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THE UNIVERSITY RULE? HOW IS IMPORTANT A GIVEN TOPIC ON THE SUCCESS OF A STUDENT ?*

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Abstract

It may seem relevant for success that a bachelor or a master degree student understand what it takes to be graduated. Common graduation requirements include several tests on different topics in order to complete all credit hours. Every program and institution can differ in the minimum grade that validates a test, as well as in the aggregation process of the various grades. To model such differences in the process, we introduce the university rule defined as follows: (i) Students must pass several topics; (ii) There are several periods (*eg.* semesters, years); (iii) For each period and each topic, there is a test; (iv) The result of the test on each topic and each period is a grade; (v) A decision process maps the result of each test to a final result indicating whether the student has obtained the required graduation or not. Understanding the process is a first step for the success of a student. A second step consists in knowing the importance of a given topic within the final result. Given the university rule, we present tools that evaluate the importance of a given topic. In particular, we extend to the university rule, the two well-known importance (power) indices of Shapley-Shubik (Shapley and Shubik [1954]) and Banzhaf (Banzhaf [1965]).

KEYWORDS: University rules; Importance (power) index; Shapley-Shubik index; Banzhaf index.

JEL Classification: C71, D71

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1 Introduction

Consider the Masters degree program "Advanced Applied Economics" offered at the University of Caen Normandy in France. The first year of the program is structured into two periods, each containing three courses (topics) : "Economic Modeling", "Econometrics" and "Thematic course" (a choice between Public Policy, Social Choice, Firms and Markets and Trade Policies). Before applying to this program, understanding what it takes to graduate may set students for success. While all programs and institutions are different, every student in a given program must fulfill all courses requirements, that is complete all credit hours (60 credits). But more importantly, the chance of success is correlated with the process of aggregation of the different test results. In the above example, there is a test for each topic at each period, and for each test a student obtains a grade between 0 and 20. One of the two following conditions must be satisfied for a student to validate the first year of the Masters degree program : i) in each period the (weighted) mean of all three grades must be equal to or above average (10 over 20); ii) the (weighted) mean of all six grades obtained during the year must be equal to or below average.

In this paper we model the process of aggregation by introducing the university rule defined as follows: (i) Students must follow several topics; (ii) There are several periods (*eg.* semesters, years); (iii) For each period and each topic, there is a test; (iv) The result of the test on each topic and each period is a grade; (v) A decision process maps the result of each test to a final result indicating whether the student has obtained the required graduation or not. The first year of the advanced Applied Economics program will then correspond to the following University rule: (i) Students must follow three topics : "Economic Modeling", "Econometrics" and "Thematic course"; (ii) There are two semesters; (iii) For each topic there is one test per semester; (iv) The grades vary from 0 to 20; (v) The decision process is introduced above. Note also that for this program, all topics do not have the same weight when we aggregate the results of the different test at each period. In the first period, "Economic Modeling" and "Econometrics" have a weight of 12 over 30 while the "Thematic course" has a weight of 6. In the second period, the "Thematic course" has a weight of 12 and the weight is 9 for the two other topics. One can then ask what the importance of "Economic modeling" is for the success of a student ? Is "Econometrics" more important than the "Thematic course"?

The succes of a student depends therefore not only on the aggregation system but

also on the different topics taught and on their weight. In that paper, given an aggregation rule (a university rule), our main question is to evaluate the importance of a given topic under the final result of a student. To answer this question, we use cooperative game theory tools, especially tools that focus on the evaluation of a player's influence (power) in a cooperative game.

There exists a large literature on cooperative game theory that studies the influence of a player within an aggregation process when the final result is dichotomous. In our context, the players are the topics and the final result is "pass" or "fail". The first kind of works evaluating the influence of a topic focuses on simple rule (called simple game in cooperative game theory) where a test can only yield two only different grades : "validate" or "not validate". The most famous theories in that context are those of the Shapley-Shubik influence index (Shapley and Shubik [1954]) and the Banzhaf influence index (Banzhaf [1965]). The readers can refer to Andjiga et al. [2003] and Laruelle and Valenciano [2008] for a detailed description of influence index when considering simple rule.

More sophisticated scenarios have considered cases in which tests may yield a wider range of grades. Felsenthal and Machover [1997, 1998, 2001] with ternary voting games and Bilbao et al. [2000] with bicooperative games introduce rules with three possible grades. A generalization of these two models are the $(j, 2)$ games introduced by Freixas and Zwicker [2003], Freixas [2005a,b], in which each topic can have a grade between j ordered possible grades.¹ Note that some models – not relevant in the context of the university rule – introduce situations in which the choices offered to the players are not totally ordered. This is for instance the case of quaternary games (Laruelle and Valenciano [2012]) or dichotomous multi-type games (Courtin et al. [2016, 2017]).

A second way to extend the standard framework of simple rules consists in considering a multi-dimensional aggregation process. Indeed in many fields, players express their choices on many questions simultaneously. For example, in situation in which a group of experts have to decide whether a candidate qualifies for a promotion, several criteria can be taken into account: performance, ability to work in a team, and leadership. Each expert gives an evaluation of the candidate for each criterion. This is also the case in our context where topics are taught in several periods. Courtin and Laruelle [2020] introduces multi-dimensional rules in which the mod-

¹ $(j, 2)$ games are derived from (j, k) games where there is k possible final outputs. See for example Kurz et al. [2021].

elling aggregation process is very close to the university rule. The main difference is that there are only two possible grades for each topic. Courtin [2022] extends and characterizes five well-known influence indices: the Shapley-Shubik influence index, the Banzhaf influence index, the Public good influence index (Holler [1982]), the Null individual free influence index (Alonso-Meijide et al. [2011]) and the Shift influence index (Alonso-Meijide and Freixas [2010]).

The university rule is thus clearly at the borderline between multi-dimensional rules and $(j, 2)$ games. The process is very similar to the multi-dimensional rule, with the difference that there is a large set of grades for each topic, as in the $(j, 2)$ rules.

In this paper, we choose to evaluate the importance of a given topic following the approach of Shapley-Shubik and Banzhaf. This paper is structured as follows. Section 2 introduces the general framework of the university rule. Section 3 defines and characterizes the Banzhaf influence index for the university rule, while Section 4 discusses some particular aggregation process. Section 5 introduces the Shapley-Shubik influence index for the university rule and Section 6 concludes.

2 General Framework

Configuration

Let $N = \{1, \dots, n\}$ be a set of n topics (a generic topic will be denoted by i, j, k), and $M = \{1, \dots, m\}$ be a set of m periods (a generic period will be denoted by a, b, c). Let G be a set of totally ordered distinct grades with the highest grade being 1 and the lowest 0.² The total number of grades is denoted g and a generic grade will be γ . Each grade γ is associated to a level of grade l_γ , with $l_\gamma \in \{1, 2, \dots, g\}$. Given two grades γ and γ' , $\gamma > \gamma'$ is equivalent to $l_\gamma < l_{\gamma'}$. For each pair of topic and period there is a test and a grade associated. We assume here that all the topics are taught at each period, therefore there are $n.m$ tests. We denote by i^a a test on a given topic at a given period, and by N^* (with $|N^*| = n.m$) the set of all tests. The result of a test on topic i in period a is denoted s_i^a with $s_i^a \in G$. For a given grade γ , let γs^a denotes the total number of tests with the grade γ for the period a : we have $\gamma s^a = |\{i^a | s_i^a = \gamma\}|$.

Let \mathbb{Z}_n^m denote the set of $n \times m$ matrices of grades. A configuration is a matrix

²We normalize grades between 0 and 1 in order to represent the various grading systems (apart from grading system with qualitative grades). For example, in the french system, grades are reals between 0 and 20.

$\mathbf{S} \in \mathbb{Z}_n^m$ where the element of row i and column a is s_i^a :

$$\mathbf{S} = \begin{pmatrix} s_1^1 & s_1^m \\ & \\ & \\ s_n^1 & s_n^m \end{pmatrix}.$$

Matrix \mathbf{I}_n^m is the matrix where $s_i^a = 1$ for all tests i^a , and $\mathbf{0}_n^m$ is the matrix where $s_i^a = 0$ for all tests i^a . Denote two configurations by \mathbf{S} and \mathbf{T} . Then we will write $\mathbf{S} \leq \mathbf{T}$ if $s_i^a \leq t_i^a$ for any $i \in N$, $a \in M$. For period a and topic i , we denote by $\mathbf{S}_{\downarrow i^a}$ the configuration obtained from \mathbf{S} shifting s_i^a to the immediately lower level grade, while all other grades remain unchanged. For any \mathbf{S} , let $\tilde{\mathbf{S}}_{i \leftrightarrow j}^{a \leftrightarrow b} \in \mathbb{Z}_n^m$ denote a configuration where a test i^a and a test j^b permute their grades: $\tilde{s}_k^c = s_k^c$ for any $k \neq i, j$ and for any $c \neq a, b$; $\tilde{s}_i^a = s_j^b$ and $\tilde{s}_j^b = s_i^a$. We define a permutation π as $\pi\mathbf{S} = \{\tilde{\mathbf{S}}_{i \leftrightarrow j}^{a \leftrightarrow b} \in \mathbb{Z}_n^m \text{ for any } a, b \text{ and for any } i, j\}$.

University rule

A university rule specifies the configurations of \mathbb{Z}_n^m that lead the student to pass and those that lead the student to fail.

Definition 1. *A university rule with n topics and m periods is a function*

$$\begin{aligned} \mathbf{w} : \mathbb{Z}_n^m &\rightarrow \{0, 1\} \\ \mathbf{S} &\mapsto \mathbf{w}(\mathbf{S}) \end{aligned}$$

with: $\mathbf{w}(\mathbf{I}_n^m) = 1$; $\mathbf{w}(\mathbf{0}_n^m) = 0$; $\mathbf{w}(\mathbf{S}) = 1 \Rightarrow \mathbf{w}(\mathbf{T}) = 1$ for any $\mathbf{S} \leq \mathbf{T}$.

We denote by \mathbb{W}_n^m , the set of all university rules. We refer to the configurations that leads the student to pass ($\mathbf{w}(\mathbf{S}) = 1$) as winning configurations. All other configurations are referred to as losing configurations ($\mathbf{w}(\mathbf{S}) = 0$). The set of winning configurations is denoted $\mathcal{W}(\mathbf{w})$. A configuration \mathbf{S} is minimal winning if $\mathbf{w}(\mathbf{S}) = 1$ and for any configuration $\mathbf{T} \neq \mathbf{S}$ with $\mathbf{T} \leq \mathbf{S}$ we have $\mathbf{w}(\mathbf{T}) = 0$. The set of minimal winning configurations is denoted $\mathcal{M}(\mathbf{w})$. Note that a university rule can also be represented by its set of minimal winning configurations as illustrated by the two following examples.

Example 1. \mathbf{w}_1 rule

Consider a university program with two periods ($m = 2$). Three topics ($n = 3$) are taught at each period. The set of grades is $G = \{1, \frac{1}{2}, 0\}$. The decision process is the following: in the first period, one must obtain at least a grade $\frac{1}{2}$ for the first topic; in the second period, one must obtain at least a grade $\frac{1}{2}$ in at least two topics (any two topics). More formally, $\mathbf{w}_1 : \mathbb{Z}_3^2 \rightarrow \{0, 1\}$ with

$$\mathbf{w}_1(\mathbf{S}) = \begin{cases} 1 & \text{if } s_1^1 \geq \frac{1}{2} \text{ and } |\{i^2 \mid s_i^2 \geq \frac{1}{2}\}| \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$\mathcal{M}(\mathbf{w}_1) = \left\{ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \right\}.$$

Example 2. \mathbf{w}_2 rule

Consider a university program with three periods ($m = 3$). Two topics ($n = 2$) are taught at each period. The set of grades is $G = \{1, \frac{1}{2}, 0\}$. The decision process is the following: In the first two periods, one must obtain at least three grades equal to or above $\frac{1}{2}$ out of four tests; in the third period, one must earn at least one grade equal to or above $\frac{1}{2}$ out of two tests. More formally, $\mathbf{w}_2 : \mathbb{Z}_2^3 \rightarrow \{0, 1\}$ with

$$\mathbf{w}_2(\mathbf{S}) = \begin{cases} 1 & \text{if } |\{i^a \mid s_i^a \geq \frac{1}{2} \text{ for } a = 1, 2\}| \geq 3 \text{ and } |\{i^3 \mid s_i^3 \geq \frac{1}{2}\}| \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{M}(\mathbf{w}_2) = \left\{ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \right. \\ \left. \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right\}.$$

Let us now introduce three specific rules.

