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The Harmonious Coloring Game*

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Abstract

A harmonious k -coloring of a graph G is a 2-distance proper k -coloring of its vertices such that each edge is uniquely identified by the colors of its endpoints. Here, we introduce its game version: the harmonious coloring game. In this two-player game, Alice and Bob alternately select an uncolored vertex and assigns to it a color in $\{1, \dots, k\}$ with the constraint that, at every turn, the set of colored vertices induces a valid partial harmonious coloring. Alice wins if all vertices are colored; otherwise, Bob wins. The harmonious game chromatic number $\chi_{hg}(G)$ is the minimum integer k such that Alice has a winning strategy with k colors. In this paper, we prove the PSPACE-hardness of three variants of this game. As a by-product, we prove that a variant introduced by Chen et al. in 1997 of the classical graph coloring game is PSPACE-hard. We also obtain lower and upper bounds for $\chi_{hg}(G)$ in graph classes, such as paths, cycles, grids and forests of stars.

1 Introduction

A k -coloring of a graph G is a function c which associates to each vertex v a “color” $c(v) \in \{1, \dots, k\}$ in such a way that adjacent vertices have distinct colors. The *chromatic number* $\chi(G)$ is the minimum k such that G has a k -coloring. The *graph coloring game* is the game version of the coloring problem: given a graph G and a positive integer k , two players Alice and Bob alternate turns, starting with Alice, in selecting a non-colored vertex to be colored by a “color” in $\{1, \dots, k\}$ not already used in one of its neighbors. Alice wins if every vertex is successfully colored at the end; otherwise, Bob wins. The *game chromatic number* $\chi_g(G)$ is the minimum k such that Alice has a winning strategy. Clearly, $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$ where $\Delta(G)$ is the maximum degree of G .

The graph coloring game was introduced by Brams in the context of coloring maps and was described by Gardner in 1981 in his “*Mathematical Games*” column of Scientific American [13]. It remained unnoticed until Bodlaender [2] reinvented it in 1991 as the “Coloring Construction Game”, and its computational complexity was left as “*an interesting open problem*”.

After this, many papers were published investigating the graph coloring game and the game chromatic number [3, 6, 10, 17]. Only in 2020, it was finally proved that the graph coloring game (deciding if $\chi_g(G) \leq k$) is PSPACE-complete [5]. Thereafter, many games based on classical variants of the coloring problem were proved PSPACE-complete, such as the greedy coloring game [19] and the connected coloring game [20].

In this paper, we propose the game version of the harmonious coloring of a graph. A *harmonious k -coloring* of G is a k -coloring c of G such that $\{c(u), c(v)\} \neq \{c(x), c(y)\}$ for every two edges uv and xy of G [12]. That is, each edge is uniquely determined by the colors of its endpoints. The *harmonious chromatic number* $\chi_h(G)$ is the smallest k such that G admits a harmonious k -coloring. The computational complexity of determining the harmonious chromatic number was considered in [9] where it has been shown that the problem is already NP-complete in the class of forest of stars. General

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upper bounds on this parameter were provided in [8] and particular graph classes were considered such as paths, cycles [21, 14], bounded degree trees [7], etc. (e.g., see the recent work [11]).

In order to define the harmonious coloring game, we observe that a harmonious coloring is not only a proper coloring, but also a 2-distance proper coloring. That is, vertices at distance at most two must receive distinct colors. For example, in the path P_3 with vertices v_1, v_2, v_3 , the vertices v_1 and v_3 must receive distinct colors, independently of the color of v_2 , since otherwise the edges v_1v_2 and v_2v_3 will have the same colors in their endpoints.

We then say that a *partial coloring* of a graph is *harmonious* if there is no pair of colored vertices at distance at most two with the same color¹ and there is no pair of edges whose endpoints are colored and the sets of colors of the endpoints are equal. With this, we can define the *harmonious coloring game* (HCG). Given a graph G and a positive integer k , Alice and Bob (starting with Alice) alternately colors uncolored vertices of G in such a way that the partial coloring obtained in each turn is harmonious.

In other words, at each turn, the current player chooses an uncolored vertex v and assigns to it a color $c(v) \in \{1, \dots, k\}$ such that: (a) there is no vertex colored $c(v)$ at distance at most two from v , and (b) there is no edge whose endpoints are already colored with colors $c(v)$ and $c(v')$ for every colored neighbor v' of v .

Alice wins if every vertex is successfully colored at the end; otherwise, Bob wins. The *harmonious game chromatic number* $\chi_{hg}(G)$ is the least k such that Alice wins the harmonious coloring game in G with k colors. Clearly, $\chi(G) \leq \chi_h(G) \leq \chi_{hg}(G)$.

Our Contributions. We start by focusing in Section 2 on the computational complexity of the harmonious coloring game, that is, the problem of deciding if $\chi_{hg}(G) \leq k$. Although we were unable to prove that the original problem is PSPACE-complete, we were able to prove that three variants are. The first variant is related to a variant of the graph coloring game, named as *sequential coloring construction game* by Bodlaender [2] in 1981: the vertices must be colored following a predefined given sequence, with players only being able to choose the color to be used. Bodlaender proved that this variant is PSPACE-complete [2] and we use this fact to prove in Section 2 that the *sequential harmonious coloring game* is also PSPACE-complete even for split graphs.

The second variant considers that some vertices of the input graph are pre-colored before the beginning of the game. We then prove that the *pre-colored harmonious coloring game* is PSPACE-complete even for split graphs in which the pre-colored vertices induce a maximum clique. The third variant is related to another variant of the graph coloring game, named as *Chromatic Game II* by Chen [4] in 1997, in which Bob can color a vertex v with a new color (that was not used previously) only if v cannot be colored with an already used color. Although this game was introduced almost thirty years ago, to the best of our knowledge, it was not proved PSPACE-hard before. We then prove that the Chromatic Game II is PSPACE-complete and use this to prove that the corresponding variant of the harmonious coloring game is also PSPACE-complete.

