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# Semi-classical limit for Klein-Gordon equation toward relativistic Euler equations via an adapted modulated energy method

Tony Salvi

## Abstract

We show the convergence of the solutions to the massive nonlinear Klein-Gordon equation toward solutions to a relativistic Euler with potential type system in the semi-classical limit. In particular, the momentum and the density of Klein-Gordon converge to the momentum and the density of the relativistic Euler system in Lebesgue norms. The relativistic Euler with potential is equivalent to the usual relativistic Euler with pressure up to a rescaling. The proof relies on the modulated energy method adapted to the wave equation and the relativistic setting.

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## 1 Introduction

In this paper, we are interested in the semi-classical version of the massive Klein-Gordon (KG) equation with a potential

$$\varepsilon^2 \square \Phi^\varepsilon = \Phi^\varepsilon + 2V'(|\Phi^\varepsilon|^2) \Phi^\varepsilon. \quad (1)$$

We set ourselves in the Minkowski spacetime on  $\mathbb{R}^{1+3}$ . The complex function  $\Phi^\varepsilon$  is the wave function,  $V'$  is a potential<sup>1</sup> and  $\varepsilon$  a small parameter. In particular, we look at the behaviour of the solutions when  $\varepsilon$  goes to 0, the semi-classical limit. We show that in this limit the momentum  $\mathbf{J}^\varepsilon$  and the density  $\rho^\varepsilon$  associated to  $\Phi^\varepsilon$  converge to the momentum  $\mathbf{J}$  and the density  $\rho$  associated to a relativistic Euler with potential type system (REP), the hydrodynamic limit. In fact, the REP system is equivalent to the usual relativistic Euler so that the semi-classical limit of KG is the usual relativistic Euler with pressure up to a rescaling. We give a discussion on this relation and on other possible extensions at the end of the paper, see sections 8 and 9.

The first inspirations of this work is the book [4] on the semi-classical limit of Schrodinger and the sequence of papers [16], [14] and [15] on the hydrodynamic limits of Klein-Gordon. Indeed, these works are linked because the Klein-Gordon equation is the relativistic "equivalent" of the Schrodinger equation, see [23], [20] or [13] for rigorous proofs of the non-relativistic limit of Klein-Gordon to Schrodinger. Nonetheless, mathematically speaking those two equations are quite different and the methods of [4] must be adapted to the wave equation and the relativistic setting. The results of [14] find themselves in the middle of the relativistic and the non relativistic setting and do not exhibit the structure we show in this paper. Moreover, the proof of the semi-classical limit of Klein-Gordon in [14] drastically relies on a rescaling in time and strong uniform assumptions on the energy, this masks the true relativistic nature of the hydrodynamic limit system. More details on the consistency of our results and theirs are given in section 8. We make there the link between the REP system and the usual relativistic Euler with pressure and we also examine their behaviour in the non-relativistic limit.

As done in the previously cited papers, we introduce a modulated energy that controls the convergence of the momentum and the density. That is a functional  $H^\varepsilon$  that depends on the time  $t$  and that controls  $L^p$  norms of the difference between the momentum (resp. the density) for KG and the momentum (resp. the density) for the REP system. If the initial data of the REP system are well prepared (see definition 4.4) and are close to the initial data of (1) (so that  $H^\varepsilon(0) = O(\varepsilon^2)$ ), then the modulated energy propagates its smallness (with respect to  $\varepsilon$ ) on a finite time interval.

We adapt the modulated energy of [4] directly to the case of wave equations. In [14], the authors use a modulated energy on the "modulated" Klein-Gordon equation, i.e., a Schrodinger equation with a source term whose unknown is, up to a rescaling, the wave function solution to (1). The use of the modulated energy method goes back to [3] on the quasi neutral limit for Vlasov-Poisson and was quickly adapted to the Schrodinger equation. See [14] for an overview of the history of this method on Schrodinger affiliated systems.

The arguments given in our proof rely on the existence of sufficiently regular and global solution to the nonlinear Klein-Gordon equation. Such solutions exist in the subcritical case. This is a standard result, see [19] for example. In dimension 3, the subcritical exponent for the  $\dot{H}^1$  norm corresponds to  $\gamma = 2$ , see section 3.

## 1.1 Context on the Klein Gordon equation

We introduce the massive nonlinear Klein-Gordon equation in its physical scaling

$$\hbar^2 \square_g \Phi = c^2 m^2 \Phi + 2V'(|\Phi|^2) m \Phi, \quad (2)$$

with  $c$  the speed of light,  $\hbar$  the Planck constant divided by  $2\pi$ ,  $m > 0$  the mass (set to 1 in what follows) and  $g = \text{Diag}(-c^2, 1, 1, 1)$  the Minkowski metric so that  $\square_g = -\frac{1}{c^2} \partial_{tt}^2 + \Delta$ . In the absence of a potential the Klein-Gordon equation is directly derived via the relativistic dispersion relation (the energy-momentum relation) as the Schrödinger equation is derived via the non relativistic dispersion relation for mass, energy and momentum. The wave function solution to (2) with  $V' = 0$  describes the evolution of a relativistic massive spinless free particle. We add the potential to model more complex phenomena, in this paper we are only interested in the defocusing case with a polynomial non-linearity,  $V'(x) = x^{\gamma-1}$  and  $\gamma \geq 2$ . With  $\gamma = 2$  we recover the Higgs-Boson equation, see [11].

By setting  $c = 1$  and  $\varepsilon = \hbar$  we recover (1). Indeed, in the relativistic context  $c$  is normalized to 1 and in semi classical physics  $\hbar$  is considered a very small constant.

One can also recover (1) using the natural units  $c = \hbar = 1$  and rescaling the solution  $\Phi$  to (2) as  $\Phi^\varepsilon(t, x) = \Phi(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ . Thus, the solutions  $\Phi^\varepsilon$  to (1) are seen as high frequency, see section 1.2 for more details.

---

<sup>1</sup>We restrict ourselves to polynomial defocusing nonlinearities  $V'(x) = x^{\gamma-1}$  for  $\gamma \geq 2$ .

With the previous remark, we see that the solution  $\Phi$  is rescaled so that the time interval on which the  $\Phi^\varepsilon$  are defined shrinks as  $\varepsilon$  approaches 0. This is not a problem for global solutions, if  $\Phi$  is defined on  $\mathbb{R}_+ \times \mathbb{R}^3$  then so is  $\Phi^\varepsilon$  for any  $\varepsilon > 0$ . It is important because we want the convergence of  $\mathbf{J}^\varepsilon$  (resp.  $\rho^\varepsilon$ ) to  $\mathbf{J}$  (resp.  $\rho$ ) to make sense on a fixed time interval (uniform in  $\varepsilon$ ). In other words, we want a family  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  of solution to (1) and the solution to the REP system to exist on a common time interval with enough regularity to manage the proof of convergence. From a mathematical point of view, the Klein-Gordon equation is a nonlinear wave equation. Thus, a rich literature can be used to study it. See [28] or [19] for example.

## 1.2 WKB approximation

The solutions to (1) are truly high frequency because of the scaling of the equation. It is natural to use the WKB expansion to get approximate solutions and to understand their behaviours. We refer to [25] and [26] (with applications on Maxwell's equations) for a general presentation of the WKB method and to [10] for an application on the semi classical Schrodinger equation.

**Proposition 1.1.** *Let  $\Phi_1^\varepsilon = e^{i\frac{\omega}{\varepsilon}} A$  with  $\omega$  a smooth real phase and  $A$  a smooth complex amplitude satisfying*

$$\begin{cases} \nabla^\alpha \omega \nabla_\alpha \omega = -1 - 2V'(|A|^2), \\ 2\nabla^\alpha \omega \nabla_\alpha A + A \square \omega = 0. \end{cases} \quad (3)$$

*Then,  $\Phi_1^\varepsilon$  is an approximate solution of order 2 to (1), that is*

$$\varepsilon^2 \square \Phi_1^\varepsilon - \Phi_1^\varepsilon - 2V'(|\Phi_1^\varepsilon|^2) \Phi_1^\varepsilon = O(\varepsilon^2). \quad (4)$$

*Proof.* By direct calculations we get

$$\varepsilon^2 \square \Phi_1^\varepsilon - \Phi_1^\varepsilon - 2V'(|\Phi_1^\varepsilon|^2) \Phi_1^\varepsilon = e^{i\frac{\omega}{\varepsilon}} (-\nabla^\alpha \omega \nabla_\alpha \omega - 1 - 2V'(|A|^2)) + \varepsilon i e^{i\frac{\omega}{\varepsilon}} (2\nabla^\alpha \omega \nabla_\alpha A + A \square \omega) + \varepsilon^2 e^{i\frac{\omega}{\varepsilon}} (\square A) = O(\varepsilon^2). \quad (5)$$

□

**Remark 1.2.** *The equation*

$$\nabla^\alpha \omega \nabla_\alpha \omega = -1 - 2V'(|A|^2) \quad (6)$$

*is called the eikonal equation. In the case of plane waves,  $\omega = x^\alpha \mathbf{k}_\alpha$  with  $\mathbf{k}$  and  $A$  constant, we recover the dispersion relation*

$$\mathbf{k}^\alpha \mathbf{k}_\alpha = -1 - 2V'(|A|^2). \quad (7)$$

**Proposition 1.3.** *Let  $(\omega, A)$  be solution to (3), then  $(U_\alpha, \rho) = (\nabla_\alpha \omega, |A|^2)$  is solution to*

$$\begin{cases} U^\alpha \nabla_\alpha U_\beta + \nabla_\beta V'(\rho) = 0, \\ U^\alpha \nabla_\alpha \rho + \rho \nabla_\alpha U^\alpha = 0, \end{cases} \quad (8)$$

*and  $U$  is a timelike vector field with the normalization*

$$U^\alpha U_\alpha = -1 - 2V'(\rho). \quad (9)$$

*Proof.* By direct calculations, we use (6) to get

$$U^\alpha \nabla_\alpha U_\beta + \nabla_\beta V'(\rho) = \nabla^\alpha \omega \nabla_\alpha \nabla_\beta \omega + \nabla_\beta V'(|A|^2) = \frac{1}{2} \nabla_\beta (\nabla^\alpha \omega \nabla_\alpha \omega + 1 + 2V'(|A|^2)) = 0, \quad (10)$$

and, with the second equation of (3),

$$U^\alpha \nabla_\alpha \rho + \rho \nabla_\alpha U^\alpha = \nabla^\alpha \omega \nabla_\alpha |A|^2 + |A|^2 \square \omega = A (\nabla^\alpha \omega \nabla_\alpha \bar{A} + \frac{1}{2} \bar{A} \square \omega) + \bar{A} (\nabla^\alpha \omega \nabla_\alpha A + \frac{1}{2} A \square \omega) = 0. \quad (11)$$

Moreover,  $U^\alpha$  satisfies the equation (9) by definition and is timelike because  $V'(\rho)$  is positive ( $\rho$  is positive by definition and  $V'(x) = x^{\gamma-1}$ ). □

**Remark 1.4.** With the definitions 3.2 and 3.3 of  $\mathcal{J}^\varepsilon$  and  $\rho^\varepsilon$ , we see that for  $\Phi_1^\varepsilon$  from proposition 1.1 we have

$$\mathcal{J}^\varepsilon = \frac{i}{2}(\Phi_1^\varepsilon \overline{\nabla \Phi_1^\varepsilon} - \overline{\Phi_1^\varepsilon} \nabla \Phi_1^\varepsilon) = \nabla \omega |A|^2 + O(\varepsilon) = U\rho + O(\varepsilon), \quad (12)$$

$$\rho^\varepsilon = |\Phi_1^\varepsilon|^2 = |A|^2 = \rho, \quad (13)$$

so that  $U\rho$  and  $\rho$  give a good approximation of the momentum and the density of  $\Phi_1^\varepsilon$  when  $\varepsilon$  is small.

### 1.3 First version of the theorem

We give a rough version and a comment on the main results of this paper.

**Theorem 1.1.** Let  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1}$  and  $(\mathcal{U}, \varrho)$  be initial data for (1) and (8) respectively. We assume that

1. the initial data are sufficiently regular,
2. the energy for the family  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1}$  is uniformly bounded,
3. the  $(\mathcal{U}, \rho)$  are well prepared, see definition 4.4,
4. the modulated energy  $H^\varepsilon(0)$  is of the size  $O(\varepsilon^2)$ .

Then, the solution  $(\mathbf{U}, \rho)$  with initial data  $(\mathcal{U}, \varrho)$  is the semi-classical (hydrodynamic) limit of the family of solutions  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  to (17) with initial data  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1}$  when  $\varepsilon$  goes to 0. More precisely, the modulated energy remains small on the time of existence of the solutions. This implies that the momentum and the density of the solutions to Klein-Gordon converge in the right Lebesgue spaces to the momentum and the density of the solution of the REP system.

In the case  $\gamma = 2$ , the functions  $(\mathcal{U}, \varrho)$  and  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  are defined on a common time interval  $[0, T]$  where  $T$  only depends on  $(\mathcal{U}, \varrho)$ . Thus, the convergence has a strong meaning.

**Theorem 1.2.** Under the assumptions of the previous theorem, the semi classical (hydrodynamic) limit of the massive Klein-Gordon with a potential is the relativistic Euler with pressure up to a convenient rescaling.