First, let π be any permutation of the set N^* , then the multi-dimensional rule $\pi\mathbf{w}$ is given by $\pi\mathbf{w}(\mathbf{S}) = \mathbf{w}(\pi\mathbf{S})$.

Second, given two rules $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{W}_n^m$, we have: rule $\mathbf{w} \vee \bar{\mathbf{w}}$ defined as $(\mathbf{w} \vee \bar{\mathbf{w}})(\mathbf{S}) = \max\{\mathbf{w}(\mathbf{S}), \bar{\mathbf{w}}(\mathbf{S})\}$; and rule $\mathbf{w} \wedge \bar{\mathbf{w}}$ defined as $(\mathbf{w} \wedge \bar{\mathbf{w}})(\mathbf{S}) = \min\{\mathbf{w}(\mathbf{S}), \bar{\mathbf{w}}(\mathbf{S})\}$. Note that $\mathcal{W}(\mathbf{w} \vee \bar{\mathbf{w}}) = \mathcal{W}(\mathbf{w}) \cup \mathcal{W}(\bar{\mathbf{w}})$ and $\mathcal{W}(\mathbf{w} \wedge \bar{\mathbf{w}}) = \mathcal{W}(\mathbf{w}) \cap \mathcal{W}(\bar{\mathbf{w}})$.

Third, for any configuration $\mathbf{S} \neq \mathbf{0}_n^m$, the unanimity university rule $\mathbf{u}_{\mathbf{S}}$ is defined as

$$\mathbf{u}_{\mathbf{S}}(\mathbf{T}) = \begin{cases} 1 & \text{if } \mathbf{S} \leq \mathbf{T} \\ 0 & \text{otherwise} \end{cases}$$

The following proposition shows that a university rule can be derived from a combination of university unanimity rules.

Proposition 1. *Let \mathbf{w} be a university rule with $\mathcal{M}(\mathbf{w}) = \{\mathbf{S}_1, \dots, \mathbf{S}_k\}$ the set of minimal winning configurations. \mathbf{w} can be expressed as $\mathbf{w} = u_{\mathbf{S}_1} \vee \dots \vee u_{\mathbf{S}_k}$.*

Proof. Consider a configuration $\mathbf{T} \in \mathbb{Z}_n^m$.

Suppose that $\mathbf{w}(\mathbf{T}) = 1$ then, there exists at least one minimal winning configuration \mathbf{S}_1 such that $\mathbf{S}_1 \leq \mathbf{T}$. Therefore, $\mathbf{u}_{\mathbf{S}_1}(\mathbf{T}) = 1$ and $\mathbf{w}(\mathbf{T}) = u_{\mathbf{S}_1}(\mathbf{T}) \vee \dots \vee u_{\mathbf{S}_k}(\mathbf{T}) = 1$

Suppose that $\mathbf{w}(\mathbf{T}) = 0$ then, $\mathbf{S}_l \not\leq \mathbf{T}$ for all $l = 1, \dots, k$. Therefore, $\mathbf{u}_{\mathbf{S}_l}(\mathbf{T}) = 0$ for all $l = 1, \dots, k$ and then $\mathbf{w}(\mathbf{T}) = u_{\mathbf{S}_1}(\mathbf{T}) \vee \dots \vee u_{\mathbf{S}_k}(\mathbf{T}) = 0$ \square

Influence index

Let \mathbf{w} be a university rule. An influence index is a mapping ϕ which assigns to the university rule \mathbf{w} a vector $\phi(\mathbf{w}) = (\phi_1(\mathbf{w}), \dots, \phi_n(\mathbf{w}))$. For a given topic i , $\phi_i(\mathbf{w})$ can be seen as the influence of this topic on the final result. We make the choice of presenting the evaluation of influence from the topic's perspective. However, the tools we use in this paper would also be relevant in evaluating the influence of a given period. We also denote by ${}_{\gamma}\phi_{i^a}(\mathbf{w})$ the influence of the topic i for the period a and the grade γ . Since we assume that all the tests for a given topic at any periods and for any grade are *a priori* equally influent, we have $\phi_i(\mathbf{w}) = \sum_{a=1}^m \sum_{\gamma \in G} \gamma \phi_{i^a}(\mathbf{w})$. We implicitly assume here that all periods have the same weight for a given topic. A given period can thus give more influence than another solely on account of the framework of the university rule. Note that this is a particular class of influence indices. Other less restrictive classes could be explored in further research.

3 Banzhaf evaluation

In order to extend the standard Banzhaf influence index (Banzhaf [1965]) to the context of a university rule, we first extend the notion of a swing for a topic i . In words, a topic is in position of a swing when the fall of one of its grade to the immediately lower grade level changes the result from pass to fail. More formally a ${}_{\gamma}i^a$ -swing for

a topic i , for a period a and a given grade ($\gamma \neq 0$) is defined as a pair of configurations of the form $(\mathbf{S}, \mathbf{S}_{\downarrow i^a})$ such that $\mathbf{w}(\mathbf{S}) = 1$ and $\mathbf{w}(\mathbf{S}_{\downarrow i^a}) = 0$. Topic i is in position of a swing if there exists one γi^a -swing for at least one $a \in M$ and one grade γ . For each topic i at a given period a , and for a given grade γ , we denote by ${}_{\gamma}\eta_i^a(\mathbf{w})$ the number of γi^a -swings for the university rule \mathbf{w} , i.e. ${}_{\gamma}\eta_i^a(\mathbf{w}) = \sum_{\substack{\mathbf{S} \in \mathbb{Z}_n^m \\ s_i^a = \gamma}} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S}_{\downarrow i^a})]$.

We shall write $\eta_i(\mathbf{w})$ the number of swings for topic i for the university rule \mathbf{w} i.e. $\eta_i(\mathbf{w}) = \sum_{a=1}^m \sum_{\gamma \in G} {}_{\gamma}\eta_i^a(\mathbf{w})$. Finally we have $\overline{{}_{\gamma}\eta_i^a(\mathbf{w})}$ the total number of γi^a -swings for any grade γ and any period a for the university rule \mathbf{w} , i.e. $\overline{{}_{\gamma}\eta_i^a(\mathbf{w})} = \sum_{i=1}^n {}_{\gamma}\eta_i^a(\mathbf{w})$, and $\overline{\eta(\mathbf{w})}$ the total number of swings for the university rule \mathbf{w} , i.e. $\overline{\eta(\mathbf{w})} = \sum_{a=1}^m \sum_{\gamma \in G} \overline{{}_{\gamma}\eta_i^a(\mathbf{w})}$.

Definition 2. Let \mathbf{w} be a university rule. The Banzhaf influence index for a topic $i \in N$ is given by :

$$BZ_i(\mathbf{w}) = \sum_{a \in M} \sum_{\gamma \in G} \gamma BZ_{i^a}(\mathbf{w})$$

with

$${}_{\gamma}BZ_{i^a}(\mathbf{w}) = \frac{{}_{\gamma}\eta_i^a(\mathbf{w})}{g^{n.m-1}} \text{ for each } \gamma \in G$$

The Banzhaf index can be seen as the probability that in a randomly chosen configuration, topic i will be in a position to change the outcome by altering its grade. Notice that with two grades, the Banzhaf influence index for a university rule - repetition is equivalent to the Banzhaf influence index introduced by Banzhaf [1965] when there is a single period, and equivalent to the Banzhaf influence index for multi-dimensional rules introduced by Courtin [2022] when there is more than one period. Likewise, for a single period and a number of grade greater or equal to three, the Banzhaf influence index for a university rule is equivalent to the Banzhaf influence index for $(j, 2)$ rules introduced by Freixas [2005a].

Characterization

We now present four properties in order to characterize the previous index. These properties are derived from the Banzhaf characterization (Banzhaf [1965]).

Axiom 1. (*Banzhaf Efficiency*)

For any $\mathbf{w} \in \mathbb{W}_m^n$,

$$\sum_{i=1}^n \gamma \phi_{i^a}(\mathbf{w}) = \frac{\overline{\gamma \eta^a(\mathbf{w})}}{g^{n,m-1}} \text{ for each } \gamma \in G \text{ and each } a \in M.$$

The property of Banzhaf efficiency states that the total aggregated influence of the test with the same level of grade is a fraction of the total number of swings for this level of grade. Since $\phi_i(\mathbf{w}) = \sum_{a=1}^m \sum_{\gamma \in G} \gamma \phi_{i^a}(\mathbf{w})$ and $\overline{\eta(\mathbf{w})} = \sum_{a=1}^m \sum_{\gamma \in G} \overline{\gamma \eta^a(\mathbf{w})}$ this efficiency axiom implies that $\sum_{i=1}^n \phi_i(\mathbf{w}) = \sum_{i=1}^n \sum_{a=1}^m \sum_{\gamma \in G} \gamma \phi_{i^a}(\mathbf{w}) = \sum_{a=1}^m \sum_{\gamma \in G} \frac{\overline{\gamma \eta^a(\mathbf{w})}}{g^{n,m-1}} = \frac{\overline{\eta(\mathbf{w})}}{g^{n,m-1}}$.

A topic i for a period a and a grade γ is a γi^a -dummy for \mathbf{w} if $\gamma \eta_i^a(\mathbf{w}) = 0$. If for every $a \in M$ and for every $\gamma \in G, \gamma \neq 0$ a topic i is γi^a -dummy, then it is a dummy topic for a university rule \mathbf{w} , i.e. $\eta_i(\mathbf{w}) = 0$.

Axiom 2. (Dummy)

For any $\mathbf{w} \in \mathbb{W}_m^n$, if a topic i for a period a and a grade γ is a γi^a -dummy, then:

$$\gamma \phi_{i^a}(\mathbf{w}) = 0.$$

A topic i for a period a and a grade γ that is γi^a -dummy has no influence on the final result. A consequence of this axiom is that a topic i that is γi^a -dummy for each period a and each grade γ has no influence on the final result, i.e. $\phi_i(\mathbf{w}) = 0$.

Axiom 3. (Anonymity)

Let $\mathbf{w} \in \mathbb{W}_m^n$. For any permutation π of N^* , and any test $i^a \in N^*$,

$$\gamma \phi_{\pi(i^a)}(\pi \mathbf{w}) = \gamma \phi_{i^a}(\mathbf{w})$$

According to the anonymity axiom, the influence of a test does not depend on its label or relative position in N^* .

Axiom 4. (Transfer)

For any $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{W}^{n \times m}$

$$\phi(\mathbf{w}) + \phi(\bar{\mathbf{w}}) = \phi(\mathbf{w} \vee \bar{\mathbf{w}}) + \phi(\mathbf{w} \wedge \bar{\mathbf{w}})$$

Axiom 4 is an extension of the transfer axiom introduced by Dubey and Shapley [1979] in the context of the university rule. When considering two university rules, Axiom 4 means that the total influence of a topic is simply the sum of influences available in the min rule and in the max rule.

The Banzhaf influence index for the university rule is the only index that simultaneously satisfies all four previous axioms.

Theorem 1. Let ϕ be an index for the university rule. Then, ϕ satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 4 if and only if $\phi = \text{BZ}$.

Proof. Let \mathbf{w} be a university rule. For any configuration \mathbf{S} , consider the unanimity

$$\text{university rule } \mathbf{u}_{\mathbf{S}} \text{ defined by } \mathbf{u}_{\mathbf{S}}(\mathbf{T}) = \begin{cases} 1 & \text{if } \mathbf{S} \leq \mathbf{T}, \\ 0 & \text{otherwise.} \end{cases}$$

Every test i^a such that $s_i^a = 0$ is a $0i^a$ -dummy in $\mathbf{u}_{\mathbf{S}}$. Therefore, by Axiom 2, ${}_{\gamma}\phi_{i^a}(\mathbf{u}_{\mathbf{S}}) = 0$ for each test $i^a \in N^*$ such that $s_i^a = \gamma = 0$. This implies that $\phi_i(\mathbf{u}_{\mathbf{S}})$ reduces to $\sum_{a=1}^m \sum_{\gamma \neq 0} \gamma \phi_{i^a}(\mathbf{u}_{\mathbf{S}})$.

If π is the permutation that interchanges i^a and j^b , for some $i, j \in N$ and some $a, b \in N$ such that $s_i^a = s_j^b = \gamma$, and leaves the other test unchanged then, $\pi\mathbf{u}_{\mathbf{S}} = \mathbf{u}_{\mathbf{S}}$ and thus, by Axiom 3, it follows that ${}_{\gamma}\phi_{j^b}(\pi\mathbf{u}_{\mathbf{S}}) = {}_{\gamma}\phi_{j^b}(\mathbf{u}_{\mathbf{S}}) = {}_{\gamma}\phi_{i^a}(\mathbf{u}_{\mathbf{S}})$.

