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# Multi-phase high frequency solutions to Klein-Gordon-Maxwell equations in Lorenz gauge in (3+1) Minkowski spacetime

Tony Salvi

## Abstract

We study a 1-parameter family  $(A_\lambda, \Phi_\lambda)_\lambda$  of multi-phase high frequency solutions to Klein-Gordon-Maxwell equations in Lorenz gauge in the (3+1)-dimensional Minkowski spacetime. This family is based on an initial ansatz. We prove that for  $\lambda$  small enough the family of solutions exists on an interval uniform in  $\lambda$  only function of the initial ansatz. These solutions are close to an approximate solution constructed by geometric optics. The initial ansatz needs to be regular enough, to satisfy a polarization condition and to satisfy the constraints for Maxwell null-transport in Lorenz gauge, but there is no need for smallness of any kind. The phases need to interact in a coherent way. We also observe that the limit  $(A_0, \Phi_0)$  is not solution to Klein-Gordon-Maxwell equations but to a Klein-Gordon-Maxwell null-transport type system.

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# 1 Introduction

In this article, we construct a family of specific solutions to the Klein-Gordon Maxwell (KGM) equations

$$\begin{cases} \partial_{\alpha} F^{\alpha\beta} = -Im(\Phi \overline{\partial^{\beta} \Phi}) + A^{\beta} |\Phi|^2, \\ (\partial^{\alpha} + iA^{\alpha})(\partial_{\alpha} + iA_{\alpha})\Phi = 0, \end{cases} \quad (1)$$

in the Minkowski spacetime on  $\mathbb{R}^{3+1}$ . The unknowns are the charged scalar field  $\Phi$  (a complex function) and the electromagnetic four-potential  $A$ . We also define the Faraday tensor (the electromagnetic field) as  $F = dA$ , that is  $F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}$  in coordinates. This is a system of semilinear second order partial differential equations. In general, the solutions are given in a particular gauge (a representative of an equivalence class), see section 1.1. Our specific solutions  $(A_{\lambda}, \Phi_{\lambda})_{\lambda}$  are based on a multi phase<sup>1</sup> high frequency initial ansatz of definition 2.21 for which  $\lambda$  represents the period of oscillation (as  $\lambda$  goes to 0 we recover high frequency oscillations) and is tuned to the amplitude. We are interested in the behaviour of these solutions when  $\lambda$  is very small : the limit  $(A_0, \Phi_0)$  and the convergence of the different part of the solutions as  $\lambda$  goes to 0. Indeed, the solutions admit a decomposition in an approximate solution (approximated with respect to  $\lambda$ ) plus an error term. The approximation itself is constructed by geometric optics as a low order phases-amplitude WKB expansion<sup>2</sup>. For that to make sense, we want the family of solutions to live on a uniform time interval with respect to  $\lambda$ . In particular, the time does not shrink when  $\lambda$  goes to 0.

A work of the same type has been done in [10] and [9] on Yang Mills equation (generalising KGM) using different methods. The author constructs exact solutions that require higher order expansion for a mono phase ansatz, see section 7.2. In particular, the author gives a precise description of optic geometric expansions for any order and differentiate the transported part, the polarized part and the "free" part<sup>3</sup> of the system. Nonetheless, the author does not give a description of the initial data. In this article, we are interested in construction for lower order

<sup>1</sup>We are speaking here about any finite number of phases.

<sup>2</sup>See section 1.3, on geometric optics.

<sup>3</sup>Due to the gauge freedom, see section 1.1.

expansion (and thus lower order approximation) with a multi phase ansatz and in the precise manner the phases interact. We also give a description of the multi phase high frequency initial data. The multi phase high frequency (MPHF) solutions have been studied for generic first order symmetric hyperbolic<sup>4</sup> systems in [19]. Note that this also requires higher order expansions.

Another particularly interesting and related case for high frequency expansion is the one of the Einstein equation in vacuum. It is a system of quasilinear second order partial differential equations whose solutions are spacetime metric also defined up to a gauge. The behaviours of (multi phase or not) high frequency solutions to this equation are known, see [3] for the beginning of it or [8], [6], [5], [17], [7], [24], [23] and [25] for the most recent works. Indeed, the latter papers are dedicated to the proof of the Burnett conjecture (or its reverse counterpart) enunciated in [2]. This conjecture states that the limits of such high frequency solutions<sup>5</sup> are solutions to Einstein coupled with a massless Vlasov field, i.e., there is a backreaction (an effective stress energy tensor in that case). The backreaction is linked with the structure of the nonlinearities of the equation and is clearly observable using geometric optics with the right scaling for the amplitude and the frequency. For example, the presence of nonlinearities with the so called **null structure**<sup>6</sup> prevents the apparition of backreactions. We are also interested in understanding such structures on the KGM system in view of geometric optics.

As done in the previously cited papers, we use the WKB expansion method (which leads to geometric optics) to construct our approximate solution. Then we construct the error term to have an exact solution. The approximation is understood in the not obvious sense of definition 3.1.

Indeed, the WKB expansion is commonly used for hyperbolic system which is more subtle here. Since, to recover the hyperbolicity, we choose the Lorenz gauge (see (7)). However, having a good approximate solution to the hyperbolic reduced system and the real system is not equivalent, see remark 4.9. In Lorenz gauge, all the parameters in the expansion obey transport or wave equations which is very convenient. In particular, we have access to the Strichartz estimates that we use widely in section 6.3 to estimate the error term. The final family of solutions is in an exact Lorenz gauge which is in contrast with the corresponding wave gauge for Einstein equations [23], this is possible mainly because the gauge equation for KGM is linear.

Another reason why we define approximation as in definition 3.1 is because the oscillating terms and the non oscillating terms are not treated the same way as a reminder. The important oscillating terms come from interactions between two different phases and have a good behaviour.

We restrict the type of interactions between two phases, either everywhere characteristic or nowhere characteristic. This is the commonly used strong coherence assumption<sup>7</sup> restricted to the interactions between two phases only. We deal with an approximate solution of order one so we do not see any other degree of interactions. The resonant interactions<sup>8</sup> are not treated in [6] and [25], it is possible here because of the scaling of the equation which shifts the hierarchy of the equations for the different parameters. It can be noted that if all the phases are plane waves then they fit the strong coherence case of the theorem.

In general, dealing with high frequency solutions with high amplitude makes poor uniform Sobolev norms with respect to  $\lambda$ . The error term has a high amplitude (because it is constructed above an approximation of order 1) and only a half derivative uniformly bounded. We use the fact that only certain directions contribute to this bad behaviour and can be isolated by splitting the error in different parts. These different parts obey a coupled system of wave/transport equations. We resort to the use of **auxiliary functions** to handle a loss of derivative in this system. To show the uniform estimates for  $\lambda$  small we use a bootstrap argument, see section 6.2.3.

The initial data for the KGM system must satisfy the Maxwell constraint equation and a gauge condition. This implies that the initial ansatz cannot be prescribed freely, we give the precise conditions in definition 2.24. This condition is directly linked with the system given in section 1.2. Naturally, there are also constraint equations for the error term such that the full initial data satisfy the Maxwell constraints. Moreover, because we cut the error term in different parts with a nice structure and use auxiliary functions, we have other requirements on the initial data. We show how to construct a generic set of initial data for the error term satisfying these constraints and

---

<sup>4</sup>In particular, without gauge choice.

<sup>5</sup>The assumptions are that the convergence of the derivative of the metric is only a weak convergence, see [2] for a more precise statement.

<sup>6</sup>Which is not the case in the Einstein equations or in the KGM equations.

<sup>7</sup>See [19].

<sup>8</sup>When interactions give rise to characteristic phases in the sense of geometric optics.

having the structure that we want. Each element of the set leads to a 1-parameter family of MPHf solutions to KGM. These initial data and the estimates in sections 6.3.2, 6.3.3 and 6.1.1 are the key to the construction of our exact solutions.

## 1.1 Generalities on Klein-Gordon Maxwell

The KGM equations

$$\begin{cases} \partial_\alpha F^{\alpha\beta} = -Im(\Phi\overline{\partial^\beta\Phi}) + A^\beta|\Phi|^2, \\ (\partial^\alpha + iA^\alpha)(\partial_\alpha + iA_\alpha)\Phi = 0 \end{cases} \quad (2)$$

are a model of quantum electrodynamics for charged spinless particles (in our case it is also massless). The quantity

$$\mathcal{J}^\beta = -Im(\Phi\overline{\partial^\beta\Phi}) + A^\beta|\Phi|^2 \quad (3)$$

is the flux of charge density and we have

$$\partial_\alpha \mathcal{J}^\alpha = 0, \quad (4)$$

that is the conservation of charge. Because the Faraday tensor  $F_{\alpha\beta}$  is antisymmetric, the 0 component of the Maxwell equation gives

$$\partial_i F^{i0} = -Im(\Phi\overline{\partial^0\Phi}) + A^0|\Phi|^2 = \mathcal{J}^0, \quad (5)$$

the divergence of the electric field is equal to the density of charge.

A solution to (2) is a critical point of the scalar quantum electrodynamics action

$$\mathcal{L} = \int_{\mathbb{R}^{3+1}} -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}(\partial_\alpha + iA_\alpha)\Phi(\partial^\alpha + iA^\alpha)\Phi dxdt, \quad (6)$$

so that (2) corresponds to the Euler-Lagrange equations of this lagrangian.

The system is invariant by change of gauge. Indeed, let  $\chi$  be a real function on  $\mathbb{R}^{3+1}$  and let

$$A'_\alpha = A_\alpha + \partial_\alpha\chi, \quad \Phi' = e^{-i\chi}\Phi,$$

then  $(A', \Phi')$  is a solution to KGM if and only if  $(A, \Phi)$  is a solution too. We are free to fix a gauge and remove the indeterminacy by adding one equation. In this paper, we use the **Lorenz gauge**

$$\partial_\alpha A^\alpha = 0 \quad (7)$$

to recover a system of semilinear wave equation. With that choice (2) becomes

$$\begin{cases} \square A^\beta = -Im(\Phi\overline{\partial^\beta\Phi}) + A^\beta|\Phi|^2, \\ \square\Phi = -2iA^\alpha\partial_\alpha\Phi + A^\alpha A_\alpha\Phi. \end{cases} \quad (8)$$

We call this reduced system KGML and we call the reduced Maxwell and reduced Klein-Gordon equations Maxwell and Klein-Gordon equations to simplify.

Other gauge choices which are very used are the Coulomb gauge and the temporal gauge. For example, these are used to obtain global existence results or low-regularity class of solution in [13], [14] and [11].

### 1.1.1 Lorenz gauge

To solve (2) one can first prescribe initial data  $(A^\alpha|_{t=0} = a^\alpha, \partial_t A^\alpha|_{t=0} = \dot{a}^\alpha, \Phi|_{t=0} = \phi, \partial_t\Phi|_{t=0} = \dot{\phi})$  satisfying the Lorenz gauge condition and the Maxwell constraint equation from (5), that is

$$\begin{cases} \Delta a^0 + \partial_i \dot{a}^i = Im(\phi\overline{\dot{\phi}}) + a^0|\phi|^2, \\ \partial_i a^i + \dot{a}^0 = 0. \end{cases} \quad (9)$$

Then, find a solution to the KGML system (8) and propagate the Lorenz gauge via the following argument. We take the divergence of the Maxwell equation in (8) so that

$$\square\partial_\beta A^\beta = -\partial_\beta A^\beta|\Phi|^2, \quad (10)$$

where we use the Klein-Gordon equation in (8) to get this result. From there, and because we choose initial data accordingly with  $\partial_\alpha A^\alpha|_{t=0} = 0$  and  $\partial_t(\partial_\alpha A^\alpha)|_{t=0} = 0$  (due to the constraints), we deduce that the equation (7) remains true for all time, i.e., the gauge propagates.

### 1.1.2 Plane wave

Before discussing exact multi phase high frequency solutions we wish to present quickly the plane wave solutions to the linearized Klein-Gordon Maxwell system around the vacuum state

$$\begin{cases} \partial_\alpha F_{p.w.}^{\alpha\beta} = \square A_{p.w.}^\beta - \partial^\beta \partial_\alpha A_{p.w.}^\alpha = 0, \\ \square \Phi_{v.p.} = 0. \end{cases} \quad (11)$$

The plane waves solutions are constant along planes perpendicular to the direction of propagation, we write them under the form

$$A_{p.w.}^\alpha = P^\beta \cos(k_\alpha x^\alpha) + Q^\beta \sin(k_\alpha x^\alpha), \quad \Phi_{p.w.} = \Psi e^{ik'_\alpha x^\alpha}$$

respectively for the four-potential and the wave function. The quantities  $P, Q, \Psi, k$  and  $k'$  are constants. We search for solutions to (11) so this leads us to parameters<sup>9</sup> satisfying

$$k_\alpha k^\alpha = k'_\alpha k'^\alpha = 0, \quad k_\alpha P^\alpha = k_\alpha Q^\alpha = 0.$$

The waves propagate in a null direction (at the speed of light) and the amplitudes of the potential are polarized<sup>10</sup>. It is not surprising that our initial ansatz is also polarized the same way, see definition 2.22.

## 1.2 Discussion on Klein-Gordon Maxwell null-transport

We introduce the KGM null-transport system

$$\begin{cases} \partial_\alpha F^{\alpha\beta} = -Im(\Phi \overline{\partial^\beta \Phi}) + A^\beta |\Phi|^2 + \sum_{j=1}^N V_{(j)}^\beta \rho_{(j)}^2, \\ (\partial^\alpha + iA^\alpha)(\partial_\alpha + iA_\alpha)\Phi = 0, \\ V_{(j)}^\alpha \partial_\alpha V_{(j)}^\beta = 0, \\ V_{(j)}^\alpha V_{(j)\alpha} = 0, \\ 2V_{(j)}^\alpha \partial_\alpha \rho_{(j)} + \partial_\alpha V_{(j)}^\alpha \rho_{(j)} = 0, \end{cases} \quad (12)$$

with  $V_{(j)}$  a four-velocity vector field and  $\rho_{(j)}$  a charge density both defined for  $j$  an integer smaller than a finite  $N$ . The quantities  $A$  and  $\Phi$  are the four-potential for the electromagnetic field and the wave function. The system (12) is close to the null-dust (or discrete Vlasov) Klein-Gordon Maxwell system. The difference is that the charges  $\rho_{(j)}$  influence the electromagnetic field but their trajectories (following the flow lines of  $V_{(j)}$ ) do not see the electromagnetic field.

We claim that a part of the parameters<sup>11</sup> of the first order MPHF approximation are solution to this system. These parameters (and by extension all the parameters of the approximate solution) are called the background. In other words, the limit in  $L^2$ , the background terms  $(A_0, \Phi_0)$ , of the high frequency family  $(A_\lambda, \Phi_\lambda)$  is solution to KGM null-transport and not KGM, there is a backreaction. This gives us an equivalent of the results of the Burnett conjecture.

One can check that this system is also gauge invariant and that the gauge propagates with the same technique as for KGM in section 1.1.1.

## 1.3 Geometric optics

We refer to [19] and [11] for very complete descriptions of the subject. Geometric optics is a high frequency approximation of solution. It comes from optics which is meant to describe the propagation of light, that is electromagnetic waves obeying the Maxwell equations. The propagation occurs along rays and fits the particle-like view point of light. This high frequency approximation is generalised to first order symmetric hyperbolic systems of the form

$$M(U)^\alpha \partial_\alpha U = F(U), \quad (13)$$

<sup>9</sup>One can check that there are other plane wave solutions to the Maxwell vacuum equation with a non characteristic phase, i.e.,  $k_\alpha k^\alpha \neq 0$ . These solutions are pure gauge solution, they are in particular invisible in the Faraday tensor.

<sup>10</sup>The Faraday tensor itself is polarized, the electromagnetic waves oscillates perpendicularly to the direction of propagation.

<sup>11</sup>The rest of the parameters obey decoupled equations.

for  $U$  a vectorial quantity defined on  $\mathbb{R}^{3+1}$ . Typically, by analogy with light, the amplitudes are propagated at the speeds given by the eigenvalues of the system and are polarized in the eigenspaces. In the monophasic case, we look at approximate solutions<sup>12</sup> as such a (WKB) expansion

$$U_{app}(t, x) = \lambda^p \sum_{i \geq 0} \lambda^{i/2} U_i(t, x, \frac{u}{\lambda}), \quad (14)$$

where  $\lambda$  is the typical period or wavelength,  $p$  a tuning parameter,  $u$  a phase and where the  $U_i(t, x, \theta)$  are periodic with respect to their third variable. In general,  $U_{app}$  is a good approximation of order  $m$  if for  $0 \leq i \leq m$  the  $U_i$  are solutions to a cascade of equation<sup>13</sup> that depends on the phase  $u$ .

The two main features of these equations are the so called eikonal equation<sup>14</sup> for the phase

$$\det(M^\alpha \partial_\alpha u) = 0, \quad (15)$$

with the associated polarisation condition

$$U_0 \in \ker(M^\alpha \partial_\alpha u). \quad (16)$$

A phase is said to be characteristic if it solves the eikonal equation. We do not detail the transport equation for the amplitudes here. In the case of a linear system there is, by definition, no interactions and no creation of harmonics. Thus, one can look at a more precise profile

$$U_{app}(t, x) = \lambda^p \sum_{i \geq 0} \lambda^{i/2} e^{i \frac{u}{\lambda}} U_i(t, x). \quad (17)$$

In general, in the nonlinear case<sup>15</sup> there is no reason to be able to have such an easy description, harmonics might play an important role. In fact, it depends on the parameter  $p$ , if the amplitude is big ( $p$  small) then the nonlinearity appears at the leading order and produces harmonics, it is called weakly nonlinear optics. We see that the geometric optics really takes into account the scaling of problems. However, some structures in the nonlinearities might still prevent the apparition of harmonics, for example the null structures, see [16].

In the multi phase case, the general ansatz is

$$U_{app}(t, x) = \lambda^p \sum_{i \geq 0} \lambda^{i/2} U_i(t, x, \frac{\mathcal{U}}{\lambda}), \quad (18)$$

where  $\mathcal{U}$  is the vector containing all the phases. Recall that we only look at a finite number of phases, this implies that the group of frequency (representative of the interaction between phases) is isomorphic to  $\mathbb{Z}^d$  for  $d$  finite. To be able to apply geometric optics we need to qualify the interactions between phases. A typical restriction is the strong coherence assumption, that is, for any  $z \in \mathbb{Z}^d$  and any phase  $u_z = z \cdot \mathcal{U}$  we have

$$\forall (t, x) \in \mathbb{R}^{3+1} \det(M^\alpha \partial_\alpha u_z)(t, x) = 0 \quad \text{or} \quad \forall (t, x) \in \mathbb{R}^{3+1} \det(M^\alpha \partial_\alpha u_z)(t, x) \neq 0. \quad (19)$$

The phase  $u_z$  is either everywhere characteristic or nowhere characteristic. Under this assumption one can generally build approximate solutions at any order, see [19] for more details.

The next step is to study the stability of such approximate solutions, that is the existence of an exact solution close to the approximated ones on a fixed interval (independent of  $\lambda$ ). One can construct the exact solution as the sum of the approximate solution (truncation of the WKB expansion (14) at order  $m$ ) plus an error term  $Z_\lambda$ , that is the parametrix

$$U_\lambda(t, x) = \lambda^p \sum_{i \geq 0}^m \lambda^{i/2} U_i(t, x, \frac{\mathcal{U}}{\lambda}) + Z_\lambda. \quad (20)$$

It is mathematically interesting to see up to which order of expansion one has to go to be able to build the exact solution. For example, in the KGM case regularity is lost at each order of expansion, see section 7.3, so being able to construct the error term at low order allows us to treat less regular initial ansatz. More generally, higher orders

<sup>12</sup>With respect to  $\lambda$ .

<sup>13</sup>Each level is meant to cancel a certain order of  $\lambda$ .

<sup>14</sup>Written here for linear systems.

<sup>15</sup>In this paper we only speak about the semilinear case, here it corresponds to  $M^\alpha$  being independant of  $U$ .

require the description of bigger hierarchical system of equations.

In this paper we start with the initial ansatz

$$a_{1\lambda}^\alpha = a_0^\alpha + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (p_{\mathcal{A}}^\alpha \cos(\frac{v_{\mathcal{A}}}{\lambda}) + q_{\mathcal{A}}^\alpha \sin(\frac{v_{\mathcal{A}}}{\lambda})), \quad \phi_{1\lambda} = \phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (\psi_{\mathcal{A}} e^{i\frac{v_{\mathcal{A}}}{\lambda}}) \quad (21)$$

and we search for solutions to (1)<sup>16</sup> based on it. Firstly, we want an approximate solution of order one to (1) (which are very close to approximate solutions to (8) at first order). This typically gives rise to high frequency expansions such as (14). In our case this simplifies to

$$A_{1\lambda}^\alpha = A_0^\alpha + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (P_{\mathcal{A}}^\alpha \cos(\frac{u_{\mathcal{A}}}{\lambda}) + Q_{\mathcal{A}}^\alpha \sin(\frac{u_{\mathcal{A}}}{\lambda})), \quad \Phi_{1\lambda} = \Phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (\Psi_{\mathcal{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}}). \quad (22)$$

It is described in sections 2.3 and 4. We remark that this ansatz contains only the first harmonic of each phase, as in the linear case (17). Moreover, there are  $O(1)$  non oscillating terms. This is the same type of expansion as in [25] and [6] for the Einstein case, the only difference is the scaling of  $\lambda$ . The amplitude is bigger here because the nonlinearities are weaker (quadratic with only one derivative).

Let starts the discussion by considering only one phase. Despite the bigger amplitude we remark that the parameters in the expansion obey linear transport equations<sup>17</sup>, see [19]. Indeed, in comparison with Einstein there is a shift in the cascade of equation, for example the lower order interaction of the first parameter with itself is of order  $O(1)$  and its transport equation of order  $O(\lambda^{-1/2})$ . In fact, there is an even stronger property : there is no creation of harmonics below the order  $O(\lambda)$ . This is due to the structure of the nonlinearity. It is called **transparency**, see [12] and [16]. Nonetheless, we observe that even if there was creation of harmonics in the equation at the lowest order (that is  $O(1)$ ), the first order expansion could still be described as (22) because of the shift in the hierarchy of equation. This is a simpler case than the Einstein one where one has to use the polarization of higher order term to treat the harmonics, see [23].

The absence of harmonics at big amplitudes is not equivalent to the absence of interaction at big amplitude. Indeed, there is an **O(1) non-oscillating** term in the Maxwell equation coming from the interaction of the oscillating part of  $\Phi_{1\lambda}$  with itself. In other words, there is a backreaction (in the language of [2]) or a rectification (in the language of geometric optics [19]). There is an effective flux of charge density appearing (in comparison with Einstein where there is an effective stress energy tensor). It is visible because of the fine tuning between the amplitude and the frequency of the oscillation with respect to the structure of the nonlinearities.

Now, considering the different phases we see that there are **O(1) oscillating terms** coming from the interaction of two phases. Under the strong coherence assumption (19), there are two possibilities for each new arising phase. If a phase is non characteristic, then it is easy to deal with it as done in [6] for example, see section 6.1.3. Indeed, it behaves like the reminder of a very small amplitude oscillating term plugged in the equation (1).

If the phase is characteristic, then we can absorb it with a small amplitude parameter, this is due to the shift in the cascade of equation. This cannot be done for the Einstein case in [6] and [25]. All of this allows us to treat the approximate solution under the form of a simple expansion (22). Then, we are able to construct an exact solution close to the approximated one by adding an error term. The error term absorbs directly all the  $O(\lambda^{1/2})$  reminder, we do not see interactions between more than two phases at all. This would not be the case if the error term was constructed after a higher order expansion. One can observe that we did not talk about the non hyperbolicity of the system or the choice of gauge. This is because it is almost invisible too. Indeed, the error term can absorb the "gauge defect" of the approximate solution (22) for free at this order if the initial data are well chosen, see remark 4.9 and sections 5.1 and 6.3.6.

With our previous discussion we see that the most important nonlinearities<sup>18</sup> are

$$Im(\Phi \overline{\partial^\beta \Phi}), \quad 2iA^\alpha \partial_\alpha \Phi,$$

coming respectively from the Maxwell equation and the Klein-Gordon equation. Both contain a hidden **null structure up to a Riesz operator**, this is shown in the Coulomb gauge in [15]<sup>19</sup>. The transparency is often

<sup>16</sup>We can observe that it is a system of second order partial differential equations so the things are slightly different.

<sup>17</sup>Usually, the bigger is the amplitude the more we see the nonlinearities.

<sup>18</sup>In the sense that the cubic nonlinearities are basically invisible in a low order approximation.

<sup>19</sup>In Lorentz gauge the KG nonlinearity is also a null-structure in the sense of counting "bad derivatives".



carried by the null structures but it is not the case here. For the Maxwell equation, the  $O(1)$  harmonics are canceled by the nature of complex number but there still remains the backreaction. This effect guarantees the absence of a complete null structure. For the KG equation, the polarization of the profiles perpendicularly to the direction of propagation kills totally the  $O(1)$  interactions. This is closely related to the **compatibility conditions** described in [16].

## 1.4 Rough statement of the theorems

**Theorem 1.1.** *Let an initial ansatz (21) and the parameters that composed it have the following properties :*

- *To be a solution to the constraint equations for KGM null transport in Lorenz gauge*
- *To have a polarized first parameter for the electromagnetic potential*
- *To have phases two by two fully collinear or fully separated (strong coherence)*
- *To be regular enough without any assumptions about its size*

*then there exists  $\lambda_0 > 0$  and  $T > 0$  such that for all  $\lambda < \lambda_0$  there exists a family of MPHf solution to KGM in Lorenz gauge  $(A_\lambda, \Phi_\lambda)_{\lambda < \lambda_0}$  on  $[0, T]$ . The solutions are decomposed as a first order expansion (22) (the background terms) plus an error term. As  $\lambda$  goes to 0 the family converges to  $(A_0, \Phi_0)$ , a solution KGM null-transport in Lorenz gauge.*

We see that the conditions on the initial ansatz are the same as in [25], the main difference is that there is no assumption about the size of the initial ansatz. Moreover, the limit  $(A_0, \Phi_0)$  is not here solution to a physically relevant system.

**Theorem 1.2.** *Under the same assumptions, the  $(A_\lambda, \Phi_\lambda)_{\lambda < \lambda_0}$  is uniformly bounded in  $H^{1/2}$  and the error term is small in this norm.*

*The first order expansion corresponds to the truncation of a WKB expansion of order 1, that is an approximate solution of KGM in Lorenz gauge in the sense of definition 3.1.*

We observe in section 7.2 that these results are comparable to the ones of [10] in terms of smallness and stability of KGM for the monophasic case with higher order expansion.

## 1.5 Ideas of the proof

The proof of both theorems follows from the construction of the MPHf solution (point 1 of Theorem 3.1), thus we only detail this part. In particular, we detail the structure of the error term that we cut in different parameters with different roles.

Given an initial ansatz which satisfies the assumptions 1.1 the five main steps of the construction are :

- (i) To construct an approximate solution based on the initial ansatz via geometric optics.
- (ii) To provide well prepared initial data for the error parameters.
- (iii) To construct an exact solution to KGML based on the previous approximation with the precise error term. The different parameters of the error term are given by a wave/transport system.
- (iv) To show that the Lorenz gauge propagates and to recover a solution to KGM.
- (v) To show that the time of existence of solutions is independent of  $\lambda < \lambda_0$  by a bootstrap argument.

We do not detail here the construction of the approximate solution as it is straightforward once we have the initial ansatz and a common problem, see section 4. We start with a discussion on the hyperbolic system of equations KGML, we do not consider yet the problem of the gauge.

The structure of both equations in Lorenz gauge is captured by this schematic equation for  $\mathbf{F}_\lambda$ ,

$$\square \mathbf{F}_\lambda = \mathbf{F}_\lambda \partial \mathbf{F}_\lambda + (\mathbf{F}_\lambda)^3, \quad (23)$$

with

$$\mathbf{F}_\lambda = \mathbf{F}_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\mathcal{A}} + \mathbf{Z}_\lambda, \quad (24)$$

where  $\mathbf{Z}_\lambda$  is the **error term** and the other terms form a **first order expansion**,  $\mathbf{F}_0$  and  $\mathbf{F}_{\mathcal{A}}$  are the **background** of section 4. The background **does not depend on**  $\lambda$ , it needs to be solution to a system of equations given in 4.1 to have a good approximate solution (in the sense given in definition 3.1) and to be able to construct the exact family of solutions.

In this paper we only need an almost approximate solution 3.1 of order 1 to build the error term and have exact solutions. In particular, the order 1 is lower than what can be allowed in [10] to construct an exact solution for only one phase, see section 7.2. Searching the error at a low order does not give a lot of smallness. It is harder to improve the estimates in a bootstrap argument. For example, at order 1 we do not have pointwise uniform bounds in  $\lambda$  on compact sets. The error term is too big in the Sobolev norms<sup>20</sup>. Meanwhile, searching it at a higher order brings more difficulties with the interaction of phases. The two classes of interaction are dealt here with very different methods (treating the d'Alembertian as an elliptic operator when there is no resonance and reintroducing a parameter with smaller amplitude when there is). Implementing those methods at higher order approximations would require a more general profile for the expansion and more conditions on the initial ansatz.

In general, we want some norms to be small in  $\lambda$  but big enough in terms of control of the regularity of  $\mathbf{Z}_\lambda$  to control the RHS of its evolution equation. With a classical approximate solution of order 1 we must have

$$\square \mathbf{Z}_\lambda = O(\lambda^{1/2}) + [\dots], \quad (25)$$

where the  $O(\lambda^{1/2})$  comes from the background and where the  $[\dots]$  contain terms involving  $\mathbf{Z}_\lambda$ . We could think of having heuristically<sup>21</sup>

$$\sum_{k \leq 1} (\lambda^k \|\mathbf{Z}_\lambda(t)\|_{H^{k+1}} + \lambda^k \|\partial_t \mathbf{Z}_\lambda(t)\|_{H^k}) + \lambda \|\partial_{tt}^2 \mathbf{Z}_\lambda(t)\|_{L^2} \sim O(\lambda^{1/2}),$$

with natural energy estimates. Thus, be in "good enough norms" in terms of control and smallness to close the bootstrap, for example with lemmas 8.17 and 8.18.

There are obstacles to that which arise from the interaction of phases and the  $\mathbf{F}_\lambda \partial \mathbf{F}_\lambda$  term on the RHS. The problems they pose are solved with the precise structure of the error term

$$\mathbf{Z}_\lambda = \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \mathbf{F}_{\lambda \mathcal{A}}^+ e^{i \frac{u_{\mathcal{A}}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+ e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} + \mathcal{E}_\lambda^{ell} + \mathcal{E}_\lambda^{evo}. \quad (26)$$

Let us detail those problems

- 1) We do not have an approximate solution in the usual sense, i.e., there are high frequency interaction terms in  $O(1)$ .
- 2) The term  $\mathbf{F}_\lambda \partial \mathbf{F}_\lambda$  contains schematically  $\mathbf{Z}_\lambda \lambda^{-1/2}$ . This is also a problem for the mono phase case. It is treated in [10] with product estimates for higher order approximation only.
- 3) There is a loss of derivative due to the  $\square \mathbf{F}_{\lambda \mathcal{A}}^+$  term appearing in the equation.
- 4) The term  $\mathbf{F}_\lambda \partial \mathbf{F}_\lambda$  carries  $\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}$  that cannot be treated in terms of smallness with Sobolev product estimates and embeddings.
- 5) The term  $\mathbf{F}_\lambda \partial \mathbf{F}_\lambda$  carries  $i \mathcal{E}_\lambda^{evo} \sum_{\mathcal{A} \in \mathbb{A}} \partial u_{\mathcal{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\lambda \mathcal{A}}^+$  that cannot be treated neither.

For 1) there are two cases<sup>22</sup> :

- A part of the  $O(1)$  interactions are oscillating with a characteristic phase<sup>23</sup> and are absorbed by  $\check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+$  (more details are given in section 6.1.2).

<sup>20</sup>In particular, the  $\mathcal{E}_\lambda^{evo}$  defined after.

<sup>21</sup>We ignore the fact that we need some weighted Sobolev norms.

<sup>22</sup>Due to the restriction on the phases possible interactions.

<sup>23</sup>Introducing a new parameter in  $\lambda$  also brings a  $\lambda^{-1}$  term via the d'Alembertian which is identically zero if and only if the phase is characteristic.

- The other part of the interaction, that are oscillating with a non-characteristic phase, are inverted by hand. Indeed, the d'Alembertian is an elliptic operator in the non characteristic direction. The defect of this inversion gives  $O(\lambda^1)$  terms that are harmless. This is done with  $\mathcal{E}_\lambda^{ell}$  in section 6.1.3.

The quantity  $\mathcal{E}_\lambda^{ell}$  is directly given by the background and  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$  is solution to the equation

$$\mathcal{L}_{\mathcal{A}\pm\mathcal{B}}\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+ = O(1) + [\dots], \quad (27)$$

which only depends on the background, the transport operator is defined in 2.3.

This equation is linear in  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$ . We consider these two **decoupled error parameters** as background terms in the sense that they only depend on it and their time of existence too.