The assumption 1 is present for obvious technical purposes. The second one 2 is not restrictive in the sense that the typical solutions we are interested in satisfy it. Indeed, even explicit high frequency solutions<sup>2</sup>, for which the derivatives behave badly, have energies that are uniformly bounded in  $\varepsilon$ . This comes from the presence of  $\varepsilon$  in the definition of the energy (21) to balance the size of the derivatives. The assumptions 3 and 4 are more generic than it seems. The assumption 3 on well-prepared initial data refers to 4.4. It states that the profile  $(\mathcal{U}, \varrho)$  must satisfy the eikonal equation,  $\mathcal{U}$  must be future directed and  $\varrho \geq 0$ . We observe that it is coherent with the approximation in section 1.2. There are also requirements on the regularity of the initial data that we discuss below.

Typically, one can think of building exact solutions  $\Phi^\varepsilon$  to (1) based on mono phase high frequency approximations  $\Phi_1^\varepsilon$  and with initial data  $(\varphi_1^\varepsilon, \dot{\varphi}_1^\varepsilon)_{0 < \varepsilon < 1} = (ae^{\frac{v}{\varepsilon}}, \dot{a}e^{\frac{v}{\varepsilon}} + i\frac{\dot{v}}{\varepsilon}a^\varepsilon e^{\frac{v}{\varepsilon}})$ . In particular, the initial data for the phase  $(\omega, \partial_t \omega)|_{t=0} = (v, \dot{v})$  are solutions to the eikonal equation (6). Then, one can pick  $(\mathcal{U}_0, \mathcal{U}_i) = (\dot{v}, \nabla_i v)$  and  $\varrho = |a|^2$ . If the approximate solutions are stable then, for each exact solution based on them, the quantities  $(\mathcal{U}, \varrho)$  described above are natural profiles, they fit the criteria 3 and 4 of the theorem<sup>3</sup>. In other words, the mono phase high frequency solutions made by WKB expansion correspond to the assumptions of the theorem.

Moreover, we observe in that even if  $v = 0$  ( $\varphi_1^\varepsilon$  is not initially highly oscillating) then (with the eikonal equation (6))  $\dot{v}$  is not zero. The solution becomes instantaneously highly oscillatory. This remark is made in [4] on the same type of initial data for the semi-classical Schrodinger equation.

More generally, we see that the solutions "have to be" highly oscillating in at least one direction because of the scaling of the equation, the derivatives of  $\Phi^\varepsilon$  have to be big in comparison with its amplitude. The assumption 4 (and more generally the modulated energy approach) morally means that there is only one direction in which the solution is high frequency, this is captured by the vector field  $\mathbf{U}$ . We discuss this point in section 9 and we compare the modulated energy with other approaches.

Finally, we require some regularity for  $\sqrt{\varrho}$  to have well prepared initial data. This is useful for the proof of the well-posedness of the REP system in proposition 4.7. This restriction makes sense because  $\rho$  is thought to be an approximation of  $\rho^\varepsilon = |\Phi^\varepsilon|^2$  from which we expect that  $\sqrt{\rho}$  has the regularity of  $|\Phi^\varepsilon|$ .

<sup>2</sup>See for example the ansatz for the approximate solution of section 1.2.

<sup>3</sup>If they have the good regularity, see below.

It is already interesting to note that the properties of the well prepared  $(\mathcal{U}, \varrho)$  propagate, see proposition 4.6. In particular, we take advantage of the eikonal equation (the normalization of the vector field  $\mathbf{U}$  in proposition 4.6) during the proof of proposition 4.7 for the well posedness of the REP system and in section 7.3 to prove the propagation of smallness. We note also that this particular normalization is a direct equivalent of the  $-1$  normalization for the usual relativistic Euler, see section 8.

The second theorem is a direct consequence of the first and the equivalence of the REP system and the relativistic Euler equations given by proposition 8.1.

## 1.4 Idea of the proof

Firstly, we need to show the existence of sufficiently regular solutions to (17) and (38) with initial data  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1}$  and  $(\mathcal{U}, \varrho)$ .

For the semi classical Klein-Gordon equation (17), it is easy to prove local well-posedness with energy methods but we need global well-posedness to have a uniform time of existence for the family of solution  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  and to recover the strong statement iii) of the Theorem 6.1. The global well-posedness is sketched in section 3 for  $\gamma = 2$ , this is a common result. For the REP system, we show in section 4 that it is a symmetrizable hyperbolic system and so it is locally well-posed for sufficiently regular initial data. Moreover,  $(\mathbf{U}, \rho)$  must be more regular than what  $\Phi^\varepsilon$  needs to be and must be of size  $O(1)^4$  to perform the operation in section 7.2 and section 7.3.

Then, the rest of the proof concerns the convergence of the momentum and the density. The core of the argument is to construct a quantity  $H^\varepsilon$  that

- 1) Controls the convergence in the sense that

$$H^\varepsilon(t) = O(\varepsilon^2) \implies \|\mathbf{J}^\varepsilon(t) - \mathbf{U}\rho(t)\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|\mathbf{J}^\varepsilon(t) - \mathbf{U}\rho(t)\|_{L^{\gamma+L^1}} + \|\rho^\varepsilon - \rho\|_{L^\gamma} + \|V(\rho^\varepsilon) - V(\rho)\|_{L^1} = O(\varepsilon^\delta), \quad (14)$$

for some  $\delta > 0$ . This is the **coercivity property**.

- 2) Propagates the smallness in the sense that

$$H^\varepsilon(0) = O(\varepsilon^2) \implies H^\varepsilon(t) = O(\varepsilon^2), \quad (15)$$

for  $t$  finite such that the solutions exist and remain regular. This is the **propagation property**.

We can directly construct an equivalent (see proposition 7.8) of the modulated energy of [4] made for the semi classical Schrodinger equation. This equivalent is also based (by definition) on the energy of the semi classical Klein-Gordon equation. We show in proposition 7.8 that it actually satisfies the point 1). Nonetheless, this first candidate eventually fails to satisfy the point 2) with the techniques used here.

To fix that, we build a new "modulated energy"  $H^\varepsilon$  based on the full stress energy tensor that is equivalent to the previous one in terms of control and coercivity but with a better structure for point 2). This is proved in proposition 7.11.

The proof follows as the ones of [4] or [14], that is to show that

$$\frac{d}{dt}H^\varepsilon \leq H^\varepsilon + O(\varepsilon^2). \quad (16)$$

Another novelty is that we have to use the eikonal equation (the unusual normalization of the four velocity  $\mathcal{U}$ ) to show (16). We also use the fact that the potential is polynomial to get the inequality (16) in a straightforward way. The Lebesgue spaces in which the convergences occur are directly linked to the degree  $\gamma - 1$  of the polynomial  $V'$ . It is logical that the convergence of the density takes place in  $L^\gamma$  (the norm of the potential energy density) but we can observe that the convergence of the momentum depends on the convergence of the density. In fact, if there is no potential we cannot prove the convergence of the momentum, see remark 7.9.

## 1.5 Acknowledgments

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<sup>4</sup>With respect to  $\varepsilon$ .

## 1.6 Outline of the paper

- In section 3 we give all the details on the semi classical Klein-Gordon equation.
- In section 4 we give all the details on the REP system.
- In section 5 we give the basic definition of the new modulated energy.
- In section 6 we give the main theorem.
- In section 7 we give the proof of the theorem.
- In section 8 we show the equivalence between the REP system and the relativistic Euler.
- In section 9 we conclude and we make some general remarks and comparisons.

## 2 Notations

**Notation 2.1.** For  $p, q \in [1, \infty]$ , we define the Banach space  $L^p + L^q = \{u = a + b \mid a \in L^p, b \in L^q\}$  with the norm  $\|u\|_{L^p+L^q} = \inf\{\|a\|_{L^p} + \|b\|_{L^q} \mid a \in L^p, b \in L^q, a + b = u\}$ .

**Notation 2.2.** The symbol  $\nabla$  alone denotes the spacetime gradient and we note  $|\nabla f|^2 = \nabla f \cdot \overline{\nabla f} = \partial_t f \overline{\partial_t f} + \nabla f \cdot \overline{\nabla f}$  with  $\nabla$  the space gradient and  $\cdot$  the usual scalar products of both  $\mathbb{R}^3$  and  $\mathbb{R}^4$ .

**Notation 2.3.** We note with bold letters the four-vectors, such as  $\mathbf{X}$ , and with roman letters,  $X$  with  $X^i = \mathbf{X}^i$ , the three-dimensional vectors made of its space components. We note  $X^0 = \mathbf{X}^0$  its time component.

## 3 Klein-Gordon equation

We give the details on the semi-classical version of the massive Klein-Gordon equation with a potential

$$\varepsilon^2 \square \Phi^\varepsilon = \Phi^\varepsilon + 2V'(|\Phi^\varepsilon|^2)\Phi^\varepsilon. \quad (17)$$

**Definition 3.1.** The stress energy tensor associated to (17) is

$$T^{KG}[\Phi^\varepsilon]_{\alpha\beta} = \frac{\varepsilon^2}{2}(\nabla_\alpha \Phi^\varepsilon \nabla_\beta \overline{\Phi^\varepsilon} + \nabla_\alpha \overline{\Phi^\varepsilon} \nabla_\beta \Phi^\varepsilon) - \frac{1}{2}g_{\alpha\beta}(\varepsilon^2 \nabla_\gamma \overline{\Phi^\varepsilon} \nabla^\gamma \Phi^\varepsilon + m^2 |\Phi^\varepsilon|^2 + 2mV(|\Phi^\varepsilon|^2)), \quad (18)$$

its divergence is equal to 0

$$\nabla_\alpha (T^{KG})^\alpha{}_\beta = 0. \quad (19)$$

We also have

$$\int T^{KG}[\Phi^\varepsilon]_{00} dx = \mathcal{E}^{KG}[\Phi^\varepsilon], \quad (20)$$

where  $\mathcal{E}^{KG}$  is the associated conserved energy

$$\mathcal{E}^{KG}[\Phi^\varepsilon] = \int_{\mathbb{R}^3} \frac{\varepsilon^2 |\nabla \Phi^\varepsilon|^2}{2} + \frac{|\Phi^\varepsilon|^2}{2} + V(|\Phi^\varepsilon|^2) dx. \quad (21)$$

**Definition 3.2.** The momentum (or charge current density) associated to (17) is

$$\mathbf{J}_\alpha^\varepsilon = \frac{\varepsilon i}{2}(\Phi^\varepsilon \overline{\nabla_\alpha \Phi^\varepsilon} - \overline{\Phi^\varepsilon} \nabla_\alpha \Phi^\varepsilon). \quad (22)$$

We also set  $J^\varepsilon$  from  $\mathbb{R}^{1+3}$  to  $\mathbb{R}^3$  as  $J_i^\varepsilon = \mathbf{J}_i^\varepsilon$  and  $J_0^\varepsilon = \mathbf{J}_0^\varepsilon$ . We have the equations

$$\partial_t J_0^\varepsilon = \nabla \cdot \mathbf{J}^\varepsilon, \quad (23)$$

written equivalently as

$$\nabla_\alpha (\mathbf{J}^\varepsilon)^\alpha = 0, \quad (24)$$

and

$$\partial_t J^\varepsilon = \varepsilon i(\partial_t \Phi^\varepsilon \overline{\nabla \Phi^\varepsilon} - \overline{\partial_t \Phi^\varepsilon} \nabla \Phi^\varepsilon) + \nabla J_0^\varepsilon. \quad (25)$$

**Notation 3.3.** To simplify the notation we set the density  $\rho^\varepsilon = |\Phi^\varepsilon|^2$ .

**Remark 3.4.** We see that the density  $\rho^\varepsilon$  and the charge density (the component 0 of the momentum four-vector)  $(J^\varepsilon)^0$  are different quantities. In the non-relativistic case (for the Schrödinger equation) they are the same quantity, the probability density. We say that  $(J^\varepsilon)^0$  is a charge density because it is not positive definite but its total charge is conserved as the total probability for Schrödinger. We have

$$\frac{d}{dt} \int_{\mathbb{R}^3} (J^\varepsilon)^0 dx = 0. \quad (26)$$

Moreover, we have

$$\frac{d}{dt} \langle x \rangle = \frac{d}{dt} \int_{\mathbb{R}^3} (J^\varepsilon)^0 x dx = \int_{\mathbb{R}^3} J^\varepsilon dx, \quad (27)$$

which is an analogous of the Ehrenfest theorem applied to the position operator.

We add the following useful equations to our list.