Therefore ${}_{\gamma}\phi_{i^a}(\mathbf{u}_{\mathbf{S}})$ (and then $\phi_i(\mathbf{u}_{\mathbf{S}})$) is uniquely determined if ϕ exists, and for each $\gamma \in G$ and each $a \in M$, is given by

$${}_{\gamma}\phi_{i^a}(\mathbf{u}_{\mathbf{S}}) = \begin{cases} \overline{\frac{{}_{\gamma}\eta^a(\mathbf{u}_{\mathbf{S}})}{g^{n,m-1} \cdot \gamma s^a}} & \text{if } s_i^a = \gamma, \gamma \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

which, using Axiom 1, is also equal to:

$${}_{\gamma}\phi_{i^a}(\mathbf{u}_{\mathbf{S}}) = \begin{cases} \frac{\prod_{h=0, h \neq g-l_{\gamma}}^{g-1} (g-h)^{|i^a: s_i^a = \gamma' \text{ and } l_{\gamma'} = g-h|} \times l_{\gamma'}^{(\gamma s^a - 1)}}{g^{n,m-1}} & \text{if } s_i^a = \gamma, \gamma \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

since

$$\overline{{}_{\gamma}\eta^a(\mathbf{u}_{\mathbf{S}})} = \gamma s^a \times \prod_{h=0, h \neq g-l_{\gamma}}^{g-1} (g-h)^{|i^a: s_i^a = \gamma' \text{ and } l_{\gamma'} = g-h|} \times l_{\gamma'}^{(\gamma s^a - 1)}$$

Now every game $\mathbf{w} \in \mathbb{W}_n^m$ has a finite number of minimal winning configurations, $\mathcal{M}(\mathbf{w}) = \{\mathbf{S}_1, \dots, \mathbf{S}_k\}$ and they completely determine \mathbf{w} , since $\mathbf{w}(\mathbf{T}) = 1$ if and only if $\mathbf{S}_l \leq \mathbf{T}$ for at least one $l = 1, \dots, k$. For any configuration \mathbf{S}_1 , we define the unanimity rule $\mathbf{u}_{\mathbf{S}_1}$ by $\mathbf{u}_{\mathbf{S}_1}(\mathbf{T}) = \begin{cases} 1 & \text{if } \mathbf{S}_1 \in \mathbf{T} \\ 0 & \text{otherwise} \end{cases}$. By Proposition 1, \mathbf{w} can be written as $\mathbf{w} = \mathbf{u}_{\mathbf{S}_1} \vee \dots \vee \mathbf{u}_{\mathbf{S}_k}$, where the right hand side is defined associatively.

Now if $\mathbf{w} \in \mathbb{W}_n^m$ is not in the form $\mathbf{u}_{\mathbf{S}}$, then $k > 1$, so \mathbf{w} can be written $\mathbf{w}' \vee \mathbf{w}''$, where \mathbf{w}' and \mathbf{w}'' are rules with fewer winning configurations than \mathbf{w} . For example, let $\mathbf{w}' = \mathbf{u}_{\mathbf{S}_1}$ and $\mathbf{w}'' = \mathbf{u}_{\mathbf{S}_2} \vee \dots \vee \mathbf{u}_{\mathbf{S}_k}$. Of course the game $\mathbf{w}' \wedge \mathbf{w}''$ has even fewer

winning coalitions. So we can perform an induction on the number of winning configurations, using Axiom 4:

$$\phi(\mathbf{w}) = \phi(\mathbf{w}' \vee \mathbf{w}'') = \phi(\mathbf{w}') + \phi(\mathbf{w}'') - \phi(\mathbf{w}' \wedge \mathbf{w}''),$$

and it follows that $\phi(\mathbf{w})$ is uniquely determined.

Therefore, we have proved that $\phi(\mathbf{w})$ is unique for any feasible number of winning configurations, *i.e.* for all $\mathbf{w} \in \mathbb{W}_n^m$.

Existence. It is obvious that BZ satisfies the efficiency, the null player, the symmetry axioms and the transfer axiom. \square

Let us finished this section by the evaluation of the Banzhaf influence index of the topics in examples w_1 and w_2 .

Example 3. Banzhaf evaluation of w_1

a) Topic 1 for period 1 is in position of $\frac{1}{2}1^1$ -swing when : i) its grade is $\frac{1}{2}$ in the first period, there are 9 possibilities; ii) in the second period the conditions to pass are satisfied, *i.e.* there is no case with at least two zero grade (there are $3^3 = 27$ total cases and seven cases with at least two zero grades), there are 20 possibilities. Thus $\frac{1}{2}BZ_{1^1}(\mathbf{w}_1) = \frac{20 \times 9}{3^{3 \times 2 - 1}} = \frac{180}{243}$.

Topic 1 for period 1 is $\gamma 1^1$ -dummy for $\gamma \in \{0, 1\}$ then $\gamma BZ_{1^1}(\mathbf{w}_1) = 0$ for $\gamma = 0, 1$.

Topics 2 and 3 for period 1 are γi^1 -dummy for all γ , then $\gamma BZ_{1^2}(\mathbf{w}_1) = \gamma BZ_{1^3}(\mathbf{w}_1) = 0$ for all γ .

b) topic i for period 2 is in position of $\frac{1}{2}i^2$ -swing when: i) the grade of i^2 is $\frac{1}{2}$ and there is at least one 0 grade, there are 4 cases; the grade of 1^1 is 1 or $\frac{1}{2}$ in the first period, there are 18 cases. Thus for all i , $\frac{1}{2}BZ_{i^2}(\mathbf{w}_1) = \frac{18 \times 4}{3^{3 \times 2 - 1}} = \frac{72}{243}$. For all i , i^2 is γi^2 -dummy for $\gamma \in \{0, 1\}$ then $\gamma BZ_{i^2}(\mathbf{w}_1) = 0$ for $\gamma \in \{0, 1\}$.

c) Finally we have $BZ_1(\mathbf{w}_1) = \frac{1}{2}BZ_{1^1}(\mathbf{w}_1) + \frac{1}{2}BZ_{1^2}(\mathbf{w}_1) = \frac{252}{243}$ and $BZ_2(\mathbf{w}_1) = BZ_3(\mathbf{w}_1) = \frac{1}{2}BZ_{i^2}(\mathbf{w}_1) = \frac{72}{243}$.

Example 4. Banzhaf evaluation of w_2

a) Topic 1 for period 1 is in position of $\frac{1}{2}1^1$ -swing when the two situations arise: its grade is $\frac{1}{2}$ in period 1 and there is one grade 0 in the two first periods, there are 12 cases; in the third period the conditions to pass are satisfied, *i.e.* there is no case with at least two zero grade, there are 8 cases. Thus $\frac{1}{2}BZ_{1^1}(\mathbf{w}_2) = \frac{12 \times 8}{3^{3 \times 2 - 1}} = \frac{96}{243}$.

Topics 1 and 2 for period 1 and period 2 have the same influence, then $\frac{1}{2}BZ_{11}(\mathbf{w}_2) = \frac{1}{2}BZ_{12}(\mathbf{w}_2) = \frac{1}{2}BZ_{21}(\mathbf{w}_2) = \frac{1}{2}BZ_{22}(\mathbf{w}_2) = \frac{96}{243}$.

b) Topic i in period 3 is in position of $\frac{1}{2}i^3$ -swing when the two situations arise: i) i^3 has a $\frac{1}{2}$ grade and the topics different from i has a 0 grade, there is only 1 case; ii) the conditions to pass are satisfied, i.e there is at most one grade 0 in the two first periods (there are $3^4 = 81$ cases in total, and 33 cases with at least two zeros): we have $81 - 33 = 48$ cases. Thus $\frac{1}{2}BZ_{i^3}(\mathbf{w}_2) = \frac{48 \times 1}{3^{3 \times 2 - 1}} = \frac{48}{243}$.

c) For all i , for $a = 1, 2, 3$ and for all $\gamma \in \{0, 1\}$, topic i in period a is γi^a -dummy, therefore $\gamma BZ_{i^a}(\mathbf{w}_2) = 0$.

d) Finally, since both topics have the same influence, $BZ_i(\mathbf{w}_2) = \frac{1}{2}BZ_{i^1}(\mathbf{w}_2) + \frac{1}{2}BZ_{i^2}(\mathbf{w}_2) + \frac{1}{2}BZ_{i^3}(\mathbf{w}_2) = \frac{96}{243} + \frac{96}{243} + \frac{48}{243} = \frac{240}{243}$.

4 Calculus of the Banzhaf influence index

We have seen that the total number of swings for a university rule plays an important role in the formal axiomatization of the Banzhaf influence index. In this section we present the exact results for three different university rules.

4.1 A given grade to a given number of tests

The first class of rules we consider is the following. A student obtains his or her diploma if he or she passes an *a priori* number of tests (say x) over the $n.m$ tests by obtaining at each of these tests at least a given grade $\bar{\gamma}$. For example, $\bar{\gamma}$ can correspond to the middle grade. In the french system the middle grade is equal to 10 over 20.

Proposition 2. Let G be a set of grades, with $\bar{\gamma} \in G$, let x be a real number, with $0 < x \leq nm$ and define the university rule \mathbf{w}_3 by $\mathbf{w}_3(\mathbf{S}) = 1$ if $|\{i^a \mid s_i^a \geq \bar{\gamma}\}| \geq x$. The Banzhaf influence index is given by:

$$BZ_i(\mathbf{w}_3) = m \cdot \frac{\frac{(nm-1)!}{(x-1)!(nm-x)!} \cdot (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}} \text{ for all } i$$

Proof. a) For a period a and a grade $\gamma = \bar{\gamma}$, i is in position of $\bar{\gamma} i^a$ -swing when $(x-1)$ tests, not including test i^a have a grade greater than or equal to $\bar{\gamma}$: the number of combinations of $(x-1)$ tests in $(nm-1)$ tests is given by $C_{nm-1}^{x-1} = \frac{(nm-1)!}{(x-1)!(nm-x)!}$ and the

number of times that $(x - 1)$ tests have a grade greater or equal to $\bar{\gamma}$ is $(l_{\bar{\gamma}})^{x-1}(g - l_{\bar{\gamma}})^{nm-x}$. We thus have a number of occurrences of $\frac{(nm-1)!}{(x-1)!(nm-x)!} \cdot (l_{\bar{\gamma}})^{x-1}(g - l_{\bar{\gamma}})^{nm-x}$. Thus

$$\bar{\gamma}BZ_{i^a}(\mathbf{w}_3) = \frac{(nm-1)!}{(x-1)!(nm-x)!} \cdot \frac{(l_{\bar{\gamma}})^{x-1}(g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}}.$$

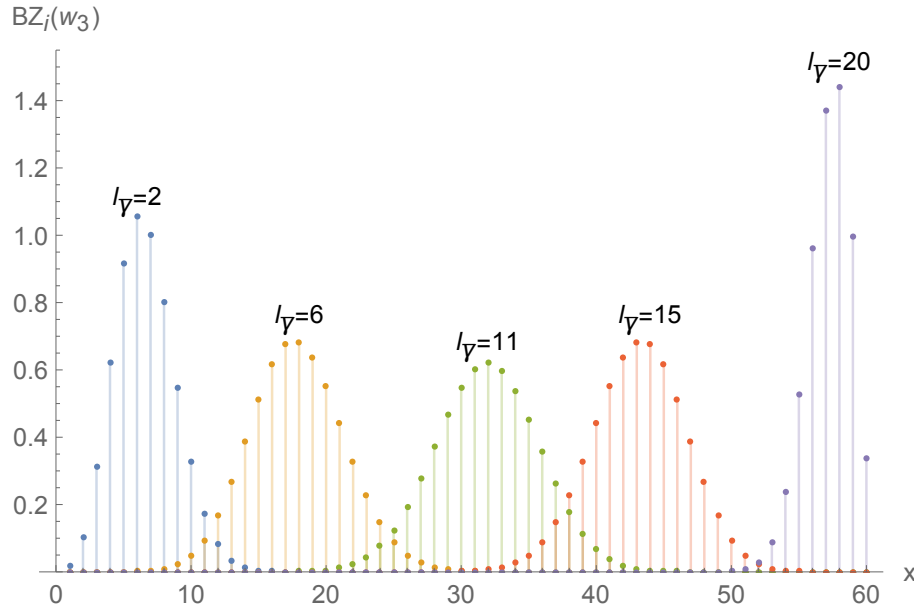
b) For a period a and a grade $\gamma \neq \bar{\gamma}$, i is γi^a -dummy, then $\gamma BZ_{i^a}(\mathbf{w}_3) = 0$ for all $\gamma \neq \bar{\gamma}$.

$$\text{Therefore for all } i \text{ we have } BZ_i(\mathbf{w}_3) = \sum_{a \in M} \bar{\gamma} BZ_{i^a}(\mathbf{w}_3) = m \cdot \frac{(nm-1)!}{(x-1)!(nm-x)!} \cdot \frac{(l_{\bar{\gamma}})^{x-1}(g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}}$$

□

Figure 1 illustrates Proposition 2 with $n = 10$, $m = 6$, $g = 21$, for various values of $l_{\bar{\gamma}}$ and x .

