Blocks of Ariki-Koike algebras and level-rank duality

Nicolas Jacon, David Declercq

BLOCKS OF ARIKI-KOIKE ALGEBRAS AND LEVEL-RANK DUALITY

DAVID DECLERCQ AND NICOLAS JACON

Abstract. We study the blocks for Ariki-Koike algebras using a general notion of core for l-partitions. We interpret the action of the affine symmetric group on the blocks in the context of level rank duality and study the orbits under this action.

1. Introduction

One of the main (and still open) problem in the modular representation of the symmetric group $\mathfrak{S}_n$ (that is over a field k of characteristic $p > 0$) is the determination of the decomposition matrix. This matrix describes the decomposition of certain remarkable modules, the Specht modules, into simples via a process of modular reduction. The rows are indexed by the partitions of n (which naturally index the Specht modules) where as the columns are indexed by certain partitions, named the p-regular partitions (which index the simple modules of $k\mathfrak{S}_n$.) The decomposition matrix is in fact a block diagonal matrix and we may try to study these "blocks" independently one from the other. This leads to fundamental problems which have both combinatorial and geometric flavor:

- The distributions of the partitions (and the p-regular partitions) into blocks may be described using the notion of p-core and p-quotient of a partition. The complexity of the block is measured by a combinatorial datum, the weight of the block.
- There is a strong relation between some of the blocks by the works and Scopes [15] and Chuang-Rouquier [3]. In particular, an action of the affine symmetric group on the set of blocks induces derived equivalences between objects associated to these blocks. In some cases, these equivalences are Morita equivalences and the associated block decomposition matrices are just the same.

This paper is concerned with the Ariki-Koike algebra which may be seen as a generalization of the group algebra of the symmetric group. It is in fact a deformation of the complex reflection group of type $G(l, 1, n)$. This algebra has been intensively studied during the past decades (see [1, 4]). In particular, we still have a notion of decomposition matrix in this case. The analogues of the Specht modules are this time indexed by l-tuple of partitions ("l-partitions"), and the set of e-regular partitions by the so called Uglov l-partitions. A generalized version of the notion of weight has been provided by Fayers in [2] and the distribution into blocks has been studied by Lyle and Mathas in [10]. More recently, a notion of Scopes equivalence has been studied by Lyle [9], Webster [17] (in a more general context) and in [14]. In [7], the second author together with C.Lecouvey have introduced a generalization of the notion of p-core partitions in this context and show that some of the results known in the case of the modular representation theory of the symmetric group are still verified in the context of Ariki-Koike algebras. The aim of this paper is to go deeper in the study of the notion of blocks using this generalized notion of core. We describe precisely the action of the affine symmetric group using this notion. In particular, generalizing works by Gerber and Lecouvey, we interpret the induced action on the Uglov l-partitions in the context of level-rank duality, and show that this action may be obtained using the notion of crystal isomorphism as studied in [7]. We also show that the notion of Scopes equivalence can be easily generalized in this context using the notion of core. Contrary to the case of the symmetric group, two blocks with the same weight may not be in the same orbit. We then study and classify the different orbits under the action. We will see that the generalized notion of core nicely simplify the different classical properties of blocks.

The paper will be organized as follows. First, we describe the combinatorial notions which are needed in this paper: this includes the notion of partitions, symbols, abaci, core and quotient (in the case of the

2020 *Mathematics Subject Classification*: 20C08,05E10,20C20

symmetric group). The third part first generalizes this notion in the context of Ariki-Koike algebras. We in particular introduce the concept of level rank duality. This section contains several new results on the generalized core (see Proposition 3.3 and 3.6). In the fourth and fifth part, we study the blocks of Ariki-Koike algebras and the action of the affine Weyl group together with the analogue of Scopes equivalence (see Prop. 5.1) and we interpret this action using the level-rank duality in Prop. 5.3. The last part is finally devoted to the classification of the orbits under this action.

2. Partitions, symbols and abaci

In this part, we introduce the combinatorial notions needed to deal with the modular representation theory of Ariki-Koike algebras and to introduce the notion of blocks.

2.1. **Partitions.** By definition, a *partition* of n is a sequence of non increasing positive integers $(\lambda_1, \dots, \lambda_r)$ of total sum n. We denote by:

- Π^l the set of all l-partitions, that is, the set of all l-tuples $(\lambda^1, \dots, \lambda^l)$ of partitions.
- $\Pi = \Pi^1$ the set of all partitions.

For any subset E of Π^l and any $n \in \mathbb{Z}_{\geq 0}$, we denote $E(n)$ the set of l-partitions in E with total size n.

