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The Convex Set Forming Game*

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Abstract

In 1984, Frank Harary introduced the first graph convexity game, focused on the geodesic convexity. A set $S \subseteq V$ of vertices of a graph $G = (V, E)$ is convex if every shortest path between two vertices of S is also included in S . We introduce the Convex Set Forming Game CFG: two players alternately select vertices in such a way that the set of selected vertices is always a convex set. In the normal (resp., misère) variant, the last player to be able to select a vertex wins (resp., loses). We also define a new graph invariant $\text{gc}(G)$ as the largest integer k such that the first player has a strategy ensuring that, at the end of the game, at least k vertices of the graph G have been selected. We first show that the problems of deciding the outcome (does the first player win?) of the game in both variants (normal and misère), as well as the problem of deciding whether $\text{gc}(G) \geq k$, are PSPACE-complete. As a by-product, we prove that the optimization variant of the classical NODE KAYLES game is PSPACE-complete. Then, we focus on convexable graphs, i.e., n -node graphs G for which $\text{gc}(G) = n$. For this purpose, we say that a set $S = \{v_1, \dots, v_{|S|}\} \subseteq V$ in a graph G admits a Convex Elimination Ordering (CEO) if $\{v_1, \dots, v_i\}$ is convex for every $1 \leq i \leq |S|$. We show that the class of graphs whose vertex-set admits a CEO coincides with the chordal graphs and that this class strictly contains the convexable graphs. Moreover, every graph which is Ptolemaic (distance-hereditary chordal) or unit interval is convexable. Finally, we give a polynomial-time algorithm for computing a

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largest set admitting a CEO in outerplanar graphs, which gives upper bounds on $gc(G)$ in outerplanar graphs G .

Keywords: Convexity number, geodesic games, PSPACE-hardness, chordal graphs.

1 Convexity games

Convexity is a classical topic, studied in many different branches of mathematics. The study of convexities applied to graphs has started about 50 years ago. The 1972 paper of Erdős et al. [11] is one of the first in this topic, focused in tournaments. Accordingly to Duchet [10], the first paper on general graphs, published in English, is the 1981 paper “Convexity in graphs” from Frank Harary and Juhani Nieminem [14]. The main convexity that has been studied is the *geodesic convexity*, in which a set $S \subseteq V$ of vertices of a graph $G = (V, E)$ is *convex* if, for every $u, v \in S$, then every $w \in V$ on a shortest path between u and v also belongs to S . Typical questions are to compute a largest convex proper subset of V [7], or to compute a smallest hull set (subset of vertices whose convex hull is the whole graph) [12].

In 1984, Harary introduced the first graph convexity games in his abstract “Convexity in graphs: achievement and avoidance games” [13]. These games were investigated in a sequence of five papers [4, 5, 13, 15, 20] from 1985 to 2003, all of them focused on the geodesic convexity. After a gap of twenty years, it was proved in 2024 the first PSPACE-hardness results on graph convexity games [1]. Recently, another game related to hull sets has been studied in [2].

In this paper, we introduce a new natural two-player game related to convexity: the Convex Set Forming game (CFG). Two players, Alice and Bob, select vertices of a connected graph G in such a way that the set of selected vertices is always a convex set. More precisely, Alice and Bob alternately (starting with Alice) select any not already selected vertex v such that $S \cup \{v\}$ is a convex set of G , where S is the set of previously selected vertices. The game ends when no such a vertex exists (i.e., when $S \cup \{v\}$ is not convex for every $v \notin S$). In the normal variant, denoted CFG-NORMAL, the last to play wins. In the misère variant, denoted CFG-MISÈRE, the last to play loses. In the optimization variant, denoted CFG-OPT, it is also given an integer k in the instance and Alice wins if the game stops with at least k selected vertices, and Bob wins otherwise. From the classical Zermelo-von Neumann theorem [22], one of the two players has a winning strategy in each one of these variants, since they are finite perfect-information games without draw. So, the decision problem of these games is whether Alice has a winning strategy.

The study of CFG-OPT naturally leads us to define a new graph invariant $gc(G)$ which is the largest integer k such that Alice has a strategy ensuring (whatever Bob does) that, at the end of the game, at least k vertices of the graph G have been

selected. Hence, CFG-OPT is equivalent to the problem of deciding, given a graph G and $k \in \mathbb{N}$, whether $\text{gc}(G) \geq k$.

As an example, consider a cycle C_4 with four vertices. Alice first selects any vertex v . Then Bob must select a neighbour of v (note that, when G is connected, the set of selected vertices must always induce a connected subgraph) and the game stops since any set with three vertices in C_4 is not convex. Therefore, Bob wins the normal variant, Alice wins the misère variant, and Alice wins the optimization variant if and only if $k \leq 2$, i.e., $\text{gc}(C_4) = 2$.

Our Contributions. We first focus on the complexity of deciding the outcome of the game. The Convex Set Forming game (CFG) is similar to the classical Clique Forming game and Node Kayles game, both proved PSPACE-hard [21] in 1978. The difference is that in the Clique Forming game (resp. Node Kayles) the selected vertices must always induce a clique (resp. independent set), instead of a convex set. In Section 2, we prove that the three variants of CFG are PSPACE-complete, by reductions from the Clique Forming game. We remark that these results also hold in the case of the *monophonic convexity*¹.

As a by-product, we also prove that the optimization variants of the Clique Forming and Node Kayles games are PSPACE-complete, and consequently the corresponding parameters *game clique number* $\omega_g(G)$ and *game independence number* $\alpha_g(G)$ are PSPACE-hard to compute.

In Section 3, we then study the *convexable* graphs, i.e., the connected n -node graphs G such that $\text{gc}(G) = n$. As an example, consider any tree T . The set of selected vertices during the game always induces a subtree T' of T and can always be enlarged, except if $T' = T$. Then, every tree is convexable. Note that, for a graph G to be convexable, there must be an ordering (v_1, \dots, v_n) of its vertices such that $\{v_1, \dots, v_i\}$ is convex for every $1 \leq i \leq n$. Given a graph $G = (V, E)$ and $S \subseteq V$, we say that S admits a *Convex Elimination Ordering* (CEO) if S can be ordered $(v_1, \dots, v_{|S|})$ such that $\{v_1, \dots, v_i\}$ is convex for every $1 \leq i \leq |S|$. A graph is said *CEO* if its vertex-set admits a CEO. We show that the class of CEO graphs actually coincides with the class of chordal graphs and that this class strictly contains the class of convexable graphs. Then, we show that the class of convexable graphs strictly contains the classes of Ptolemaic graphs (distance hereditary chordal) and of unit interval graphs. We leave the question of the complexity of recognizing a convexable graph as an open problem.

Finally, in Section 4, we show that a largest set admitting a CEO can be computed in polynomial time in the class of outerplanar graphs. Note that the size of a largest set admitting a CEO in a graph G provides an upper bound on $\text{gc}(G)$.

¹In the monophonic convexity, a set S of vertices of a graph G is *m-convex* if, for every $u, v \in S$, then every vertex w on an induced path between u and v also belongs to S [18].

We conclude with several open questions.

2 PSPACE-hardness of Convex Set Forming game

In this section, we prove that the normal, misère and optimization variants of the Convex Set Forming game CFG are PSPACE-complete. As mentioned before, we consider the games as decision problems: does Alice have a winning strategy? In the normal and misère variants, the instance is a graph and, in the optimization variant, the instance is a graph and a positive integer k . Since the number of turns is at most n and, in each turn, the number of possible vertices to label is at most n , all these game variants are polynomially bounded two player games, which implies that they are in PSPACE [16].

We obtain reductions from the normal, misère and optimization variants of the CLIQUE FORMING game. The normal variant was proved PSPACE-complete [21] in 1978 and the misère variant was proved PSPACE-Complete [6] recently in 2024. The optimization variant of the CLIQUE FORMING game was not proved PSPACE-hard before and we did this at the end of this section. In the CLIQUE FORMING game, two players, Alice and Bob, starting by Alice, alternately select vertices and the subset of the chosen vertices must induce a clique at any turn. As usual, the last to play wins in the normal variant and loses in the misère variant. Moreover, in the optimization variant, Alice wins if the resulting clique has size at least k at the end. This is related to the NODE KAYLES game, in which the objective is to get an independent set, instead of a clique. The CLIQUE FORMING game is the NODE KAYLES game played on the complement of the graph, and vice-versa.

Theorem 2.1. *The normal, misère and optimization variants of the Convex Set Forming game CFG are PSPACE-complete even in graphs with diameter two.*

Proof. We first obtain a reduction for CFG-NORMAL. Let G be an instance of the normal variant of the CLIQUE FORMING game. We may assume that G is not complete.