In Section 3, we obtain lower and upper bounds for general graphs. In Section 4, we obtain for paths and cycles the lower bound $4 \lfloor \frac{n}{12} \rfloor + 1$ and the upper bound $\frac{2(n+3)}{3}$ for the harmonious game chromatic number. In Section 5, we use our results in paths to obtain lower and upper bounds for the Cartesian grid, the triangular grid and the king grid. Finally, motivated by the fact that computing the harmonious chromatic number of forest of stars is NP-complete [9], we give, in Section 6, lower and upper bounds for the harmonious game chromatic number in this graph class. We conclude with some open problems.

2 PSPACE-hardness of three variants

2.1 Sequential variant and Pre-colored variant

When Bodlaender first introduced the graph coloring game, he also defined a sequential version of the problem that he called *Sequential Coloring Construction Game* [2]. In this variant, the instance is a graph G , a positive integer k and an ordering (v_1, v_2, \dots, v_n) of the vertices of G . Alice and Bob alternate color the vertices of G in such a way that, on turn i , the current player (Alice if i is odd and

¹The reason for adding this constraint is that, without it, a simple winning strategy for Bob (in any connected graph with diameter at least two) would be to color a vertex at distance two from the first vertex colored by Alice, with the same color.

Bob if i is even) must color the vertex v_i with a color in $\{1, \dots, k\}$ distinct from the ones appearing on its neighborhood. As usual, Alice wins if all the vertices of G are colored, and Bob wins otherwise. The goal is to decide whether Alice has a winning strategy. Bodlaender [2] proved that this problem is PSPACE-complete even for $k = 3$.

Here we introduce the *sequential harmonious coloring game* similarly except that the coloring constructed by Alice and Bob must always induce a harmonious partial coloring of G . The next theorem proves PSPACE-hardness even for split graphs, which are graphs whose vertex set has a partition (K, S) , where K is a maximum clique and S is an independent set.

Theorem 1. *Deciding if Alice wins the sequential harmonious coloring game is PSPACE-complete even when the input graph is a split graph with partition (K, S) and $k = |K| + 3$.*

Proof. We obtain a reduction from the sequential coloring construction game, which is PSPACE-complete [2]. Given a graph $G = (V, E)$, let \widehat{G} be the graph obtained from G by subdividing each edge exactly once and adding an edge between each pair of new vertices. Note that \widehat{G} is a split graph with partition $(K_{|E|}, V)$, such that each vertex of the clique is adjacent to exactly 2 vertices of the independent set. This construction was proposed previously in [1] to prove that deciding if $\chi_h(G) \leq k$ is NP-complete for split graphs.

Let \mathcal{I} be an instance of the sequential coloring construction game in a graph $G = (V, E)$ with ordering $(v_1, v_2, v_3, \dots, v_n)$ and $k = 3$ colors. We assume that G has an even number of edges. If this is not the case, one can add an isolated edge to G and append both of its endpoints to the end of the ordering without changing the outcome of \mathcal{I} .

We claim that Alice has a winning strategy for the sequential harmonious coloring game on \widehat{G} with ordering $(e_1, e_2, e_3, \dots, e_{|E|}, v_1, v_2, v_3, \dots, v_n)$ with $|E| + 3$ colors if and only if she also has a winning strategy on \mathcal{I} , where $e_1, e_2, e_3, \dots, e_{|E|}$ is any ordering of the vertices in the clique $K_{|E|}$.

Suppose that Alice has a winning strategy on \mathcal{I} . When playing the sequential harmonious coloring game on \widehat{G} , for the first $|E|$ turns each player has to color his\her vertex using a new color, since $\{e_1, \dots, e_{|E|}\}$ is a clique in \widehat{G} . In turn $|E| + 1$, Alice starts coloring v_1 (recall that E is even) and use the same strategy of the sequential coloring construction game to color the remaining vertices with the 3 remaining colors.

That is possible because, given that the clique is already completely colored, no vertex v_i can receive a color already used in any vertex e_j of the clique, since they are at distance at most 2. Furthermore, if a pair of vertices v_i and v_j are adjacent in G , they share a common neighbor in \widehat{G} , namely the vertex resulting of the subdivision of the edge with endpoints v_i and v_j , and must receive distinct colors on a harmonious coloring of \widehat{G} . Since these are the only restrictions when coloring the vertices of the independent set of \widehat{G} , Alice and Bob can make exactly the same moves as if they were playing on \mathcal{I} .

The same arguments of the last two paragraphs can be applied in the case when Bob has a winning strategy on \mathcal{I} . Thus, in that case, Bob also wins when playing the sequential harmonious coloring game on \widehat{G} with ordering $(e_1, e_2, e_3, \dots, e_{|E|}, v_1, v_2, v_3, \dots, v_n)$ with $|E| + 3$ colors. \square

In the proof of the last theorem, note that it is not necessary to have an ordering among the vertices of the clique. In fact, since the coloring game, without the ordering, has been proven PSPACE-complete [5], we could use the same arguments, just requiring that the vertices of the clique are pre-colored at the beginning of the game, to prove that the harmonious coloring game is also PSPACE-complete when some vertices are pre-colored.

Theorem 2. *Given a split graph G with a pre-coloring of the vertices of its maximum clique and a positive integer k , deciding if Alice has a winning strategy for the harmonious coloring game on G with k colors is PSPACE-complete.*

2.2 Bob restricted variant

In 1997, Chen et al. [4] introduced a new variation of the classical graph coloring game, that they called *Chromatic Game II*. This version is played exactly the same way except that, in each turn, Bob can color a vertex v with a new color (that was not used previously) only if v cannot be colored with an already used color. The minimum number of colors for which Alice has a winning strategy on a graph G in this variation of the game is denoted by $\chi_g^*(G)$. Clearly $\chi(G) \leq \chi_g^*(G) \leq \chi_g(G)$ since Bob has more constraints.

Although this variant was introduced almost thirty years ago, to the best of our knowledge, it was not proved PSPACE-hard before. The next theorem obtains this result and later we use this fact to prove the PSPACE-hardness of the corresponding variant of the harmonious coloring game.