For 2), we remark that the previously cited bad term is highly oscillating only with **characteristic** phases

$$\mathbf{F}_\lambda \partial \mathbf{F}_\lambda = \mathbf{Z}_\lambda \lambda^{-1/2} \sum_{\mathcal{A} \in \mathbb{A}} \partial u_{\mathcal{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\mathcal{A}} + [\dots]. \quad (28)$$

Thus, it can be absorbed by introducing again a new parameter,  $\mathbf{F}_{\lambda\mathcal{A}}^+$ , that also oscillates with these phases. In fact the transport equation for  $\mathbf{F}_{\lambda\mathcal{A}}^+$  is only coupled to  $\mathcal{E}_\lambda^{evo}$  because the other part of  $\mathbf{Z}_\lambda$  are already better with the splitting. See sections 6.1 and 6.3 for the construction.

Finally,  $\mathcal{E}_\lambda^{evo}$  absorbs all the remaining. In particular, all the high frequency interactions between the error parameters and the first order expansion  $\mathbf{F}_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\mathcal{A}}$  are above the minimum smallness required which is  $O(\lambda^{1/2})$ .

The quantities  $\mathcal{E}_\lambda^{evo}$  and  $\mathbf{F}_{\lambda\mathcal{A}}^+$  must satisfy a coupled system

$$\begin{cases} \square \mathcal{E}_\lambda^{evo} = \mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda - \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \square \mathbf{F}_{\lambda\mathcal{A}}^+ e^{i \frac{u_{\mathcal{A}}}{\lambda}} + [\dots], \\ \mathcal{L}_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ = \lambda^{-1/2} \partial u_{\mathcal{A}} \mathcal{E}_\lambda^{evo} \mathbf{F}_{\mathcal{A}} + [\dots]. \end{cases} \quad (29)$$

Now, we have to actually prove that this problem is well posed. Classical energy estimates for a wave-transport system give us

$$\partial \mathcal{E}_\lambda^{evo} \sim \partial \mathbf{F}_{\lambda\mathcal{A}}^+, \quad (30)$$

$$\mathbf{F}_{\lambda\mathcal{A}}^+ \sim \mathcal{E}_\lambda^{evo}, \quad (31)$$

in terms of derivatives.

This leads us to problem 3) on the loss of derivatives. We use an auxiliary variable to fix it. Indeed, knowing that  $\mathbf{F}_{\lambda\mathcal{A}}^+$  is solution to a transport equation, we can improve the regularity of  $\square \mathbf{F}_{\lambda\mathcal{A}}^+$  (if the initial data are well chosen, see section 5.2) by deriving a transport equation for  $\mathbf{G}_{\lambda\mathcal{A}}^+ := \square \mathbf{F}_{\lambda\mathcal{A}}^+$ . The auxiliary variable  $\mathbf{G}_{\lambda\mathcal{A}}^+$  satisfies better estimates than two standard derivatives of  $\mathbf{F}_{\lambda\mathcal{A}}^+$ . So, instead of searching a solution to (23), we search for a solution to

$$\begin{cases} \square \mathcal{E}_\lambda^{evo} = \mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda - \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \mathbf{G}_{\lambda\mathcal{A}}^+ e^{i \frac{u_{\mathcal{A}}}{\lambda}} + [\dots], \\ \mathcal{L}_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ = \lambda^{-1/2} \partial u_{\mathcal{A}} \mathcal{E}_\lambda^{evo} \mathbf{F}_{\mathcal{A}} + [\dots], \\ \mathcal{L}_{\mathcal{A}} \mathbf{G}_{\lambda\mathcal{A}}^+ = [\mathcal{L}_{\mathcal{A}}, \square] \mathbf{F}_{\lambda\mathcal{A}}^+ + \square(\mathcal{L}_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+). \end{cases} \quad (32)$$

The estimates on the commutator are given in section 8.2. The quantities  $\mathcal{E}_\lambda^{evo}$ ,  $\mathbf{F}_{\lambda\mathcal{A}}^+$  and  $\mathbf{G}_{\lambda\mathcal{A}}^+$  are the **coupled error parameters**.

The last term that needs a special care is  $\mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda$ , this brings us to problem 4) and 5) with the terms  $\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}$  and  $i \mathcal{E}_\lambda^{evo} \sum_{\mathcal{A} \in \mathbb{A}} \partial u_{\mathcal{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\lambda\mathcal{A}}^+$ .

For 4) we use the Strichartz estimates for wave equation, see section 6.3.4.

For 5) it is more complicated. It is a quadratic term of the form  $\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+$  that does not have any extra smallness usable with Strichartz estimates.<sup>24</sup>

<sup>24</sup>We can use the Strichartz estimates on  $\mathbf{F}_{\lambda\mathcal{A}}^+$  due to the control of  $\mathbf{G}_{\lambda\mathcal{A}}^+$ .

Nonetheless, it has a good structure as it oscillates with respect to characteristic phases. So, instead of letting it in the RHS of  $\square \mathcal{E}_\lambda^{evo}$  we can choose to divide it and place it in the transport equation for  $\mathbf{F}_{\lambda\mathcal{A}}^+$  as done for the first bad term 2).

- On the RHS of  $\square \mathcal{E}_\lambda^{evo}$  this term is lacking smallness (as it is nonlinear we need a smallness of  $O(\lambda^{1/2+\varepsilon})$  for some  $\varepsilon > 0$  to close the bootstrap, see b) of section 6.2) but is largely regular enough (it holds no derivatives and is only quadratic).
- On the RHS of  $\mathcal{L}_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+$  it has enough smallness but it is missing regularity. Indeed, because we have a transport equation for  $\square \mathbf{F}_{\lambda\mathcal{A}}^+$  by commuting the one of  $\mathbf{F}_{\lambda\mathcal{A}}^+$  with  $\square$ , we get the source term  $\partial \mathbf{F}_{\lambda\mathcal{A}}^+ \partial \mathcal{E}_\lambda^{evo}$  which is not in  $H^1$  (where  $\square \mathbf{F}_{\lambda\mathcal{A}}^+$  needs to be) since  $\partial \mathbf{F}_{\lambda\mathcal{A}}^+$  and  $\partial \mathcal{E}_\lambda^{evo}$  are in  $H^1$ .

We split the term with a projector on frequency

$$\Pi_- f(x) = \mathcal{F}^{-1}(\mathbf{1}_{|\xi| \leq \frac{1}{\lambda^\kappa}} \hat{f}(\xi))(x), \quad (33)$$

for a small  $\kappa > 0$  and  $\Pi_+ = 1 - \Pi_-$ .

We put the high frequency part in  $\square \mathcal{E}_\lambda^{evo}$ . With  $\kappa$  we recover the  $\lambda^\varepsilon$  missing, we exchange regularity for smallness. In the equation for  $\square \mathbf{F}_{\lambda\mathcal{A}}^+$ , we put the low frequency. We miss regularity but the price of a derivative is very low ( $\kappa$  can be taken as small as we want) so we can cover it with the natural smallness of  $\mathcal{E}_\lambda^{evo}$ , we exchange smallness for regularity. We give more details in the bootstrap part 6.3.5.

The full system of equations is given in section 6.1.1.

We remark that

$$\begin{aligned} \|\mathbf{Z}_\lambda\|_{L^2} &\sim O(\lambda^{1/2}), \\ \|\mathbf{Z}_\lambda\|_{H^1} &\sim O(1), \end{aligned}$$

and also that the error term is smaller in  $H^{1/2}$  than what it is using only the interpolation of the previous estimates, that is

$$\|\mathbf{Z}_\lambda\|_{H^{1/2}} \sim O(\lambda^{1/2}).$$

The error term seems not small enough to handle the estimates of the RHS of (23) (which contains cubic terms). But if taken with its good structure ( $e^{i\frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\lambda\mathcal{A}}^+$  and  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+ e^{i\frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}}$  carrying all the smallness defect) this is sufficient to estimates all the RHS terms of  $\square \mathcal{E}_\lambda^{evo}$ . In the theorems, we see that no smallness is required for the initial ansatz (and thus the background initial data), a small  $\lambda$  is sufficient to close the bootstrap.

The background is just one order more regular than the error term (and thus the full solution) which is mainly due to the  $\square \mathbf{F}_{\mathcal{A}}$  in  $O(\lambda)$  that is absorbed by the error term. The error term itself is at the lowest level possible in term of regularity to have a local well posedness result for a system of equations of the type of KGM in Lorenz gauge using the energy method. Well posedness in lower regularity class has been proved in [13] for KGM using Coulomb gauge and the structure of the equation.

This ends the discussion on finding an exact solution to KGML with a good structure for the bootstrap.

The previous arguments rely in fact on the choice of **well prepared error initial** data (which we give in the definition 5.9) and with the **required smallness** 5.12. By well prepared we morally mean with the right regularity and such that the initial data for the auxiliary function  $\mathbf{G}_{\lambda\mathcal{A}}^+$  make sense. Moreover, we want an exact solution to KGM. This requires an exact solution to KGML in Lorenz gauge, that is with

$$\partial_\alpha A_\lambda^\alpha = 0. \quad (34)$$

This also relies on the initial data, with definition 5.11, i.e., we need initial data that satisfy the **constraint equation for KGM in Lorenz gauge**. The gauge propagation is direct after that, see section 1.1.1.

Even with these conditions, for each  $\lambda > 0$  and each initial ansatz, there is still freedom of choice on the error initial data and thus on the solutions to the system. The constraints are underdetermined. We exhibit a set of generic error initial data fitting these three definitions. Each elements of the set leads to a 1-parameter family of MPHf solutions to KGM. To construct the set :

- We construct **initial data**  $\mathbf{z}_\lambda$  for the **error term**  $\mathbf{Z}_\lambda$  :
  - We choose the free part of the initial data for the error term with compact support and generic smallness properties, adding one more smallness assumption in proposition 5.14 to control the first time derivative of the gauge in definition 5.12.
  - We find a solution to the constraint equations for Maxwell in Lorenz gauge with respect to the initial ansatz and the previously chosen free initial data. In particular, solving the Maxwell constraints corresponds to inverting a perturbed Laplacian, which is done in some weighted Sobolev spaces defined in 2.1. The RHS of the equation corresponds to the RHS of the Maxwell evolution equation, we thus recover the same kind of problem of smallness as described above. Nonetheless, because the equation is not coupled and because we have an elliptic operator instead of a hyperbolic one everything is simplified.
- We cut the **error term initial data**  $\mathbf{z}_\lambda$  into initial data for the **error parameters** (see the **precise error term** (26)) with the right properties and we use the freedom of choice to deal with technical problems :
  - We express  $\mathcal{E}_\lambda^{ell}|_{t=0}$  with respect to the initial background.
  - We choose  $\mathbf{F}_{\lambda\mathcal{A}}^+|_{t=0} = 0$  and  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+|_{t=0} = 0$  to deal with the smallness of  $\mathcal{E}_\lambda^{evo}$  and the smallness and the regularity of  $\mathbf{G}_{\lambda\mathcal{A}}^+$ .
  - We set  $\mathcal{E}_\lambda^{evo}|_{t=0}$  with respect to the previous choices and  $\mathbf{z}_\lambda$  so that the initial data for the precise error term give a solution to the constraints.

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## 1.7 Outline of the paper

- In section 2 we give some notations and definitions on the functional spaces, on the phases and on the multi phase high frequency initial ansatz.
- In section 3 we set the two main theorems.
- In section 4 we derive the evolution equations for the background to have an almost approximate solution 3.1, the geometric optics approximation.
- In section 5 we give the details on the initial data for the error term.
- In section 6 we give the details on the error term, the existence of a local in time solution to KGML, the propagation of gauge and finally the uniform time of existence with the smallness results and the bootstrap.
- In section 7 we conclude and put everything together. We do some other general remarks
- Appendix

## 2 Notations and definitions

### 2.1 General notations

**Definition 2.1.** For  $m \in \mathbb{N}$ ,  $1 \leq p < +\infty$  and  $\delta \in \mathbb{R}$  we define the weighted Sobolev space  $W_{\delta,p}^m$  as the closure of  $C_0^\infty$  for the norm

$$\|u\|_{W_{\delta,p}^m} = \left( \sum_{i \leq m} \|(1 + |x|^2)^{\frac{\delta+i}{2}} \partial^i u\|_{L^p}^p \right)^{\frac{1}{p}},$$

with  $H_\delta^k = W_{2,\delta}^k$  and  $L_\delta^p = W_{p,\delta}^0$ . These are Banach spaces.

**Definition 2.2.** For  $m \in \mathbb{N}$  and  $\delta \in \mathbb{R}$  we define the weighted  $C_\delta^m$  as the Banach space of  $m$ -times continuously differentiable functions for the norm

$$\|u\|_{C_\delta^m} = \sum_{i \leq m} \sup_{x \in \mathbb{R}^3} |(1 + |x|^2)^{\frac{\delta+i}{2}} \partial^i u(x)|.$$

**Notation 2.3.** We write the transport operator  $2\partial^\alpha u_{\mathcal{A}} \partial_\alpha + \square u_{\mathcal{A}}$  as  $\mathcal{L}_{\mathcal{A}}$ .

**Notation 2.4.** We note by  $\text{Supp}(f)$  the support in space of a function of space. For a function of space and time, defined on the interval  $[0, t]$ , we abuse some notation by writing  $\text{Supp}(f) = U_{s \in [0, t]} \text{Supp}(f(s))$ .

**Notation 2.5.** For  $\Omega$  a compact set of  $\mathbb{R}^3$  we define  $\mathcal{I}(t, \Omega) := \{x \in \mathbb{R}^3 | \exists y \in \Omega, |x - y| \leq t\}$ .

## 2.2 Phases

### 2.2.1 Generalities

Here we define all the properties that concern the phases.

**Definition 2.6.** A phase  $u(x, t)$  is defined as a scalar function of space and time whose spacetime gradient  $\mathbf{d}u^\#$  is never 0, it has no critical point. In this paper, it is said to be **characteristic**<sup>25</sup> if it satisfies the so-called **eikonal** equation for a metric  $g$  associated to the wave operator

$$\partial^\alpha u \partial_\alpha u = g(\mathbf{d}u^\#, \mathbf{d}u^\#) = 0. \quad (35)$$

The phase is **isotrope** for the metric. For the Minkowski metric in Cartesian coordinates, it is equivalent to

$$|\nabla u|^2 - (\partial_t u)^2 = 0. \quad (36)$$

Moreover, the phase is future directed if

$$g(\mathbf{d}u^\#, \partial_t) < 0. \quad (37)$$

**Proposition 2.7.** If the phase  $u$  is characteristic for a metric  $g$  then its gradient  $\mathbf{d}^\# u$  satisfies the geodesic equation

$$g^{\alpha\beta} \partial_\beta u \partial_{\alpha\gamma}^2 u - \Gamma_{\gamma\alpha}^\beta \partial^\alpha u \partial_\beta u = 0. \quad (38)$$

*Proof.* We have

$$0 = \partial_\gamma (g^{\alpha\beta} \partial_\beta u \partial_\alpha u) = 2g^{\alpha\beta} \partial_\beta u \partial_{\alpha\gamma}^2 u - 2g^{\alpha\mu} \Gamma_{\mu\beta}^\gamma \partial_\alpha u \partial_\gamma u.$$

□

**Remark 2.8.** We wrote the equation for any metric  $g$  and for any system of coordinates. For the Minkowski metric in Cartesian coordinates, the Christoffel symbols are 0.

Now we only consider the Minkowski metric.

**Proposition 2.9.** From any  $(v, \dot{v}) \in (C^2(\mathbb{R}^3), C^1(\mathbb{R}^3))$  **eikonal initial data**, i.e., where

$$(\dot{v})^2 = |\nabla v|^2, \quad (39)$$

with  $\nabla v \neq 0$  and  $\dot{v} < 0$  we can construct a unique future directed characteristic phase  $u(t, x) \in C^2([0, T_v] \times \mathbb{R}^3)$  with initial data  $u|_{t=0} = v$  and  $\partial_t u|_{t=0} = \dot{v}$ , where  $T_v$  is maximal.

*Proof.* This is a classical result as we have a self transport equation. We use the representation formula

$$\mathbf{d}^\# u(\chi(y, t), t) = |\nabla v(y)|(1, \xi(y)), \quad (40)$$

where we set  $\xi = \frac{\nabla v}{|\nabla v|}$  and  $\chi(y, t) = y + t\xi(y)$ .

The solution exists as long as  $\chi(t)$  is a diffeomorphism from  $\mathbb{R}^3$  to itself. □

<sup>25</sup>We refer to the section on geometric optics for more details, this eikonal equation is associated to the d'Alembertian.

In this article, we consider only two types of interactions that contain all the plane waves possible interactions<sup>26</sup>. We consider either that two phases are fully collinear or that they are fully angularly separated.

**Definition 2.10.** Two phases  $u_{\mathcal{A}}$  and  $u_{\mathcal{B}}$  are called **fully angularly separated** (there is a non resonant interaction if the phases are characteristic) at time  $t$  if

$$\forall x \in \mathbb{R}^3 \frac{|\partial_i u_{\mathcal{A}} \partial^i u_{\mathcal{B}}|}{|\nabla u_{\mathcal{A}}| |\nabla u_{\mathcal{B}}|}(x, t) < 1.$$

**Definition 2.11.** Two phases  $u_{\mathcal{A}}$  and  $u_{\mathcal{B}}$  are called **fully collinear** (there is a resonant interaction if the phases are characteristic) at time  $t$  if their gradient are collinear at every point,

$$\forall x \in \mathbb{R}^3 \frac{|\partial_i u_{\mathcal{A}} \partial^i u_{\mathcal{B}}|}{|\nabla u_{\mathcal{A}}| |\nabla u_{\mathcal{B}}|}(x, t) = 1.$$

**Remark 2.12.** Without any mention of  $t$ , fully angularly separated and fully collinear mean that it is true for all  $t$  on the intersection of the interval of definition of the two phases.

**Proposition 2.13.** If  $u_{\mathcal{A}}$  and  $u_{\mathcal{B}}$  are fully angularly separated at 0 and characteristic then they are fully angularly separated for all  $t$ .

If  $u_{\mathcal{A}}$  and  $u_{\mathcal{B}}$  are fully collinear at 0 and characteristic then they are fully collinear for all  $t$ .

*Proof.* We suppose that the two phases exist on  $[0, T^*]$  and are fully collinear at time  $t = 0$  so that  $\exists k(x) > 0$  such that  $\forall x \in \mathbb{R}^3$

$$\begin{aligned} \nabla v_{\mathcal{A}}(x) &= k(x) \nabla v_{\mathcal{B}}(x) \\ \Leftrightarrow \nabla u_{\mathcal{A}}(x, 0) &= k(x) \nabla u_{\mathcal{B}}(x, 0) \\ \Leftrightarrow \forall t \in [0, T^*], \nabla u_{\mathcal{A}}(\chi_{\mathcal{A}}(x, t), t) &= k(x) \nabla u_{\mathcal{B}}(\chi_{\mathcal{A}}(x, t), t) \\ \Leftrightarrow \forall y \in \mathbb{R}^3, \forall t \in [0, T^*], \nabla u_{\mathcal{A}}(y, t) &= k(\chi_{\mathcal{A}}^{-1}(y, t)) \nabla u_{\mathcal{B}}(y, t), \end{aligned}$$

where we use the representation formula and its inverse for both phases. In particular, we see that we substitute  $\chi_{\mathcal{B}}(x, t)$  with  $\chi_{\mathcal{A}}(x, t)$  because  $\chi_{\mathcal{B}}(x, t)$  only depends on  $\xi_{\mathcal{B}} = \frac{\nabla v_{\mathcal{B}}}{|\nabla v_{\mathcal{B}}|} = \frac{\nabla v_{\mathcal{A}}}{|\nabla v_{\mathcal{A}}|} = \xi_{\mathcal{A}}$  and so  $\chi_{\mathcal{B}}(x, t) = \chi_{\mathcal{A}}(x, t)$ . We see that, at a time  $t$  two phases are collinear if and only if they were collinear at time 0. This is due to the constant speed of propagation. The reverse argument apply for fully angularly separated phases.  $\square$

## 2.2.2 Phase set

**Definition 2.14.** We call  $\{v_{\mathcal{A}}, \dot{v}_{\mathcal{A}} | \mathcal{A} \in \mathbb{A}\}$  an **initial phase set** where each  $v_{\mathcal{A}}$  and  $\dot{v}_{\mathcal{A}}$  are defined on  $\mathbb{R}^3$  with  $\nabla v \neq 0$ . To simplify, we forbid the existence of two phases such that  $\nabla v_{\mathcal{A}} = \nabla v_{\mathcal{B}}$ .

In the case where  $(v_{\mathcal{A}}, \dot{v}_{\mathcal{A}})_{\mathcal{A} \in \mathbb{A}}$  are **eikonal initial data**, we call  $\{u_{\mathcal{A}} | \mathcal{A} \in \mathbb{A}\}$  the associated **phase set** emerging naturally from the initial phase set. Each  $u_{\mathcal{A}}$  is a characteristic phase defined on  $[0, T_{\min}[\times \mathbb{R}^3$  for  $T_{\min}$  given by the intersection of all the intervals of existence. We set  $N = |\mathbb{A}|$  the number of phases and  $\mathcal{U}_{\mathbb{A}}$  the vector of size  $N$  containing each phase.

**Definition 2.15.** We call **angularly adapted** an initial phase set  $\{v_{\mathcal{A}}, \dot{v}_{\mathcal{A}} | \mathcal{A} \in \mathbb{A}\}$  (respectively a phase set  $\{u_{\mathcal{A}} | \mathcal{A} \in \mathbb{A}\}$ ) if for any  $(v_{\mathcal{A}}, v_{\mathcal{B}})$  (respectively  $(u_{\mathcal{A}}, u_{\mathcal{B}})$ ) with  $\mathcal{A}, \mathcal{B} \in \mathbb{A}$  the phases are either fully angularly separated or fully collinear.

**Definition 2.16.** For a given phase set  $\{u_{\mathcal{A}} | \mathcal{A} \in \mathbb{A}\}$ , let  $\mathcal{C} \subset \mathbb{A} \times \mathbb{A}$  be the set of couple of distinct phases that are fully collinear and  $\mathcal{S} \subset \mathbb{A} \times \mathbb{A}$  be the set of couple of phases that are fully angularly separated. In particular, if the phase set is angularly adapted then  $\mathcal{C} \cup \mathcal{S} \cup_{\mathcal{A} \in \mathbb{A}} (u_{\mathcal{A}}, u_{\mathcal{A}}) = \mathbb{A} \times \mathbb{A}$ .

If the phase set is angularly adapted with characteristic phases we get a good control of the smallness of important quantities.

<sup>26</sup>See the geometric optics part 1.3.

**Proposition 2.17.** For a given angularly adapted phase set  $\{u_{\mathcal{A}} | \mathcal{A} \in \mathbb{A}\}$  with characteristic phases, for any compact set  $\Omega \subset \mathbb{R}^3$  and any  $\tau > 0$  there exists  $\eta_0$  which depend on  $(\Omega, \tau)$  such that

$$\forall(x, t) \in \Omega \times [0, \tau], \quad \min_{\mathcal{A} \in \mathbb{A}} |\partial^0 u_{\mathcal{A}}(x, t)| > \eta_0, \quad (41)$$

$$\forall(x, t) \in \Omega \times [0, \tau], \quad \min_{\mathcal{A} \in \mathbb{A}} |\nabla u_{\mathcal{A}}(x, t)| > \eta_0, \quad (42)$$

$$\forall(x, t) \in \Omega \times [0, \tau], \quad \min_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}} |\partial_{\beta}(u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^{\beta}(u_{\mathcal{A}} \pm u_{\mathcal{B}})(x, t)| > \eta_0, \quad (43)$$

$$\forall(x, t) \in \Omega \times [0, \tau], \quad \min_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}} |\partial_i(u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^i(u_{\mathcal{A}} \pm u_{\mathcal{B}})(x, t)| > \eta_0, \quad (44)$$

$$\forall(x, t) \in \Omega \times [0, \tau], \quad \min_{(\mathcal{A}, \mathcal{B}) \in \mathcal{C}, \mathcal{A} \neq \mathcal{B}} |\partial_i(u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^i(u_{\mathcal{A}} \pm u_{\mathcal{B}})(x, t)| > \eta_0. \quad (45)$$

*Proof.* The first two lines come from the fact that  $\mathbf{d}^{\#} u_{\mathcal{A}}$  is never 0 and characteristic. The two last lines come from the fact that the phases are all different with no critical points (their space gradient is never 0).

Finally, the third line is a consequence of the following proposition. □

**Proposition 2.18.** For  $(\mathcal{A}, \mathcal{B}) \in \mathcal{S}$  the phase  $u_{\mathcal{A}} \pm u_{\mathcal{B}}$  is not characteristic.

*Proof.* We can write in coordinates  $\partial_0 u_{\mathcal{A}} = k \partial_0 u_{\mathcal{B}}$  and  $\partial_i u_{\mathcal{A}} = k \partial_i u_{\mathcal{B}} + l_i$  where  $l$  is a spacelike vector field and  $k > 0$  a function. We have

$$\partial_{\beta}(u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^{\beta}(u_{\mathcal{A}} \pm u_{\mathcal{B}}) = \pm 2 \partial_{\beta} u_{\mathcal{A}} \partial^{\beta} u_{\mathcal{B}}.$$

We suppose that this phase is characteristic, then

$$\begin{aligned} 0 &= \partial_{\alpha} u_{\mathcal{A}} \partial^{\alpha} u_{\mathcal{A}} = 2k \partial_i u_{\mathcal{B}} l^i + l^i l_i, \\ 0 &= \partial_{\alpha} u_{\mathcal{A}} \partial^{\alpha} u_{\mathcal{B}} = \partial_i u_{\mathcal{B}} l^i. \end{aligned}$$

By assumption, the phases are separated which implies  $l^i l_i > 0$  so that there is a contradiction between line 1 and 2. The new  $u_{\mathcal{A}} \pm u_{\mathcal{B}}$  phase cannot be characteristic. □

**Remark 2.19.** If the initial phase set is composed of sufficiently regular eikonal initial data 2.9 and is angularly adapted then its associated phase set is angularly adapted for all  $t \in [0, T_{\min}[$ . We recover the **strong coherence assumption**, the interactions are either everywhere characteristic or never characteristic.

### 2.3 Initial ansatz

We introduce some material to have a better understanding of the conditions on the initial data for the background and the initial ansatz.

**Definition 2.20.** We call a **background initial data set** the following set of functions  $(a_0^{\alpha}, \dot{a}_0^{\alpha}, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^{\alpha})$  for  $\mathcal{A} \in \mathbb{A}$ . It has its associated **initial phase set**  $\{v_{\mathcal{A}}, \dot{v}_{\mathcal{A}} | \mathcal{A} \in \mathbb{A}\}$ . We write  $p_{\mathcal{A}}^{\alpha} + i q_{\mathcal{A}}^{\alpha} = w_{\mathcal{A}}^{\alpha}$  for convenience.

**Definition 2.21.** For a given  $\lambda$ , we define the **initial ansatz** as

$$\begin{aligned} a_{1\lambda}^{\alpha} &= a_0^{\alpha} + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (p_{\mathcal{A}}^{\alpha} \cos(\frac{v_{\mathcal{A}}}{\lambda}) + q_{\mathcal{A}}^{\alpha} \sin(\frac{v_{\mathcal{A}}}{\lambda})), \\ \phi_{1\lambda} &= \phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (\psi_{\mathcal{A}} e^{i \frac{v_{\mathcal{A}}}{\lambda}}). \end{aligned}$$

We say that  $a_{1\lambda}^{\alpha}$  is the first order approximation of the electromagnetic vector potential and  $\phi_{1\lambda}$  is the first order approximation of the wave function. The function  $\phi_0, \phi_{1\lambda}, \psi_{\mathcal{A}}$  and  $w_{\mathcal{A}}^{\alpha}$  are complex functions.



**Definition 2.22.** We call the **background constraint equations** the following system for a given background initial data set

$$-\Delta a_0^0 - \partial_i(\dot{a}_0^i) = -Im(\phi_0 \overline{\dot{\phi}_0}) - a_0^0 |\phi_0|^2 + \sum_{\mathcal{A}} \dot{v}_{\mathcal{A}} |\psi_{\mathcal{A}}|^2, \quad (46)$$

$$\dot{a}_0^0 = -\partial_i a_0^i, \quad (47)$$

$$\forall \mathcal{A} \in \mathbb{A}, (\dot{v}_{\mathcal{A}})^2 = |\nabla v_{\mathcal{A}}|^2, \quad (48)$$

$$\forall \mathcal{A} \in \mathbb{A}, \partial_i v_{\mathcal{A}} w_{\mathcal{A}}^i + \dot{v}_{\mathcal{A}} w_{\mathcal{A}}^0 = 0. \quad (49)$$

**Remark 2.23.** The first three equations are the constraints for the initial data of a solution to the KGM-null dust system in Lorenz gauge whose flows are given by  $(|\nabla v_{\mathcal{A}}|, \nabla v_{\mathcal{A}})$  and are irrotational. In particular, the first equation is the Maxwell constraint, the second is the Lorenz gauge condition and the third is the eikonal equation. Finally, the fourth equation is a polarization condition. The first order term in the expansion is orthogonal to the direction of propagation, we discuss it in sections 1.1.2 and 1.3.

The construction of the full high frequency ansatz relies on a given **admissible** background initial data set.

**Definition 2.24.** A background initial data set  $(a_0^\alpha, \dot{a}_0^\alpha, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^\alpha)$  is said to be **admissible** if it satisfies the background constraint equations (46), (47), (48) and (49), if its initial phase set is angularly adapted (2.15) and if it has the following regularity

$$\begin{aligned} & \|\phi_0\|_{H^3} + \|a_0^i\|_{H^3} + \|\dot{a}_0^i\|_{H^2} + \|\dot{\phi}_0\|_{H^2} + \|a_0^0\|_{H_{\delta_0}^3} + \|\dot{a}_0^i\|_{H_{\delta_0}^2} \leq c'_0, \\ & \max_{\mathcal{A} \in \mathbb{A}} (\|\psi_{\mathcal{A}}\|_{H^3} + \|w_{\mathcal{A}}\|_{H^3} + \|v_{\mathcal{A}}\|_{H_{\delta_1}^5} + \|\dot{v}_{\mathcal{A}}\|_{H_{\delta_1+1}^4}) \leq c'_0, \end{aligned}$$

for<sup>27</sup>  $-3/2 < \delta_0$  and  $\delta_1 < -5/2$ . There is no need for  $c'_0 > 0$  to be small. Moreover, we impose that

$$Supp(a_0^i, \dot{a}_0^i, \phi_0, \dot{\phi}_0, \psi_{\mathcal{A}}, w_{\mathcal{A}}^\alpha) \subset B_{S'},$$

for some  $S' \in \mathbb{R}_+$ . Only the phases, their gradients,  $\dot{a}_0^0$  and  $a_0^0$  are not compactly supported, and in particular not in  $L^2$ . By Sobolev injection we see that  $v_{\mathcal{A}} \in C^2(\mathbb{R}^3)$ .

**Remark 2.25.** The weight  $-3/2 < \delta_0$  comes from the typical space on which the Laplacian in  $\mathbb{R}^3$  is inverted and the weight  $\delta_1 < -5/2$  allows the plane waves, that is,  $\nabla v$  is constant.

We are interested in finding exact solutions to Klein-Gordon Maxwell in Lorenz gauge based on the initial ansatz of definition 2.21. For that, we add an error which is defined with respect to the background initial data in section 5.

**Definition 2.26.** The **initial parametrix** has the shape

$$\begin{aligned} a_\lambda^\alpha &= a_{1\lambda}^\alpha + z_\lambda^\alpha, \\ \phi_\lambda &= \phi_{1\lambda} + \zeta_\lambda, \end{aligned}$$

where  $(z_\lambda^\alpha, \zeta_\lambda)$  are the **error initial data**.

### 3 Main results

Theorem 3.1 and 3.2 together imply that the KGM system is stable in the sense of geometric optics for multi phase high frequency ansatz. The approximate solution remains close to an exact solution for  $\lambda$  small enough.

<sup>27</sup>We defined the weighted Sobolev norms in 2.1.

### 3.1 Main theorems

**Theorem 3.1.** *Let  $(a_0^\alpha, \dot{a}_0^\alpha, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^\alpha)$  be an admissible background initial data set 2.24 and let  $-3/2 < \delta < -1/2$  with  $\delta \leq \delta_0$ . For  $0 < \lambda < \lambda_0$ , with  $\lambda_0$  depending on the background and  $\delta$  :*

1. *There exists one family of high frequency solutions to KGM in Lorenz gauge. We have  $(A_\lambda^i, \Phi_\lambda) \in (C^0([0, T], H^2) \cap C^1([0, T], H^1) \cap C^2([0, T], L^2))^2$  and  $A_\lambda^0 \in C^0([0, T], H_\delta^2) \cap C^1([0, T], H_\delta^1) \cap C^2([0, T], L_\delta^2)$  based on the initial ansatz 2.21 with the time  $T > 0$  independant of  $\lambda < \lambda_0$  and fixed by the background initial data set.*
2. *Such a solution can be written as the following **parametria***

$$A_\lambda^\alpha = A_0^\alpha + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} \left( \cos\left(\frac{u_{\mathcal{A}}}{\lambda}\right) P_{\mathcal{A}}^\alpha + \sin\left(\frac{u_{\mathcal{A}}}{\lambda}\right) Q_{\mathcal{A}}^\alpha \right) + Z_\lambda^\alpha,$$

$$\Phi_\lambda = \Phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}} + \mathcal{Z}_\lambda,$$

where  $(A_0^\beta, W_{\mathcal{A}}^\beta, \Phi_0, \Psi_{\mathcal{A}}^\beta, u_{\mathcal{A}})$  is the **background**<sup>28</sup> and where  $(Z_\lambda, \mathcal{Z}_\lambda)$  is the **error term**.