2.2. **Abaci and symbols.** Let $m \in \mathbb{Z}$. By definition, a *β-set* or *symbol* of *charge* $m \in \mathbb{Z}$ is an infinite sequence of integers $X = (\beta_i)_{i<m}$ such that:

(1) For all $i < m$, we have $\beta_{i-1} < \beta_i$ (that is X is a strictly increasing sequence)
(2) There exists $N < m$ such that for all $j \leq N$, we have $\beta_j = j$.

We denote by $\mathrm{Symb}(m)$ the set of symbols of charge m. If $m \in \mathbb{Z}$ then the symbol $X = (\beta_i)_{i<m}$ such that $\beta_i = i$ for all $i < m$ is called the *trivial symbol* of charge m.

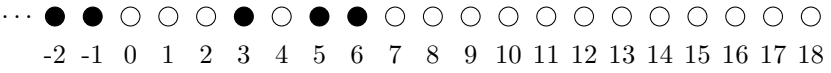
If $X = \mathrm{Symb}(m)$ and if $r \in \mathbb{Z}$, we write $X[r]$ for the symbol $(\gamma)_{j<r+m} \in \mathrm{Symb}(m+r)$ such that for all $j < r + m$, we have $\gamma_j = \beta_{j-r} + r$. We write

$$X \simeq X[r]$$

This is an equivalence relation on the set of symbols and each equivalence class has at most one element with a fixed charge.

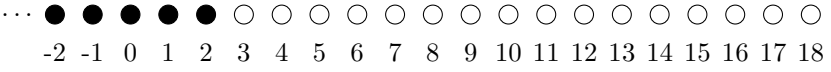
A symbol may be conveniently represented using its abacus configuration. In this way, we associate to a symbol X an horizontal runner full of (an infinite number of) beads numerated in $\mathbb{Z}$. A bead numerated by $a \in \mathbb{Z}$ is colored in black if and only if $a \in X_j$. The others are written in white.

Example 2.1. The abacus associated to $X = (\dots, -3, -2, -1, 3, 5, 6)$ is



The charge of a symbol X may be conveniently read in the associated abacus as follows. For each black bead, if there exists a white bead at its left then replace the leftmost one with a black bead and the black bead itself with a white bead. We obtain the abacus of a trivial symbol. The charge of X is then the charge of the trivial symbol, that is the number associated with the leftmost white bead.

Example 2.2. Keeping the above example, if we perform the above manipulation, we obtain



which is the abacus of the trivial symbol of charge 3.

2.3. **l-Symbol.** An *l-symbol* is a collection of l β-sets:

$$\mathbf{X} = (X_0, \dots, X_{l-1})$$

The *multicharge* of the symbol is the l-tuple $(m_0, \dots, m_{l-1}) \in \mathbb{Z}^l$ where for all $j = 0, \dots, l-1$, the number m_j is the charge of $X_j = (\beta_i^j)_{i<m_j}$. It will sometimes be convenient in the following to write such a symbol

from bottom to top, taking into account the gaps between the charges of each symbol (see the example below).

$$\mathbf{X} = \begin{pmatrix} X_l \\ \vdots \\ X_1 \end{pmatrix}$$

If $\mathbf{X}$ is an l-symbol and if $r \in \mathbb{N}$, we write $\mathbf{X}[r]$ for the symbol $(X_1[r], \dots, X_l[r])$. Again, we write $\mathbf{X} \simeq \mathbf{X}[r]$.

Example 2.3. Let us consider the 3-symbol:

$$\mathbf{X} = ((\dots, -1, 0, 2, 4, 6), (, \dots, -1, 0, 3, 4), (\dots, -1, 0, 2, 5)).$$

Then we can write it as follows:

$$\mathbf{X} = \begin{pmatrix} \dots & -1 & 0 & 2 & 5 & \\ \dots & -1 & 0 & 3 & 4 & \\ \dots & -1 & 0 & 2 & 4 & 6 \end{pmatrix}$$

Then the associated multicharge is $(4, 3, 3)$. The 3-symbol

$$\mathbf{Y} = ((\dots, 0, 1, 2, 3, 5, 7, 9), (\dots, 0, 1, 2, 3, 6, 7), (\dots, 0, 1, 2, 3, 5, 8))$$

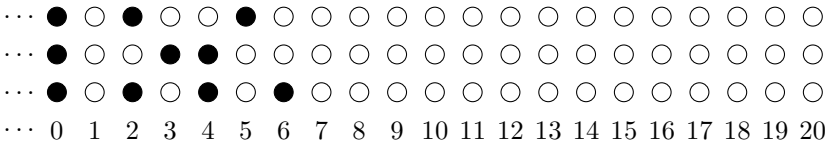
has multicharge $(7, 6, 6)$ and we have $\mathbf{Y} = \mathbf{X}[3]$ so that $\mathbf{X} \simeq \mathbf{Y}$.

