$  is a point process,  $\mathbf{S}$  has likely infinite mass; the regularity of  $\mathbf{S}$  around 0 is related to the long range dependency of  $\mathbf{M}$ .

**Example 1** (Discrete fields). If  $\mathbf{E} = \mathbb{Z}^d$ , rather consider  $\mathbf{X}_{\mathbf{m}} := I_{\mathbf{M}}(1_{\{\mathbf{m}\}})$ ,  $\mathbf{m} \in \mathbb{Z}^d$ , and up to an affine transformation impose the convention  $\mathbf{E}\mathbf{X}_0 = 0$ ,  $\text{Var}(\mathbf{X}_0) = 1$ . Then

$$c(\mathbf{m}) := \mathbf{E}(\mathbf{X}_0 \mathbf{X}_{\mathbf{m}})$$

is bounded by  $c(0) = 1$ , hence tempered. Since  $c$  is positive definite,  $\mathbf{S} := \mathcal{F}(c\mathcal{L})$  satisfies (4) but with  $\mathbb{T}^d$  as integration domain.

**Example 2** (Point processes). Point processes are the prominent examples motivating the current work and related line of literature about rigidity. A configuration is a locally finite counting measure on  $\mathbf{E} = \mathbb{R}^d$ , and a random point process is a random variable  $\mathbf{P}$  in the space of configurations endowed with the counting  $\sigma$ -algebra generated by mappings  $\mathbf{P} \mapsto \#\mathbf{P} \cap A$  for  $A$  compact. The integrability assumption means  $\mathbf{E}(\#\mathbf{P} \cap A^2) < \infty$  for bounded measurable  $A$ . When  $\mathbf{P}$  has no multiple point, we assimilate  $\mathbf{P}$  as the random set formed by its support to use set-related notation, as there is a one-to-one correspondance.



For instance if  $\mathbf{P}$  is the unit intensity homogeneous Poisson process,  $\mathbf{C} = \delta_0$ ,  $\mathbf{S} = \mathcal{L}$  the Lebesgue measure on  $\mathbb{R}^d$ , but this example is not of great interest for us because, due to its spatial independence,  $\mathbf{P}$  does not experience rigidity or hyperuniformity.

For point processes, long range interaction is measured through the *truncated correlation measure*  $\rho_{tr}^{(2)}(dt)$  satisfying

$$\mathbf{C} = \rho_{tr}^{(2)} + \delta_0.$$

The Dirac mass  $\delta_0$  comes from the diagonal terms in the double summation of the covariance, reflecting the purely atomic nature of the point process. Informally,  $\rho_{tr}^{(2)}(dx)$  describes the probability that there is a point close to  $x$  given that there is a point at 0.

**Example 3** (Gaussian process). Let  $\hat{\mathbf{E}} = \mathbb{R}^d$  if  $\mathbf{E} = \mathbb{R}^d$ , or  $\hat{\mathbf{E}} = \mathbb{T}^d$  if  $\mathbf{E} = \mathbb{Z}^d$ . Given any non-negative measure  $\mathbf{S}$  with finite mass on  $\hat{\mathbf{E}}$ , the spectral representation theorem ([Adler and Taylor, 2007, Th. 5.4.2]) gives a stationary centred Gaussian process  $\mathbf{X} = \{\mathbf{X}(x), x \in \mathbf{E}\}$  such that  $\mathbf{S}$  is the spectral measure of the random measure  $I_{\mathbf{M}}(f) = \int f(x)\mathbf{X}(x)\mu(dx)$ , where  $\mu$  is either Lebesgue or the counting measure.  $\mathbf{X}$  is furthermore strongly stationary in the sense where  $\tau_y\mathbf{X} := \{\mathbf{X}(x+y); x \in \mathbf{E}\}$  has the same law as  $\mathbf{X}$ , for  $y \in \mathbf{E}$ . Since  $\mathbf{S}$  is finite,  $\mathbf{C} = (2\pi)^{-d}\hat{\mathcal{F}}\mathbf{S} = \mathbf{c}\mathcal{L}$  where the covariance function  $\mathbf{c}$  is defined by

$$\mathbf{E}(\mathbf{X}(0)\mathbf{X}(x)) = \mathbf{c}(x), x \in \mathbf{E}.$$

## 2 Rigidity of random stationary measures

For  $A \subset \mathbf{E}$ , the notation  $\mathbf{M}_A$  indicates the collection  $I_{\mathbf{M}}(f)$  for  $f \in C_c^b(A)$ . The general problem of rigidity for a random signed measure  $\mathbf{M}$  is to be able to infer a functional  $F(\mathbf{M}_A)$  knowing only  $\mathbf{M}_{A^c}$ . The *mass rigidity*, for instance, called *number rigidity* for point processes, means that  $\mathbf{M}(A) \in \sigma(\mathbf{M}_{A^c})$ . *Maximal rigidity* means that any bounded functional  $F$   $\mathbf{M}_A$ -measurable can be predicted by  $\mathbf{M}_{A^c}$ , or equivalently

$$\sigma(\mathbf{M}_A) \subset \sigma(\mathbf{M}_{A^c}).$$

For  $\gamma : \mathbf{E} \rightarrow \mathbb{C}$ , bounded with bounded support, we are interested here in predicting  $I_{\mathbf{M}}(\gamma)$ , i.e. in determining if  $I_{\mathbf{M}}(\gamma) \in \sigma(\mathbf{M}_{A^c})$ , called  $\gamma$ -rigidity on  $A$ , and maximal rigidity (MR) on  $A$  holds if  $\gamma$ -rigidity holds for all such  $\gamma$ . Define furthermore *linear*  $\gamma$ -rigidity if  $I_{\mathbf{M}}(\gamma)$  can be approximated by linear statistics of  $\mathbf{M}_{A^c}$ , i.e. if a.s. and in  $L^2(\mathbf{P})$ , for some  $h_n \in C_c^b(A^c)$ ,  $n \geq 1$ ,

$$I_{\mathbf{M}}(\gamma) = \lim_n I_{\mathbf{M}}(h_n),$$

which we write  $I_{\mathbf{M}}(\gamma) \in \sigma_{lin}(\mathbf{M}_{A^c})$ , and *maximal linear rigidity* if  $I_{\mathbf{M}}(\gamma) \in \sigma_{lin}(\mathbf{M}_{A^c})$  for all  $\gamma$  bounded. The term *linear* is sometimes omitted in this article, but all methods employed here pertain to linear rigidity.

We are particularly interested in (linear)  $\mathbf{k}$ -rigidity for  $\mathbf{k} = (k_i) \in \mathbb{N}^d$ , i.e.  $\gamma_{\mathbf{k}}$ -rigidity for  $\gamma_{\mathbf{k}}(t) = t^{\mathbf{k}}$  with  $t^{\mathbf{k}} = \prod_{i=1}^d t_i^{k_i}$ ,  $t \in \mathbb{R}^d$ . For  $k \in \mathbb{N}$ , say that  $\mathbf{M}$  is  $k$ -rigid if it is  $\mathbf{k}$ -rigid for every  $\mathbf{k}$  with  $|\mathbf{k}| \leq k$  (in particular, number rigidity corresponds to 0-rigidity).

Call *convex body*  $A$  a compact convex set with non-empty interior. In the heart of the paper, Theorem 7 provides an abstract necessary and sufficient condition for linear



$k$ -rigidity on  $A$  in terms of  $S$ , and shows in particular that linear  $\gamma$ -rigidity only depends on the *spectral density*  $\mathcal{J}$ , the density of the continuous part of  $S$ ;  $\mathcal{J}$  is symmetric and non-negative, as  $S$ . For this reason, we sometimes talk about the rigidity of  $\mathcal{J}$  instead of the linear rigidity of  $M$ , but it means exactly the same thing.

## 2.1 Local sufficient condition

We consider a random stationary measure  $M$  on  $\mathbb{R}^d$  with spectral density  $\mathcal{J}$ . The crucial concept is that of a  $k$ -pole.

**Definition 1.** Given  $k \in \mathbb{N}^d$ ,  $s : \mathbb{R}^d \rightarrow \mathbb{R}_+$ , say that  $\mathcal{J}^{-1}$  has a pole of order  $k$  in zero, or  $k$ -pole, if it is locally  $u^k$ -incompatible around 0: for every polynomial  $Q = \sum_m a_m u^m$  such that  $\int_{B(0,\varepsilon)} |Q|^2 \mathcal{J}^{-1} < \infty$  for some  $\varepsilon > 0$ , we have  $a_k = 0$  (with  $1/0 = \infty$ ).

For  $k, k' \in \mathbb{N}^d$ , say that  $k' \preceq k$  if  $k'_i \leq k_i$  for  $i = 1, \dots, d$ . Remark that, through multiplication of  $Q$  by  $u^{k-k'}$ , a  $k$ -pole is a  $k'$ -pole for  $k' \preceq k$ . Let us state our most general sufficient condition.

**Theorem 1.** If  $\mathcal{J}^{-1}$  has a  $k$ -pole in 0,  $M$  is  $k$ -rigid on any bounded measurable  $A \subset \mathbb{R}^d$ .

All the proofs of results of this subsection are at Section 4.4. In many situations, 0 being a  $k$ -pole is equivalent to

$$\int_{B(0,\varepsilon)} u^{2k} \mathcal{J}^{-1}(u) du = \infty \quad (5)$$

for all  $\varepsilon > 0$ :

- In dimension  $d = 1$ , we can also say that  $\mathcal{J}$  has a *zero of multiplicity*  $2k$ : for every non-zero polynomial  $Q$  on  $\mathbb{R}$ , we have  $Q(u) = au^q(1 + o(1))$  as  $u \rightarrow 0$  for some  $q \in \mathbb{N}, a \neq 0$ , and  $Q \in L^2(\mathcal{J}^{-1}; B(0, \varepsilon))$  if and only if  $\int_{B(0,\varepsilon)} u^{2q} \mathcal{J}^{-1} < \infty$ , a  $k$ -pole indeed means (5).
- If  $k = 0$  (number rigidity), a 0-pole, or just *pole*, is indeed equivalent to (5) because if  $Q(0) \neq 0, Q(u) \sim Q(0)$  as  $u \rightarrow 0$ .

This is unfortunately not always the case:

**Example 4.** Let  $\mathcal{J}(u) = (u_1 - u_2)^2, u \in \mathbb{R}^2$ . Hence  $\int_{B(0,\varepsilon)} u_1^2 \mathcal{J}^{-1}(u) du = \infty$  for  $\varepsilon > 0$ , but it does not mean that  $\mathcal{J}^{-1}$  has a  $(1, 0)$ -pole because  $Q(u) := (u_1 - u_2)$  satisfies  $\partial_1 Q(0) \neq 0$  and  $\int_{B(0,\varepsilon)} Q^2 \mathcal{J}^{-1} < \infty$ .

We have the following more general proposition, allowing in particular to treat isotropic spectral measures.

**Proposition 1.** Define  $\tilde{\mathcal{J}}(u) = \sup_{v: \|v\|=\|u\|} \mathcal{J}(v)$ . Let  $k \in \mathbb{N}^d$ . Then  $M$  is  $k$ -rigid on any bounded measurable  $A$  if for  $\varepsilon > 0$

$$\int_{B(0,\varepsilon)} \tilde{\mathcal{J}}^{-1}(u) u^{2k} du = \infty \text{ or equivalently } \int_{B(0,\varepsilon)} \tilde{\mathcal{J}}^{-1}(u) \|u\|^{2|k|} du = \infty, \quad (6)$$

where  $|k| = \sum_i k_i$ .

Let us illustrate this result by deriving the two seminal results about number rigidity by [Ghosh and Peres \[2017\]](#).

**Example 5.** Let  $\mathbf{P}$  be either  $\mathbf{P}_{\text{Gin}}$ , the infinite Ginibre ensemble or  $\mathbf{P}_{\text{GAF}}$ , the zero set of the planar Gaussian analytic function (see [Ghosh and Peres \[2017\]](#) for precise definitions).  $\mathbf{P}$  is an isotropic process in dimension  $d = 2$  where  $\mathbf{C}$  has a density decaying exponentially fast, hence

$$\mathfrak{J}(u) = \mathfrak{J}(0) + \frac{1}{2}\|u\|^2 \int \|t\|^2 \mathbf{C}(dt) + O(\|u\|^4).$$

[Ghosh and Peres \[2017\]](#) start from the facts that for a twice differentiable function  $f \in \mathcal{C}_c^b(\mathbb{R}^2)$ , for some finite  $C$ ,

$$\text{Var}(I_{\mathbf{P}_{\text{Gin}}}(f)) \leq C \int \|\nabla f\|^2,$$

$$\text{Var}(I_{\mathbf{P}_{\text{GAF}}}(f)) \leq C \int |\Delta f|^2.$$

Since  $\text{Var}(I_{\mathbf{P}}(f)) = \int |\hat{f}(u)|^2 \mathfrak{J}(u) du$ , this readily implies that in both cases  $\mathfrak{J}(0) = 0$  (hyperuniformity), hence by [Theorem 1](#) for  $k = 0$ ,  $\mathfrak{J}^{-1}$  has a 0-pole in 0 and both processes are number rigid. For  $\mathbf{P} = \mathbf{P}_{\text{GAF}}$ , it also implies that  $\int \|t\|^2 \mathbf{C}(dt) = 0$ , hence

$$\int \|u\|^2 \mathfrak{J}^{-1}(u) du = \infty,$$

and  $\mathfrak{J}$  is isotropic. Using also [Proposition 1](#) it has a pole of order 1 in 0, hence  $\mathbf{P}_{\text{GAF}}$  is 1-rigid (also proved in [Ghosh and Peres \[2017\]](#)).

Up to date,  $\mathbf{P}_{\text{GAF}}$  and its “toy model” ([Sodin and Tsirelson \[2004\]](#)) are to the author’s knowledge the only non-MR stationary point processes for which it is rigorously proven that  $\mathfrak{J}(u) = O(\|u\|^4)$ . We give at [Section 2.4](#) examples for which  $\mathfrak{J}(u) = O(\|u\|^{2n})$  for  $n$  a prime number which are not MR. The process  $\mathbf{P}_{\text{Gin}}$  is an instance of the class of projector determinantal point processes, the general case is treated at [Section 3.1](#).