3. *The background is itself solution to a system close to KGM specified in proposition 4.1 and section 1.2. The quantities  $A_0$  and  $\Phi_0$  (to which the solution converges in the sense given below) are the electromagnetic potential and the wave function of this system.*
4. *We have the convergence*

$$\forall t \in [0, T], (A_\lambda^i, \Phi_\lambda)(t) \xrightarrow{L^2} (A_0^i, \Phi_0)(t), \quad A_\lambda^0(t) \xrightarrow{L_\delta^2} A_0^0(t),$$

$$\forall t \in [0, T], (\partial A_\lambda^i, \partial \Phi_\lambda)(t) \xrightarrow{L^2} (\partial A_0^i, \partial \Phi_0)(t), \quad \partial A_\lambda^0(t) \xrightarrow{L^2} \partial A_0^0(t).$$

5. *We do not have uniform convergence on compact set but we have for  $1 \leq p < \infty$*

$$(A_\lambda^\alpha, \Phi_\lambda) \xrightarrow{L^p([0, T], L^\infty)} (A_0^\alpha, \Phi_0).$$

The second theorem makes a link between [10] and this paper. See section 7.2.

**Theorem 3.2.** *For  $0 < \lambda < \lambda_0$ , there exists  $C > 0$  such that*

1. *The family of solutions given by the previous theorem 3.1 is uniformly bounded in  $H^{1/2}$ ,*

$$\forall \lambda < \lambda_0, \forall t \in [0, T], \|A_\lambda^0(t)\|_{H_\delta^{1/2}} + \|A_\lambda^i(t)\|_{H^{1/2}} + \|\Phi_\lambda(t)\|_{H^{1/2}} < C.$$

2. *In particular, the error term is small in that norm,*

$$\forall \lambda < \lambda_0, \forall t \in [0, T], \|Z_\lambda^0(t)\|_{H_\delta^{1/2}} + \|Z_\lambda^i(t)\|_{H^{1/2}} + \|\mathcal{Z}_\lambda(t)\|_{H^{1/2}} < \lambda^{1/2} C,$$

and more generally

$$\forall \lambda < \lambda_0, \forall s \in [0, 1/2], \forall t \in [0, T], \|Z_\lambda^0(t)\|_{H_\delta^{1-s}} + \|Z_\lambda^i(t)\|_{H^{1-s}} + \|\mathcal{Z}_\lambda(t)\|_{H^{1-s}} < \lambda^s C.$$

3. *The **first order WKB expansion***

$$A_{1\lambda}^\alpha = A_0^\alpha + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} \left( \cos\left(\frac{u_{\mathcal{A}}}{\lambda}\right) P_{\mathcal{A}}^\alpha + \sin\left(\frac{u_{\mathcal{A}}}{\lambda}\right) Q_{\mathcal{A}}^\alpha \right),$$

$$\Phi_{1\lambda} = \Phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}}$$

is an almost approximate solution of order 1 to KGM in Lorenz gauge in sense given below.

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<sup>28</sup>We write  $W_{\mathcal{A}}^\beta = P_{\mathcal{A}}^\beta + iQ_{\mathcal{A}}^\beta$  for convenience.

**Definition 3.1.** We say that  $(A_{m\lambda}, \Phi_{m\lambda})$  is an **almost approximate solution** of order  $m$  in  $\lambda$  to KGM in Lorenz gauge if

$$\begin{cases} \square A_{m\lambda}^\beta + \text{Im}(\overline{\phi_{m\lambda}(\partial^\beta + iA_{m\lambda}^\beta)}\phi_{m\lambda}) = \lambda^{m/2}\Xi_{\lambda A}^\beta + \lambda^{m/2-1/2}\tilde{\Xi}_{\lambda A}^\beta, \\ (\partial^\alpha + iA_{m\lambda}^\alpha)(\partial_\alpha + iA_{\alpha m\lambda})\phi_{m\lambda} = \lambda^{m/2}\Xi_{\lambda\Phi} + \lambda^{m/2-1/2}\tilde{\Xi}_{\lambda\Phi}, \\ \partial_\alpha A_{m\lambda}^\alpha = \lambda^{m/2}\Xi_{\lambda L} \end{cases} \quad (50)$$

We set the  $\Xi_\lambda$  terms to be in  $O(1)$  and the  $\tilde{\Xi}_\lambda$  to be a sum of high frequency terms :

$$\lambda^{m-1/2} \sum_{N' \leq i \geq 0} e^{i\frac{u_i}{\lambda}} \nu_i,$$

for some finite number of phases  $(u_i)_{i \in [0, N']}$  that are either everywhere characteristic or nowhere characteristic and for some functions  $\nu_i$ .

**Remark 3.2.** In the monophasic case, the author of [10] uses the same type of definition for approximate solution (a sub case of this definition as there is only harmonics of the same characteristic phase) to build his exact solution on.

## 4 First order approximation

### 4.1 Evolution equations for the background

In this section we construct the first order approximation based on the initial ansatz to have an almost approximate solution. The first order approximation is made of the **background** terms which **do not depend on  $\lambda$** .

**Proposition 4.1.** For a given admissible background initial data set 2.24 :

1. There exists a **background**  $(A_0^\alpha, \Phi_0, u_{\mathcal{A}}, \Psi_{\mathcal{A}}, W_{\mathcal{A}}^\alpha)$  and a maximal time  $T$  such that it is solution to

$$\begin{cases} \square A_0^\beta = -\text{Im}(\overline{\Phi_0 \partial^\beta \Phi_0}) + A_0^\beta |\Phi_0|^2 + \sum_{\mathcal{A}} \partial^\beta u_{\mathcal{A}} |\Psi_{\mathcal{A}}|^2, \\ \square \Phi_0 + 2iA_0^\alpha \partial_\alpha \Phi_0 - A_0^\alpha A_{0\alpha} \Phi_0 = 0, \\ \mathcal{L}_{\mathcal{A}} \Psi_{\mathcal{A}} = -i2A_0^\alpha \partial_\alpha u_{\mathcal{A}} \Psi_{\mathcal{A}}, \\ \partial^\beta u_{\mathcal{A}} \partial_\beta u_{\mathcal{A}} = 0, \\ \mathcal{L}_{\mathcal{A}} W_{\mathcal{A}}^\beta = i\partial^\beta u_{\mathcal{A}} \overline{\Psi_{\mathcal{A}}} \Phi_0, \end{cases} \quad (51)$$

with initial data

$$\begin{aligned} (A_0^\alpha|_{t=0} = a_0, \partial_t A_0^\alpha|_{t=0} = \dot{a}_0, \Phi_0|_{t=0} = \phi_0, \partial_t \Phi_0|_{t=0} = \dot{\phi}_0), \\ (u_{\mathcal{A}}|_{t=0} = v_{\mathcal{A}}, \partial_t u_{\mathcal{A}}|_{t=0} = \dot{v}_{\mathcal{A}}, \Psi_{\mathcal{A}}|_{t=0} = \psi_{\mathcal{A}}, W_{\mathcal{A}}^\alpha|_{t=0} = w_{\mathcal{A}}^\alpha), \end{aligned} \quad (52)$$

and with the regularity

$$\begin{aligned} \sum_{j=0}^3 (\|\Phi_0\|_{C^j([0, T], H^{3-j})} + \|A_0^i\|_{C^j([0, T], H^{3-j})} + \|A_0^0\|_{C^j([0, T], H_{\delta_0}^{3-j})}) \leq c_0, \\ \max_{\mathcal{A} \in \mathbb{A}} \left( \sum_{j=0}^3 (\|\Psi_{\mathcal{A}}\|_{C^j([0, T], H^{3-j})} + \|W_{\mathcal{A}}\|_{C^j([0, T], H^{3-j})}) \right) \leq c_0, \\ \max_{\mathcal{A} \in \mathbb{A}} \sum_{j=0}^5 \|u_{\mathcal{A}}\|_{C^j([0, T], H_{\delta_1+j}^{5-j})} \leq c_0, \end{aligned}$$

where  $c_0$  only depends on  $c'_0$ . We observe that by finite speed of propagation we get

$$\text{Supp}(A_0^i, \Phi_0, \Psi_{\mathcal{A}}, W_{\mathcal{A}}^\alpha) \subset \mathcal{I}(T, B_{S'}) \subset B_S,$$

for  $S = S' + T$  and also that only the phases and  $A_0^0$  are not compactly supported, and in particular not in  $L^2$ .

2. We have the **polarization**

$$\partial_\beta u_{\mathcal{A}} W_{\mathcal{A}}^\beta = 0, \quad (53)$$

and the **Lorenz gauge** condition

$$\partial_\alpha A_0^\alpha = 0. \quad (54)$$

3. Moreover,  $(A_{1\lambda}, \Phi_{1\lambda})$  defined as the first order expansion

$$A_{1\lambda}^\alpha = A_0^\alpha + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (P_{\mathcal{A}}^\alpha \cos(\frac{u_{\mathcal{A}}}{\lambda}) + Q_{\mathcal{A}}^\alpha \sin(\frac{u_{\mathcal{A}}}{\lambda})), \quad \Phi_{1\lambda} = \Phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (\Psi_{\mathcal{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}}) \quad (55)$$

is an almost approximate solution of order 1 to KGM in Lorenz gauge on this time interval.

**Remark 4.2.** We see that the system (51) has a triangular structure. The phases can be determined first with proposition 2.9, then the first 3 equations form a coupled system. As noted in section 1.2, these 4 equations together are equivalent to KGML null-transport (12) with irrotational flows given by the phase gradient  $\mathbf{d}^\# u_{\mathcal{A}}$  and the corresponding charge density  $|\Psi_{\mathcal{A}}|$ . Indeed,  $u_{\mathcal{A}}$  is a characteristic phase so its spacetime gradient is a null vector field and obeys the geodesic equation. We have

$$\mathcal{L}_{\mathcal{A}} |\Psi_{\mathcal{A}}| = \frac{\Psi_{\mathcal{A}}}{2|\Psi_{\mathcal{A}}|} (\mathcal{L}_{\mathcal{A}} \overline{\Psi_{\mathcal{A}}}) + \frac{\overline{\Psi_{\mathcal{A}}}}{2|\Psi_{\mathcal{A}}|} (\mathcal{L}_{\mathcal{A}} \Psi_{\mathcal{A}}) = \frac{1}{2} (2iA_0^\alpha \partial_\alpha u_{\mathcal{A}}) - \frac{1}{2} (2iA_0^\alpha \partial_\alpha u_{\mathcal{A}}) = 0.$$

The interaction of the phases with themselves produces a backreaction, an  $O(1)$  non oscillating term that we interpret as a charge density flux. Finally, the equation for  $W_{\mathcal{A}}^\beta$  is decoupled and propagates the polarization with the help of the geodesic equation for  $\mathbf{d}^\# u_{\mathcal{A}}$ .

**Remark 4.3.** In the case where  $\Psi_{\mathcal{A}}|_{t=0}$  is equal to zero then it remains 0 (propagation of charge density) and so the  $(A_0, \Phi_0)$  are solution of KGM in Lorenz gauge.

**Remark 4.4.** Due to the polarization condition there is no backreaction in the Klein-Gordon equation.

**Notation 4.5.** We note by  $C_0 = C_0(c_0, \frac{1}{\eta_0}, N, T)$  any constants that are a polynomial in  $c_0, \frac{1}{\eta_0}, N = |\mathbb{A}|$  and  $T$ .

*Proof of 1 of proposition 4.1.* Following remark 4.2, we first determine the phases. For that we use the geodesic and the method of characteristics as explained in proposition 2.9. We have sufficient regularity for proposition 2.9. Then, we can use classical local existence results from energy estimates for a wave/transport system, here the first three equations and the last equation. The regularity propagates on the time of existence.  $\square$

*Proof of 2 of proposition 4.1.* The initial data are admissible 2.24 so we have the polarization condition and the Lorenz gauge condition at  $t = 0$ .

To recover the propagation of the polarization condition, we contract equation 5 with the vector field  $\mathbf{d}^\# u_{\mathcal{A}}$  and we use the geodesic equation.

For the propagation of the gauge, we use the argument given in section 1.1.1. We take the divergence of the Maxwell equation and we get a linear wave equation for the divergence of the potential with 0 initial data.  $\square$

## 4.2 Almost approximate solution

Now we want to prove that the solution to (51) put under the form (55) is an almost approximate solution of order 1 to KGM.

*Proof of 3 of proposition 4.1.* The polarization condition (53) and the Lorenz gauge (56) for  $A_0$  imply that

$$\partial_\alpha A_{1\lambda}^\alpha = \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (\partial_\alpha P_{\mathcal{A}}^\alpha \cos(\frac{u_{\mathcal{A}}}{\lambda}) + \partial_\alpha Q_{\mathcal{A}}^\alpha \sin(\frac{u_{\mathcal{A}}}{\lambda})).$$

We just proved that

$$\partial_\alpha A_{1\lambda}^\alpha = O(\lambda^{1/2}). \quad (56)$$

We plug the truncated WKB expansions (55) in KGML (8), we get

$$\begin{aligned} \square A_0^\beta &= -\underbrace{Im(\Phi_0 \overline{\partial^\beta \Phi_0}) + A_0^\beta |\Phi|^2 + \sum_{\mathcal{A}} |\Psi_{\mathcal{A}}|^2 \partial^\beta u_{\mathcal{A}}}_{= \square A_0^\beta} \\ &+ \lambda^{-1/2} \sum_{\mathcal{A}} \sin\left(\frac{u_{\mathcal{A}}}{\lambda}\right) \underbrace{(\square u_{\mathcal{A}} P_{\mathcal{A}}^\beta + 2\partial^\alpha u_{\mathcal{A}} \partial_\alpha P_{\mathcal{A}}^\beta + \partial^\beta u_{\mathcal{A}} Im(\Phi_0 \overline{\Psi_{\mathcal{A}}}))}_{=0} \\ &- \lambda^{-1/2} \sum_{\mathcal{A}} \cos\left(\frac{u_{\mathcal{A}}}{\lambda}\right) \underbrace{(\square u_{\mathcal{A}} Q_{\mathcal{A}}^\beta + 2\partial^\alpha u_{\mathcal{A}} \partial_\alpha Q_{\mathcal{A}}^\beta - \partial^\beta u_{\mathcal{A}} Re(\Phi_0 \overline{\Psi_{\mathcal{A}}}))}_{=0} \\ &+ \lambda^{-3/2} \sum_{\mathcal{A}} \underbrace{\partial^\alpha u_{\mathcal{A}} \partial_\alpha u_{\mathcal{A}}}_{=0} \left( \cos\left(\frac{u_{\mathcal{A}}}{\lambda}\right) P_{\mathcal{A}}^\beta + \sin\left(\frac{u_{\mathcal{A}}}{\lambda}\right) Q_{\mathcal{A}}^\beta \right) \\ &+ \underbrace{\sum_{\mathcal{A}, \mathcal{B} \in \mathbb{A}, \mathcal{A} \neq \mathcal{B}} \partial^\beta u_{\mathcal{B}} Im(e^{i\frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}} \overline{(ie^{i\frac{u_{\mathcal{B}}}{\lambda}} \Psi_{\mathcal{B}})})}_{= \tilde{\Xi}_{\lambda A}} + \lambda^{1/2} \Xi_{\lambda A}^\beta \end{aligned} \quad (57)$$

and

$$\begin{aligned} \square \Phi_0 &= -\underbrace{2iA_0^\alpha \partial_\alpha \Phi_0 + A_0^\alpha A_{\alpha 0} \Phi_0}_{= \square \Phi_0} \\ &- \sum_{\mathcal{A}} \lambda^{-1/2} i e^{i\frac{u_{\mathcal{A}}}{\lambda}} \underbrace{(\square u_{\mathcal{A}} \Psi_{\mathcal{A}} + 2\partial^\alpha u_{\mathcal{A}} \partial_\alpha \Psi_{\mathcal{A}} + 2iA_0^\alpha \partial_\alpha u_{\mathcal{A}} \Psi_{\mathcal{A}})}_{=0} \\ &+ \lambda^{-3/2} \sum_{\mathcal{A}} \underbrace{\partial^\alpha u_{\mathcal{A}} \partial_\alpha u_{\mathcal{A}}}_{=0} e^{i\frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}} \\ &- \underbrace{\sum_{\mathcal{A}, \mathcal{B} \in \mathbb{A}, \mathcal{A} \neq \mathcal{B}} i2 \left( \cos\left(\frac{u_{\mathcal{A}}}{\lambda}\right) P_{\mathcal{A}}^\alpha + \sin\left(\frac{u_{\mathcal{A}}}{\lambda}\right) Q_{\mathcal{A}}^\alpha \right) \partial_\alpha u_{\mathcal{B}} i e^{i\frac{u_{\mathcal{B}}}{\lambda}} \Psi_{\mathcal{B}} + \lambda^{1/2} \Xi_{\lambda \Phi}}_{= \tilde{\Xi}_{\lambda \Phi}}. \end{aligned} \quad (58)$$

The two first equations of (51) cancel the  $O(1)$  non oscillating terms, including the backreaction. Equations 3 and 5 of (51) cancel the  $O(\lambda^{-1/2})$  terms respectively in the equation for Klein-Gordon and Maxwell. Then, with the eikonal equation, we get rid of the  $O(\lambda^{-3/2})$  terms that come from the d'Alembertian. The remaining terms are in  $O(1)$ , composed of the high frequency interactions, or in  $O(\lambda^{1/2})$ . We clearly see that the latter type corresponds to the  $\Xi_{\lambda \Phi}$  and  $\Xi_{\lambda A}$  and we prove in the next proposition that the former corresponds to the  $\tilde{\Xi}_{\lambda \Phi}$  and  $\tilde{\Xi}_{\lambda A}$  of definition 3.1.  $\square$

**Proposition 4.6.** *The  $O(1)$  high frequency interaction terms  $\tilde{\Xi}_{\lambda A}$  of (57) and  $\tilde{\Xi}_{\lambda \Phi}$  of (58) can be written and separated as follows :*

$$\tilde{\Xi}_{\lambda A}^\beta = \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{C}, \mathcal{A} < \mathcal{B}} Re(\overline{K_{A(\mathcal{A} \pm \mathcal{B})}^\beta(\mathbf{B}'_0)} e^{i\frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}}) + \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} Re(\overline{S_{A(\mathcal{A} \pm \mathcal{B})}^\beta(\mathbf{B}'_0)} e^{i\frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}}), \quad (59)$$

for Maxwell and

$$\tilde{\Xi}_{\lambda \Phi} = \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{C}, \mathcal{A} < \mathcal{B}} e^{i\frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} K_{\phi(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0) + \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i\frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} S_{\Phi(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0), \quad (60)$$

for Klein-Gordon and thus are the form of the  $\tilde{\Xi}$ s terms of definition 3.1, i.e., high frequency terms oscillating with phases that are either always characteristic or never characteristic. The  $<$  denotes whatever total order on  $\mathbb{A}$ . The  $K$ s correspond to terms emerging from collinear interactions and the  $S$ s to the ones emerging from separated interactions.

**Notation 4.7.** We use the schematic notation for both

$$\tilde{\Xi}_\lambda = \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{C}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{B}} \pm u_{\mathcal{A}}}{\lambda}} K_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0) + \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{B}} \pm u_{\mathcal{A}}}{\lambda}} S_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0).$$

*Proof of proposition 4.6.* By direct calculation using trigonometric formulas. This ends the proof of propositions 4.6 and 4.1.  $\square$

We give the properties of the interaction terms.

**Proposition 4.8.** For  $K_{\mathcal{A} \pm \mathcal{B}}$  and  $S_{\mathcal{A} \pm \mathcal{B}}$  defined as in proposition 4.6 :

1. We have the following estimates

$$\begin{aligned} \forall t \in [0, T] \sum_{k \leq 3} \|K_{\mathcal{A} \pm \mathcal{B}}(t)\|_{H^k} dt &\leq C_0, \\ \forall t \in [0, T] \sum_{k \leq 3} \|S_{\mathcal{A} \pm \mathcal{B}}(t)\|_{H^k} dt &\leq C_0. \end{aligned}$$

2. The quantity  $K_{A(\mathcal{A} \pm \mathcal{B})}^\beta$  is orthogonal to  $\partial^\beta u_{\mathcal{A}} \pm \partial^\beta u_{\mathcal{B}}$ .

*Proof of 1 of proposition 4.8.* By direct calculations as  $H^3$  is an algebra for  $n = 3$  and as all the terms in the definition are at least in  $H^3$ .  $\square$

*Proof of 2 of proposition 4.8.* From proposition 4.6 and the definition of  $\tilde{\Xi}_{\lambda A}$  in (57) we get that  $\forall (\mathcal{A}, \mathcal{B}) \in \mathcal{C}, \mathcal{A} < \mathcal{B}$

$$\begin{aligned} &\partial^\beta u_{\mathcal{B}} \text{Im}(e^{i \frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}}(\overline{ie^{i \frac{u_{\mathcal{B}}}{\lambda}} \Psi_{\mathcal{B}}})) + \partial^\beta u_{\mathcal{A}} \text{Im}(e^{i \frac{u_{\mathcal{B}}}{\lambda}} \Psi_{\mathcal{B}}(\overline{ie^{i \frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}}})) \\ &= -(\partial^\beta u_{\mathcal{B}} + \partial^\beta u_{\mathcal{A}}) \text{Re}(2e^{i \frac{u_{\mathcal{A}} - u_{\mathcal{B}}}{\lambda}} \Psi_{\mathcal{A}} \overline{\Psi_{\mathcal{B}}}) \\ &= \text{Re}(e^{i \frac{u_{\mathcal{A}} - u_{\mathcal{B}}}{\lambda}} K_{A(\mathcal{A} - \mathcal{B})}^\beta(\mathbf{B}'_0)). \end{aligned}$$

We clearly have that  $K_{A(\mathcal{A} - \mathcal{B})}^\beta(\mathbf{B}'_0)$  is orthogonal to  $\partial^\alpha u_{\mathcal{A}} - \partial^\alpha u_{\mathcal{B}}$  (the phases are collinear and characteristic) and  $K_{A(\mathcal{A} + \mathcal{B})}^\beta(\mathbf{B}'_0) = 0$ .  $\square$

**Remark 4.9.** With the definition of an almost approximate solution 3.1 and the fact that the  $\Xi_L$  term is high frequency the Maxwell equation is a bit worse than the rest of the system. Indeed, the Maxwell equation comes from the divergence of the Faraday tensor  $F_{1\lambda}^{\alpha\beta} = \partial^\alpha A_{1\lambda}^\beta - \partial^\beta A_{1\lambda}^\alpha$  which includes one derivative of  $\partial_\alpha A_{1\lambda}^\alpha$ . We showed that we have

$$\square A_{1\lambda}^\beta + \text{Im}(\overline{\phi_{1\lambda}(\partial^\beta + iA_{1\lambda}^\beta)\phi_{1\lambda}}) = \lambda^{1/2} \Xi_{\lambda A}^\beta(\lambda) + \tilde{\Xi}_{\lambda A}^\beta(\lambda)$$

and so the Maxwell equation really is

$$\partial_\alpha F_{1\lambda}^{\alpha\beta} + \text{Im}(\overline{\phi_{1\lambda}(\partial^\beta + iA_{1\lambda}^\beta)\phi_{1\lambda}}) = \lambda^{1/2} \Xi_{\lambda A}^\beta + \lambda^0 \tilde{\Xi}_{\lambda A}^\beta - \lambda^{1/2} \partial^\beta \Xi_L = O(\lambda^{-1/2}).$$

It is in particular relevant because the constraint equation has this RHS and we have to solve it to have an exact solution. However, because all the terms contributing to the  $\lambda^{-1/2}$  are high frequency and the operator of the constraint is elliptic we can recover the mandatory smallness.

In general, we can add a parameter at order  $\lambda^{3/2}$  in the expansion whose polarization absorbs the problematic term. It is done in [10] for example. This term would be of the size of the error term in our case so we do it directly with the error term instead.

## 5 Initial data for exact solutions

Here we give all the details and list all the requirements on the **initial data** for the **error term** to prove the theorems 3.1 and 3.2. In particular, we search initial data for the **precise error term** (26). Indeed, as explain in the introduction 1.5 and in section 6.1, we describe the evolution of the error term with a system of coupled equations (78) to have the precise structure to close. This requires to prescribe initial data to all the different **error parameters** under certain **admissibility conditions** given in section 5.2.

In sections 5.3 and 5.4, we exhibit for any  $\lambda > 0$  a **generic set** of initial data fulfilling these requirements, see proposition 5.16. To construct it, we first search for small solutions to the constraint equations for KGM in Lorenz gauge (63) and then we separate them in the different initial data for the error parameters taking care of the different properties needed.

Because we have more variables than we have constraint equations we have some freedom of choice that we will exploit the less possible in that specific construction.

Before that we give the definition of the constraint equations and the admissibility criteria.

### 5.1 Equations of constraint for KGM in Lorenz gauge

The **initial data set for Klein-Gordon Maxwell** is composed of  $(a_\lambda^\alpha, \dot{a}_\lambda^\alpha, \phi_\lambda, \dot{\phi}_\lambda)$ .

In the case of high frequency solution, we expand it as the **initial parametrix**

$$\begin{aligned} a_\lambda^\alpha &= a_{1\lambda}^\alpha + z_\lambda^\alpha, & \dot{a}_\lambda^\alpha &= \dot{a}_{1\lambda}^\alpha + \dot{z}_\lambda^\alpha, \\ \phi_\lambda &= \phi_{1\lambda} + \zeta_\lambda, & \dot{\phi}_\lambda &= \dot{\phi}_{1\lambda} + \dot{\zeta}_\lambda, \end{aligned} \quad (61)$$

where  $(z_\lambda, \zeta_\lambda, \dot{z}_\lambda, \dot{\zeta}_\lambda)$  is the **error initial data** for the **error term**  $(Z_\lambda, \mathcal{Z}_\lambda)$ .

**Definition 5.1.** For a given admissible background initial data set  $(a_0^\alpha, \dot{a}_0^\alpha, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^\alpha)$  we define the initial ansatz  $(a_{1\lambda}^\alpha, \phi_{1\lambda})$  directly with the background as in definition 2.21.

We define  $(\dot{a}_{1\lambda}^\alpha, \dot{\phi}_{1\lambda})$  as  $((\partial_t A_{1\lambda}^\alpha)|_{t=0}, (\partial_t \Phi_{1\lambda})|_{t=0})$  where  $(A_{1\lambda}^\alpha, \Phi_{1\lambda})$  are almost approximate solution to KGM and are given using proposition 4.1. The quantities  $((\partial_t A_{1\lambda}^\alpha)|_{t=0}, (\partial_t \Phi_{1\lambda})|_{t=0})$  can be expressed only in terms of the background initial data and their space derivatives.

**Remark 5.2.** We recall that the constraints for  $(a_\lambda^\alpha, \dot{a}_\lambda^\alpha, \phi_\lambda, \dot{\phi}_\lambda)$  to Klein-Gordon-Maxwell in Lorenz gauge are

$$\begin{cases} \dot{a}_\lambda^0 = -\partial_i a_\lambda^i, \\ -\Delta a_\lambda^0 - \partial_i (\dot{a}_\lambda^i) = -\text{Im}(\phi_\lambda \overline{\dot{\phi}_\lambda}) - a_\lambda^0 |\phi_\lambda|^2. \end{cases} \quad (62)$$

**Remark 5.3.** If we fix  $(a_{1\lambda}^\alpha, \phi_{1\lambda}, \dot{a}_{1\lambda}^\alpha, \dot{\phi}_{1\lambda})$  with definition 5.1, it remains to determine the **error initial data**  $(z_\lambda, \zeta_\lambda, \dot{z}_\lambda, \dot{\zeta}_\lambda)$  to have initial data for the exact solution  $(a_\lambda^\alpha, \dot{a}_\lambda^\alpha, \phi_\lambda, \dot{\phi}_\lambda)$ . That is 10 variables (as  $z_\lambda$  and  $\dot{z}_\lambda$  are four-vectors). There are only 2 constraint equations so 8 variables are free.

The next proposition gives the constraints and the regularity for the **error initial data**. We solve it in section 5.3.

**Proposition 5.4.** For a given admissible background initial data set  $(a_0^\alpha, \dot{a}_0^\alpha, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^\alpha)$ , let  $(a_{1\lambda}^\alpha, \dot{a}_{1\lambda}^\alpha, \phi_{1\lambda}, \dot{\phi}_{1\lambda})$  be given by the definition 5.1 with  $\lambda > 0$  and let the error initial data set  $(z_\lambda^0, z_\lambda^i, \zeta_\lambda, \dot{z}_\lambda^0, \dot{z}_\lambda^i, \dot{\zeta}_\lambda) \in H_\delta^2 \times H^2 \times H^2 \times H_\delta^1 \times H^1 \times H^1$  satisfies

$$\begin{cases} \dot{z}_\lambda^0 + \partial_i z_\lambda^i = -\lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} \text{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} (\partial_\alpha \overline{W_{\mathcal{A}}^\alpha})|_{t=0}), \\ -\Delta z_\lambda^0 + |\phi_{1\lambda} + \zeta_\lambda|^2 z_\lambda^0 - \partial_i z_\lambda^i = \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (\partial_t \text{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} \overline{\partial_\alpha W_{\mathcal{A}}^\alpha})|_{t=0}) + \lambda^{1/2} \xi_{\lambda\mathcal{A}}^0(\mathbf{B}_0) + \tilde{\xi}_{\lambda\mathcal{A}}^0(\mathbf{B}_0), \\ -\text{Im}(\phi_{1\lambda} \overline{\dot{\zeta}_\lambda}) - \text{Im}(\zeta_\lambda \overline{\dot{\phi}_{1\lambda}}) - \text{Im}(\zeta_\lambda \overline{\dot{\zeta}_\lambda}) - |\zeta_\lambda|^2 a_{1\lambda}^0 - (\zeta_\lambda \overline{\dot{\phi}_{1\lambda}} + \overline{\zeta_\lambda} \dot{\phi}_{1\lambda}) a_{1\lambda}^0, \end{cases} \quad (63)$$

where  $\lambda^{1/2} \xi_{\lambda\mathcal{A}}^0(\mathbf{B}_0) = \lambda^{1/2} \Xi_{\lambda\mathcal{A}}^0(\mathbf{B}_0)|_{t=0}$  and  $\tilde{\xi}_{\lambda\mathcal{A}}^0(\mathbf{B}_0) = \tilde{\Xi}_{\lambda\mathcal{A}}^0(\mathbf{B}_0)|_{t=0}$  are from the RHS of the almost approximate solution in section 4.2. Then, with  $(a_\lambda^\alpha, \dot{a}_\lambda^\alpha, \phi_\lambda, \dot{\phi}_\lambda)$  defined as (61), we have a solution to (62) with enough regularity for our purposes.

*Proof.* First we look at the Lorenz gauge condition, we want

$$\dot{a}_\lambda^0 + \partial_i a_\lambda^i = \dot{a}_{1\lambda}^0 + \partial_i a_{1\lambda}^i + \dot{z}_\lambda^0 + \partial_i z_\lambda^i = 0,$$

and we have

$$\dot{a}_{1\lambda}^0 + \partial_i a_{1\lambda}^i = \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} (\partial_\alpha \overline{W_{\mathcal{A}}^\alpha})|_{t=0}),$$

from definition 2.22, so we get the first equation.

For the second equation, we first calculate

$$\begin{aligned} & -\Delta(\lambda^{1/2} \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} \overline{W_{\mathcal{A}}^0})) - \partial_{jt}^2(\lambda^{1/2} \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} \overline{W_{\mathcal{A}}^j})) \\ &= -\lambda^{-1/2} \operatorname{Re}(ie^{i\frac{u_{\mathcal{A}}}{x}} (\Delta u_{\mathcal{A}} \overline{W_{\mathcal{A}}^0} + 2\partial_j u_{\mathcal{A}} \partial_j \overline{W_{\mathcal{A}}^0} + \partial_t(\partial_j u_{\mathcal{A}} \overline{W_{\mathcal{A}}^j}) + \partial_t u_{\mathcal{A}} \partial_j \overline{W_{\mathcal{A}}^j})) \\ & -\lambda^{1/2} \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} (\Delta W_{\mathcal{A}}^0 + \partial_{tt}^2 \overline{W_{\mathcal{A}}^j})) + \lambda^{-3/2} \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} (\partial_j u_{\mathcal{A}} \partial_j u_{\mathcal{A}} \overline{W_{\mathcal{A}}^0} + \partial_j u_{\mathcal{A}} \partial_t u_{\mathcal{A}} \overline{W_{\mathcal{A}}^j})) \\ &= -\lambda^{-1/2} \operatorname{Re}(ie^{i\frac{u_{\mathcal{A}}}{x}} (\Delta u_{\mathcal{A}} \overline{W_{\mathcal{A}}^0} - \partial_{tt}^2 u_{\mathcal{A}} \overline{W_{\mathcal{A}}^0} - \partial_t u_{\mathcal{A}} \partial_t \overline{W_{\mathcal{A}}^0} + 2\partial_j u_{\mathcal{A}} \partial_j \overline{W_{\mathcal{A}}^0} - \partial_t u_{\mathcal{A}} \partial_t \overline{W_{\mathcal{A}}^0})) \\ & -\lambda^{1/2} \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} (\Delta \overline{W_{\mathcal{A}}^0} - \partial_{tt}^2 \overline{W_{\mathcal{A}}^0})) - \partial_t(\operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} \lambda^{1/2} \partial_\alpha \overline{W_{\mathcal{A}}^\alpha})) \\ &= -\square(\lambda^{1/2} \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} \overline{W_{\mathcal{A}}^0})) - \partial_t(\operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} \lambda^{1/2} \partial_\alpha \overline{W_{\mathcal{A}}^\alpha})), \end{aligned}$$

where we use that  $\partial_\alpha u_{\mathcal{A}} W_{\mathcal{A}}^\alpha = 0$ .