An *l-symbol* $\mathbf{X} = (X_0, \dots, X_{l-1})$ can be conveniently represented using its abacus configuration. In this way, we associate to each X_j from $j = 0$ to $l - 1$ an abacus as above and we write them from bottom to top so that the bead in the same column are numerated by the same integer. We call the associated object an *l-abacus* (or simply abacus if $l = 1$).

Example 2.4. Let $l = 3$ and let us consider the following 3-symbol:

$$X = ((\dots, -1, 0, 2, 4, 6), (\dots, -1, 0, 3, 4), (\dots, -1, 0, 2, 5)).$$

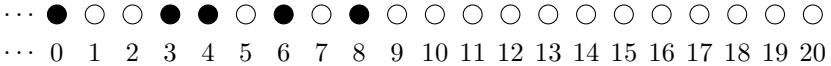
The associated 3-abacus is:



We can recover the associated multicharge by moving the black beads at the left in each runner.

2.4. **Symbol of a multipartition.** To each symbol $X = (\beta_i)_{i<m}$ (and thus to each abacus) of charge m we can canonically associate a partition $\lambda(X) = (\lambda_1, \dots, \lambda_r)$ such that for all $i \geq 1$, we have $\lambda_i = \beta_{m-i} + i - m$. Note that if $k >> 1$ then $\lambda_k = 0$. Regarding the abacus associated to the set of β-numbers, the parts of the partition are easily obtained by counting the numbers of white beads at the left of each black bead.

Example 2.5. Let $X := (\dots, -1, 0, 3, 4, 6, 8)$. The associated charge is $m = 5$. Then we have $\lambda(X) = (4, 3, 2, 2)$. The abacus configuration gives:



It is easy to see that $\lambda(X) = \lambda(Y)$ if and only if $X \simeq Y$. Conversely, to any partition $(\lambda_1, \dots, \lambda_r)$, we can associate a set of β-numbers (and thus an abacus). Let $m \in \mathbb{Z}$. Then we define

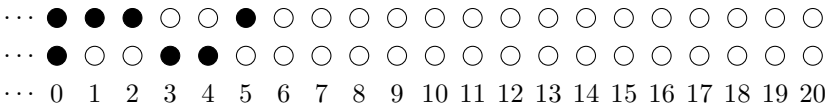
$$X^m(\lambda) = (\beta_i)_{i<m}$$

where for all $i = 1, \dots, m$, we have $\beta_{m-i} = \lambda_i - i + m$. Note that for all $(m_1, m_2) \in \mathbb{Z}^2$, we have $X^{m_1}(\lambda) \simeq X^{m_2}(\lambda)$.

To each l-symbol $\mathbf{X} = (X_0, \dots, X_{l-1})$, we can associate an l-ipartition $\boldsymbol{\lambda}(X) = (\lambda(X_0), \dots, \lambda(X_{l-1}))$ attached to this symbol together with a multicharge $\mathbf{s}(X) \in \mathbb{Z}^l$ which is the multicharge of the symbol.

Reciprocally, to each multipartition $\boldsymbol{\lambda}^l = (\lambda^0, \dots, \lambda^{l-1})$ [1] and multicharge $\mathbf{s} = (s_0, \dots, s_{l-1})$, one can attach an l-symbol $\mathbf{X}^{\mathbf{s}}(\boldsymbol{\lambda}^l) = (X^{s_0}(\lambda^0), \dots, X^{s_{l-1}}(\lambda^{l-1}))$.

Example 2.6. Let $X = ((, \dots, 0, 3, 4), (, \dots, 0, 1, 2, 5))$ then we have $\lambda(X) = ((2, 2), (2))$ and $\mathbf{s}(X) = (3, 4)$. The abacus configuration gives:



2.5. **Core and quotient.** With all these notions in mind, we will now be able to define the core and the quotient of a partition. We fix $m \in \mathbb{Z}$. Importantly, our notion of e-quotient will depend on this choice. Let $e \in \mathbb{N}$ and let $n \in \mathbb{N}$ and let $\lambda \in \Pi(n)$.