Let G' be the graph obtained from G by adding four new vertices u_1, u'_1, u_2 and u'_2 adjacent to all vertices of G , and the edges $u_1u'_1$ and $u_2u'_2$. See Figure 1. Notice that G' has diameter 2. We prove that Alice has a winning strategy in the normal CLIQUE FORMING game on G if and only if she has a winning strategy in the normal variant of CFG on G' . Notice that the set of selected vertices must induce a clique, since u_1, u'_1, u_2 and u'_2 are in shortest paths in G' between any two non-adjacent vertices of G . Suppose that Alice has a winning strategy on the normal CLIQUE FORMING game on G . If Bob selects u_1 (resp. u_2), she selects u'_1 (resp. u'_2). Otherwise, she follows her winning strategy on G . Since she is the last

to play in the CLIQUE FORMING game on G , then she is also the last to play in CFG on G' . Similarly if Bob has a winning strategy.

Regarding CFG-MISÈRE, we have exactly the same reduction, but from the misère variant of the CLIQUE FORMING game, which was proved PSPACE-hard [6] in 2024. The arguments are very similar.

Regarding CFG-OPT, we also have the same reduction from the graph G to the graph G' , but now it is given a positive integer k in the CLIQUE FORMING game. So let $k' = k + 2$ be the positive integer which is part of the instance of CFG-OPT together with G' . Similarly from the arguments before, Alice has a winning strategy in the optimization variant of the CLIQUE FORMING game on G with k vertices if and only if she has a winning strategy in CFG-OPT on G' with $k' = k+2$ vertices. \square

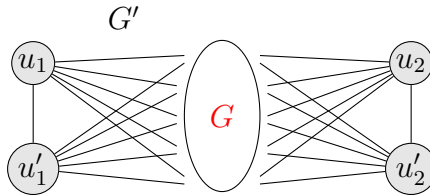


Figure 1: Graph G of the reductions in Theorem 2.1 on the normal and misère variants of CFG.

Finally, we prove the PSPACE-hardness of the optimization variants of the CLIQUE FORMING game and the NODE KAYLES game. We first reproduce in Theorem 2.2 the 1978 PSPACE-hardness proof of the normal NODE KAYLES game [21], since our proof in Theorem 2.3 is strongly based on it.

Theorem 2.2. [Schaefer, 1978] *The normal NODE KAYLES game is PSPACE-complete.*

Proof. Reduction from the TQBF problem: it is given a totally quantified CNF formula Φ with n variables x_1, \dots, x_n and m clauses B_1, \dots, B_m , where the quantifiers are alternately \exists and \forall starting from x_n to x_1 . This can be seen as a game in which Alice and Bob alternately set true or false to the variables x_n, x_{n-1}, \dots, x_1 in this order. Alice wins if the formula is true at the the end; otherwise, Bob wins.

Consider the example $\Phi = \exists x_3 \forall x_2 \exists x_1 : B_1 \wedge B_2 \wedge B_3$, with the following clauses: $B_1 = (x_1 \vee \bar{x}_1)$, $B_2 = (x_1 \vee \bar{x}_2)$ and $B_3 = (\bar{x}_1 \vee x_2 \vee x_3)$. In this example, Alice wins, since Alice sets true to the variable x_3 in the first turn (in order to satisfy clause B_3), Bob then sets true to the variable x_2 (trying to make B_2 unsatisfied) and Alice must set true to variable x_1 , satisfying all clauses and winning the game.

Schaefer [21] assumes that the number n of variables is odd (otherwise we can add a new variable which does not appear in any clause) and the first clause B_1 is

$(x_1 \vee \bar{x}_1)$, which is always satisfied and does not change the outcome of the game. Given a TQBF formula Φ , the constructed graph $G = G(\Phi)$ is described below.

For $i \in [n]$, create in G the vertices x_i and \bar{x}_i associated to the variable x_i . For $k \in [m]$, create the vertex $x_{0,k}$ associated to the clause B_k . For $i \in [n]$ and $j \in \{0, \dots, i-1\}$, create the vertex $y_{i,j}$. Let X_i be the set with all the vertices x_i , \bar{x}_i and $y_{i,j}$, and make X_i be a clique in G . Let X_0 be the set with the vertices $x_{0,1}, \dots, x_{0,m}$ associated to the clauses, and make X_0 be a clique in G .

If the literal x_i is in the clause B_k , create the edge $x_i x_{0,k}$. If the literal \bar{x}_i is in the clause B_k , create the edge $\bar{x}_i x_{0,k}$. For $i \in [n]$ and $j \in \{0, \dots, i-1\}$, create the edge $y_{i,j} w$ for every vertex $w \in (X_0 \cup \dots \cup X_{i-1}) - X_j$. This ends the reduction of [21]. See Figure 2 for the formula Φ mentioned above, which is the same example in [21].

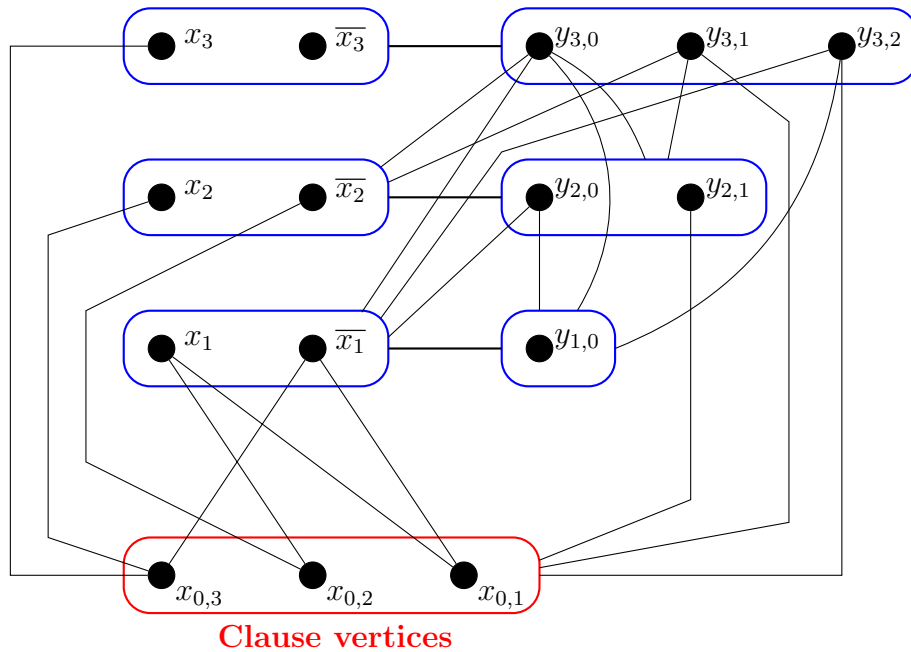


Figure 2: Example of the construction of Theorem 3.2 of [21] from the input formula $\exists x_3 \forall x_2 \exists x_1 : (x_1 \vee \bar{x}_1) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$. A line leading to an enclosure represents multiple edges leading to each vertex in the enclosure. Also, within each enclosure, all vertices are mutually joined by edges, which are not shown.

Schaefer [21] defines a game as legitimate if for $i = 1, \dots, n$ the vertex selected at move i is either x_{n-i+1} or \bar{x}_{n-i+1} . If a player does an illegitimate move when all previous moves were legitimate, then the other player wins immediately. In order to prove this, fix $i \in [n]$ and assume that the first $n - i$ moves were legitimate, that is, the vertices selected so far are one of each pair $\{x_n, \bar{x}_n\}, \dots, \{x_{i+1}, \bar{x}_{i+1}\}$. Clearly,

from the construction of G , no vertex of $X_n \cup X_{n-1} \cup \dots \cup X_{i+1}$ can be played now. Suppose that the player on move $n - i + 1$ plays illegitimately, by selecting a vertex from $X_i \cup \dots \cup X_0$ other than x_i or \bar{x}_i . If the player selects a node in X_k , for $k < i$, then the opponent wins by choosing $y_{i,k}$ and no other vertex is playable, since any vertex of X_k is adjacent to the illegitimately played vertex and any vertex of $(X_0 \cup \dots \cup X_i) - X_k$ is adjacent to $y_{i,k}$. Finally, if the illegitimate move is in X_i , it must be $y_{i,k}$ for some $k < i$ and therefore the opponent wins by selecting x_k if $k > 0$ or $x_{0,1}$ if $k = 0$, which is playable because of the assumption about B_1 . As before, no other vertex is playable.

As a consequence, we may consider from now on that the players move legitimately. Then every move on the TQBF game in Φ has an obvious corresponding move on the NODE KAYLES game in $G(\Phi)$, and vice-versa: setting x_i true (resp. false) in TQBF corresponds to selecting the vertex x_i (resp. \bar{x}_i) in NODE KAYLES, and vice-versa.

If Alice has a winning strategy on a formula Φ of TQBF, Alice can play the corresponding moves on the graph $G(\Phi)$ of Node Kayles. After the last move of TQBF, all clauses are satisfied (because of the winning strategy of Alice on TQBF) and then no clause vertex is playable on Node Kayles. Since n is odd, Alice is the last to play, winning the NODE KAYLES game.