Theorem 3. *The Chromatic Game II is PSPACE-complete. That is, given a graph G and an integer k , deciding if $\chi_g^*(G) \leq k$ is PSPACE-complete.*

Proof. We obtain a reduction from the graph coloring game, which was proved PSPACE-complete in [5]. Let (G, k) be an instance of the graph coloring game and let G' be the graph obtained from G by adding, for every $1 \leq i \leq k$, an induced P_4 $Q_i = (a_i, b_i, c_i, d_i)$ such that b_i and d_i are adjacent to all vertices of G and every two vertices in distinct P_4 's Q_i and Q_j are adjacent. We prove that $\chi_g(G) = k$ if and only if $\chi_g^*(G') = 2k$. That is, Alice has a winning strategy in the graph coloring game with k colors in G if and only if she also has in the Chromatic Game II with $2k$ colors in G' .

Note that if a_i and d_i have the same color for some i at any moment of the Chromatic Game II in G' , Alice has already lost the game, since Q_i will receive at least 3 distinct colors and any other P_4 Q_j will have at least 2 distinct colors each. Therefore, at least $2k + 1$ colors will be needed to finish the coloring.

Suppose that Alice wins the graph coloring game on G using k colors. Let us describe a winning strategy for her on G' on the Chromatic Game II with $2k$ colors. Alice starts coloring a vertex of G using her winning strategy (for the graph coloring game). From now on, each time Bob colors a vertex of Q_i , Alice colors the closest non-neighbor of it in Q_i with the same color. Otherwise, if Bob colors a vertex of G , Alice proceeds playing according to her winning strategy in G . From this, Alice assures that each Q_i is colored with exactly 2 colors and one of those colors are available to be used on the vertices of G . Since Alice wins the graph coloring game in G with k colors, she can complete the coloring of the vertices of G in G' by using her winning strategy.

Now suppose that Bob wins the graph coloring game on G using k colors. The main idea of Bob's strategy to win on G' is to ensure that (except possibly for her first or second move) Alice will always be forced to use a new color in the P_4 's Q_i so that, after her $(k + 1)$ -th move, k distinct colors are used on a_i or c_i for every $1 \leq i \leq k$, and those colors are available for Bob to play in G .

If Alice first colors a_i (resp. d_i), Bob colors d_i (resp. a_i) with the same color and wins. If Alice first colors c_i , then Bob colors d_i , forcing Alice to color a_i or b_i , since otherwise Bob can color a_i with the same color of d_i , winning the game. If Alice first colors b_i , then Bob colors a_i , forcing Alice to color d_i with the color of b_i or to color c_i or a vertex of G with the color of a_i , since otherwise Bob can color d_i with the same color of a_i , winning the game. If Alice's first move is to color some vertex of G , then Bob colors a_i , forcing Alice to color c_i or d_i on her next move.

Thus, from Bob's second move until his k -th move, he colors for some Q_j with no colored vertices the vertex a_j with a new color, since a_j is adjacent to all colors used previously. From this, he forces Alice to color either c_j or d_j .

After Alice's $(k + 1)$ -th turn, a_j or d_j is colored for every $1 \leq j \leq k$, and therefore there are exactly k colors already used in the game, which are now available to color the vertices of G . Furthermore either (i) Alice has colored a vertex of G on her first two moves and there is exactly 2 uncolored vertices in each Q_j or (ii) she did not color any vertex in G and there is one P_4 with exactly one uncolored vertex while all the others have exactly two.

After that, if (i) Alice has colored a vertex of G on her first or second move, then Bob follows his winning strategy on G , since all k colors are now available to him. If Alice plays on a vertex outside G at some turn, Bob colors any uncolored vertex outside of G , which is possible since the number of uncolored vertices outside G is always even after Alice's turn.

If (ii) Alice has colored no vertex in G after her $(k + 1)$ -th turn, Bob colors an uncolored vertex in some Q_j , until Alice finally colors a vertex in G , which is possible since the number of uncolored vertices outside of G is odd after Alice's turn. After Alice colors a vertex in G , Bob just use the same strategy of the previous paragraph.

Consequently, Alice has a winning strategy for the graph coloring game with k colors in G if and only if $\chi^*(G') = 2k$. Since deciding if Alice has a winning strategy with k colors in G is PSPACE-complete [5], we are done. \square

Motivated by this variant by Chen et al. [4] of the graph coloring game, we introduce the *Restricted Harmonious Coloring Game* (HCG*). This variant of the game plays exactly as the harmonious

coloring game except that Bob can color a vertex v with a new color (that was not used previously) only if v cannot be colored with an already used color. The minimum number of colors for which Alice has a winning strategy on a graph G on HCG* is denoted by $\chi_{hg}^*(G)$. Clearly, $\chi_h(G) \leq \chi_{hg}^*(G) \leq \chi_{hg}(G)$. We remark that the results of the upcoming Lemma 2 and Theorems 5, 6 and 7 also hold for $\chi_{hg}^*(G)$, since Bob's strategy to achieve the lower bounds is to use the same color repeatedly.

We use Theorem 3 to prove that this version of the harmonious coloring game is also PSPACE-complete even for split graphs.

Theorem 4. *The Restricted Harmonious Coloring Game HCG* is PSPACE-complete even in split graphs. That is, given a split graph G and an integer k , deciding if $\chi_{hg}^*(G) \leq k$ is PSPACE-complete.*

Proof. We obtain a reduction from the Chromatic Game II, which was proved PSPACE-hard in Theorem 3. Let $G = (V, E)$ and k be an instance of Chromatic Game II. Let \widehat{G} be the split graph obtained from G as in the proof of Theorem 1 (subdividing each edge exactly once and adding an edge between each pair of new vertices). Again, we can assume that $|E|$ is even. We prove that Alice has a winning strategy in the Chromatic Game II on G with k colors if and only if she also has in HCG* on \widehat{G} with $k + |E|$ colors.

Let C be the clique of \widehat{G} with the new vertices. Notice that every color of a vertex v of C is used only once, since every vertex of \widehat{G} is at distance at most 2 from v .