For  $k = 0$ , i.e. for number rigidity, our sufficient condition is

$$\int_{B(0,\varepsilon)} \frac{1}{\mathfrak{J}(u)} du = \infty. \tag{7}$$

Let us explain why it allows to recover the sufficient condition of [Ghosh and Lebowitz \[2017\]](#) in dimensions  $d = 1, 2$ . In dimension 1, they prove that a (unit intensity) point process  $\mathbf{P}$  is number rigid if the truncated correlation measure  $\rho_{tr}^{(2)}$  has a density  $\mathfrak{e} = \mathcal{F}(\mathfrak{J} - 1)$  satisfying  $\int \mathfrak{e} = -1$  (hyperuniformity) and

$$|\mathfrak{e}(t)| \leq c(1 + |t|)^{-2}, t \in \mathbb{R}.$$

It implies that  $\mathfrak{J}$  is Lipschitz in 0 :

$$2|\mathfrak{J}(u)| = |\mathfrak{J}(u) + \mathfrak{J}(-u)| = \int (e^{itu} + e^{-itu})\mathfrak{e}(t) dt + 2 \int \mathfrak{e}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}} |e^{itu} + e^{-itu} - 2| |\mathfrak{c}(t)| dt \\
&\leq \int 4c \frac{\sin(tu/2)^2}{(1+|t|)^2} dt \\
&= 4c|u| \int \frac{\sin(z/2)^2}{(|z|+|u|)^2} dz \\
&\leq C|u| \text{ with } C = 4c \int \frac{\sin(z/2)^2}{|z|^2} dz < \infty.
\end{aligned}$$

Hence (7) is satisfied. In dimension 2, with similar definitions, the condition is

$$|\mathfrak{c}(t)| \leq c(1 + \|t\|)^{-4-\varepsilon}$$

which gives that  $\|t\|^2 \mathfrak{c}(t) \in L^1(\mathbb{R}^2)$ , hence  $\mathfrak{s} := \hat{c} + 1$  is twice differentiable in 0, and by symmetry  $\mathfrak{s}'(0) = 0$ , meaning  $\mathfrak{s}(u) \leq c\|u\|^2$ , and (7) holds again.

Bufetov et al. [2018] also give a sufficient condition, but depending on the structure factor of the discretised version, which is not the same as the original one, see the discussion at Section 2.5.

Let us conclude this subsection with an example illustrating the possible complexity of the concept of  $\mathfrak{k}$ -pole:

**Example 6.** Assume that some  $\mathfrak{s} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  satisfies  $\mathfrak{s}(u) \sim u_1^4$  as  $u \rightarrow 0$ . Let  $Q$  a non-zero polynomial in  $L^2(\mathfrak{s}^{-1}, B(0, \varepsilon))$  for some  $\varepsilon > 0$ . We put  $Q$  under the form

$$Q(u) = P(u_2) + u_1 R(u_2) + u_1^2 S(u_1, u_2)$$

where  $P, R, S$  are polynomials with one or two arguments. We have  $u_1^2 S(u) \in L^2(\mathfrak{s}^{-1}, B(0, 1))$ , hence so is  $P(u_2) + u_1 R(u_2)$ . By isolating dominating terms of  $P$  and  $R$ , there are exponents  $p, q \in \mathbb{N}$  and constants  $c_P, c_R$  such that in polar coordinates

$$\int_0^\varepsilon \left[ \int_0^{2\pi} \frac{(c_P \rho^p \sin(\theta)^p + c_R \rho \cos(\theta) \rho^q \sin(\theta)^q)^2}{\rho^4 \cos(\theta)^4} d\theta \right] \rho d\rho < \infty.$$

Examining the point  $\theta = \pi/2$  in the inner integral, it first implies  $c_P = 0$ , and then  $c_R = 0$ , which means  $P = R = 0$ .

We indeed proved that for a polynomial  $Q = \sum a_m u^m$  in  $L^2(\mathfrak{s}^{-1}, B(0, \varepsilon))$ , all terms of exponent  $\mathfrak{m} = (0, m), m \in \mathbb{N}$  or  $\mathfrak{m} = (1, m), m \in \mathbb{N}$  must vanish. This means that 0 is a pole of order  $(1, m)$  for all  $m \in \mathbb{N}$  (and a fortiori of order  $(0, m)$ ), and it does not admit other orders.

## 2.2 Necessary conditions

We investigate what non-rigidity implies on the spectral density  $\mathfrak{s}$ , and in particular whether a  $\mathfrak{k}$ -pole is necessary for  $\mathfrak{k}$ -rigidity. Maximal rigidity really depends on the behaviour of  $\mathfrak{s}$  on all  $\mathbb{R}^d$ , and is rather studied in Lachièze-Rey [2024+]. We nevertheless give a context where maximal rigidity and  $\mathfrak{k}$ -rigidity are easier to handle.

Say that a pole  $u_0 \in \mathbb{R}^d$  has finite order for  $\mathfrak{s}^{-1}$  if for some  $\varepsilon > 0, q \in \mathbb{N}$ ,

$$\int_{B(u_0, \varepsilon)} \frac{\|u - u_0\|^{2q}}{\mathfrak{s}(u)} du < \infty.$$

**Definition 2.** Say that  $\mathfrak{s}$  is *simple* if  $\mathfrak{s}^{-1}$  has finitely many poles, all with finite order, and  $\mathfrak{s}(u) \geq c(1 + \|u\|)^{-p}$  for some  $p \in \mathbb{N}, c > 0$ , at some distance  $\varepsilon > 0$  from these poles.

**Proposition 2.** Assume that  $\mathfrak{s}$  is *simple*. Then  $\mathbf{M}$  is not linearly maximally rigid on  $B(0, \eta)$  for  $\eta > 0$ . Furthermore, for  $\mathbf{k} \in \mathbb{N}^d$ ,  $\mathbf{M}$  is linearly  $\mathbf{k}$ -rigid on  $B(0, \eta)$  if and only if 0 is a pole of order  $\mathbf{k}$ .

The proof is at Section 4.5.1. This proposition in particular serves to treat stationary Determinantal Point Processes as their structure factor vanishes only in 0, see Section 3.1.

Apart from the *simple* case, we can give converse statements under stronger structural assumptions: for  $\mathfrak{s}$  *isotropic* (i.e.  $\mathfrak{s}(u)$  only depends on  $\|u\|$ ) or *separable* (i.e.  $\mathfrak{s}(u) = \mathfrak{s}_1(u_1) \dots \mathfrak{s}_d(u_d), u \in \mathbb{R}^d$  for some even functions  $\mathfrak{s}_i : \mathbb{R} \rightarrow \mathbb{R}_+$ ) and by assuming upfront that maximal rigidity does not hold, which often can be proved by other means. Remark 5 yields that if rigidity does not hold for some  $\mathfrak{s}' \leq \mathfrak{s}$ , then it does not hold either for  $\mathfrak{s}$ , hence in the general case one can investigate the largest separable/isotropic function  $\mathfrak{s}' \leq \mathfrak{s}$ .

**Theorem 2.** Let  $\mathbf{M}$  with spectral density  $\mathfrak{s}$  with either

- $\mathfrak{s}$  is isotropic and  $A = B(0, R), R > 0$
- $\mathfrak{s}$  is separable and  $A = [-R, R]^d, R > 0$ .

Assume  $\mathbf{M}$  is not maximally rigid on  $A$  and (5) does not hold. Then for  $\eta > 0$ ,  $\mathbf{M}$  is not  $\mathbf{k}$ -rigid on  $A^{+\eta}$ .

Proof at Section 4.6 .

**Remark 1.** Under these hypotheses, if it can be established by other means that  $\mathbf{M}$  is linearly 0-rigid or 1-rigid, one can reverse the reasoning of Example 5 and deduce bounds on the variance of linear statistics similar to those of  $\mathbf{P}_{\text{GAF}}$  and  $\mathbf{P}_{\text{Gin}}$ .

In view of Proposition 2, Theorems 1 and 2, (5) is necessary and sufficient for  $\mathbf{k}$ -rigidity on  $A$  in the following cases

- $\mathfrak{s}$  is “simple”,
- $\mathbf{M}$  is not LMR and  $\mathfrak{s}$  is quasi-isotropic, i.e. there is  $c_-, c_+ > 0$  such that

$$c_- \mathfrak{s}(\|u\|) \leq \mathfrak{s}(u) \leq c_+ \mathfrak{s}(\|u\|), u \in \mathbb{R}^d,$$

- $\mathbf{M}$  is not LMR and  $\mathfrak{s}$  is quasi-separable, i.e.  $0 \leq c_- \mathfrak{s}' \leq \mathfrak{s} \leq c_+ \mathfrak{s}'$  for some separable  $\mathfrak{s}'$ .

**Remark 2** (No phase transition). For  $\mathbf{M}$  which is not LMR on any convex body, linear number rigidity happens either on every convex body or none.

On the other hand, we give in Lachièze-Rey [2024+] examples of Gaussian measures which are maximally rigid for  $R \leq R_c$  for some  $R_c > 0$ , and not even 0-rigid for  $R > R_c$ . There are examples of processes obtained as perturbations of hyperplane networks, such as in Klatt et al. [2020], Peres and Sly [2014], where number rigidity occurs in dimension

$d \geq 3$ , but it does not enter in the current framework as it can be proved that this rigidity is non-linear, and the tools employed strongly depend on the underlying network structure. We are not aware of models which are rigid but not linearly rigid and not derived from a lattice construction. We think that for some continuous random fields, the set of zeros is rigid but not linearly, this will be the object of future research.

The structural conditions on  $\mathfrak{J}$  are necessary as there exists random measures which are linearly  $k$ -rigid without  $\mathfrak{J}$  vanishing around 0. We give an example for mass rigidity.

**Proposition 3.** Any random measure which spectral density satisfies  $\mathfrak{J}(u) \leq cu^2$  in some neighbourhood of  $(1, 0)$  is 0-rigid on any convex body  $A$ .

Proof at Section 4.5.2. We can therefore build examples which are completely standard, i.e. with fast decay of the correlations, linear variance, and which are rigid; by the previous results, such examples will never be isotropic or separable. To give a concrete example, take a mixture of an isotropic and a separable density

$$\mathfrak{J}(u) = \begin{cases} 1 & \text{if } \|u\| \leq 1/2 \\ \frac{u_2^2}{1+u_2^{10}} \frac{1}{1+u_1^{10}} & \text{otherwise,} \end{cases}$$

not LMR because  $\psi(u) := u_2(1 + \|u\|^{100})^{-1} \in L^2(\mathfrak{J}^{-1})$ . To *realise*  $\mathfrak{J}$ , i.e. find  $\mathbf{M}$  for which such  $\mathfrak{J}$  is indeed the spectral density, use for instance Gaussian fields with Example 3. Whether there are such number rigid point processes with standard second order behaviour which are not hyperuniform depends on our ability to have such spectral densities for random atomic measures, this relates to the generally difficult problem of *realisability of point processes*.

## 2.3 $k$ -tolerance

Ghosh and Peres [2017] show that the infinite Ginibre ensemble  $\mathbf{P}_{\text{Gin}}$  is not rigid at an order higher than 0, in the sense that if one determines  $N := \mathbf{P}_{\text{Gin}}(A)$  from  $\mathbf{P}_{\text{Gin}}1_{A^c}$ , then conditionnally to  $\mathbf{P}_{\text{Gin}}1_{A^c}$  and  $N$ ,  $\mathbf{P}_{\text{Gin}}1_A$ 's law is continuous wrt  $\mathbf{U}_N = \{U_1, \dots, U_N\}$ , consisting of i.i.d. uniform variables on  $A$ . They similarly show that the zero set  $\mathbf{P}_{\text{GAF}}$  of the planar GAF is 1-rigid and not more, in the sense that conditionnally to

$$(N, U, \Phi) := (\mathbf{P}_{\text{GAF}}(A), \sum_{x \in \mathbf{P}_{\text{GAF}} \cap A} x, \mathbf{P}_{\text{GAF}} \cap A^c),$$

$\mathbf{P}_{\text{GAF}}1_A$  has a continuous law wrt to the law of  $\mathbf{U}_N$  conditioned to  $\sum_{i=1}^N U_i = U$ . A similar result is proved by Dereudre et al. [2020] for the sine $_{\beta}$  processes  $\mathbf{P}_{\beta}, \beta > 0$ .

These strong distributional results rely on the determinantal nature of  $\mathbf{P}_{\text{Gin}}$ , the Gaussian nature of  $\mathbf{P}_{\text{GAF}}$ , or the DLR equations defining  $\mathbf{P}_{\beta}$ , and cannot be extended in the current general context. We can still investigate how ‘‘linearly tolerant’’ is a process which is not rigid, i.e. whether the random moments  $m_{\mathbf{m}}(A) := \int_A t^{\mathbf{m}} d\mathbf{M}(t)$ ,  $\mathbf{m} \in \mathbb{N}^d$  are tied by a finite linear relation

$$\sum_{\mathbf{m}} a_{\mathbf{m}} m_{\mathbf{m}}(A) \in \sigma_{\text{lin}}(\mathbf{M}; A^c)$$

or equivalently if we have  $Q$ -rigidity on  $A$  for the polynomial  $Q(t) = \sum_m a_m t^m$ . We show that this holds for what we call a  $k$ -polynomial  $Q$ , i.e. such that for some  $k_0 \succeq k$ ,  $a_{k_0} \neq 0$  and the non-vanishing terms  $a_m \neq 0$  are such that  $m \preceq k_0$ .

**Proposition 4.** Assume that 0 is not a pole of order  $k$  and the assumptions of Theorem 2 or of Proposition 2 hold. Then  $M$  is not  $Q$ -rigid for any  $k$ -polynomial  $Q$ .

The proof of this proposition is actually included in the proofs of resp. Proposition 2 and Theorem 2. As is apparent from the proofs, we will in fact have no  $Q$ -rigidity for almost all polynomials having at least a non-vanishing term of order  $k_0 \succeq k$ , i.e. for a linear combination  $Q = \sum_i \alpha_i Q_i$  where the  $Q_i$  are  $k$ -polynomials, but some particular combinations of  $\alpha_i$  are harder to discard.

## 2.4 Point processes with arbitrary rigidity

For systems which have a non-zero spectral measure, the highest order of rigidity that is encountered in existing models is the 1-rigidity of the zeros of the planar GAF, see Ghosh and Peres [2017], it corresponds to a structure factor decaying in 0 as  $\|u\|^4$  in dimension  $d = 2$  (see Example 5). Sodin and Tsirelson [2004] provide a toy model with a simpler structure that exhibits similar properties, in particular for the rigidity order and decay of the structure factor. Until now there was no generalisation of this model (or another model) with an arbitrary order of rigidity.