Finally, if Bob has a winning strategy on a formula Φ of TQBF, Bob can play the corresponding moves on the graph $G(\Phi)$ of Node Kayles. After the last move of TQBF, there is at least one non-satisfied clause (because of the winning strategy of Bob on TQBF) and then there is a playable clause vertex on Node Kayles. Since n is odd, Alice selected either x_1 or \bar{x}_1 and Bob can select a playable clause vertex in the next move, winning the NODE KAYLES game. \square

Now we are able to prove the PSPACE-hardness of the optimization variants of the CLIQUE FORMING and NODE KAYLES games, and the corresponding parameters *game clique number* $\omega_g(G)$ and *game independence number* $\alpha_g(G)$. The *game clique number* $\omega_g(G)$ is the maximum k such that Alice has a winning strategy with at least k vertices in the CLIQUE FORMING game. The *game independence number* $\alpha_g(G)$ is the maximum k such that Alice has a winning strategy with at least k vertices in the NODE KAYLES game. Clearly, $\omega_g(G) \leq \omega(G)$, $\alpha_g(G) \leq \alpha(G)$ and $\omega_g(G) = \alpha_g(\bar{G})$, where $\omega(G)$ and $\alpha(G)$ are the clique number and the independence number, respectively.

Theorem 2.3. *The optimization variants of the CLIQUE FORMING game and the NODE KAYLES game are PSPACE-complete. Consequently, the game clique number $\omega_g(G)$ and the game independence number $\alpha_g(G)$ are PSPACE-hard to compute.*

Proof. Since the optimization variant of the CLIQUE FORMING game is the optimization variant of the NODE KAYLES game played on the complement of the graph (and

vice-versa), we will only prove that the optimization variant of the NODE KAYLES game is PSPACE-hard.

We obtain a reduction from a variant of the TQBF problem, which we call TQBF', based on the reduction of [21] for the NODE KAYLES game (see Theorem 2.2 above). In TQBF', Alice wants to make the formula false and Bob wants to make the formula true. This is the same as TQBF played on the complement of the formula. Thus, TQBF' is also PSPACE-Complete.

Following the proof of Theorem 2.2, instead of assuming that the number n of variables is odd, we assume that n is even. As in the proof of Theorem 2.2, we can do this since it is possible to add a new variable which does not appear in any clause. We also assume that $B_1 = (x_1 \vee \bar{x}_1)$.

Given a TQBF' formula Φ , let $G = G(\Phi)$ be the graph constructed in the proof of Theorem 2.2. Let $G' = G'(\Phi)$ be the graph obtained from G by replacing for $0 \leq i < n/2$ every vertex $y_{n-2i,j}$ and $y_{n-2i-1,j}$ by independent sets $Y_{n-2i,j}$ and $Y_{n-2i-1,j}$ with $n - 2i - 1$ vertices each (respecting the adjacencies of $y_{n-2i,j}$ and $y_{n-2i-1,j}$). See Figure 3.

Moreover, the positive number of the optimization variant instance is $n + 1$, where n is the number of variables. That is, Alice wants to obtain an independent set of size greater than n , and Bob wants to avoid this.

As in the proof of Theorem 2.2, we can define a move i as legitimate if the selected vertex is either x_{n-i+1} or \bar{x}_{n-i+1} .

We also prove that, if an illegitimate move is done after all previous moves were legitimate, then the other player wins the optimization variant of the NODE KAYLES game. For this, fix $i \in [n]$ and assume that the first $n - i$ moves were legitimate. From the construction of G' , no vertex of $X_n \cup X_{n-1} \cup \dots \cup X_{i+1}$ can be played now. Suppose that the player on move $n - i + 1$ plays illegitimately, by selecting a vertex from $X_i \cup \dots \cup X_0$ other than x_i or \bar{x}_i . If the player selects a node in X_k , for $k < i$, then the opponent wins by choosing a vertex of $Y_{i,k}$ and then only the other vertices of $Y_{i,k}$ are playable, since any vertex of X_k is adjacent to the illegitimately played vertex and any vertex of $(X_0 \cup \dots \cup X_i) - X_k$ is adjacent to $Y_{i,k}$. We explain better this winning argument in the next paragraph. Finally, if the illegitimate move is in X_i , it must be in a vertex of $Y_{i,k}$ for some $k < i$ and therefore the opponent wins by selecting x_k if $k > 0$ or $x_{0,1}$ if $k = 0$, which is playable because of the assumption about B_1 . As before, if the illegitimate move was in $Y_{i,k}$, then only the other vertices of $Y_{i,k}$ are playable.

Recall that n is even. Notice that, if i is even, then the illegitimate move was from Alice and then the number of selected vertices is $(n - i) + 1 + |Y_{i,k}| = n - i + 1 + (i - 1) = n$ for some k . Moreover, if i is odd, then the illegitimate move was from Bob and then the number of selected vertices is $(n - i) + 1 + |Y_{i,k}| = n - i + 1 + i = n + 1$ for some k .

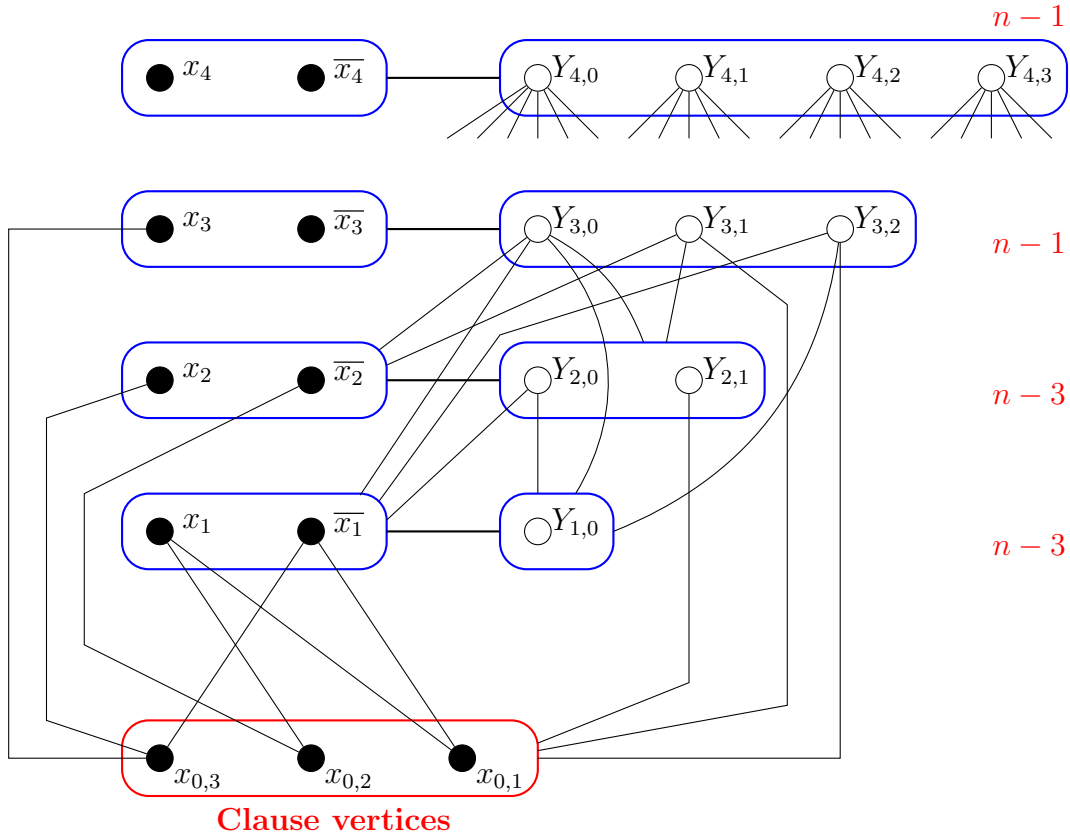


Figure 3: Example of the modification on the construction of Theorem 3.2 of [21] from the input formula $\exists x_3 \forall x_2 \exists x_1 : (x_1 \vee \bar{x}_1) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$. The numbers in red are the sizes of the corresponding independent sets $Y_{i,j}$, where $n = 4$.

As a consequence, we may consider from now on that the players move legitimately in the optimization variant of the NODE KAYLES game. Then every move on the TQBF' game in Φ has an obvious corresponding move on the optimization variant of the NODE KAYLES game in $G'(\Phi)$, and vice-versa: setting x_i true (resp. false) in TQBF' corresponds to selecting the vertex x_i (resp. \bar{x}_i) in NODE KAYLES, and vice-versa.

If Bob has a winning strategy on a formula Φ of TQBF', he can play the corresponding moves on the graph $G'(\Phi)$ of Node Kayles. Recall that Bob wants to satisfy all clauses here. After the last move of TQBF', all clauses are satisfied (because of the winning strategy of Bob on TQBF') and then no clause vertex is playable on Node Kayles, implying n selected vertices.