We have

$$-\Delta a_{1\lambda}^0 - \partial_i(\dot{a}_{1\lambda}^i) + \operatorname{Im}(\phi_{1\lambda} \overline{\dot{\phi}_{1\lambda}}) + a_{1\lambda}^0 |\phi_{1\lambda}|^2 = \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} \partial_t(\operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{x}} \partial_\alpha \overline{W_{\mathcal{A}}^\alpha})|_{t=0}) + \lambda^{1/2} \xi_{\lambda A}^0(\mathbf{B}_0) + \tilde{\xi}_{\lambda A}^0(\mathbf{B}_0) \quad (64)$$

because of the previous calculations and the fact that the background initial data are admissible 2.24 and so satisfy the constraint equation (46).

All the other terms are obtained by direct calculations. □

## 5.2 Definition of the criteria for admissible error initial data

As explained in the introduction 1.5, the error term is divided in different error parameters to recover the necessary structure for the bootstrap argument of section 6.3. We give more details on the splitting in section 6.1. The **error parameters** are  $((E^{evo})_\lambda^\alpha, \mathcal{E}_{\lambda^0}^{evo}, W_{\lambda^0}^{+\alpha}, \check{W}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}, \Psi_{\lambda^0}^+, \check{\Psi}_{\mathcal{A} \pm \mathcal{B}}^+, G_{W_{\lambda^0}^+}^{+\alpha}, G_{\Psi_{\lambda^0}^+}^+, (E_\lambda^{ell})^\alpha, \mathcal{E}_\lambda^{ell})$ , their corresponding **error parameters initial data** are

$$((e_\lambda^{evo})^\alpha, \epsilon_\lambda^{evo}, (\dot{e}_\lambda^{evo})^\alpha, \dot{\epsilon}_\lambda^{evo}, w_{\lambda^0}^{+\alpha}, \check{w}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}, \psi_{\lambda^0}^+, \check{\psi}_{\mathcal{A} \pm \mathcal{B}}^+, g_{W_{\lambda^0}^+}^{+\alpha}, g_{\Psi_{\lambda^0}^+}^+, (e_\lambda^{ell})^\alpha, \epsilon_\lambda^{ell}, (\dot{e}_\lambda^{ell})^\alpha, \dot{\epsilon}_\lambda^{ell}).$$

In our construction, the quantities  $((e_\lambda^{evo})^\alpha, \epsilon_\lambda^{evo}, (\dot{e}_\lambda^{evo})^\alpha, \dot{\epsilon}_\lambda^{evo}, w_{\lambda^0}^{+\alpha}, \psi_{\lambda^0}^+, g_{W_{\lambda^0}^+}^{+\alpha}, g_{\Psi_{\lambda^0}^+}^+)$  are initial data for the system (78),  $(\check{w}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}, \check{\psi}_{\mathcal{A} \pm \mathcal{B}}^+)$  are initial data for the **decoupled equations** (87) and  $((e_\lambda^{ell})^\alpha, \epsilon_\lambda^{ell}, (\dot{e}_\lambda^{ell})^\alpha, \dot{\epsilon}_\lambda^{ell})$  must be given **directly by the background**. Details on these equations are given in section 6.1.

The next definition explains how from the **error parameters initial data** one can build **error initial data** under the form (61).

**Definition 5.5.** *For a given admissible background initial data and error parameters initial data we can define  $(a_\lambda^\alpha, \dot{a}_\lambda^\alpha, \phi_\lambda, \dot{\phi}_\lambda)$ , the initial data for KGM, with*

$$\begin{aligned} a_\lambda^\alpha &= a_{1\lambda}^\alpha + z_\lambda^\alpha, & \dot{a}_\lambda^\alpha &= \dot{a}_{1\lambda}^\alpha + \dot{z}_\lambda^\alpha, \\ \phi_\lambda &= \phi_{1\lambda} + \zeta_\lambda, & \dot{\phi}_\lambda &= \dot{\phi}_{1\lambda} + \dot{\zeta}_\lambda, \end{aligned}$$

where  $(a_{1\lambda}^\alpha, \dot{a}_{1\lambda}^\alpha, \phi_{1\lambda}, \dot{\phi}_{1\lambda})$  can be expressed only in terms of the **background initial data** as in definition 5.1 and where the **precise error term initial data**  $(z_\lambda^\alpha, \dot{z}_\lambda^\alpha, \zeta_\lambda, \dot{\zeta}_\lambda)$  can be expressed in terms of the **error parameters**



initial data as

$$\left\{ \begin{aligned}
z_\lambda^\alpha &= \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \operatorname{Re}(e^{i\frac{v_\mathcal{A}}{\lambda}} \overline{w_{\lambda\mathcal{A}}^{+\alpha}}) + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \operatorname{Re}(e^{i\frac{v_\mathcal{A} \pm v_\mathcal{B}}{\lambda}} \overline{\check{w}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}}) + (e_\lambda^{\text{evo}})^\alpha + (e_\lambda^{\text{ell}})^\alpha, \\
\zeta_\lambda &= \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \psi_{\lambda\mathcal{A}}^+ e^{i\frac{v_\mathcal{A}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \check{\psi}_{\mathcal{A} \pm \mathcal{B}}^+ e^{i\frac{v_\mathcal{A} \pm v_\mathcal{B}}{\lambda}} + \epsilon_\lambda^{\text{evo}} + \epsilon_\lambda^{\text{ell}}, \\
\dot{z}_A^\alpha &= \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \operatorname{Re}(e^{i\frac{v_\mathcal{A}}{\lambda}} (\partial_t \overline{W_{\lambda\mathcal{A}}^{+\alpha}})|_{t=0}) + \sum_{\mathcal{A}} \dot{v}_\mathcal{A} \operatorname{Re}(ie^{i\frac{v_\mathcal{A}}{\lambda}} \overline{w_{\lambda\mathcal{A}}^{+\alpha}}) + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \operatorname{Re}(e^{i\frac{v_\mathcal{A} \pm v_\mathcal{B}}{\lambda}} (\partial_t \overline{\check{W}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}})|_{t=0}) \\
&+ \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} (\dot{v}_\mathcal{A} \pm \dot{v}_\mathcal{B}) \operatorname{Re}(e^{i\frac{v_\mathcal{A} \pm v_\mathcal{B}}{\lambda}} \overline{\check{w}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}}) + (\dot{\epsilon}_\lambda^{\text{evo}})^\alpha + (\dot{\epsilon}_\lambda^{\text{ell}})^\alpha, \\
\dot{\zeta}_\lambda &= \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 (\partial_t \Psi_{\lambda\mathcal{A}}^+)|_{t=0} e^{i\frac{v_\mathcal{A}}{\lambda}} + \sum_{\mathcal{A}} \dot{v}_\mathcal{A} \psi_{\lambda\mathcal{A}}^+ ie^{i\frac{v_\mathcal{A}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \partial_t (\check{\Psi}_{\mathcal{A} \pm \mathcal{B}}^+)|_{t=0} e^{i\frac{v_\mathcal{A} \pm v_\mathcal{B}}{\lambda}} \\
&+ \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} (\dot{v}_\mathcal{A} \pm \dot{v}_\mathcal{B}) \check{\psi}_{\mathcal{A} \pm \mathcal{B}}^+ ie^{i\frac{v_\mathcal{A} \pm v_\mathcal{B}}{\lambda}} + \dot{\epsilon}_\lambda^{\text{evo}} + \dot{\epsilon}_\lambda^{\text{ell}}.
\end{aligned} \right. \tag{65}$$

**Remark 5.6.** We observe that  $(g_{W_{\lambda\mathcal{A}}^+}^{+\alpha}, g_{\Psi_{\lambda\mathcal{A}}^+}^{+\alpha})$  do not appear in these parametriz. Indeed, these two quantities are the initial data for **auxiliary functions** which are here to deal with regularity to have local well posedness, see section 6.3.3.

**Remark 5.7.** The quantities  $(\partial_t \overline{W_{\lambda\mathcal{A}}^{+\alpha}})_{t=0}$  and  $(\partial_t \Psi_{\lambda\mathcal{A}}^+)_{t=0}$  are calculated directly with the evolution equations (86) and depend on  $((e_\lambda^{\text{evo}})^\alpha, \epsilon_\lambda^{\text{evo}}, w_{\lambda\mathcal{A}}^{+\alpha}, \psi_{\lambda\mathcal{A}}^+, \nabla w_{\lambda\mathcal{A}}^{+\alpha}, \nabla \psi_{\lambda\mathcal{A}}^+)$  (a part of the **initial data** for the **error parameters**) and the background.

The same remark holds for  $(\partial_t \check{\Psi}_{\mathcal{A} \pm \mathcal{B}}^+)_{t=0}$  and  $(\partial_t \check{W}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha})_{t=0}$ .

We define the three criteria required for the initial data to prove the theorems 3.1 and 3.2. First, a special definition is made for  $((e_\lambda^{\text{ell}})^\alpha, \epsilon_\lambda^{\text{ell}}, (\dot{\epsilon}_\lambda^{\text{ell}})^\alpha, \dot{\epsilon}_\lambda^{\text{ell}})$ .

**Definition 5.8.** We define  $((e_\lambda^{\text{ell}})^\alpha, \epsilon_\lambda^{\text{ell}}, (\dot{\epsilon}_\lambda^{\text{ell}})^\alpha, \dot{\epsilon}_\lambda^{\text{ell}})$  as

$$\begin{aligned}
(e_\lambda^{\text{ell}})^\alpha &= ((E_\lambda^{\text{ell}})^\alpha(t))|_{t=0}, & \epsilon_\lambda^{\text{ell}} &= (\mathcal{E}_\lambda^{\text{ell}}(t))|_{t=0}, \\
(\dot{\epsilon}_\lambda^{\text{ell}})^\alpha &= \partial_t (E_\lambda^{\text{ell}})^\alpha(t)|_{t=0}, & \dot{\epsilon}_\lambda^{\text{ell}} &= (\partial_t \mathcal{E}_\lambda^{\text{ell}}(t))|_{t=0},
\end{aligned}$$

with  $((E_\lambda^{\text{ell}})^\alpha, \mathcal{E}_\lambda^{\text{ell}})$  from definition 6.9. As we expect to have a solution to an evolution system, the time derivatives are computed with the transport equations of proposition 4.1. To be more precise, with the initial ansatz we can construct a background solution to (51) and take the time derivatives of these background terms at time  $t = 0$ , they are functions of the **background initial data** and their space derivatives.

Moreover, we directly get the following regularity and smallness from section 6.1.3

$$\sum_{k \leq 2} \lambda^k (\|e_\lambda^{\text{ell}}\|_{H^k} + \|\epsilon_\lambda^{\text{evo}}\|_{H^k}) + \sum_{k \leq 1} \lambda^{k+1} (\|\dot{\epsilon}_\lambda^{\text{ell}}\|_{H^k} + \|\dot{\epsilon}_\lambda^{\text{ell}}\|_{H^k}) \leq \lambda^2 C_0. \tag{66}$$

**Definition 5.9.** We say that the **error parameters initial data** are **admissible for KGML** with respect to a **background initial data** if we have

$$g_{W_{\lambda\mathcal{A}}^+}^{+\alpha} = (\square W_{\lambda\mathcal{A}}^{+\alpha})|_{t=0}, \quad g_{\Psi_{\lambda\mathcal{A}}^+}^{+\alpha} = (\square \Psi_{\lambda\mathcal{A}}^+)|_{t=0},$$

and

$$\begin{aligned}
\forall \mathcal{A} \in \mathbb{A}, w_{\lambda\mathcal{A}}^{+\alpha} &\in H^2, & \forall (\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}, \check{w}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha} &\in H^3, \\
\forall \mathcal{A} \in \mathbb{A}, \psi_{\lambda\mathcal{A}}^+ &\in H^2, & \forall (\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}, \check{\psi}_{\mathcal{A} \pm \mathcal{B}}^+ &\in H^3, \\
\forall \mathcal{A} \in \mathbb{A}, g_{W_{\lambda\mathcal{A}}^+}^{+\alpha} &\in H^1, & \forall \mathcal{A} \in \mathbb{A}, g_{\Psi_{\lambda\mathcal{A}}^+}^{+\alpha} &\in H^1, \\
(e_\lambda^{\text{evo}})^i &\in H^2, & (\dot{\epsilon}_\lambda^{\text{evo}})^i &\in H^1, \\
(e_\lambda^{\text{evo}})^0 &\in H_\delta^2, & (\dot{\epsilon}_\lambda^{\text{evo}})^0 &\in H_{\delta+1}^1, \\
\epsilon_\lambda^{\text{evo}} &\in H^2, & \dot{\epsilon}_\lambda^{\text{evo}} &= H^1,
\end{aligned}$$

and  $((e_\lambda^{ell})^\alpha, \epsilon_\lambda^{ell}, (\dot{e}_\lambda^{ell})^\alpha, \dot{\epsilon}_\lambda^{ell})$  given by the definition 5.8. We also impose that the functions  $(g_{W_{\lambda\mathcal{A}}}^\alpha, g_{\Psi_{\lambda\mathcal{A}}}^\alpha, w_{\lambda\mathcal{A}}^{+\alpha}, \psi_{\lambda\mathcal{A}}^+, \check{w}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}, \check{\psi}_{\mathcal{A}\pm\mathcal{B}}^+, (e_\lambda^{evo})^i, (\dot{e}_\lambda^{evo})^i, \epsilon_\lambda^{evo}, \dot{\epsilon}_\lambda^{evo})$  are compactly supported<sup>29</sup> in  $B_{S'}$ .

**Remark 5.10.** The second derivatives,  $(\partial_{tt}^2 W_{\lambda\mathcal{A}}^{+\alpha})_{t=0}$  and  $(\partial_{tt}^2 \Psi_{\lambda\mathcal{A}}^+)_{t=0}$ , are obtained by taking one time derivative on the transport equation. They can be expressed with respect to  $((e_\lambda^{evo})^\alpha, \epsilon_\lambda^{evo}, (\dot{e}_\lambda^{evo})^\alpha, \dot{\epsilon}_\lambda^{evo}, w_{\lambda\mathcal{A}}^{+\alpha}, \psi_{\lambda\mathcal{A}}^+, \nabla w_{\lambda\mathcal{A}}^{+\alpha}, \nabla \psi_{\lambda\mathcal{A}}^+)$ ,  $((\partial_t W_{\lambda\mathcal{A}}^{+\alpha})_{t=0}, (\partial_t \Psi_{\lambda\mathcal{A}}^+)_{t=0}, \nabla(\partial_t W_{\lambda\mathcal{A}}^{+\alpha})_{t=0}, \nabla(\partial_t \Psi_{\lambda\mathcal{A}}^+)_{t=0})$  and the background. Taking into account remark 5.7, we see that the quantities  $(\square \Psi_{\lambda\mathcal{A}}^+)_{t=0}$  and  $(\square W_{\lambda\mathcal{A}}^{+\alpha})_{t=0}$  can be expressed in terms of the **error parameters initial data** and the background.

**Definition 5.11.** We say that the **error parameters initial data set** is **admissible for KGM in Lorenz gauge** with respect to a **background initial data** if  $(z_\lambda^\alpha, \dot{z}_\lambda^\alpha, \zeta_\lambda, \dot{\zeta}_\lambda)$  from (65) is solution to the constraint equations (63).

**Definition 5.12.** We define the **required smallness** for the coupled parameters as

$$\sum_{k \leq 1} \lambda^k (\|\epsilon_\lambda^{evo}\|_{H^{k+1}} + \|\dot{\epsilon}_\lambda^{evo}\|_{H^k} + \|(e_\lambda^{evo})^i\|_{H^{k+1}} + \|(\dot{e}_\lambda^{evo})^i\|_{H^k} + \|(e_\lambda^{evo})^0\|_{H_\delta^{k+1}} + \|(\dot{e}_\lambda^{evo})^0\|_{H_\delta^k}) \leq \lambda^{1/2} C_0,$$

$$\max_{\mathcal{A} \in \mathbb{A}} \left( \sum_{k \leq 1} \lambda^k (\|w_{\lambda\mathcal{A}}^{+\alpha}\|_{H^{k+1}} + \|g_{W_{\lambda\mathcal{A}}}^{+\alpha}\|_{H^k} + \|\psi_{\lambda\mathcal{A}}^+\|_{H^{k+1}} + \|g_{\Psi_{\lambda\mathcal{A}}}^+\|_{H^k}) \right) \leq C_0,$$

and for the other parameters<sup>30</sup> as

$$\max_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} (\|\check{w}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}\|_{H^3} + \|\check{\psi}_{\mathcal{A}\pm\mathcal{B}}^+\|_{H^3}) \leq C_0.$$

**Remark 5.13.** Initial data fulfilling the three criteria 5.9, 5.11 and 5.12 are admissible. They have the sufficient conditions to build exact solutions that allows to close the bootstrap argument in section 6.3.5 and prove the main theorems 3.1 and 3.2.

The next two sections are dedicated to the construction of a **generic set of error parameters initial data** fitting these definitions.

### 5.3 Inversion of the perturbed Laplacian and smallness for the error term

This section gives the basis for the construction of the **generic set** of admissible initial data of proposition 5.16. It guarantees that our **precise error initial data** (65) satisfy the constraint system (63), we use these results in the next section. We are also searching for solutions with a small error term with respect to  $\lambda$  so we need to have small initial data for it. Thus, in the next proposition we choose the **error initial data**  $(z_\lambda^i, \zeta_\lambda, \dot{z}_\lambda^i, \dot{\zeta}_\lambda)$  with some smallness assumptions and we determine  $(z_\lambda^0, \dot{z}_\lambda^0)$  through the equations. Moreover, we need to add some more smallness to  $\partial_t \dot{z}_\lambda^i$  to recover the required smallness on  $z_\lambda^0$ . This is possible because we have freedom of choice and this makes the precise error initial data a little less generic.

**Proposition 5.14.** Let  $(z_\lambda^i, \zeta_\lambda, \dot{z}_\lambda^i, \dot{\zeta}_\lambda) \in H^2 \times H^2 \times H^1 \times H^1$  be compactly supported<sup>31</sup> in  $B_{S'}$  and have the following smallness

$$\sum_{k \leq 1} \lambda^k (\|\zeta_\lambda\|_{H^{k+1}} + \lambda^k \|\dot{\zeta}_\lambda\|_{H^k} + \lambda^k \|z_\lambda^i\|_{H^{k+1}} + \lambda^k \|\dot{z}_\lambda^i\|_{H^k}) \leq \lambda^{1/2} C_0, \quad (67)$$

with

$$\|\partial_t \dot{z}_\lambda^i\|_{L^2} \leq \lambda^{1/2} C_0, \quad (68)$$

then there exists  $(z_\lambda^0, \dot{z}_\lambda^0)$  solution to (63) such that

$$\sum_{k \leq 1} \lambda^k (\|z_\lambda^0\|_{H_\delta^{k+1}} + \lambda^k \|\dot{z}_\lambda^0\|_{H_\delta^k}) \leq \lambda^{1/2} C_0, \quad (69)$$

where the weighted norms are defined in 2.1.

<sup>29</sup>We pick the same  $S'$  as for the background in proposition 4.1 without loss of generality.

<sup>30</sup>The quantities  $(\check{w}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}, \check{\psi}_{\mathcal{A}\pm\mathcal{B}}^+)$  are treated as background terms in the evolution scheme as the evolution equation (87) is decoupled. See section 6.1.2.

<sup>31</sup>We pick the same  $S'$  as for the background in proposition 4.1 without loss of generality.

*Proof.* For  $z_\lambda^0$ , we use the first equation of (63) and we see directly the result.

**Remark 5.15.** We see that the polarization condition (49) is mandatory to cancel the  $\lambda^{-1/2}$  potentially harmful terms in the first equation.

For  $z_\lambda^0$ , we look at the second equation with the operator  $-\Delta + |\phi_{1\lambda} + \zeta_\lambda|^2$ .

We see that if we invert this perturbed Laplacian directly<sup>32</sup> with 8.1 we will have  $\|z_\lambda^0\|_{H_\delta^2} \sim O(\lambda^{-1/2})$ <sup>33</sup> because of the derivative of the divergence of  $W_{\mathcal{A}}$  (see remark 4.9), the  $O(1)$  high frequency interactions and  $Im(\zeta_\lambda \overline{\phi_{1\lambda}})$ . This latter term contains  $-\sum_{\mathcal{A} \in \mathbb{A}} Im(i\zeta_\lambda \lambda^{-1/2} \dot{v}_{\mathcal{A}} \overline{\psi}_{\mathcal{A}} e^{-i\frac{v_{\mathcal{A}}}{\lambda}})$ . It is a term of the type of 2) given in the introduction in 1.5, it is treated by introducing  $\mathbf{F}_{\lambda, \mathcal{A}}^+$  in the evolution equations.

To deal with that, we can invert by hand the  $\lambda^{-1/2}$  and  $\lambda^0$  terms. We set

$$z_\lambda^{0BIS} = \lambda^{3/2} \sum_{\mathcal{A} \in \mathbb{A}} \left[ \frac{\dot{v}_{\mathcal{A}} Re(e^{i\frac{v_{\mathcal{A}}}{\lambda}} (\partial_\alpha \overline{W_{\mathcal{A}}^\alpha}))|_{t=0}}{\partial^i v_{\mathcal{A}} \partial_i v_{\mathcal{A}}} \right] - \lambda^{3/2} \sum_{\mathcal{A} \in \mathbb{A}} \left[ \frac{Im(i\zeta_\lambda \dot{v}_{\mathcal{A}} \overline{\psi}_{\mathcal{A}} e^{-i\frac{v_{\mathcal{A}}}{\lambda}})}{\partial^i v_{\mathcal{A}} \partial_i v_{\mathcal{A}}} \right], \quad (70)$$

$$+ \lambda^2 \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{C}, \mathcal{A} < \mathcal{B}} \frac{Re(K_{A(\mathcal{A} \pm \mathcal{B})}^\beta(\mathbf{B}'_0) e^{i\frac{v_{\mathcal{A}} \pm v_{\mathcal{B}}}{\lambda}})}{\partial^i (v_{\mathcal{A}} \pm v_{\mathcal{B}}) \partial_i (v_{\mathcal{A}} \pm v_{\mathcal{B}})} + \lambda^2 \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} \frac{Re(S_{A(\mathcal{A} \pm \mathcal{B})}^\beta(\mathbf{B}'_0) e^{i\frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}})}{\partial^i (v_{\mathcal{A}} \pm v_{\mathcal{B}}) \partial_i (v_{\mathcal{A}} \pm v_{\mathcal{B}})}. \quad (71)$$

By itself,  $z_\lambda^{0BIS}$  has enough smallness and is compactly supported, we have

$$\sum_{k \leq 1} \lambda^k \|z_\lambda^{0BIS}\|_{H^{k+1}} \leq \lambda^{1/2} C_0, \quad (72)$$

where we use proposition 2.17 to have control of the smallness of the denominator.

Then, we search for  $z_\lambda^{0TER}$  as a solution to

$$\begin{aligned} (-\Delta + |\phi_{1\lambda} + \zeta_\lambda|^2) z_\lambda^{0TER} - \partial_i \dot{z}_\lambda^i &= (\Delta - |\phi_{1\lambda} + \zeta_\lambda|^2) z_\lambda^{0BIS} + \lambda^{1/2} \zeta_{\lambda A}^0(\mathbf{B}_0) + \tilde{\zeta}_{\lambda A}^0(\mathbf{B}_0) \\ &- Im(\phi_{1\lambda} \overline{\dot{\zeta}_\lambda}) - Im(\zeta_\lambda \overline{\dot{\phi}_{1\lambda}}) - Im(\zeta_\lambda \overline{\dot{\zeta}_\lambda}) - |\zeta_\lambda|^2 a_{1\lambda}^0 - (\zeta_\lambda \overline{\phi_{1\lambda}} + \overline{\zeta_\lambda} \phi_{1\lambda}) a_{1\lambda}^0 \\ &+ \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} (\partial_t Re(e^{i\frac{u_{\mathcal{A}}}{\lambda}} (\partial_\alpha \overline{W_{\mathcal{A}}^\alpha})))|_{t=0}. \end{aligned} \quad (73)$$

The RHS is in  $O(\lambda^{1/2})$  and well defined in terms of regularity<sup>34</sup> and the quantity  $|\phi_{1\lambda} + \zeta_\lambda|^2$  is in  $H^2$  and compactly supported. This allows us to use the lemma 8.14 to get

$$\|z_\lambda^{0TER}\|_{H_\delta^2} \leq \lambda^{1/2} C_0.$$

In particular, the term  $Im(\zeta_\lambda \overline{\dot{\zeta}_\lambda})$  (which is dangerous in the evolution equations, see 4) of section 1.5, but harmless here because of size  $C_0$  by assumption) is controlled in  $L^2$  as follow

$$\|Im(\zeta_\lambda \overline{\dot{\zeta}_\lambda})\|_{L^2} \leq \|\zeta_\lambda\|_{H^1} \|\dot{\zeta}_\lambda\|_{H^{1/2}} \leq \lambda^{1/2} C_0,$$

using the product estimate (172) and the interpolation inequality (163). All the other terms are easily controlled in  $L^2$ , we use the extra smallness (68) for  $\partial_i \dot{z}_\lambda^i$ .

Finally, we pick  $z_\lambda^0 = z_\lambda^{0BIS} + z_\lambda^{0TER}$  and we get

$$\sum_{k \leq 1} \lambda^k \|z_\lambda^0\|_{H_\delta^{k+1}} \leq \lambda^{1/2} C_0,$$

which ends the proof.  $\square$

<sup>32</sup>The perturbed Laplacian operator is only invertible in some weighted Sobolev spaces.

<sup>33</sup>The constant in lemma 8.14 only depends on  $L^2$  norms of  $C_0$  size initial data.

<sup>34</sup>In particular, the Laplacian of  $\zeta_\lambda$  appears and needs to be in  $L^2$  which is the case because  $\zeta_\lambda$  is in  $H^2$ . The other terms involve only background terms which are more regular.

## 5.4 Construction of the precise error initial data

In the previous section, we show how to construct small **error initial data**  $(z_\lambda, \zeta_\lambda, \dot{z}_\lambda, \dot{\zeta}_\lambda)$  to have exact solutions to the constraints (63). However, if we search directly a solution to (76) whose initial data are  $(z_\lambda, \zeta_\lambda, \dot{z}_\lambda, \dot{\zeta}_\lambda)$  we will **not have the right structure** to close the bootstrap. In this section, we treat the construction of a generic set of **initial data** for the **error parameters** that are **admissible** (remark 5.13) and based on the previously cited **error initial data** of proposition 5.14.

**Proposition 5.16.** *For a given admissible background initial data set  $(a_0^\alpha, \dot{a}_0^\alpha, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^\alpha)$  and for  $\lambda > 0$ , we pick  $(z_\lambda^0, z_\lambda^i, \zeta_\lambda, \dot{z}_\lambda^0, \dot{z}_\lambda^i, \dot{\zeta}_\lambda)$  from proposition 5.14 and  $(e_\lambda^{ell}, \epsilon_\lambda^{ell}, \dot{e}_\lambda^{ell}, \dot{\epsilon}_\lambda^{ell})$  from definition 5.8. Then, for*

$$\begin{aligned} (e_\lambda^{evo})^\alpha &= z_\lambda^\alpha - (e_\lambda^{ell})^\alpha, & \epsilon_\lambda^{evo} &= \zeta_\lambda - \epsilon_\lambda^{ell}, \\ \forall \mathcal{A} \in \mathbb{A}, w_{\lambda\mathcal{A}}^{+\alpha} &= 0, & \forall (\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}, \check{w}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha} &= 0, \\ \forall \mathcal{A} \in \mathbb{A}, \psi_{\lambda\mathcal{A}}^+ &= 0, & \forall (\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}, \check{\psi}_{\mathcal{A}\pm\mathcal{B}}^+ &= 0, \\ \forall \mathcal{A} \in \mathbb{A}, g_{W_{\lambda\mathcal{A}}}^{+\alpha} &= (\square W_{\lambda\mathcal{A}}^{+\alpha})|_{t=0}, & \forall \mathcal{A} \in \mathbb{A}, g_{\Psi_{\lambda\mathcal{A}}}^+ &= (\square \Psi_{\lambda\mathcal{A}}^+)|_{t=0}, \end{aligned}$$

$$\begin{aligned} (\dot{e}_\lambda^{evo})^\alpha &= \dot{z}_\lambda^\alpha - \lambda \sum_{\mathcal{A}} \text{Re}(e^{-\frac{v_{\mathcal{A}}}{\lambda}} \overline{(\partial_t W_{\lambda\mathcal{A}}^{+\alpha})}|_{t=0}) - \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda \text{Re}(e^{-\frac{v_{\mathcal{A}\pm\mathcal{B}}}{\lambda}} \overline{(\partial_t \check{W}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha})}|_{t=0}) - (\dot{e}_\lambda^{ell})^\alpha, \\ \dot{\epsilon}_\lambda^{evo} &= \dot{\zeta}_\lambda - \lambda \sum_{\mathcal{A}} (\partial_t \check{\Psi}_{\lambda\mathcal{A}}^+)|_{t=0} e^{i\frac{v_{\mathcal{A}}}{\lambda}} - \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda \partial_t (\check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+)|_{t=0} e^{i\frac{v_{\mathcal{A}\pm\mathcal{B}}}{\lambda}} - \dot{\epsilon}_\lambda^{ell}, \end{aligned}$$

the initial data for the **error parameters** are **admissible for KGML**, **admissible for KGM in Lorenz gauge** and have the **required smallness**.

**Remark 5.17.** *This set is generic as we do not use any freedom of choice on  $(z_\lambda^0, z_\lambda^i, \zeta_\lambda, \dot{z}_\lambda^0, \dot{z}_\lambda^i, \dot{\zeta}_\lambda)$ , we just ask for a convenient smallness and regularity with an extra condition on the smallness of  $\partial_i \dot{z}_\lambda^i$  for technical purposes. We already remark, with the estimates (67) and (69), that the error terms initial data  $(z_\lambda^0, z_\lambda^i, \zeta_\lambda, \dot{z}_\lambda^0, \dot{z}_\lambda^i, \dot{\zeta}_\lambda)$  have better estimates than the error terms  $(Z_\lambda, \mathcal{Z}_\lambda)$  (see point 2 of the Theorem 3.2) in terms of smallness. This is due to the choice of the parameters  $(w_{\lambda\mathcal{A}}^{+\alpha}, \psi_{\lambda\mathcal{A}}^+, \check{w}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}, \check{\psi}_{\mathcal{A}\pm\mathcal{B}}^+) = 0$ . It gives extra smallness to  $(z_\lambda^0, z_\lambda^i, \zeta_\lambda, \dot{z}_\lambda^0, \dot{z}_\lambda^i, \dot{\zeta}_\lambda)$  and gives the sufficient regularity to  $(g_{W_{\lambda\mathcal{A}}}^{+\alpha}, g_{\Psi_{\lambda\mathcal{A}}}^+)$ . At time  $t \neq 0$  there is no reason to have  $(W_{\lambda\mathcal{A}}^{+\alpha}, \Psi_{\lambda\mathcal{A}}^+, \check{W}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}, \check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+) = 0$ , the extra smallness of the error initial data does not propagate but the regularity of  $(g_{W_{\lambda\mathcal{A}}}^{+\alpha}, g_{\Psi_{\lambda\mathcal{A}}}^+)$  does.*

Overall, we can say that we use the freedom of choice only on the extra parameters that we introduced. The 10 true initial data are let with their 2 constraints and their 8 real degrees of freedom.

*Proof of proposition 5.16.* For the KGML admissibility, there is only the regularity to prove. We show that we have the regularity and the required smallness with estimates of the RHS of the error parameters initial data. We only need to be careful with the time derivatives on  $W_{\lambda\mathcal{A}}^{+\alpha}, \check{W}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}, \Psi_{\lambda\mathcal{A}}^+$  and  $\check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+$ , the estimates for  $(z_\lambda, \zeta_\lambda, \dot{z}_\lambda, \dot{\zeta}_\lambda)$  and  $(e_\lambda^{ell}, \epsilon_\lambda^{ell}, \dot{e}_\lambda^{ell}, \dot{\epsilon}_\lambda^{ell})$  are direct using proposition 5.14 and the inequalities (69), (67) and (66). In particular, for  $g_{W_{\lambda\mathcal{A}}}^{+\alpha}$  and  $g_{\Psi_{\lambda\mathcal{A}}}^+$ , we use remark 5.10 and the fact that we chose  $w_{\lambda\mathcal{A}}^{+\alpha} = 0$  and  $\psi_{\lambda\mathcal{A}}^+ = 0$  to deduce that even if the  $\square$  terms seems to holds 2 derivatives it is not the case. Indeed, the two derivatives in space are 0 and the two derivatives in time only depend on quantities that hold one derivative and have the right regularity. We have

$$\begin{aligned} (\partial_t W_{\lambda\mathcal{A}}^{+\alpha})|_{t=0} &= \lambda^{-1/2} f_w(\mathbf{B}'_0) \epsilon_\lambda^{evo}, \\ (\partial_t \Psi_{\lambda\mathcal{A}}^+)|_{t=0} &= \lambda^{-1/2} f_\psi(\mathbf{B}'_0) (e_\lambda^{evo})^\alpha, \\ (\partial_{tt}^2 W_{\lambda\mathcal{A}}^{+\alpha})|_{t=0} &= L_w^\alpha(\nabla(\partial_t W_{\lambda\mathcal{A}}^{+\alpha})|_{t=0}, \lambda^{-1/2} \dot{\epsilon}_\lambda^{evo}, \mathbf{B}'_0) + Q_w^\alpha(\epsilon_\lambda^{evo}, e_\lambda^{evo}) + [\dots], \\ (\partial_{tt}^2 \Psi_{\lambda\mathcal{A}}^+)|_{t=0} &= L_\psi^\alpha(\nabla(\partial_t \Psi_{\lambda\mathcal{A}}^+)|_{t=0}, \lambda^{-1/2} \dot{e}_\lambda^{evo}, \mathbf{B}'_0) + Q_\psi^\alpha(e_\lambda^{evo}, e_\lambda^{evo}) + [\dots], \end{aligned}$$

where the  $[\dots]$  involve lower order more regular terms or smaller terms. The  $f_w$  and  $f_\psi$  contain no derivatives of  $\mathbf{B}'_0$ ,  $L_w$  and  $L_\psi$  are linear in their 2 first parameters and  $Q_w$  and  $Q_\psi$  contain a product with respect to their 2 parameters. We can estimate these terms using (41)<sup>35</sup> and all the previous estimates from proposition 5.14 and

<sup>35</sup>As we have to divide by  $\partial^0 u_{\mathcal{A}}$ .

definition 2.24.