Figure 1: Influence of topic i for w_3



Consider for example $l_{\bar{\gamma}} = 20$ where the number of grades that pass the test is large. When the number x of grades above $\bar{\gamma}$ needed to obtain the diploma is small, the influence of a given topic is small since the validation of many other topics can suffice to obtain the diploma. When the number of tests x to pass increases, passing becomes more important and then its influence increases.

By contrast, when the number of grades that ensure passing the test is small (for example if $l_{\bar{\gamma}} = 2$, only two grades yield a "pass" result), then if x is large, it is very hard do obtain the diploma. Therefore, a given topic has a small influence since being in position of a swing involves that the students passes a large number

of other topics, and the likelihood of such an event is very small. In that case the influence is greater for small values of x since the likelihood that one passes a small number of topics is sufficiently large.

One can also notice that, regardless of $l_{\bar{\gamma}}$ the influence of a topic first increases and then decreases with respect to x . Indeed, for a small value of x , the probability that the student has passed enough topics to obtain the diploma is greater than for a large value of x . This implies also that the argmax is greater for a greater $l_{\bar{\gamma}}$. For example for $l_{\bar{\gamma}} = 2$ the argmax is $x = 6$ whereas for $l_{\bar{\gamma}} = 20$ the argmax is $x = 58$.

To summarize, the influence of a topic mainly depends on the number of tests already passed: it must be sufficient to allow a given topic to be in position of a swing, but not too high, otherwise the diploma is already obtained.

4.2 One test is compulsory

We consider a university rule such that one test is compulsory (say the first test at the first period (1^1)) and the student must pass at least x tests over the $n.m$ tests. Once again, we assume that $\bar{\gamma}$ is the minimum grade to pass a test.

Proposition 3. *Let G be a set of grades, with $\bar{\gamma} \in G$, let x be a real number, with $0 < x \leq nm$ and define the university rule \mathbf{w}_4 by $\mathbf{w}_4(\mathbf{S}) = 1$ if $s_1^1 \geq \bar{\gamma}$ and $|\{i^a \mid s_i^a \geq \bar{\gamma}\}| \geq x - 1$, for $i^a \neq 1^1$. The Banzhaf index is given by*

For $i = 1$:

$$BZ_1(\mathbf{w}_4) = \frac{\sum_{k=x}^{nm} \frac{(nm-1)!}{(k-1)!(nm-k)!} (l_{\bar{\gamma}})^{k-1} (g - l_{\bar{\gamma}})^{nm-k}}{g^{nm-1}} + (m-1) \frac{\frac{(nm-2)!}{(x-2)!(nm-x)!} (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}}$$

For $i \neq 1$

$$BZ_i(\mathbf{w}_4) = m \cdot \frac{\frac{(nm-2)!}{(x-2)!(nm-x)!} (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}}$$

Proof. a) For a period a and a grade $\gamma = \bar{\gamma}$, i is in position of $\bar{\gamma} 1^1$ -swing when at least one of the following situation arises:

- $(x - 1)$ tests, not including test i^a , have a grade greater than or equal to $\bar{\gamma}$. The number of combinations of $(x - 1)$ tests over $(nm - 1)$ is $C_{nm-1}^{x-1} \cdot (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}$.

- x tests different from 1^1 have a grade greater or equal to $\bar{\gamma}$. The number of occurrences is $C_{nm-1}^x \cdot (l_{\bar{\gamma}})^x (g - l_{\bar{\gamma}})^{nm-x-1}$.
- ...
- $(nm-1)$ tests different from 1^1 have a grade greater or equal to $\bar{\gamma}$. The number of occurrences is $C_{nm-1}^{nm-1} \cdot (l_{\bar{\gamma}})^{nm-1} (g - l_{\bar{\gamma}})^{nm-nm}$.

If we sum up all these situations, we obtain $\sum_{k=x}^{nm} \frac{(nm-1)!}{(k-1)!(nm-k)!} (l_{\bar{\gamma}})^{k-1} (g - l_{\bar{\gamma}})^{nm-k}$. Thus

$$\bar{\gamma}BZ_{1^1}(\mathbf{w}_4) = \frac{\sum_{k=x}^{nm} \frac{(nm-1)!}{(k-1)!(nm-k)!} (l_{\bar{\gamma}})^{k-1} (g - l_{\bar{\gamma}})^{nm-k}}{g^{nm-1}}.$$

b) For a period a , a grade $\gamma = \bar{\gamma}$, and $i^a \neq 1^1$, i is in position of $\bar{\gamma}i^a$ -swing when $(x-1)$ tests including 1^1 have a grade greater or equal to $\bar{\gamma}$. The number of occurrences is $C_{nm-2}^{x-2} \cdot (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x} = \frac{(nm-2)!}{(x-2)!(nm-x)!} (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}$. Thus $\bar{\gamma}BZ_{i^a}(\mathbf{w}_4) = \frac{\frac{(nm-2)!}{(x-2)!(nm-x)!} (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}}$ for all $i^a \neq 1^1$.

c) For all i and all a , topic i in period a is γi^a -dummy for all $\gamma \neq \bar{\gamma}$, then $\gamma BZ_{i^a}(\mathbf{w}_4) = 0$.

Therefore for topic 1 we have $BZ_1(\mathbf{w}_4) = \bar{\gamma}BZ_{1^1}(\mathbf{w}_4) + \sum_{a \neq 1, a \in M} \bar{\gamma}BZ_{1^a}(\mathbf{w}_4)$

$$= \frac{\sum_{k=x}^{nm} \frac{(nm-1)!}{(k-1)!(nm-k)!} (l_{\bar{\gamma}})^{k-1} (g - l_{\bar{\gamma}})^{nm-k}}{g^{nm-1}} + (m-1) \frac{\frac{(nm-2)!}{(x-2)!(nm-x)!} (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}}}.$$

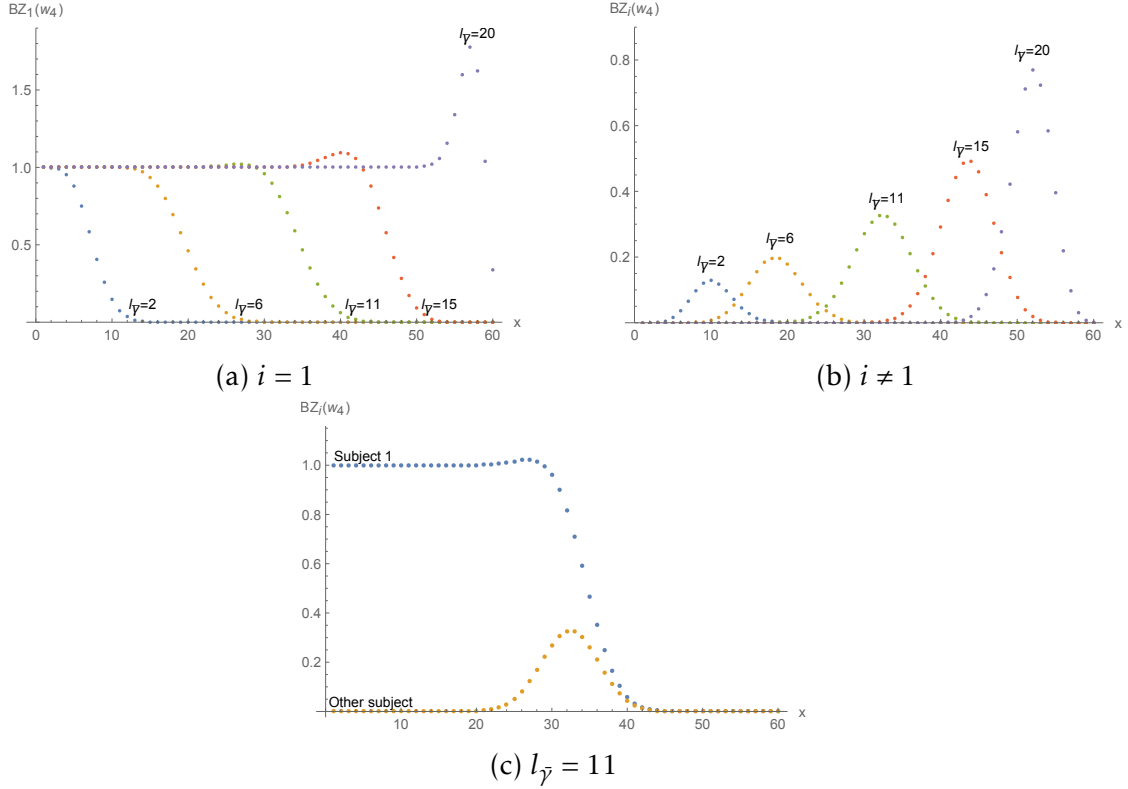
And for topic $i \neq 1$, we have $BZ_i(\mathbf{w}_4) = \sum_{a \in M} \bar{\gamma}BZ_{i^a}(\mathbf{w}_4) = m \cdot \frac{\frac{(nm-2)!}{(x-2)!(nm-x)!} (l_{\bar{\gamma}})^{x-1} (g - l_{\bar{\gamma}})^{nm-x}}{g^{nm-1}}$

□

Figure 2 illustrates Proposition 3 with $n = 10$, $m = 6$, $g = 21$, for various values of $l_{\bar{\gamma}}$ and x .

When $x = 1$, the student must pass only the compulsory test. Topic 1 thus has all the influence while all other topics have no influence. As x increases, the influence of a non-compulsory topic increases up to a maximum and then decreases. The argmax increases with respect to $l_{\bar{\gamma}}$ both for the compulsory topic and the other topics. Indeed, the probability that the compulsory test has a grade greater than or equal to $\bar{\gamma}$ increases, which increases the probability that other topics be in position of a swing.

Figure 2: Influence of the topics for w_4



4.3 The overall average

The university rule considered here implies that the student obtains the overall average i.e. $\sum_{i^a} s_i^a \geq \frac{nm}{2}$. In the following, let $\bar{s} = \sum_{i^a} s_i^a$ and $\bar{s}^{-i^a} = \sum_{j^b, j^b \neq i^a} s_j^b$.

We will consider two situations, one with three grades and one with four grades.

Proposition 4. Let $G = \{1, \frac{1}{2}, 0\}$ and define the university rule \mathbf{w}_5 by $\mathbf{w}_5(\mathbf{S}) = 1$ if $\bar{s} \geq \frac{nm}{2}$. The Banzhaf index for all i is given by:

- For nm even:

$$BZ_i(\mathbf{w}_5) = m \cdot \frac{(nm-1)! \sum_{y=0}^{\frac{nm-2}{2}} \frac{\frac{nm}{2} + 1 + y}{(\frac{nm}{2} - 1 - y)! (\frac{nm}{2} - y)! (1+2y)!}}{3^{nm-1}}$$

- For nm odd:

$$BZ_i(\mathbf{w}_5) = m \cdot \frac{(nm-1)! \sum_{y=0}^{\frac{nm-1}{2}} \frac{\frac{nm+1}{2}+y}{(\frac{nm-1}{2}-y)!(\frac{nm+1}{2}-y)!(2y)!}}{3^{nm-1}}$$

Proof. We develop the proof for $i = 1$ and $a = 1$. The proof is similar for all other topics and periods.

a) For a period $a = 1$ and a grade $\gamma = 1$, 1 is in position of 11^1 -swing when $\bar{s}^{-1} = \frac{nm}{2} - 1$. We must distinguish between nm even and nm odd.

i) nm even: We enumerate first the total number of occurrences by considering first the configurations with the maximum number of grades $\gamma = 1$. Since nm is even, this maximum is $\frac{nm}{2} - 1$, and it is sufficient to obtain $\bar{s}^{-1} = \frac{nm}{2} - 1$. There is $C_{nm-1}^{\frac{nm}{2}-1}$ such configurations.