Finally, if Alice has a winning strategy on a formula Φ of TQBF', she can play the corresponding moves on the graph $G'(\Phi)$ of Node Kayles. After the last move

of TQBF', there is at least one non-satisfied clause (because of the winning strategy of Alice on TQBF') and then there is a playable clause vertex on the optimization variant of NODE KAYLES. Thus, the last player must select a playable clause vertex in the next move, implying $n + 1$ selected vertices. \square

3 Convexable Graphs

Let us now say that a graph is *convexable* if $gc(G) = n$: Alice can ensure that all vertices are selected in CFG. In this section, we aim at characterizing the convexable graphs.

3.1 Convex Elimination Orderings and Preliminaries

A *Convex Elimination Ordering* (CEO) of a graph G is an ordering (v_1, \dots, v_n) of $V(G)$ such that $G_i = G[v_1, \dots, v_i]$ induces a convex set in \mathcal{C} for every $1 \leq i \leq n$. Note that the concept of CEO is closely related to the Convex Set Forming Game CFG, since two players collaborating in the game would manage to eventually select all the vertices of a graph G if and only if G admits a CEO (i.e., it corresponds to the one-player Convex Set Forming Game). Clearly, if a graph does not have a CEO, then $gc(G) < n$.

In the following, we relate Convex Elimination Orderings to Perfect Elimination Orderings (PEO), which are orderings (v_1, \dots, v_n) of $V(G)$ such that v_i is *simplicial* in $G_{\geq i} = G[v_i, \dots, v_n]$ for every $1 \leq i \leq n$, i.e., $N_{G_{\geq i}}(v_i)$ induces a complete graph. It is well known that a graph is chordal if and only if it has a PEO [17].

Lemma 3.1. *Let G be a connected graph. An ordering (v_1, \dots, v_n) of $V(G)$ is a CEO if and only if (v_n, \dots, v_1) is a PEO.*

Proof. We begin by proving that any CEO is the reverse of a PEO. By contrapositive, assume that (v_n, \dots, v_1) is not a PEO and let $j > 1$ be any integer such that v_j is not simplicial in $G[v_{j-1}, \dots, v_1]$. Then, there exist two non-adjacent vertices in $N(v_j) \cap \{v_{j-1}, \dots, v_1\}$. These vertices are linked by a shortest path containing v_j . Therefore $G[v_1, \dots, v_{j-1}]$ is not convex, so (v_1, \dots, v_n) is not a CEO.

Now, assume that (v_n, \dots, v_1) is a PEO (and so the graph G is chordal). We will show that the ordering (v_1, \dots, v_n) is a CEO. For contradiction, assume that (v_1, \dots, v_j) is not convex on \mathcal{C} for some $1 < j < n$. Then there are non-adjacent vertices $a, b \in \{v_1, \dots, v_j\}$ with an induced a - b path P having all internal vertices in $\{v_{j+1}, \dots, v_n\}$. Since v_{j+1} is simplicial in $G[v_1, \dots, v_j]$ and $N(v_{j+1}) \cap \{v_1, \dots, v_j\}$ is a minimal separator [17], then $a, b \in N(v_{j+1})$ and consequently a and b are adjacent, a contradiction. \square

This has strong implications on which graphs can admit a CEO. Since admitting a PEO is a characterization of chordal graphs, we can now state the following corollary.

Corollary 3.2. *Let G be a connected graph. Then, G admits a CEO if and only if G is chordal. Consequently, deciding if a graph admits a CEO is polynomial time solvable. Moreover, if G is not chordal, then $\text{gc}(G)$ cannot be n .*

By definition of a CEO, it follows that the class of convexable graphs is a subclass of graphs admitting a CEO and so, by Corollary 3.2, a subclass of chordal graphs.

Corollary 3.3. *Every convexable graph admits a CEO and, consequently, is chordal.*

The converse of Corollary 3.3 is not true since the 6-fan F_6 , depicted in Figure 4, clearly admits a CEO but it is not convexable (Lemma 3.4). Recall that the k -fan with $k \geq 1$ is the graph F_k obtained from a path (v_1, \dots, v_k) with k vertices by adding a universal vertex v_0 to it.

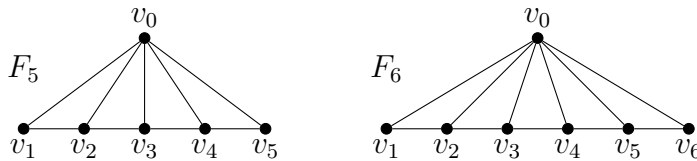


Figure 4: F_5 is convexable, but F_6 is not.

Lemma 3.4. *F_5 is convexable, but F_k for $k \geq 6$ is not convexable.*

Proof. First consider the 5-fan F_5 . Alice first selects v_3 . Then Bob can only select v_0 , v_2 or v_4 . If Bob selects v_0 , Alice selects v_2 . If Bob selects v_2 or v_4 , Alice selects v_0 . After this, note that the extreme points v_1 and v_5 cannot be selected before their neighbors v_2 and v_4 has been selected. Thus, all vertices are selected and, consequently, F_5 is convexable.

Now consider the 6-fan F_6 . First suppose that Alice selects v_3 in her first move. Then Bob selects v_2 , Alice is forced to select v_0 and thus Bob selects v_6 , ensuring that v_4 and v_5 cannot be selected during the game. If Alice selects v_2 in her first move, Bob selects v_3 and we have the same situation. Now suppose that Alice selects v_1 in her first move. Then Bob selects v_2 , Alice is forced to select v_0 and thus Bob selects v_5 , ensuring that v_3 and v_4 cannot be selected during the game. Analogously, by symmetry, if Alice selects v_4 , v_5 or v_6 in her first move. Therefore, F_6 is not convexable. We can easily extend it to prove that the k -fan F_k is not convexable for $k \geq 6$. \square

The following lemma shows that there are convexable interval graphs containing non-convexable interval graphs as induced subgraph. For this, let F'_7 be the graph of Figure 5 obtained from the 7-fan F_7 by adding one extra universal vertex v'_0 . Notice that F_6 and F_7 are induced subgraphs of F'_7 .

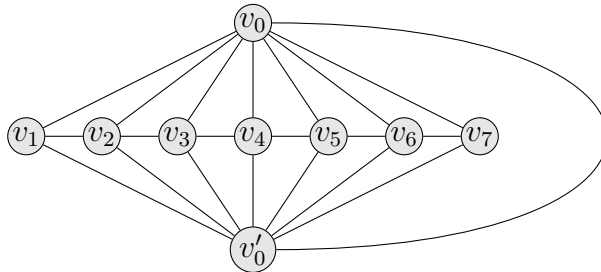


Figure 5: Convexable graph F'_7 containing F_7 (and so F_6) as induced subgraphs.

Lemma 3.5. *The graph F'_7 of Figure 5 is convexable.*

Proof. Alice first selects v_4 . Then Bob can only select v_0, v'_0, v_3 or v_5 . If Bob selects v_0 or v'_0 , Alice selects v_3 . If Alice selects v_3 or v_5 , Bob selects v_0 . After this, Bob is forced to select v_0 or v'_0 , and then Alice selects v_3 or v_5 in her next move.

Thus, the selected vertices at this point are exactly v_0, v'_0, v_3, v_4 and v_5 . Bob cannot select neither v_1 nor v_7 . Then he is forced to select v_2 or v_6 , and Alice can ensure that all vertices will be selected. \square

Monophonic convexity. Regarding the monophonic convexity (based on induced paths, instead of shortest paths), the converse of Corollary 3.3 is true. That is, we say that a set S of vertices of a graph G is *m-convex* if, for every $u, v \in S$, every w on an induced u - v path also belongs to S [18]. We say that a graph is *m-convexable* if it is convexable in the monophonic convexity. Also let CFG_m be the Convex Set Forming Game in the monophonic convexity. We will use the following lemma regarding chordal graphs. It is known that every minimal separator of a chordal graph is a clique [9].

Lemma 3.6 ([19]). *If S is a minimal u - v separator of a chordal graph G , then the connected components of $G - S$ containing u or v have a vertex that is adjacent to all vertices of S .*

Theorem 3.7. *A graph is m-convexable if and only if it is chordal.*

Proof. It is easy to see that Lemma 3.1 and Corollaries 3.2 and 3.3 also work on the monophonic convexity. Let us now prove the converse. Let G be a connected

chordal graph. We prove that, for every m -convex set $K \subsetneq V(G)$, there is a vertex $v \in V(G) \setminus K$ such that $K \cup \{v\}$ is m -convex. Let C be a connected component of $G - K$ and let $S = N(C) \subseteq K$. Notice that S is a clique, since otherwise there is an induced path between non-adjacent vertices of S whose internal vertices are in C , a contradiction because K is m -convex. Let $c \in C$ and let G' be the chordal graph obtained from G by adding a new vertex b adjacent to all vertices of S . Then S is a minimal b - c separator of G' and, from Lemma 3.6, there is a vertex $v \in C$ adjacent to every vertex of S . Therefore $K \cup \{v\}$ is m -convex in G . Finally, if the chordal graph is not connected, the game proceeds in its components and every vertex of each component is selected during the game. \square

The above proof implies:

Corollary 3.8. *Let G be a chordal graph. Alice wins the normal (resp. misère) variant of the Convex Set Forming Game in the monophonic convexity (CFG_m) if and only if the number of vertices is odd (resp. even). Moreover, $\text{gc}_m(G) = n$, where $\text{gc}_m(G)$ is the game convexity number related to the monophonic convexity.*

In what follows, we return to the geodesic convexity and identify subclasses of connected chordal graphs that are convexable.