Let  $\rho > 0$  and  $n \in \mathbb{N}^*$ . Define  $P_{\rho,n}$  as the process obtained by putting at distance  $\rho$  of each point  $m$  of  $\mathbb{Z}^d$  the  $n$ -th order roots of unity rotated by a random independent quantity  $\theta_m$ , and applying a global stationarising shift  $U$ :

$$P_{\rho,n} = \sum_{m \in \mathbb{Z}^d} \sum_{z \in \mathbf{U}_n} \delta_{U+m+\rho e^{i\theta_m} z}$$

where  $\mathbf{U}_n = \{e^{2\pi i k/n}; k = 1, \dots, n\} \subset \mathbb{C}$ ,  $U$  is an independent uniform random variable in  $[0, 1]^2$ ,  $\theta_m$  are i.i.d. uniform numbers on  $[0, 2\pi]$ . The structure factor  $S$  is defined as before by assimilating  $\mathbb{C}$  to  $\mathbb{R}^2$ .

**Theorem 3.** For  $n \geq 3$  a prime number, for some  $c > 0$ ,  $S(u) \leq c|u|^{2n}$  as  $u \rightarrow 0, u \in \mathbb{C}$ .  $P_{\rho,n}$  is  $k$ -rigid for  $k \leq n-1$  on any convex body  $A$ , and not maximally rigid on  $B(0, \rho+1)$ .

Any other model with a spectral gap instead of  $\mathbb{Z}^d + U$  (i.e. a *stealthy process*) would give the same results. The model of Sodin and Tsirelson [2004] is obtained by applying independent random scalings  $\rho_m, m \in \mathbb{Z}^d$  instead of a global deterministic one, hence only the terms up to order 4 vanish, yielding 1-rigidity, but we need to have a deterministic  $\rho$  if we want higher order terms to vanish.

For the proof, we establish a general formula for the structure factor of a unit intensity stationary point process  $\Phi \subset \mathbb{R}^d$  perturbed by an arbitrarily cluster of points. Let  $c_0 = \sum_{k=1}^N \delta_{X_k}$  a random finite point process with law  $\mu$ , where  $N \in \mathbb{N}$  and the  $X_i \in \mathbb{R}^d, i \geq 1$ , are random. The average number of points is denoted by  $\kappa = \mathbf{E}(N)$ . Let  $c_m, m \in \Phi$  i.i.d. copies of  $c_0$  (cond. to  $\Phi$ ) and

$$\Phi_\mu := \sum_{m \in \Phi} \tau_m c_m$$

where points are counted according to multiplicity. We call such a process a *lattice of clusters* with *cluster law*  $\mu$ . Remark that  $\Phi_\mu$  has intensity

$$\mathbf{E}(\Phi_\mu(A)) = \kappa \mathcal{L}(A), \quad A \subset \mathbb{R}^d,$$

with possibly multiple points.

**Proposition 5.** Let  $\Phi$  a  $L^2$  wide sense stationary unit intensity point process with structure factor  $\mathbf{S}_\Phi$  and  $\mu$  such that  $\mathbf{E}(\#\mathbf{c}_0^2) < \infty$ . Let  $\varphi(u) = \mathcal{F}\mathbf{c}_0(u) = \int e^{-i\langle u, t \rangle} \mathbf{c}_0(dt)$ . Then the structure factor of  $\Phi_\mu$  is

$$\mathbf{S}(du) = (\mathbf{E}|\varphi(u)|^2 - |\mathbf{E}\varphi(u)|^2)\mathcal{L}(du) + |\mathbf{E}\varphi(u)|^2\mathbf{S}_\Phi(du).$$

This result, of independent interest, is proved at Section 4.7. It generalises and strenghtens the result for lattices perturbed by i.i.d. points when a.s.  $\mathbf{c}_1(\mathbb{R}^d) = 1$  ([Coste, 2021, Th. 9.3]).

*Proof of Theorem 3.* In the case of  $\mathbf{P}_{\rho, n}$ ,

$$\Phi = \sum_{m \in \mathbb{Z}^d} \delta_{U+m}, \quad \mathbf{c}_0 = \sum_{k=1}^n \delta_{\rho e^{i k \alpha + \theta}}$$

where  $\alpha = 2\pi/n$  and  $\theta$  is a random element of  $[0, 2\pi]$ . We define for  $u \in \mathbb{C}, \theta \in [0, 2\pi]$ ,

$$\varphi_\theta(u) = \sum_{k=1}^n \exp(-i\langle e^{i(k\alpha + \theta)}, u \rangle),$$

so that  $\varphi(u) = \varphi_\theta(\rho u)$  when  $\theta$  is random. We use the Taylor expansion with the Lagrange form of the remainder: for  $u \in \mathbb{R}$

$$\varphi_\theta(u) = \sum_{p=0}^{2n-1} \frac{(-iu)^p}{p!} \sum_{k=1}^n \cos(k\alpha + \theta)^p u^p + u^{2n} R_n(u, \theta)$$

where  $R_n(u, \theta) = O(\|\varphi_\theta^{(2n)}\|_{\mathbb{R}})$  is uniformly bounded over  $u, \theta \in \mathbb{R}$ . We also use the fact that for some real coefficients  $\lambda_{m,p}$ ,

$$\begin{aligned} \sum_{k=1}^n \cos(k\alpha + \theta)^p &= \sum_{k=1}^n \sum_{m=0}^p \lambda_{m,p} \cos(m(k\alpha + \theta)) \\ &= \sum_{m=0}^p \lambda_{m,p} \mathcal{R}\left(\sum_{k=1}^n e^{i m k \alpha} e^{i m \theta}\right). \end{aligned}$$

For  $p \in \{1, \dots, 2n-1\}$ , the terms  $m \notin \{0, n\}$  vanish because  $\sum_{k=1}^n e^{i m k \alpha} = 0$ , because  $m$  is prime with  $n$ . We have for  $p \leq 2n-1$

$$\sum_{k=1}^n \cos(k\alpha + \theta)^p = n\lambda_{0,p} + \mathbf{1}[p \geq n] \lambda_{n,p} \mathcal{R}(n e^{i n \theta}) = n\lambda_{0,p} + \mathbf{1}[p \geq n] n\lambda_{n,p} \cos(n\theta).$$



Finally,

$$\varphi_\theta(u) = \sum_{p=0}^{2n-1} \underbrace{\frac{(-iu)^p}{p!} n\lambda_{0,p}}_{c_p} u^p + \cos(n\theta) \sum_{p=n}^{2n-1} \underbrace{\frac{(-iu)^p}{p!} n\lambda_{n,p}}_{a_p} u^p + u^{2n} R_n(u, \theta).$$

From that we can deduce the expansion of  $|\varphi_\theta(u)|^2 = \sum_{l=0}^{2n-1} \gamma_l(\theta) u^l + u^{2n} S_n(u, \theta)$ , where  $S_n(u, \theta)$  is uniformly bounded for the same reasons. Hence we have

$$\begin{aligned} \mathbf{E}|\varphi(u)|^2 &= \sum_{\substack{p,q=0 \\ p+q < 2n}}^{2n-1} c_p \bar{c}_q (\rho u)^{p+q} + \mathbf{E}(\cos(n\theta))^2 u^{2n} \sum_{\substack{p,q=n \\ p+q < 2n}}^{2n-1} a_p \bar{a}_q \rho^{p+q} u^{p+q-2n} \\ &\quad + \mathbf{E}(\cos(n\theta)) \sum_{p=0}^{2n-1} \sum_{q=n}^{2n-p} 2\mathcal{R} c_p \bar{a}_q \rho^{p+q} u^{p+q} + (\rho u)^{2n} S_n(\rho u, \theta) \\ |\mathbf{E}\varphi(u)|^2 &= \left| \sum_{p=0}^{2n-1} c_p \rho^p u^p + (\rho u)^{2n} \mathbf{E}(R_n(\rho u, \theta)) \right|^2 = \sum_{\substack{p,q=0 \\ p+q < 2n}}^{2n-1} c_p \bar{c}_q \rho^{p+q} u^{p+q} + O(u^{2n}) \end{aligned}$$

whence indeed, as claimed, using  $\mathbf{E}(\cos(n\theta)) = 0$ ,  $\mathbf{E}|\varphi(u)|^2 - |\mathbf{E}\varphi(u)|^2 = O(u^{2n})$ . The conclusion comes from the fact that  $\mathbf{S}_\Phi = \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{0\}} \delta_{\mathbf{m}}$  (Coste [2021]).

The rigidity result is then a straightforward consequence of Proposition 1. Let us prove the non-rigidity result. With positive probability, the cluster  $\tau_U \mathbf{c}_0$  is contained in  $B(0, \rho + 1)$ , hence there is no way to recover the value of  $\theta_0$  from the observation of  $\mathbf{P} \cap B(0, \rho + 1)^c$ , proving that  $\mathbf{P}$  is not MR on  $B(0, \rho + 1)$ .  $\square$

**Remark 3** (Higher dimensions). It is possible to lift this example to higher dimensions in a rather artificial way: attach to each point of  $\mathbb{Z}^d + U$  a cluster which points are the  $(u_1, u_2, 0, \dots, 0)$  where  $(u_1, u_2)$  is an atom of an independent version of  $\mathbf{c}_0$ .

Whether there is a full dimensional analogue of  $\mathbf{U}_n$  in  $\mathbb{R}^3$  that would cause the same decay of the structure factor is intriguing. Preliminary computations regarding Platonic solids show that icosahedron and dodecahedron might give a decay of  $\mathcal{J}(u)$  in  $\|u\|^4$ , but not above, and other solids do not have enough symmetries. In dimension 4, some tested  $p$ -cells do not work either.

## 2.5 Rigidity for discrete processes

Let us derive similar results for a centred discrete process  $\mathbf{X} = \{\mathbf{X}_{\mathbf{m}}; \mathbf{m} \in \mathbb{Z}^d\}$  with the linear statistics

$$\mathbf{X}(\gamma) := \sum_{\mathbf{m}} \mathbf{X}(\mathbf{m}) \gamma(\mathbf{m}).$$

We assume  $\mathbf{X}$  is wide-sense  $L^2$ -stationary with spectral measure  $\mathbf{S}$ , i.e. for  $\gamma$  with bounded support

$$\text{Var}(\mathbf{X}(\gamma)) = (2\pi)^{-d} \int_{\mathbb{T}^d} |\hat{\gamma}(u)|^2 \mathbf{S}(du)$$

where  $\mathbb{T} = [-\pi, \pi]$ , for the complex-valued trigonometric polynomial

$$\hat{\gamma}(u) = \sum_{\mathbf{m} \in \mathbb{Z}^d} e^{-i(\mathbf{m}, u)} \gamma(\mathbf{m}).$$

For  $A \subset \mathbb{Z}^d$ , the class of such trigonometric polynomials where  $\gamma(\mathbf{m}) = 0$  for  $\mathbf{m} \notin A$  is denoted by  $\mathcal{E}(A)$ , by analogy with entire functions of exponential type (see Section 4).

**Theorem 4.** Let  $\mathfrak{s}(u)du$  the continuous component of  $S$ .

- $X$  is not LMR on  $\llbracket m \rrbracket^d$  if and only if there is  $\psi \in \mathcal{E}(\llbracket m \rrbracket^d)$  such that

$$\int \frac{|\psi(u)|^2}{\mathfrak{s}(u)} du < \infty. \quad (8)$$

- For  $\mathbf{k} \in \mathbb{N}^d$ ,  $X$  is not  $\mathbf{k}$ -rigid if and only if there is  $\psi \in \mathcal{E}(\llbracket m \rrbracket^d) \cap L^2(\mathfrak{s}^{-1})$  such that  $\partial^{\mathbf{k}}\psi(0) \neq 0$ , where  $\partial^{\mathbf{k}} = \partial_1^{k_1} \dots \partial_d^{k_d}$ .
- If 0 is a pole of order  $\mathbf{k}$  for  $\mathfrak{s}^{-1}$ , then  $X$  is  $\mathbf{k}$ -rigid.

**Proposition 6** (Specialization to time series in dimension  $d = 1$ ).  $X$  is not LMR on  $\llbracket m \rrbracket$  if and only if the number of poles of  $\mathfrak{s}^{-1}$  on  $[-\pi, \pi]$  counted with multiplicity is  $\leq 2m$ . Also,  $X$  is not 0-rigid if and only if there are  $2m$  poles or less and 0 is not a pole.

The proof is at Section 4.1. We discuss the LMR result below, the other statements seem to be the first dealing with partial rigidity, i.e. when one is concerned not in retrieving all of  $X$  on  $\llbracket m \rrbracket^d$ , but only its restricted moments, by analogy with the continuous case.

The line of research of linear interpolation in the 20th century was mainly concerned with time series, and authors noted that for a polynomial satisfying (8), perfect linear interpolation is impossible, see (10.28) and Theorem 10.3 in [Rozanov \[1967\]](#), with also an extension to processes taking values in  $\mathbb{R}^q$ . Much later, [Lyons and Steif \[2003\]](#) study the related problem of the strong full K property for  $\{0, 1\}$ -valued  $X$ , i.e. what does it imply on the  $X(\mathbf{m})$ ,  $|\mathbf{m}| \leq m$ , if we impose  $X(\mathbf{m}) = 1$  for  $m \leq |\mathbf{m}| < m + k$  for  $m, k$  arbitrarily large? It is clear indeed that if no polynomial  $\psi$  is in  $L^2(\mathfrak{s}^{-1})$ , then  $X$  is LMR for all  $m$ , and as such for  $k$  large enough the external conditioning completely determines  $X(\mathbf{m})$ ,  $|\mathbf{m}| \leq m$ . This strong resemblance explains why condition (iii) in their theorem 7.7 is exactly the same condition. None of these works seems to acknowledge that  $m$  must be the degree of  $\psi$ . This result also completes the answer of [Ghosh \[2015\]](#) to the Question 9.9 of [Lyons and Steif \[2003\]](#) regarding rigidity on  $A = \{0\}$ .

[Bufetov et al. \[2018\]](#) studies the rigidity of continuous models through a discretisation procedure. Given a stationary random measure  $M$  on  $\mathbb{R}^d$ , one can consider the associated stationary discrete process

$$X_{\mathbf{m}} = M(\mathbf{m} + [-1/2, 1/2]^d), \mathbf{m} \in \mathbb{Z}^d.$$

[Bufetov et al. \[2018\]](#) show number rigidity of  $M$  on  $[-1/2, 1/2]^d$  by showing the rigidity of  $X$  on  $\{0\}$ , and derive a necessary condition in terms of the covariance decay for  $X$ .