If a vertex $v \in V$ is colored, then its color cannot be used in any vertex of the clique C and any $u \in N_G(v)$ since v and u have a common neighbor in C (the new vertex corresponding to their edge). Therefore, the partial coloring after each move must induce a coloring of G .

Furthermore, if Bob can color a vertex $v \in V$ with a new color c , it means that for every color a already used on a vertex of V there is an edge with endpoints colored a and b for some color b appearing in the neighborhood of v in \widehat{G} . Since the neighborhood of v is composed only by the vertices of the clique, which have a unique color each, that means that for every color a already used in the vertices of V there is a vertex $u \in N_G(v)$ with color a . Therefore Bob is restricted when he colors a vertex of V in the HCG* on \widehat{G} in the exact same fashion he is on the Chromatic Game II on G .

If Alice has a winning strategy for the Chromatic Game II on G with k colors, then, to win in HCG* on \widehat{G} with $k + |E|$ colors, she always colors a vertex of G following her winning strategy on it, unless Bob just colored a vertex of the clique C on his last turn, which, in this case, Alice just colors another vertex of the clique. Consequently, Alice never allows Bob to “pass” his turn in G .

The case in which Bob has a winning strategy for the Chromatic Game II on G with k colors is analogous to the one in the previous paragraph, except that it is Bob that uses his winning strategy to color the vertices of V . \square

3 General upper bounds

Fact. $\chi_{hg}(G) \leq n$ for every graph G with n vertices.

Proof. At her turn, Alice may always use the smallest unused color (i.e., a new color). \square

We say that Alice plays *greedily* if, at her turn, she can choose an arbitrary uncolored vertex and colors it with any possible (i.e., satisfying the constraints of the game) color.

We will often use the following easy result for the lower bounds.

Lemma 1. *If, at the end of a game, there exists a monochromatic set X of vertices and all vertices in $N(X)$ have been colored, then at least $|N(X)| + 1$ colors have been used.*

Next we obtain an upper bound on χ_{hg} in general graphs with bounded maximum degree. This result will be used later in particular graph classes.

Lemma 2. *Let $y \geq \max \left\{ \frac{\Delta \cdot (n + \Delta + 1)}{\Delta + 1}, \Delta \cdot (\Delta - 1) + 1 \right\}$. Then Alice greedily wins the harmonious coloring game with y colors in a graph G with n vertices and maximum degree Δ .*

Proof. Alice plays greedily until she cannot play anymore. We show that actually all vertices have been colored. By contradiction, suppose that Alice loses. Let Y be the set of colored vertices after the end of the game. Then $|Y| < n$, since some vertices remain uncolored and cannot receive any available color.

Let v be an uncolored vertex. If there are no colored vertices in $N(v)$, then the set of colors forbidden to v are exactly the ones used at distance 2 from v . Since $y \geq \Delta(\Delta - 1) + 1$, there's at least one color available for v , a contradiction.

Otherwise, let $C \neq \emptyset$ be the set of colors received by the neighbors of v . Let $X \subseteq V(G)$ be the set of all vertices colored with a color in C . Since v cannot be colored, for each color $c \notin C$, there must be a vertex colored with c adjacent to a vertex of X . Note that there are at least $y - |X| \geq y - \Delta$ such colors c and so $|N(X)| \geq y - \Delta$. Since the vertices in X have degree at most Δ , then $|X| \geq \frac{y - \Delta}{\Delta}$. Furthermore, each color not appearing on $N(v)$ was used at least once. Hence, we obtain a contradiction:

$$|Y| \geq |N(X)| + |X| \geq y - \Delta + \frac{y - \Delta}{\Delta} = \frac{(\Delta + 1)(y - \Delta)}{\Delta} \geq \frac{(\Delta + 1)(\frac{\Delta(n + \Delta + 1)}{\Delta + 1} - \Delta)}{\Delta} = n.$$

□

Corollary 1. *Let G be any n -node graph with maximum degree Δ , then*

$$\chi_{hg}(G) \leq \max \left\{ \frac{(n + \Delta + 1)\Delta}{\Delta + 1}, (\Delta - 1)\Delta + 1 \right\}.$$

4 Paths and cycles

Let P_n and C_n be the path and the cycle with n vertices, respectively.

Lemma 3. $\chi_{hg}(P_n) \geq 4 \lfloor \frac{n}{12} \rfloor + 1$

Proof. The strategy for Bob is to create a color class with size at least $2 \lfloor \frac{n}{12} \rfloor$. Indeed, assume that Bob manages to create a monochromatic set X of size at least $2 \lfloor \frac{n}{12} \rfloor$. By Lemma 1, at least $|N(X)| + 1$ colors are used. Below, we show how Bob creates such a set X with all vertices with degree two, leading to the use of at least $4 \lfloor \frac{n}{12} \rfloor + 1$ colors.

Let $P_n = (u_1, u_2, \dots, u_n)$. Assume, for simplicity's sake, that n is divisible by 12 and let $P^i = (v_1^i, v_2^i, \dots, v_{12}^i)$ be the i -th subpath of order 12 of P_n , with $1 \leq i \leq \frac{n}{12}$, that is $v_k^i = u_{12(i-1)+k}$. Bob's strategy is the following (note that Bob colors a vertex of some subpath P^i if and only if Alice has just colored a vertex of P^i):

- i) If Alice plays on P^i for the first time on some vertex v_j^i with color $c \neq 1$, then Bob colors v_{j-2}^i with color 1 if he can do so and $j - 2 > 1$. Otherwise, Bob colors v_{j+2}^i with color 1.
- ii) If Alice plays on P^i for the first time on some vertex v_j^i with color $c = 1$, then Bob colors v_{j-3}^i with color 1 if he can do so and $j - 3 > 1$. Otherwise, Bob colors v_{j+3}^i with color 1.
- iii) If Alice plays on some P^i that already has colored vertices, then Bob colors any vertex $v \in V(P^i) \setminus \{v_1^i, v_{12}^i\}$ with color 1 if he can do so. Otherwise, Bob colors any uncolored vertex $v \in V(P^i)$ with a color $c \neq 1$.
- iv) If Alice plays on some P^i , the only vertex left to color on P^i is $v \in \{v_1^i, v_{12}^i\}$ and the only color available for v is 1, then Bob colors v with 1.