Assume now that we remove one grade $\gamma = 1$. In order to obtain $\frac{nm}{2} - 1$ we can replace this grade by two grades $\gamma = \frac{1}{2}$. There are $C_{nm-1}^{(\frac{nm}{2}-1)-1} C_{\frac{nm}{2}+1}^{2,1}$ such configurations.

Secondly assume that we remove two grades $\gamma = 1$ from the initial situation, and replace them by four grades $\frac{1}{2}$. There are $C_{nm-1}^{(\frac{nm}{2}-1)-2} C_{\frac{nm}{2}+2}^{2,2}$ such configurations.

Note that we can replace each grade $\gamma = 1$ by two grades $\frac{1}{2}$ until all $\frac{nm}{2} - 1$ grades $\gamma = 1$ have been replaced. We can generalize the previous methods for all $n.m$ even. Denote by y the number of grades $\gamma = 1$ that we remove and replace by two $\gamma = \frac{1}{2}$.

We obtain $\sum_{y=0}^{\frac{nm}{2}-1} C_{nm-1}^{(\frac{nm}{2}-1)-y} C_{\frac{nm}{2}+y}^{2,y}$ situations.

Thus for $n.m$ even we have:

$${}_1BZ_{1^1}(\mathbf{w}_5) = \frac{\sum_{y=0}^{\frac{nm}{2}-1} C_{nm-1}^{(\frac{nm}{2}-1)-y} C_{\frac{nm}{2}+y}^{2,y}}{3^{nm-1}} \cdot x \quad (1)$$

ii) nm odd: The maximum number of grades $\gamma = 1$ in a configuration is $(\frac{nm}{2} - 1) - \frac{1}{2} = \frac{nm}{2} - \frac{3}{2}$. To achieve $\frac{nm}{2} - 1$ we must add one grade $\gamma = \frac{1}{2}$. There are $C_{nm-1}^{(\frac{nm-1}{2}-1)} C_{\frac{nm+1}{2}}^1$ situations.

We can remove one grade $\gamma = 1$ and replace it with two grades $\gamma = \frac{1}{2}$. There are $C_{nm-1}^{(\frac{nm-1}{2}-1)-1} C_{(\frac{nm+1}{2})+1}^{1+2,1}$ such configurations. Once again we can replace the grade $\gamma = 1$ by two grades $\gamma = \frac{1}{2}$ until all $\gamma = 1$ grades are replaced. Therefore we have

$\sum_{y=0}^{\frac{nm-1}{2}-1} C_{nm-1}^{(\frac{nm-1}{2}-1)-y} C_{\frac{nm+1}{2}+y}^{1+2.y}$ configurations.

Thus for nm odd we have:

$${}_1BZ_{1^1}(\mathbf{w}_5) = \frac{\sum_{y=0}^{\frac{nm-1}{2}-1} C_{nm-1}^{(\frac{nm-1}{2}-1)-y} C_{\frac{nm+1}{2}+y}^{1+2.y}}{3^{nm-1}}. \quad (2)$$

b) For a period $a = 1$, a grade $\gamma = \frac{1}{2}$, 1 is in position of $\frac{1}{2}1^1$ -swing when $\bar{s}^{-1^1} = \frac{nm-1}{2}$. Once again there are two cases: $n.m$ even and $n.m$ odd. Since the proofs are similar to those in a) they are omitted.

For $n.m$ odd and $n.m$ even, the total number of occurrences is respectively given by $\sum_{y=0}^{\frac{nm-1}{2}} C_{nm-1}^{(\frac{nm-1}{2})-y} C_{\frac{nm-1}{2}+y}^{2.y}$ and $\sum_{y=0}^{\frac{nm}{2}-1} C_{nm-1}^{(\frac{nm}{2}-1)-y} C_{\frac{nm}{2}+y}^{1+2.y}$.

Therefore for nm even, we have:

$${}_{\frac{1}{2}}BZ_{1^1}(\mathbf{w}_5) = \frac{\sum_{y=0}^{\frac{nm}{2}-1} C_{nm-1}^{(\frac{nm}{2}-1)-y} C_{\frac{nm}{2}+y}^{1+2.y}}{3^{nm-1}}. \quad (3)$$

while for nm odd we have:

$${}_{\frac{1}{2}}BZ_{1^1}(\mathbf{w}_5) = \frac{\sum_{y=0}^{\frac{nm-1}{2}} C_{nm-1}^{(\frac{nm-1}{2})-y} C_{\frac{nm-1}{2}+y}^{2.y}}{3^{nm-1}}. \quad (4)$$

c) For a period $a = 1$, 1 is ${}_01^1$ -dummy for $\gamma = 0$ then ${}_0BZ_{1^1}(\mathbf{w}_5) = 0$.

d) Finally for nm even, Banzhaf of 1^1 is given by (1) + (3), that is:

$$BZ_{1^1}(\mathbf{w}_5) = {}_1BZ_{1^1}(\mathbf{w}_5) + {}_{\frac{1}{2}}BZ_{1^1}(\mathbf{w}_5) = \frac{(nm-1)! \sum_{y=0}^{\frac{nm-2}{2}} \frac{\frac{nm}{2}+1+y}{(\frac{nm-1}{2}-y)! (\frac{nm}{2}-y)! (1+2y)!}}{3^{nm-1}}.$$

Likewise for nm odd, Banzhaf is the sum of (2) + (4), that is:

$$BZ_{1^1}(\mathbf{w}_5) = {}_1BZ_{1^1}(\mathbf{w}_5) + {}_{\frac{1}{2}}BZ_{1^1}(\mathbf{w}_5) = \frac{(nm-1)! \sum_{y=0}^{\frac{nm-1}{2}} \frac{\frac{nm+1}{2}+y}{(\frac{nm-1}{2}-y)! (\frac{nm+1}{2}-y)! (2y)!}}{3^{nm-1}}.$$

e) Thus for nm even:

$$BZ_1(\mathbf{w}_5) = \sum_{a \in M} \sum_{\gamma \in G} \gamma BZ_{1^1}(\mathbf{w}_5) = m \cdot BZ_{1^1}(\mathbf{w}_5) = m \cdot \frac{(nm-1)! \sum_{y=0}^{\frac{nm-2}{2}} \frac{\frac{nm}{2}+1+y}{(\frac{nm}{2}-1-y)! (\frac{nm}{2}-y)! (1+2y)!}}{3^{nm-1}},$$

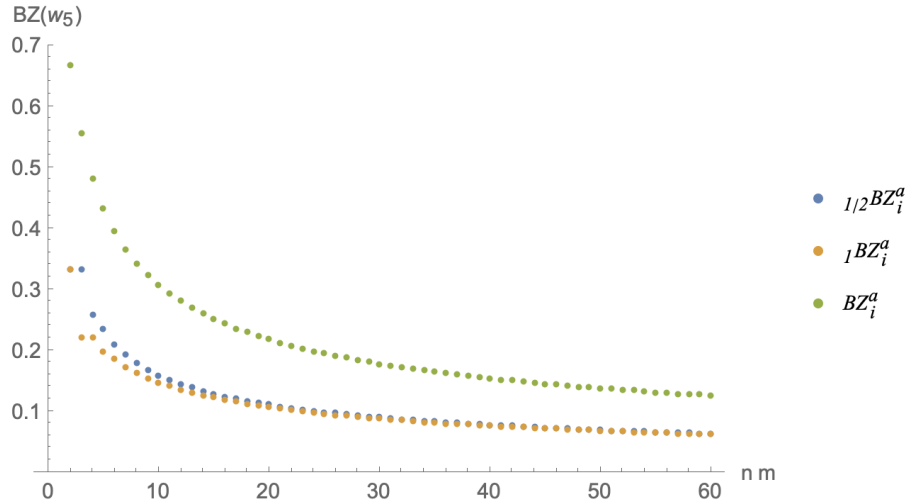
and for nm odd:

$$BZ_1(\mathbf{w}_5) = \sum_{a \in M} \sum_{\gamma \in G} \gamma BZ_{1^1}(\mathbf{w}_5) = m \cdot BZ_{1^1}(\mathbf{w}_5) = m \cdot \frac{(nm-1)! \sum_{y=0}^{\frac{nm-1}{2}} \frac{\frac{nm+1}{2}+y}{(\frac{nm-1}{2}-y)! (\frac{nm+1}{2}-y)! (2y)!}}{3^{nm-1}}.$$

□

Figure 3 illustrates Proposition 4 for various values of nm .

Figure 3: Influence of a test for w_5



$\frac{1}{2} BZ_{i^a}$, $\frac{1}{3} BZ_{i^a}$ and therefore BZ_{i^a} decrease as the number of tests increases since a given test has a smaller probability to influence the average.

We now present results for four grades.

Proposition 5. Let $G = \{1, \frac{2}{3}, \frac{1}{3}, 0\}$ and define the university rule \mathbf{w}_6 by $\mathbf{w}_6(\mathbf{S}) = 1$ if $\bar{s} \geq \frac{n \cdot m}{2}$. The Banzhaf index for all i is given by:

- For nm even: $BZ_i(\mathbf{w}_6) =$

$$m(nm-1)! \frac{\sum_{k=0}^{\frac{nm-2}{4} \min(3k+2, \frac{nm}{2}-k)} \sum_{z=0}^{(3k-z+1)(3k-z+2)(-k+\frac{nm}{2}+z+1)+(3k-z+2)(-k+\frac{nm}{2}-z)(-2k+\frac{nm}{2}+2z)+(-2k+\frac{nm}{2}-1)(-k+\frac{nm}{2}-z)(-k+\frac{nm}{2}+z)}{(-2k+\frac{nm}{2}-1)!(2z+1)!(3k-z+2)!(-k+\frac{nm}{2}-z)!}}{4^{nm-1}}$$

• For nm odd: $BZ_i(\mathbf{w}_6) =$

$$m(nm-1)! \frac{\sum_{k=0}^{\frac{nm-1}{4} \min(3k+2, \frac{nm-1}{2}-k)} \sum_{z=0}^{(\frac{nm-1}{2}-2k)(3k-z+2)(-k+\frac{nm-1}{2}+z+1)+(2z+1)(3k-z+2)(k+\frac{nm+1}{2}-z)+2(\frac{nm-3}{2}-2k)(\frac{nm-1}{2}-2k)(-k+\frac{nm-1}{2}-z)}{(\frac{nm-1}{2}-2k)!(2z+1)!(3k-z+2)!(-k+\frac{nm-1}{2}-z)!}}{4^{nm-1}}$$

Proof. Once again, we present the proof for $i = 1$ and $a = 1$.

a) In order to evaluate when 1 is in position of 1^1 -swing we must distinguish between nm even and nm odd.

i) nm even: for a period $a = 1$ and a grade $\gamma = 1$, 1 is in position of 1^1 -swing when $\bar{s}^{-1} = \frac{nm}{2} - 1$. We enumerate the total number of occurrences by considering first the configurations with the maximum number of grades $\gamma = 1$. Since $n.m$ is even, this maximum is $\frac{nm}{2} - 1$, and it is sufficient to obtain $\bar{s}^{-1} = \frac{nm}{2} - 1$. There are $C_{nm-1}^{\frac{nm}{2}-1}$ such configurations.

i-a) Assume that we remove an odd number of grades $\gamma = 1$. Firstly consider that we remove one grade $\gamma = 1$. In order to obtain $\frac{nm}{2} - 1$ we can replace this grade by one grade $\gamma = \frac{2}{3}$ and one grade $\gamma = \frac{1}{3}$. There are $C_{nm-1}^{(\frac{nm}{2}-1)-1} (C_{(nm-1)-(\frac{nm}{2}-1)-1}^{3.1-1} C_{(nm-1)-(\frac{nm}{2}-1)-1-\frac{3.1-1}{2}}^1)$ $= C_{nm-1}^{(\frac{nm}{2}-1)-1} (C_{\frac{n}{2}+1}^{\frac{3.1-1}{2}} C_{\frac{n-1}{2}}^1)$ such configurations. From such configurations, we can remove one grade $\gamma = \frac{2}{3}$ and replace it by two grades $\gamma = \frac{1}{3}$. There are $C_{nm-1}^{(\frac{nm}{2}-1)-1} (C_{\frac{nm}{2}+1}^{\frac{3.1-1}{2}-1} C_{\frac{nm-1-1}{2}+1}^{1+2.1})$ situations.