For  $\dot{\epsilon}_\lambda^{evo}$  and  $\dot{\epsilon}_\lambda^{evo}$  we have

$$\sum_{k \leq 1} \lambda^k \|\lambda e^{i \frac{v_{\mathcal{A}}}{\lambda}} (\partial_t W_{\lambda \mathcal{A}}^+) |_{t=0}\|_{H^k} \leq \lambda C_0, \quad \sum_{k \leq 1} \lambda^k \|\lambda e^{i \frac{v_{\mathcal{A}}}{\lambda}} (\partial_t \Phi_{\lambda \mathcal{A}}^+) |_{t=0}\|_{H^k} \leq \lambda C_0,$$

and the estimates for the other terms are direct. For  $g_{W_{\lambda \mathcal{A}}^+}^{+\alpha}$  and  $g_{\Phi_{\lambda \mathcal{A}}^+}^+$  we have

$$\sum_{k \leq 1} \lambda^k \|(\partial_{tt}^2 W_{\lambda \mathcal{A}}^+) |_{t=0}\|_{H^k} \leq C_0, \quad \sum_{k \leq 1} \lambda^k \|(\partial_{tt}^2 \Phi_{\lambda \mathcal{A}}^+) |_{t=0}\|_{H^k} \leq C_0,$$

so that we have the required smallness and the required regularity of definition 5.12. In particular, the quantities  $g_{W_{\lambda \mathcal{A}}^+}^{+\alpha}$  and  $g_{\Phi_{\lambda \mathcal{A}}^+}^+$  are more regular than two standard derivatives of  $W_{\lambda \mathcal{A}}^+$  and  $\Phi_{\lambda \mathcal{A}}^+$  are expected to be.

For the KGM in Lorenz gauge admissibility, because our **precise error initial data** are constructed on the **error initial data** from definition 5.5 we clearly see that  $(a_\lambda^\alpha, \dot{a}_\lambda^\alpha, \phi_\lambda, \dot{\phi}_\lambda)$  is solution to the constraint equations (62).  $\square$

## 6 Exact multi phase high frequency solutions

### 6.1 More detail on the error term

#### 6.1.1 Parametrix and evolution equations for the error term

We want to have exact solutions to KGM in Lorenz gauge under the form of theorem 3.1 :

$$A_\lambda^\alpha = A_{1\lambda}^\alpha + Z_\lambda^\alpha, \quad \Phi_\lambda = \Phi_{1\lambda} + \mathcal{Z}_\lambda, \quad (74)$$

where the  $(A_{1\lambda}^\alpha, \Phi_{1\lambda})$  are **almost approximate solution of order 1** to KGM in Lorenz gauge<sup>36</sup> and  $(Z_\lambda^\alpha, \mathcal{Z}_\lambda)$  the **error terms**.

**Notation 6.1.** We can write  $\cos(\frac{u_{\mathcal{A}}}{\lambda})P_{\mathcal{A}}^\alpha + \sin(\frac{u_{\mathcal{A}}}{\lambda})Q_{\mathcal{A}}^\alpha$  as  $\text{Re}(e^{i \frac{u_{\mathcal{A}}}{\lambda}} \overline{W_{\mathcal{A}}^\beta})$  and so both parametrix have the schematic shape

$$\mathbf{F}_\lambda = \mathbf{F}_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\mathcal{A}} + \mathbf{Z}_\lambda. \quad (75)$$

In general, when we use the bold letters as  $\mathbf{Z}_\lambda$  it means that we are speaking about both the potential and the wave function at the same time on a schematic level. We also write  $\mathbf{B}_0$  for the background terms ( $\mathbf{F}_0$  and  $\mathbf{F}_{\mathcal{A}}$ ) and  $\mathbf{B}'_0$  when we also mean  $\partial^\alpha u_{\mathcal{A}}$  to shorten the notation.

The error terms  $(Z_\lambda, \mathcal{Z}_\lambda)$ , both given as  $\mathbf{Z}_\lambda$ , must be schematically solution to

$$\square \mathbf{Z}_\lambda = \mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda + \mathcal{L}_1 \left( \mathbf{Z}_\lambda, \partial \mathbf{B}_0, \mathbf{B}'_0, \frac{\mathcal{U}_{\mathbb{A}}}{\lambda} \right) + \mathcal{L}_2 \left( \partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{\mathcal{U}_{\mathbb{A}}}{\lambda} \right) + \mathcal{H} \left( \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{\mathcal{U}_{\mathbb{A}}}{\lambda} \right) + \lambda^{1/2} \mathbf{\Xi}_\lambda + \mathbf{\tilde{\Xi}}_\lambda, \quad (76)$$

where  $\mathcal{L}_1$  (resp.  $\mathcal{L}_2$ ) are linear in their first 3 (resp. first 2) parameters and oscillate at high frequency for some phases  $u$ , the quantity  $\mathcal{H}$  is at least quadratic and at most cubic in its first parameter and also oscillate at high frequency with respect to  $\mathcal{U}_{\mathbb{A}}$ . Finally,  $\mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda$  is explicit and the last terms are already given and only depend on the background.

This system (represented by only one equation here) is clearly locally well posed for  $\lambda > 0$  and initial data such  $(z_\lambda, \dot{z}_\lambda) \in H^2 \times H^1$  but it is not direct at all that we can recover a time of existence uniform in  $\lambda$ .

To show that, we define more precisely the error term under the form

$$\mathbf{Z}_\lambda = \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \mathbf{F}_{\lambda \mathcal{A}}^+ e^{i \frac{u_{\mathcal{A}}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \mathbf{F}_{\mathcal{A} \pm \mathcal{B}}^+ e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} + \mathbf{\mathcal{E}}_\lambda^{ell} + \mathbf{\mathcal{E}}_\lambda^{evo}. \quad (77)$$

<sup>36</sup>See remark 4.9.

The quantities  $(\mathcal{E}_\lambda^{evo}, \mathbf{F}_{\lambda\mathcal{A}}^+)$  plus the auxiliary function  $\mathbf{G}_{\lambda\mathcal{A}}^+$  (introduced in 3) to handle a loss of derivative) are the **coupled evolution parameters** and are solution to the coupled system of equation

$$\left\{ \begin{array}{l} \square \mathcal{E}_\lambda^{evo} = \mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo} + \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} (i \partial u_{\mathcal{A}} \Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+) - \lambda^1 \mathbf{G}_{\lambda\mathcal{A}}^+) + [\dots], \\ \mathcal{L}_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ = \lambda^{-1/2} \partial u_{\mathcal{A}} \mathcal{E}_\lambda^{evo} \mathbf{B}_0 + \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+) + \partial u_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ \mathbf{B}_0, \\ \mathcal{L}_{\mathcal{A}} \mathbf{G}_{\lambda\mathcal{A}}^+ = [\mathcal{L}_A, \square] \mathbf{F}_{\lambda\mathcal{A}}^+ + \lambda^{-1/2} \partial u_{\mathcal{A}} \square \mathcal{E}_\lambda^{evo} \mathbf{B}_0 + \square (\partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)) \\ + \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{G}_{\lambda\mathcal{A}}^+) - \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \square \mathbf{F}_{\lambda\mathcal{A}}^+) + \partial u_{\mathcal{A}} \mathbf{G}_{\lambda\mathcal{A}}^+ \mathbf{B}_0 + [\dots], \end{array} \right. \quad (78)$$

where the  $[\dots]$  depend on  $\mathcal{E}_\lambda^{ell}$ ,  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$ , the background or lower order terms. The quantities  $\mathcal{E}_\lambda^{ell}$  and  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$  are the **decoupled evolution parameters** and are given by

$$\mathcal{E}_\lambda^{ell} = -\lambda^2 \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{B}} \pm u_{\mathcal{A}}}{\lambda}} \frac{\mathbf{S}_{(\mathcal{A}\pm\mathcal{B})}(\mathbf{B}'_0)}{\partial_\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}})}, \quad (79)$$

$$\forall (\mathcal{A}, \mathcal{B}) \in \mathbb{C}, \mathcal{A} < \mathcal{B}, \mathcal{L}_{\mathcal{A}\pm\mathcal{B}} \check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+ = \mathbf{K}_{(\mathcal{A}\pm\mathcal{B})}(\mathbf{B}'_0) + (\partial u_{\mathcal{A}} \pm \partial u_{\mathcal{B}}) \mathbf{B}_0 \check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+, \quad (80)$$

for  $\mathbf{S}_{(\mathcal{A}\pm\mathcal{B})}(\mathbf{B}'_0)$  and  $\mathbf{K}_{(\mathcal{A}\pm\mathcal{B})}(\mathbf{B}'_0)$  given by proposition 4.6 and notations 4.7.

We give details on the choices and the constraints for the initial data in section 5 and also in section 6.1.2 for  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$ .

**Remark 6.2.** *Non schematically the parametrix is written*

$$\begin{aligned} A_\lambda^\alpha &= A_0^\alpha + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} \text{Re}(e^{i \frac{u_{\mathcal{A}}}{\lambda}} \overline{W_{\mathcal{A}}^\alpha}) + \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \text{Re}(e^{i \frac{u_{\mathcal{A}}}{\lambda}} \overline{W_{\lambda\mathcal{A}}^{+\alpha}}) + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \text{Re}(e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \overline{W_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}}) \\ &+ (E_\lambda^{evo})^\alpha + (E_\lambda^{ell})^\alpha, \end{aligned} \quad (81)$$

$$\Phi_\lambda = \Phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}} + \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \Psi_{\lambda\mathcal{A}}^+ e^{i \frac{u_{\mathcal{A}}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+ e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} + \mathcal{E}_\lambda^{evo} + \mathcal{E}_\lambda^{ell}, \quad (82)$$

and we have

$$(E_\lambda^{ell})^\alpha = -\lambda^2 \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} \frac{\text{Re}(\overline{S_{A(\mathcal{A}\pm\mathcal{B})}^\alpha(\mathbf{B}'_0)} e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}})}{\partial_\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}})}, \quad (83)$$

$$\mathcal{E}_\lambda^{ell} = -\lambda^2 \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} \frac{S_{\Phi(\mathcal{A}\pm\mathcal{B})}(\mathbf{B}'_0) e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}}}{\partial_\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}})}. \quad (84)$$

**Remark 6.3.** *We write the transport equations for  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$  and  $\mathbf{F}_{\lambda\mathcal{A}}^+$  in a non schematic way as*

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathcal{A}\pm\mathcal{B}} \check{W}_{\mathcal{A}\pm\mathcal{B}}^{+\beta} = i(\partial^\beta u_{\mathcal{A}} \pm \partial^\beta u_{\mathcal{B}}) \overline{\check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+} \Phi_0 + iK_{A(\mathcal{A}\pm\mathcal{B})}^\beta(\mathbf{B}'_0), \\ \mathcal{L}_{\mathcal{A}\pm\mathcal{B}} \check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+ = -i2A_0^\alpha (\partial^\alpha u_{\mathcal{A}} \pm \partial^\alpha u_{\mathcal{B}}) \check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+ - iK_{\Phi(\mathcal{A}\pm\mathcal{B})}(\mathbf{B}'_0), \end{array} \right. \quad (85)$$

where the RHS of the equation for  $\check{W}_{\mathcal{A}\pm\mathcal{B}}^{+\beta}$  is orthogonal to  $(\partial_\beta u_{\mathcal{A}} \pm \partial_\beta u_{\mathcal{B}})$  (see proposition 4.8), and

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathcal{A}} W_{\lambda\mathcal{A}}^{+\beta} = i\partial^\beta u_{\mathcal{A}} \overline{\Psi_{\lambda\mathcal{A}}^+} \Phi_0 + \lambda^{-1/2} i\partial^\beta u_{\mathcal{A}} \overline{\Psi_{\mathcal{A}}^{evo}} \mathcal{E}_\lambda^{evo} + i\partial^\beta u_{\mathcal{A}} \Pi_-(\overline{\Psi_{\lambda\mathcal{A}}^+} \mathcal{E}_\lambda^{evo}), \\ \mathcal{L}_{\mathcal{A}} \Psi_{\lambda\mathcal{A}}^+ = -i2A_0^\alpha \partial_\alpha u_{\mathcal{A}} \Psi_{\lambda\mathcal{A}}^+ - \lambda^{-1/2} i2(E_\lambda^{evo})^\alpha \partial_\alpha u_{\mathcal{A}} \Psi_{\mathcal{A}} - i2\partial_\alpha u_{\mathcal{A}} \Pi_-(\overline{(E_\lambda^{evo})^\alpha} \Psi_{\lambda\mathcal{A}}^+), \end{array} \right. \quad (86)$$

where the RHS of the equation for  $W_{\lambda\mathcal{A}}^{+\beta}$  is orthogonal to  $\partial_\beta u_{\mathcal{A}}$ .

We can already give all the details on the decoupled evolution parameters  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$  and  $\mathcal{E}_\lambda^{ell}$  as they only depend on the background and not on the other coupled parameters.

### 6.1.2 Information on $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+$

**Definition 6.4.** For a given admissible background initial data, let  $B'_0$  be the background given by proposition 4.1. We define the transport equation for  $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+$  as

$$\forall \mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}, \quad \mathcal{L}_{\mathcal{A}\pm\mathcal{B}} \check{F}_{\mathcal{A}\pm\mathcal{B}}^+ = K_{(\mathcal{A}\pm\mathcal{B})}(B'_0) + (\partial u_{\mathcal{A}} \pm \partial u_{\mathcal{B}}) B_0 \check{F}_{\mathcal{A}\pm\mathcal{B}}^+. \quad (87)$$

**Remark 6.5.** We use  $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+$  to absorb the  $\sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{C}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{B}} \pm u_{\mathcal{A}}}{\lambda}} K_{(\mathcal{A}\pm\mathcal{B})}(B'_0)$  in  $\check{\Xi}_\lambda$ , i.e., the  $O(1)$  high-frequency interaction terms that oscillate in a null directions<sup>37</sup>, it does **not depend on**  $\lambda$ . The equation (87) is **decoupled** from the the system (78). We use our freedom to choose  $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+|_{t=0} = 0$  in proposition 5.16 to simplify. More details on the initial data are given in section 5.

**Remark 6.6.** The RHS term

$$(\partial u_{\mathcal{A}} \pm \partial u_{\mathcal{B}}) B_0 \check{F}_{\mathcal{A}\pm\mathcal{B}}^+ \quad (88)$$

comes from  $\mathcal{L}_2(\partial Z, B_0, \frac{u_{\mathcal{A}}}{\lambda})$  in (76) and the term

$$K_{(\mathcal{A}\pm\mathcal{B})}(B'_0) \quad (89)$$

from the cutting in two of  $\check{\Xi}_\lambda$  in 4.6.

The next proposition gives the smallness and regularity properties of  $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+$  given by the equation (87).

**Proposition 6.7.** Let  $\check{f}_{\mathcal{A}\pm\mathcal{B}}^+ \in H^3$  with  $\text{Supp}(\check{f}_{\mathcal{A}\pm\mathcal{B}}^+) \subset \Omega$  for  $\Omega$  a compact set in  $\mathbb{R}^3$ . There exists a unique local solution  $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+ \in \cap_{j=0}^3 (C^j([0, T], H^{3-j}))$  for  $T$  the time of existence of the background and equal to  $\check{f}_{\mathcal{A}\pm\mathcal{B}}^+ \in H^3$  at time  $t = 0$ . Moreover, we have  $\text{Supp}(\check{F}_{\mathcal{A}\pm\mathcal{B}}^+) \subset \mathcal{I}(T, \Omega)$  and the estimates

$$\max_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \sum_{k \leq 3} (\|\partial_t^k \check{F}_{\mathcal{A}\pm\mathcal{B}}^+\|_{L^\infty([0, T], H^{3-k})}) \leq C_0. \quad (90)$$

*Proof.* We refer to section 8.1, in particular we use first proposition 8.4<sup>38</sup> to get the right  $H^3$  norms and we use the fact that  $\check{f}_{\mathcal{A}\pm\mathcal{B}}^+$  is in  $H^3$ . Then, we plug it in proposition 8.5 to get the control on the time derivatives. The finite speed of propagation implies the property of the support.  $\square$

**Remark 6.8.** We see that  $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+$  is in  $H^3$  for  $t \in [0, T]$  as the background and is more regular than  $F_{\lambda\mathcal{A}}^+$  and  $\mathcal{E}_\lambda^{evo}$ . This regularity is useful because in the RHS of (117) is present the term  $\square \check{F}_{\mathcal{A}\pm\mathcal{B}}^+$ . In general, we see that  $\check{F}_{\mathcal{A}\pm\mathcal{B}}^+$  is a **decoupled error parameter** and only depends on the background so it is treated as a **background term**.

### 6.1.3 Information on $\mathcal{E}_\lambda^{ell}$

**Definition 6.9.** For a given admissible background initial data, let  $B'_0$  be the background given by proposition 4.1. We set  $\mathcal{E}_\lambda^{ell}$  with the equation

$$\mathcal{E}_\lambda^{ell} = -\lambda^2 \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{I}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{B}} \pm u_{\mathcal{A}}}{\lambda}} \frac{S_{(\mathcal{A}\pm\mathcal{B})}(B'_0)}{\partial_\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}}) \partial^\alpha (u_{\mathcal{A}} \pm u_{\mathcal{B}})}. \quad (91)$$

**Remark 6.10.** These expressions make sense because the denominator is not 0, see proposition 2.17. As it is a direct expression of the background,  $\mathcal{E}_\lambda^{ell}$  is **treated as a background term**<sup>39</sup>.

<sup>37</sup>Along a characteristic phase emerging from resonant interaction.

<sup>38</sup>The linear term on the RHS of (153) is harmless and controlled like the other linear term in the evolution equation.

<sup>39</sup>Even if its expression depends on  $\lambda$ .

**Remark 6.11.** We use  $\mathcal{E}_\lambda^{ell}$  to invert the  $O(1)$  high-frequency interaction terms that oscillate in a non null direction<sup>40</sup> from  $\tilde{\Xi}_\lambda$ . More precisely,

$$\square \mathcal{E}_\lambda^{ell} - \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \mathbf{S}_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0) = O(\lambda^1) f(\mathbf{B}'_0), \quad (92)$$

and at the same time  $\mathcal{E}_\lambda^{ell}$  is small enough as an error parameter, see the next proposition.

**Proposition 6.12.** We have  $\text{Supp}(\mathcal{E}_\lambda^{ell}) \subset B_S$  and

$$\sum_{k \leq 2} \lambda^k \|\mathcal{E}_\lambda^{ell}\|_{L^\infty([0, T], H^k)} + \sum_{k \leq 1} \lambda^{k+1} \|\partial_t \mathcal{E}_\lambda^{ell}\|_{L^\infty([0, T], H^k)} + \lambda^2 \|\partial_{tt}^2 \mathcal{E}_\lambda^{ell}\|_{L^\infty([0, T], L^2)} \leq \lambda^2 C_0, \quad (93)$$

and

$$\sum_{k \leq 1} \lambda^k \|\square \mathcal{E}_\lambda^{ell} - \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \mathbf{S}_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0)\|_{L^\infty([0, T], H^k)} \leq \lambda^1 C_0. \quad (94)$$

*Proof.* Using the definition 6.9 and the previous remark 6.11, it is trivial for the support and we get the estimates by direct calculation with (43) to have control on the smallness of the denominator.  $\square$

## 6.2 Main propositions

We give here the schematic propositions that compose the proof of the Theorems 3.1 and 3.2 after the construction of the approximate solution step (i) of 1.5. The three propositions correspond to the steps (iii), (iv) and (v). Each proposition relies on conditions on the initial data for **the error parameters**, see the definitions 5.9, 5.11 and 5.12. We give a generic set of initial data fulfilling these conditions with the specific choice of initial data of proposition 5.16, this corresponds to point (ii). In general, in this section and the following ones all the propositions and definitions concern the error terms for  $\lambda > 0$  with respect to a given **initial ansatz** and the corresponding background, given in proposition 4.1.

### 6.2.1 Exact solutions to KGML

**Proposition 6.13** (Exact solutions to KGML). *Let  $(\mathbf{f}_{\lambda \mathcal{A}}^+, \mathbf{g}_{\lambda \mathcal{A}}^+, \check{\mathbf{f}}_{\mathcal{A} \pm \mathcal{B}}^+, \mathbf{e}_\lambda^{evo}, \dot{\mathbf{e}}_\lambda^{evo}, \mathbf{e}_\lambda^{ell}, \dot{\mathbf{e}}_\lambda^{ell})$  be initial data for the error parameters admissible for KGML 5.9 with respect to a certain background initial data and a  $\lambda > 0$ . We give  $\mathcal{E}_\lambda^{ell}$  and  $\check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+$  as explained in section 6.1.3 and 6.1.2. Then,*

1. *the system (78) is well-posed and there exists a time  $t_\lambda$  for which the solution  $(\mathbf{F}_{\lambda \mathcal{A}}^+, \mathbf{G}_{\lambda \mathcal{A}}^+, \mathcal{E}_\lambda^{evo})$  exists and stays in its class of regularity.*
2. *the solution put under the form<sup>41</sup> of the parametrix  $\mathbf{F}_\lambda$  (75) is solution to KGML and has a good structure for proposition 6.18.*

### 6.2.2 Exact solutions to KGM in Lorenz gauge

**Proposition 6.14** (Exact solutions to KGM in Lorenz gauge). *Let  $(\mathbf{F}_{\lambda \mathcal{A}}^+, \mathbf{G}_{\lambda \mathcal{A}}^+, \mathcal{E}_\lambda^{evo}, \mathcal{E}_\lambda^{ell}, \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+)$  be given by the previous proposition 6.13. If the initial data for error parameters is also admissible for KGM in Lorenz gauge 5.11 then the parametrix (75) is solution to KGM in Lorenz gauge, i.e., the gauge propagates.*

### 6.2.3 Assumptions

If we search directly for an error term solution to (76) with classical well posedness techniques its time of existence  $t_\lambda$  tends to 0 as  $\lambda$  goes to 0 due to the fact that we do not have good uniform bounds for the norms. To show that the time  $t_\lambda$  is not optimal (for  $\lambda$  small enough) we make some assumptions about the size of the norms of the coupled evolution parameters on  $[0, T]$  and we show that they can be improved so that no explosion occurs.

<sup>40</sup>Along a non characteristic phase emerging from non resonant interactions.

<sup>41</sup>That is  $(A_\lambda, \Phi_\lambda)$  in the non schematic way.



**Notation 6.15.** We write  $\underline{\mathcal{E}}_\lambda^{evo}$  for all the parts of the parameters  $\mathcal{E}_\lambda^{evo}$  that are not  $(E_\lambda^{evo})^0$  in the parametrix written in 6.2. The quantity  $(E_\lambda^{evo})^0$  is a part of the error term associated to  $A_\lambda^0$ , we find it by solving the constraint equation (62). To invert the Laplacian we usually use weighted Sobolev norms, so that  $(E_\lambda^{evo})^0$  it is not in  $L^2$ .

**Assumptions 6.16.** There exists  $c_1 > 0$  and  $c_2 > 0$  such that for  $\tau \in [0, T]$

$$\begin{aligned} \max_{\mathcal{A} \in \mathbb{A}} \sum_{k \leq 1} \lambda^k (\| \mathbf{F}_{\lambda, \mathcal{A}}^+(\tau) \|_{H^{k+1}} + \| \partial_t \mathbf{F}_{\lambda, \mathcal{A}}^+(\tau) \|_{H^k}) + \lambda \| \partial_{tt}^2 \mathbf{F}_{\lambda, \mathcal{A}}^+(\tau) \|_{L^2} &\leq c_1 e^{\tau c_2}, \\ \sum_{k \leq 1} \lambda^k (\| \underline{\mathcal{E}}_\lambda^{evo}(\tau) \|_{H^{k+1}} + \| \partial_t \underline{\mathcal{E}}_\lambda^{evo}(\tau) \|_{H^k}) + \lambda \| \partial_{tt}^2 \underline{\mathcal{E}}_\lambda^{evo}(\tau) \|_{L^2} &\leq \lambda^{1/2} c_1 e^{\tau c_2}, \\ \sum_{k \leq 1} \lambda^k (\| (E_\lambda^{evo})^0(\tau) \|_{H_\delta^{k+1}} + \| \partial_t (E_\lambda^{evo})^0(\tau) \|_{H_{\delta+1}^k}) + \lambda \| \partial_{tt}^2 (E_\lambda^{evo})^0(\tau) \|_{L_{\delta+2}^2} &\leq \lambda^{1/2} c_1 e^{\tau c_2}, \\ \max_{\mathcal{A} \in \mathbb{A}} \sum_{k \leq 1} (\lambda^k \| \mathbf{G}_{\lambda, \mathcal{A}}^+(\tau) \|_{H^k}) + \lambda \| \partial_t \mathbf{G}_{\lambda, \mathcal{A}}^+(\tau) \|_{L^2} &\leq c_1 e^{\tau c_2}. \end{aligned}$$

**Notation 6.17.** We note by  $C_{0,1,2,\tau} = C_{0,1,2,\tau}(c_0, \frac{1}{\eta_0}, N, T, c_1, c_2, \tau)$  any constants that are polynomial in  $c_1 e^{\tau c_2}$ ,  $\frac{c_1}{c_2} e^{\tau c_2}$  and the background constants  $c_0, \frac{1}{\eta_0}, N = |\mathbb{A}|$  and  $T$ .

To improve the estimates, there are three possibilities or a combination of them term by term

- We show that our estimates depend in fact on some background norms that do not explode on  $[0, T]$  ( $\leq C_0$ )
- We show that our estimates depend on the a priori estimates but with a bigger power in  $\lambda$ , so that, for  $\lambda$  small enough, we improve them. ( $\leq \lambda^{>0} C_{0,1,2,\tau}$ )
- We show that the source term only depends linearly on the coupled evolution parameters ( $C_0 \frac{c_1}{c_2} e^{\tau c_2}$ )

### 6.2.4 Uniform in $\lambda$ time of existence

**Proposition 6.18** (Uniform in  $\lambda$  time of existence). *Let  $(\mathbf{F}_{\lambda, \mathcal{A}}^+, \mathbf{G}_{\lambda, \mathcal{A}}^+, \underline{\mathcal{E}}_\lambda^{evo}, \mathcal{E}_\lambda^{ell}, \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+)$  be given by the proposition 6.13. If the initial data for the **error parameters** have also **the required smallness** then there exists a certain  $\lambda_0$ , only function of the background, such that for all  $\lambda < \lambda_0$  the parametrix  $\mathbf{F}_\lambda$  exists and stays in its class of regularity on  $[0, T]$  (the existence time of the background). Thus, the family of solutions to KGML  $(\mathbf{F}_\lambda)_{\lambda < \lambda_0}$  has a uniform time of existence.*

## 6.3 System of evolution

We recall that our goal is to have a family of multi phase high frequency  $(\mathbf{F}_\lambda)_{\lambda < \lambda_0}$  solutions to KGM based on the initial ansatz 2.21 with a uniform time of existence with respect to  $\lambda$ . We show in proposition 4.1 how to construct an almost approximate solution of order 1 as

$$A_{1\lambda}^\alpha = A_0^\alpha + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} \text{Re}(e^{i \frac{u_{\mathcal{A}}}{\lambda}} \overline{W_{\mathcal{A}}^\alpha}), \quad \Phi_{1\lambda} = \Phi_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \Psi_{\mathcal{A}}. \quad (95)$$

To have an exact solution we extend schematically the parametrix as

$$\mathbf{F}_\lambda = \mathbf{F}_0 + \lambda^{1/2} \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\mathcal{A}} + \mathbf{Z}_\lambda, \quad (96)$$

with

$$\mathbf{Z}_\lambda = \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \mathbf{F}_{\lambda, \mathcal{A}}^+ e^{i \frac{u_{\mathcal{A}}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+ e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} + \mathcal{E}_\lambda^{ell} + \mathcal{E}_\lambda^{evo}, \quad (97)$$

solution to

$$\square \mathbf{Z}_\lambda = \mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda + \mathcal{L}_1(\mathbf{Z}_\lambda, \partial \mathbf{B}_0, \mathbf{B}'_0, \frac{\mathcal{U}_{\mathbb{A}}}{\lambda}) + \mathcal{L}_2(\partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{\mathcal{U}_{\mathbb{A}}}{\lambda}) + \mathcal{H}(\mathbf{Z}_\lambda, \mathbf{B}_0, \frac{\mathcal{U}_{\mathbb{A}}}{\lambda}) + \lambda^{1/2} \Xi + \check{\Xi}, \quad (98)$$

where the terms are more detailed in section 6.1.1.

We discuss all the construction of the initial data in section 5 and the notion of admissibility in the sense of the three definitions 5.9, 5.11 and 5.12.

The existence, the regularity and the smallness of  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$  and  $\mathcal{E}_\lambda^{ell}$  are given in sections 6.1.2 and 6.1.3. It remains to show the local well posedness of

$$\left\{ \begin{array}{l} \square \mathcal{E}_\lambda^{evo} = \mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo} + \sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} (i\partial u_{\mathcal{A}} \Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+) - \lambda^1 \mathbf{G}_{\lambda\mathcal{A}}^+) + [\dots], \\ \mathcal{L}_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ = \lambda^{-1/2} \partial u_{\mathcal{A}} \mathcal{E}_\lambda^{evo} \mathbf{B}_0 + \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+) + \partial u_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ \mathbf{B}_0, \\ \mathcal{L}_{\mathcal{A}} \mathbf{G}_{\lambda\mathcal{A}}^+ = [\mathcal{L}_{\mathcal{A}}, \square] \mathbf{F}_{\lambda\mathcal{A}}^+ + \lambda^{-1/2} \partial u_{\mathcal{A}} \square \mathcal{E}_\lambda^{evo} \mathbf{B}_0 + \square (\partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)), \\ + \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{G}_{\lambda\mathcal{A}}^+) - \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \square \mathbf{F}_{\lambda\mathcal{A}}^+) + \partial u_{\mathcal{A}} \mathbf{G}_{\lambda\mathcal{A}}^+ \mathbf{B}_0 + [\dots], \end{array} \right. \quad (99)$$

from which we determine  $\mathcal{E}_\lambda^{evo}$  and  $\mathbf{F}_{\lambda\mathcal{A}}^+$ , to show that  $\mathbf{Z}_\lambda$  (97) is in fact solution to (98) and to close the bootstrap. Recall that  $\mathbf{G}_{\lambda\mathcal{A}}^+$  is a auxiliary function introduced to deal with a loss of derivative.

In the following sections we first show the local well-posedness of (99). Then, for each of  $\mathcal{E}_\lambda^{evo}$ ,  $\mathbf{F}_{\lambda\mathcal{A}}^+$  and  $\mathbf{G}_{\lambda\mathcal{A}}^+$  we detail their roles, their evolution equations and their RHS. In particular, how to estimate them in  $H^k$  norms to obtain energy estimates and improve the a priori ones 6.16. Finally, gathering all the information on the different parameters we show that we have a solution to (2) and that there exists  $\lambda_0$  such that for  $\lambda < \lambda_0$  the time of existence  $T$  is uniform in  $\lambda$ .

### 6.3.1 Well-posedness

We reformulate point 1 of proposition 6.13 more precisely as follows :

**Proposition 6.19.** *For a given admissible background initial data set 2.24, let  $\mathbf{B}'_0$  be the background given by proposition 4.1 and let  $\lambda > 0$ . Let  $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$  be given by proposition 6.7 and  $\mathcal{E}_\lambda^{ell}$  be given by definition 6.9 from  $(\mathbf{f}_{\lambda\mathcal{A}}^+, \mathbf{g}_{\lambda\mathcal{A}}^+, \check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+, \mathbf{e}^{evo}, \dot{\mathbf{e}}^{evo}, \mathbf{e}^{ell}, \dot{\mathbf{e}}^{ell})$  admissible for KGML 5.9. Then, the problem (99) is well posed for the initial data  $(\mathbf{f}_{\lambda\mathcal{A}}^+, \mathbf{g}_{\lambda\mathcal{A}}^+, \mathbf{e}^{evo}, \dot{\mathbf{e}}^{evo})$ . There exists a time  $t_\lambda$  such that there exists a solution  $(\mathbf{F}_{\lambda\mathcal{A}}^+, \mathbf{G}_{\lambda\mathcal{A}}^+, \mathcal{E}_\lambda^{evo})$  with the following regularity<sup>42</sup>*

$$\forall \mathcal{A} \in \mathbb{A}, \mathbf{F}_{\lambda\mathcal{A}}^+ \in \bigcap_{j=0}^2 C^j([0, t_\lambda], H^{2-j}), \quad \forall \mathcal{A} \in \mathbb{A}, \mathbf{G}_{\lambda\mathcal{A}}^+ \in \bigcap_{j=0}^1 C^j([0, t_\lambda], H^{1-j}),$$

$$\underline{\mathcal{E}}_\lambda^{evo} \in \bigcap_{j=0}^2 C^j([0, t_\lambda], H^{2-j}), \quad (E^{evo})^0 \in \bigcap_{j=0}^2 C^j([0, t_\lambda], H_{\delta+i}^{2-j}).$$

Moreover, we have  $Supp(\mathbf{F}_{\lambda\mathcal{A}}^+, \mathbf{G}_{\lambda\mathcal{A}}^+, \underline{\mathcal{E}}_\lambda^{evo}) \subset B_S$ .