Claim 1. *If, at the end of the game, $c(v_{12}^{i-1}) \neq 1$ and $c(v_1^{i+1}) \neq 1$ then, after Bob's second move in P^i , there are at least 2 vertices with color 1 in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$.*

Proof of Claim. Observe that initially, since $c(v_{12}^{i-1}) \neq 1$ and $c(v_1^{i+1}) \neq 1$, all 10 vertices in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$ can receive color 1.

If Alice's first move on P^i is to color a non-extremity vertex of P^i with color 1, Bob will apply rule *ii*) and we are done.

If Alice's first move on P^i is to color a vertex x with a color $c \neq 1$ she forbids color 1 for at most 3 vertices (x and its neighbors) in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$. Furthermore, after Bob applies rule *i*) (let z be the vertex he colored 1), he forbids color 1 for at most 3 more vertices (z , the neighbor y of z that is not a neighbor of x and the other neighbor of y) in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$. After that, on Alice second move in P^i if she colors a non-extremity vertex of P^i with color 1 we are done. If she colors a vertex with

a color distinct from 1 she forbids at most 3 more vertices from receiving color 1 in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$, therefore there is at least one vertex that can still receive color 1 and it will receive this color after Bob applies rule *iii*).

Now suppose that Alice colors v_1^i (resp., v_{12}^i) with color 1 in her first move in P^i . Then, in his first move on P^i , Bob will color v_4^i (resp., v_9^i) and on his second move either Alice will color a vertex in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$ with color 1 or, after her move, there will be at least one vertex that can be colored with 1 in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$ (even if Alice colored the other extremity of P^i with color 1). If the later is true then Bob applies rule *iii*) on his second move in P^i and colors such vertex with 1. The argument is analogous if we assume that Alice colors a extremity of the path with 1 only on her second move on P^i .

Therefore, at the end of the game, for each P^i such that $c(v_{12}^{i-1}) \neq 1$ and $c(v_1^{i+1}) \neq 1$, there are at least 2 vertices with color 1 in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$. \diamond

Observe that, in the case $c(v_{12}^{i-1}) \neq 1$ and $c(v_1^{i+1}) \neq 1$, if Alice or Bob colors either v_1^i or v_{12}^i with color 1, then P^i will have at least 3 vertices colored 1. If Alice and Bob color both v_1^i and v_{12}^i with color 1, then P^i will have exactly 4 vertices colored 1.

Claim 2. *If, at the end of the game, $c(v_{12}^{i-1}) = 1$ or $c(v_1^{i+1}) = 1$, then after Bob first move in P^i , there is at least 1 vertex colored 1 in $V(P^i) \setminus \{v_1^i, v_{12}^i\}$.*

Proof of Claim. Assume that, in her first move on P^i , Alice colors a vertex v with color 1. If $v \neq v_{12}^i$ and $v \neq v_1^i$ we are done. Otherwise, Alice's move forbids color 1 either for the vertices v_{11}^i and v_{10}^i or v_2^i and v_3^i . After that Bob, will apply rule *ii*) and color v_9^i with color 1 in the first case or color v_4^i with color 1 on the second case.

If in her first move on P^i Alice colors a vertex v_j^i with a color $c \neq 1$, she forbids color 1 for at most 3 vertices on $V(P^i) \setminus \{v_1^i, v_{12}^i\}$. After that, Bob certainly can apply rule *i*) and color v_{j-2}^i or v_{j+2}^i with color 1. \diamond

Let $c_1(P^i)$ denote the number of vertices with color 1 in P^i at the end of the game. From both claims, by the end of the game, for each $1 \leq i \leq \frac{n}{12}$, either $c_1(P^i) \geq 2$ or ($c_1(P^i) = 1$ and ($c(v_{12}^{i-1}) = 1$ or $c(v_1^{i+1}) = 1$)).

Assume that $c_1(P^i) = 1$ and, w.l.o.g., that $c(v_1^{i+1}) = 1$. Therefore either $c_1(P^{i+1}) \geq 3$ or $c_1(P^{i+1}) = 2$ and $c(v_1^{i+2}) = 1$. Since the graph is finite the second case cannot happen indefinitely. Let $k > i$ be the smallest integer such that $c_1(P^k) \geq 3$. Then,

$$\begin{aligned} \sum_{j=i}^k c_1(P^j) &= c_1(P^i) + \sum_{j=i+1}^{k-1} c_1(P^j) + c_1(P^k) \\ &\geq 1 + 2(k - i - 1) + 3 = 2(k - i + 1) \end{aligned}$$

Consequently, for each path in P^i, \dots, P^k we can "count" at least 2 vertices with color 1. Furthermore, this bound holds even if $k = \frac{n}{2}$ and $c(v_{12}^k) \neq 1$, for that reason, all these vertices have degree 2.

Now suppose that there is a path $P^{i'}$, with $i < k < i'$, such that $c_1(P^{i'}) = 1$, $c(v_{12}^{i-1}) = 1$ and k is the greatest integer smaller than i' such that $c_1(P^k) \geq 3$. By the same reasons of the last paragraph, we can conclude that for every $k < j < i'$, $c_1(P^j) = 2$ and $c(v_{12}^k) = 1$. Since $c(v_1^k) = 1$ and $c(v_{12}^k) = 1$ then $c_1(P^k) = 4$. For that reason,

$$\begin{aligned} \sum_{j=i}^{i'} c_1(P^j) &= c_1(P^i) + \sum_{j=i+1}^{k-1} c_1(P^j) + c_1(P^k) + \sum_{j=k+1}^{i'-1} c_1(P^j) + c_1(P^{i'}) \\ &= 1 + 2(k - i - 1) + 4 + 2(i' - k - 1) + 1 = 2(i' - i + 1) \end{aligned}$$

Therefore, for each path in $P^i, \dots, P^{i'}$ we can "count" exactly 2 vertices with color 1.