Secondly assume that we remove three grades $\gamma = 1$ from the initial situation, and replace them by four grades $\gamma = \frac{2}{3}$ and one grade $\frac{1}{3}$. There are $C_{nm-1}^{(\frac{nm}{2}-1)-3} (C_{\frac{nm}{2}+3}^{\frac{3.3-1}{2}} C_{\frac{nm-1-3}{2}}^1)$ such configurations. Once again we can remove one grade $\gamma = \frac{2}{3}$ and replace it by two grades $\gamma = \frac{1}{3}$. There are $C_{n-1}^{(\frac{n}{2}-1)-3} (C_{\frac{nm}{2}+3}^{\frac{3.3-1}{2}-1} C_{\frac{nm-1-3}{2}+1}^{1+2.1})$ configurations. We can also replace two grades $\gamma = \frac{2}{3}$ by four grades $\gamma = \frac{1}{3}$. There are $C_{nm-1}^{(\frac{nm}{2}-1)-3} (C_{\frac{nm}{2}+3}^{\frac{3.3-1}{2}-2} C_{\frac{nm-1-3}{2}+2}^{1+2.2})$ such configurations. Note that we can replace grades $\gamma = \frac{2}{3}$ by grades $\gamma = \frac{1}{3}$ until all $\frac{2}{3}$ grades are replaced or until there are not enough tests in the configuration to replace a $\gamma = \frac{2}{3}$ grade by two $\gamma = \frac{1}{3}$ grades. We can generalize our method for all $n.m$ even when we remove of an odd number of grades $\gamma = 1$. Let $(2.k + 1)$, with k a positive integer, be the numbers of grade $\gamma = 1$ we remove. Let z be the number of grades $\gamma = \frac{2}{3}$ removed and replaced by two $\gamma = \frac{1}{3}$. We obtain the following

formula $\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm}{2}-1)-(2k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+(2k+1)}^{3(2k+1)-z} C_{\frac{nm-1-(2k+1)}{2}+z}^{2z+1} \right]$ which is equal to

$$\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{\frac{nm}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+2k+1}^{3k+1-z} C_{\frac{nm}{2}-(k+1)+z}^{2z+1} \right] \cdot 3$$

i-b) Assume now that we remove an even number of grades $\gamma = 1$. For example, we can replace two grades $\gamma = 1$ by three grades $\gamma = \frac{2}{3}$. The number of such configurations is $C_{nm-1}^{(\frac{nm}{2}-1)-2} (C_{(nm-1)-(\frac{nm}{2}-2)-1}^{\frac{3 \cdot 2}{2}} C_{(nm-1)-(\frac{nm}{2}-1)-2-\frac{3 \cdot 2}{2}}^0) = C_{nm-1}^{(\frac{n}{2}-1)-2} (C_{\frac{nm}{2}+2}^{\frac{3 \cdot 2}{2}} C_{\frac{nm-2}{2}}^0)$. From this situation, we can replace one grade $\gamma = \frac{2}{3}$ by two grades $\gamma = \frac{1}{3}$. There are $C_{nm-1}^{(\frac{n}{2}-1)-2} (C_{\frac{nm}{2}+2}^{\frac{3 \cdot 2}{2}-1} C_{\frac{nm-2}{2}+1}^{2 \cdot 1})$ such configurations. We can also replace two grades $\gamma = \frac{2}{3}$ by four grade $\gamma = \frac{1}{3}$, and so on. Now, let $2 \cdot k$ be the number of grades $\gamma = 1$ we remove.

Then the general formula is given by: $\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm}{2}-1)-2k} \left[\sum_{z=0}^{\min\{3k, \frac{n}{2}-k\}} C_{\frac{nm}{2}+2k}^{3k-z} C_{\frac{nm}{2}-k+z}^{2z} \right]$.

Then the total number of occurrences of $\bar{s}^{-1} = \frac{nm}{2} - 1$ for $n \cdot m$ even is given by:

$$\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm}{2}-1)-2k} \left[\sum_{z=0}^{\min\{3k, \frac{nm}{2}-k\}} C_{\frac{nm}{2}+2k}^{3k-z} C_{\frac{nm}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-2}{4}} C_{n-1}^{\frac{nm}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+2k+1}^{3k+1-z} C_{\frac{nm}{2}-(k)+z}^{2z+1} \right]$$

Therefore for $n \cdot m$ even, we have ${}_1BZ_{1^1}(\mathbf{w}_6) =$

$$\frac{\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm}{2}-1)-2k} \left[\sum_{z=0}^{\min\{3k, \frac{nm}{2}-k\}} C_{\frac{nm}{2}+2k}^{3k-z} C_{\frac{nm}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-2}{4}} C_{n-1}^{\frac{nm}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+2k+1}^{3k+1-z} C_{\frac{nm}{2}-(k)+z}^{2z+1} \right]}{4^{nm-1}} \quad (5)$$

ii) $n \cdot m$ odd: for a period $a = 1$ and a grade $\gamma = 1$, 1 is in position of ${}_11^1$ -swing when $\bar{s}^{-1} = \frac{n \cdot m}{2} - 1 + \frac{1}{6} = \frac{3(nm)-5}{6}$.⁴ The logic of the proof is similar to $n \cdot m$ even and omitted. The total number of occurrences is given by:

$$\sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-(2k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{n-1}{2}-k\}} C_{\frac{nm+1}{2}+2k}^{3k+1-z} C_{\frac{nm-1}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{nm-1}{2}-k\}} C_{\frac{nm+1}{2}+2k+1}^{3k+2-z} C_{\frac{nm-1}{2}-k+z}^{2z+1} \right]$$

³Note that the maximum number of grades $\gamma = \frac{2}{3}$ replaced is equal to $3k + 1$, the maximum of $\frac{2}{3}$ grades ; or $\frac{nm}{2} - (k + 1)$, the number where there is not enough remaining test in the configuration in order to replace the $\gamma = \frac{2}{3}$ grades by two $\frac{1}{3}$ grades.

⁴Note that for $n \cdot m$ odd, $\frac{nm}{2} - 1$ is not a multiple of $\frac{1}{3}$ then we must add $+\frac{1}{6}$. For example for $n \cdot m = 3$, 1^1 is in position of ${}_11^1$ -swing when $\bar{s}^{-1} = \frac{3}{2} - 1 + \frac{1}{6} = \frac{2}{3}$.

Therefore for nm odd, we have ${}_1BZ_{1^1}(\mathbf{w}_6) =$

$$\frac{\sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-(2k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{n-1}{2}-k\}} C_{\frac{nm+1}{2}+2k}^{3k+1-z} C_{\frac{nm-1}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{nm-1}{2}-k\}} C_{\frac{nm+1}{2}+2k+1}^{3k+2-z} C_{\frac{nm-1}{2}-k+z}^{2z+1} \right]}{4nm-1} \quad (6)$$

b) For a period $a = 1$ and a grade $\gamma = \frac{2}{3}$, 1 is in position of $\frac{2}{3}1^1$ -swing when $\bar{s}^{-1} = \frac{nm}{2} - \frac{2}{3}$ for nm even; and $\bar{s}^{-1} = \frac{nm}{2} - \frac{2}{3} + \frac{1}{6} = \frac{3(nm)-3}{6}$ for nm odd. The proof is similar to the previous one and omitted.

For $n.m$ even, the total number of occurrences is given by:

$$\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm}{2}-1)-2k} \left[\sum_{z=0}^{\min\{3k, \frac{nm}{2}-k-1\}} C_{\frac{nm}{2}+2k}^{3k-z} C_{\frac{nm}{2}-k+z}^{2z+1} \right] + \sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{\frac{nm}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+2k+1}^{3k+2-z} C_{\frac{nm}{2}-(k+1)+z}^{2z} \right]$$

For $n.m$ odd, the total number of occurrences is given by:

$$\sum_{k=0}^{\frac{nm-1}{4}} C_{nm-1}^{\frac{n-1}{2}-2k} \left[\sum_{z=0}^{\min\{3k, \frac{nm-1}{2}-k\}} C_{\frac{nm-1}{2}+2k}^{3k-z} C_{\frac{nm-1}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-(2k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm-3}{2}-2k\}} C_{\frac{nm-1}{2}+2k+1}^{3k+1-z} C_{\frac{nm-1}{2}-k+z}^{2z+1} \right]$$

Therefore for $n.m$ even, we have $\frac{2}{3}BZ_{1^1}(\mathbf{w}_6) =$

$$\frac{\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm}{2}-1)-2k} \left[\sum_{z=0}^{\min\{3k, \frac{nm}{2}-k-1\}} C_{\frac{nm}{2}+2k}^{3k-z} C_{\frac{nm}{2}-k+z}^{2z+1} \right] + \sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{\frac{nm}{2}-2(z+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+2k+1}^{3k+2-z} C_{\frac{nm}{2}-(k+1)+z}^{2z} \right]}{4nm-1} \quad (7)$$

and for $n.m$ odd we have $\frac{2}{3}BZ_{1^1}(\mathbf{w}_6) =$

$$\frac{\sum_{k=0}^{\frac{nm-1}{4}} C_{nm-1}^{\frac{n-1}{2}-2k} \left[\sum_{z=0}^{\min\{3k, \frac{nm-1}{2}-k\}} C_{\frac{nm-1}{2}+2k}^{3k-z} C_{\frac{nm-1}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-(2k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm-3}{2}-2k\}} C_{\frac{nm-1}{2}+2k+1}^{3k+1-z} C_{\frac{nm-1}{2}-k+z}^{2z+1} \right]}{4nm-1} \quad (8)$$

c) For a period $a = 1$ and a grade $\gamma = \frac{1}{3}$, 1 is in position of a $\frac{1}{3}1^1$ -swing when $\bar{s}^{-1} = \frac{nm}{2} - \frac{1}{3}$ for $n.m$ even; and $\bar{s}^{-1} = \frac{nm}{2} - \frac{1}{3} + \frac{1}{6} = \frac{3(nm)-1}{6}$ for $n.m$ odd. The proof is similar to the previous one and therefore omitted.

For nm even, the total number of occurrences is given by

$$\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm}{2}-1)-2k} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm}{2}-k-1\}} C_{\frac{nm}{2}+2k}^{3k+1-z} C_{\frac{nm}{2}-k-1+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{\frac{nm}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+2k+1}^{3k+2-z} C_{\frac{nm}{2}-(k+1)+z}^{2z+1} \right]$$

For $n.m$ odd, the total number of occurrence is given by:

$$\sum_{k=0}^{\frac{nm-3}{4}} C_{n-1}^{\frac{nm-1}{2}-(2k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{n-1}{2}-k\}} C_{\frac{nm+1}{2}+2k}^{3k+1-z} C_{\frac{nm-1}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{n-1}{2}-k\}} C_{\frac{nm+1}{2}+2k+1}^{3k+2-z} C_{\frac{nm-1}{2}-k+z}^{2z+1} \right]$$

Therefore for $n.m$ even, we have $\frac{1}{3}BZ_{1^1}(\mathbf{w}_6) =$

$$\frac{\sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{(\frac{nm-1}{2})-2k} \left[\sum_{z=0}^{\min\{3k+1, \frac{nm}{2}-k-1\}} C_{\frac{nm}{2}+2k}^{3k+1-z} C_{\frac{nm}{2}-k+1+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-2}{4}} C_{nm-1}^{\frac{nm}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{nm}{2}-(k+1)\}} C_{\frac{nm}{2}+2k+1}^{3k+2-z} C_{\frac{nm}{2}-(k+1)+z}^{2z+1} \right]}{4^{nm-1}} \quad (9)$$

and for $n.m$ odd we have $\frac{1}{3}BZ_{1^1}(\mathbf{w}_6) =$

$$\frac{\sum_{k=0}^{\frac{nm-3}{4}} C_{n-1}^{\frac{nm-1}{2}-(2k+1)} \left[\sum_{z=0}^{\min\{3k+1, \frac{n-1}{2}-k\}} C_{\frac{nm+1}{2}+2k}^{3k+1-z} C_{\frac{nm-1}{2}-k+z}^{2z} \right] + \sum_{k=0}^{\frac{nm-3}{4}} C_{nm-1}^{\frac{nm-1}{2}-2(k+1)} \left[\sum_{z=0}^{\min\{3k+2, \frac{n-1}{2}-k\}} C_{\frac{nm+1}{2}+2k+1}^{3k+2-z} C_{\frac{nm-1}{2}-k+z}^{2z+1} \right]}{4^{nm-1}} \quad (10)$$

d) Finally for $n.m$ even, the Banzhaf influence index of 1^1 is given by (5) +(7)+(9). After simplification, we obtain: $BZ_{1^1}(\mathbf{w}_6) =$

$$\frac{(nm-1)! \sum_{k=0}^{\frac{nm-2}{4}} \sum_{z=0}^{\min(3k+2, \frac{nm}{2}-k)} \frac{(3k-z+1)(3k-z+2)(-k+\frac{nm}{2}+z+1)+(3k-z+2)(-k+\frac{nm}{2}-z)(-2k+\frac{nm}{2}+2z)+(-2k+\frac{nm}{2}-1)(-k+\frac{nm}{2}-z)(-k+\frac{nm}{2}+z)}{(-2k+\frac{nm}{2}-1)!(2z+1)!(3k-z+2)!(-k+\frac{nm}{2}-z)!}}{4^{nm-1}}$$