*Proof.* In terms of derivatives we get

$$\begin{aligned} \partial \mathcal{E}_\lambda^{evo} &\sim \partial \mathcal{E}_\lambda^{evo}, \quad \mathbf{G}_{\lambda\mathcal{A}}^+, \quad \partial \mathbf{F}_{\lambda\mathcal{A}}^+, \\ \mathbf{F}_{\lambda\mathcal{A}}^+ &\sim \mathbf{F}_{\lambda\mathcal{A}}^+, \quad \mathcal{E}_\lambda^{evo}, \\ \mathbf{G}_{\lambda\mathcal{A}}^+ &\sim \mathbf{G}_{\lambda\mathcal{A}}^+, \quad \partial \mathbf{F}_{\lambda\mathcal{A}}^+, \quad \partial \mathcal{E}_\lambda^{evo}. \end{aligned}$$

To give more details, in the equation for  $\mathbf{G}_{\lambda\mathcal{A}}^+$ , we use the evolution equation to replace  $\square \mathcal{E}_\lambda^{evo}$  by its RHS to get linear terms in  $\partial \mathcal{E}_\lambda^{evo}$ ,  $\partial \mathbf{F}_{\lambda\mathcal{A}}^+$  or  $\mathbf{G}_{\lambda\mathcal{A}}^+$  at most, all the other terms are in  $H^2$  and thus in  $L^\infty$ . We also use the results of section 8.2 on the commutator and we treat the nonlinearity  $\Pi_-(\partial \mathbf{F}_{\lambda\mathcal{A}}^+ \partial \mathcal{E}_\lambda^{evo})$  using the projector on lower frequency to make it as regular as we want. All the other nonlinearities of the system pass with the regularity described in the proposition. In fact, a lot of terms are estimated in details in the bootstrap parts 6.3.2, 6.3.3 and 6.3.4.

Finally, for the support property, we know that with initial data admissible for KGML we have

$Supp(\mathbf{f}_{\lambda\mathcal{A}}^+, \mathbf{g}_{\lambda\mathcal{A}}^+, \underline{\mathbf{e}}^{evo}, \dot{\underline{\mathbf{e}}}^{evo}) \subset B_{S'}$  and so by finite speed of propagation we recover the results.  $\square$

<sup>42</sup>We make a difference between  $\underline{\mathcal{E}}_\lambda$  and  $(E_\lambda^{evo})^0$  as in 6.15.

**Remark 6.20.** With the previous propositions 6.12, 6.7 and 6.19 we have  $\text{Supp}(\mathbf{F}_{\lambda\mathcal{A}}^+, \mathbf{G}_{\lambda\mathcal{A}}^+, \check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+, \underline{\boldsymbol{\mathcal{E}}}_{\lambda}^{evo}, \boldsymbol{\mathcal{E}}_{\lambda}^{ell}) \subset B_S$ . The quantity  $(E_{\lambda}^{evo})^0$  is not compactly supported but in  $H_{\delta}^2$ . Nonetheless, all the right hand side terms of the evolution equations (99) are compactly supported because  $A^0$  and  $(E_{\lambda}^{evo})^0$  are always in products with compactly supported terms. So, when doing estimates in sections 6.3.2, 6.3.3 and 6.3.4, the  $\|\cdot\|_{L^p}$  are really  $\|\cdot\|_{L^p(B_S)}$ . We also have

$$\|\cdot\|_{H^m(B_S)} \leq C_S \|\cdot\|_{H_{\delta}^m}.$$

Thus, we abuse notation to write all the weighted norms as classical Sobolev norms in the following estimates.

**Remark 6.21.** When we use the Strichartz estimates 8.2 and the lemmas 8.17 and 8.18 on  $(E_{\lambda}^{evo})^0$  in the following sections we need bounds on  $\|\partial_t(E_{\lambda}^{evo})^0(0)\|_{L^2}$  and  $\|(E_{\lambda}^{evo})^0(0)\|_{\dot{H}^1}$ . Because  $(E_{\lambda}^{evo})^0|_{t=0}$  (resp.  $\partial_t(E_{\lambda}^{evo})^0|_{t=0}$ ) is only bounded in  $H_{\delta}^2$  (resp.  $H_{\delta+1}^1$ ) for  $-3/2 < \delta < -1/2$ , we have no information on the previous norms. Nonetheless, we remark in 6.20 that  $(E_{\lambda}^{evo})^0$  appears always in products with compactly supported terms. In particular, in the following section, the equations only see  $(E_{\lambda}^{evo})^0|_{B_S}$ . The latter coincides with  $E_{\lambda}^0$  given by

$$\begin{cases} \square E_{\lambda}^0 = \square(E_{\lambda}^{evo})^0, \\ E_{\lambda}^0(t=0) = (E_{\lambda}^{evo})^0(t=0)|_{\mathcal{I}(T, B_S)}, \\ \partial_t E_{\lambda}^0(t=0) = (\partial_t(E_{\lambda}^{evo})^0)(t=0)|_{\mathcal{I}(T, B_S)}, \end{cases} \quad (100)$$

where  $\mathcal{I}(T, B_S)$  is the propagation of  $B_S$  defined in 2.5. Thus, we can use the Strichartz estimates on  $E_{\lambda}^0$ , which is in the right norms with its compact support.

### 6.3.2 Estimates for $\mathbf{F}_{\lambda\mathcal{A}}^+$

The transport equation for  $\mathbf{F}_{\lambda\mathcal{A}}^+$  is

$$\mathcal{L}_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ = \lambda^{-1/2} \partial u_{\mathcal{A}} \boldsymbol{\mathcal{E}}_{\lambda}^{evo} \mathbf{B}_0 + \partial u_{\mathcal{A}} \Pi_{-}(\boldsymbol{\mathcal{E}}_{\lambda}^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+) + \partial u_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ \mathbf{B}_0. \quad (101)$$

**Remark 6.22.** We introduce  $\mathbf{F}_{\lambda\mathcal{A}}^+$  specifically to absorb  $\lambda^{-1/2} \partial u_{\mathcal{A}} \boldsymbol{\mathcal{E}}_{\lambda}^{evo} \mathbf{B}_0$ , the  $\lambda^{-1/2}$  high frequency terms oscillating in the directions of the set of phases and arising from the interaction of the error term with the background. The equation (101) is **coupled** with the equations (109) and (117).

We give the detail of the RHS terms and their estimates in the next proposition.

**Proposition 6.23.** *The RHS terms*

$$\lambda^{-1/2} \partial u_{\mathcal{A}} \boldsymbol{\mathcal{E}}_{\lambda}^{evo} \mathbf{B}_0, \quad \partial u_{\mathcal{A}} \Pi_{-}(\boldsymbol{\mathcal{E}}_{\lambda}^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+) \quad \text{and} \quad \partial u_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ \mathbf{B}_0 \quad (102)$$

respectively come from  $\mathcal{L}_1(\mathbf{Z}_{\lambda}, \partial \mathbf{B}_0, \mathbf{B}'_0, \frac{\mathcal{Q}_{\lambda}}{\lambda})$ ,  $\mathbf{Z}_{\lambda} \partial \mathbf{Z}$  and  $\mathcal{L}_2(\partial \mathbf{Z}_{\lambda}, \mathbf{B}_0, \frac{\mathcal{Q}_{\lambda}}{\lambda})$ . Under the assumptions 6.16 and for any  $\tau \in [0, T]$ , we have the estimates

$$I := \sum_{k \leq 1} \int_0^{\tau} \lambda^k \|\partial u_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ \mathbf{B}_0(t)\|_{H^{k+1}} dt \leq C_0 \frac{c_1}{c_2} e^{\tau c_2}, \quad (103)$$

$$II := \sum_{k \leq 1} \int_0^{\tau} \lambda^k \|\lambda^{-1/2} \partial u_{\mathcal{A}} \boldsymbol{\mathcal{E}}_{\lambda}^{evo}(t)\|_{H^{k+1}} dt \leq C_0 \frac{c_1}{c_2} e^{\tau c_2}, \quad (104)$$

$$III := \sum_{k \leq 1} \int_0^{\tau} \lambda^k \|\partial u_{\mathcal{A}} \Pi_{-}(\boldsymbol{\mathcal{E}}_{\lambda}^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)(t)\|_{H^{k+1}} dt \leq C_{0,1,2,\tau} \lambda^{>0}. \quad (105)$$

*Proof.* We obtain the first two inequalities, for I and II, by direct calculations with Sobolev inequalities of section 8.3, the product estimate (172) and the assumptions 6.16. It fits the case c) as the terms are linear in the error parameters.

To get the inequality for III, we do not need to use the projector to have extra smallness (see case b)). We have

first

$$\begin{aligned}
& \int_0^\tau \|\partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+)(t)\|_{H^1} dt \\
& \leq \|\partial u_{\mathcal{A}}\|_{L^\infty([0,\tau], W^{1,\infty})} \int_0^\tau \|\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+(t)\|_{L^2} + \|\nabla \mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+(t)\|_{L^2} + \|\mathcal{E}_\lambda^{evo} \nabla \mathbf{F}_{\lambda \mathcal{A}}^+(t)\|_{L^2} dt \\
& \leq C_0 \left( \underbrace{\|\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)}}_{III.1} + \underbrace{\|\nabla \mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)}}_{III.2} + \underbrace{\|\mathcal{E}_\lambda^{evo} \nabla \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)}}_{III.3} \right).
\end{aligned}$$

For III.1 we used classical product inequality (172) to obtain

$$III.1 \leq C_{0,1,2,\tau} \lambda^{1/2}.$$

We use Lemma 8.18 to recover

$$\begin{aligned}
III.2 & \leq C_0 \|\nabla \mathcal{E}_\lambda^{evo}\|_{L^p([0,\tau], L^2)} (\|\mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^q([0,\tau], H^2)})^\theta (\|\square \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)} + \|\mathbf{F}_{\lambda \mathcal{A}}^+(0)\|_{\dot{H}^1} + \|\partial_t \mathbf{F}_{\lambda \mathcal{A}}^+(0)\|_{L^2})^{1-\theta} \\
& \leq C_{0,1,2,\tau} \lambda^{1/2} \lambda^{-\theta} (\|\mathbf{G}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)} + C_0)^{1-\theta} \leq C_{0,1,2,\tau} \lambda^{1/2-\theta} \leq C_{0,1,2,\tau} \lambda^{>0},
\end{aligned}$$

$$\begin{aligned}
III.3 & \leq C_0 \|\nabla \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^p([0,\tau], L^2)} (\|\mathcal{E}_\lambda^{evo}\|_{L^q([0,\tau], H^2)})^\theta (\|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], L^2)} + \|\mathcal{E}_\lambda^{evo}(0)\|_{\dot{H}^1} + \|\partial_t \mathcal{E}_\lambda^{evo}(0)\|_{L^2})^{1-\theta} \\
& \leq C_{0,1,2,\tau} \lambda^{-\theta/2} (\|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], L^2)} + C_0 \lambda^{1/2})^{1-\theta} \leq C_{0,1,2,\tau} \lambda^{1/2-\theta} \leq C_{0,1,2,\tau} \lambda^{>0},
\end{aligned}$$

for  $\theta \in (0, 1/2)$  and where  $p$  and  $q$  depend on  $\theta$ .

The inequality  $\|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], L^2)} \leq C_{0,1,2,\tau} \lambda^{1/2}$  comes from the naive estimates of proposition 6.33.

We have secondly

$$\begin{aligned}
& \lambda \int_0^\tau \|\partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+)(t)\|_{H^2} dt \leq \lambda \|\partial u_{\mathcal{A}}\|_{L^\infty([0,\tau], L^\infty)} \int_0^\tau \|\nabla \nabla (\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+)(t)\|_{L^2} dt \\
& + \lambda \int_0^\tau \underbrace{\|\partial u_{\mathcal{A}}(t)\|_{W^{1,\infty}} \|\nabla (\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+)(t)\|_{L^2} + \|\nabla \nabla \partial u_{\mathcal{A}}(t)\|_{L^2} \|\mathcal{E}_\lambda^{evo}(t)\|_{L^\infty} \|\mathbf{F}_{\lambda \mathcal{A}}^+(t)\|_{L^\infty} + \|\partial u_{\mathcal{A}} \mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+(t)\|_{H^1}}_{III.4} dt \\
& \leq III.4 + C_0 \lambda \underbrace{\|\nabla \mathcal{E}_\lambda^{evo} \nabla \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)}}_{III.5} + C_0 \lambda \underbrace{\|\nabla \nabla \mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)}}_{III.6} + C_0 \lambda \underbrace{\|\mathcal{E}_\lambda^{evo} \nabla \nabla \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^1([0,\tau], L^2)}}_{III.7}.
\end{aligned}$$

For III.4, we use the previous calculations, the Sobolev embeddings of proposition 8.7 and the product estimate of proposition 8.16 to get for a small  $0 < \varepsilon < 1/2$

$$III.4 \leq C_{0,1,2,\tau} \lambda^{1/2-\varepsilon}.$$

For III.6 and III.7 we also use Lemma 8.18 as with III.2 and III.3, losing one  $\lambda$  in the operation. To estimate III.5, we use lemma 8.17 to obtain

$$\begin{aligned}
III.5 & \leq \lambda C_0 \|\nabla \mathbf{F}_{\lambda \mathcal{A}}^+\|_{L^p([0,\tau], H^{1/2-\nu})} (\|\square \nabla \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], L^2)} + \|\nabla \mathcal{E}_\lambda^{evo}(0)\|_{\dot{H}^1} + \|\partial_t \nabla \mathcal{E}_\lambda^{evo}(0)\|_{L^2}) \\
& \leq C_{0,1,2,\tau} \lambda^{1/2+\nu} (\|\square \nabla \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], L^2)} + \lambda^{-1/2} C_0) \\
& \leq C_{0,1,2,\tau} \lambda^{>0},
\end{aligned}$$

for  $\nu > \varepsilon'$ , where  $\varepsilon'$  is from the naive estimates of proposition 6.33 and where  $p$  depends on  $1/2 > \nu > 0$ .

Finally, we obtain

$$III \leq C_0 (III.1 + III.2 + III.3 + III.4 + III.5 + III.6 + III.7) \leq C_{0,1,2,\tau} \lambda^{>0}.$$

□

### 6.3.3 Estimates for $\mathbf{G}_{\lambda\mathcal{A}}^+$

The transport equation for  $\mathbf{G}_{\lambda\mathcal{A}}^+$  is

$$\mathcal{L}_{\mathcal{A}}\mathbf{G}_{\lambda\mathcal{A}}^+ = [(2\partial^\alpha u_{\mathcal{A}}\partial_\alpha + \square u_{\mathcal{A}}), \square]\mathbf{F}_{\lambda\mathcal{A}}^+ + \lambda^{-1/2}\partial u_{\mathcal{A}}\square\mathcal{E}_\lambda^{evo}\mathbf{B}_0 + \square(\partial u_{\mathcal{A}}\Pi_-(\mathcal{E}_\lambda^{evo}\mathbf{F}_{\lambda\mathcal{A}}^+)) \quad (106)$$

$$+ \partial u_{\mathcal{A}}\Pi_-(\mathcal{E}_\lambda^{evo}\mathbf{G}_{\lambda\mathcal{A}}^+) - \partial u_{\mathcal{A}}\Pi_-(\mathcal{E}_\lambda^{evo}\square\mathbf{F}_{\lambda\mathcal{A}}^+) + \partial u_{\mathcal{A}}\mathbf{G}_{\lambda\mathcal{A}}^+\mathbf{B}_0 + [\dots]. \quad (107)$$

$$(108)$$

**Remark 6.24.** This *auxiliary function*  $\mathbf{G}_{\lambda\mathcal{A}}^+$  is here to estimate  $\square\mathbf{F}_{\lambda\mathcal{A}}^+$  and fight the loss of derivatives. The equation (106) is obtained by commuting the transport equation for  $\mathbf{F}_{\lambda\mathcal{A}}^+$  with the d'Alembertian, that is

$$\mathcal{L}_{\mathcal{A}}\mathbf{G}_{\lambda\mathcal{A}}^+ = [(2\partial^\alpha u_{\mathcal{A}}\partial_\alpha + \square u_{\mathcal{A}}), \square]\mathbf{F}_{\lambda\mathcal{A}}^+ + \square(\mathcal{L}_{\mathcal{A}}\mathbf{F}_{\mathcal{A}}), \quad (109)$$

where we replace all the appearances of  $\square\mathbf{F}_{\lambda\mathcal{A}}^+$  by  $\mathbf{G}_{\lambda\mathcal{A}}^+$ . The quantities  $\square\mathbf{F}_{\lambda\mathcal{A}}^+$  and  $\mathbf{G}_{\lambda\mathcal{A}}^+$  have the same initial data, see definition 5.9, and obey the same transport equation so they are equal. The equation (106) is **coupled** with the equations (101) and (117).

We give the detail of the RHS terms and their estimates in the next proposition.

**Proposition 6.25.** The  $[\dots]$  terms have the shape

$$[\dots] = \square(\partial u_{\mathcal{A}}\mathbf{F}_{\lambda\mathcal{A}}^+\mathbf{B}_0) - \partial u_{\mathcal{A}}\square\mathbf{F}_{\lambda\mathcal{A}}^+\mathbf{B}_0 + \square(\lambda^{-1/2}\partial u_{\mathcal{A}}\mathcal{E}_\lambda^{evo}\mathbf{B}_0) - \lambda^{-1/2}\partial u_{\mathcal{A}}\square\mathcal{E}_\lambda^{evo}\mathbf{B}_0, \quad (110)$$

the other terms are explicit. Under the assumptions 6.16 and for any  $\tau \in [0, T]$ , for the  $[\dots]$  we have

$$IV := \sum_{k \leq 1} \int_0^\tau \lambda^k \|[\dots](t)\|_{H^k} dt \leq C_0 \frac{c_1}{c_2} e^{\tau c_2}, \quad (111)$$

for the commutator we have

$$V := \sum_{k \leq 1} \int_0^\tau \lambda^k \|[(2\partial^\alpha u_{\mathcal{A}}\partial_\alpha + \square u_{\mathcal{A}}), \square]\mathbf{F}_{\mathcal{A}}(t)\|_{H^k} dt \leq C_0 \frac{c_1}{c_2} e^{\tau c_2} + C_{0,1,2,\tau} \lambda^{>0}, \quad (112)$$

for the remaining terms without a projector we have

$$VI := \sum_{k \leq 1} \int_0^\tau \lambda^k \|\partial u_{\mathcal{A}}\mathbf{G}_{\lambda\mathcal{A}}^+\mathbf{B}_0(t)\|_{H^k} dt \leq C_0 \frac{c_1}{c_2} e^{\tau c_2}, \quad (113)$$

$$VII := \sum_{k \leq 1} \int_0^\tau \lambda^k \|\lambda^{-1/2}\partial u_{\mathcal{A}}\square\mathcal{E}_\lambda^{evo}\mathbf{B}_0(t)\|_{H^k} dt \leq C_0 \frac{c_1}{c_2} e^{\tau c_2} + C_0 + C_{0,1,2,\tau} \lambda^{>0}, \quad (114)$$

and finally, for the terms involving the projector on low frequencies we have

$$VIII := \sum_{k \leq 1} \int_0^\tau \lambda^k \|\partial u_{\mathcal{A}}\Pi_-(\mathcal{E}_\lambda^{evo}\mathbf{G}_{\lambda\mathcal{A}}^+)(t)\|_{H^k} dt \leq C_{0,1,2,\tau} \lambda^{>0}, \quad (115)$$

$$IX := \sum_{k \leq 1} \int_0^\tau \lambda^k \|\square(\partial u_{\mathcal{A}}\Pi_-(\mathcal{E}_\lambda^{evo}\mathbf{F}_{\lambda\mathcal{A}}^+))(t) - \partial u_{\mathcal{A}}\Pi_-(\mathcal{E}_\lambda^{evo}\square\mathbf{F}_{\lambda\mathcal{A}}^+)(t)\|_{H^k} dt \leq C_{0,1,2,\tau} \lambda^{>0}. \quad (116)$$

*Proof.* We obtain the  $[\dots]$  directly from remark 6.24. To derive the estimates on IV we use directly the assumptions 6.16. There are only linear terms in  $\mathcal{E}_\lambda^{evo}$  and  $\mathbf{F}_{\lambda\mathcal{A}}^+$  with at most one derivative, we get the case c). We use the better regularity of the background.

For V, using (161) we get

$$\begin{aligned} & \sum_{k \leq 1} \int_0^\tau \lambda^k \|[(2\partial^\alpha u_{\mathcal{A}}\partial_\alpha + \square u_{\mathcal{A}}), \square]\mathbf{F}_{\mathcal{A}}(t)\|_{H^k} dt \\ & \leq C_0 \sum_{k \leq 1} \int_0^\tau \lambda^k (\|\mathbf{G}_{\lambda\mathcal{A}}^+(t)\|_{H^k} + \|\partial(2\partial^\alpha u_{\mathcal{A}}\partial_\alpha + \square u_{\mathcal{A}})\mathbf{F}_{\lambda\mathcal{A}}^+(t)\|_{H^k} + \|\mathbf{F}_{\lambda\mathcal{A}}^+(t)\|_{H^{k+1}}) dt. \end{aligned}$$

Then, with the estimates on  $\mathcal{L}_{\mathcal{A}} \partial_{\alpha} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+}$  of proposition 6.23 and the assumptions 6.16 we get the desired inequality directly.

For VI the inequality is direct with assumption 6.16, we recover the case c). The quantity VII is estimated with the improved estimate on  $\|\square \mathcal{E}_{\lambda}^{evo}\|_{L^1([0,\tau], H^k)}$  given all along section 6.3.4. In both cases we use the better regularity of the background<sup>43</sup>.

For VIII and IX, we use the projector<sup>44</sup> and the product inequality (172) to recover extra smallness (case b)). Firstly, for VIII

$$\begin{aligned}
VIII &\leq \sum_{k \leq 1} \lambda^k \|\partial u_{\mathcal{A}}\|_{L^{\infty}([0,\tau], W^{1,\infty})} \|\Pi_{-}(\mathbf{G}_{\lambda_{\mathcal{A}}}^{+} \mathcal{E}_{\lambda}^{evo})\|_{L^1([0,\tau], H^k)} \\
&\leq C_0 \sum_{k \leq 1} \lambda^k \lambda^{-\kappa/2} \|\mathbf{G}_{\lambda_{\mathcal{A}}}^{+} \mathcal{E}_{\lambda}^{evo}\|_{L^1([0,\tau], H^{k-1/2})} \\
&\leq C_0 \sum_{k \leq 1} \lambda^{k-\kappa/2} \|\mathcal{E}_{\lambda}^{evo}\|_{L^{\infty}([0,\tau], H^1)} \|\mathbf{G}_{\lambda_{\mathcal{A}}}^{+}\|_{L^1([0,\tau], H^k)} \\
&\leq C_{0,1,2,\tau} \lambda^{1/2-\kappa} \leq C_{0,1,2,\tau} \lambda^{>0},
\end{aligned}$$

where we use the assumption 6.16 and the product inequality from proposition 8.16 for both  $k = 0$  and  $k = 1$ . For IX, we first develop the term as

$$\begin{aligned}
IX &\leq \underbrace{\sum_{k \leq 1} \lambda^k \|\square \partial u_{\mathcal{A}} \Pi_{-}(\mathcal{E}_{\lambda}^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+})\|_{L^1([0,\tau], H^k)}}_{IX.1} + \underbrace{\sum_{k \leq 1} \lambda^k \|\partial \partial u_{\mathcal{A}} \partial \Pi_{-}(\mathcal{E}_{\lambda}^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+})\|_{L^1([0,\tau], H^k)}}_{IX.2} \\
&+ \underbrace{\sum_{k \leq 1} \lambda^k \|\partial u_{\mathcal{A}} \Pi_{-}(\partial \mathcal{E}_{\lambda}^{evo} \partial \mathbf{F}_{\lambda_{\mathcal{A}}}^{+})\|_{L^1([0,\tau], H^k)}}_{IX.3} + \underbrace{\sum_{k \leq 1} \lambda^k \|\partial u_{\mathcal{A}} \Pi_{-}(\square \mathcal{E}_{\lambda}^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+})\|_{L^1([0,\tau], H^k)}}_{IX.4},
\end{aligned}$$

then for IX.1 and IX.2

$$\begin{aligned}
IX.1 &\leq \sum_{k \leq 1} \lambda^k \|\square \partial u_{\mathcal{A}}\|_{L^{\infty}([0,\tau], H^1)} \|\Pi_{-}(\mathcal{E}_{\lambda}^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+})\|_{L^1([0,\tau], H^{k+1/2+\varepsilon})} \\
&\leq C_0 \sum_{k \leq 1} \lambda^{k-\kappa-\varepsilon\kappa} \|\mathcal{E}_{\lambda}^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+}\|_{L^1([0,\tau], H^{k-1/2})} \\
&\leq C_0 \sum_{k \leq 1} \lambda^{k-\kappa-\varepsilon\kappa} \|\mathcal{E}_{\lambda}^{evo}\|_{L^2([0,\tau], H^1)} \|\mathbf{F}_{\lambda_{\mathcal{A}}}^{+}\|_{L^2([0,\tau], H^k)} \\
&\leq C_{0,1,2,\tau} \lambda^{1/2-\kappa-\varepsilon\kappa},
\end{aligned}$$

$$\begin{aligned}
IX.2 &\leq \sum_{k \leq 1} \lambda^k \|\partial \partial u_{\mathcal{A}}\|_{L^{\infty}([0,\tau], H^1)} \|\Pi_{-}(\partial(\mathcal{E}_{\lambda}^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+}))\|_{L^1([0,\tau], H^{k+1/2+\varepsilon})} \\
&\leq C_0 \sum_{k \leq 1} \lambda^{k-\kappa-\varepsilon\kappa} \|\partial(\mathcal{E}_{\lambda}^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^{+})\|_{L^1([0,\tau], H^{k-1/2})} \\
&\leq C_0 \sum_{k \leq 1} \lambda^{k-\kappa-\varepsilon\kappa} (\|\partial \mathcal{E}_{\lambda}^{evo}\|_{L^{\infty}([0,\tau], H^k)} \|\mathbf{F}_{\lambda_{\mathcal{A}}}^{+}\|_{L^1([0,\tau], H^1)} + \|\partial \mathbf{F}_{\lambda_{\mathcal{A}}}^{+}\|_{L^{\infty}([0,\tau], H^k)} \|\mathcal{E}_{\lambda}^{evo}\|_{L^1([0,\tau], H^1)}) \\
&\leq C_{0,1,2,\tau} \lambda^{1/2-\kappa-\varepsilon\kappa},
\end{aligned}$$

where  $\varepsilon > 0$  is small (but not 0 as we have to control the  $L^{\infty}$  norm).

<sup>43</sup>One derivative of  $\mathbf{B}'_0$  is in  $L^{\infty}$ .

<sup>44</sup>In particular, the fact that the cost a derivative is very low,  $\kappa$  is as small as we want.

For IX.3 and IX.4, we have

$$\begin{aligned}
IX.3 &\leq \sum_{k \leq 1} \lambda^k \|\partial u_{\mathcal{A}}(t)\|_{L^\infty([0,\tau], W^{1,\infty})} \|\Pi_- (\partial \mathcal{E}_\lambda^{evo} \partial \mathbf{F}_{\lambda \mathcal{A}}^+) \|_{L^1([0,\tau], H^k)} \\
&\leq C_0 \sum_{k \leq 1} \lambda^{k-\kappa} \|\partial \mathcal{E}_\lambda^{evo} \partial \mathbf{F}_{\lambda \mathcal{A}}^+ \|_{L^1([0,\tau], H^{k-1})} \\
&\leq C_0 \sum_{k \leq 1} \lambda^{k-\kappa} \|\partial \mathcal{E}_\lambda^{evo}\|_{L^2([0,\tau], H^k)} \|\partial \mathbf{F}_{\lambda \mathcal{A}}^+(t)\|_{L^2([0,\tau], H^{1/2})} \\
&\leq C_0 \lambda^{1/2-\kappa},
\end{aligned}$$

$$\begin{aligned}
IX.4 &\leq \sum_{k \leq 1} \lambda^k \|\partial u_{\mathcal{A}}\|_{L^\infty([0,\tau], W^{1,\infty})} \|\Pi_- (\square \mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+) \|_{L^1([0,\tau], H^k)} \\
&\leq C_0 \sum_{k \leq 1} \lambda^k \lambda^{-\kappa/2} \|\square \mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+ \|_{L^1([0,\tau], H^{k-1/2})} \\
&\leq C_0 \sum_{k \leq 1} \lambda^k \lambda^{-\kappa/2} \|\mathbf{F}_{\lambda \mathcal{A}}^+ \|_{L^\infty([0,\tau], H^1)} \|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], H^k)} \\
&\leq C_{0,1,2,\tau} \lambda^{1/2-\kappa/2},
\end{aligned}$$

where we use the naive estimates 6.33 for  $\|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], H^k)}$  with  $k = 0, 1$ .

This gives

$$IX \leq IX.1 + IX.2 + IX.3 + IX.4 \leq C_{0,1,2,\tau} \lambda^{1/2-\kappa-\varepsilon} \leq C_{0,1,2,\tau} \lambda^{>0},$$

and it ends the proof.  $\square$

**Remark 6.26.** *The only term that really requires the use of the projector on low frequency is  $\partial \mathcal{E}_\lambda^{evo} \partial \mathbf{F}_{\lambda \mathcal{A}}^+$  which is a null form. The others can be treated with the lemmas 8.17 and 8.18.*

### 6.3.4 Estimates for $\mathcal{E}_\lambda^{evo}$

The wave equation for  $\mathcal{E}_\lambda^{evo}$  is

$$\square \mathcal{E}_\lambda^{evo} = \mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo} + \sum_{\mathcal{A} \in \mathbb{A}} e^{i \frac{u_{\mathcal{A}}}{\lambda}} (i \partial u_{\mathcal{A}} \Pi_+ (\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+) - \lambda^1 \mathbf{G}_{\lambda \mathcal{A}}^+) + [\dots]. \quad (117)$$

**Remark 6.27.** *We use  $\mathcal{E}_\lambda^{evo}$  to absorb all the remaining terms of the equation (98). The equation (117) is **coupled** with the equations (101) and (106).*

The next propositions give the details on the  $[\dots]$  term and all the estimates of the RHS of  $\square \mathcal{E}_\lambda^{evo}$ .