Finally, we can conclude that using the strategy above, by the end of the game $\sum_{i=1}^{\frac{n}{12}} c_1(P^i) \geq 2 \frac{n}{12}$. \square

The above proof can be easily adapted for C_n . Note also that the strategy described for Bob also works if Alice is allowed to skip her turn, i.e., Bob can ensure a monochromatic set of size at least $2 \frac{n}{12}$ even if Alice is allowed to skip some of her turns.

Theorem 5.

$$4 \left\lfloor \frac{n}{12} \right\rfloor + 1 \leq \chi_{hg}(P_n) \leq \frac{2(n+3)}{3},$$

and

$$4 \left\lfloor \frac{n}{12} \right\rfloor + 1 \leq \chi_{hg}(C_n) \leq \frac{2(n+3)}{3}.$$

Proof. The upper bounds directly follow from Corollary 1. The lower bound for paths follows Lemma 3 and the lower bound for cycles follows a direct adaptation of the proof of Lemma 3. \square

5 Grids

A grid with n rows and m columns is a graph with vertex set $V = \{v_{i,j} | 1 \leq i \leq n \wedge 1 \leq j \leq m\}$. We say that the vertex $v_{i,j}$ is on row i and column j . For the Cartesian, triangular and king grids the set of edges are respectively:

- $E_{\square} = \{v_{i,j}v_{l,k} | (i = l \wedge j - k = \pm 1) \vee (j = k \wedge i - l = \pm 1)\}$
- $E_{\Delta} = E_{\square} \cup \{v_{i,j}v_{l,k} | i - l = k - j = 1\}$
- $E_{\text{king}} = E_{\Delta} \cup \{v_{i,j}v_{l,k} | i - l = k - j = -1\}$

The results on paths may be somehow extended to grids. On this section we use the some of the bounds already obtained to show bounds for the harmonious coloring game on Cartesian, triangular and king grids.

Lemma 4. *Let G be a grid graph with n rows and m columns, then:*

- $\chi_{hg}(G) \geq 8 \left\lfloor \frac{n-2}{5} \right\rfloor \left\lfloor \frac{m}{12} \right\rfloor + 1$, if G is a Cartesian grid;
- $\chi_{hg}(G) \geq 12 \left\lfloor \frac{n-4}{5} \right\rfloor \left\lfloor \frac{m}{12} \right\rfloor + 1$, if G is a triangular grid;
- $\chi_{hg}(G) \geq 16 \left\lfloor \frac{n-6}{5} \right\rfloor \left\lfloor \frac{m}{12} \right\rfloor + 1$, if G is a king grid.

Proof. For each type of grid, let i and j_i be the integers defined below:

- $1 \leq i \leq \left\lfloor \frac{n-2}{5} \right\rfloor$ and $j_i = 5i - 3$ for the Cartesian grid;
- $1 \leq i \leq \left\lfloor \frac{n-4}{5} \right\rfloor$ and $j_i = 5i - 2$ for the triangular grid;
- $1 \leq i \leq \left\lfloor \frac{n-6}{5} \right\rfloor$ and $j_i = 5i - 1$ for the king grid;

Let P^i be the path induced by the vertices on row j_i of the grid and let v_k^i be the vertex of P^i on the k -th column of the grid.

To achieve the bounds described above, Bob plays using his strategy for the paths, described on Lemma 3, on the vertices of each P^i . That is, for each of these paths, Bob wants to have at least $2 \left\lfloor \frac{m}{12} \right\rfloor$ internal vertices with color 1.

Notice that, in the proof of Lemma 3, Bob's strategy is only concerned in the vertices for which the color 1 is forbidden. Furthermore is worth noticing that Bob's strategy works even when Alice can skip her turn, since this cannot decrease the number of vertices colored with 1 by the end of the game. In the case Alice does so, Bob can just choose an internal vertex of any of the subpaths and color it, preferably with color 1, such that the number of vertices that can still receive color 1 in the subpath are maximized after Bob's move.

Observe that if Alice colors a vertex v at distance at most 2 from a vertex of some P^i , Bob's strategy is not affected on any P^k when $k \neq i$. Therefore, when Alice colors a vertex v with color c at distance at most 2 from the path P^i , Bob can use a strategy analogous to the one on Lemma 3 as if Alice had played on a vertex of P^i . Obviously if Alice colors a vertex of P^i , Bob just follows his strategy as in Lemma 3.

More precisely, there are three cases depending on the type of the considered grid.

Cartesian Grid. There are 2 possibilities when Alice colors a vertex at distance 1 or 2 from P^i on the Cartesian grid:

- If Alice colors a vertex on column k on rows $j_i \pm 1$ with any color c or on rows $j_i \pm 2$ with color 1, she forbids, on the worst case, the vertices v_k^i , v_{k+1}^i and v_{k-1}^i from receiving color 1. Therefore, for Bob's strategy on P^i , is equivalent for her to have colored v_k^i with a color $c' \neq 1$.
- If Alice colors a vertex on rows $j_i \pm 2$ with color $c \neq 1$, she does not forbid the use of color 1 for any new vertices on P^i . Bob plays in P^i as if Alice had passed her turn.

Clearly this strategy leads to at least $2 \lfloor \frac{m}{12} \rfloor$ vertices with color 1 for each P^i on the Cartesian grid.

Triangular Grid. For the triangular grid, there are 3 possibilities when Alice colors a vertex at distance 1 or 2 from P^i :

- If Alice colors a vertex on column k on rows $j_i \pm 1$ with color $c \neq 1$ or on rows $j_i \pm 2$ with $c = 1$, she forbids, in the worst case, the color 1 from being used on the vertices v_{k-2}^i , v_{k-1}^i and v_k^i . Therefore, for Bob's strategy on P^i , is equivalent for her to have colored the vertex v_{k-1}^i with a color $c' \neq 1$.
- If Alice colors a vertex on column k on rows $j_i \pm 1$ with color $c = 1$, she forbids color 1 for the vertices v_{k-2}^i , v_{k-1}^i , v_k^i and v_{k+1}^i . Then Bob can just assume that Alice colored v_k^i with color 1 and follow his strategy on P^i . Note that, in this case, the vertex colored by Alice and v_k^i have the same degree.
- If Alice colors a vertex on rows $j_i \pm 2$ with color $c \neq 1$, she does not forbid the use of color 1 for any new vertices on P^i . Bob plays in P^i as if Alice had passed her turn.