3.2 Ptolemaic graphs

Ptolemaic graphs are defined as distance-hereditary chordal graphs. They correspond exactly to chordal graphs without the 4-fan F_4 (also called gem) as an induced subgraph. This class is incomparable with proper interval graphs as there exist proper interval graphs that are not Ptolemaic (the gem), and Ptolemaic graphs that are not proper interval (the claw).

Theorem 3.9. *Every Ptolemaic graph G is convexable.*

Proof. Let G be a Ptolemaic graph. We actually prove a stronger result: for every convex set $K \subsetneq V(G)$, there exists a vertex $v \in V(G) \setminus K$ such that $K \cup \{v\}$ is convex (i.e., every convex set K is the prefix of a CEO for G). In other words, in the Convex Set Forming game (CFG), Alice can always find a vertex to be selected to extend the convex set of selected vertices and, whatever be her choice, Bob will have an available vertex to select next.

Let C be a connected component of $G - K$ and let $S = N(C) \subseteq K$. If $S = \{a\}$, then any neighbour v of a in C is such that $K \cup \{v\}$ is convex. Hence, let us assume that $|S| > 1$. Let a and b be two vertices in S . First, let us assume that a and b are not adjacent and consider such a and b as close as possible (while not adjacent). Let P be a shortest a - b path and Q be an induced a - b path whose internal vertices are in

C . Since K is convex, then $V(P) \subseteq K$ and Q is longer than P . If all internal vertices of P are in $K \setminus S$, then $P \cup Q$ is an induced cycle of size at least 5, contradicting the fact that G is chordal. Then, by the minimality of the choice of a and b , $V(P) \subseteq S$ and consequently (by minimality of the distance between a and b) $P = (a, x, b)$ has exactly three vertices for some $x \in S$. Since G is chordal, all internal vertices of Q are adjacent to x . Therefore, $Q \cup \{x\}$ contains a 4-fan (gem) as induced subgraph, a contradiction.

Thus a and b are adjacent and, consequently, S is a clique. Let $c \in C$ and let G' be the chordal graph obtained from G by adding a new vertex b adjacent to all vertices of S . Then S is a minimal b - c separator of G' and, from Lemma 3.6, there is a vertex $v \in C$ adjacent to every vertex of S . Therefore $K \cup \{v\}$ is convex in G .

Finally, if the Ptolemaic graph is not connected, the game proceeds in its components and every vertex of each component is selected during the game. \square

The above proof implies:

Corollary 3.10. *Let G be a Ptolemaic graph. Alice wins the normal (resp. misère) variant of the Convex Set Forming Game (CFG) if and only if the number of vertices is odd (resp. even). Moreover, $\text{gc}(G) = n$.*

We show in the next lemma that the above result cannot be extended to F_5 -free chordal graphs. Figure 6 shows a non-convexable chordal graph that contains no induced F_5 .

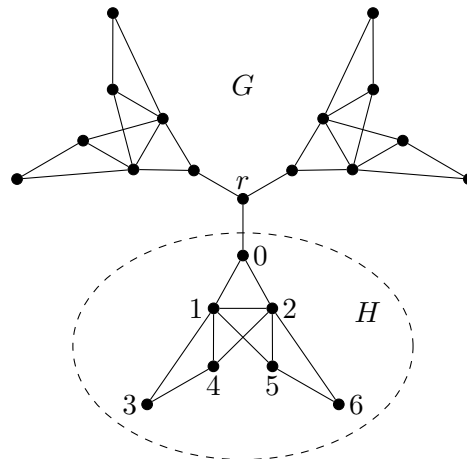


Figure 6: Non-convexable graph containing induced 4-fans but no induced 5-fan.

Lemma 3.11. *There are connected chordal graphs without induced 5-fan that are not convexable.*

Proof. First consider the game in the graph H , with vertices $V = \{0, \dots, 6\}$, shown inside the dotted circle of Figure 6. If Bob starts by selecting the vertex 0 of H , then Alice has to answer 1 (or symmetrically 2) and Bob can select 3 (resp., 6) and then none of 2 or 4 (resp., 1 or 5) can be selected anymore.

Let G be obtained from three copies X, Y and Z of H by adding a new vertex r adjacent to each of the vertices 0 of the copies of H (Figure 6). It can be checked that G is connected chordal and F_5 -free.

Let us assume that Alice does not play first in X . Let us consider the first step when r is selected. If it is by Alice, then Bob selects the vertex 0 of Y . Otherwise, at his next turn, Bob selects the vertex 0 of the copy Y or Z that has not been played yet (say it is Y). Then, by the parity of the number of the vertices, Bob may ensure that Alice is the next one to play in Y . Following the reasoning as above, all vertices of Y cannot be selected. Hence, G is not convexable. \square

In another direction, we know several convexable graphs that are not Ptolemaic. For example, the 5-fan F_5 (see Lemma 3.4), which belongs to the class of Interval graphs, considered in the following section.

3.3 Interval graphs

An *interval graph* is the intersection graph of real intervals. In this section, we are interested in CFG-OPT in interval graphs: the optimization version of the geodesic Convex Set Forming Game.

A *unit interval graph* G is the intersection graph of a set of real intervals of unit length. Precisely, given a set $V = \{ [h_i, h_i + 1] : i = 1, \dots, n \}$ of unit-length real intervals ($h_i \in \mathbb{R}$ for $i = 1, \dots, n$), its intersection graph G with vertices v_1, \dots, v_n is the graph such that v_i and v_j are adjacent if and only if $[h_i, h_i + 1]$ and $[h_j, h_j + 1]$ have non-empty intersection. Note that we may assume that all h_i 's are distinct, i.e., G is also a *proper interval graph* (no interval is contained in another). Actually, a graph is a proper interval graph if and only if it is a unit interval graph [3]. Note that recognizing a unit interval graph and obtaining an interval representation of it can be done in linear time [8].

Note that any k -fan is an interval graph. However, it is a unit interval graph if and only if $k \leq 4$. In particular, no unit interval graph contains an induced 5-fan F_5 . From now on, let $[a, b]$ with $a < b \in \mathbb{N}$ denote the set $\{a, a + 1, \dots, b - 1, b\}$.

Theorem 3.12. *Every connected unit interval graph is convexable.*

Proof. Let G be a connected unit interval graph with vertex-set ordered (v_1, \dots, v_n) such that $h_1 < h_2 < \dots < h_n$. We will describe a strategy for Alice ensuring that all vertices are eventually selected.

The idea is to define the notion of *extendable* convex set and to show that, if the set of selected vertices (after Alice's turn) is extendable, then after a finite number of steps, Alice can reach a larger extendable set (after her turn). Moreover, once v_1 and v_n belong to the set of selected vertices that is extendable, then all remaining vertices can be added in any order. The first move of Alice is always to select the vertex v_1 which (as it can be seen below) is an extendable set. So the proof below is by induction on the size of the extendable set of selected vertices after some moves of Alice.

We say that a convex set S is *extendable* if there exists $m \geq 1$ and m intervals $R = [l_1 = 1, r_1] \cup \dots \cup [l_m, r_m]$ (with $l_j \leq r_j < l_{j+1} - 1$ for any $1 \leq j < m$ and $l_m \leq r_m \leq n$), and $I_j \subseteq \{v_\ell : \ell \in [r_{j-1} + 1, l_j - 1]\}$ for any $1 < j \leq m$ such that:

1. $S = \{v_i\}_{i \in R} \cup \bigcup_{1 < j \leq m} I_j$;
2. $\{v_\ell : \ell \in [r_{j-1}, l_j]\}$ induces a complete graph for any $1 < j \leq m$, and
3. $N(\{v_{r_{j+1}}, \dots, v_n\}) \cap \{v_1, \dots, v_{r_j}\} \subseteq \{v_{l_j}, \dots, v_{r_j}\} \subseteq S$ for every $1 \leq j \leq m$.