Likewise for nm odd, the Banzhaf influence index of 1^1 is the sum of (6)+(8)+(10), and is given by $BZ_{1^1}(\mathbf{w}_6) =$

$$= \frac{(nm-1)! \sum_{k=0}^{\frac{nm-1}{4}} \sum_{z=0}^{\min(3k+2, \frac{nm-1}{2}-k)} \frac{(\frac{nm-1}{2}-2k)(3k-z+2)(-k+\frac{nm-1}{2}+z+1)+(2z+1)(3k-z+2)(k+\frac{nm+1}{2}-z)+2(\frac{nm-3}{2}-2k)(\frac{nm-1}{2}-2k)(-k+\frac{nm-1}{2}-z)}{(\frac{nm-1}{2}-2k)!(2z+1)!(3k-z+2)!(-k+\frac{nm-1}{2}-z)!}}{4^{nm-1}}$$

e) Thus for $n.m$ even: $BZ_1(\mathbf{w}_6) = \sum_{a \in M} \sum_{\gamma \in G} \gamma BZ_{1^1}(\mathbf{w}_6) = m.BZ_{1^1}(\mathbf{w}_6) =$

$$m(nm-1)! \sum_{k=0}^{\frac{nm-2}{4}} \sum_{z=0}^{\min(3k+2, \frac{nm}{2}-k)} \frac{(3k-z+1)(3k-z+2)(-k+\frac{nm}{2}+z+1)+(3k-z+2)(-k+\frac{nm}{2}-z)(-2k+\frac{nm}{2}+2z)+(-2k+\frac{nm}{2}-1)(-k+\frac{nm}{2}-z)(-k+\frac{nm}{2}+z)}{(-2k+\frac{nm}{2}-1)!(2z+1)!(3k-z+2)!(-k+\frac{nm}{2}-z)!}}{4^{nm-1}}$$

and for $n.m$ odd: $BZ_1(\mathbf{w}_6) = \sum_{a \in M} \sum_{\gamma \in G} \gamma BZ_{1^1}(\mathbf{w}_6) = m.BZ_{1^1}(\mathbf{w}_6) =$

$$= \frac{m(nm-1)! \sum_{k=0}^{\frac{nm-1}{4}} \sum_{z=0}^{\min(3k+2, \frac{nm-1}{2}-k)} \frac{(\frac{nm-1}{2}-2k)(3k-z+2)(-k+\frac{nm-1}{2}+z+1)+(2z+1)(3k-z+2)(k+\frac{nm+1}{2}-z)+2(\frac{nm-3}{2}-2k)(\frac{nm-1}{2}-2k)(-k+\frac{nm-1}{2}-z)}{(\frac{nm-1}{2}-2k)!(2z+1)!(3k-z+2)!(-k+\frac{nm-1}{2}-z)!}}{4^{nm-1}}$$

□

5 Shapley evaluation

In order to extend the Shapley-Shubik influence index (Shapley and Shubik [1954]) to the university rule, we extend the notion of a "roll call" introduced by Felsenthal and Machover [1997]. A roll call can be seen as the tests of topic i queuing up in a random order (to be passed in a given order); and at the same time each test i^a has an equal probability of $\frac{1}{g}$ to have a given grade γ . Note that in some university programs, students can try the tests in the order they want.

More formally, let C^N be the probability space consisting of the set \mathbb{Z}_n^m of all configurations of N , with each configuration assigned the same probability: $\frac{1}{g^{n.m}}$. A given configuration of C^N will be denoted by C . Let Q^{N^*} be the probability space consisting of the set of all queues Q of N^* , with each queue assigned probability $\frac{1}{n.m!}$. Note that N^* is derived from N . A queue denoted Q of N^* is simply a bijection from N^* to the set $I_{N^*} = \{1^1, \dots, n^m\}$ with $|I_{N^*}| = n.m$. We denote by a roll call of N an ordered pair $R = \langle Q, C \rangle$, where Q is a queue of N^* and C a configuration of N . We denote by $Q^{N^*} \times C^N$ the probability space consisting of the set of all roll calls of N , with each roll call assigned the same probability: $\frac{1}{(n.m)!g^{n.m}}$.

A topic i at a period a with a grade γ is a γi^a -pivot for the roll-call R and the rule \mathbf{w} , denoted $\gamma piv^a(R, \mathbf{w})$, if one of the two following excluding conditions is satisfied:

1. regardless of the grades of all remaining tests, the student obtains his or her diploma if it obtains a grade γ to this test ;
2. regardless of the grade of all the remaining tests, the student cannot obtain his or her diploma with a grade γ to this test.

For each topic i at a given period a , and for a given grade γ , we denote by $\gamma p_i^a(\mathbf{w})$ the number of γi^a -pivots for the university rule \mathbf{w} , *i.e.* $\gamma p_i^a(\mathbf{w}) = |\{R \in Q^{N^*} \times C^N : \gamma piv^a(R, \mathbf{w}) = i^a\}|$. A pivot for a topic i is defined as a γi^a -pivot for at least one $a \in M$ and one grade γ . We shall denote by $p_i(\mathbf{w})$ the number of pivots for topic i for the university rule \mathbf{w} *i.e.* $p_i(\mathbf{w}) = \sum_{a=1}^m \sum_{\gamma \in G} \gamma p_i^a(\mathbf{w})$. Finally we have $\overline{\gamma p^a(\mathbf{w})}$ the total number of γi^a -pivots for a grade γ and a period a for the university rule \mathbf{w} , *i.e.* $\overline{\gamma p^a(\mathbf{w})} = \sum_{i=1}^n \gamma p_i^a(\mathbf{w})$, and $\overline{p(\mathbf{w})}$ the total number of pivots for the university rule \mathbf{w} , *i.e.*

$\overline{p(\mathbf{w})} = \sum_{a=1}^m \sum_{\gamma \in G} \overline{\gamma p^a(\mathbf{w})} = (n.m)!g^{(n.m)}$, since for each roll call there exists a pivot topic.

Definition 3. Let \mathbf{w} be a university rule The Shapley influence index for a topic $i \in N$ is given by :

$$SH_i(\mathbf{w}) = \sum_{a \in M} \sum_{\gamma \in G} \gamma SH_{ia},$$

with

$$\gamma SH_{ia}(\mathbf{w}) = \frac{|\{R \in Q^{N^*} \times C^N : \gamma piv^a(R, \mathbf{w}) = i^a\}|}{(n.m)!g^{(n.m)}} = \frac{\gamma p_i^a(\mathbf{w})}{(n.m)!g^{(n.m)}} \text{ for each } \gamma \in G.$$

As suggested by Felsenthal and Machover [1997] for ternary game, for the calculus we can use $\gamma SH_{ia} = P(i^a \text{ is } \gamma i^a - \text{pivot})$ where P is the (discrete and uniform) probability distribution in the roll call space $Q^{N^*} \times C^N$. This influence index generalizes both the Shapley influence index for multi-dimensional rules introduced by Courtin [2022] when there are more than two grades, and the Shapley influence index for $(j, 2)$ rules introduced by Freixas [2005b] when there is only one period.

Characterization

Since the Shapley influence index does not satisfy the Banzhaf efficiency axiom, we introduce a new axiom which states that the total aggregated influence of all topics is equal to one.

Axiom 5. (*Shapley Efficiency*)

For all $\mathbf{w} \in \mathbb{W}_m^n$,

$$\sum_{i=1}^n \gamma \phi_{ia}(\mathbf{w}) = \frac{\overline{\gamma p^a(\mathbf{w})}}{(n.m)!g^{(n.m)}} \text{ for each } \gamma \in G \text{ and each } a \in M.$$

Since $\phi_i(\mathbf{w}) = \sum_{a=1}^m \sum_{\gamma \in G} \gamma \phi_{ia}(\mathbf{w})$ and $\overline{p(\mathbf{w})} = \sum_{a=1}^m \sum_{\gamma \in G} \overline{\gamma p^a(\mathbf{w})} = (n.m)!g^{(n.m)}$, this efficiency

axiom implies that $\sum_{i=1}^n \phi_i(\mathbf{w}) = \sum_{i=1}^n \sum_{a=1}^m \sum_{\gamma \in G} \gamma \phi_{ia}(\mathbf{w}) = \sum_{a=1}^m \sum_{\gamma \in G} \frac{\overline{\gamma p^a(\mathbf{w})}}{g^{n.m-1}} = \frac{(n.m)!g^{(n.m)}}{(n.m)!g^{(n.m)}} = 1$

Theorem 2. Let ϕ be an index for the university rule. Then, ϕ satisfies Axiom 2, Axiom 3, Axiom 4 and Axiom 5 if and only if $\phi = SH$.

Proof. The beginning and the end of the proof are similar to the proof of Theorem 1 and therefore omitted. Only the part concerning the determination of $\gamma \phi_{ia}(\mathbf{u}_S)$ must be replaced by:

$\gamma\phi^a(\mathbf{u}_S)$ (and then $\phi_i(\mathbf{u}_S)$) is uniquely determined, if ϕ exists, and for each $\gamma \in G$, and each $a \in M$, using Axiom 5, is given by

$$\gamma\phi_{i^a}(\mathbf{u}_S) = \begin{cases} \frac{\gamma P^a(\mathbf{u}_S)}{(n,m)!g^{(n,m)}_{\gamma s^a}} & \text{if } s_i^a = \gamma \\ 0 & \text{otherwise} \end{cases}$$

□

To conclude this section, let us illustate Shapley in Examples 1 and 2.

Example 5. Shapley evaluation of w_1

Topics 2 and 3 at period 1 play a "dummy" role, and therefore they have no influence. Hence we may only consider roll calls of four topics (whithout considering the tests 2^1 and 3^1).

Topic 1 at period 1 is pivotal when :

Table 1: ${}_01^1$ pivot

j^2	1rst	2nd	3nd	4nd	
–	☒				
$s_j^2 = \gamma$		☒			
$s_j^2 > 0$			☒	☒	
$s_j^2 > 0$ and $s_k^2 = 0$ for one $k \neq j$			☒	☒	
P	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{27} + \frac{1}{27}$	$\frac{2}{81} + \frac{1}{27}$	$\frac{49}{162}$

Column 1 represents the result of the tests different from 1^1 before the student takes test 1^1 . Column 2, 3, 4 and 5 represent the order of passage of test 1^1 . $s_j^2 > 0$ and $s_k^2 = 0$ for one $k \neq j$ means that one test of the second period has a zero grade, and the other tests of period 2 have a grade different from 0. The last line is the probability that each situation occurs.

Table 2: ${}_\gamma 1^1$ pivot for $\gamma = \frac{1}{2}, 1$

j^2	3nd	4nd	
$s_j^2 > 0$	☒	☒	
$s_j^2 > 0$ and $s_k^2 = 0$ for one $k \neq j$		☒	
P	$\frac{1}{27}$	$\frac{2}{81} + \frac{1}{27}$	$\frac{8}{81}$

Thus $SH_{1^1}(\mathbf{w}_1) = {}_0SH_{1^1}(\mathbf{w}_1) + \frac{1}{2}SH_{1^1}(\mathbf{w}_1) + {}_1SH_{1^1}(\mathbf{w}_1) = \frac{49}{162} + \frac{8}{81} + \frac{8}{81} = \frac{1}{2}$.

Topic 1 at period 2 is pivotal in cases described in Tables 3 and 4. Thus $SH_{1^2}(\mathbf{w}_1) = {}_0SH_{1^2}(\mathbf{w}_1) + \frac{1}{2}SH_{1^2}(\mathbf{w}_2) + {}_1SH_{1^2}(\mathbf{w}_1) = \frac{11}{162} + \frac{4}{81} + \frac{4}{81} = \frac{1}{6}$.

Moreover topics 1^2 , 2^2 and 3^2 have the same influence.