Clearly this strategy leads to at least $2 \lfloor \frac{m}{12} \rfloor$ vertices with color 1 for each P^i on the triangular grid.

King Grid. There are 3 possibilities when Alice colors a vertex at distance 1 or 2 from P^i on the king grid:

- If Alice colors a vertex on column k on rows $j_i \pm 1$ or $j_i \pm 2$ with color $c = 1$, she forbids color 1 for the vertices v_{k-2}^i , v_{k-1}^i , v_k^i , v_{k+1}^i and v_{k+2}^i . Then Bob can just assume that Alice colored v_k^i with color 1 and follow his strategy on P^i . Note that, in this case, the vertex colored by Alice and v_k^i have the same degree.
- If Alice colors a vertex on column k on rows $j_i \pm 1$ with color $c \neq 1$, she forbids, on the worst case, the vertices v_k^i , v_{k+1}^i and v_{k-1}^i from receiving color 1. Therefore, for Bob's strategy on P^i , is equivalent for her to have colored v_k^i with a color $c' \neq 1$.
- If Alice colors a vertex on rows $j_i \pm 2$ with color $c \neq 1$, she does not forbid the use of color 1 for any new vertices on P^i . Bob plays in P^i as if Alice had passed her turn.

Clearly this strategy leads to at least $2 \lfloor \frac{m}{12} \rfloor$ vertices with color 1 for each P^i on the king grid.

Since each of the vertices colored with 1 for each P^i have degrees 4, 6 and 8 for the Cartesian, triangular and king grids respectively, we obtain the desired bounds by Lemma 1. \square

Theorem 6. *Let G be a grid graph with $n > 3$ rows and $m > 1$ columns, then:*

- $8 \lfloor \frac{n-2}{5} \rfloor \lfloor \frac{m}{12} \rfloor + 1 \leq \chi_{hg}(G) \leq \frac{4nm}{5} + 4$, if G is a Cartesian grid;
- $12 \lfloor \frac{n-4}{5} \rfloor \lfloor \frac{m}{12} \rfloor + 1 \leq \chi_{hg}(G) \leq \frac{6nm}{7} + 6$, if G is a triangular grid;
- $16 \lfloor \frac{n-6}{5} \rfloor \lfloor \frac{m}{12} \rfloor + 1 \leq \chi_{hg}(G) \leq \frac{8nm}{9} + 8$, if G is a king grid;

Proof. The upper bounds directly follow Corollary 1. The lower bounds come from Lemma 4. \square

6 Forest of Stars

We are interested in this graph class because it is known that computing the harmonious chromatic number of a forest of stars is NP-complete [9]. For any graph $G = (V, E)$ and $v \in V$, let $\deg(v)$ denote the degree of the vertex v .

Claim 3. *Let S and S' be two stars with n and m leaves respectively, $n \geq m$. Then, $\chi_{hg}(S) = n + 1$ and $\chi_{hg}(S \cup S') = n + 1$ if $n > m$ and $\chi_{hg}(S \cup S') = n + 2$ otherwise.*

Proof of Claim. Since any harmonious coloring must be proper at distance 2, and $\chi_{hg}(G) \leq |V(G)|$ for any graph G , we get $\chi_{hg}(S) = n + 1$.

In the disjoint union $S \cup S'$, Alice starts by coloring a leaf of S with color 1. If, at his first turn, Bob has not colored the center of S' , Alice colors it with the smallest possible color. If Bob has colored the center of S' with a color distinct from 1, then Alice colors a leaf of S' with 1. From now, Alice plays greedily. This strategy is winning with $n + 1$ (or $n + 2$ if $n = m$) colors which is optimal since it equals the harmonious number of $S \cup S'$. \diamond

Lemma 5. *Let $G = S_1 \cup S_2 \cup S_3 \cup \dots \cup S_n$ where each S_i is a star with center s_i such that $\deg(s_1) \geq \deg(s_2) \geq \deg(s_3) \geq \dots \geq \deg(s_n)$. Then:*

$$\chi_{hg}(G) \geq 1 + \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \deg(s_{2i-1})$$

Proof. Let 1 be the color used by Alice at her first turn. If Alice colored the center of a star at her first turn, let $c = 1$ and let $c = 2$ otherwise. While it is possible, the strategy of Bob consists of coloring the vertex s_j with color c , where j is the smallest integer such that s_j is uncolored.

Claim 4. *After the j -th move of Bob ($0 < j \leq \lceil \frac{n}{2} \rceil$), s_1 is colored with color c and, for every $0 < i \leq j$, at least $i - 1$ vertices are colored with c in $\{s_2, s_3, \dots, s_{2j-1}\}$.*

Proof of Claim. The proof is by induction on j . If $j = 1$, if Alice's first move colors a center s (with color 1), then $c = 1$ and either $s = s_1$ or $s \neq s_1$ and Bob's first move consists in coloring s_1 with c . If Alice does not color a center, then Bob colors s_1 with c . In all cases, we are done.