Claim 3.13. *Let S be extendable and let S' be a convex set with $S \subseteq S'$. Then, for every $i \leq r_m$, $S' \cup \{v_i\}$ is convex. Moreover, if S' is extendable, then $S' \cup \{v_i\}$ is extendable.*

Proof of Claim. If $v_i \in S$, the result holds directly. So, assume that $v_i \notin S$. Then there is a unique $1 < j \leq m$ such that $v_i \in \{v_{r_{j-1}+1}, \dots, v_{l_j-1}\} \setminus I_j$. For contradiction, suppose that $S' \cup \{v_i\}$ is not convex. Therefore, there exists a shortest path $P = (u_0 = v_i, u_1, \dots, u_p = w)$ from v_i to some vertex w in $S' \cap \{v_1, \dots, v_{r_{j-1}}\}$ (the case in $S' \cap \{v_{l_j}, \dots, v_n\}$ is symmetric) with $u_1 \notin S'$. Then, by Property 3, $u_1 \in \{v_{r_{j-1}+1}, \dots, v_{l_j-1}\} \setminus I_j$. By Property 2 and the fact that G is a unit interval graph, then $P' = (v_{l_j}, u_1, \dots, u_p = w)$ is also a shortest path. Indeed, if $(v_{l_j}, u'_1, u'_2, \dots, w)$ were a path shorter than P' from v_{l_j} to w , then, because G is a unit interval graph, $(v_i, u'_1, u'_2, \dots, w)$ would also be a path contradicting the fact that P is a shortest v_i - w path. This contradicts the fact that S' is convex.

Then, $S' \cup \{v_i\}$ is extendable by just adding v_i to I_j . ◇

In particular, the previous claim implies that if, at some step, the set of selected vertices is extendable and contains v_1 and v_n (i.e., $r_m = v_n$), then Alice eventually ensures that all vertices will be selected.

Claim 3.14. *If S is extendable, then $S \cup \{v_{r_m+1}\}$ is extendable.*

Proof of Claim. Since each interval in the interval representation of G has unit length, v_{r_m+1} is adjacent to every vertex in $N(\{v_{r_m+1}, \dots, v_n\}) \cap \{v_1, \dots, v_{r_m}\}$. Therefore, every shortest path from v_{r_m+1} to a vertex in S has its first vertex in $N(\{v_{r_m+1}, \dots, v_n\}) \cap \{v_1, \dots, v_{r_m}\} \subseteq S$. Hence, $S \cup \{v_{r_m+1}\}$ is convex.

To show that it is extendable, it is sufficient to replace r_m by $r_m + 1$ and all three properties are satisfied for $S \cup \{v_{r_m+1}\}$ since S is extendable. \diamond

Now, let us assume (by induction on $r_m = i \geq 1$) that, after Alice's turn, the set S of selected vertices is extendable (which is the case after the first move of Alice who starts by selecting v_1). By the previous claims, if Bob selects a vertex v_j with $j \leq i + 1$ and Alice selects v_{i+1} (if $j \leq i$) or v_{i+2} (if $j = i + 1$), then the induction hypothesis is satisfied for a larger extendable set of selected vertices.

Hence, let us assume that Bob selects v_j with $j > i + 1$ (so $S \cup \{v_j\}$ is convex).

Claim 3.15. *If $j = n$ or v_j is a cut-vertex (separating the set $\{v_1, \dots, v_{j-1}\}$ from $\{v_{j+1}, \dots, v_n\}$), then the set of selected vertices is extendable.*

Proof of Claim. By setting $j = r_{m+1} = l_{m+1}$ and adding v_j to R , properties 1 and 3 are satisfied. Since $S \cup \{v_j\}$ is convex, it induces a connected subgraph and so v_j must be adjacent to $v_{r_m} = v_i$. Therefore, because G is a unit interval graph, $\{v_i, v_{i+1}, \dots, v_j\}$ induces a complete graph and Property 2 also holds. \diamond

So, if $j = n$, by Claim 3.13, all vertices will eventually be selected. Moreover, if v_j is a cut-vertex, then, by selecting v_{j+1} , Alice leads to a larger extendable set by Claim 3.14. Hence, we may assume that $j < n$ and v_j is not a cut vertex.

Note that, since G is a connected unit interval graph, then $\{v_i v_j\} \in E$. Let $k > j$ be the smallest integer such that $\{v_j v_k\} \notin E$ (if k does not exist, let us set $k = n + 1$). Note that $k > j + 1$ since G is connected. Note that, because G is an interval graph, $\{v_i, \dots, v_j\}$ and $\{v_j, \dots, v_{k-1}\}$ induce complete graphs.

Let $i < h < j$ be the smallest integer such that v_h is adjacent to some vertex v_p with $p > j$ (h exists because v_j is not a cut-vertex).

Claim 3.16. *$S \cup \{v_j, v_h\}$ is convex.*

Proof of Claim. By contradiction, assume that there is a shortest path $(u_0 = v_h, u_1, \dots, v_p)$ from v_h to $v_p \in S$ with $u_1 \notin S \cup \{v_j, v_h\}$. Then, (v_j, u_1, \dots, v_p) is also a shortest path, contradicting that $S \cup \{v_j\}$ is convex. \diamond

Then, Alice can select v_h and does it.

Claim 3.17. *Any subset of $\{v_1, \dots, v_j\}$ containing $S \cup \{v_h, v_j\}$ is convex.*

Proof of Claim. For the vertices $v_t \notin S \cup \{v_h, v_j\}$ with $t < i$, it follows from Claim 3.13. For the vertices $x \in \{v_i, \dots, v_j\}$, it comes from the fact that $\{v_i, \dots, v_j\}$ induces a clique. Indeed, if adding x would not induce a convex set, then adding v_j would not have induced a convex set (the proof is similar to the one of Claim 3.16).

\diamond

Claim 3.18. *Let $S' \subseteq \{v_1, \dots, v_j\}$ be a convex super-set of $S \cup \{v_h, v_j\}$ such that $\{v_h, \dots, v_j\} \setminus S' \neq \emptyset$. Then, $S' \cup \{v_t\}$ with $t > j$ is not convex.*

Proof of Claim. Let $v_k \in \{v_h, \dots, v_j\} \setminus S'$. Let $P = (u_0 = v_t, u_1, \dots, u_p)$ be any shortest path from v_t to a vertex $u_p \in S'$. Note that $u_1 \in \{v_h, \dots, v_j\}$ by definition of h and because $S' \cup \{v_t\}$ must be convex. If u_2 is adjacent to v_k , then $(u_0 = v_t, v_k, \dots, u_p)$ is also a shortest path showing that $S' \cup \{v_t\}$ is not convex. Hence, u_2 is not a neighbour of v_k , but then u_2 is not adjacent to v_j either (G being a unit interval graph). Therefore $(v_j, v_k, u_2, \dots, u_p)$ is also a shortest path and so S' is not convex, a contradiction. \diamond

Claim 3.19. *Let $S' \subseteq \{v_1, \dots, v_j\}$ be a convex super-set of $S \cup \{v_h, \dots, v_j\}$. Then, S' is extendable.*

Proof of Claim. By construction, Property 1 of the definition of a extendable set is satisfied by S' . Let us set $h = l_{m+1}$ and $j = l_{m+1}$. Properties 2 and 3 are satisfied for every $j' \leq m$ because S is extendable and by Claim 3.13 for the vertices $v_{i'}$ with $i' \leq r_m = i$. Property 3 for $m + 1$ is satisfied by definition of h . Finally, Property 2 for $m + 1$ is satisfied, since v_j is adjacent to v_{r_m} and G is a unit interval graph. \diamond

From Claims 3.17 and 3.18, it follows that the vertices in $\{v_h, \dots, v_j\}$ must (and can) be selected before any vertex v_p can be selected with $p > j$. Moreover, once the vertices of $\{v_h, \dots, v_j\}$ are selected, we reach a larger extendable subset by Claim 3.19. If Alice was the one to complete that, then the induction hypothesis (with a larger extendable set) is reached. Otherwise, she can select v_{j+1} by Claim 3.14 and, again, we are done. \square

Remark. *The converse of the Theorem 3.12 is not true since the claw (star with three leaves) is convexable, but is not a proper interval graph. More generally, we have seen that Ptolemaic graphs are convexable and many of them are not proper interval graphs. In particular:*

Corollary 3.20. *Every tree and, more generally, every block graph (every two-connected component is a clique) is convexable.*

4 Largest CEO set in Outerplanar graphs

Given a graph G , a set $S \subseteq V(G)$ is called a CEO set if there exists an ordering $(v_1, \dots, v_{|S|})$ of S such that, for every $1 \leq i \leq |S|$, $G[v_1, \dots, v_i]$ induces a convex subgraph of G . We say that v_1 (resp., v_1 and v_2) is (are) the *starting* vertex (vertices) of S . Here, we show that computing a largest CEO set of outerplanar graph can be

done in polynomial time. This problem can be seen as the optimization version of the Convex Set Forming Game CFG when Bob wants to help Alice.

A graph is *outerplanar* if it admits a planar embedding (a planar drawing without crossing edges) such that all vertices lie on the outerface. An important property that will be used is that every internal face of an outerplanar graph corresponds to an isometric cycle.