**Remark 4.** A reformulation of 0-rigidity in dimension 1 is the following:  $X$  is 0-rigid if  $X$  is LMR or 0 is a pole, which means that either  $\int_{B(0,\varepsilon)} \mathfrak{J}^{-1} = \infty$  or that the poles are away from zero and strictly more than  $m$  (counted with multiplicity), i.e. if one considers only  $m$  points, necessarily at least one pole is not covered, i.e. for  $u_1, \dots, u_m \in \mathbb{C}^d \setminus \{1\}$ ,

$$\int \frac{\prod_i |e^{iu} - e^{iu_i}|^2}{\mathfrak{J}(u)} du = \infty.$$

This seems to correspond to the content of Remark 2.1 of [Bufetov et al. \[2018\]](#) for continuous measures  $\mathbf{S} = \mathfrak{J}\mathcal{L}$ .

At the contrary of the continuous case,  $k$ -rigidity is not monotonous in  $k$ , in the sense that there are examples of 1-rigid processes that are not 0-rigid in dimension 1, due to the constraint on the degree:

**Example 7.** Consider  $m = 1$  in dimension  $d = 1$ , and  $\mathfrak{J}(u) = (u - 1)^2(u + 1)^2, u \in \mathbb{T}$ . Let  $\psi$  a degree 1 trigonometric polynomial of  $L^2(\mathfrak{J}^{-1})$ . In particular, there is a polynomial  $P$  of degree 2 such that

$$e^{iu}\psi(u) = P(e^{iu}) = a(e^{iu} - z_1)(e^{iu} - z_2), u \in \mathbb{T},$$

for some  $a \in \mathbb{C} \setminus \{0\}, z_1, z_2 \in \mathbb{C}$ . Examining the neighbourhoods of 1 and  $-1$  of  $\mathfrak{J}^{-1}$ , we necessarily have  $z_1 = e^i, z_2 = e^{-i}$  (or the other way around), this is also sufficient for  $\psi \in L^2(\mathfrak{J}^{-1})$ , showing that  $L^2(\mathfrak{J}^{-1}) \cap \mathcal{E}(\llbracket 1 \rrbracket) \neq \{0\}$ , hence  $\mathfrak{J}$  is not LMR. It also implies that every such  $\psi$  is symmetric, hence  $\psi'(0) = 0$ , meaning that  $\mathfrak{J}$  is 1-rigid. On the other hand,  $P(0) \neq 0$ , hence  $\mathfrak{J}$  is not 0-rigid.

## 3 Applications

### 3.1 Determinantal processes and completeness

Determinantal points processes (DPPs), introduced in the context of quantum mechanics, have gained popularity as many classes of essential models in Random Matrices, Statistical Physics, Combinatorics and other fields have proven to be determinantal, see [Hough et al. \[2009\]](#). In the Euclidean context, we follow [Soshnikov \[2000\]](#): a simple point process  $\mathbf{P}$  on  $\mathbb{R}^d$  is determinantal with kernel  $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$  if for every  $k \in \mathbb{N}^*$ ,

$$\rho_k(x_1, \dots, x_k) := \det((K(x_i, x_j))_{1 \leq i, j \leq k}) \geq 0$$

is the  $k$  point correlation function of  $\mathbf{P}$ , i.e. for any non-negative  $\varphi : (\mathbb{R}^d)^k \rightarrow \mathbb{R}$ , we have

$$\mathbf{E} \left( \sum_{\substack{\neq \\ x_1, \dots, x_k \in \mathbf{P}}} \varphi(x_1, \dots, x_k) \right) = \int \rho_k \varphi,$$

where the sum runs over  $k$ -tuples of pairwise distinct points. We assume up to a scalar rescaling that  $K(0, 0) = 1$  to deal with unit intensity processes. Stationarity yields that  $\rho_2(x, y) = \rho_2(x - y)$  with an abuse of notation. Not all functions  $K$  give rise to a DPP and in particular we will require that  $K$  is Hermitian and positive definite, so that

$$\rho_2(x - y) = 1 - \kappa(x - y)^2 \in \mathbb{R}_+$$

where  $\kappa(x) = |K(0, x)|$ ,  $0 \leq \kappa(x) \leq 1$ , we also have to assume that  $\kappa$  is locally square integrable for definiteness issues. See [Hough et al., 2009, Section 4.5] for unicity and existence questions, and Soshnikov [2000] for a treatment of such stationary DPPs, it is shown in particular that they are mixing and ergodic, and the trace-class property yields that  $\hat{\kappa}$  is integrable and  $0 \leq \hat{\kappa} \leq 1$ . The main result of this section is a characterisation of  $\mathbf{k}$ -rigidity in the continuous setting:

**Corollary 1.** *Let  $\mathbf{P}$  be a stationary DPP on  $\mathbb{R}^d$  with Hermitian positive kernel  $K$  and  $\kappa(x) = |K(0, x)|$  square integrable. Then for  $\mathbf{k} \in \mathbb{N}^d$ ,  $\mathbf{P}$  is  $\mathbf{k}$ -rigid on a convex body  $A$  if and only if  $(1 - \hat{\kappa}^2)^{-1}$  has a pole of order  $\mathbf{k}$  in 0. When it is not the case,  $\mathbf{P}$  is  $\mathbf{k}$ -tolerant in the sense that it is not  $Q$ -rigid on  $A$  for any  $\mathbf{k}$ -polynomial  $Q$ .*

We see that  $\mathfrak{s} := (1 - \hat{\kappa}^2)$  must vanish in 0 to have some rigidity. That is because, by Soshnikov [2000],  $\mathfrak{s}(u)du$  is indeed the structure factor of  $\mathbf{P}$ , and  $\mathfrak{s}(0) = 0$  exactly means that  $\mathbf{P}$  is HU (see (2)); hyperuniform DPPs are a very important class in Statistical Physics and Random Matrices, and are also used for tasks of numerical integration Bardenet and Hardy [2020] or machine learning Kulesza and Taskar [2012].

*Proof.* Since  $\mathfrak{s}(u) \geq 0$ ,  $\int \kappa^2 \leq 1$ , and  $\mathfrak{s}(u) = 0$  is only possible if  $u = 0$ . Since  $\kappa^2$  is integrable, the Riemann-Lebesgue lemma ([Duistermaat and Kolk, 2006, Th. 14.2]) yields that  $|\hat{\kappa}^2(u)| < 1/2$  for  $|u| \geq T$  for some  $T$ . By Theorem 1,  $\mathbf{P}$  is  $\mathbf{k}$ -rigid on  $A$  if  $\mathfrak{s}^{-1}$  has a pole of order  $\mathbf{k}$  in 0. If it is not satisfied, it follows by Proposition 4 that  $\mathbf{M}$  is not  $Q$ -rigid for a  $\mathbf{k}$ -polynomial  $Q$ . □

Linear number rigidity is therefore equivalent to the non-integrability of  $(1 - \hat{\kappa}^2)^{-1}$ . For many examples,  $\hat{\kappa}^2(u) = 1 - \sigma^2 u^2 + o(u^2)$  for some  $\sigma > 0$ , hence there is  $\mathbf{k}$ -rigidity if and only if  $\mathbf{k} = 0$  and  $d \in \{1, 2\}$ , and no rigidity in higher dimensions. This applies for instance to the infinite Ginibre ensemble on  $\mathbb{R}^2 \approx \mathbb{C}$ , defined by

$$K(x, y) = e^{x\bar{y} - \frac{1}{2}|x|^2 - \frac{1}{2}|y|^2}; x, y \in \mathbb{C},$$

i.e.  $\kappa(x) = e^{-\frac{|x|^2}{2}}$ , retrieving the result of Ghosh and Peres [2017], or the result of Ghosh [2015], stating that the sine process in dimension 1 and Ginibre ensemble in dimension 2 are number rigid. Tensor kernels give non rigid examples: Bufetov et al. [2018] proves non-rigidity for the DPP which kernel is  $\kappa(x) = \text{sinc}(x_1)\text{sinc}(x_2)$  on  $A = [-1/2, 1/2]^2$  with a discretisation (see Section 2.5). Theorem 1 yields that this  $\mathbf{P}$  is not linear number rigid in the continuous sense on any convex body  $A$ , any discrete average will not be linearly rigid either.

This result also bears a connection to the question of finding random countable sets of exponential functions spanning  $L^2(A)$  for a compact  $A \subset \mathbb{R}^d$ . For  $\chi \subset \mathbb{R}^d$ , call

$$E_\chi = \{t \mapsto e^{tx}; x \in \chi\}.$$

Say that  $E_\chi$  is *complete* if functions of  $L^2([-\pi, \pi])$  are  $L^2$  approximable by functions of  $E_\chi$ . Lyons and Steif [2003] investigated the question of completeness for a random set  $\chi \subset \mathbb{R}^d$ , and more particularly if  $\chi$ 's law is a DPP, leaving open several questions. Ghosh [2015] established a connection between number rigidity and completeness, and a corollary of [Ghosh, 2015, Th. 1.3] and Corollary 1 is the following result:

**Corollary 2.** *If  $\mathbf{P}$  is a DPP, then  $E_{\mathbf{P}}$  spans  $L^2([-\pi, \pi])$  if  $(1 - \widehat{\kappa}^2)^{-1}$  is not integrable in 0.*

In particular it retrieves Th. 1.5 of Ghosh [2015], which shows number rigidity in dimension 1 if  $\hat{\kappa}$  is the indicator function of an interval (i.e.  $\hat{\kappa} = \hat{\kappa}^2$ ): Parseval equality yields

$$\widehat{\kappa}^2(0) = \int \kappa^2 = \int \hat{\kappa}^2 = \int \hat{\kappa} = \kappa(0) = 1,$$

hence  $\mathcal{J}(0) = 0$  (meaning  $\mathbf{P}$  is hyperuniform), and  $\widehat{\kappa}^2$  is clearly Lipschitz, hence  $\mathcal{J}(u) = O(|u|)$  and we have number rigidity.

### 3.2 Consequences for the covariance of Gibbs measures

Gibbs measures are also a prominent class of point processes, especially in statistical physics Lewin [2022]. It is known that in the case of short range interactions, such processes are not hyperuniform, see Dereudre and Flimmel [2024], hence  $\mathbf{S}$  will not vanish around 0 and the process is most likely not rigid. On the other hand, those with a very strong dependency at long range, such as Coulomb gases, or more generally Riesz gases, form a very important class of models in statistical physics, and are expected to be hyperuniform, see [Lewin, 2022, VI.C] and references therein.

Such models are usually defined for a finite but large number of particles interacting through some physical equations, and defining an infinite stationary model compatible with these equations is already a challenge, there can in many instances be distinct infinite models compatible with local conditions. In dimension 1, Valkó and Viràg [2009] have reached a universal explicit limit of 1D  $\text{sine}_\beta$  log gases, further studied in Dereudre et al. [2020], Chhaibi and Najnudel [2018], where it is proved that it is number rigid. Not much seems to be known in higher dimensions on the rigidity (or existence) of stationary models.

Summing up the current main results in this context,  $k$ -rigidity of a hyperuniform isotropic (or 1D) model is equivalent to (5), which is already a rigorous statement on the second order properties of the process through  $\mathcal{J}$ , though sometimes difficult to connect with the properties of the Fourier transform  $\mathbf{C}$  if the correlations are not known to decay fast. In particular, proving hyperuniformity is already a challenge, the only rigorous result in dimension  $d \geq 2$  is to my knowledge the work of Leblé [2023], showing that 2D Coulomb gases are hyperuniform, but the obtained estimate is not sufficient to prove rigidity.

In the other direction, Dereudre et al. [2020] and Dereudre and Vasseur [2023] have established non-rigidity results, the latter managed to prove the existence of an infinite stationary and isotropic model in all dimension  $d \geq 1$  compatible with the  $s$ -Riesz DLR equations for  $d - 1 < s < d$ , the *circular Riesz gas*, which is not number rigid. Hence we can use the necessary condition of Theorem 2 to deduce rigorous statements about  $\mathbf{S}$ , and therefore  $\mathbf{C} = \mathcal{F}\mathbf{S}$ . The fact that Riesz gases are hyperuniform is expected, see an argument in [Lewin, 2022, VI.C], but there does not seem to be a rigorous proof.

**Corollary 3.** *Let  $\mathbf{M}$  a hyperuniform non-number rigid isotropic random stationary measure in dimension  $d \in \{1, 2\}$ . Then*

$$\int |t|^d |\mathbf{C}|(dt) = \infty.$$

*Proof.* If

$$\int_{\mathbb{R}} |t| |\mathbf{C}(dt)| < \infty$$

in dimension 1,  $\mathbf{S} = (2\pi)^{-d} \dot{\mathcal{F}} \mathbf{C}$  has a differentiable density  $\mathfrak{J}$  satisfying  $\mathfrak{J}(0) = 0$  (hyperuniformity), and

$$|\mathfrak{J}(u)| \leq \int |u| |t| |\mathbf{C}(t)| dt,$$

hence  $\mathfrak{J}^{-1}$  is not integrable in 0, and according to Theorem 1,  $\mathbf{M}$  is number rigid, we reach a contradiction.

In dimension 2, if

$$\int_{\mathbb{R}^2} \|t\|^2 |\mathbf{C}(dt)| < \infty,$$

$\mathbf{S}$  is twice differentiable, and by symmetry  $\partial_1 \mathfrak{J}(0) = \partial_2 \mathfrak{J}(0) = 0$ , hence

$$\mathfrak{J}(u) \leq c \|u\|^2,$$

similarly  $\mathfrak{J}(u)^{-1}$  is not integrable in 0,  $\mathbf{M}$  is number rigid, which is again a contradiction.  $\square$

The  $\text{sine}_\beta$  gases in 1D are proved by Dereudre et al. [2020], Chhaibi and Najnudel [2018] to be number rigid, and the proof of Chhaibi and Najnudel [2018] proves that this rigidity is linear. Theorem 2 immediately gives the following result.

**Corollary 4.** *For all  $\beta > 0$ , the spectral density  $\mathfrak{J}_\beta$  of the  $\text{sine}_\beta$  process satisfies*

$$\int_{B(0,\varepsilon)} \frac{1}{\mathfrak{J}_\beta(u)} du = \infty.$$

That seems to be in agreement with predictions and existing results, see [Lewin, 2022, IV-A-2d].

Dereudre et al. [2020] prove that it is not further rigid, and in particular not (linearly) 1-rigid. This is actually expected, but it still rigourously prevents some behaviours.