**Proposition 6.28.** *The  $[\dots]$  are separated and estimated term by term. Under the assumptions 6.16 and for any*

$\tau \in [0, T]$ , we have

$$X := \sum_{k \leq 1} \int_0^\tau \lambda^k \|(\mathcal{L}_1(\mathbf{Z}_\lambda, \partial \mathbf{B}_0, \mathbf{B}'_0, \frac{u}{\lambda}) - \lambda^{-1/2} \sum_{\mathcal{A} \in \mathbb{A}} i \partial u_{\mathcal{A}} \mathcal{E}_\lambda^{evo} \mathbf{B}_0 e^{i \frac{u_{\mathcal{A}}}{\lambda}})(t)\|_{H^k} dt \leq \lambda^{1/2} C_0 \frac{c_1}{c_2} e^{\tau c_2}, \quad (118)$$

$$XI := \sum_{k \leq 1} \int_0^\tau \lambda^k \|(\mathcal{L}_2(\partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda}) - A - B)(t)\|_{H^k} dt \leq \lambda^{1/2} C_0 \frac{c_1}{c_2} e^{\tau c_2}, \quad (119)$$

$$\text{for } A = \sum_{\mathcal{A} \in \mathbb{A}} i e^{i \frac{u_{\mathcal{A}}}{\lambda}} \partial u_{\mathcal{A}} \mathbf{F}_{\lambda \mathcal{A}}^+ \mathbf{B}_0 \text{ and } B = \sum_{(\mathcal{A}, \mathcal{B}) \in \mathbb{C}, \mathcal{A} < \mathcal{B}} i e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} (\partial u_{\mathcal{A}} \pm \partial u_{\mathcal{B}}) \mathbf{B}_0 \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+,$$

$$XII := \sum_{k \leq 1} \int_0^\tau \lambda^k \|\mathcal{H}(\mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda})(t)\|_{H^k} dt \leq \lambda^{>1/2} C_{0,1,2,\tau}, \quad (120)$$

$$XIII := \sum_{k \leq 1} \int_0^\tau \lambda^k \|(\mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda - C - D)(t)\|_{H^k} dt \leq \lambda^{>1/2} C_{0,1,2,\tau} + \lambda^{1/2} C_0 \frac{c_1}{c_2} e^{\tau c_2}, \quad (121)$$

$$\text{for } C = \mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo} \text{ and } D = \sum_{\mathcal{A} \in \mathbb{A}} i e^{i \frac{u_{\mathcal{A}}}{\lambda}} \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda \mathcal{A}}^+),$$

$$XIV := \sum_{k \leq 1} \int_0^\tau \lambda^k \left\| \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \square \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+(t) \right\|_{H^k} dt \leq \lambda^1 C_0, \quad (122)$$

$$XV := \sum_{k \leq 1} \int_0^\tau \lambda^k \left\| \square \mathcal{E}_\lambda^{ell} - \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \mathbf{S}_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0)(t) \right\|_{H^k} dt \leq \lambda^{1/2} C_0, \quad (123)$$

$$XVI := \sum_{k \leq 1} \int_0^\tau \lambda \|\lambda^{1/2} \check{\Xi}_\lambda(t)\|_{H^k} dt \leq \lambda^{1/2} C_0, \quad (124)$$

$$(125)$$

where the functions  $\mathcal{H}$ ,  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are given in (98). Overall, we have

$$\sum_{k \leq 1} \int_0^\tau \lambda^k \|[\dots](t)\|_{H^k} dt \leq C_0 \lambda^{1/2} + \lambda^{>1/2} C_{0,1,2,\tau} + \lambda^{1/2} C_0 \frac{c_1}{c_2} e^{\tau c_2}. \quad (126)$$

$$(127)$$

*Proof.* We recall that

$$\square \mathbf{Z}_\lambda = \mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda + \mathcal{L}_1(\mathbf{Z}_\lambda, \partial \mathbf{B}_0, \mathbf{B}'_0, \frac{u}{\lambda}) + \mathcal{L}_2(\partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda}) + \mathcal{H}(\mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda}) + \lambda^{1/2} \check{\Xi}_\lambda + \check{\Xi}_\lambda. \quad (128)$$

First, we can write

$$\begin{aligned} & \sum_{(\mathcal{A}, \mathcal{B}) \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda \square (e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+) - \mathcal{L}_2(\partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda}) - \check{\Xi}_\lambda = \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \square \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+, \\ & - \mathcal{L}_2(\partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda}) + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} i e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} (\partial u_{\mathcal{A}} \pm \partial u_{\mathcal{B}}) \mathbf{B}_0 \check{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+ - \check{\Xi}_\lambda, \\ & + \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \mathbf{K}_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0), \end{aligned}$$

from remark 6.6 and we can write from proposition 4.6 that

$$\check{\Xi}_\lambda - \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \mathbf{K}_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0) = \sum_{(\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}} e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \mathbf{S}_{(\mathcal{A} \pm \mathcal{B})}(\mathbf{B}'_0).$$



Secondly, we can write

$$\begin{aligned}
& \sum_{\mathcal{A} \in \mathbb{A}} \lambda \square (e^{i\frac{u_{\mathcal{A}}}{\lambda}} \mathbf{F}_{\lambda\mathcal{A}}^+) - \mathcal{L}_1(\mathbf{Z}_\lambda, \partial \mathbf{B}_0, \mathbf{B}'_0, \frac{u}{\lambda}) - \mathcal{L}_2(\partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda}) - \mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda = \sum_{\mathcal{A} \in \mathbb{A}} \lambda e^{i\frac{u_{\mathcal{A}}}{\lambda}} \square \mathbf{F}_{\lambda\mathcal{A}}^+ \\
& - \mathcal{L}_1(\mathbf{Z}_\lambda, \partial \mathbf{B}_0, \mathbf{B}'_0, \frac{u}{\lambda}) + \lambda^{-1/2} \sum_{\mathcal{A} \in \mathbb{A}} i \partial u_{\mathcal{A}} \mathcal{E}_\lambda^{evo} \mathbf{B}_0 e^{i\frac{u_{\mathcal{A}}}{\lambda}} - \mathcal{L}_2(\partial \mathbf{Z}_\lambda, \mathbf{B}_0, \frac{u}{\lambda}) + \sum_{\mathcal{A} \in \mathbb{A}} i e^{i\frac{u_{\mathcal{A}}}{\lambda}} \partial u_{\mathcal{A}} \mathbf{F}_{\lambda\mathcal{A}}^+ \mathbf{B}_0 \\
& - \mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda + \sum_{\mathcal{A} \in \mathbb{A}} i e^{i\frac{u_{\mathcal{A}}}{\lambda}} \partial u_{\mathcal{A}} \Pi_-(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+),
\end{aligned}$$

by proposition 6.23. We plug the parametrix (97) in the equation (98). Then, we assemble the previous calculations, we replace the  $\square \mathbf{F}_{\lambda\mathcal{A}}^+$  by  $\mathbf{G}_{\lambda\mathcal{A}}^+$ , we separate  $\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}$  and  $\sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} (i \partial u_{\mathcal{A}} \Pi_+ \mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)$  from  $\mathbf{Z}_\lambda \partial \mathbf{Z}_\lambda$  and we pass  $\square \mathcal{E}_\lambda^{ell}$  to the RHS. We recover the full equation for  $\mathcal{E}_\lambda^{evo}$  and in particular the [...].

The quantities X and XI are linear in the coupled evolution parameters (case c)) so we get the estimates directly by assumptions and using the regularity of the background.

The quantity XII is at least quadratic in terms holding no derivatives, thus we have for free extra smallness with the product estimate 8.16 (see case b)).

For XIII, we see that we took off the problematic terms. One can remark that there is a product between  $\mathcal{E}_\lambda^{evo}$  and  $\tilde{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+$  without any  $\lambda$  but as the latter is treated as a background term (a decoupled error parameter) it is in fact linear in  $\mathcal{E}_\lambda^{evo}$  (a coupled error parameter). Again, because  $\tilde{\mathbf{F}}_{\mathcal{A} \pm \mathcal{B}}^+$  works as a background term we obtain directly the inequality for XIV by the assumptions 6.16 (case c)).

For XV, we use 6.12.

The estimate for XVI is direct by definition of  $\Xi$  in 3.1, it is the case a).

Finally, for [...] we just add all the estimates established above.  $\square$

**Proposition 6.29.** *Under the assumptions 6.16 and for any  $\tau \in [0, T]$ , for  $-\sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} \lambda^1 \mathbf{G}_{\lambda\mathcal{A}}^+$  we have the inequality*

$$XVII := \sum_{k \leq 1} \int_0^\tau \lambda^k \left\| \sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} \lambda^1 \mathbf{G}_{\lambda\mathcal{A}}^+(t) \right\|_{H^k} dt \leq \lambda^{>1/2} C_{0,1,2,\tau}. \quad (129)$$

*Proof.* By direct calculations using the assumptions 6.16 (case c)) on  $\mathbf{G}_{\lambda\mathcal{A}}^+$ .  $\square$

**Proposition 6.30.** *Under the assumptions 6.16 and for any  $\tau \in [0, T]$ , for  $\sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} i \partial u_{\mathcal{A}} \Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)$  we have*

$$XVIII := \sum_{k \leq 1} \int_0^\tau \lambda^k \left\| \sum_{\mathcal{A} \in \mathbb{A}} e^{i\frac{u_{\mathcal{A}}}{\lambda}} i \partial u_{\mathcal{A}} \Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)(t) \right\|_{H^k} dt \leq \lambda^{>1/2} C_{0,1,2,\tau}. \quad (130)$$

*Proof.* For the  $L^2$  norm, we have

$$\begin{aligned}
XVIII.1 & := \int_0^\tau \|\partial u_{\mathcal{A}} \Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)(t)\|_{L^2} dt \leq \|\partial u_{\mathcal{A}}\|_{L^\infty([0,\tau], W^{1,\infty})} \int_0^\tau \|\Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)(t)\|_{L^2} dt \\
& \leq C_0 \int_0^\tau \left( \int_{\mathbb{R}^3} |\Pi_+(\widehat{\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+})(t)|^2(\xi) \frac{|\xi|}{|\xi|} d\xi \right)^{1/2} dt \\
& \leq C_0 \int_0^\tau \left( \int_{(1/\lambda)^\kappa}^\infty |(\widehat{\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+})(t)|^2(\xi) \frac{|\xi|}{|\xi|} |\xi|^2 d\xi \right)^{1/2} dt \\
& \leq C_0 \lambda^{\kappa/2} \int_0^\tau \|(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda\mathcal{A}}^+)(t)\|_{H^{1/2}} dt \\
& \leq C_0 \lambda^{\kappa/2} \int_0^\tau \|\mathcal{E}_\lambda^{evo}(t)\|_{H^1} \|\mathbf{F}_{\lambda\mathcal{A}}^+(t)\|_{H^1} dt \leq C_{0,1,2,\tau} \lambda^{>1/2},
\end{aligned}$$

where we use the Plancherel theorem, the product estimate 8.16 and the assumptions 6.16 on  $\mathcal{E}_\lambda^{evo}$  and  $\mathbf{F}_{\lambda\mathcal{A}}^+$ . We see that without the extra smallness  $\lambda^{\kappa/2}$  we cannot close the bootstrap as we have a quadratic term in  $O(\lambda^{1/2})$ .

On the other hand, for the  $H^1$  norm we do not need to use the projector to get the extra smallness (case b)), we have

$$\begin{aligned}
XVIII.2 &:= \lambda^1 \int_0^\tau \|\partial u_{\mathcal{A}} \Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^+)(t)\|_{H^1} dt \leq \lambda^1 \|\partial u_{\mathcal{A}}\|_{L^\infty([0,\tau],W^{1,\infty})} \int_0^\tau \|\Pi_+(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^+)(t)\|_{H^1} dt \\
&\leq C_0 \lambda^1 \int_0^\tau \|(\mathcal{E}_\lambda^{evo} \mathbf{F}_{\lambda_{\mathcal{A}}}^+)(t)\|_{H^1} dt \\
&\leq C_0 \lambda^1 \int_0^\tau \|\mathcal{E}_\lambda^{evo}(t)\|_{H^{5/4}} \|\mathbf{F}_{\lambda_{\mathcal{A}}}^+(t)\|_{H^{5/4}} dt \\
&\leq C_{0,1,2,\tau} \lambda^1 \lambda^{1/4} \lambda^{-1/4} \leq C_{0,1,2,\tau} \lambda^{>1/2},
\end{aligned}$$

by the product estimate 8.16 and interpolation 8.10.

We conclude that  $XVIII \leq XVIII.1 + XVIII.2 \leq C_{0,1,2,\tau} \lambda^{>1/2}$ .  $\square$

It remains to estimate the quadratic term  $\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}$ .

**Proposition 6.31.** *Under the assumptions 6.16 and for any  $\tau \in [0, T]$ , for  $\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}$  we have*

$$XIX := \sum_{k \leq 1} \int_0^\tau \lambda^k \|\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}(t)\|_{H^k} dt \leq \lambda^{>1/2} C_{0,1,2,\tau}. \quad (131)$$

**Remark 6.32.** *Using only product estimates and Sobolev interpolation (8.16 and 8.10) we are missing  $\lambda^\varepsilon$  to close the bootstrap (case b)). Indeed, for  $0 < \alpha < 3/2$ ,  $0 < \beta \leq 1$  and  $\alpha + \beta = 3/2$  we have*

$$\|\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}\|_{L^2} \leq \|\mathcal{E}_\lambda^{evo}\|_{H^\alpha} \|\partial \mathcal{E}_\lambda^{evo}\|_{H^\beta},$$

which leads to the cases

$$1/2 \leq \alpha \leq 1, \quad \|\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}(\tau)\|_{L^2} \leq \lambda^{-1/2+\alpha} (c_1)^2 e^{2\tau c_2}, \quad 1 < \alpha < 3/2, \quad \|\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}(\tau)\|_{L^2} \leq \lambda^{1/2} (c_1)^2 e^{2\tau c_2},$$

by interpolation. Moreover, for the  $\dot{H}^1$  we use the same techniques and we get

$$\|\nabla \mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}(\tau)\|_{L^2} \leq \lambda^{-1/2} C_{0,1,2,\tau}, \quad \|\mathcal{E}_\lambda^{evo} \nabla \partial \mathcal{E}_\lambda^{evo}(\tau)\|_{L^2} \leq \lambda^{-1/2-\varepsilon} C_{0,1,2,\tau},$$

where we lose a small  $\varepsilon$  because we need to control the  $L^\infty$  norm of  $\mathcal{E}_\lambda^{evo}$ . This gives us at best

$$XIX \leq C_{0,1,2,\tau} \lambda^{1/2-\varepsilon}.$$

We observe that we did not use at all the integral in time.

**Proposition 6.33.** *We assume 6.16. With all the previous estimates (6.28, 6.29 and 6.30) and using the bad product estimates from the latter remark 6.32 on  $\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}$ , we have, for  $0 < \varepsilon'$  (as small as we want), the naive estimates*

$$\|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau],L^2)} \leq C_{0,1,2,\tau} \lambda^{1/2} + C_0 \lambda^{1/2}, \quad \|\square \nabla \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau],L^2)} \leq C_{0,1,2,\tau} \lambda^{-1/2-\varepsilon'} + C_0 \lambda^{-1/2}, \quad (132)$$

where the  $C_0 \frac{c_1}{c_2} e^{\tau c_2}$  can be controlled by  $C_{0,1,2,\tau}$ . We see from remark 6.20 that even  $\square(E_\lambda^{evo})^0$  is in  $L^2$  in space.

*Proof of 6.31.* Let

$$XIX.1 := \|\mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau],L^2)}, \quad XIX.2 := \lambda \|\mathcal{E}_\lambda^{evo} \nabla \partial \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau],L^2)}, \quad XIX.3 := \lambda \|\nabla \mathcal{E}_\lambda^{evo} \partial \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau],L^2)}.$$

We use Lemma 8.18 to get

$$\begin{aligned}
XIX.2 &\lesssim \lambda \|\nabla \partial \mathcal{E}_\lambda^{evo}\|_{L^p([0,\tau],L^2)} (\|\mathcal{E}_\lambda^{evo}\|_{L^q([0,\tau],H^2)})^\theta (\|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau],L^2)} + \|\mathcal{E}_\lambda^{evo}(0)\|_{\dot{H}^1} + \|\partial_t \mathcal{E}_\lambda^{evo}(0)\|_{L^2})^{(1-\theta)} \\
&\leq C_{0,1,2,\tau} \lambda^{1/2} \lambda^{-\theta/2} (\|\square \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau],L^2)} + c_1 \lambda^{1/2})^{(1-\theta)} \\
&\leq C_{0,1,2,\tau} \lambda^{>1/2},
\end{aligned}$$

for  $\theta \in (0, 1/2)$  and  $p$  and  $q$  functions of  $\theta$ . We used the naive estimates of 6.33 to obtain the last line. This method is directly applicable to XIX.1 to have

$$XIX.1 \leq C_{0,1,2,\tau} \lambda^{>1/2}.$$

Now, we use Lemma 8.17 to get

$$\begin{aligned} XIX.3 &\lesssim \lambda \|\partial \mathcal{E}_\lambda^{evo}\|_{L^p([0,\tau], H^{1/2-\nu})} (\|\square \nabla \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], L^2)} + \|\nabla \mathcal{E}_\lambda^{evo}(0)\|_{\dot{H}^1} + \|\partial_t \nabla \mathcal{E}_\lambda^{evo}(0)\|_{L^2}) \\ &\leq \lambda^{1+\nu} (\|\square \nabla \mathcal{E}_\lambda^{evo}\|_{L^1([0,\tau], L^2)} + c_1 \lambda^{-1/2}) \\ &\leq C_{0,1,2,\tau} \lambda^{>1/2}, \end{aligned}$$

for  $\nu > \varepsilon'$ , where  $\varepsilon'$  is from proposition 6.33 and where  $p$  depends on  $1/2 > \nu > 0$ . Finally,

$$XIX \lesssim XIX.1 + XIX.2 + XIX.3 \leq C_{0,1,2,\tau} \lambda^{>1/2}.$$

This ends the proof of proposition 6.31.  $\square$

### 6.3.5 Exact solution to KGML and bootstrap

The next propositions are the consequences of the previous sections, they gather the different results on the existence of multi phase high frequency solutions. We reformulate 2 of proposition 6.13 independently and non schematically.

**Proposition 6.34.** *For a given admissible background initial data set  $(a_0^\alpha, \dot{a}_0^\alpha, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^\alpha)$ , let  $\lambda > 0$  and let  $((e_\lambda^{evo})^\alpha, \epsilon_\lambda^{evo}, (\dot{e}_\lambda^{evo})^\alpha, \dot{\epsilon}_\lambda^{evo}, w_{\lambda\mathcal{A}}^+, \check{w}_{\mathcal{A}\pm\mathcal{B}}^+, \psi_{\lambda\mathcal{A}}^+, \check{\psi}_{\mathcal{A}\pm\mathcal{B}}^+, g_{W_{\lambda\mathcal{A}}^+}, g_{\Psi_{\lambda\mathcal{A}}^+}, (e_\lambda^{evo})^\alpha, \epsilon_\lambda^{ell}, (\dot{e}_\lambda^{ell})^\alpha, \dot{\epsilon}_\lambda^{ell})$  be **initial data admissible for KGML**. Then, there exists a time  $t_\lambda$  for which we have a local exact multi-phase high frequency solution to KGML  $(A_\lambda^\alpha, \Phi_\lambda)$  for these initial data assembled as in definition 5.5. We have*

$$\begin{aligned} A_\lambda^0 &= A_{1\lambda}^0 + Z_\lambda^0 \in \bigcap_{j=0}^2 C^j([0, t_\lambda], H_{\delta+i}^{2-j}), & A_\lambda^i &= A_{1\lambda}^i + Z_\lambda^i \in \bigcap_{j=0}^2 C^j([0, t_\lambda], H^{2-j}), \\ \Phi_\lambda &= \Phi_{1\lambda} + \mathcal{Z}_\lambda \in \bigcap_{j=0}^2 C^j([0, t_\lambda], H^{2-j}), \end{aligned}$$

where  $(A_{1\lambda}^\alpha, \Phi_{1\lambda})$  are almost approximate solution of order 1 to KGM in Lorenz gauge and where  $(Z_\lambda^\alpha, \mathcal{Z}_\lambda)$  is the precise error term with

$$Z_\lambda^\alpha = \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \operatorname{Re}(e^{i\frac{u_{\mathcal{A}}}{\lambda}} \overline{W_{\lambda\mathcal{A}}^{+\alpha}}) + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \operatorname{Re}(e^{i\frac{u_{\mathcal{A}\pm\mathcal{B}}}{\lambda}} \overline{W_{\mathcal{A}\pm\mathcal{B}}^{+\alpha}}) + (E_\lambda^{evo})^\alpha + (E_\lambda^{ell})^\alpha, \quad (133)$$

$$\mathcal{Z}_\lambda = \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \Psi_{\lambda\mathcal{A}}^+ e^{i\frac{u_{\mathcal{A}}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \check{\Psi}_{\mathcal{A}\pm\mathcal{B}}^+ e^{i\frac{u_{\mathcal{A}\pm\mathcal{B}}}{\lambda}} + \mathcal{E}_\lambda^{evo} + \mathcal{E}_\lambda^{ell}. \quad (134)$$

*Proof.* We do it schematically. For initial data admissible for KGML there exists a time  $t_\lambda$  for which we have a solution  $(\mathbf{F}_{\lambda\mathcal{A}}^+, \mathbf{G}_{\lambda\mathcal{A}}^+, \mathcal{E}_\lambda^{evo})$  to the system (98) by proposition 6.19 ( $\check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+$  is given by proposition 6.7 and  $\mathcal{E}_\lambda^{ell}$  is given by definition 6.9).

With remarks 6.6, 6.11, 6.27 (and in particular proposition 6.28) we see that

$$\mathbf{Z}_\lambda = \sum_{\mathcal{A} \in \mathbb{A}} \lambda^1 \mathbf{F}_{\lambda\mathcal{A}}^+ e^{i\frac{u_{\mathcal{A}}}{\lambda}} + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda^1 \check{\mathbf{F}}_{\mathcal{A}\pm\mathcal{B}}^+ e^{i\frac{u_{\mathcal{A}\pm\mathcal{B}}}{\lambda}} + \mathcal{E}_\lambda^{ell} + \mathcal{E}_\lambda^{evo}$$

is solution to (98). This equation is given with respect to the background in proposition 4.1, it is constructed so that if  $\mathbf{Z}_\lambda$  is a solution then  $\mathbf{F}_\lambda = \mathbf{F}_{1\lambda} + \mathbf{Z}_\lambda$  is solution to (23), i.e.,  $(A_\lambda = A_{1\lambda} + Z_\lambda, \Phi_\lambda = \Phi_{1\lambda} + \mathcal{Z}_\lambda)$  is a solution to KGML (8). The regularity follows directly from propositions 6.19, 6.7 and 6.12.  $\square$

**Proposition 6.35.** *For a given admissible background initial data set 2.24, let  $\mathbf{B}'_0$  be given by proposition 4.1 with size  $c_0$ . Let  $\lambda > 0$  and let  $(\mathbf{f}_{\lambda\mathcal{A}}^+, \mathbf{g}_{\lambda\mathcal{A}}^+, \check{\mathbf{f}}_{\mathcal{A}\pm\mathcal{B}}^+, \epsilon_\lambda^{evo}, \dot{\epsilon}_\lambda^{evo}, \epsilon_\lambda^{ell}, \dot{\epsilon}_\lambda^{ell})$  be initial data **admissible for KGML** 5.9*

and have the **required smallness** 5.12. Then, under the assumptions 6.16 and for any  $\tau \in [0, T]$ , the solution  $(\mathbf{F}_{\lambda\mathcal{A}}^+, \mathbf{G}_{\lambda\mathcal{A}}^+, \mathcal{E}_{\lambda}^{evo})$  to (98) given by proposition 6.19 obey the following estimates<sup>45</sup>

$$\begin{aligned} \sum_{k \leq 1} \lambda^k (\|\underline{\mathcal{E}}_{\lambda}^{evo}(\tau)\|_{H^{k+1}} + \|\partial_t \underline{\mathcal{E}}_{\lambda}^{evo}(\tau)\|_{H^k}) + \lambda \|\partial_{tt}^2 \underline{\mathcal{E}}_{\lambda}^{evo}(\tau)\|_{L^2} &\leq C_{0,1,2,\tau} \lambda^{>1/2} + \lambda^{1/2} C_0 \frac{c_1}{c_2} e^{\tau c_2} + \lambda^{1/2} C_0, \\ \sum_{k \leq 1} \lambda^k (\|(E_{\lambda}^{evo})^0(\tau)\|_{H_{\delta}^{k+1}} + \|\partial_t (E_{\lambda}^{evo})^0(\tau)\|_{H_{\delta+1}^k}) + \lambda \|\partial_{tt}^2 (E_{\lambda}^{evo})^0(\tau)\|_{L_{\delta+2}^2} &\leq C_{0,1,2,\tau} \lambda^{>1/2} + \lambda^{1/2} C_0 \frac{c_1}{c_2} e^{\tau c_2} + \lambda^{1/2} C_0, \\ \max_{\mathcal{A} \in \mathbb{A}} \sum_{k \leq 1} \lambda^k (\|\mathbf{F}_{\lambda\mathcal{A}}^+(\tau)\|_{H^{k+1}} + \|\partial_t \mathbf{F}_{\lambda\mathcal{A}}^+(\tau)\|_{H^k}) + \lambda \|\partial_{tt}^2 \mathbf{F}_{\lambda\mathcal{A}}^+(\tau)\|_{L^2} &\leq C_{0,1,2,\tau} \lambda^{>0} + C_0 \frac{c_1}{c_2} e^{\tau c_2} + C_0, \\ \max_{\mathcal{A} \in \mathbb{A}} \sum_{k \leq 1} (\lambda^k \|\mathbf{G}_{\lambda\mathcal{A}}^+(\tau)\|_{H^k}) + \lambda \|\partial_t \mathbf{G}_{\lambda\mathcal{A}}^+(\tau)\|_{L^2} &\leq C_{0,1,2,\tau} \lambda^{>0} + C_0 \frac{c_1}{c_2} e^{\tau c_2} + C_0, \end{aligned}$$

for constants defined in 4.5 and 6.17.

*Proof.* Firstly, we add all the estimates from sections 6.3.2, 6.3.3 and 6.3.4 and we use propositions 8.4 and 8.1<sup>46</sup> with proposition 8.2 and the smallness of the initial data 5.12 to have control of the Sobolev norms in space. Then, we use propositions 8.3 and 8.5 to get the control on the time derivatives. The results on the smallness are immediate.  $\square$

**Lemma 6.36.** For  $\varepsilon > 0$ ,  $c_0 > 0$ ,  $\frac{1}{\eta_0} > 0$ ,  $N \in \mathbb{N}$ ,  $T > 0$  and any  $C_0 = C_0(c_0, \frac{1}{\eta_0}, N, T)$  and  $C_{0,1,2,\tau}(\cdot, \cdot, \cdot) = C_{0,1,2,\tau}(c_0, \frac{1}{\eta_0}, N, T, \cdot, \cdot, \cdot)$  from 4.5 and 6.17 there exists  $\lambda_0 > 0$ ,  $c_1$  and  $c_2$ , such that for any  $\lambda < \lambda_0$  and for any  $0 \leq \tau \leq T$  we have

$$C_0 \frac{c_1}{c_2} e^{\tau c_2} + C_0 + C_{0,1,2,\tau}(c_1, c_2, \tau) \lambda^{\varepsilon} \leq \frac{1}{2} c_1 e^{\tau c_2}.$$

*Proof.* We pick any  $c_2$  such that  $\frac{C_0}{c_2} \leq \frac{1}{6}$  then we pick  $c_1$  such that  $C_0 \leq \frac{1}{6} c_1$  and  $\lambda_0$  such that  $C_{0,1,2,\tau}(c_1, c_2, T) \lambda^{\varepsilon} \leq \frac{1}{6} c_1$ .  $\square$

### 6.3.6 Exact solution to KGM in Lorenz gauge and gauge propagation

We prove the proposition 6.14. First, we rewrite it more precisely as follows :

**Proposition 6.37.** For a given admissible background initial data set 2.24  $(a_0^{\alpha}, \dot{a}_0^{\alpha}, \phi_0, \dot{\phi}_0, v_{\mathcal{A}}, \dot{v}_{\mathcal{A}}, \psi_{\mathcal{A}}, w_{\mathcal{A}}^{\alpha})$ , let  $\lambda > 0$  and let  $((e_{\lambda}^{evo})^{\alpha}, \epsilon_{\lambda}^{evo}, (\dot{e}_{\lambda}^{evo})^{\alpha}, \dot{\epsilon}_{\lambda}^{evo}, w_{\lambda\mathcal{A}}^{+\alpha}, \check{w}_{\lambda\mathcal{A}\pm\mathcal{B}}^{+\alpha}, \psi_{\lambda\mathcal{A}}^+, \check{\psi}_{\lambda\mathcal{A}\pm\mathcal{B}}^+, g_{W^+}^{+\alpha}, g_{\Psi^+}^{+\alpha}, (e_{\lambda}^{ell})^{\alpha}, \epsilon_{\lambda}^{ell}, (\dot{e}_{\lambda}^{ell})^{\alpha}, \dot{\epsilon}_{\lambda}^{ell})$  be **KGML admissible** and **KGM in Lorenz gauge admissible**. Let  $(\Phi_{\lambda}, A_{\lambda}^{\alpha})$  be the solution to KGML on  $[0, t_{\lambda}]$  given by proposition 6.34 for these initial data assembled as in definition 5.5. Then, it is also a solution to KGM in Lorenz gauge. We have for all  $t \in [0, t_{\lambda}]$

$$\partial_{\alpha} A_{\lambda}^{\alpha}(t) = 0, \tag{135}$$

$$\partial_{\alpha} A_0^{\alpha}(t) = 0, \tag{136}$$

$$\forall \mathcal{A} \in \mathbb{A}, \partial_{\alpha} u_{\mathcal{A}} W_{\mathcal{A}}^{\alpha}(t) = 0, \tag{137}$$

$$\tag{138}$$

and if the parameters are initially polarized as

$$\forall \mathcal{A} \in \mathbb{A}, \partial_{\alpha} u_{\mathcal{A}} w_{\lambda\mathcal{A}}^{+\alpha} = 0, \quad \forall (\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}, (\partial_{\alpha} u_{\mathcal{A}} \pm \partial_{\alpha} u_{\mathcal{B}}) \check{w}_{\mathcal{A}\pm\mathcal{B}}^{+\alpha} = 0, \tag{139}$$

$$\tag{140}$$

<sup>45</sup>We make a difference between  $\underline{\mathcal{E}}_{\lambda}^{evo}$  and  $(E_{\lambda}^{evo})^0$  as in 6.15.

<sup>46</sup>The linear term in the RHS of (146) and (153) is not a problem and is dealt with as the other linear terms (case c)).

then we have

$$\forall \mathcal{A} \in \mathbb{A}, \partial_\alpha u_{\mathcal{A}} W_{\lambda \mathcal{A}}^{+\alpha}(t) = 0, \quad \forall (\mathcal{A}, \mathcal{B}) \in \mathcal{S}, \mathcal{A} < \mathcal{B}, (\partial_\alpha u_{\mathcal{A}} \pm \partial_\alpha u_{\mathcal{B}}) \check{W}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}(t) = 0, \quad (141)$$

$$(142)$$

$$\begin{aligned} & \sum_{\mathcal{A} \in \mathbb{A}} (\lambda^{1/2} \operatorname{Re}(e^{i \frac{u_{\mathcal{A}}}{\lambda}} \overline{\partial_\alpha W_{\mathcal{A}}^{+\alpha}}) + \lambda \operatorname{Re}(e^{i \frac{u_{\mathcal{A}}}{\lambda}} \overline{\partial_\alpha W_{\lambda \mathcal{A}}^{+\alpha}}))(t) + \sum_{\mathcal{A}, \mathcal{B} \in \mathbb{C}, \mathcal{A} < \mathcal{B}} \lambda \operatorname{Re}(e^{i \frac{u_{\mathcal{A}} \pm u_{\mathcal{B}}}{\lambda}} \overline{\partial_\alpha \check{W}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}})(t) \\ & + \partial_\alpha (E_{\lambda \mathcal{A}}^{ell})^\alpha(t) + \partial_\alpha (E_{\lambda \mathcal{A}}^{evo})^\alpha(t) = 0. \end{aligned} \quad (143)$$

*Proof.* We use the argument of section 1.1.1, we have initial data satisfying the constraints so we take the divergence of the Maxwell equation to recover the propagation of gauge via a wave equation with zero initial data.

For the decomposition of the divergence of  $A_\lambda$ , we already have the equalities on the second line from (53) and (54). For the propagation of polarization, we observed in remark 6.3 that the RHS of the evolution equation for  $W_{\lambda \mathcal{A}}^{+\alpha}$  (resp.  $\check{W}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha}$ ) are orthogonal to  $\partial_\alpha u_{\mathcal{A}}$  (resp.  $\partial_\beta u_{\mathcal{A}} \pm \partial_\beta u_{\mathcal{B}}$ ). We contract the equations with their respective vector fields and use the geodesic equation for the characteristic phases to conclude.

Finally, the last line comes from all the remaining term.  $\square$

**Remark 6.38.** *The set we exhibit in proposition 5.16 is polarized as  $\check{w}_{\mathcal{A} \pm \mathcal{B}}^{+\alpha} = w_{\lambda \mathcal{A}}^{+\alpha} = 0$ .*

## 7 Conclusion

We put everything together and give some general remarks.

### 7.1 Proof of the theorems

For the first theorem 3.1, we only detail the proof of the point 1. Indeed, everything boils down to the construction of the MPHf solutions, the other points are direct consequences of our construction and the point 5 is proved using the Strichartz estimates for waves. The statement of point 1 is that for a given initial ansatz there exists  $\lambda_0$  such that below it there is a family of multi phase high frequency solutions to KGM  $(A_\lambda, \Phi_\lambda)_{\lambda < \lambda_0}$  on  $[0, T]$ . More precisely, for each  $\lambda > 0$  we can build an almost approximate solution arising from the initial ansatz and an exact solution based on this almost approximate solution. We follow the steps of 1.5. The approximation (step (i)) is done in proposition 4.1. The rest of the theorem follows from the main propositions 6.13 (step (iii)), 6.14 (step (iv)) and 6.18 (step (v)). The construction of the initial data (step (ii)) is considered after.

From proposition 6.13, we have for any  $\lambda > 0$  a well structured multi phase high frequency local solution to KGML on  $[0, t_\lambda]$ . The part 1 of this proposition is proved in proposition 6.19 and its part 2 in proposition 6.34. This requires initial data admissible for KGML 5.9. Then, with the proposition 6.14, we recover a true solution to KGM in Lorenz gauge if the initial data are admissible for KGM in Lorenz gauge 5.11, this is proved in proposition 6.37. Finally, the good structure of the previously cited solutions is exploited to prove the proposition 6.18. This proposition states that for  $\lambda < \lambda_0$  the family of solutions to KGM exists on  $[0, T]$ . For  $\lambda > 0$ , we have either  $t_\lambda > T$  and the proof is done or, if  $\lambda$  is small enough, that  $t_\lambda$  can be improved up to  $T$ . To demonstrate the second case, we use the following method. We assume some estimates 6.16 from which we deduce the proposition 6.35. Then, using the lemma 6.36, we exhibit  $\lambda_0$  and we show that the estimates of proposition 6.35 are in fact better than the assumptions 6.16 for  $\lambda < \lambda_0$ . This shows a contradiction,  $t_\lambda$  cannot be the maximum time of existence, it can be pushed up to  $T$ . This requires a certain smallness for the error initial data 5.12.

We exhibit one generic set of initial data which fits the three criteria in proposition 5.16. This concludes the proof of the first theorem.