A *tree-decomposition* of a graph $G = (V, E)$ consists of a pair (T, \mathcal{X}) where T is a tree and $\mathcal{X} = (X_t)_{t \in V(T)}$ is a family of subsets (called *bags*) of vertices of G indexed by the nodes of T such that :

- $\bigcup_{t \in V(T)} X_t = V$;
- for every $\{u, v\} \in E$, there exists $t \in V(T)$ with $u, v \in X_t$, and
- for every $u \in V$, $\{t \in V(T) \mid u \in X_t\}$ induces a subtree of T .

It can easily be shown by induction on the number of faces that any outerplanar graph $G = (V, E)$ admits a tree-decomposition (T, \mathcal{X}) where T is a tree rooted in some node $r \in V(T)$ and every bag either induces a tree (called a *tree-bag*) or a face (called a *face-bag*) such that, for every tree-bag $X_t \in \mathcal{X}$, each child t' (and also the parent) of t corresponds to a face-bag and $X_t \cap X_{t'}$ is a single vertex and, for every face-bag $X_t \in \mathcal{X}$ with a face-bag child $X_{t'}$, X_t and $X_{t'}$ intersect in a single vertex or in an edge.

For every node $t \in V(T)$ of T (rooted at r), let T_t be the subtree of T rooted in t and let $G_t = G[\bigcup_{t' \in T_t} X_{t'}]$.

We describe a recursive algorithm that takes a node $t \in V(T)$ as input (let $X^t = X_t \cap X_{t'}$ with t' the parent of t , and $X^t = \emptyset$ if $t = r$) and computes $(S_{opt}^t, S_x^t, S_y^t, S_{xy}^t)$ satisfying the following:

- S_{opt}^t is a largest CEO set in G_t ;
- if $X^t = \{x\}$, S_x^t is a CEO set in G_t , such that $x \in S_x^t$ and S_x^t is largest among such CEO sets; and $S_y^t = S_{xy}^t = \emptyset$; Moreover, x is a starting vertex of S_x^t .
- if $X^t = \{x, y\}$ (in which case $\{x, y\} \in E(G)$):
 - S_x^t is a CEO set in G_t , such that $x \in S_x^t$ and $y \notin S_x^t$ and S_x^t is largest among such CEO sets; Moreover, x is a starting vertex of S_x^t .
 - S_y^t is a CEO set in G_t , such that $y \in S_y^t$ and $x \notin S_y^t$ and S_y^t is largest among such CEO sets; Moreover, y is a starting vertex of S_y^t .
 - S_{xy}^t a CEO set in G_t , such that $x, y \in S_{xy}^t$ and S_{xy}^t is largest among such CEO sets; Moreover, x and y are starting vertices of S_{xy}^t .

The algorithm proceeds as follows.

- If X_t is a leaf-bag (i.e., t is a leaf of T), let $X^t = \{x\}$ or $X^t = \{x, y\}$ (where $X^t = X_t \cap X_{t'}$ with t' the parent of t) or $X^t = \emptyset$ if $t = r$:
 - if X_t is a tree-bag, then $S_{opt} = S_x = X_t$ and $S_y^t = S_{xy}^t = \emptyset$.
 - if X_t is a face-bag, let (v_0, v_1, \dots, v_p) be the corresponding face with, if $X_t \neq \emptyset$, $v_0 = x$ and (if $X^t = \{x, y\}$) $v_1 = y$. Then, $S_{opt}^t = S_x^t = \{v_0, v_p, v_{p-1}, \dots, v_{\lfloor \frac{p}{2} \rfloor + 1}\}$, $S_y^t = \{v_1, v_2, \dots, v_{\lfloor \frac{p}{2} \rfloor + 1}\}$, $S_{xy}^t = \{v_0, v_1, \dots, v_{\lfloor \frac{p}{2} \rfloor}\}$ (and $S_y^t = S_{xy}^t = \emptyset$ if y is not defined).
- Now, let $t \in V(T)$ with $d \geq 1$ children t_1, \dots, t_d and assume that, for every $1 \leq i \leq d$, $(S_{opt}^{t_i}, S_{x_i}^{t_i}, S_{y_i}^{t_i}, S_{x_i y_i}^{t_i})$ has been computed recursively (where $X^{t_i} = \{x_i\}$ or $X^{t_i} = \{x_i, y_i\}$).
 - if X_t is a tree-bag, then let $S_x^t = X_t \cup \bigcup_{1 \leq i \leq d} S_{x_i}^{t_i}$. Let $1 \leq j \leq d$ be such that $S_{opt}^{t_j}$ is a largest set among the sets $S_{opt}^{t_i}$, $1 \leq i \leq d$. If $|S_x^t| \geq |S_{opt}^{t_j}|$, then $S_{opt}^t = S_x^t$ and $S_{opt}^t = S_{opt}^{t_j}$ otherwise.
 - if X_t is a face-bag, let $F = (v_0, v_1, \dots, v_p)$ be the corresponding face. Set $\mathcal{Y} = \emptyset$.
 - * For every $a, b \in V(F)$ (except if the distance between a and b equals $\frac{p+1}{2}$ and p is odd), let P_{ab} be the (unique) shortest a - b path (P_{ab} lies in F). Set $Y_{ab} = P_{ab}$ (in the particular case when F is a triangle, then $P_{ab} = V(F)$). For every $1 \leq i \leq d$, if $X^{t_i} \cap P_{ab} = \{x_i\}$ then add $S_{x_i}^{t_i}$ to Y_{ab} ; if $X^{t_i} \cap P_{ab} = \{y_i\}$ then add $S_{y_i}^{t_i}$ to Y_{ab} ; and if $X^{t_i} \cap P_{ab} = \{x_i, y_i\}$ then add $S_{x_i, y_i}^{t_i}$ to Y_{ab} . Add Y_{ab} to \mathcal{Y} . (note that, if F is a triangle, $|\mathcal{Y}| = 1$).
 - * Let S_x^t be any largest set in \mathcal{Y} that contains x (but not y if y is defined); let S_y^t be any largest set in \mathcal{Y} that contains y but not x and let S_{xy}^t be any largest set in \mathcal{Y} that contains both x and y .
 - * Let $1 \leq j \leq d$ be such that $S_{opt}^{t_j}$ is a largest set among the sets $S_{opt}^{t_i}$, $1 \leq i \leq d$. If $S_{opt}^{t_j}$ is largest than any set in \mathcal{Y} , then $S_{opt}^t = S_{opt}^{t_j}$; otherwise let S_{opt}^t be any largest set in \mathcal{Y} .

Theorem 4.1. *Let G be any n -node outerplanar graph. The above algorithm computes a largest CEO-set of G in time $O(n^3)$.*

Proof. Let (T, \mathcal{X}) be a tree-decomposition of G as described above (it can be computed in linear-time). In particular, T is rooted at some node r .

Since the number of bags of the tree-decomposition is $O(n)$ and that the computation time, for each bag, is dominated by the fact of considering each pair of vertices (for a face-bag), the time complexity is $O(n^3)$.

We show by induction that, for every bag X_t , the computed sets $(S_{opt}^t, S_x^t, S_y^t, S_{xy}^t)$ satisfy the desired properties. Then, S_{opt}^r will be the desired solution.

The result clearly holds for any leaf bag. So let $t \in V(T)$ with $d \geq 1$ children (t_1, \dots, t_d) and assume, by induction, that $(S_{opt}^{t_i}, S_{x_i}^{t_i}, S_{y_i}^{t_i}, S_{x_i y_i}^{t_i})$ satisfy the desired properties for every $1 \leq i \leq d$.

- Let us first consider the case when X_t is a tree-bag. Recall that, for every $1 \leq i \leq d$, $X^{t_i} = \{x_i\}$ where x_i is a cut vertex of G . Let S be a largest CEO of G_t intersecting X_t .

Claim 4.2. $X_t \subseteq S$.

Proof of Claim. Otherwise, let $v \in (X_t \setminus S) \cap N(S)$, then $S \cup \{v\}$ is convex and so it is a larger CEO (by placing v at the end of the ordering) contradicting the maximality of S . \diamond

Claim 4.3. For every $1 \leq i \leq d$, $S \cap V(G_{t_i})$ is a largest CEO set of G_{t_i} among those sets that contains x_i .

Proof of Claim. For purpose of contradiction, assume that this is not true for some $1 \leq i \leq d$. Let $(v_1, \dots, v_{j-1}, v_j = x_i, v_{j+1}, \dots, v_h)$ be the CEO (induced by the one of S) for the set $S \setminus (V(G_{t_i}) \setminus \{x_i\})$ and let O_i be the CEO of a largest CEO-set of G_{t_i} containing x_i . Note that O_i can be chosen such that it start from x_i (since by induction, $S_{x_i}^{t_i}$ is a largest CEO-set of G_{t_i} containing x_i and it can start from x_i). Then, Let $(v_1, \dots, v_{j-1}) \odot O_i \odot (v_{j+1}, \dots, v_h)$ is a CEO for G_t , largest than S , a contradiction. \diamond

Now, let $\{x\} = X^t$ (unless $t = r$) and let \mathcal{O} be a CEO of X_t starting in x (this clearly exists since X_t is a tree) and, for every $1 \leq i \leq d$, let O_i be the ordering of $S_{x_i}^{t_i}$. Then, $\mathcal{O} \odot O_1 \odot \dots \odot O_d$ is a maximum CEO set of G_t containing x . And x is a starting vertex.