**Corollary 5.** *Let  $\mathbf{M}$  a hyperuniform process in dimension 1 that is not 1-rigid. Then*

$$\int |t|^3 |\mathbf{C}(dt)| = \infty \text{ or } \int t^2 \mathbf{C}(t) dt \neq 0.$$

At first sight, the second moment vanishing seems unlikely, but this actually occurs for the model  $\mathbf{P}_{\text{GAF}}$  (see Example 5), and hyperuniformity itself is the result of the vanishing of a moment (of order 0). Such moment cancellations, are sometimes called *sum rules* by physicists, and whether they occur and yield higher order rigidity is a fascinating topic of mathematics, see again [Lewin, 2022, VI.C] for references.

## 4 Proofs

For  $\mathbf{S}$  a non-negative measure on  $\hat{\mathbf{E}} = \mathbb{R}^d$  or  $\hat{\mathbf{E}} = \mathbb{T}^d$  and  $A \subset \hat{\mathbf{E}}$ , write  $L^2(\mathbf{S}; A)$  as the space of functions  $\psi : A \rightarrow \mathbb{C}$  satisfying

$$\int_A |\psi|^2 d\mathbf{S} < \infty,$$

simply note  $L^2(\mathbf{S}) = L^2(\mathbf{S}; \hat{\mathbf{E}})$ .

We start with the discrete case, which does not require high level Fourier technology, and can serve as a model for the proof of the main theoretical result, Theorem 7.

### 4.1 Discrete processes ( $\hat{\mathbf{E}} = \mathbb{T}^d$ )

*Proof of Theorem 4.* For  $A \subset \mathbb{Z}^d$ , recall that  $\mathcal{E}(A)$  is the space of trigonometric polynomials  $\sum_{\mathbf{m} \in A} a_{\mathbf{m}} e^{i\langle \mathbf{m}, u \rangle}$  with  $a_{\mathbf{m}} \in \mathbb{C}$ . For  $\gamma : \mathbb{Z}^d \rightarrow \mathbb{C}$ ,  $\gamma$ -rigidity means that for some  $h_n$  compactly supported in  $A^c$ , a.s. and in  $L^2(\mathbf{P})$ ,

$$\mathbf{X}(\gamma) = \lim_n \mathbf{X}(h_n),$$

and this is equivalent to

$$\inf_{h: (\llbracket m \rrbracket^d)^c \rightarrow \mathbb{C}} \text{Var}(\mathbf{X}(\gamma) - \mathbf{X}(h)) = \inf_{h: (\llbracket m \rrbracket^d)^c \rightarrow \mathbb{C}} \int_{\mathbb{T}^d} \mathbf{S}(du) |\hat{\gamma}(u) - \hat{h}(u)|^2 = 0,$$

where  $\hat{\gamma} \in \mathcal{E}(\llbracket m \rrbracket^d)$  and  $\hat{h} \in \mathcal{E}((\llbracket m \rrbracket^d)^c)$ . The orthogonal space of  $\mathcal{E}((\llbracket m \rrbracket^d)^c)$  in  $L^2(\mathbf{S})$  is

$$H_m^\perp := \left\{ \varphi \in L^2(\mathbf{S}) : \int_{\mathbb{T}^d} \bar{\varphi} \hat{h} d\mathbf{S} = 0; h : (\llbracket m \rrbracket^d)^c \rightarrow \mathbb{C} \text{ with finite support} \right\}.$$

For such  $\varphi$ ,  $\Psi(du) := \varphi(u) \mathbf{S}(du)$  satisfies

$$c_{\mathbf{m}} := \int \bar{\Psi}(du) e^{i\langle \mathbf{m}, u \rangle} = \begin{cases} \int_{\mathbb{T}^d} \bar{\varphi}(u) e^{i\langle \mathbf{m}, u \rangle} \mathbf{S}(du) \leq \|\varphi\|_{L^2(\mathbf{S})} & \text{for } |\mathbf{m}| \leq m, \\ 0 & \text{if } |\mathbf{m}| > m \end{cases}$$

recalling that  $\mathbf{S}(\mathbb{T}^d) = \text{Var}(\mathbf{X}_0) = 1$ . It means that  $\Psi$  coincides as inverse Fourier transform with

$$\sum_{|\mathbf{m}| \leq m} c_{\mathbf{m}} e^{i\langle \mathbf{m}, u \rangle}.$$

In particular  $\Psi$  has a density  $\psi \in \mathcal{E}(\llbracket m \rrbracket^d)$ , and the negligible set supporting the singular part  $\mathbf{S}_s$  of  $\mathbf{S}$  is not charged (i.e.  $\varphi \mathbf{S}_s = 0$ ). We also have  $\varphi = \psi \mathcal{J}^{-1} \in L^2(\mathbf{S})$ , hence

$$\int \frac{|\psi|^2}{\mathcal{J}} < \infty.$$

We proved indeed that in  $L^2(\mathbf{S}, \mathbb{C})$ ,

$$H_m^\perp \subset \{ \psi \mathcal{J}^{-1} : \psi \in \mathcal{E}(\llbracket m \rrbracket^d) \cap L^2(\mathcal{J}^{-1}) \}.$$



For the converse, we clearly have for such  $\psi$  vanishing on the negligible singular part's support,  $\varphi := \psi \mathcal{J}^{-1} \in L^2(\mathbf{S})$ , and for  $h \in \mathcal{E}(\llbracket m \rrbracket^d)^c$

$$\int \bar{\varphi} \hat{h} \mathcal{J} = \int \bar{\psi} \hat{h} = 0,$$

hence the inclusion is an equality. Since LMR is equivalent to  $H_m^\perp = \{0\}$ , this part of the proof is complete.

Mass rigidity (or 0-rigidity) means  $\gamma$ -rigidity for  $\gamma = 1_{\llbracket m \rrbracket^d}$ , i.e.  $\langle \varphi, \hat{\gamma} \rangle = 0$  for  $\varphi \in H_m^\perp$ . Hence it means, for all  $\psi = \sum_{\mathbf{m}} c_{\mathbf{m}} e^{i\mathbf{u} \cdot \mathbf{m}} \in \mathcal{E}(\llbracket m \rrbracket^d) \cap L^2(\mathcal{J}^{-1})$ ,

$$0 = \langle \psi \mathcal{J}^{-1}, \hat{\gamma} \rangle_{\mathbf{S}} = \left\langle \sum_{\mathbf{m} \in \llbracket m \rrbracket^d} \hat{\delta}_{\mathbf{m}}, \psi \right\rangle = \int_{\mathbb{T}^d} \sum_{\mathbf{m}} e^{-i\langle \mathbf{m}, \mathbf{u} \rangle} \sum_{\mathbf{m}'} c_{\mathbf{m}'} e^{i\mathbf{m}' \cdot \mathbf{u}} d\mathbf{u} = \sum_{\mathbf{m}} c_{\mathbf{m}} = \psi(0).$$

$\mathbf{k}$ -rigidity means the same with

$$0 = \left\langle \sum_{\mathbf{m}} \mathbf{m}^{\mathbf{k}} \hat{\delta}_{\mathbf{m}}, \psi \right\rangle = \sum_{\mathbf{m}} \mathbf{m}^{\mathbf{k}} c_{\mathbf{m}} = i^{\mathbf{k}} \partial^{\mathbf{k}} \psi(0).$$

Assume now that  $\mathcal{J}^{-1}$  has a pole of order  $\mathbf{k}$ . Let  $\varphi \in H_m^\perp \setminus \{0\}$ ,  $\psi = \varphi \mathcal{J} \in \mathcal{E}(\llbracket m \rrbracket^d)$ . In particular,  $\int_{B(0, \varepsilon)} |\psi|^2 \mathcal{J}^{-1} < \infty$ . By Lemma 1 below, there is a polynomial  $Q$  equivalent to  $\psi$  in 0 with same derivatives up to order  $\mathbf{k}$ . Therefore,

$$\int |Q|^2 \mathcal{J}^{-1} < \infty$$

and since 0 is a  $\mathbf{k}$ -pole,  $\partial^{\mathbf{k}} \psi(0) = \partial^{\mathbf{k}} Q(0) = 0$ . We indeed proved  $\mathbf{k}$ -rigidity.  $\square$

**Lemma 1.** Let  $\psi(u) = \sum_{\mathbf{m}} a_{\mathbf{m}} u^{\mathbf{m}}$  an analytic function on  $\mathbb{C}^d$ . There exists  $Q$  a polynomial equivalent to  $\psi$  in the neighbourhood of 0 with  $\partial^{\mathbf{k}} Q(0) = \partial^{\mathbf{k}} \psi(0)$ .

*Proof.* Let  $I = \{\mathbf{m} : a_{\mathbf{m}} \neq 0\} \subset \mathbb{N}^d$ . It is easy to see that there is a finite set  $I_0 \subset I$  that dominates  $I$  in the sense that for all  $\mathbf{m}' \in I$ , there is some  $\mathbf{m} \in I_0$  with  $\mathbf{m}' \preceq \mathbf{m}$ . Said differently, there does not exist infinite  $I_0 \subset \mathbb{N}^d$  made up of extremal points, i.e. such that every  $\mathbf{m} \in I_0$  is not  $\preceq$ -smaller than all others  $\mathbf{m}' \in I_0$ .

Define

$$Q(u) = \sum_{\mathbf{m} \in I_0 \cup \{\mathbf{k}\}} a_{\mathbf{m}} u^{\mathbf{m}}$$

so that  $Q, \psi$  have the same term of order  $\mathbf{k}$ . Also, by uniform convergence of the series, as  $u \rightarrow 0$ ,

$$\psi(u) = \sum_{\mathbf{m} \in I_0} [a_{\mathbf{m}} u^{\mathbf{m}} (1 + o(1))] = Q(u) (1 + o(1)).$$

$\square$

*Proof of Proposition 6.* Assume the poles of  $\mathcal{J}^{-1}$  in  $[-\pi, \pi)$  have finite orders  $k_i$ , i.e. for  $\varepsilon > 0$  sufficiently small,

$$\int_{u_i - \varepsilon}^{u_i + \varepsilon} \frac{|u - u_i|^{2k_i}}{\mathcal{J}(u)} du < \infty$$

or equivalently

$$\int_{u_i-\varepsilon}^{u_i+\varepsilon} \frac{|e^{iu} - e^{iu_i}|^{2k_i}}{\mathcal{J}(u)} du < \infty$$

with  $\sum_i k_i \leq 2m$ . Since  $\mathcal{J}$  is symmetric, for  $u_i$  a pole,  $-u_i$  is also a pole (with same order). Let  $u_1, \dots, u_p$  the poles  $\neq 0$  (repeated according to multiplicity), with  $p \leq m$ . If  $p = m$ , 0 is not a pole otherwise there are more than  $2m$  poles. If  $p < m$ , if 0 is a pole, its order is  $2(m - p)$  or less by the hypothesis. We hence define  $\psi \in \mathcal{E}(\llbracket m \rrbracket)$  by

$$\psi(u) = |e^{iu} - 1|^{2(m-p)} \prod_{i=1}^p (e^{iu} - e^{iu_i})(e^{-iu} - e^{iu_i}).$$

$\psi$  is by construction of degree  $m$ , non null, and in  $L^2(\mathcal{J}^{-1})$ , hence  $\mathbf{X}$  is not LMR by Theorem 4.

Conversely, assume not LMR: by Theorem 4 there is

$$\psi(u) = \sum_{|m| \leq m} c_m e^{i\langle m, u \rangle} \neq 0$$

in  $L^2(\mathcal{J}^{-1})$ , and since  $\mathcal{J}(u) = \mathcal{J}(-u)$ ,  $\psi(-u) \in L^2(\mathcal{J}^{-1})$ . Now, either  $\psi(u) = -\psi(-u)$ , or  $\tilde{\psi}(u) := \psi(u) + \psi(-u)$  is a non-zero even function of  $L^2(\mathcal{J}^{-1})$ . In any case, we can assume  $|\psi(u)| = |\psi(-u)|$  without loss of generality. Let

$$P(z) = z^m \sum_m c_m z^m = \prod_{i=1}^{2m} (z - z_i)$$

for some  $z_1, \dots, z_{2m} \in \mathbb{C}$ , which is a polynomial of degree  $2m$ , and  $\psi(u) = P(e^{iu})e^{-imu}$ .

A zero of  $\psi$  is necessarily a root of  $P$  with norm 1,  $z_i = e^{iu_i}$ , and  $-u_i$  is also a zero, hence  $e^{-iu_i}$  is also a root of  $P$ . Since  $P$  has at most  $2m$  roots,  $\psi$  has at most  $2m$  zeros, which means that  $\mathcal{J}$  has at most  $2m$  poles (all counted with multiplicity).

For the converse sense for 0-rigidity, assume that  $\mathbf{X}$  is not LMR, meaning there are less than  $2m$  poles. If 0 is not a pole, then indeed with the previous notation  $\psi(1) = P(e^{iu \cdot 0}) \neq 0$ , whence  $|\psi(u)| = |P(e^{iu})|$  is in  $\mathcal{E}(\llbracket m \rrbracket) \cap L^2(\mathcal{J}^{-1}) \setminus \{0\}$  with

$$\left\langle \sum_{|k| \leq m} \delta_k, \hat{\psi} \right\rangle = \psi(1) \neq 0,$$

it gives the equivalence for 0-rigidity. □

## 4.2 Schwartz' Paley-Wiener Theorem

The Fourier transform used here relies on the space of tempered distributions, denoted by  $\mathcal{S}'(\mathbb{R}^d)$  as the dual of the Schwart space  $\mathcal{S}(\mathbb{R}^d)$  of  $C^\infty$  functions which derivatives have superpolynomial decay; there is no need to enter into technical details in this article, see for instance [Duistermaat and Kolk \[2006\]](#) for a theoretical exposition. We only consider

in this paper complex-valued measures, that is a distribution  $\Psi$  on  $\mathbb{R}^d$  such that for some non-negative measure  $|\Psi|$  on  $\mathbb{R}^d$ ,

$$|\Psi(f)| \leq \int |f| |\Psi|(dx), f \in C_c^b(\mathbb{R}^d).$$

Such  $\Psi$  is indeed tempered if for some  $p > 0$

$$\int_{\mathbb{R}^d} (1 + \|u\|)^{-p} |\Psi|(du) < \infty, \quad (9)$$

it is in particular locally finite. We only consider measures of the form  $\Psi = \psi \mathbf{S}$  where  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$  and  $\mathbf{S}$  is a non-negative measure.