Let us assume by induction that the result holds for the j th turn. If, at the $(j + 1)$ -th turn of Bob, s_{2j} or s_{2j+1} can be colored with c , then Bob colors a vertex s_k , $k \leq 2j + 1$, with color c . Then, the result holds. Otherwise (if none of s_{2j} or s_{2j+1} can be colored with c), it implies that Alice has already colored one vertex in each of S_{2j} and S_{2j+1} . Therefore, some star in $\{S_2, \dots, S_{2j-1}\}$ has no vertex colored yet. Then, Bob colors a vertex s_k , $k \leq 2j - 1$, with color c . This concludes the proof of the claim. \diamond

It follows that the centers colored with color c are (all together) adjacent to at least $\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \deg(s_{2i-1})$ vertices. Therefore, $\chi_{hg}(G) \geq 1 + \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \deg(s_{2i-1})$. \square

Lemma 6. *Let $G = S_1 \cup S_2 \cup S_3 \cup \dots \cup S_n$ where each S_i is a star with center s_i such that $\deg(s_1) \geq \deg(s_2) \geq \deg(s_3) \geq \dots \geq \deg(s_n)$. Then:*

$$\chi_{hg}(G) \leq 2 + \left\lfloor \frac{n}{2} \right\rfloor + \deg(s_1) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \deg(s_{2i})$$

Proof. Note that we may assume that $\deg(s_j) > 0$ for any $j \leq n$ since isolated vertices do not create any constraint. Hence, $\deg(s_j) \geq 1$ for every $1 \leq j \leq n$.

Let $y = 2 + \lfloor \frac{n}{2} \rfloor + \deg(s_1) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \deg(s_{2i})$ and let us show that Alice has a winning strategy using y colors. Note that $y \geq n$.

Alice first colors s_1 with color 1. While there is an uncolored center, at her turn:

- if Bob has just colored a center s_i with $\{i, i'\} = \{2k, 2k + 1\}$ and $s_{i'}$ is uncolored, then Alice colors $s_{i'}$ with a new color (a color she has not used yet).
- otherwise, Alice colors the uncolored center with smallest index with a new (a color she has not used yet) color (note that this color could have been used by Bob).

Once all centers have been colored, Alice greedily colors any uncolored vertex with the smallest possible color.

First, let us show that, while all centers are not colored, then the Alice's strategy is valid, i.e., there is a color available for Alice. Consider such a turn j (so $1 < j \leq n$) when Alice has to color some center s_i . Up to this turn, at most $j - 1$ leaves may have been colored (by Bob, since Alice only colored centers until now) and so at most $j - 1 < n \leq y$ forbidden pairs of colors are forbidden. Therefore, there must be a color available for Alice.

Note that the same argument holds for Bob, and so, while not all centers have been colored, Bob can always play.

For purpose of contradiction, let us assume that the Alice's strategy is not winning. So, at some turn j , Alice (or Bob) cannot color any vertex with any of the y colors. By above paragraph, all centers must have been colored (and so $j > n$). Let f be any uncolored leaf and let s_i be the center of the star containing f and let c be the color of s_i . Let $X = X_c \cup X_f$ be the set of vertices colored with c before turn j where X_c is the set of centers colored with c and X_f is the set of leaves colored with c (at most one vertex per star with the center not colored c). The number of forbidden pairs of colors containing c is then at most $1 + |X_f| + \sum_{s_k \in X_c} \deg(s_k) - 1$. Note that $|X_c| + |X_f| \leq n$.

Note that Alice's strategy ensures that, for every color c and every integer $i \in [1, \lfloor \frac{n}{2} \rfloor]$ but at most one such integer, $c(s_{2i}) \neq c$ or $c(s_{2i+1}) \neq c$. Note also that for $c = 1$ all $i \in [1, \lfloor \frac{n}{2} \rfloor]$ satisfies such condition.

Therefore, by previous sentence,

$$|X_f| + \sum_{s_k \in X_c} \deg(s_k) - 1 \leq |X_f| + \deg(s_1) + \sum_{i=1}^{|X_c|} \deg(s_{2i}) - 1$$

Hence, since $|X_c| + |X_f| \leq n$,

$$|X_f| + \sum_{s_k \in X_c} \deg(s_k) - 1 \leq n - |X_c| + \deg(s_1) + \sum_{i=1}^{|X_c|} \deg(s_{2i}) - 1$$

Therefore, because $\deg(s) > 0$ for every center s ,

$$|X_f| + \sum_{s_k \in X_c} \deg(s_k) - 1 \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 + \deg(s_1) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \deg(s_{2i}),$$

and so

$$|X_f| + \sum_{s_k \in X_c} \deg(s_k) - 1 \leq \left\lfloor \frac{n}{2} \right\rfloor + \deg(s_1) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \deg(s_{2i})$$

and so

$$1 + |X_f| + \sum_{s_k \in X_c} \deg(s_k) - 1 < y$$

which contradicts the fact that Alice (or Bob) cannot color a vertex at her turn. \square

Theorem 7. *Let $n \geq 3$. For any forest G of n stars with centers with degree $d_1 \geq d_2 \geq \dots \geq d_n$:*

$$1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} d_{2i-1} \leq \chi_{hg}(G) \leq 2 + \left\lfloor \frac{n}{2} \right\rfloor + d_1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} d_{2i}$$

7 Conclusions and Open Problems

In this paper, we have introduced the Harmonious Coloring Game (HCG), a new two player game related to the harmonious coloring problem. We proved that three variants of this game are PSPACE-complete: Sequential HCG, Pre-Colored HCG and Bob-Restricted HCG. As a by-product, we also proved that the *Chromatic Game II*, a game introduced in 1997 by Chen et al. [4], is also PSPACE-complete.

Despite that, we were unable to prove the computational complexity of the original HCG. One of the difficulties is that, for a given graph $G = (V, E)$ and the constructed graph \widehat{G} in the proof of Theorems 1 and 4, it is not true that $\chi_g(G) = k$ if and only if $\chi_{hg}(\widehat{G}) = k + |E|$, since Bob can, at each of his turns on \widehat{G} , use a new color on a vertex of V . If k is small enough compared to $|E|$, Bob can use $k + 1$ distinct colors on the vertices of V at some point of the game and the clique cannot be fully colored anymore. Nevertheless, the following seems reasonable.

Conjecture 1. *The harmonious coloring game is PSPACE-complete. That is, given a graph G and an integer k , deciding if $\chi_{hg}(G) \leq k$ is PSPACE-complete.*

We also obtained lower and upper bounds for general graphs and for some graph classes. However, we were unfortunately unable to determine the exact value of the harmonious game chromatic number even for paths and cycles.

Question 1. *What is the exact value of $\chi_{hg}(P_n)$ and $\chi_{hg}(C_n)$ for every $n \geq 1$?*

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