Hence, S_x^t satisfies the desired properties. The fact that S_{opt}^t also satisfies the desired properties trivially holds (since by induction, all $S_{opt}^{t_i}$ are maximum in G_{t_i} and, if they do not intersect X_t , they are inclusion maximal in G_t).

- Second, assume that X_t is a face-bag and let $F = (v_0, \dots, v_p)$ be the corresponding face. Recall that, for every $1 \leq i \leq d$, $X^{t_i} = \{x_i\}$ where x_i is a cut vertex of G or $X^{t_i} = \{x_i, y_i\}$ where $\{x_i, y_i\}$ is an edge separator. Let S be a inclusion maximal CEO of G_t intersecting X_t .

Claim 4.4. If X_t is a triangle, i.e., if $p = 2$, then $X_t \subseteq S$.

Proof of Claim. Otherwise, let $v \in (X_t \setminus S) \cap N(S)$, then $S \cup \{v\}$ is convex and so it is a larger CEO (by placing v at the end of the ordering) contradicting the maximality of S . \diamond

Claim 4.5. *If $p > 2$, then there exist $a, b \in V(F)$ (and with the distance between a and b is $< \frac{p+1}{2}$ if p is odd) such that $S \cap X_t = P_{ab}$ where P_{ab} is the unique shortest path between a and b .*

Proof of Claim. First, $S \cap X_t$ must be connected by convexity of S . Second, let $O = (u_1, \dots, u_{|S|})$ be a CEO of S and let v_j be the last vertex of $S \cap X_t$ appearing in O . If $S \cap X_t = F$, then (v_1, \dots, v_{j-1}) is not convex (because $p > 2$), a contradiction. Hence, $S \cap X_t$ must be a path P_{ab} between two vertices $a, b \in F$ and P_{ab} must be the unique shortest path between a and b (recall that F induces an isometric cycle in G) since otherwise, (v_1, \dots, v_j) would not be convex. \diamond

Claim 4.6. *For every $1 \leq i \leq d$, let $Y_i = X_t \cap S \subseteq X^{t_i}$. Then, $S \cap V(G_{t_i})$ is a largest CEO set of G_{t_i} among those CEO sets S' of G_{t_i} such that $S' \cap X^{t_i} = Y_i$.*

Proof of Claim. The proof is similar to the one of Claim 4.3. \diamond

The end of the proof of this case is a direct adaptation of the end of the proof of the previous case (when X_t is a tree-bag). \square

5 Conclusions and Open Problems

In this paper, we have introduced a new two player game related to (geodesic) convexity, the Convex Set Forming Game CFG, and show that determining its outcome is PSPACE-complete in general. It would be interesting to determine a large class of graphs in which the problem can be solved in polynomial time.

Regarding the optimization variant of the CFG, the class of convexable graphs seems particularly interesting. We have exhibited several classes of convexable graphs and the situation is summarised in the next proposition, and illustrated in Figure 7.

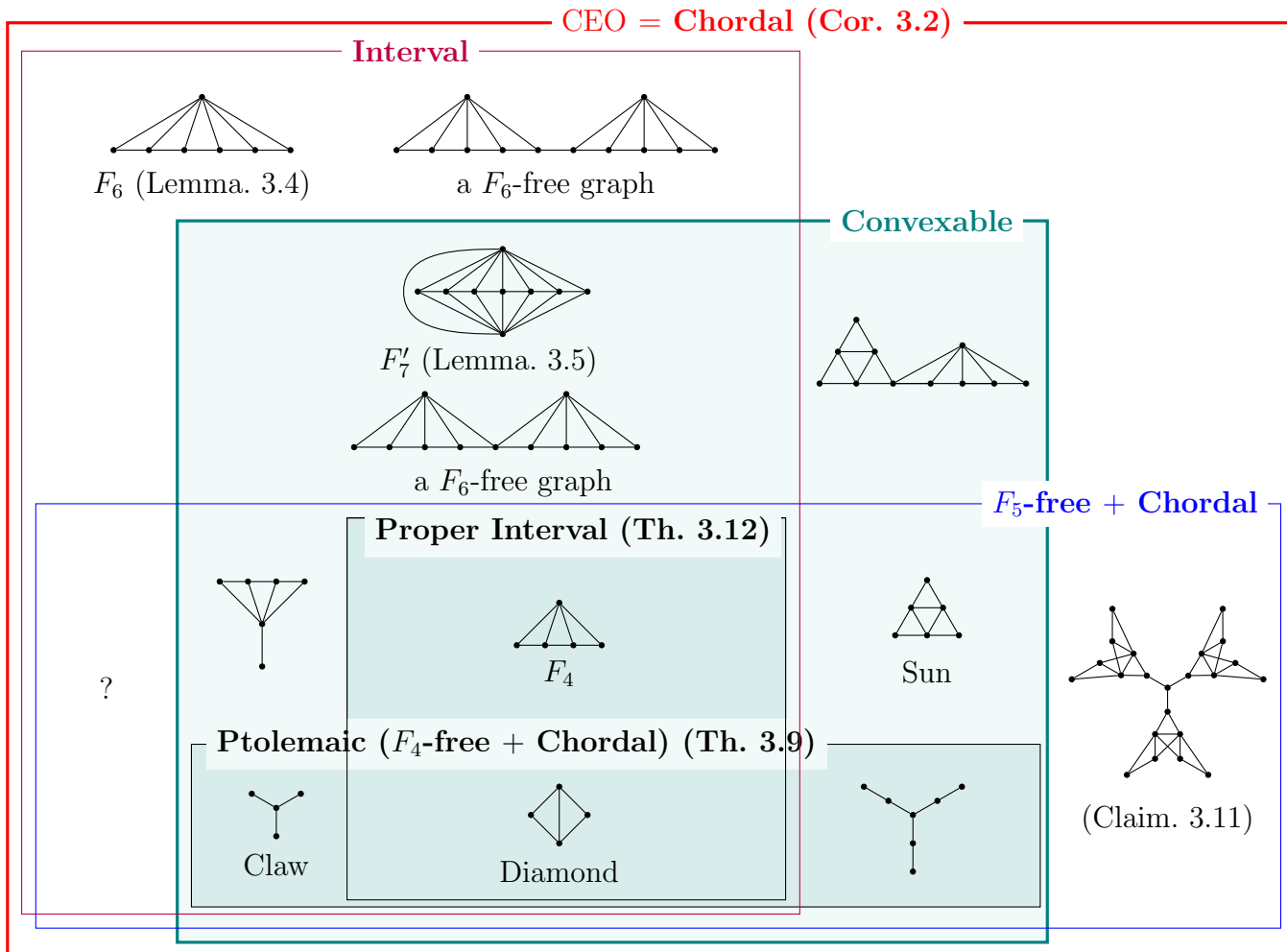


Figure 7: Partial characterization of Convexable graphs: a visual summary. All graphs in the classes depicted in “dark green” (Proper Interval \cup Ptolemaic) are convexable. Graphs depicted in the “light green” areas are examples of convexable graphs (to illustrate that these area are not empty). Graphs depicted in the “white” areas are examples of non convexable graphs. The question mark indicates that we do not know whether there exist Interval F_5 -free not convexable graphs.

Proposition 5.1. *The following inclusions hold:*

- $Ptolemaic \subsetneq Convexable \subsetneq Chordal$,
- $Connected\ Proper\ interval \subsetneq Convexable \subsetneq Chordal$,

- $Connected\ Interval \setminus Convexable \neq \emptyset$, and
- $(Chordal \setminus Interval) \cap Convexable \neq \emptyset$.

Proof. The first inclusion is the Theorem 3.9 and it is proper because F_5 is convexable. The second inclusion is the Corollary 3.3 and it is proper from Lemma 3.4. The third inclusion comes from Theorem 3.12. The fact that it is a proper inclusion comes from Corollary 3.20. \square

We were unfortunately unable to fully characterize the class of convexable graphs. For instance,

Question 1. *Are there interval F_5 -free connected graphs that are not convexable?*

Actually, the case of fans is intriguing. For instance (see Figure 7), we have examples with 2 copies of F_5 as induced subgraphs that are convexable and some examples that are not: take two copies of F_5 . In one case identify their vertices v_1 , and in the second case make their vertices v_1 adjacent. This seems to relate (but not only) with the “parity of the distance between the 2 copies”.

Question 2. *What is the complexity of deciding whether a graph is convexable?*

On our way to characterize convexable graphs, we defined the notion of sets of vertices admitting a CEO.

Question 3. *Given a graph G and $k \in \mathbb{N}$, what is the complexity of deciding whether $G = (V, E)$ contains a subset $S \subseteq V$, $|S| \geq k$, such that S admits a CEO?*

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