For  $\Psi$  tempered, one can define the Fourier transform  $\mathcal{F}\Psi \in \mathcal{S}'(\mathbb{R}^d)$  through

$$\mathcal{F}\Psi(\varphi) = \int \hat{\varphi} d\Psi$$

for  $\varphi \in C_c^\infty(\mathbb{R}^d)$  and  $\hat{\varphi}(u) = \int_{\mathbb{R}^d} e^{-i\langle u, t \rangle} dt$  is the usual Fourier transform on  $L^2(\mathbb{R}^d)$ . Recall the Fourier inversion:

**Theorem 5** (Duistermaat and Kolk [2006], Th.14.18). For  $\Psi \in \mathcal{S}'(\mathbb{R}^d)$ ,

$$\mathcal{F}(\mathcal{F}\dot{\Psi}) = (2\pi)^d \Psi$$

where  $\dot{\Psi} = \Psi(-\cdot)$  in the distributional sense. Call *spectrum* of  $\Psi$ , denoted by  $\text{sp}(\Psi)$ , the support of  $\mathcal{F}\Psi$  as a distribution, i.e. the largest closed set  $A$  such that  $\langle \mathcal{F}\Psi, \varphi \rangle = 0$  for  $\varphi$  supported by  $A^c$ .

For a compact  $A$ , define for  $\zeta \in \mathbb{R}^d$

$$s_A(\zeta) = \sup_{x \in A} \langle \zeta, x \rangle.$$

In the case  $A = B(0, R)$ , we have  $s_A(\zeta) = R\|\zeta\|$ . One can also notice that  $s_A = s_{\text{conv}(A)}$  characterises the closed convex hull  $\text{conv}(A)$  of  $A$ .

Call analytic function on  $\mathbb{C}^d$  any function

$$\psi(z) = \sum_{\mathbf{m}} a_{\mathbf{m}} z_1^{m_1} \dots z_d^{m_d}, z \in \mathbb{C}^d$$

where  $a_{\mathbf{m}} \in \mathbb{C}$  and the series is absolutely convergent on  $\mathbb{C}^d$ . Recall that by Hartog's theorem (see [Krantz, 1992, Sec.0.2]), this is equivalent to the analyticity of  $\psi$  in each variable  $z_i$  separately. The scalar product on  $\mathbb{C}^d$  is  $\langle x, z \rangle = \sum_i \bar{x}_i z_i$ . For  $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ , write  $\|z\|_{\mathbb{C}}^2 = \langle \bar{z}, z \rangle$ , not to be mistaken with the entire function  $\|u\|^2 = \sum_i u_i^2$ , mostly used on  $\mathbb{R}^d$ .

**Theorem 6.** [Schwartz' Paley-Wiener Theorem] Let  $A \subset \mathbb{R}^d$  bounded. Let  $\Psi$  a complex-valued measure on  $\mathbb{R}^d$ . Then  $\Psi$  is tempered with  $\text{sp}(\Psi) \subset \text{conv}(A)$  if and only if  $\Psi$  has a tempered density  $\psi$  wrt  $\mathcal{L}$  on  $\mathbb{R}^d$  that can be extended as an analytic function on  $\mathbb{C}^d$  such that for some finite  $C$ ,

$$|\psi(z)| \leq C \exp(s_A(\mathcal{F}z)), z \in \mathbb{C}^d.$$

Such functions are denoted by  $\mathcal{E}(A)$ , which is therefore a subclass of the class of entire functions of  $\mathbb{R}^d$ , and also a subclass of the class of (restrictions to  $\mathbb{R}^d$  of) analytic functions with exponential type, and the extension to  $\mathbb{C}^d$  is still denoted  $\psi$ .

An important observation is that for  $\psi \in \mathcal{E}(A)$ , for a polynomial  $Q$ ,  $Q\psi \in \mathcal{E}(A)$ , hence it still has spectrum in  $A$ , and if  $\psi Q^{-1}$  is analytic, it is also in  $\mathcal{E}(A)$ .

A more general version of the previous theorem for all  $\mathcal{S}'(\mathbb{R}^d)$  is proved at [Duistermaat and Kolk, 2006, Th. 17.1, Theorems 17.3]. See Lachièze-Rey [2024+] for a proof of a result slightly more general than the direct implication.

The construction of some functions of  $\mathcal{E}(A)$  rely on the following technique, involving  $J_d$  the Fourier transform of the unit sphere indicator:

$$J_d(u) := \frac{B_{d/2}(\|u\|)}{\|u\|^{d/2}}, u \in \mathbb{R}^d, \quad (10)$$

where  $B_{d/2}$  is the Bessel function of the first kind of order  $d/2$ .

**Lemma 2.** Let  $\psi \in \mathcal{E}(A)$  for some bounded  $A$ . Then for every non-zero polynomial  $P$  and  $\eta > 0, p \geq 0$ , the function

$$\tilde{\psi}(u) = P(u)\psi(u)J_d(\eta u/M)^M$$

with  $M = \frac{2}{d+1}(\deg(P) + p)$  satisfies  $|\tilde{\psi}(u)| \leq c|\psi(u)|\|u\|^{-p}$  for some finite  $c$  and  $u \in \mathbb{R}^d$ , and  $\psi \in \mathcal{E}(A^{+\eta})$  where  $A^{+\eta} := \cup_{t \in A} B(t, \eta)$ .

*Proof.* By Theorem 6,  $J_d$  is an analytic function with spectrum in  $B(0, 1)$ , hence  $J_d(\eta \cdot /M)^M \in \mathcal{E}(B(0, \eta))$ . Hence  $\tilde{\psi}$  is analytic as well, and using again Theorem 6, it proves that

$$|\tilde{\psi}(z)| \leq c \exp(s_A(z)) \exp(\eta \|z\|).$$

Then, for some  $x_0 \in A$ ,

$$s_A(z) + \eta \|z\| = \langle x_0, z \rangle + \eta \|z\| \leq \sup_{x \in A, t \in B(0, \eta)} \langle x + t, z \rangle = s_{A+\eta}(z)$$

hence  $\psi \in \mathcal{E}(A^{+\eta})$ . We have the classical equivalent as  $u \rightarrow \infty$ , for some  $c, c_d > 0$ ,

$$J_d(u) = c\|u\|^{-(d+1)/2} \cos(\|u\| - c_d)(1 + O(\|u\|^{-1})), \quad (11)$$

which gives  $|\tilde{\psi}(u)| \leq c|\psi(u)|\|u\|^{-p}$ . □

### 4.3 Characterisation of $\gamma$ -rigidity

Let us translate the general rigidity problem in the Fourier space. Let  $A \subset \mathbb{R}^d$ . For  $\gamma \in \mathcal{C}_c^b(\mathbb{R}^d)$ ,  $\gamma$ -rigidity on  $A$  means

$$\inf_{h \in \mathcal{C}_c^\infty(A^c)} \text{Var} \left( \int_A \gamma(t) \mathbf{M}(dt) - \int_{A^c} h(t) \mathbf{M}(dt) \right) = 0.$$

It implies that for some sequence  $h_n, I_{\mathbf{M}}(\gamma) - I_{\mathbf{M}}(h_n) \rightarrow 0$  in  $L^2(\mathbf{P})$ , hence for some subsequence we have a.s.  $I(\gamma) = \lim_{n' \rightarrow \infty} I(h_{n'})$ , meaning  $I(\gamma) \in \sigma_{lin}(\mathbf{M}_{A^c})$ . By (4),  $\gamma$ -rigidity is equivalent to

$$\inf_{h \in \mathcal{C}_c^\infty(A^c)} \int \mathbf{S}(du) |\hat{\gamma}(u) - \hat{h}(u)|^2 = 0.$$

The heart of this article lies in the current characterisation of  $\gamma$ -rigidity. Recall that the spectral density  $\mathfrak{s} \geq 0$  is the density of the continuous part of  $\mathbf{S}$ .

**Theorem 7.** Let  $A$  a convex body.  $\mathbf{M}$  is linearly  $\gamma$ -rigid on  $A$  if and only if for all  $\psi \in \mathcal{E}(A) \cap L^2(\mathfrak{s}^{-1})$ ,

$$\int_A \hat{\gamma} \bar{\psi} = 0.$$

Before the proof, an instructive immediate corollary:

**Remark 5.** For  $0 \leq \mathfrak{s} \leq \mathfrak{s}'$ ,  $\gamma$ -rigidity for  $\mathfrak{s}'$  on  $A$  implies  $\gamma$ -rigidity for  $\mathfrak{s}$  on  $A$ .

We also need a lemma:

**Lemma 3.** For any spectral measure  $\mathbf{S}$  of a  $L^2$  wide-sense stationary random measure  $\mathbf{M}$ ,

$$\int_{\mathbb{R}^d} (1 + \|u\|)^{-2(d+1)} \mathbf{S}(du) < \infty.$$

In particular,  $\mathbf{S}$  is a tempered measure.

*Proof.* Let  $J_d$  the Fourier transform of the indicator of the unit ball (see (10)), satisfying in particular (11). Let  $z_0 = (\pi/2, \dots, \pi/2)$  and

$$\psi(u) = J_d(u)^2 + J_d(u + z_0)^2.$$

The function  $\cos(\|u - c_d\|)^2 + \cos(\|u - c_d + z_0\|)^2$  is larger than some  $\kappa > 0$ , hence for some  $R > 0$ ,  $\psi$  satisfies for  $\|u\| \geq R$

$$\frac{c_-}{\|u\|^{d+1}} \leq \psi(u) \leq \frac{c_+}{\|u\|^{d+1}}.$$

Also  $\hat{\psi}$  is bounded by 2 and  $\text{sp}(\psi) \subset \text{sp}(J_d^2) \subset B(0, 2)$ . Therefore, by (4)

$$\begin{aligned} \int \mathbf{S}(du) (1 + \|u\|)^{-2(d+1)} &\leq \mathbf{S}(B(0, R)) + \int_{B(0, R)^c} c \mathbf{S}(du) \psi(u)^2 \\ &\leq \mathbf{S}(B(0, R)) + c(2\pi)^d \text{Var} \left( I_{\mathbf{M}}(\hat{\psi}) \right) < \infty. \end{aligned}$$

□

*Proof of Theorem 7.* Denote by  $\bar{H}^{\mathbf{S}}$  the closure of some subspace  $H$  of  $L^2(\mathbf{S})$ , and let

$$H_A = \{\hat{h} : h \in \mathcal{C}_c^\infty(A^c)\}.$$

This is indeed a subspace of  $L^2(\mathbf{S})$  because for  $h \in \mathcal{C}_c^\infty(A^c)$ ,  $\hat{h} \in \mathcal{S}(\mathbb{R}^d)$  has rapid decay, and

$$\int_{\mathbb{R}^d} |\hat{h}|^2 \mathbf{S} \leq c \int_{\mathbb{R}^d} (1 + \|u\|)^{-2(d+1)} \mathbf{S}(du) < \infty$$

by Lemma 3.

Hence  $\mathbf{M}$  is  $\gamma$ -rigid if and only if  $\hat{\gamma} \in \bar{H}_A^\mathbf{S}$ , if and only if  $\int \hat{\gamma} \bar{\varphi} \mathbf{S} = 0$  for  $\varphi \in H_A^\perp$  where

$$H_A^\perp = \{\varphi \in L^2(\mathbf{S}) : \int_{\mathbb{R}^d} \bar{\varphi} \hat{h} d\mathbf{S} = 0; h \in \mathcal{C}_c^\infty(A^c)\}.$$

The proof of the theorem is concluded by the following lemma:

**Lemma 4.** Let  $A \subset \mathbb{R}^d$  bounded measurable.

$$H_A^\perp = \{\psi \mathcal{J}^{-1} : \psi \in \mathcal{E}(\text{conv}(A)) \cap L^2(\mathcal{J}^{-1})\}.$$

*Proof of Lemma 4.* Let  $\varphi \in H_A^\perp$ . Let  $\Psi = \varphi \mathbf{S}$ . The measure  $\Psi$  is tempered because by Cauchy-Schwarz inequality in  $L^2(\mathbf{S})$ ,

$$\int_{\mathbb{R}^d} |\varphi(u)| \mathbf{S}(du) (1 + \|u\|)^{-(d+1)} \leq \|\varphi\|_{L^2(\mathbf{S})} \sqrt{\int (1 + \|u\|)^{-2(d+1)} \mathbf{S}(du)}$$

and the latter is finite with Lemma 3. Let  $\mathcal{F}\Psi$  its Fourier transform in the sense of  $\mathcal{S}'(\mathbb{R}^d)$ . Let  $h \in \mathcal{C}_c^\infty(A^c) \subset \mathcal{S}(\mathbb{R}^d)$ , then  $\hat{h} \in \mathcal{S}(\mathbb{R}^d) \cap L^2(\mathbf{S})$ . By definition of  $H_A^\perp$ ,

$$0 = \int \hat{h} \bar{\varphi} d\mathbf{S} = \int \hat{h} d\bar{\Psi} = \overline{\mathcal{F}\Psi(h)}.$$

Hence  $\text{sp}(\Psi) \subset A$ . By Theorem 6,  $\Psi$  has a tempered density  $\psi$  :

$$\Psi = \psi \mathcal{L} = \varphi \mathcal{J} \mathcal{L}$$

and  $\psi$  is a function of  $\mathcal{E}(\text{conv}(A))$ . Also, since  $\varphi \in L^2(\mathbf{S})$ ,

$$\int \frac{\psi^2}{\mathcal{J}} = \int \varphi^2 \mathcal{J} = \int \varphi^2 \mathbf{S} < \infty,$$

indeed  $\psi \in L^2(\mathcal{J}^{-1})$  (with  $1/0 = \infty$ ).

For the converse, let  $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(\text{conv}(A))$ , in particular  $\text{sp}(\psi) \subset \text{conv}(A)$ , and let  $\varphi = \psi \mathcal{J}^{-1} \in L^2(\mathbf{S})$ . For  $h \in \mathcal{C}_c^\infty(A^c) \subset \mathcal{S}(\mathbb{R}^d)$ ,

$$0 = \mathcal{F}\psi(h) = \langle \psi, \hat{h} \rangle = \int \bar{\varphi} \hat{h} d\mathbf{S},$$

indeed  $\varphi \in H_A^\perp$ . □

□

## 4.4 Proof of Theorem 1

We assume here that  $\mathcal{J}$  has a pole of order  $k$  in 0. Let us prove  $t^k$ -rigidity. It is enough to prove  $\gamma$ -rigidity for some  $\gamma \in \mathcal{S}(\mathbb{R}^d)$  such that  $\gamma(t) = t^k$  for  $t \in A$ . We actually prove that it is  $\gamma$ -rigid on  $\text{conv}(A)$ , which in turns implies  $k$ -rigidity on  $A$  : if  $\gamma \in \sigma_{lin}(\mathbf{M}_{\text{conv}(A)^c})$ ,

$$\int_A t^k \mathbf{M}(dt) = \int_{\text{conv}(A)} \gamma d\mathbf{M} - \int_{\text{conv}(A) \setminus A} \gamma d\mathbf{M} \in \sigma_{lin}(\mathbf{M}_{A^c}).$$

Hence without loss of generality we assume  $A$  is convex. We use Theorem 7: we must prove that  $\hat{\gamma}$  is orthogonal to all  $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A)$ . We use the following lemma.