**Proposition 3.5.** Let  $\Phi^\varepsilon$  be a wave function then we have

$$\frac{(\mathbf{J}^\varepsilon)^\alpha (\mathbf{J}^\varepsilon)_\alpha}{\rho^\varepsilon} = -\varepsilon^2 \nabla_\alpha \sqrt{\rho^\varepsilon} \nabla^\alpha \sqrt{\rho^\varepsilon} + \varepsilon^2 \nabla_\alpha \Phi^\varepsilon \overline{\nabla^\alpha \Phi^\varepsilon}, \quad (28)$$

$$\frac{\mathbf{J}^\varepsilon \cdot \mathbf{J}^\varepsilon}{\rho^\varepsilon} = -\varepsilon^2 \nabla \sqrt{\rho^\varepsilon} \cdot \nabla \sqrt{\rho^\varepsilon} + \varepsilon^2 |\nabla \Phi^\varepsilon|^2, \quad (29)$$

and if  $\Phi^\varepsilon$  is a solution to (17), then the momentum  $\mathbf{J}^\varepsilon$  and the density  $\rho^\varepsilon$  are solutions to

$$\frac{(\mathbf{J}^\varepsilon)^\alpha (\mathbf{J}^\varepsilon)_\alpha}{\rho^\varepsilon} = \varepsilon^2 \sqrt{\rho^\varepsilon} \square \sqrt{\rho^\varepsilon} - \rho^\varepsilon - 2V'(\rho^\varepsilon) \rho^\varepsilon. \quad (30)$$

*Proof.* By direct calculations, we get

$$\begin{aligned} \frac{(\mathbf{J}^\varepsilon)^\alpha (\mathbf{J}^\varepsilon)_\alpha}{|\Phi^\varepsilon|^2} &= -\frac{\varepsilon^2 (\Phi^\varepsilon \nabla_\alpha \overline{\Phi^\varepsilon} \Phi^\varepsilon \overline{\nabla^\alpha \Phi^\varepsilon} + \overline{\Phi^\varepsilon} \nabla_\alpha \Phi^\varepsilon \overline{\Phi^\varepsilon} \nabla^\alpha \Phi^\varepsilon - 2 \nabla^\alpha \Phi^\varepsilon \overline{\nabla_\alpha \Phi^\varepsilon} |\Phi^\varepsilon|^2)}{4|\Phi^\varepsilon|^2} \\ &= -\varepsilon^2 \left( \frac{\nabla_\alpha |\Phi^\varepsilon| \Phi^\varepsilon \overline{\nabla^\alpha \Phi^\varepsilon}}{2|\Phi^\varepsilon|} - \frac{\overline{\Phi^\varepsilon} \nabla^\alpha \Phi^\varepsilon \Phi^\varepsilon \overline{\nabla_\alpha \Phi^\varepsilon}}{4|\Phi^\varepsilon|^2} + \frac{\nabla_\alpha |\Phi^\varepsilon| \overline{\Phi^\varepsilon} \nabla^\alpha \Phi^\varepsilon}{2|\Phi^\varepsilon|} - \frac{\overline{\Phi^\varepsilon} \nabla^\alpha \Phi^\varepsilon \Phi^\varepsilon \overline{\nabla_\alpha \Phi^\varepsilon}}{4|\Phi^\varepsilon|^2} - \frac{2 \nabla^\alpha \Phi^\varepsilon \overline{\nabla_\alpha \Phi^\varepsilon} |\Phi^\varepsilon|^2}{4|\Phi^\varepsilon|^2} \right) \\ &= -\varepsilon^2 \nabla_\alpha |\Phi^\varepsilon| \nabla^\alpha |\Phi^\varepsilon| + \varepsilon^2 \nabla_\alpha \Phi^\varepsilon \overline{\nabla^\alpha \Phi^\varepsilon}, \end{aligned}$$

for the first equality. The second is obtained with the same techniques. Then, we have

$$\begin{aligned} \frac{(\mathbf{J}^\varepsilon)^\alpha (\mathbf{J}^\varepsilon)_\alpha}{|\Phi^\varepsilon|^2} &= -\varepsilon^2 \nabla^\alpha (\nabla_\alpha |\Phi^\varepsilon| |\Phi^\varepsilon|) + \varepsilon^2 \square |\Phi^\varepsilon| |\Phi^\varepsilon| + \varepsilon^2 \frac{\nabla^\alpha (\nabla_\alpha \Phi^\varepsilon \overline{\Phi^\varepsilon}) + \nabla^\alpha (\Phi^\varepsilon \nabla_\alpha \overline{\Phi^\varepsilon})}{2} - \varepsilon^2 \frac{\square \Phi^\varepsilon \overline{\Phi^\varepsilon} + \Phi^\varepsilon \square \overline{\Phi^\varepsilon}}{2} \\ &= -\varepsilon^2 \frac{\nabla^\alpha (\Phi^\varepsilon \nabla_\alpha \overline{\Phi^\varepsilon} + \Phi^\varepsilon \nabla_\alpha \overline{\Phi^\varepsilon})}{2} + \varepsilon^2 \square |\Phi^\varepsilon| |\Phi^\varepsilon| + \varepsilon^2 \frac{\nabla^\alpha (\nabla_\alpha \Phi^\varepsilon \overline{\Phi^\varepsilon}) + \nabla^\alpha (\Phi^\varepsilon \nabla_\alpha \overline{\Phi^\varepsilon})}{2} - \varepsilon^2 \frac{\square \Phi^\varepsilon \overline{\Phi^\varepsilon} + \Phi^\varepsilon \square \overline{\Phi^\varepsilon}}{2} \\ &= \varepsilon^2 \sqrt{\rho^\varepsilon} \square \sqrt{\rho^\varepsilon} - \varepsilon^2 \frac{\square \Phi^\varepsilon \overline{\Phi^\varepsilon} + \Phi^\varepsilon \square \overline{\Phi^\varepsilon}}{2}, \end{aligned}$$

so that if  $\Phi^\varepsilon$  is a solution to (17) we recover

$$\frac{(\mathbf{J}^\varepsilon)^\alpha (\mathbf{J}^\varepsilon)_\alpha}{|\Phi^\varepsilon|^2} = \varepsilon^2 \sqrt{\rho^\varepsilon} \square \sqrt{\rho^\varepsilon} - \rho^\varepsilon - 2V'(\rho^\varepsilon) \rho^\varepsilon.$$

□

In the next proposition we state the global well-posedness result.

**Proposition 3.6.** Let  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)$  be in  $H^2 \times H^1$  with

$$\int_{\mathbb{R}^3} \frac{\varepsilon^2 |\nabla \varphi^\varepsilon|^2}{2} + \frac{\varepsilon^2 |\dot{\varphi}^\varepsilon|^2}{2} + \frac{|\varphi^\varepsilon|^2}{2} + V(|\varphi^\varepsilon|^2) dx \leq C_\varepsilon, \quad (31)$$

then, for  $\gamma = 2$  there exists a global solution to (17) with initial data  $(\Phi^\varepsilon, \partial_t \Phi^\varepsilon)|_{t=0} = (\varphi^\varepsilon, \dot{\varphi}^\varepsilon)$  and with  $\Phi^\varepsilon \in \bigcap_{j=0}^2 C^{2-j}(\mathbb{R}_+, H^j)$ .



*Proof.* We show the global well-posedness<sup>5</sup> of

$$\begin{cases} \square\Phi = \Phi + 2V'(|\Phi|^2)\Phi \\ (\Phi, \partial_t\Phi)|_{t=0} = (\varphi, \dot{\varphi}), \end{cases} \quad (32)$$

and we get the results of the proposition by rescaling,  $(\varphi^\varepsilon(\varepsilon x), \varepsilon\dot{\varphi}^\varepsilon(\varepsilon x)) = (\varphi(x), \dot{\varphi}(x))$  and  $\Phi^\varepsilon(\varepsilon x) = \Phi(x)$ . The energy<sup>6</sup> of (32) is

$$\mathcal{E}^{KG}(\Phi) = \int_{\mathbb{R}^3} \frac{|\nabla\Phi|^2}{2} + \frac{|\Phi|^2}{2} + V(|\Phi|^2)dx = \int_{\mathbb{R}^3} \frac{|\nabla\Phi|^2}{2} + \frac{|\Phi|^2}{2} + |\Phi|^4 dx = C_e, \quad (33)$$

where  $C_e$  is a constant, the energy is conserved.

To prove the global well posedness we work by contradiction. First, we know that we can show the local well posedness easily by the a priori energy estimates method, we have the existence of a solution to (32) on an interval  $[0, T^*]$ . Then, we suppose that the time  $T^*$  is the maximal time of existence. This means that

$$\lim_{t \rightarrow T^*} \|\Phi(t)\|_{H^2} + \|\partial_t\Phi(t)\|_{H^1} + \|\partial_{tt}^2\Phi(t)\|_{L^2} = +\infty. \quad (34)$$

We remark that the energy controls the norms  $\|\Phi\|_{H^1}$  and  $\|\partial_t\Phi\|_{L^2}$ . Thus the explosion can only occur at the level of  $\|\Phi\|_{\dot{H}^2}$ ,  $\|\partial_t\Phi\|_{\dot{H}^1}$  and  $\|\partial_{tt}^2\Phi(t)\|_{L^2}$ . We apply the classical energy estimates to get

$$\|\nabla\nabla\Phi(t)\|_{L^2} + \|\nabla\Phi(t)\|_{L^2} \lesssim \|\nabla\nabla\Phi(0)\|_{L^2} + \|\nabla\Phi(0)\|_{L^2} + \int_0^t \|\Phi(s)\|_{L^6}^2 \|\nabla\Phi(s)\|_{L^6} ds.$$

Then, by Sobolev embedding, we know that  $H^1 \hookrightarrow L^6$  in dimension 3, so

$$\|\nabla\Phi\|_{L^6} \lesssim \|\nabla\Phi\|_{H^1} \lesssim \|\nabla\Phi\|_{L^2} + \|\nabla\nabla\Phi\|_{L^2} \lesssim C_e^{1/2} + \|\nabla\nabla\Phi\|_{L^2}, \quad \|\Phi\|_{L^6} \lesssim \|\Phi\|_{H^1} \lesssim C_e^{1/2}. \quad (35)$$

Finally, we use the Gronwall lemma to get

$$\|\nabla\nabla\Phi(t)\|_{L^2} + \|\nabla\Phi(t)\|_{L^2} \lesssim C(C_e, t), \quad (36)$$

where  $C(C_e, t)$  grows exponentially with respect to  $t$ . This implies that  $\|\Phi\|_{\dot{H}^2}$  and  $\|\partial_t\Phi\|_{\dot{H}^1}$  stay bounded for all finite  $t$ . To get the same result for the norm  $\|\partial_{tt}^2\Phi(t)\|_{L^2}$  we use the equation (32). Thus,

$$\lim_{t \rightarrow T^*} \|\Phi(t)\|_{H^2} + \|\partial_t\Phi(t)\|_{H^1} + \|\partial_{tt}^2\Phi(t)\|_{L^2} < +\infty, \quad (37)$$

and we reach a contradiction,  $T^*$  cannot be finite, we have a global solution.  $\square$

## 4 Relativistic Euler with potential

We introduce the relativistic Euler system with potential (REP system)

$$\begin{cases} \mathbf{U}^\alpha \nabla_\alpha \mathbf{U}_\beta + \frac{\nabla_\beta B}{\rho} = 0, \\ \mathbf{U}^\alpha \nabla_\alpha \rho + \nabla_\alpha \mathbf{U}^\alpha \rho = 0, \\ B = B(\rho), \end{cases} \quad (38)$$

where  $\rho$  is the energy density,  $B$  the potential and  $\mathbf{U}$  the four-velocity vector field. We denote by  $U$  the vector field from  $\mathbb{R}^{1+3}$  to  $\mathbb{R}^3$  with  $U_i = \mathbf{U}_i$ , the space components, and by  $U_0 = \mathbf{U}_0$  the time component. We compare the REP system to the usual relativistic Euler with pressure in section 8.

<sup>5</sup>This is a standard result which we give here for the sake of completeness, see [19] for example.

<sup>6</sup>By this rescaling, one can see that the energy for  $\Phi$  is not uniformly bounded in  $\varepsilon$  if the energies for the family of  $\Phi^\varepsilon$  are only uniformly bounded. Indeed,  $\Phi$  depends implicitly on  $\varepsilon$  and we have  $C_e \leq c_0 \varepsilon^{-3}$  under the assumptions of theorem 6.1. Nonetheless, the global well posedness remains true for large initial data.

**Definition 4.1.** *The stress energy tensor associated to (38) is*

$$T^{EP}[\mathbf{U}, \rho]_{\alpha\beta} = \rho \mathbf{U}_\alpha \mathbf{U}_\beta + g_{\alpha\beta} B, \quad (39)$$

*its divergence is equal to 0*

$$\nabla_\alpha (T^{EP})^\alpha{}_\beta = 0. \quad (40)$$

*We also have*

$$\int T^{EP}[\mathbf{U}, \rho]_{00} dx = \mathcal{E}^{EP}[\mathbf{U}, \rho], \quad (41)$$

*where  $\mathcal{E}^{EP}$  is the associated conserved energy*

$$\mathcal{E}^{EP}[\mathbf{U}, \rho] = \int_{\mathbb{R}^3} \mathbf{U}_0 \mathbf{U}_0 \rho - B dx. \quad (42)$$

**Definition 4.2.** *The momentum associated to (38) is*

$$\mathbf{J}_\alpha = \rho \mathbf{U}_\alpha, \quad (43)$$

*with*

$$\nabla_\alpha \mathbf{J}^\alpha = 0. \quad (44)$$

*We also set  $J$  from  $\mathbb{R}^{1+3}$  to  $\mathbb{R}^3$  as  $J_i = \mathbf{J}_i$  and  $J_0 = \mathbf{J}_0$ . We have the conservation law*

$$\frac{d}{dt} \int_{\mathbb{R}^3} J_0 dx = \frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{U}_0 \rho dx = 0. \quad (45)$$

**Remark 4.3.** *The solution  $\rho$  to the conservative transport*

$$\mathbf{U}^\alpha \nabla_\alpha \rho + \nabla_\alpha \mathbf{U}^\alpha \rho = 0 \quad (46)$$

*is given implicitly by*

$$\rho(\chi(\tau, y)) = \rho(0, y) e^{-\int_0^\tau \nabla_\alpha \mathbf{U}^\alpha(\chi(\theta, y)) d\theta}, \quad (47)$$