Table 3: 01^2 pivot

j^a		2nd	3nd	4nd	
for $a = 1$	for $a = 2$				
–	$ j^2 = 1, s_j^2 = 0$	☒			
–	$ j^2 = 2, s_j^2 = 0$ and $s_k^2 > 0, j \neq k$		☒		
$s_1^1 > 0$	$ j^2 = 1, s_j^2 > 0$		☒		
$s_1^1 > 0$	$ j^2 = 2, s_j^2 = 0$ and $s_k^2 > 0, j \neq k$			☒	
P		$\frac{1}{54}$	$\frac{1}{81} + \frac{1}{81}$	$\frac{2}{81}$	$\frac{11}{162}$

Table 4: $\gamma 1^2$ pivot for $\gamma = \frac{1}{2}, 1$

j^a		3nd	4nd	
for $a = 1$	for $a = 2$			
$s_1^1 > 0$	$ j^2 = 1, s_j^2 > 0$	☒		
$s_1^1 > 0$	$ j^2 = 2, s_j^2 = 0$ and $s_k^2 > 0, j \neq k$		☒	
P		$\frac{2}{81}$	$\frac{2}{81}$	$\frac{4}{81}$

Then we have: $SH_1(\mathbf{w}_1) = SH_{11}(\mathbf{w}_1) + SH_{12}(\mathbf{w}_1) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ and $SH_2(\mathbf{w}_1) = SH_3(\mathbf{w}_1) = \frac{1}{6}$.

Example 6. Shapley evaluation of w_2

Topic 1 at period 1 is pivotal in cases described in Tables 5 and 6. Thus $SH_{11}(\mathbf{w}_2) = {}_0SH_{11}(\mathbf{w}_2) + \frac{1}{2}SH_{11}(\mathbf{w}_2) + {}_1SH_{11}(\mathbf{w}_2) = \frac{353}{3645} + \frac{188}{3645} + \frac{188}{3645} = \frac{1}{5}$.

Moreover topic 2 at period 1 and topics 1 and 2 at period 2 have the same influence as topic 1 at period 1.

Topic 1 at period 3 is pivotal in cases described in Tables 7 and 8.

Thus $SH_{13}(\mathbf{w}_2) = {}_0SH_{13}(\mathbf{w}_2) + \frac{1}{2}SH_{13}(\mathbf{w}_2) + {}_1SH_{13}(\mathbf{w}_2) = \frac{313}{7290} + \frac{104}{3645} + \frac{104}{3645} = \frac{1}{10}$.

Moreover topic topics 1 and 2 at period 3 have a symmetric role, hence have the same influence.

Then we have $SH_1(\mathbf{w}_2) = SH_{11}(\mathbf{w}_2) + SH_{12}(\mathbf{w}_2) + SH_{13}(\mathbf{w}_2) = \frac{1}{5} + \frac{1}{5} + \frac{1}{10} = \frac{1}{2}$ and $SH_2(\mathbf{w}_2) = \frac{1}{2}$.

Table 5: 01^1 pivot

j^a		2nd	3nd	4nd	5nd	6nd	
for $a = 1, 2$	for $a = 3$						
$s_j^a = 0$	—	☒					
$s_j^a = 0$ and $s_k^a > 0$ for one $k \neq j$	—		☒	☒			
$s_j^a = 0$	$ j^3 = 1, s_j^3 = \gamma$		☒				
$s_j^a = 0$ and $s_k^a > 0$ for one $k \neq j$	$ j^3 = 1, s_j^3 = \gamma$			☒			
$s_j^a = 0$	$ j^3 = 2, s_j^3 = 0$ and $s_k^3 > 0$ for one $k \neq j$			☒			
$s_j^a = 0$	$ j^3 = 2, s_j^3 > 0$			☒			
$s_j^a = 0$ and $s_k^a > 0$ for two $k \neq j$	$ j^3 = 1, s_j^3 > 0$				☒		
$s_j^a = 0$ and $s_k^a > 0$ for two $k \neq j$	$ j^3 = 1, s_j^3 = 0$				☒		
$s_j^a = 0$ and $s_k^a > 0$ for one $k \neq j$	$ j^3 = 2, s_j^3 > 0$				☒		
$s_j^a = 0$ and $s_k^a > 0$ for one $k \neq j$	$ j^3 = 2, s_j^3 > 0$ and $s_k^3 = 0$ for one $k \neq j$				☒		
$s_j^a = 0$ and $s_k^a > 0$ for two $k \neq j$	$ j^3 = 2, s_j^3 > 0$ and $s_k^3 = 0$ for one $k \neq j$					☒	
$s_j^a = 0$ and $s_k^a > 0$ for two $k \neq j$	$ j^3 = 2, s_j^3 > 0$					☒	
P		$\frac{1}{90}$	$\frac{1}{135} + \frac{1}{90}$	$\frac{1}{405} + \frac{2}{135} + \frac{1}{405} + \frac{1}{405}$	$\frac{8}{1215} + \frac{4}{1215} + \frac{8}{1215} + \frac{8}{1215}$	$\frac{8}{729} + \frac{8}{729}$	$\frac{353}{3645}$

Table 6: $\gamma 1^1$ pivot for $\gamma = \frac{1}{2}, 1$

j^a		4nd	5nd	6nd	
for $a = 1, 2$	for $a = 3$				
$s_j^a > 0$	$ j^3 = 1, s_j^3 > 0$	☒			
$s_j^a = 0$ and $s_k^a > 0$ for two $k \neq j$	$ j^3 = 1, s_j^3 > 0$		☒		
$s_j^a > 0$	$ j^3 = 2, s_j^3 > 0$		☒		
$s_j^a > 0$	$ j^3 = 2, s_j^3 > 0$ and $s_k^3 = 0$ for one $k \neq j$		☒		
$s_j^a = 0$ and $s_k^a > 0$ for one $k \neq j$	$ j^3 = 2, s_j^3 > 0$ and $s_k^3 = 0$ for one $k \neq j$			☒	
$s_j^a = 0$ and $s_k^a > 0$ for two $k \neq j$	$ j^3 = 2, s_j^3 > 0$			☒	
P		$\frac{4}{405}$	$\frac{8}{1215} + \frac{8}{1215} + \frac{8}{1215}$	$\frac{8}{729} + \frac{8}{729}$	$\frac{188}{3645}$

6 Concluding discussion

To conclude, let us summarize all the characterizations in the following tables.

SH	BZ
Transfer	Transfer
Anonymity	Anonymity
Banzhaf Efficiency	Shapley Efficiency
Dummy	Dummy

The dummy, anonymity and transfer properties appear in both characterizations presented above. The Shapley efficiency characterizes Shapley but not Banzhaf where

Table 7: 01^3 pivot

j^a		2nd	3nd	4nd	5nd	6nd	
for $a = 1, 2$	for $a = 3$						
$s_j^a = \gamma$	$ j^3 = 1, s_j^3 = 0$	☒	☒				
$s_j^a > 0$	$ j^3 = 1, s_j^3 = 0$			☒	☒	☒	
$s_j^a > 0$ and $s_k^a = 0$ for one $k \neq j$	$ j^3 = 1, s_j^3 = 0$			☒	☒	☒	
P		$\frac{1}{270}$	$\frac{1}{135}$	$\frac{2}{405} + \frac{2}{405}$	$\frac{16}{3645} + \frac{8}{1215}$	$\frac{8}{2187} + \frac{16}{2187}$	$\frac{313}{7290}$

Table 8: $\gamma 1^3$ pivot for $\gamma = \frac{1}{2}, 1$

j^a		4nd	5nd	6nd	
for $a = 1, 2$	for $a = 3$				
$s_j^a > 0$	—	☒	☒		
$s_j^a > 0$ and $s_k^a = 0$ for one $k \neq j$	—		☒		
$s_j^a > 0$	$ j^3 = 1, s_j^3 = 0$		☒	☒	
$s_j^a > 0$ and $s_k^a = 0$ for one $k \neq j$	$ j^3 = 1, s_j^3 = 0$			☒	
P		$\frac{8}{1215}$	$\frac{8}{3645} + \frac{16}{3645} + \frac{16}{3645}$	$\frac{8}{2187} + \frac{16}{2187}$	$\frac{104}{3645}$

a modified efficiency property appears.

There are several ways in which the university rules could be explored in further research. We can study, first, the generalization of the Shapley and Banzhaf influence indices to university rules where some topics are not taught at every periods. Another interesting direction is the extension of the university rule when considering that some topics are *a priori* linked (Microeconomics Principle and Advanced Microeconomics for example) in the spirit of Owen [1977] and Owen [1981]. Following the works of Tchantcho et al. [2008], we can also derive the desirability relation to qualitatively compare the *a priori* influence of topics in a university rule.

References

- J.M. Alonso-Meijide and J. Freixas. A new power index based on minimal winning coalitions without any surplus. *Decision support system*, 49:70–76, 2010.
- J.M. Alonso-Meijide, M. Álvarez-Mozos, F. Ferreira, and A. Pinto. Two new power indices based on winning coalitions. *Journal of Difference Equations and Applications*, 17:1095–1100, 2011.

- N. Andjiga, F. Chantreuil, and D. Lepelley. La mesure du pouvoir de vote. *Mathématiques et Sciences Humaines*, 163:111–145, 2003.
- J.F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.
- J.M. Bilbao, J.R. Fernández, A. Jiménez Losada, and E. Lebron. Bicooperative games. In J.M. Bilbao, editor, *Cooperative Games on Combinatorial Structures*. Kluwer, Dordrecht, 2000.
- S. Courtin. Evaluation of decision power in multi-dimensional rules. *Mathematical Social Science*, 115:27–36, 2022.
- S. Courtin and A. Laruelle. Multi-dimensional rules. *Mathematical Social Science*, 103:1–7, 2020.
- S. Courtin, Z. Nganmeni, and B. Tchantcho. The Shapley-Shubik power index for dichotomous multi-type games. *Theory and Decision*, 81:413–426, 2016.
- S. Courtin, Z. Nganmeni, and B. Tchantcho. Dichotomous multi-type games with a coalition structure. *Mathematical Social Sciences*, 86:9–17, 2017.
- P. Dubey and L.S. Shapley. Mathematical properties of the banzhaf power index. *Mathematics of Operations Research*, 4:99–131, 1979.
- D.S. Felsenthal and M. Machover. Ternary voting games. *International Journal of Game Theory*, 26:335–351, 1997.
- D.S. Felsenthal and M. Machover. *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*. Edward Elgar Publishing Limited, 1st edition, 1998.
- D.S. Felsenthal and M. Machover. Models and reality: The curious case of the absent abstention. In M.J. Holler and G. Owen, editors, *Power Indices and Coalition Formation*. Kluwer Academic Press, Dordrecht, 2001.
- J. Freixas. The Banzhaf index for games with several levels of approval in the input and output. *Annals of Operations Research*, 137:45–66, 2005a.
- J. Freixas. The Shapley Shubik power index for games with several levels of approval in the input and output. *Decision Support Systems*, 39:185–195, 2005b.

- J. Freixas and W.S Zwicker. Weighted voting, abstention, and multiple levels of approval. *Social Choice and Welfare*, 21:399–431, 2003.
- M.J. Holler. Forming coalitions and measuring voting power. *Political Studies*, 30: 262–271, 1982.
- S. Kurz, I. Moyouwou, and H. Touyem. Axiomatizations for the shapley?shubik power index for games with several levels of approval in the input and output. *Social Choice and Welfare*, 56:569–594, 2021.
- A. Laruelle and F. Valenciano. *Voting and collective decision-making*. Cambridge University Press, 1st edition, 2008.
- A. Laruelle and F. Valenciano. Quaternary dichotomous voting rules. *Social Choice and Welfare*, 38:431–454, 2012.
- G. Owen. Values of games with a priori unions. In R. Hein and O. Moeschlin, editors, *Essays in Mathematical Economics and Game Theory*. Springer, New York, 1977.
- G. Owen. Modification of the Banzhaf-Coleman index for games with a priori unions. In M.J. Holler, editor, *Power, Voting, and Voting Power*. Physica-Verlag, Wurzburg, 1981.
- L.S. Shapley and M. Shubik. A model for evaluating the distribution of power in a committee system. *Rutgers Law Review*, 48:787–792, 1954.
- B. Tchantcho, L. Dikko Lambo, R. Pongou, and B. Mbama Engoulou. Voters’ power in voting games with abstention: Influence relation and ordinal equivalence of power theories. *Games and Economic Behavior*, 64:335–350, 2008.