**Lemma 5.** Let  $\gamma \in \mathcal{S}(\mathbb{R}^d)$  coinciding with  $t^k$  on  $A$  and  $\psi \in \mathcal{E}(A)$ . Then with  $k = |\mathbf{k}|$ ,

$$\int \psi \hat{\gamma} = i^{-k} \partial^k \psi(0).$$

*Proof.* Let  $\tilde{\psi} = \psi \star f \in \mathcal{S}(\mathbb{R}^d)$  for some regularising function  $f \in \mathcal{S}(\mathbb{R}^d)$ . In particular,  $\text{sp}(\tilde{\psi}) \subset A$  and the Fourier inversion formula yields,

$$\begin{aligned} \partial^k \tilde{\psi}(0) &= \int_A \partial^k|_{u=0} e^{i\langle t, u \rangle} (2\pi)^{-d} \widehat{\tilde{\psi}}(t) dt = (2\pi)^{-d} \int_A (it)^k e^{i\langle t, u \rangle} \widehat{\tilde{\psi}}(t) dt \\ &= (2\pi)^{-d} i^k \int_A \gamma(t) \widehat{\tilde{\psi}}(t) dt = i^k (2\pi)^{-d} \int_{\mathbb{R}^d} \gamma(t) \widehat{\tilde{\psi}}(t) dt = i^k \int \hat{\gamma} \tilde{\psi}. \end{aligned}$$

Letting  $f$  converge to  $\delta_0$  yields the result. □

Since  $\psi \in L^2(\mathcal{J}^{-1})$  and is analytic, we have by Lemma 1 for some polynomial  $Q$  with  $\partial^k Q(0) = \partial^k \psi(0)$  and for  $\varepsilon$  sufficiently small

$$\int_{B(0, \varepsilon)} \frac{|Q(u)|^2}{\mathcal{J}(u)} du < \infty,$$

by definition of  $k$ -incompatibility it forces  $\partial^k Q(0) = 0 = \partial^k \psi(0) = \int \psi \hat{\gamma}$ , for all  $\psi \in \mathcal{E}(A) \cap L^2(\mathcal{J}^{-1})$ . We proved that  $\mathbf{M}$  is  $k$ -rigid.

*Proof of Proposition 1.* Let  $Q \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon))$  a polynomial,  $q$  the degree of lowest order terms, meaning  $Q = Q_h + R$  with  $Q_h$  a non-null homogeneous polynomial, i.e. verifying for some  $q \in \mathbb{N}$ ,  $u \neq 0$ ,

$$Q_h(u) = \|u\|^q Q_h(\theta), \theta = \frac{u}{\|u\|} \in \mathbb{S}^{d-1},$$

and  $R(u) = o(\|u\|^q)$  as  $u \rightarrow 0$ . We have

$$\begin{aligned} \int_{B(0, \varepsilon)} \mathcal{J}^{-1}(u) |Q|^2(u) du &\geq \int_{B(0, \varepsilon)} \tilde{\mathcal{J}}(\|u\|)^{-1} \|u\|^{2q} (|Q_h(\theta)| + o(1))^2 du \\ &\geq \sigma_d \int_0^\varepsilon \tilde{\mathcal{J}}(\rho)^{-1} \rho^{2q} \left( \int_{\mathbb{S}^{d-1}} |Q_h(\theta)|^2 d\theta + o(1) \right) \rho^{d-1} d\rho. \end{aligned}$$

Therefore, if  $q \leq k$ , indeed  $\mathcal{J}^{-1}|Q|^2$  is not integrable around 0, which means  $Q$  cannot have terms of degree  $\leq k$ , and  $\mathcal{J}$  is  $k$ -rigid for  $|\mathbf{k}| \leq k$ . □



## 4.5 Necessity lemma

The necessity proofs rely on the following technique.

**Lemma 6.** Let  $Q$  a  $k$ -polynomial (for instance  $Q(u) = u^k$ ),  $A \subset \mathbb{R}^d$  bounded measurable. Assume for some  $\varepsilon > 0$  there is  $\psi \in L^2(\mathcal{J}^{-1}, \mathbb{R}^d \setminus B(0, \varepsilon)) \cap \mathcal{E}(A)$  and  $\psi(0) \neq 0$ , and that 0 is not a pole of order  $k$  for  $\mathcal{J}^{-1}$ . Then  $M$  is not  $Q$ -rigid on  $A^{+\eta}$  for  $\eta > 0$ .

*Proof.* Since 0 is not a pole of order  $k$ , it means that there exists a polynomial  $P = \sum_{k'} b_{k'} u^{k'}$  with  $b_k \neq 0$  and for some  $\varepsilon > 0$

$$\int_{B(0, \varepsilon)} |P|^2 \mathcal{J}^{-1} < \infty.$$

If there is also a term  $b_{k'} \neq 0$  for some  $k' \prec k$ , (i.e.  $k' \preceq k, k' \neq k$ ), we have

$$\tilde{P} = u^{k-k'} P \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon)),$$

we can assume without loss of generality that there is no term strictly smaller than  $k$ . Let then

$$\tilde{\psi}(u) = \psi(u) J_d(u\eta / \deg(P))^{\deg(P)},$$

satisfying  $\tilde{\psi}(0) \neq 0$ . By Lemma 2,  $P\tilde{\psi} \in \mathcal{E}(B(0, A^{+\eta}))$  and  $|P\tilde{\psi}| \leq c|\psi|$ , hence also  $P\tilde{\psi} \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon)^c)$ . We also have  $P\tilde{\psi} \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon))$  because  $P\psi(u) \sim P(u)$  as  $u \rightarrow 0$ , hence  $P\tilde{\psi} \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A^{+\eta})$ .

Since  $Q$  is a  $k$ -polynomial,  $Q(u) = \sum_{m \preceq k_0} a_m u^m$  for some  $k_0 \succeq k$  with  $a_{k_0} \neq 0$ . Since 0 is also a pole of order  $k_0$  (because  $k_0 \succeq k$ ), the whole proof can be done with  $k_0$  instead of  $k$ , and we assume without loss of generality  $k = k_0$ .

By lemma 5, for some  $\gamma \in \mathcal{S}(\mathbb{R}^d)$  coinciding with  $Q$  on  $A^{+\eta}$ ,

$$\partial^Q(P\tilde{\psi})(0) = i^k \int \hat{\gamma} P\tilde{\psi},$$

hence by Theorem 7 the proof is complete if we prove that  $\partial^Q(P\tilde{\psi})(0) \neq 0$ , because then we have found an element of  $L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A^{+\eta})$  which is not orthogonal to  $\hat{\gamma}$  (and  $\gamma$ -rigidity on  $A^{+\eta}$  is equivalent to  $Q$ -rigidity on  $A^{+\eta}$ ).

We have

$$\partial^Q(P\tilde{\psi})(0) := \sum_{m \preceq k} \sum_{k' \neq k} b_{k'} a_m \partial^m(u^{k'} \tilde{\psi})(0) = a_k b_k \partial^k(u^k \tilde{\psi})(0) + \sum_{m \preceq k, k' \neq k} b_{k'} a_m \partial^m(u^{k'} \tilde{\psi})(0).$$

For  $k' \neq k, m \preceq k$ , we have  $k' \not\preceq m$ , hence the conclusion comes from the fact that  $\tilde{\psi}(0) \neq 0$  and the following identity: We have for  $k' \neq m$

$$\partial^m(u^{k'} \tilde{\psi})(0) = \delta_{m=k'} m! \tilde{\psi}(0)$$

with  $m! = m_1 \dots m_d$ . Let us finally prove this identity.

The assumption yields that we either have  $\mathbf{m} = \mathbf{k}'$ , or  $\mathbf{m}_i < \mathbf{k}'_i$  for some  $i$ . Then the Leibniz formula for functions of one argument yields, with the notation  $\mathbf{k}'_i = (\mathbf{k}'_1, \dots, \underbrace{0}_i, \dots, \mathbf{k}'_d)$ ,

$$u_{\hat{i}} = (u_1, \dots, \underbrace{0}_i, \dots, u_d)$$

$$\begin{aligned} \partial^{\mathbf{m}}(u^{\mathbf{k}'} \tilde{\psi}(u)) &= \left( \prod_{j \neq i} \partial_j^{\mathbf{m}_j} \right) \partial_i^{\mathbf{m}_i} (u_i^{\mathbf{k}'_i} u_{\hat{i}}^{\mathbf{k}'_i} \tilde{\psi}(u_i, u_{\hat{i}})) \\ &= \left( \prod_{j \neq i} \partial_j^{\mathbf{m}_j} \right) \left( \sum_{a > 0} \lambda_a u_i^a \tilde{\psi}_a(u) + 1_{\mathbf{k}'_i = \mathbf{m}_i} u_i^{\mathbf{k}'_i} \mathbf{m}_i! \tilde{\psi}(u_i, u_{\hat{i}}) \right) \text{ for some } \lambda_a \in \mathbb{R}, \text{ functions } \tilde{\psi}_a, \\ &= \sum_{a > 0} \lambda_a u_i^a \left( \prod_{j \neq i} \partial_j^{\mathbf{m}_j} \right) (\tilde{\psi}_a(u)) + 1_{\mathbf{k}'_i = \mathbf{m}_i} \mathbf{m}_i! \left( \prod_{j \neq i} \partial_j^{\mathbf{m}_j} \right) (u_i^{\mathbf{k}'_i} \tilde{\psi}(u)) \\ \partial^{\mathbf{m}}(u^{\mathbf{k}'} \tilde{\psi}(u))|_{u=0} &\begin{cases} = 0 \text{ if } \mathbf{m}_i < \mathbf{k}'_i \\ = \mathbf{m}_i! \tilde{\psi}(0) \text{ if } \mathbf{m} = \mathbf{k}' \text{ (with an induction on } j). \end{cases} \end{aligned}$$

□

#### 4.5.1 Proof of Proposition 2

Let  $u_1, \dots, u_m \in \mathbb{R}^d$  the poles of  $\mathcal{J}^{-1}$ , let  $\varepsilon > 0, q$  such that  $P(u) := \prod_i \|u - u_i\|^{2q}$  is in  $L^2(\mathcal{J}^{-1}, B(u_i, \varepsilon))$  for all  $i$ . Introduce

$$\psi(u) = P(u) J_d(u/\eta M)^M$$

with  $M = \frac{d+1}{2}(\deg(P) + p + d)$ . Lemma 2 (with  $1 \in \mathcal{E}(\{0\})$ ) yields  $\psi \in \mathcal{E}(B(0, \eta))$  and  $|\psi(u)| \leq c(1 + \|u\|)^{-p-d}$  ( $\psi$  is smooth hence bounded around 0). We have

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|\psi(u)|^2}{\mathcal{J}(u)} du &\leq c \int_{\mathbb{R}^d \setminus \cup_i B(u_i, \varepsilon)} |\psi(u)|^2 (1 + \|u\|)^p + \sum_i c_{i, \varepsilon} \int_{B(u_i, \varepsilon)} \frac{\|u - u_i\|^{4q}}{\mathcal{J}(u)} du \\ &\leq c \int (1 + \|u\|)^{-2d-2p} (1 + \|u\|)^p du + C \\ &< \infty, \end{aligned}$$

hence  $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(B(0, \eta)) \setminus \{0\}$ . Let  $\gamma = \widehat{\psi}$ , bounded and supported by  $B(0, \eta)$ .  $M$  is not  $\gamma$ -rigid on  $B(0, \eta)$  by Theorem 7 because with Parseval's identity

$$\int \tilde{\gamma} \widehat{\psi} = \int |\psi|^2 \neq 0.$$

In particular,  $M$  is not linearly maximally rigid.

The sufficiency part about  $\mathbf{k}$ -rigidity is Theorem 1. For the necessity, define instead

$$\tilde{\psi}(u) = \prod_{i: u_i \neq 0} \|u - u_i\|^{2q} J_d(u/\eta M)^M.$$

A similar reasoning shows that  $\tilde{\psi}$  is in  $L^2(\mathcal{J}^{-1}, \mathbb{R}^d \setminus B(0, \varepsilon)) \cap \mathcal{E}(B(0, \eta))$  for some  $\varepsilon > 0$  with  $\tilde{\psi}(0) \neq 0$ . We can then conclude with Lemma 6.

### 4.5.2 Proof of Proposition 3

The proof consists in showing that  $\mathbf{M}$  is  $\gamma$ -rigid for any  $\gamma \in \mathcal{S}(\mathbb{R}^d)$  taking value 1 on  $A$ , hence 0-rigid on  $A$ . For that we must prove by Theorem 7 that  $\int \hat{\gamma} \psi = 0$  for  $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A)$ . Since for some  $\varepsilon > 0$ , for  $u_1 \in [-\varepsilon, \varepsilon]$ ,

$$\int_{B((1+u_1, 0), \varepsilon)} \frac{1}{\mathcal{J}(u)} du \geq \int_{[1+u_1-\varepsilon, 1+u_1+\varepsilon] \times [-\varepsilon, \varepsilon]} cu_2^{-2} du_1 du_2 = \infty,$$

$\psi$  must vanish on  $\{(1 + u_1, 0); |u_1| \leq \varepsilon\}$ . Hence  $\lambda : v \mapsto \psi(v, 0)$  is an entire function vanishing on an interval. Therefore  $\lambda \equiv 0$ , in particular  $\psi(0) = 0$ . Lemma 5 then yields the conclusion.

## 4.6 Proof of Theorem 2

### Isotropic case

We prove that there is no  $Q$ -rigidity for some polynomial  $Q = \sum_{m \leq k_0} a_m u^m$  for some  $k_0 \succeq k$ . For  $k$ -rigidity, simply take  $Q(u) = u^k$ . Since  $\mathbf{M}$  is not LMR, there exists  $\psi_0 \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A) \setminus \{0\}$ . The proof relies on a self-contained lemma, which will be useful elsewhere.

**Lemma 7.** For  $\mathcal{J}$  isotropic, if there is  $\psi_0 \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(B(0, R)) \setminus \{0\}$ , then for  $\eta > 0$  there is  $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(B(0, R + \eta)) \setminus \{0\}$  that is isotropic.