*where  $\chi$  are the flow line of  $\mathbf{U}$ , solution to*

$$\begin{cases} \dot{\chi}^\alpha(\tau, y) = \mathbf{U}^\alpha(\chi(\tau, y)) \\ \chi(0, y) = (0, y). \end{cases} \quad (48)$$

*We observe that if  $\rho(0) \geq 0$  then  $\rho(t) \geq 0$ . Moreover, if  $\mathbf{U}$  is future directed ( $\mathbf{U}^0 > 0$ ) then  $J^0$  is positive definite.*

*We are looking for the specific case  $B = m(\gamma - 1)V(\rho)$  with the normalization  $\mathbf{U}^\alpha \mathbf{U}_\alpha + 1 + 2V'(\rho) = 0$ .*

**Definition 4.4.** *Let  $(\mathcal{U}, \varrho)$  be initial data for (38) with  $B = (\gamma - 1)V(\rho)$ . The initial data are well-prepared<sup>7</sup> if*

$$\varrho \geq 0, \quad \mathcal{U}^0 > 0 \quad \mathcal{U}^\alpha \mathcal{U}_\alpha + 1 + 2V'(\varrho) = 0, \quad (49)$$

*and if  $(\sqrt{\varrho}, \mathcal{U}) \in (H^4)^2$ . Moreover, if  $\gamma \in (2, 3) \cup (3, 4) \cup (4, 5)$  the quantity  $\sqrt{\varrho}^{\gamma-1}$  needs to be in  $H^4$ .*

**Remark 4.5.** *We chose  $\mathcal{U}^0 > 0$  ( $\mathcal{U}$  future directed) as a convention but everything can be done with  $\mathcal{U}^0 < 0$  ( $\mathcal{U}$  past directed).*

**Proposition 4.6.** *Let  $(\mathbf{U}, \rho)$  be a continuous solution to (38) with  $B = (\gamma - 1)V(\rho)$  defined on  $[0, T^*]$  with  $(\mathcal{U}, \varrho)$  well prepared initial data from definition 4.4. Then, for all  $t \in [0, T^*]$*

$$\rho(t) \geq 0, \quad \mathbf{U}^0(t) > 0, \quad \mathbf{U}^\alpha(t) \mathbf{U}_\alpha(t) + 1 + 2V'(\rho(t)) = 0, \quad (50)$$

*so that  $\mathbf{U}$  is future directed and in particular  $\mathbf{U}^0 \geq 1$ .*

<sup>7</sup>See the restrictions and the explanation of the definition of well prepared initial data and the other assumptions of the theorem in section 1.3.

*Proof.* We show that the quantity  $\mathbf{U}^\alpha(t)\mathbf{U}_\alpha(t) + 1 + 2V'(\rho(t))$  is constant along the flow lines. We have

$$\mathbf{U}^\beta \nabla_\beta (\mathbf{U}^\alpha \mathbf{U}_\alpha + 1 + 2V'(\rho)) = 2(\mathbf{U}^\alpha \mathbf{U}^\beta \nabla_\beta \mathbf{U}_\alpha + \mathbf{U}^\beta \nabla_\beta V'(\rho)) = 2(-\mathbf{U}^\beta \frac{B'}{\rho} \nabla_\beta \rho + \mathbf{U}^\beta V''(\rho) \nabla_\beta \rho) = 0.$$

Then, with the formula (47) we get that  $\rho \geq 0$ . Finally, we have from (50) that

$$\mathbf{U}^0 = \sqrt{1 + 2V'(\rho) + \mathbf{U}^i \mathbf{U}_i} \geq 1. \quad (51)$$

□

**Proposition 4.7.** *Let  $B = (\gamma - 1)V(\rho)$ , then (38) is equivalent to the following system*

$$\begin{cases} \mathbf{U}^\alpha \nabla_\alpha \mathbf{U}_\beta + \nabla_\beta V'(\rho) = 0, \\ \mathbf{U}^\alpha \nabla_\alpha \rho + \nabla_\alpha \mathbf{U}^\alpha \rho = 0. \end{cases} \quad (52)$$

*This system is locally well posed for  $(\mathcal{U}, \varrho)$  well prepared initial data from definition 4.4 and with  $(\mathcal{U}, \varrho) \in H^n \times H^n$  with  $n \geq 4$ .*

*Proof.* The goal is to put the system under a quasilinear symmetrized hyperbolic form and apply the classical well posedness result using energy methods. We can observe that the system is almost symmetric but the natural symmetrizer  $\text{Diag}(\rho, \rho, \rho, \rho, V'')$  is not uniformly coercive. It is a classical problem in fluid dynamics, it corresponds to the presence of vacuum. As done in [4], we chose to write the system (52) in terms of  $\mathbf{U}$  and  $f = \sqrt{\rho}^{\gamma-1}$ , thus we need to assume the regularity of  $\sqrt{\rho}$ . The well prepared initial data have this feature, see 4.4. Written with respect to  $(\mathbf{U}, f = \sqrt{\rho}^{\gamma-1})$ , the system (52) is equivalent to

$$\begin{cases} \mathbf{U}^\alpha \nabla_\alpha \mathbf{U}_\beta + 2f \nabla_\beta f = 0, \\ \frac{4}{\gamma-1} \mathbf{U}^\alpha \nabla_\alpha f + 2 \nabla_\alpha \mathbf{U}^\alpha f = 0, \end{cases} \quad (53)$$

and with the argument of 4.3, we see that  $\rho$  and  $f$  stay positive so the relation between  $f$  and  $\rho$  is invertible. The system (53) is symmetric, we have

$$\begin{cases} A^\alpha \nabla_\alpha \begin{pmatrix} \mathbf{U} \\ f \end{pmatrix} = 0, \end{cases} \quad (54)$$

where the  $A^\alpha$  are symmetric matrices. Nonetheless,  $A^0$  does not fit the coercivity criteria ( $A^0(X, X) > c|X|^2$  for some  $c > 0$ ) to apply the energy method (in fact  $A^0$  is not positive definite). However, with the well prepared initial data we know from 4.6 that the normalization propagates and thus

$$U^0 = \mathbf{U}^0 = \sqrt{\mathbf{U}^i \mathbf{U}_i + 1 + 2f^2} = \sqrt{U^i U_i + 1 + 2f^2},$$

so that the system (53) is overdetermined. Thus, we search for solutions to the reduced system

$$\begin{cases} B^\alpha \nabla_\alpha \begin{pmatrix} U \\ f \end{pmatrix} = 0, \\ U^0 = \mathbf{U}^0(U, f), \end{cases} \quad (55)$$

with

$$B^0 = \begin{pmatrix} U^0 & 0 & 0 & 0 \\ 0 & U^0 & 0 & 0 \\ 0 & 0 & U^0 & 0 \\ 0 & 0 & 0 & 4U^0(\frac{1}{\gamma-1} + \frac{f^2}{(U^0)^2}) \end{pmatrix}, \quad B^i = \begin{pmatrix} U^i & 0 & 0 & 2f\delta_1^i \\ 0 & U^i & 0 & 2f\delta_2^i \\ 0 & 0 & U^i & 2f\delta_3^i \\ 2fM_1^i & 2fM_2^i & 2fM_3^i & 4U^i(\frac{1}{\gamma-1} - \frac{f^2}{(U^0)^2}) \end{pmatrix},$$

where  $M_{ij} = \delta_{ij} - \frac{U_i U_j}{(U^0)^2}$ . One can check that the solutions to the previous system (55) are equivalent to the solutions to (53). The matrix  $M$  is positive definite. In particular,  $1 > \delta_{ij} - \frac{U_i U_j}{(U^0)^2} = \frac{1+2f^2}{1+U^i U_i + 2f^2} > c$  for some  $c > 0$ . Indeed,  $U^i \leq \|U_i\|_{H^2} \leq C_0$  and so  $(U^i U_i)$  stays finite. We define the  $4 \times 4$  symmetrizer

$$S = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that the  $SB^\alpha$  are symmetric matrices with  $SB^0$  uniformly coercive ( $U^0 \geq 1$  and  $4(\frac{1}{\gamma-1} + \frac{f^2}{(U^0)^2}) \geq \frac{4}{\gamma-1}$ ). We deduce that the system (55) is well posed for initial data  $(U, f)$  in  $H^n$  for  $n \geq 4$ , this is the case for well prepared initial data. Then, we calculate  $U^0$  with the equation (50) and we get all the component of  $\mathbf{U}$ .  $\square$

## 5 Definition of the new modulated energy

We give the basic definitions for the new modulated energy made with the stress energy tensor (modulated stress-energy tensor).

**Proposition 5.1.** *Let  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  be solutions to (17) and  $(\mathbf{U}, \rho)$  be a solution to (38) with well prepared initial data from definition 4.4, let  $T_{\alpha\beta}^{KG}$  and  $T_{\alpha\beta}^{EP}$  be their respective stress energy tensor field 3.1 and 4.1, then we have*

$$T_{\alpha\beta}^{KG} - T_{\alpha\beta}^{EP} = h_{\alpha\beta}^\varepsilon + I_{\alpha\beta}^\varepsilon, \quad (56)$$

with

$$\begin{aligned} h_{\alpha\beta}^\varepsilon &= \frac{1}{2}((\varepsilon \nabla_\alpha - i \mathbf{U}_\alpha) \Phi^\varepsilon \overline{(\varepsilon \nabla_\beta - i \mathbf{U}_\beta) \Phi^\varepsilon} + \overline{(\varepsilon \nabla_\alpha - i \mathbf{U}_\alpha) \Phi^\varepsilon} (\varepsilon \nabla_\beta - i \mathbf{U}_\beta) \Phi^\varepsilon) \\ &\quad - \frac{1}{2} g_{\alpha\beta} ((\varepsilon \nabla_\gamma - i \mathbf{U}_\gamma) \Phi^\varepsilon \overline{(\varepsilon \nabla^\gamma - i \mathbf{U}^\gamma) \Phi^\varepsilon}) - g_{\alpha\beta} (V(\rho^\varepsilon) - V(\rho) - V'(\rho)(\rho^\varepsilon - \rho)), \end{aligned}$$

and

$$I_{\alpha\beta}^\varepsilon = -\mathbf{U}_\alpha \mathbf{U}_\beta (\rho - \rho^\varepsilon) + \mathbf{U}_\alpha (\mathbf{J}_\beta^\varepsilon - \mathbf{U}_\beta \rho^\varepsilon) + \mathbf{U}_\beta (\mathbf{J}_\alpha^\varepsilon - \mathbf{U}_\alpha \rho^\varepsilon) - g_{\alpha\beta} (\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon) \mathbf{U}^\gamma. \quad (57)$$

*Proof.* By direct calculation and using the normalization (50), we have

$$\begin{aligned} &\frac{\varepsilon^2}{2} (\nabla_\alpha \Phi^\varepsilon \nabla_\beta \overline{\Phi^\varepsilon} + \nabla_\alpha \overline{\Phi^\varepsilon} \nabla_\beta \Phi^\varepsilon) - \frac{1}{2} g_{\alpha\beta} (\varepsilon^2 \nabla_\gamma \overline{\Phi^\varepsilon} \nabla^\gamma \Phi^\varepsilon + |\Phi^\varepsilon|^2 + 2V(|\Phi^\varepsilon|^2)) - \rho \mathbf{U}_\alpha \mathbf{U}_\beta - g_{\alpha\beta} V(\rho)(\gamma - 1) \\ &= \frac{\varepsilon^2}{2} (\nabla_\alpha \Phi^\varepsilon \nabla_\beta \overline{\Phi^\varepsilon} + \nabla_\alpha \overline{\Phi^\varepsilon} \nabla_\beta \Phi^\varepsilon) - 2\mathbf{U}_\alpha \mathbf{U}_\beta |\Phi^\varepsilon|^2 + \mathbf{U}_\alpha \mathbf{U}_\beta |\Phi^\varepsilon|^2 - \mathbf{U}_\alpha \mathbf{U}_\beta (\rho - |\Phi^\varepsilon|^2) \\ &\quad - \frac{1}{2} g_{\alpha\beta} (\varepsilon^2 \nabla_\gamma \overline{\Phi^\varepsilon} \nabla^\gamma \Phi^\varepsilon + |\Phi^\varepsilon|^2 + 2V(|\Phi^\varepsilon|^2) - (\mathbf{U}^\gamma \mathbf{U}_\gamma + 1 + 2V'(\rho)) |\Phi^\varepsilon|^2 + 2V(\rho)(\gamma - 1)) \\ &= \frac{1}{2} ((\varepsilon \nabla_\alpha - i \mathbf{U}_\alpha) \Phi^\varepsilon \overline{(\varepsilon \nabla_\beta - i \mathbf{U}_\beta) \Phi^\varepsilon} + \overline{(\varepsilon \nabla_\alpha - i \mathbf{U}_\alpha) \Phi^\varepsilon} (\varepsilon \nabla_\beta - i \mathbf{U}_\beta) \Phi^\varepsilon) + \mathbf{J}_\alpha^\varepsilon \mathbf{U}_\beta + \mathbf{J}_\beta^\varepsilon \mathbf{U}_\alpha \\ &\quad - 2\mathbf{U}_\alpha \mathbf{U}_\beta |\Phi^\varepsilon|^2 - \mathbf{U}_\alpha \mathbf{U}_\beta (\rho - |\Phi^\varepsilon|^2) - \frac{1}{2} g_{\alpha\beta} ((\varepsilon \nabla_\gamma - i \mathbf{U}_\gamma) \Phi^\varepsilon \overline{(\varepsilon \nabla^\gamma - i \mathbf{U}^\gamma) \Phi^\varepsilon} + 2\mathbf{J}_\gamma^\varepsilon \mathbf{U}^\gamma - 2\mathbf{U}_\gamma \mathbf{U}^\gamma |\Phi^\varepsilon|^2) \\ &\quad - g_{\alpha\beta} (V(|\Phi^\varepsilon|^2) - V(\rho) - V'(\rho)(|\Phi^\varepsilon|^2 - \rho)). \end{aligned}$$