For the second theorem, the proof is direct with the structure of the solution and the structure of the error term in particular. We make some more remarks on that in the next section. The third point is detailed in section 4.2.

### 7.2 Convergence and smallness of the error term

It is interesting to compare our results to the ones of [10]. The author of [10] exhibits mono phase high frequency solutions to Yang Mills coupled with a scalar field and a spinor field. More particularly, the author shows how to

construct approximate solutions of any order with the WKB expansion

$$\mathbf{F}_{M\lambda} = \sum_{0 \leq i \leq M-1} \lambda^{i/2} (\tilde{\mathbf{F}}_i(x) + \lambda^{1/2} \mathbf{F}_i^*(x, \frac{u}{\lambda}))$$

and how to construct exact solutions

$$\mathbf{F}_\lambda = \mathbf{F}_{M\lambda} + \mathbf{Z}_\lambda$$

on a time interval uniform in  $\lambda$  and for  $M \geq n$  where  $n$  is the dimension. The function  $\mathbf{F}_i(x, \cdot)$  is decomposed as a mean part  $\tilde{\mathbf{F}}_i(x)$  and a purely oscillatory part  $\mathbf{F}_i^*(x, \cdot)$ . It has to be a more general profile than the  $\cos(\frac{u}{\lambda})$  and  $\sin(\frac{u}{\lambda})$  case of our paper to treat the harmonics. The construction of approximate solutions of [10] holds for a broader family of gauges than the only Lorenz gauge. Dealing with the non hyperbolicity requires to add a higher order parameter, see the remark 4.9. The following bounds for the Yang-Mills potential and the scalar field is obtained

$$\forall \lambda < \lambda_0, \lambda^{M/2} \|\mathbf{F}_\lambda\|_{H^{1/2}} + \|\mathbf{Z}_\lambda\|_{H^{1/2}} \leq \lambda^{M/2} C,$$

for some constant  $C$  which depends on the initial data.

The KGM system is a sub case of this more general case. On the other hand, the mono phase case of [10] is a sub case of the multi phase case treated here. We show in this paper how to construct an error term at order  $M = 1$ , that is below 3 (the dimension). Our first order approximation looks like

$$\mathbf{F}_{1\lambda} = \tilde{\mathbf{F}}_0(x) + \lambda^{1/2} \mathbf{F}_0^*(x, \frac{u}{\lambda}) := \mathbf{F}_0(x) + \lambda^{1/2} \mathbf{F}(x) e^{i\frac{u}{\lambda}}.$$

In [10], it is not possible to give directly the error term at this order. This is due to the use of some product estimates (proposition 3.1.1 in [10] and 2.2.2 in [9]) which is consistent with the results of [19] for first order hyperbolic system. To get higher order approximate solutions one needs to build a hierarchical system of equations and to take care of harmonics which arise from the interactions between phases.

### 7.3 Regularity of the background

The background  $\mathbf{B}'_0 := (A_0, \Phi_0, w_{\mathcal{A}}, \psi_{\mathcal{A}}, du_{\mathcal{A}})$  defined in proposition 4.1 is more regular than the full solution (and in particular the error term). This extra regularity is mandatory and is used to have the well posedness of (99) and (87) and along the estimates in section 6.3. For example, the equation for  $\mathcal{E}_\lambda^{ev\circ}$  has  $\square \mathbf{F}_{\mathcal{A}}$  in its RHS so that  $\partial \mathcal{E}_\lambda^{ev\circ} \sim \square \mathbf{F}_{\mathcal{A}}$  in terms of regularity. We see that  $\mathbf{B}_0 = (A_0, \Phi_0, w_{\mathcal{A}}, \psi_{\mathcal{A}})$  is one degree more regular than the error term. Moreover, the phases have the following regularity

$$\max_{\mathcal{A} \in \mathbb{A}} \sum_{j=0}^5 \|u_{\mathcal{A}}\|_{C^j([0,T], H_{\delta_1}^{5-j})} \leq c_0.$$

The phases need to be this regular to have the well-posedness of (51) with solutions in  $H^3$  and to be able to use the commutator estimates of proposition 8.6 for (106). In fact, it is doable if we only have  $\square u_{\mathcal{A}}$  in  $H^3$  and  $u_{\mathcal{A}}$  in  $H_{\delta_1}^4$ . Constructing the error term for a higher order approximation requires a more regular background.

## 8 Appendix

### 8.1 Estimates for evolution equations

**Proposition 8.1.** *For  $f$  a function defined on  $[0, T'] \times \mathbb{R}^3$  solution to*

$$\begin{cases} \square f = h, \\ f(t=0) = f_0 \in H_{\delta'}^{m+1} \quad \partial_t f(t=0) = f_1 \in H_{\delta'}^m, \end{cases} \quad (144)$$

*we have the energy estimate*

$$\|\partial f(t)\|_{H_{\delta'}^m} \leq C(\|\partial f(0)\|_{H_{\delta'}^m} + \int_0^t \|h(s)\|_{H_{\delta'}^m} + \|\partial f(s)\|_{H_{\delta'}^m} ds), \quad (145)$$

and more precisely for  $\delta' = 0$

$$\|\partial f(t)\|_{H^m} \leq C(\|\partial f(0)\|_{H^m} + \int_0^t \|h(s)\|_{H^m} ds). \quad (146)$$

*Proof.* We detail the energy estimate for  $f$  for  $m = 0$  and for any  $\delta'$

$$\square f(1 + |x|^2)^{\delta'} \partial_t f = h(1 + |x|^2)^{\delta'} \partial_t f,$$

$$\begin{aligned} \int_{\mathbb{R}^3} \square f(1 + |x|^2)^{\delta'} \partial_t f dx &= \int_{\mathbb{R}^3} h(1 + |x|^2)^{\delta'} \partial_t f dx \\ &= -\frac{d}{dt} \int_{\mathbb{R}^3} (1 + |x|^2)^{\delta'} \frac{\partial_t f^2 + |\nabla f|^2}{2} dx - 2\delta' \int_{\mathbb{R}^3} (1 + |x|^2)^{\delta'-1} x_i \partial_i f \partial_t f dx, \end{aligned}$$

where  $|(1 + |x|^2)^{\delta'-1} x_i| \leq (1 + |x|^2)^{\delta'}$ . We can control this term in  $L^2$  norm with the weight  $\delta'$ . We finally get

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\partial f(t)\|_{L_{\delta'}^2}^2 &\leq C(\|\partial f(t)\|_{L_{\delta'}^2}, \|h(s)\|_{L_{\delta'}^2}, + \|\partial f(t)\|_{L_{\delta'}^2}^2), \\ \|\partial f(t)\|_{L_{\delta'}^2} &\leq C(\|\partial f(0)\|_{L_{\delta'}^2} + \int_0^t \|h(s)\|_{L_{\delta'}^2} + \|\partial f(s)\|_{L_{\delta'}^2} ds). \end{aligned}$$

Then, we commute the equation with  $\partial_{\vec{m}}$  where  $\vec{m}$  is a multi-index with  $|\vec{m}| = m$ . We multiply by  $\partial_{i\vec{m}} f$  instead of  $\partial_t f$  and the adjusted weight to get the Sobolev norms. □

Moreover, having bounds on  $\partial f = (\partial_t f, \nabla f)$  and on  $f_0$  gives us bounds on  $f(t)$  for finite  $t$ .

**Proposition 8.2.** *With  $f$  solution to (144) we have  $\forall t \in [0, T']$*

$$\frac{d}{dt} \|f(t)\|_{L^2}(t) \leq \|\partial_t f(t)\|_{L^2}, \quad (147)$$

and so for  $m \geq 0$

$$\|f\|_{H_{\delta'-1}^{m+1}}(t) \leq (\|f_0\|_{H_{\delta'-1}^{m+1}} + (t+1) \sup_{s \in [0, T']} \|\partial f(s)\|_{H_{\delta'}^m}). \quad (148)$$

*Proof.* By direct calculation. □

**Proposition 8.3.** *The time derivatives of solutions to (144) obey the following inequalities*

$$\|\partial_t f(t)\|_{H_{\delta'}^m} \leq C(\|\partial f(0)\|_{H_{\delta'}^m} + \int_0^t \|h(s)\|_{H_{\delta'}^m} + \|\partial f(s)\|_{H_{\delta'}^m} ds), \quad (149)$$

$$\|\partial_{tt}^2 f(t)\|_{H_{\delta'+1}^{m-1}} \leq (\|f(t)\|_{H_{\delta'-1}^{m+1}} + \|h(t)\|_{H_{\delta'+1}^{m-1}}). \quad (150)$$

*Proof.* The first inequality is a specific case of (146) and the second comes directly using the equation. □

**Proposition 8.4.** *For  $f$  a function defined on  $[0, T'] \times \mathbb{R}^3$  solution to*

$$\begin{cases} 2\partial^\alpha u \partial_\alpha f + \square u f = j, \\ f(t=0) = f_0 \in H^m, \end{cases} \quad (151)$$

with  $u$  a smooth characteristic phase satisfying

$$\min_{(t,x) \in [0, T'] \times \text{Supp}(f)} |\partial_t u(t, x)| > \eta > 0 \quad \|\partial u\|_{H^{m'}} + \|\square u\|_{H^{m'}} + \|\partial_t \partial u\|_{H^{m'-1}} + \|\partial_t \square u\|_{H^{m'-1}} \leq C_u, \quad (152)$$

for  $m' = \max(m, 2)$ . We have the energy estimate

$$\|f\|_{L^\infty([0, T'], H^m)} \leq C_u(\|f(0)\|_{H^m} + \int_0^{T'} \|j(s)\|_{H^m} + \|f(s)\|_{H^m} ds). \quad (153)$$

*Proof.* In the case  $m = 0$ , we multiply by  $f$

$$\partial_\alpha(\partial^\alpha u_{\mathcal{A}} f^2) = jf,$$

so we have a conservative transport equation, by Stokes theorem<sup>47</sup> we get

$$\int_{\mathbb{R}^3} (\partial^0 u_{\mathcal{A}} f^2)(t) dx = \int_{\mathbb{R}^3} (\partial^0 u_{\mathcal{A}} f^2)(0) dx + \int_0^t \int_{\mathbb{R}^3} (jf)(s) dx ds,$$

and so

$$\eta \|f(t)\|_{L^2}^2 \leq C_u \|f(0)\|_{L^2}^2 + \|f\|_{L^\infty([0, T'], L^2)} \int_0^t \|j\|_{L^2} ds,$$

which implies

$$\|f(t)\|_{L^\infty([0, T'], L^2)} \leq C_{u, \eta} (\|f(0)\|_{L^2} + \int_0^{T'} \|j\|_{L^2} ds).$$

For  $m > 1$  we commute the equation with  $\partial_{\vec{m}}$ , where  $\vec{m}$  is a multi-index with  $|\vec{m}| = m$ , and we get

$$(2\partial^\alpha u \partial_\alpha + \square u) \partial_{\vec{m}} f = [2\partial^\alpha u \partial_\alpha + \square u, \partial_{\vec{m}}] f + \partial_{\vec{m}} j,$$

where

$$\|[2\partial^\alpha u \partial_\alpha + \square u, \partial_{\vec{m}}] f(t)\|_{L^2} \leq C (\|\partial u\|_{H^{m'}} + \|\square u\|_{H^{m'}}) \|f(t)\|_{H^m},$$

for  $m' = \max(m, 2)$ . □

**Proposition 8.5.** *The time derivatives of (151) obey the following inequalities*

$$\|\partial_t f\|_{L^\infty([0, T'], H^m)} \leq C_{u, \eta} (\|f\|_{L^\infty([0, T'], H^{m+1})} + \|j\|_{L^\infty([0, T'], H^m)}), \quad (154)$$

$$\|\partial_{tt}^2 f\|_{L^\infty([0, T'], H^{m-1})} \leq C_{u, \eta} (\|f\|_{L^\infty([0, T'], H^{m+1})} + \|j\|_{L^\infty([0, T'], H^m)}). \quad (155)$$

*Proof.* For the first inequality we use directly the transport equation. In the second case we apply one derivative on the transport equation and use the previous one to control the appearing first order in time derivatives. □

## 8.2 Commutator

**Proposition 8.6.** *Let  $g$  be a smooth Lorentzian metric, we define  $\square := D_\alpha D^\alpha$  for  $D$  the covariant derivative and we contract the indices with respect to this metric.*

*Let  $f$  defined on  $[0, T'] \times \mathbb{R}^3$  be compactly supported and solution to*

$$\begin{cases} 2\partial^\alpha u \partial_\alpha f + \square u f = j, \\ f(t=0) = f_0 \in H^2, \end{cases} \quad (156)$$

*for  $u$  a smooth characteristic phase such that there exists  $C_u$  such that for every compact set  $\Omega \subset \mathbb{R}^3$*

$$\|\square \partial u\|_{L^\infty([0, T'], L^\infty(\Omega))} + \|\square \square u\|_{L^\infty([0, T'], H^{1/2}(\Omega))} + \|\partial \partial u\|_{L^\infty([0, T'], L^\infty(\Omega))} \leq C_u. \quad (157)$$

*We suppose that*

$$(\square f)|_{t=0} = G_0, \quad G_0 \in H^1. \quad (158)$$

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<sup>47</sup>We ignore the border terms as they will be useless in our application.



Let  $G$  defined on  $[0, T'] \times \mathbb{R}^3$  be solution to

$$\begin{cases} 2\partial^\alpha u \partial_\alpha G + \square u G = \square j + [(2\partial^\alpha u \partial_\alpha + \square u), \square] f, \\ G(t=0) = G_0. \end{cases} \quad (159)$$

Then,  $G$  stays in  $H^1$  for all  $t \in [0, T']$  and

$$\|G(t)\|_{H^1} \leq C_{g,u} \int_0^t (\|\square j(s)\|_{H^1} + \|\partial j(s)\|_{H^1} + \|f(s)\|_{H^2}) ds + \|G_0\|_{H^1}, \quad (160)$$

where we use the pointwise in time estimate

$$\|[(2\partial^\alpha u \partial_\alpha + \square u), \square] f(t)\|_{L^2} \leq C_{g,u} (\|G(t)\|_{L^2} + \|\partial j(t)\|_{L^2} + \|f(t)\|_{H^1}). \quad (161)$$

The result is true independently of the metric  $g$ .

*Proof.* We see that if  $u = k^\alpha x_\alpha$  for  $k$  a constant vector, that is if  $u$  is the phase of a plane wave, we have  $\partial_\alpha u = k_\alpha$  so that

$$[(2\partial^\alpha u \partial_\alpha + \square u), \square] = [(2k^\alpha \partial_\alpha), \square] = 0, \quad (162)$$

so the result is direct.

Now we do the general case, to simplify the calculations we put ourselves in wave coordinates  $g^{\mu\nu} \Gamma_{\mu\nu}^\gamma = 0$ .

$$\begin{aligned} [(2\partial^\alpha u \partial_\alpha + \square u), \square] f &= 2\partial^\alpha u \partial_\alpha (g^{\mu\nu}) \partial_\mu \partial_\nu f - 2\partial^\alpha u \partial_\alpha (g^{\mu\nu} \Gamma_{\mu\nu}^\gamma) \partial_\gamma f - 4\square \partial^\alpha u \partial_\alpha f - 4g^{\mu\nu} \partial_\mu \partial^\alpha u \partial_\nu \partial_\alpha f - \square \square u f \\ &= -4(\Gamma_{\rho\alpha}^\mu g^{\rho\nu} \partial^\alpha u + g^{\mu\alpha} \partial_\alpha \partial^\nu u) \partial_\mu \partial_\nu f - 4(\square \partial^\alpha u) \partial_\alpha f - \square \square u f = -4(\nabla^\mu \partial^\nu u) \partial_\mu \partial_\nu f - 4(\square \partial^\alpha u) \partial_\alpha f - \square \square u f. \end{aligned}$$

We can write the metric as

$$g^{\mu\nu} = -\frac{1}{2}(\partial^\mu u \partial^\nu \bar{u} + \partial^\mu \bar{u} \partial^\nu u) + \delta^{MN} e_M^\mu e_N^\nu,$$

where  $e_{M,N}$  are spacelike vector fields orthonormal to  $\partial u$ . We set  $\partial \bar{u}$  as the only null vector field such that  $g(\partial u, \partial \bar{u}) = -2$  and  $g(E_M, \partial \bar{u}) = 0$ .

We know that  $\partial_\nu u D^\mu \partial^\nu u = 0$  (35) and  $\partial_\nu u D^\nu \partial^\mu u = 0$  (38), we deduce that locally we can write the tensor  $\nabla^\mu \partial^\nu u$  as

$$a \partial u \otimes \partial u + b(\partial u \otimes e_{M,N} + e_{M,N} \otimes \partial u) + c(e_{M,N} \otimes e_{M,N}),$$

$a, b$  and  $c$  depend on  $\partial \partial u$  and  $\partial g$ . All the other components are zero.

We conclude that schematically

$$[(2\partial^\alpha u \partial_\alpha + \square u), \square] f = f_1(\partial \partial u, \partial g) \partial(\partial^\alpha u \partial_\alpha f) + f_2(\partial \partial u, \partial g) \partial f + f_3(\partial \partial u, \partial g) \partial_M \partial_N f + \square \partial u \partial f + \square \square u f,$$

for some functions  $f_1, f_2$  and  $f_3$ .

We know that the hessian  $\partial_M \partial_N f$  can be estimated in  $L^2$  on surfaces of  $u$  and  $\bar{u}$  constant via  $\Delta_{M,N} F$  and  $\partial F$ , see [21].

We use the identity

$$\Delta_{M,N} f = \square f + \frac{1}{2}(\partial^\mu u \partial^\nu \bar{u} + \partial^\mu \bar{u} \partial^\nu u) \partial_\mu \partial_\nu f,$$

to get

$$\|\partial_M \partial_N f\|_{L^2} \leq C_{u,g} (\|\square f\|_{L^2} + \|\partial(\partial^\alpha u \partial_\alpha f)\|_{L^2} + \|\partial f\|_{L^2}).$$

Putting everything together, we deduce

$$\|[(2\partial^\alpha u \partial_\alpha + \square u), \square] f(t)\|_{L^2} \leq C_{u,g} (\|\square f\|_{L^2} + \|\partial(\partial^\alpha u \partial_\alpha f)\|_{L^2} + \|f\|_{H^1}) = C_{u,g} (\|G\|_{L^2} + \|\partial j\|_{L^2} + \|f\|_{H^1}).$$

This is the desired inequality.  $\square$

### 8.3 Sobolev inequalities

Here we recall the classical Sobolev embedding and interpolation results.

**Proposition 8.7.** *Let  $n \in \mathbb{N}^*$  be the dimension  $k > 0$ ,  $1 \leq p < \infty$ ,  $1 \leq q \leq \infty$ ,  $k > l$  and  $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{k-l}{n}$*

$$W^{k,p} \hookrightarrow W^{l,q}.$$

*Proof.* From standard results. □

**Proposition 8.8.** *Let  $n \in \mathbb{N}^*$  be the dimension  $k > 0$ ,  $1 \leq p < \infty$ ,  $1 \leq q < \infty$ ,*

$$\begin{aligned} \frac{1}{p} - \frac{k}{n} \geq 0 &\Rightarrow W^{k,p} \xhookrightarrow{c} L^q, \\ \frac{1}{p} - \frac{k}{n} < 0 &\Rightarrow W^{k,p} \xhookrightarrow{c} L^\infty. \end{aligned}$$

*Proof.* From standard results. □

**Proposition 8.9.** *We have the embedding of weighted Sobolev spaces into the weighted  $C_{\delta'}^m$  spaces for  $m \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $n/p < s < \infty$  and  $\delta' \leq \delta + \frac{n}{p}$*

$$W_{p,\delta}^{s+m} \hookrightarrow C_{\delta'}^m.$$

*Proof.* From [3]. □

**Proposition 8.10.** *For  $s_\theta, s_0, s_1 \in \mathbb{R}$ ,  $1 \leq p_\theta, p_0, p_1 \leq \infty$  and  $0 < \theta < 1$  with  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  and  $s_\theta = (1-\theta)s_0 + \theta s_1$  we have*

$$\|f\|_{W^{s_\theta, p_\theta}} \leq (\|f\|_{W^{s_0, p_0}})^{1-\theta} (\|f\|_{W^{s_1, p_1}})^\theta. \quad (163)$$

*Proof.* From standard results. □

**Proposition 8.11.** *We have the Gagliardo-Nirenberg inequality : Let  $1 \leq r, q \leq \infty$ ,  $j, m \in \mathbb{N}$  such that  $j < m$ ,  $p \geq 1$  then for*

$$\frac{1}{p} = \frac{j}{n} + \theta \left( \frac{1}{r} - \frac{m}{n} \right) + (1-\theta) \frac{1}{q}, \quad \frac{j}{m} \leq \theta \leq 1,$$

*we have*

$$\|\partial_x^j f\|_{L^p(\mathbb{R}^n)} \leq C (\|\partial_x^m f\|_{L^r(\mathbb{R}^n)})^\theta (\|f\|_{L^q(\mathbb{R}^n)})^{1-\theta}.$$

*Proof.* From standard results. □

### 8.4 Laplacian inversion

**Definition 8.12.** *We define the classical Laplacian as  $\Delta = \partial_i \partial^i$  where*

$$\Delta : W_{\delta,p}^{2+m} \rightarrow W_{\delta+2,p}^m.$$

**Theorem 8.1.** *For  $n \geq 3$ ,  $\delta \not\equiv -2 + \frac{n}{p} \pmod{\mathbb{N}}$  and  $-\delta \not\equiv \frac{n}{p} \pmod{\mathbb{N}}$  with*

$$-n/p < \delta < -2 + n/p',$$

the Laplacian operator is an isomorphism from  $W_{\delta,p}^{2+m}$  to  $W_{\delta+2,p}^m$  and is in particular invertible.

Moreover, for  $f$  solution to

$$\Delta f = h,$$

we have

$$\|f\|_{H_\delta^2} \lesssim \|h\|_{L_{\delta+2}^2}. \quad (164)$$

*Proof.* From [18]. □

**Remark 8.13.** In the case  $n = 3$ ,  $p = 2$  and  $p' = 2$  we have  $-3/2 < \delta < -1/2$ . With  $m = 0$ , we see that the source term  $h$  must be in  $L_{\delta+2}^2 \subset L^2$  ( $\delta + 2 > 0$ ) but if  $\text{Supp}(h) \subset K$  for  $K$  compact then

$$\|h\|_{L_{\delta+2}^2} \leq C \|h\|_{L^2(K)}.$$

We can deduce the following useful lemma.

**Lemma 8.14.** *Let*

$$L = -\Delta + g, \quad (165)$$

for  $g \geq 0$  a compactly supported function in  $H^2$  with  $\|g\|_{H^2} \leq C_g$ . Then, for  $\delta \in (-3/2, -1/2)$  the operator  $L$  is an isomorphism from  $H_\delta^2$  to  $L_{\delta+2}^2$  and is in particular invertible.

Moreover, for  $f$  solution to

$$Lf = h, \quad (166)$$

We have

$$\|f\|_{H_\delta^2} \lesssim \|h\|_{L_{\delta+2}^2}. \quad (167)$$

*Proof.* First, we show that the multiplication by  $g$  is a compact operator from  $H_\delta^2$  to  $L_{\delta+2}^2$ . Let  $B$  be a ball of finite radius such that  $\text{Supp}(g) \subset B$ . For  $f \in H_\delta^2$  with  $\|f\|_{H_\delta^2} \leq C$  we have

$$\|gf\|_{H^1(B)} \leq \|g\|_{H^2(B)} \|f\|_{H^2(B)} \leq c(B) \|f\|_{H_\delta^2} \|g\|_{H^2} \leq c(C, B, C_g).$$

Using the Rellich-Kondrachov theorem we deduce that  $\{gf | f \in H_\delta^2, \|f\|_{H_\delta^2} \leq C\}$  is a compact subset of  $L^2(B)$  and so a compact subset of  $L_{\delta+2}^2$ . Now we show the invertibility.

The classical Laplacian is already an isomorphism from  $H_\delta^2$  to  $L_{\delta+2}^2$  for  $\delta \in (-3/2, -1/2)$  so that it is Fredholm of index 0. Adding a compact perturbation do not modify the index of the operator.

We deduce that the operator  $L$  is Fredholm of index 0. For such an operator, the injectivity will imply the invertibility.

To prove the injectivity we use the maximum principle for Sobolev regularity.

Let  $f$  be in  $H_\delta^2$  such that

$$Lf = 0,$$

then because  $g \geq 0$  we can use the maximum principle (see chapter 8 of [4] for example) for any set  $\Omega$  of finite radius  $R$  with smooth boundary we have

$$\sup_{\Omega} f \leq \sup_{\partial\Omega} f|_{\partial\Omega}, \quad \inf_{\Omega} f \geq \inf_{\partial\Omega} f|_{\partial\Omega},$$

where the trace of  $f$  is its restriction to the border as  $f$  is continuous (by Sobolev embedding in Hölder spaces from proposition 8.9). In fact, we have even more than continuity,  $f$  also obeys the following inequality

$$|f(x)| \leq \frac{C \|f\|_{H_\delta^2}}{|x|^\beta},$$

for some  $C > 0$  and for  $0 \leq \beta < \delta + 3/2$  (here  $\delta \in (-3/2, -1/2)$ ) due to the fact that the Sobolev spaces are weighted.

We have

$$|\sup_{\Omega} f| \leq \frac{C\|f\|_{H_{\delta}^2}}{R^{\beta}}, \quad |\inf_{\Omega} f| \leq \frac{C\|f\|_{H_{\delta}^2}}{R^{\beta}}.$$

Letting the  $R$  go to  $+\infty$  we deduce that  $f$  is identically 0. This proves the injectivity of  $L$  and thus its invertibility.

To prove the inequality (167), we work by contradiction.

First, because  $L$  is asymptotic to a Laplacian in the senses of [1] (for  $q = 4$ ) we have the following inequality for every  $f$  in  $H_{\delta}^2$

$$\|f\|_{H_{\delta}^2} \leq C(\|Lf\|_{L_{\delta+2}^2} + \|f\|_{L^2(B_R)}), \quad (168)$$

which is from theorem 1.10 of [1]. The constant  $C$  and  $R$  depend on  $g$  and  $\delta$ .

If the inequality (167) is false, then there exists a sequence  $f_n$  such that

$$\|f_n\|_{H_{\delta}^2} = 1, \quad \lim_{n \rightarrow \infty} \|Lf_n\|_{L_{\delta+2}^2} = 0.$$

Because the sequence is uniformly bounded there exists a subsequence that converges weakly to  $f$  in  $H_{\delta}^2$  and that converges strongly in  $L^2(B_R)$  by Rellich-Kondrachov.

We have  $\forall \varphi \in C_c^{\infty}$

$$\lim_{n \rightarrow \infty} \int Lf_n \varphi dx = 0$$

by assumptions and

$$\int (Lf_n - Lf) \varphi dx = \int (-\Delta f_n + \Delta f) \varphi dx + \int g(f_n - f) \varphi dx \leq \int -\Delta(f_n - f) \varphi dx + \|gu\|_{L^2(\Omega)} \|f_n - f\|_{L^2(\Omega)},$$

which implies that

$$\lim_{n \rightarrow \infty} \int Lf_n \varphi dx = \int Lf \varphi dx.$$

We deduce that  $Lf = 0$  and thus  $f = 0$  by injectivity.

Now, using the inequality (168), we find out that for any  $\varepsilon > 0$  there exists  $m \in \mathbb{N}$  such that for any  $n \geq m$

$$1 = \|f_n\|_{H_{\delta}^2} \leq C(\varepsilon + \|f_n\|_{L^2(B_R)}),$$

and we have  $\lim_{n \rightarrow \infty} \|f_n\|_{L^2(B_R)} = \|f\|_{L^2(B_R)}$  as the convergence is strong. We deduce that  $f \neq 0$  which is a contradiction. Thus, we obtain the inequality (167).  $\square$

## 8.5 Strichartz

**Definition 8.15.** We say that  $(q, r)$  are Strichartz admissible if :

$$\begin{aligned} q &\geq 2, & \frac{2}{q} &\leq (n-1)\left(\frac{1}{2} - \frac{1}{r}\right), \\ (q, r, n) &\neq (2, \infty, 3), & r &< \infty. \end{aligned}$$

**Theorem 8.2.** For  $(q_1, r_1)$  et  $(q_2, r_2)$  Strichartz admissible with

$$\frac{1}{q_1} + \frac{n}{r_1} = \frac{1}{q_2} + \frac{n}{r_2} - 2 = \frac{n}{2} - \gamma, \quad (169)$$

and  $f$  solution to the Cauchy problem

$$\begin{cases} \square f = j, \\ f(0) = f_0, \partial_t f(0) = \dot{f}_0, \end{cases} \quad (170)$$

then

$$\|u\|_{L^{q_1}([0, T], L^{r_1})} \leq (\|S\|_{L^{q_2}([0, T], L^{r_2})} + \|u_0\|_{\dot{H}^{\gamma}} + \|\dot{u}_0\|_{\dot{H}^{\gamma-1}}). \quad (171)$$

*Proof.* From [20] □

## 8.6 Product estimates

We recall the following Sobolev product estimates result.

**Proposition 8.16.** *For  $1 < p < \infty$ ,  $r, s \in \mathbb{R}$ ,  $r, s < \frac{n}{p}$ ,  $r + s > \max(\frac{n}{p} - \frac{n}{p'}, 0)$  with  $\frac{1}{p} + \frac{1}{p'} = 1$  and, if  $p > 2$ ,  $\|f\|_{W^{p',-r}}$  or  $\|g\|_{W^{p',-s}}$  we have*

$$\|fg\|_{W^{p,t}} \leq \|f\|_{W^{p,r}} \|g\|_{W^{p,s}}, \quad (172)$$

for  $t = r + s - \frac{n}{p}$ .

*Proof.* From [22]. □

We give two lemmas on product estimates based the Strichartz estimates.

**Lemma 8.17.** *Let  $f$  and  $g$  be two functions defined on  $[0, T'] \times \mathbb{R}^3$  for  $T' > 0$ . For  $\nu \in (0, 1/2)$  we have the inequality*

$$\|fg\|_{L^1([0, T'], L^2)} \leq C \|f\|_{L^p([0, T'], H^{1/2-\nu})} (\|\square g\|_{L^1([0, T'], L^2)} + \|g(0)\|_{\dot{H}^1} + \|\partial_t g(0)\|_{L^2}), \quad (173)$$

with  $p$  given by  $\nu > 0$ .

*Proof.* For  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $2 < \alpha = \frac{3}{1+\nu} < 3$  and  $\frac{1}{2} = \frac{1}{\alpha} + \frac{1}{\beta}$  we have

$$\begin{aligned} \|fg\|_{L^1([0, T'], L^2)} &\leq \|f\|_{L^p([0, T'], L^\alpha)} \|g\|_{L^{p'}([0, T'], L^\beta)} \\ &\leq \|f\|_{L^p([0, T'], H^{1/2-\nu})} \|g\|_{L^{p'}([0, T'], L^\beta)}. \end{aligned}$$

We set  $(r_2 = 2, q_2 = \infty)$ , they are Strichartz admissible, we want  $p'$  such that  $(\beta, p')$  are Strichartz admissible and such that

$$\frac{1}{p'} + \frac{3}{\beta} = \frac{1}{q_2} + \frac{3}{r_2} - 2 = \frac{1}{2},$$

this implies that  $p' = \frac{\alpha}{3-\alpha}$  and so  $p = \frac{\alpha}{2\alpha-3}$ . Then, we use the Strichartz estimates on  $\|g\|_{L^{p'}([0, T'], L^\beta)}$  to get the desired inequality. □

**Lemma 8.18.** *Let  $f$  and  $g$  be two functions defined on  $[0, T'] \times \mathbb{R}^3$  for  $T' > 0$ , for  $\theta \in (0, 1/2)$  we have the inequality*

$$\|fg\|_{L^1([0, T'], L^2)} \leq \|f\|_{L^p([0, T'], L^2)} (\|g\|_{L^q([0, T'], H^2)})^\theta (\|\square g\|_{L^1([0, T'], L^2)} + \|g(0)\|_{\dot{H}^1} + \|\partial_t g(0)\|_{L^2})^{1-\theta},$$

where  $p$  and  $q$  depend on  $\theta$ .

*Proof.* We have

$$\|fg\|_{L^1([0, T'], L^2)} \leq \|f\|_{L^p([0, T'], L^2)} \|g\|_{L^{p'}([0, T'], L^\infty)}, \quad (174)$$

with Gagliardo-Nirenberg 8.11 we get

$$\|fg\|_{L^1([0, T'], L^2)} \leq \|f\|_{L^p([0, T'], L^2)} (\|g\|_{L^{p'\theta}([0, T'], H^2)})^\theta (\|g\|_{L^{p'(1-\theta)}([0, T'], L^{r_1})})^{1-\theta}, \quad (175)$$

where  $r_1 = 6(\frac{1-\theta}{\theta}) > 6$  with  $\theta \in (0, 1/2)$ . We set  $(r_2 = 2, q_2 = \infty)$  and  $p_1 = p'(1-\theta)$  and search for  $(r_1, p_1)$  Strichartz admissible with

$$\frac{1}{p_1} + \frac{3}{r_1} = \frac{1}{q_2} + \frac{3}{r_2} - 2 = \frac{1}{2}.$$

We find  $p' = \frac{2}{1-\theta}$  so that  $p = \frac{2}{1+\theta}$ . Finally, we use the Strichartz estimates on  $\|g\|_{L^{p'(1-\theta)}([0, T'], L^{r_1})}$  to recover the desired inequality. □

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