*Proof.* Let  $P$  a polynomial such that  $\psi_0(u) \sim P(u)$  as  $u \rightarrow 0$  (Lemma 1), and

$$\psi_1(u) = P(u)\psi_0(u)J_d(u\eta/\deg(P))^{\deg(P)}.$$

By Lemma 2,  $\psi_1 \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A^{+\eta})$ .

Then define on  $\mathbb{C}^d$  the rotational average, using the analytic extension of  $\psi_1$  on  $\mathbb{C}^d$ ,

$$\psi(z) = \int_{O(d)} \psi_1(\theta z) d\theta, z \in \mathbb{C}^d$$

with the Haar measure on the orthogonal group  $O(d)$  of  $\mathbb{R}^d$ . Still denote by  $\psi$  its restriction to  $\mathbb{R}^d$ , which is in particular isotropic. By definition,  $\psi_1(u) \sim P(u)^2$  as  $u \rightarrow 0$  and  $P(u)^2 \geq 0$  for  $u \in \mathbb{R}^d$ , hence  $\psi$  does not vanish identically in the neighbourhood of 0, so that  $\psi \not\equiv 0$ . We have with the triangle inequality and the Cauchy-Schwarz inequality in  $L^2(\mathcal{J}^{-1})$

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|\psi(u)|^2}{\mathcal{J}(u)} du &\leq \int_{\mathbb{R}^d} \int_{O(d) \times O(d)} \frac{|\psi_1(\theta u)| |\psi_1(\theta' u)|}{\mathcal{J}(u)} d\theta d\theta' du \\ &\leq \int_{O(d) \times O(d)} \sqrt{\int_{\mathbb{R}^d} \frac{|\psi_1(\theta u)|^2}{\mathcal{J}(u)} du} \sqrt{\int_{\mathbb{R}^d} \frac{|\psi_1(\theta' u)|^2}{\mathcal{J}(u)} du} dud\theta d\theta' \\ &\leq \int_{O(d) \times O(d)} \left( \sqrt{\int_{\mathbb{R}^d} \frac{|\psi_1(u)|^2}{\mathcal{J}(u)} du} \right)^2 d\theta d\theta' < \infty \end{aligned}$$

by isotropy of  $\mathcal{J}$ , using  $\psi_1 \in L^2(\mathcal{J}^{-1})$ .

For  $z, \zeta \in \mathbb{C}^d$ , since  $\psi_1$  is analytic on  $\mathbb{C}^d$ ,

$$\psi(z + \zeta) = \int_{O(d)} [\psi_1(\theta z) + \psi_1'(\theta z)\zeta + O(\zeta)] d\theta = \psi(z) + \left( \int \psi_1'(\theta z) d\theta \right) \zeta + O(\zeta),$$

using that the quantities involved in the  $O(\cdot)$  are locally bounded; hence  $\psi$  is analytic as well.

Also, for  $z \in \mathbb{C}^d, \theta \in O(d)$ ,

$$\|\theta z\|_{\mathbb{C}}^2 = \|\theta \mathcal{R}z\|^2 + \|\theta \mathcal{I}z\|^2 = \|z\|_{\mathbb{C}}^2.$$

Using both implications of Theorem 6 on  $A^{+\eta} = B(0, R + \eta)$ ,

$$|\psi(z)| \leq c \int_{O(d)} \exp((R + \eta)\|z\|_{\mathbb{C}}) d\theta$$

and  $\text{sp}(\psi) \subset B(0, R + \eta)$ . Finally,  $\psi$  is an isotropic element of  $L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(B(0, R + \eta)) \setminus \{0\}$ .  $\square$

Hence non isotropic terms vanish and for some  $q \in \mathbb{N}, \kappa \neq 0$ ,  $\psi(u) = \kappa \|u\|^{2q} (\sum_{l>0} a_l \|u\|^{2l})$ , where  $\|u\|^{2l} = (\sum_i u_i^2)^l$ . Hence  $\tilde{\psi} := \psi(u) \|u\|^{-2q}$  is analytic and isotropic as well, with spectrum in  $B(0, R + \eta)$  with Theorem 6, with  $\tilde{\psi}(0) \neq 0$  (which was the whole point). We then can use directly Lemma 6 to conclude that  $\mathbf{M}$  is not  $Q$ -rigid on  $B(0, R + \eta)$ .

**Separable case.** We assume that  $\mathcal{J}$  is not (linearly) maximally rigid, hence there is  $\psi \in L^2(\mathcal{J}^{-1}) \cap \mathcal{E}(A) \setminus \{0\}$ . Let us build a separable version. Recall that, as analytic function of  $\mathcal{E}(A)$ ,  $\psi$  has an extension on  $\mathbb{C}^d$ . Define for  $1 \leq i \leq d, z^i \in \mathbb{C}^{d-1}$ ,

$$\psi_i^{z^i}(z_i) = \psi(z_i; z^i), z_i \in \mathbb{C},$$

where  $(z_i; z^i)$  consists in  $z_i$  at the  $i$ -th position, surrounded by the  $d - 1$  components of  $z^i$ . Let the domain  $D \subset \mathbb{R}^{d-1}$  of  $u^i$  where  $\psi_i^{u^i}$  is not identically 0. We have

$$0 < \int \frac{|\psi|^2}{\mathcal{J}} = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{|\psi_i^{u^i}(u_i)|^2}{\mathcal{J}_i(u_i)} \frac{1}{\prod_{j \neq i} \mathcal{J}_j(u_j)} du^i du_i = \int_D \int_{\mathbb{R}} \frac{|\psi_i^{u^i}(u_i)|^2}{\mathcal{J}_i(u_i)} \frac{1}{\prod_{j \neq i} \mathcal{J}_j(u_j)} du^i du_i < \infty$$

hence there exists  $u^i \in D$  such that  $\psi_i^{u^i} \in L^2(\mathcal{J}_i^{-1}) \setminus \{0\}$ . Clearly  $\psi_i^{u^i}$  is analytic as analyticity in several complex variables implies analyticity in each variable. Let  $q_i \in \mathbb{N}, \kappa_i \neq 0$  the dominating power and coefficient in  $0 : \psi_i^{u^i}(z) \sim \kappa_i z^{q_i}$  as  $z \rightarrow 0$ . Define  $\psi_i(z) := z^{-q_i} \psi_i^{u^i}(z), z \in \mathbb{C}$ , still analytic. Define finally

$$\tilde{\psi}(z) = \prod_i \psi_i(z_i), z = (z_i) \in \mathbb{R}^d.$$

Let  $\varepsilon > 0$ . We have for some finite  $\kappa > 0$ ,  $\psi_i(z) \leq \kappa$  if  $\|z\|_{\mathbb{C}} \leq \varepsilon$ . Since for each  $i$ ,  $\psi_i^{u^i} \in L^2(\mathcal{J}_i^{-1})$ , we have  $\tilde{\psi} \in L^2(\mathcal{J}^{-1}, B(0, \varepsilon)^c)$  (as functions on  $\mathbb{R}^d$ ).

Let  $A_R = [-R, R]^d$ . It remains to show  $\psi \in \mathcal{E}(A_R)$ . We have

$$s_{A_R}(\zeta) = R \sup_{x: |x_i| \leq 1} \sum_i x_i \zeta_i = R \sum_i \text{sign}(\zeta_i) \zeta_i = R \sum_i |\zeta_i|, \zeta \in \mathbb{R}^d.$$

Since  $\psi \in \mathcal{E}(A_R)$ , for  $z \in \mathbb{C}$ ,

$$\begin{aligned} |\psi_i^{u^i}(z)| &= |\psi(z; u^i)| \leq C \exp(s_{A_R}(\mathcal{J}(z; u^i))) \\ &\leq C \exp(Rc_{u^i} + R|\mathcal{J}z|) \\ &\leq c'_{u^i} \exp(R|\mathcal{J}z|). \end{aligned}$$

Furthermore,  $\tilde{\psi}$  is analytic, satisfies  $\tilde{\psi}(0) \neq 0$  and for  $\|z\|_{\mathbb{C}} > \varepsilon$ ,

$$|\tilde{\psi}(z)| \leq \prod_i c'_i \exp(R|\mathcal{J}z_i|) = c' \exp(R \sum_i |\mathcal{J}z_i|) = c' \exp(s_{A_R}(\mathcal{J}z)).$$

Theorem 6 again yields that  $\tilde{\psi} \in \mathcal{E}(A_R)$ . Then one can conclude with Lemma 6 that we do not have  $Q$ -rigidity on  $A_{R+\eta}$ .

## 4.7 Structure factor of cluster lattices

*Proof of Proposition 5.* The strategy is to show that

$$(2\pi)^{-d} \int |\hat{\gamma}|^2 d\mathbf{S} = \text{Var}(I_{\Phi_\mu}(\gamma))$$

for  $\gamma$  the indicator function of a bounded measurable set. Since  $\mathbf{S}$  is symmetric, we assume also that test functions  $\gamma$  are symmetric. The conclusion then comes from formula (4).

Denote by  $\nu_1$  and  $\nu_2$  the (finite) first and second order intensity measures of  $\mathbf{c}_0$ , so that in particular, Fubini's theorem yields

$$\mathbf{E}\varphi(u) = \int e^{-i\langle u, x \rangle} \nu_1(dx), \quad \mathbf{E}|\varphi(u)|^2 = \int e^{-i\langle u, x-y \rangle} \nu_2(dx, dy) + \kappa.$$

We perform Fourier transform on tempered measures (see Section 4), and recall in particular that

$$\mathcal{F}(\gamma \star \nu_1) = \hat{\gamma} \mathcal{F}\nu_1 \tag{12}$$

is valid because  $\gamma$  has compact support and  $\nu_1$  is a non-negative finite measure.  $\mathcal{F}\nu_1(u) = \mathbf{E}\varphi(u)$  is actually a bounded function, hence  $\hat{\gamma} \mathcal{F}\nu_1$  and  $\gamma \star \nu_1$  are  $L^2$  functions.

We use the conditional variance formula :

$$\begin{aligned} \text{Var}(I_{\Phi_\mu}(\gamma)) &= \mathbf{E} \left( \text{Var} \left( \sum_{\mathbf{m} \in \Phi} \sum_{x \in \mathbf{c}_\mathbf{m}} \gamma(\mathbf{m} + x) \middle| \Phi \right) \right) + \text{Var} \left( \mathbf{E} \left( \sum_{\mathbf{m} \in \Phi} \sum_{x \in \mathbf{c}_\mathbf{m}} \gamma(\mathbf{m} + x) \middle| \Phi \right) \right) \\ &= \mathbf{E} \left( \sum_{\mathbf{m} \in \Phi} \text{Var} \left( \sum_{x \in \mathbf{c}_\mathbf{m}} \gamma(\mathbf{m} + x) \middle| \Phi \right) \right) \text{ by independence of the } \mathbf{c}_\mathbf{m} \\ &\quad + \text{Var} \left( \sum_{\mathbf{m} \in \Phi} \int \gamma(\mathbf{m} + x) \nu_1(dx) \right) \text{ by definition of } \nu_1 \\ &= \int \text{Var} \left( \sum_{x \in \mathbf{c}_0} \gamma(y + x) \right) dy \text{ because } \Phi \text{ has intensity } \mathcal{L} \end{aligned}$$

$$\begin{aligned}
& + (2\pi)^{-d} \int \mathbf{S}_\Phi(du) |\widehat{\gamma \star \nu_1}(u)|^2 \text{ by definition of } \mathbf{S}_\Phi \text{ and (4)} \\
& = \int_{\mathbb{R}^d} \left[ \mathbf{E} \left( \sum_{x \in \mathcal{C}_0} \gamma(y+x) \right)^2 - \left( \int \tau_y \gamma \nu_1 \right)^2 \right] dy + (2\pi)^{-d} \int \mathbf{S}_\Phi(du) |\hat{\gamma} \mathcal{F} \nu_1(u)|^2 \text{ by (12)} \\
& = \iint \gamma(y+x)^2 \nu_1(dx) dy + \int \gamma(x+y) \gamma(z+y) \nu_2(dx, dz) dy - \int \gamma \star \nu_1(y)^2 dy \\
& \quad + (2\pi)^{-d} \int \mathbf{S}_\Phi(du) |\hat{\gamma}(u)|^2 |\mathbf{E}\varphi(u)|^2 \\
& = \int \gamma^2 \int \nu_1 + \int \gamma \star \gamma(x-z) \nu_2(dx, dz) + (2\pi)^{-d} \int \mathbf{S}_\Phi(du) |\hat{\gamma}(u)|^2 |\mathbf{E}\varphi(u)|^2 \\
& \quad - (2\pi)^{-d} \int |\widehat{\gamma \star \nu_1}(u)|^2 du \text{ with Plancherel formula in } L^2(\mathbb{R}^d) \\
& = \kappa \int \gamma^2 + \int \left( \int (2\pi)^{-d} \widehat{\gamma \star \gamma}(u) e^{i\langle u, x-z \rangle} du \right) \nu_2(dx, dz) \text{ by Fourier inversion in } L^2(\mathbb{R}^d) \\
& \quad + (2\pi)^{-d} \int \mathbf{S}_\Phi(du) |\hat{\gamma}(u)|^2 |\mathbf{E}\varphi(u)|^2 - (2\pi)^{-d} \int |\hat{\gamma}(u)|^2 |\mathcal{F} \nu_1(u)|^2 du, \\
& = (2\pi)^{-d} \kappa \int \gamma^2 + (2\pi)^{-d} \int |\hat{\gamma}(u)|^2 (|\mathbf{E}\varphi(u)|^2 - \kappa) du \\
& \quad + (2\pi)^{-d} \int \mathbf{S}_\Phi(du) |\hat{\gamma}(u)|^2 |\mathbf{E}\varphi(u)|^2 - (2\pi)^{-d} \int |\hat{\gamma}(u)|^2 |\mathbf{E}\varphi(u)|^2 du \\
& = (2\pi)^{-d} \left[ \int |\hat{\gamma}(u)|^2 (|\mathbf{E}\varphi(u)|^2 - |\mathbf{E}\varphi(u)|^2) du + \int \mathbf{S}_\Phi(du) |\hat{\gamma}(u)|^2 |\mathbf{E}\varphi(u)|^2 \right].
\end{aligned}$$

Hence we have indeed  $\mathbf{S} = (|\mathbf{E}\varphi(u)|^2 - |\mathbf{E}\varphi(u)|^2)\mathcal{L} + |\mathbf{E}\varphi(u)|^2\mathbf{S}_\Phi$ .

□

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