$\square$

**Definition 5.2.** *Let  $\mathbf{X}$  be a time like future directed vector field and  $h_{\alpha\beta}^\varepsilon$  be from 5.1, we define  $\eta(\mathbf{X})^\varepsilon = h_{\alpha 0}^\varepsilon \mathbf{X}^\alpha$ . It corresponds to looking at the modulated energy flux in the reference frame of  $\mathbf{X}$ .*

**Definition 5.3.** *Let  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  be solutions to (17) and  $(\mathbf{U}, \rho)$  be a solution to (38) with well-prepared initial data from definition 4.4, we define  $H^\varepsilon$  as  $\int_{\mathbb{R}^3} \eta^\varepsilon(\mathbf{U}) dx$ , that is*

$$\begin{aligned} H^\varepsilon(\Phi^\varepsilon, \partial_t \Phi^\varepsilon, \mathbf{U}, \rho) &= \int_{\mathbb{R}^3} \mathbf{U}^\alpha \frac{1}{2} ((\varepsilon \nabla_\alpha - i \mathbf{U}_\alpha) \Phi^\varepsilon \overline{(\varepsilon \nabla_0 - i \mathbf{U}_0) \Phi^\varepsilon} + \overline{(\varepsilon \nabla_\alpha - i \mathbf{U}_\alpha) \Phi^\varepsilon} (\varepsilon \nabla_0 - i \mathbf{U}_0) \Phi^\varepsilon) dx \\ &\quad - \int_{\mathbb{R}^3} \frac{1}{2} \mathbf{U}_0 ((\varepsilon \nabla_\gamma - i \mathbf{U}_\gamma) \Phi^\varepsilon \overline{(\varepsilon \nabla^\gamma - i \mathbf{U}^\gamma) \Phi^\varepsilon}) + \mathbf{U}_0 (V(|\Phi^\varepsilon|^2) - V(\rho) - V'(\rho)(|\Phi^\varepsilon|^2 - \rho)) dx. \end{aligned}$$

## 6 Main results

**Theorem 6.1.** *Let  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1}$  be a family of initial data for (17) and let  $(\mathcal{U}, \varrho)$  be initial data for (38). We assume that :*

1. *The initial data have the regularity  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1} \in H^2 \times H^1$  and  $(\mathcal{U}, \varrho) \in (H^4)^2$ .*

2. There exists a constant  $c_0$  such that

$$\int_{\mathbb{R}^3} \frac{\varepsilon^2 |\nabla \varphi^\varepsilon|^2}{2} + \frac{\varepsilon^2 |\dot{\varphi}^\varepsilon|^2}{2} + \frac{|\varphi^\varepsilon|^2}{2} + V(|\varphi^\varepsilon|^2) dx + \|\mathcal{U}\|_{H^4}^2 + \|\varrho\|_{H^4}^2 \leq c_0, \quad (58)$$

in particular, the energy of the family  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1}$  is uniformly bounded.

3. The initial data  $(\mathcal{U}, \rho)$  are well prepared, see definition 4.4.

4. At  $t = 0$ , the modulated energy  $H^\varepsilon$  of definition 5.3 verifies

$$H^\varepsilon(\varphi^\varepsilon, \dot{\varphi}^\varepsilon, \mathcal{U}, \varrho) = O(\varepsilon^2). \quad (59)$$

Then,

i) There exists  $(\Phi^\varepsilon)_{0 < \varepsilon < 1} \in \bigcap_{j=0}^2 C^{2-j}([0, t_\varepsilon], H^j)$  solutions to (17) for some  $t_\varepsilon$  with initial data  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)$  and  $(\mathbf{U}, \rho) \in (\bigcap_{j=0}^4 C^{4-j}([0, T], H^j))^2$  a solution to (38) for some  $T < +\infty$  with initial data  $(\mathcal{U}, \varrho)$ . For  $\gamma = 2$  we can pick  $t_\varepsilon = T$ .

ii) For all  $t \in [0, \min(t_\varepsilon, T)]$  we have

$$H^\varepsilon(\Phi^\varepsilon, \partial_t \Phi^\varepsilon, \mathbf{U}, \rho)(t) = O(\varepsilon^2), \quad (60)$$

and so

$$\|\mathcal{J}^\varepsilon(t) - \mathbf{U}\rho(t)\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|\mathcal{J}^\varepsilon(t) - \mathbf{U}\rho(t)\|_{L^\gamma + L^1} + \|\rho^\varepsilon(t) - \rho(t)\|_{L^\gamma} + \|V(\rho^\varepsilon)(t) - V(\rho)(t)\|_{L^1} = O(\varepsilon^{2/\gamma}). \quad (61)$$

iii) For  $\gamma = 2$  we have the strong statement

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{J}^\varepsilon - \mathbf{U}\rho\|_{L^\infty([0, T], L^{\frac{2\gamma}{\gamma+1}})} + \|\mathcal{J}^\varepsilon - \mathbf{U}\rho\|_{L^\infty([0, T], L^\gamma + L^1)} = 0, \quad (62)$$

$$\lim_{\varepsilon \rightarrow 0} \|\rho^\varepsilon - \rho\|_{L^\infty([0, T], L^\gamma)} + \|V(\rho^\varepsilon) - V(\rho)\|_{L^\infty([0, T], L^1)} = 0. \quad (63)$$

**Theorem 6.2.** Under the assumptions of theorem 6.1,

i) For all  $t \in [0, \min(t_\varepsilon, T)]$  we have

$$\|\mathcal{J}^\varepsilon(t) - \Gamma \mathbf{u}(\mu + p)(t)\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|\mathcal{J}^\varepsilon(t) - \Gamma \mathbf{u}(\mu + p)(t)\|_{L^\gamma + L^1} = O(\varepsilon^{2/\gamma}), \quad (64)$$

$$\|\rho^\varepsilon(t) - \Gamma^2(\mu + p)(t)\|_{L^\gamma} + \|(\gamma - 1)V(\rho^\varepsilon)(t) - p(t)\|_{L^1} = O(\varepsilon^{2/\gamma}), \quad (65)$$

ii) For  $\gamma = 2$  we have the strong statement

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{J}^\varepsilon - \Gamma \mathbf{u}(\mu + p)\|_{L^\infty([0, T], L^{\frac{2\gamma}{\gamma+1}})} + \|\mathcal{J}^\varepsilon - \Gamma \mathbf{u}(\mu + p)\|_{L^\infty([0, T], L^\gamma + L^1)} = 0, \quad (66)$$

$$\lim_{\varepsilon \rightarrow 0} \|\rho^\varepsilon - \Gamma^2(\mu + p)\|_{L^\infty([0, T], L^\gamma)} + \|(\gamma - 1)V(\rho^\varepsilon) - p\|_{L^\infty([0, T], L^1)} = 0. \quad (67)$$

where  $(\mathbf{u}, \mu, p)$  are the four-velocity, the energy density and the pressure of the corresponding relativistic Euler and  $\Gamma$  a scaling factor. These quantities are given by proposition 8.1.

## 7 Proof of the Theorem 6.1

The proof of 6.1 is divided in two parts. Firstly, we gather the results on the existence and the regularity of the solutions to the equations (17) and (38). Secondly, we show the convergence of the momentum and the density (the semi classical limit). This boils down to prove the **coercivity property** (point 1)) and the **propagation property** (point 2)) of the modulated energy.

## 7.1 Existence and regularity of the solutions

**Proposition 7.1.** *Let  $(\varphi^\varepsilon, \dot{\varphi}^\varepsilon)_{0 < \varepsilon < 1}$  and  $(\mathcal{U}, \rho)$  be initial data satisfying the assumptions 1, 2 and 3 of theorem 6.1. Then, the point i) of the theorem is true, the solutions  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  and  $(\mathbf{U}, \rho)$  exist on  $[0, t_\varepsilon]$  and  $[0, T]$  respectively and there exists  $C_0(c_0)$  such that  $\forall 0 \leq t \leq \min(t_\varepsilon, T)$*

$$\int_{\mathbb{R}^3} \frac{\varepsilon^2 |\nabla \Phi^\varepsilon(t)|^2}{2} + \frac{|\Phi^\varepsilon(t)|^2}{2} + V(|\Phi^\varepsilon(t)|^2) dx + \sum_{j=0}^4 \|\mathbf{U}\|_{C^{4-j}([0, T], H^j)}^2 + \sum_{j=0}^4 \|\rho\|_{C^{4-j}([0, T], H^j)} \leq C_0. \quad (68)$$

**Notation 7.2.** *We note  $C_0$  any constant that only depends on  $C_0$ .*

*Proof.* For the well posedness of KG (17) we use the classical energy method and proposition 3.6 in the case  $\gamma = 2$  ( $t_\varepsilon$  is as big as we want). For the REP system (38) we use the proposition 4.7. The existence of  $C_0$  is direct.  $\square$

## 7.2 Coercivity property, point 1)

In this section, we consider that  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  and  $(\mathbf{U}, \rho)$  are given by proposition 7.1 under the assumptions 1, 2 and 3 of theorem 6.1. Moreover, we consider  $h_{\alpha\beta}^\varepsilon$  from proposition 5.1.

**Definition 7.3.** *We write<sup>8</sup>  $\xi_\alpha = (\varepsilon \nabla_\alpha - i \mathbf{U}_\alpha) \Phi^\varepsilon$  and  $\Theta^\varepsilon(|\Phi^\varepsilon|^2, \rho) = V(|\Phi^\varepsilon|^2) - V(\rho) - V'(\rho)(|\Phi^\varepsilon|^2 - \rho)$  so that*

$$h_{\alpha\beta}^\varepsilon = \frac{1}{2} (\xi_\alpha \overline{\xi_\beta} + \overline{\xi_\alpha} \xi_\beta) - \frac{1}{2} g_{\alpha\beta} (\xi_\gamma \overline{\xi^\gamma}) - g_{\alpha\beta} \Theta^\varepsilon. \quad (69)$$

**Proposition 7.4.** *We have*

$$h_{00}^\varepsilon = \frac{|\xi|^2}{2} + \Theta^\varepsilon, \quad (70)$$

and

$$h_{00}^\varepsilon = \varepsilon^2 \frac{|\nabla \sqrt{\rho^\varepsilon}|^2}{2} + \frac{|\mathbf{J}^\varepsilon - \rho^\varepsilon \mathbf{U}|^2}{2\rho^\varepsilon} + \Theta^\varepsilon. \quad (71)$$

*Proof.* We have (70) by direct calculation. The equation (71) is given in [14], we give a proof of this identity for the sake of completeness. We have

$$\begin{aligned} \frac{|(\varepsilon \nabla - i \mathbf{U}) \Phi^\varepsilon|^2}{2} &= \frac{|\varepsilon \nabla \Phi|^2}{2} + \frac{|\mathbf{U}|^2 |\Phi^\varepsilon|^2}{2} - \mathbf{J}^\varepsilon \cdot \mathbf{U} = \frac{\mathbf{J}^\varepsilon \cdot \mathbf{J}^\varepsilon}{2\rho^\varepsilon} + \frac{\varepsilon^2 \nabla \sqrt{\rho^\varepsilon} \cdot \nabla \sqrt{\rho^\varepsilon}}{2} + \frac{|\mathbf{U}|^2 \rho^\varepsilon}{2} - \mathbf{J}^\varepsilon \cdot \mathbf{U} \\ &= \varepsilon^2 \frac{|\nabla \sqrt{\rho^\varepsilon}|^2}{2} + \frac{|\mathbf{J}^\varepsilon - \rho^\varepsilon \mathbf{U}|^2}{2\rho^\varepsilon}, \end{aligned}$$

where we use the equation (29).  $\square$

**Proposition 7.5.** *There exists  $c_1 > 0$  such that*

$$\forall \alpha, \beta \in \llbracket 0, 3 \rrbracket \quad h_{00}^\varepsilon > c_1 |h_{\alpha\beta}^\varepsilon|. \quad (72)$$

*Proof.* By direct calculation.  $\square$

**Proposition 7.6.** *For  $x \geq 0, y \geq 0$  the function  $\Theta(x, y)$  is positive,*

$$\Theta(x, y) \gtrsim (x - y)^2 (x^{\gamma-2} + y^{\gamma-2}), \quad (73)$$

and

$$\Theta(x, y) \gtrsim |x - y|^\gamma. \quad (74)$$

---

<sup>8</sup>We write  $\xi$  instead of  $\xi^\varepsilon$  to lighten the notation.

*Proof.* These inequalities are used in [14], we give here their proofs for the sake of completeness. For the first inequality (73), we search for  $c > 0$  such that  $\Theta(x, y) \geq c(x - y)^2(x^{\gamma-2} + y^{\gamma-2})$ . The inequality is direct for  $y = 0$ , so we divide everything by  $y^\gamma$  to get the following inequality for  $z = \frac{x}{y}$

$$\frac{z^\gamma}{\gamma} - \frac{1}{\gamma} - (z - 1) \geq c(z - 1)^2(z^{\gamma-2} + 1).$$

We split it in two parts, first we have

$$\frac{z^\gamma}{\gamma} - \frac{1}{\gamma} - (z - 1) \geq \frac{1}{2}(z - 1)^2. \quad (75)$$

Indeed, the inequality is true at  $z = 1$  and by taking a derivative with respect to  $z$  in (75) we find

$$\begin{aligned} \forall z \in [0, 1), \quad z^{\gamma-1} - 1 &\leq z - 1, \\ \forall z > 1, \quad z^{\gamma-1} - 1 &\geq z - 1. \end{aligned}$$

More generally, this implies that for any  $0 < c < \frac{1}{\gamma} \leq \frac{1}{2}$  we have

$$\frac{z^\gamma}{\gamma} - \frac{1}{\gamma} - (z - 1) \geq c(z - 1)^2, \quad (76)$$

and also

$$\forall z \in [0, 1), \quad \frac{z^\gamma}{\gamma} - \frac{1}{\gamma} - (z - 1) \geq c(z - 1)^2 z^{\gamma-2},$$

it remains to prove that  $\exists c > 0$  such that

$$\forall z > 1, \quad \frac{z^\gamma}{\gamma} - \frac{1}{\gamma} - (z - 1) \geq c(z - 1)^2 z^{\gamma-2}.$$

We compute

$$\begin{aligned} &\partial_z \left( \frac{z^\gamma}{\gamma} - \frac{1}{\gamma} - (z - 1) - c(z - 1)^2 z^{\gamma-2} \right) \\ &= z^{\gamma-1} - 1 - 2c(z - 1)z^{\gamma-2} - c(\gamma - 2)z^{\gamma-3}(z - 1)^2 \\ &= z^{\gamma-2}(z(1 - c\gamma) + c\gamma) + z^{\gamma-3}(z - 1)c(\gamma - 2) - 1. \end{aligned}$$

We know that  $z^{\gamma-3}(z - 1)c(\gamma - 2) \geq 0$  and  $z^{\gamma-2}(z(1 - c\gamma) + c\gamma) > 1$  for  $z > 1$ ,  $\gamma \geq 2$  and  $c < \frac{1}{\gamma}$ . We deduce that the derivative is strictly positive and so for  $c < \frac{1}{\gamma}$

$$\frac{z^\gamma}{\gamma} - \frac{1}{\gamma} - (z - 1) \geq c(z - 1)^2 z^{\gamma-2}. \quad (77)$$

The two inequalities (76) and (77) give us the inequality (73) for  $c < \frac{1}{2\gamma}$ . For the second inequality, we just need to show that for some  $c > 0$

$$(x - y)^2(x^{\gamma-2} + y^{\gamma-2}) \geq c(x - y)^\gamma.$$

It is trivial for  $y = 0$  so we do the same reasoning with  $z = \frac{x}{y}$ . With  $c = 1$  we directly have

$$(z^{\gamma-2} + 1) \geq c(z - 1)^{\gamma-2}.$$

□

**Proposition 7.7.** *Let  $f, g \in L^\gamma$  be two positive functions, then we have*

$$\|f^\gamma - g^\gamma\|_{L^1} \leq C_\gamma \|f - g\|_{L^\gamma} (\|f\|_{L^\gamma} + 1) (\|g\|_{L^\gamma} + 1), \quad (78)$$

and

$$\|f^\gamma - g^\gamma\|_{L^1} \lesssim \|\Theta(f, g)\|_{L^1}^{\frac{1}{2}} (\|f\|_{L^\gamma} + 1) (\|g\|_{L^\gamma} + 1), \quad (79)$$

where  $C_\gamma$  only depends on  $\gamma$ .

*Proof.* First, we consider  $\gamma = n$  an integer. Let  $x, y \geq 0$ , then we have

- $|x^n - y^n| = |(x^{n/2} - y^{n/2})^2 + 2y^{n/2}(x^{n/2} - y^{n/2})|$  for  $n$  even,
- $|x^n - y^n| = |(x^{(n-1)/2} - y^{(n-1)/2})^2(x - y) - yx(y^{n-2} - x^{n-2}) - 2x^{(n-1)/2}y^{(n-1)/2}(y - x)|$  for  $n$  odd and strictly greater than 1,

so that by induction we directly get

$$|x^n - y^n| \leq \sum_{p,q,r \in \mathbb{N}, r \geq 1, r+q+p=n} c_{r,p,q} |x - y|^r x^p y^q, \quad (80)$$

where the  $c_{r,p,q}$  are constant.

Now, for a general  $\gamma = n + \epsilon$  with  $n = \lfloor \gamma \rfloor$  and  $0 < \epsilon < 1$  we have

$$|x^\gamma - y^\gamma| = |(x^n - y^n)x^\epsilon + (x^\epsilon - y^\epsilon)y^n| \leq |x^n - y^n|x^\epsilon + |x - y|^\epsilon y^n,$$

and so with (80) we have

$$|x^\gamma - y^\gamma| \leq \sum_{a,b,c \in \mathbb{R}_+, a > 0, a+b+c=\gamma} c_{a,b,c} |x - y|^a x^b y^c,$$

where the number of  $c_{a,b,c}$  which are non zero is clearly finite. Let  $f, g \in L^\gamma$  be two positive functions then

$$\|f^\gamma - g^\gamma\|_{L^1} \leq \sum_{a,b,c \in \mathbb{R}_+, a > 0, a+b+c=\gamma} c_{a,b,c} \|(|f - g|^a f^b g^c)\|_{L^1} \leq C_\gamma \|f - g\|_{L^\gamma} (\|f\|_{L^\gamma} + 1) (\|g\|_{L^\gamma} + 1).$$

using the Hölder inequality. The second inequality (79) comes from the previous result and (74).  $\square$

**Proposition 7.8.** *Let  $H_0^\epsilon = \int_{\mathbb{R}^3} h_{00}^\epsilon dx$  then it satisfies the **coercivity property**, point 1).*

*Proof.* By algebraic computation, from (74) we have for  $\gamma \geq 2$

$$\|\rho^\epsilon - \rho\|_{L^\gamma} \leq (H_0^\epsilon)^{1/\gamma}, \quad (81)$$

and from (79)

$$\|V(\rho^\epsilon) - V(\rho)\|_{L^1} \leq \frac{1}{\gamma} \|(\rho^\epsilon)^\gamma - (\rho)^\gamma\|_{L^1} \leq C_0 (H_0^\epsilon)^{1/\gamma}, \quad (82)$$

where we control the  $L^\gamma$  norms in  $O(1)$  with the energy (68) for  $\rho^\epsilon$  and with classical Sobolev embedding for  $\rho$ . Moreover,

$$\begin{aligned} \|\mathbf{J}^\epsilon - \mathbf{U}\rho\|_{L^{\frac{2\gamma}{\gamma+1}}} &\leq \|\mathbf{J}^\epsilon - \mathbf{U}\rho^\epsilon\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|\mathbf{U}\rho^\epsilon - \mathbf{U}\rho\|_{L^{\frac{2\gamma}{\gamma+1}}}, \\ &\leq \|\sqrt{\rho^\epsilon}\|_{L^{2\gamma}} \left\| \frac{\mathbf{J}^\epsilon - \mathbf{U}\rho^\epsilon}{\sqrt{\rho^\epsilon}} \right\|_{L^2} + \|\rho^\epsilon - \rho\|_{L^\gamma} \|\mathbf{U}\|_{L^{\frac{2\gamma}{\gamma-1}}} \leq C_0 ((H_0^\epsilon)^{1/2} + (H_0^\epsilon)^{\frac{1}{\gamma}}), \end{aligned}$$

$$\begin{aligned} \|\mathbf{J}^\epsilon - \mathbf{U}\rho\|_{L^\gamma + L^1} &\leq \|\mathbf{J}^\epsilon - \mathbf{U}\rho^\epsilon\|_{L^1} + \|\mathbf{U}\rho^\epsilon - \mathbf{U}\rho\|_{L^\gamma} \\ &\leq \left\| \frac{i(\Phi^\epsilon(\epsilon \nabla - i\mathbf{U})\Phi^\epsilon - \overline{\Phi^\epsilon}(\epsilon \nabla - i\mathbf{U})\Phi^\epsilon)}{2} \right\|_{L^1} + \|\mathbf{U}\|_{L^\infty} \|\rho^\epsilon - \rho\|_{L^\gamma} \\ &\leq \|\Phi^\epsilon\|_{L^2} \|(\epsilon \nabla - i\mathbf{U})\Phi^\epsilon\|_{L^2} + \|\mathbf{U}\|_{L^\infty} \|\rho^\epsilon - \rho\|_{L^\gamma} \leq C_0 ((H_0^\epsilon)^{1/2} + (H_0^\epsilon)^{\frac{1}{\gamma}}). \end{aligned}$$

Finally, we clearly recover

$$H_0^\epsilon = O(\epsilon^2) \Rightarrow \lim_{\epsilon \rightarrow 0} (\|\mathbf{J}^\epsilon - \mathbf{U}\rho\|_{L^{\frac{2\gamma}{\gamma+1}}} + \|\mathbf{J}^\epsilon - \mathbf{U}\rho\|_{L^\gamma + L^1} + \|\rho^\epsilon - \rho\|_{L^\gamma} + \|V(\rho^\epsilon) - V(\rho)\|_{L^1}) = \lim_{\epsilon \rightarrow 0} O(\epsilon^{2/\gamma}) = 0. \quad (83)$$

$\square$



**Remark 7.9.** We can observe that the convergence of the density  $\rho^\varepsilon$  totally relies on the presence of the potential  $V'$ . Moreover, the convergence of the momentum is also directly linked to the convergence of the density and thus to  $V'$ . In particular, if  $V' = 0$  we can see that the arguments of the proof of the previous proposition 7.8 do not apply anymore.

**Proposition 7.10.** Let  $\mathbf{X}$  be a time like future directed vector field such that  $\mathbf{X}$  is uniformly bounded with  $\mathbf{X}^0 > \nu$  and  $-\mathbf{X}^\alpha \mathbf{X}_\alpha > \nu$  for some  $\nu > 0$ , then there exists  $c_2(\mathbf{X})$  and  $c_3(\mathbf{X})$  such that for  $\eta^\varepsilon(\mathbf{X}) = h_{0\beta}^\varepsilon \mathbf{X}^\beta$  we have

$$c_2(\mathbf{X})h_{00} < \eta(\mathbf{X}) < c_3(\mathbf{X})h_{00}. \quad (84)$$

*Proof.* We use the notation 2.3, we have

$$\begin{aligned} h_{0\beta} \mathbf{X}^\beta &= h_{00}X^0 + h_{0i}X^i = \frac{|\boldsymbol{\xi}|^2}{2}X^0 + \frac{\xi_0\bar{\xi}_i + \bar{\xi}_0\xi_i}{2}X^i + \Theta X^0 \\ &\geq \frac{|\boldsymbol{\xi}|^2}{2}X^0 + \Theta X^0 - |\xi_0||\boldsymbol{\xi}||X| \geq \frac{|\boldsymbol{\xi}|^2}{2}(X^0 - |X|) + \Theta X^0 \\ &\geq c_2(\nu)h_{00}^\varepsilon, \end{aligned}$$

where we used the Young inequality. We have the existence of  $c_2(\mathbf{X})$ .

The existence of  $c_3(\mathbf{X})$  is a consequence of proposition 7.5 and the fact that  $\mathbf{X}$  is uniformly bounded.  $\square$

**Proposition 7.11.** Let  $H^\varepsilon = \int_{\mathbb{R}^3} \eta^\varepsilon(\mathbf{U})dx$  (defined in 5.2) then it satisfies the coercivity property point 1). The new modulated energy  $H^\varepsilon$  is equivalent to  $H_0^\varepsilon$  in terms of coercivity. We have<sup>9</sup>

$$H^\varepsilon \sim H_0^\varepsilon = \int_{\mathbb{R}^3} h_{00}^\varepsilon dx = \int_{\mathbb{R}^3} \frac{|\boldsymbol{\xi}|^2}{2} + \Theta^\varepsilon dx = \int_{\mathbb{R}^3} \varepsilon^2 \frac{|\nabla \sqrt{\rho^\varepsilon}|^2}{2} + \frac{|\mathbf{J}^\varepsilon - \rho^\varepsilon \mathbf{U}|^2}{2\rho^\varepsilon} + \Theta^\varepsilon dx. \quad (85)$$

*Proof.* For the vector field  $\mathbf{U}$ , we have  $\nu = 1$  and  $\|\mathbf{U}\|_{C^0} \leq C_0$  so that we can apply proposition 7.10 to get

$$c_2(\mathbf{U})H_0^\varepsilon < H^\varepsilon < c_3(\mathbf{U})H_0^\varepsilon, \quad (86)$$

then we use the coercivity property of  $H_0^\varepsilon$  from 7.8.  $\square$

**Remark 7.12.** The quantity  $\eta(\mathbf{U})$  corresponds to the modulated energy flux in the reference frame of  $\mathbf{U}$ .

### 7.3 Propagation property, point 2)

In this section we consider that  $(\Phi^\varepsilon)_{0 < \varepsilon < 1}$  and  $(\mathbf{U}, \rho)$  are given by proposition 7.1 under the assumptions 1, 2, 3 and 4 of theorem 6.1.

**Proposition 7.13.** Let  $H^\varepsilon = \int_{\mathbb{R}^3} \eta^\varepsilon(\mathbf{U})dx$  (defined in 5.3) then it satisfies point 2). The new modulated energy  $H^\varepsilon$  propagates its smallness.

*Proof.* First, we calculate that

$$\begin{aligned} \eta^\varepsilon(\mathbf{U}) &= h_{0\alpha}^\varepsilon \mathbf{U}^\alpha = (T_{0\alpha}^{KG} - T_{0\alpha}^{EP})\mathbf{U}^\alpha - I_{0\alpha}^\varepsilon \mathbf{U}^\alpha \\ &= (T_{0\alpha}^{KG} - T_{0\alpha}^{EP})\mathbf{U}^\alpha - \mathbf{U}_0 \mathbf{U}_\alpha (\rho - \rho^\varepsilon)\mathbf{U}^\alpha + \mathbf{U}_0 (\mathbf{J}_\alpha^\varepsilon - \mathbf{U}_\alpha \rho^\varepsilon)\mathbf{U}^\alpha \\ &\quad + \mathbf{U}_\alpha (\mathbf{J}_0^\varepsilon - \mathbf{U}_0 \rho^\varepsilon)\mathbf{U}^\alpha - \mathbf{U}_0 (\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon)\mathbf{U}^\gamma \\ &= (T_{0\alpha}^{KG} - T_{0\alpha}^{EP})\mathbf{U}^\alpha - \mathbf{U}_\alpha \mathbf{U}^\alpha (\mathbf{U}_0 \rho - \mathbf{J}_0^\varepsilon) \\ &= (T_{0\alpha}^{KG} - T_{0\alpha}^{EP})\mathbf{U}^\alpha + (\mathbf{J}_0^\varepsilon - \mathbf{U}_0 \rho) + 2V'(\rho)(\mathbf{J}_0^\varepsilon - \mathbf{U}_0 \rho). \end{aligned} \quad (87)$$

using the definition 5.2 and the proposition 4.6. Then, we compute the derivative

$$\begin{aligned} \frac{d}{dt} H^\varepsilon &= \frac{d}{dt} \int_{\mathbb{R}^3} \eta^\varepsilon(\mathbf{U})dx \\ &= - \int_{\mathbb{R}^3} (T_{\alpha\beta}^{KG} - T_{\alpha\beta}^E) \nabla^\alpha \mathbf{U}^\beta dx + 2 \int_{\mathbb{R}^3} \partial_t \rho V''(\rho) (\mathbf{J}_0^\varepsilon - \mathbf{U}_0 \rho) dx + 2 \int_{\mathbb{R}^3} V'(\partial_t \mathbf{J}_0^\varepsilon - \partial_t (\mathbf{U}_0 \rho)) dx \\ &= - \int_{\mathbb{R}^3} (h_{\alpha\beta}^\varepsilon + I_{\alpha\beta}^\varepsilon) \nabla^\alpha \mathbf{U}^\beta dx + 2 \int_{\mathbb{R}^3} \partial_t V'(\rho) (\mathbf{J}_0^\varepsilon - \mathbf{U}_0 \rho) dx + 2 \int_{\mathbb{R}^3} V'(\partial_t \mathbf{J}_0^\varepsilon - \partial_t (\mathbf{U}_0 \rho)) dx, \end{aligned}$$

<sup>9</sup>See the definition 7.3 and the remark 7.4.

by the fact that  $T_{\alpha 0}^{KG}$  and  $T_{\alpha 0}^{EP}$  have 0 divergence and the fact that the charges are conserved, see (19), (40), (26) and (45).

Moreover, because  $\mathbf{U}$  and  $\rho$  are solutions<sup>10</sup> to (38) we have

$$\begin{aligned}
& - \int_{\mathbb{R}^3} I_{\alpha\beta}^\varepsilon \nabla^\alpha \mathbf{U}^\beta dx \\
&= - \int_{\mathbb{R}^3} -\mathbf{U}_\alpha \mathbf{U}_\beta \nabla^\alpha \mathbf{U}^\beta (\rho - \rho^\varepsilon) + \mathbf{U}_\alpha (\mathbf{J}_\beta^\varepsilon - \mathbf{U}_\beta \rho^\varepsilon) (\nabla^\alpha \mathbf{U}^\beta + \nabla^\beta \mathbf{U}^\alpha) - \nabla_\alpha \mathbf{U}^\alpha (\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon) \mathbf{U}^\gamma dx \\
&= - \int_{\mathbb{R}^3} \mathbf{U}^\beta \nabla_\beta V'(\rho) (\rho - \rho^\varepsilon) - 2(\mathbf{J}_\beta^\varepsilon - \mathbf{U}_\beta \rho^\varepsilon) \nabla^\beta V'(\rho) - \nabla_\alpha \mathbf{U}^\alpha (\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon) \mathbf{U}^\gamma dx \\
&= - \int_{\mathbb{R}^3} -2\mathbf{J}_\beta^\varepsilon \nabla^\beta V'(\rho) + \mathbf{U}^\beta \nabla_\beta V'(\rho) (\rho + \rho^\varepsilon) - \nabla_\alpha \mathbf{U}^\alpha (\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon) \mathbf{U}^\gamma dx,
\end{aligned}$$

and because the charges (26) and (45) are conserved we have

$$\begin{aligned}
& 2 \int_{\mathbb{R}^3} \partial_t V'(\rho) (\mathbf{J}_0^\varepsilon - \mathbf{U}_0 \rho) dx + 2 \int_{\mathbb{R}^3} V'(\partial_t \mathbf{J}_0^\varepsilon - \partial_t (\mathbf{U}_0 \rho)) dx = 2 \int_{\mathbb{R}^3} \partial_t V'(\rho) (\mathbf{J}_0^\varepsilon - \mathbf{U}_0 \rho) dx + 2 \int_{\mathbb{R}^3} V'(\nabla^i \mathbf{J}_i^\varepsilon - \nabla^i (\mathbf{U}_i \rho)) dx \\
&= -2 \int_{\mathbb{R}^3} \nabla^\alpha V'(\rho) (\mathbf{J}_\alpha^\varepsilon - \mathbf{U}_\alpha \rho) dx.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\frac{d}{dt} H^\varepsilon &= - \int_{\mathbb{R}^3} h_{\alpha\beta}^\varepsilon \nabla^\alpha \mathbf{U}^\beta dx - \int_{\mathbb{R}^3} \mathbf{U}^\beta \nabla_\beta V'(\rho) (\rho^\varepsilon - \rho) dx + \int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha (\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon) \mathbf{U}^\gamma dx \\
&= - \underbrace{\int_{\mathbb{R}^3} h_{\alpha\beta}^\varepsilon \nabla^\alpha \mathbf{U}^\beta dx}_{\mathcal{H}_1} + \underbrace{\int_{\mathbb{R}^3} \rho \nabla_\beta \mathbf{U}^\beta V''(\rho) (\rho^\varepsilon - \rho) dx}_{\mathcal{H}_2} + \underbrace{\int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha (\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon) \mathbf{U}^\gamma dx}_{\mathcal{H}_3}.
\end{aligned}$$

We know that we have

$$\mathcal{H}_1 \leq C_0 H^\varepsilon, \quad (88)$$

using the previous calculations and the propositions 7.10 and 7.11.

At first sight, it looks like the two last terms,  $\mathcal{H}_2$  and  $\mathcal{H}_3$ , are not controllable by  $H^\varepsilon$  but only by  $(H^\varepsilon)^{1/\gamma}$  and  $(H^\varepsilon)^{1/2}$ , they are only linear in the quantities that we are looking at. In fact, because of the specific structure of these terms we can improve the apparent lack of smallness.

Firstly,

$$\mathcal{H}_3 = \underbrace{\int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \frac{(\mathbf{J}_\gamma^\varepsilon - \mathbf{U}_\gamma \rho^\varepsilon) (\mathbf{U}^\gamma \rho^\varepsilon - (\mathbf{J}^\varepsilon)^\gamma)}{2\rho^\varepsilon} dx}_{\mathcal{H}_{3.1}} + \underbrace{\int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \left( \frac{\mathbf{J}_\gamma^\varepsilon (\mathbf{J}^\varepsilon)^\gamma}{2\rho^\varepsilon} - \frac{\mathbf{U}_\gamma \mathbf{U}^\gamma \rho^\varepsilon}{2} \right) dx}_{\mathcal{H}_{3.2}},$$

with proposition 7.11 we directly get

$$\mathcal{H}_{3.1} \leq C_0 H^\varepsilon. \quad (89)$$

For  $\mathcal{H}_{3.2}$ , we use the normalization (50) and the equation (30) to get

$$\begin{aligned}
\mathcal{H}_{3.2} &= \int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \left( \frac{\varepsilon^2 \sqrt{\rho^\varepsilon} \square \sqrt{\rho^\varepsilon} - \rho^\varepsilon - 2V'(\rho^\varepsilon) \rho^\varepsilon}{2} + \frac{\rho^\varepsilon + 2V'(\rho) \rho^\varepsilon}{2} \right) dx \\
&= \underbrace{\int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \frac{\varepsilon^2 \sqrt{\rho^\varepsilon} \square \sqrt{\rho^\varepsilon}}{2} dx}_{\mathcal{H}_{3.2.1}} - \underbrace{\int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha (V'(\rho^\varepsilon) \rho^\varepsilon - V'(\rho) \rho^\varepsilon) dx}_{\mathcal{H}_{3.2.2}}.
\end{aligned}$$

<sup>10</sup>In particular, we use the derivatives of the normalization (50).

Then, we get

$$\begin{aligned}
\mathcal{H}_{3.2.1} &= -\frac{d}{dt} \int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \frac{\varepsilon^2 \sqrt{\rho^\varepsilon} \partial_t \sqrt{\rho^\varepsilon}}{2} dx + \int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \frac{\varepsilon^2 \partial_t \sqrt{\rho^\varepsilon} \partial_t \sqrt{\rho^\varepsilon}}{2} dx + \int_{\mathbb{R}^3} \partial_t \nabla_\alpha \mathbf{U}^\alpha \frac{\varepsilon^2 \sqrt{\rho^\varepsilon} \partial_t \sqrt{\rho^\varepsilon}}{2} dx \\
&\quad - \int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \frac{\varepsilon^2 \nabla \sqrt{\rho^\varepsilon} \nabla \sqrt{\rho^\varepsilon}}{2} dx - \int_{\mathbb{R}^3} \nabla \nabla_\alpha \mathbf{U}^\alpha \nabla \sqrt{\rho^\varepsilon} \frac{\varepsilon^2 \sqrt{\rho^\varepsilon}}{2} dx \\
&\leq -\frac{d}{dt} \int_{\mathbb{R}^3} \nabla_\alpha \mathbf{U}^\alpha \frac{\varepsilon^2 \sqrt{\rho^\varepsilon} \partial_t \sqrt{\rho^\varepsilon}}{2} dx + C_0 \|\varepsilon \nabla \sqrt{\rho^\varepsilon}\|_{L^2}^2 + \varepsilon C_0 \|\varepsilon \nabla \sqrt{\rho^\varepsilon}\|_{L^2} \|\sqrt{\rho^\varepsilon}\|_{L^2} \\
&\leq \underbrace{\frac{d}{dt} \int_{\mathbb{R}^3} -\nabla_\alpha \mathbf{U}^\alpha \frac{\varepsilon^2 \sqrt{\rho^\varepsilon} \partial_t \sqrt{\rho^\varepsilon}}{2} dx}_{G^\varepsilon} + C_0 H^\varepsilon + \varepsilon C_0 (H^\varepsilon)^{1/2},
\end{aligned}$$

where the mass term in the energy (21) controls  $\|\sqrt{\rho^\varepsilon}\|_{L^2}$  by  $C_0$  with (68), and by proposition 7.11

$$\begin{aligned}
\mathcal{H}_2 + \mathcal{H}_{3.2.2} &= \int_{\mathbb{R}^3} \nabla_\beta \mathbf{U}^\beta (\rho V''(\rho)(\rho^\varepsilon - \rho) - V'(\rho^\varepsilon)\rho^\varepsilon + V'(\rho)\rho^\varepsilon) dx \\
&= \int_{\mathbb{R}^3} \nabla_\beta \mathbf{U}^\beta ((\gamma - 1)V'(\rho)(\rho^\varepsilon - \rho) - \gamma V(\rho^\varepsilon) + V'(\rho)\rho^\varepsilon) dx = \int_{\mathbb{R}^3} \gamma \nabla_\beta \mathbf{U}^\beta (V'(\rho)(\rho^\varepsilon - \rho) - V(\rho^\varepsilon) + V(\rho)) dx \\
&= - \int_{\mathbb{R}^3} \gamma \nabla_\beta \mathbf{U}^\beta \Theta^\varepsilon(\rho^\varepsilon, \rho) dx \leq C_0 H^\varepsilon,
\end{aligned}$$

where, in the last calculation, we use the fact that  $V$  is a polynomial<sup>11</sup>. Gathering everything we have

$$\frac{d}{dt} H^\varepsilon \leq C_0 \left( H^\varepsilon + (H^\varepsilon)^{1/2} \varepsilon \right) + \frac{d}{dt} G^\varepsilon. \quad (90)$$

We integrate between 0 and  $t \in [0, \min(t_\varepsilon, T)]$  and use the Young inequality to get

$$H^\varepsilon(t) \leq H^\varepsilon(0) + C_0 \int_0^t (H^\varepsilon(s) + \varepsilon^2) ds + G^\varepsilon(t) - G^\varepsilon(0).$$

Now, using again the Young inequality properly and calculations that are similar to the previous ones we find that for all  $t \in [0, \min(t_\varepsilon, T)]$

$$G^\varepsilon(t) \leq C_0(\delta^{-1} \varepsilon^2 + \delta H^\varepsilon(t)),$$

for  $\delta$  as small as we want.

It gives us

$$H^\varepsilon(t) \leq H^\varepsilon(0) + C_0 \left( \int_0^t (H^\varepsilon(s) + \varepsilon^2) ds + \delta H^\varepsilon(t) + \varepsilon^2 + \delta H^\varepsilon(0) \right), \quad (91)$$

and so for  $\delta$  small in comparison to  $C_0$  we have

$$H^\varepsilon(t) \leq C_0(H^\varepsilon(0) + \int_0^t (H^\varepsilon(s) + \varepsilon^2) ds + \varepsilon^2). \quad (92)$$

By Gronwall lemma and the assumptions of theorem 6.1 we have

$$H^\varepsilon(t) \leq C_0 \varepsilon^2, \quad (93)$$

for all  $t \in [0, \min(t_\varepsilon, T)]$  and thus the **propagation property**, point 2).

□

## 7.4 Sum up

From section 7.1 we have the point i) of the main theorem. Then, with the results of section 7.2 and 7.3 we get the point ii). The smallness of the modulated energy propagates and implies the semiclassical (hydrodynamic) limit of KGM to the REP system. Moreover, in the case  $\gamma = 2$  we have the global existence of the solutions to (17) from proposition 3.6 and thus we get iii).

<sup>11</sup>In [4], this part requires more arguments because the author deal with an approximate solution, i.e., a solution to the equation with a modified nonlinearity that is not polynomial, and then pass to the limit. This is done to recover the global solutions, here this result is direct.

## 8 Relativistic Euler equation with pressure

### 8.1 Equivalence of the REP system and the relativistic Euler with pressure

We consider the usual relativistic Euler equation

$$\begin{cases} \mathbf{u}^\alpha \nabla_\alpha \mathbf{u}^\beta + (g^{\alpha\beta} + \frac{\mathbf{u}^\alpha \mathbf{u}^\beta}{c^2}) \frac{\nabla_\alpha p}{\mu + \frac{p}{c^2}} = 0, \\ \mathbf{u}^\alpha \nabla_\alpha \mu + \nabla_\alpha \mathbf{u}^\alpha (\mu + \frac{p}{c^2}) = 0, \\ p = p(\mu), \\ \mathbf{u}^\alpha \mathbf{u}_\alpha = -c^2, \end{cases} \quad (94)$$

for  $g$  the metric (here the Minkowski metric  $g = \text{Diag}(-c^2, 1, 1, 1)$ ),  $\mu$  the energy density,  $p$  the pressure,  $\mathbf{u}$  the four-velocity and  $c$  the speed of light. The constant  $c$  is not set up to 1 because we are interested in the behaviour of this system for  $c$  near  $+\infty$ . We refer to [9], [2] and [7] for a general description of the relativistic fluid and to [1] and [8] for recent development in the mathematical analysis.

The main difference between the usual relativistic Euler (94) and REP (38) is the role of the pressure and the potential. In (94), we see that the gradient of the pressure (the density of force) acts only perpendicularly to the flow, indeed the tensor

$$\Pi_{\alpha\beta} = \frac{\mathbf{u}^\alpha \mathbf{u}^\beta}{c^2} + g_{\alpha\beta} \quad (95)$$

is the projection on the orthogonal to  $\mathbf{u}$ . The effect of the spacetime gradient of the potential  $V'm$  in (38) cannot be identified with the effect of a standard relativistic pressure. In comparison, for the non relativistic case the pressure can clearly be identified with a potential. Moreover, the total energy density  $\mathbf{U}^0 \rho$  is conserved in (38) whereas the total energy  $\mathbf{u}^0 \mu$  of (94) is not and the normalization of  $\mathbf{U}$  and  $\mathbf{u}$  are different. Nonetheless, the REP system (38) and the usual relativistic Euler (94) are equivalent if we look at the right quantities.

**Proposition 8.1.** *Let  $(\mathbf{U}, \rho)$  be a solution to (38) then the quantities*

$$\mu = \rho + \frac{(\gamma + 1)V(\rho)}{c^2}, \quad \mathbf{u} = \mathbf{U}\Gamma, \quad (96)$$

with the scaling factor  $\Gamma = \frac{1}{\sqrt{1 + \frac{2V'}{c^2}}}$  and the pressure  $p = (\gamma - 1)V(\rho)$  are solutions to (94). We also have

$$\mathbf{J} = \mathbf{U}\rho = \mathbf{u}(\mu + \frac{p}{c^2})\Gamma, \quad \rho = (\mu + \frac{p}{c^2})\Gamma^2. \quad (97)$$

*Proof.* First, we have

$$\begin{aligned} \mathbf{u}^\alpha \nabla_\alpha \mathbf{u}^\beta &= \Gamma^2 \mathbf{U}^\alpha \nabla_\alpha \mathbf{U}^\beta + \Gamma \mathbf{U}^\alpha \mathbf{U}^\beta \nabla_\alpha \Gamma = \Gamma^2 (-\nabla^\beta V' - \Gamma^2 \mathbf{U}^\alpha \mathbf{U}^\beta \frac{\nabla_\alpha V'}{c^2}) \\ &= -(g^{\alpha\beta} + \frac{\mathbf{u}^\alpha \mathbf{u}^\beta}{c^2}) \frac{\nabla_\alpha V'}{1 + \frac{2V'}{c^2}} = -(g^{\alpha\beta} + \frac{\mathbf{u}^\alpha \mathbf{u}^\beta}{c^2}) \frac{(\gamma - 1)V' \nabla_\alpha \rho}{\rho + \frac{2\gamma V'}{c^2}} \\ &= -(g^{\alpha\beta} + \frac{\mathbf{u}^\alpha \mathbf{u}^\beta}{c^2}) \frac{\nabla_\alpha p}{\mu + \frac{p}{c^2}}, \end{aligned}$$

and then

$$\begin{aligned} \mathbf{u}^\alpha \nabla_\alpha \mu &= \mathbf{u}^\alpha \nabla_\alpha \rho + \mathbf{u}^\alpha \nabla_\alpha \rho (\gamma + 1) \frac{V'}{c^2} = \frac{1}{\Gamma^2} \mathbf{u}^\alpha \nabla_\alpha \rho + \mathbf{u}^\alpha \nabla_\alpha \rho (\gamma - 1) \frac{V'}{c^2} \\ &= \frac{1}{\Gamma^2} (\mathbf{u}^\alpha \nabla_\alpha \rho + \mathbf{u}^\alpha \nabla_\alpha \rho \frac{V''}{c^2} \rho \Gamma^2) = \frac{1}{\Gamma^2} (-\nabla_\alpha \mathbf{U}^\alpha \Gamma \rho - \mathbf{U}^\alpha \nabla_\alpha (\Gamma \rho)) \\ &= (1 + \frac{2V'}{c^2}) (-\nabla_\alpha \mathbf{u}^\alpha \rho) = -\nabla_\alpha \mathbf{u}^\alpha (\mu + \frac{p}{c^2}). \end{aligned}$$

□

**Corollary 8.2.** *The theorem 6.2 is a direct corollary of theorem 6.1 and proposition 8.1.*

## 8.2 Non relativistic limit

We want to look at the non relativistic limit at least formally. Let  $(\mathbf{u}_c, \mu_c)$  be the solution to (94), traditionally, we write<sup>12</sup>  $(u_c^0, u_c)$  as  $(\theta_c, \theta_c v_c)$  where  $\theta_c = \frac{1}{\sqrt{1 - \frac{v_c^2}{c^2}}}$  is the Lorentz factor and  $v_c$  the non relativistic velocity. Formally, as  $c \rightarrow \infty$ , we are expecting

$$\lim_{c \rightarrow \infty} (u_c^0, u_c, \mu_c, p_c) = (u_\infty^0, u_\infty, \mu_\infty, p_\infty) = (1, v_\infty, \mu_\infty, p_\infty), \quad (98)$$

where  $(v_\infty, \mu_\infty)$  is solution to the non relativistic Euler equations and where  $p_\infty$  is the corresponding pressure (see for example [21], [6] and [22] for a rigorous proof of this convergence in 1d).

Moreover, with  $(\mathbf{u}_c, \mu_c)$  given by 8.1, we clearly see that

$$\begin{aligned} p_c &= (\gamma - 1)V(\rho_c), & \Gamma_c &= 1 + O(c^{-2}), \\ \mathbf{u}_c &= \mathbf{U}_c + O(c^{-2}), & \mu_c &= \rho_c + O(c^{-2}), \end{aligned}$$

where we also index the solution to (38) with a  $c$  to show the dependence. Thus, we have heuristically that

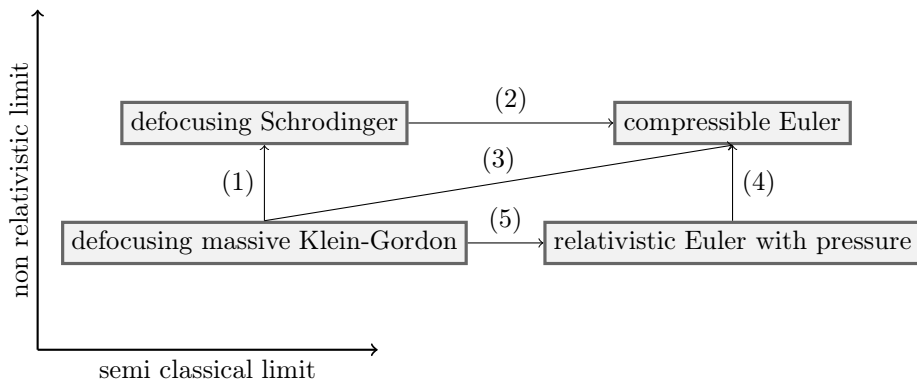
$$\lim_{c \rightarrow \infty} (U_c^0, U_c, \rho_c, (\gamma - 1)V(\rho_c)) = (1, v_\infty, \mu_\infty, p_\infty),$$

which is consistent with the non-relativistic semi-classical limit of the massive nonlinear Klein-Gordon toward the non relativistic compressible Euler given in [14]. We have

$$\begin{aligned} (J_c^\varepsilon)^0 &\xrightarrow{s.c.} U_c^0 \rho_c = u_c^0 \left( \mu_c + \frac{p_c}{c^2} \right) \Gamma_c \xrightarrow{n.r.} \mu_\infty, \\ J_c^\varepsilon &\xrightarrow{s.c.} U_c \rho_c = u_c \left( \mu_c + \frac{p_c}{c^2} \right) \Gamma_c \xrightarrow{n.r.} v_\infty \mu_\infty, \\ |\Phi_c^\varepsilon|^2 &\xrightarrow{s.c.} \rho_c = \left( \mu_c + \frac{p_c}{c^2} \right) \Gamma_c^2 \xrightarrow{n.r.} \mu_\infty. \end{aligned}$$

## 9 Conclusion

Overall, we prove in section 7 that the semiclassical (hydrodynamic) limit of the massive Klein-Gordon equation with a polynomial defocusing potential is the REP system, a relativistic Euler system with a potential. The scaling of the semi classical Klein-Gordon equation is intrinsically linked with high frequency solutions. With the modulated energy method we show that if the initial data are so that  $\Phi^\varepsilon$  has one direction of high oscillation and its modulus is not high frequency then this remains true for a finite time. We think that this work can be done for other geometry of spacetime, the arguments given in the present paper are not restricted to the Minkowski metric. Moreover, the REP system is equivalent to the usual relativistic Euler equations with pressure (see section 8). This implies that the semiclassical (hydrodynamic) limit of massive KG with a polynomial defocusing potential is the relativistic Euler equations up to a rescaling. This is summed up in the following picture



- The point (1) is proved in [23], [20] and [13] using the Strichartz estimates, the latter containing recent improvements of the formers.

<sup>12</sup>We note  $u_c^0$  the time component of  $\mathbf{u}_c$  and  $u_c$  its space components.

- The point (2) is proved in [4] via the modulated energy method and in [10] via the WKP method. One can also look at [17] for an application of modulated energy method on the study of the semi-classical limit of Gross-Pitaevskii, a very related equation.
- The point (3) is proved in [14] and [15] via modulated energy method, the double limit is performed as a simple limit by choosing  $c^{-1} = \varepsilon^\kappa$  for some  $\kappa > 0$ .
- The point (4) is proved in [21], [6] and [22] in 1d using various techniques.
- The point (5) is proved in the present paper.

The main constraints of this result are that :

- We only deal with the massive case (see for example this requirement in the proposition 7.10).
- The proof requires a potential (the convergence of the density  $\rho^\varepsilon$  is obtained with the potential, see remark 7.9).
- The solution to the REP system is supposed to be regular enough ( $\mathbf{U}$  and  $\rho$  in  $H^4$ ).
- We only deal with one direction of oscillation (we see that there is only one vector field  $\mathbf{U}$ ).

For the last point, we refer to [25] for general results on multi phase high frequency solutions or to [27] and [12] for applications on the Einstein equations. In general, we see that the modulated energy method do not say anything on the case where there is more than one direction of high oscillation. In the WKB approach, the superposition of a finite number of phases leads to (weak) solution to Vlasov type system, see [27] and [12]. To recover the semi classical kinetic (Vlasov type) limit the approach can be more direct, for example on the Schrodinger equation using the Wigner transform. We refer to [18], [24] and [5]. No works of this type have been done on the semi classical Klein-Gordon equation as far as we know.

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