First recall that we can associate to λ a set of β-numbers $X^m(\lambda)$. As usual, we take

$$X^m(\lambda) = (\beta_1, \dots, \beta_n).$$

We then define;

$$\mathbf{X} = (X_0, \dots, X_{e-1}),$$

where for all $j \in \{0, \dots, e-1\}$, X_j is the set of increasing integers obtained by reordering the set :

$$\{k \in \mathbb{N} \mid j + ke \in X^m(\lambda)\}$$

We denote

$$\mathbb{Z}^e[m] := \{(s_0, \dots, s_{e-1}) \in \mathbb{Z}^e \mid \sum_{1 \leq i \leq e-1} s_i = m\}$$

Let $\boldsymbol{\lambda}_e$ be the multipartition and $\mathbf{s}_e := (s_0, \dots, s_{e-1}) \in \mathbb{Z}^e[m]$ be such that $X^{\mathbf{s}_e}(\boldsymbol{\lambda}_e) = \mathbf{X}$ [2]. We denote

$$\tau_e(\lambda) = (\boldsymbol{\lambda}_e, \mathbf{s}_e).$$

We will use the following definitions:

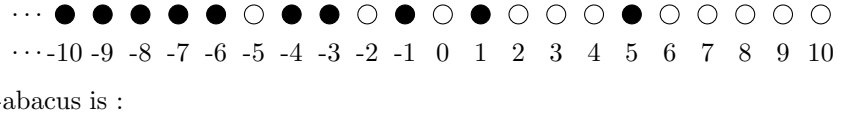
- The e-partition $\boldsymbol{\lambda}_e$ is the *e-quotient* of (λ, m).
- Let λ° be such that $X(\lambda^\circ) = X^{\mathbf{s}}(\emptyset)$. It is known as the *$e$-core partition* of (λ, m).
- The pair $(\lambda^\circ, m) \in \Pi \times \mathbb{Z}$ is the *e-core* of (λ, m) (this notations seems to be strange but it will be convenient to use it in order to be consistent with our generalizations later in the paper).
- We see that the datum of this partition is equivalent to the datum of $\mathbf{s}_e$. It will be in fact convenient for our purpose to also name the multicharge $\mathbf{s}_e \in \mathbb{Z}^e[m]$ itself as the *e-core multicharge* of λ and this is what will be done in the rest of the paper.

In fact, the notion of e-core partition does not depend on m but the e-quotient does.

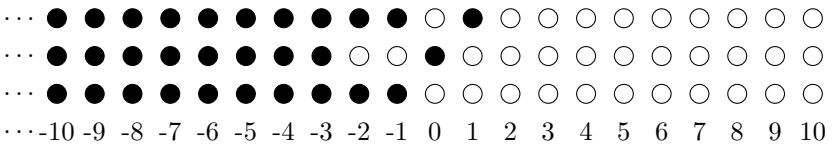
Example 2.7. Let $\lambda = (6, 3, 2, 1, 1)$. Take $e = 3$. We have:

$$X^0(\lambda) = (\dots, -7, -6, -4, -3, -1, 1, 5).$$

Then the associated abacus is given as follows:



The associated 3-abacus is :



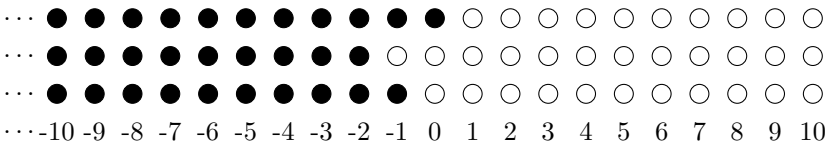
[1] The notation $\boldsymbol{\lambda}^l$ has not to be confused with λ^l which is the lth component of $\boldsymbol{\lambda}$. It will be convenient to use this notation in the following

[2] the notation $\boldsymbol{\lambda}_e$ has not be be confused with the eth part of a partition. It is convenient to use this notation because of the level rank duality which will be define later

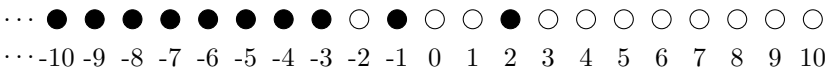
Each runner of the above 3-abacus corresponds to an abacus which itself corresponds to a partition. The 3-partition associated to this is the 3-quotient:

$$(\emptyset, (2), (1)).$$

To obtain the associated 3-core, we move all the black beads at the left end of each runner, we obtain the following 3-abacus:



We obtain the multicharge $(0, -1, 1)$ which is thus the e-core multicharge of $(\lambda, 0)$. We then come back to the 1-abacus associated to this 3-abacus :

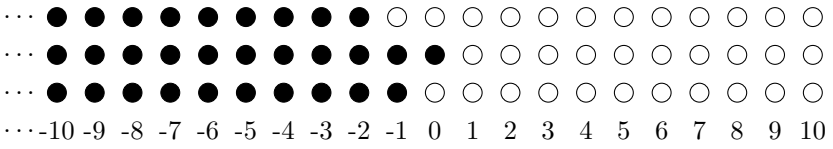


We have $\lambda^\circ = (3, 1)$ which is the e-core partition of $(\lambda, 0)$. The e-core of $(\lambda, 0)$ is $\lambda^\circ, 0)$.

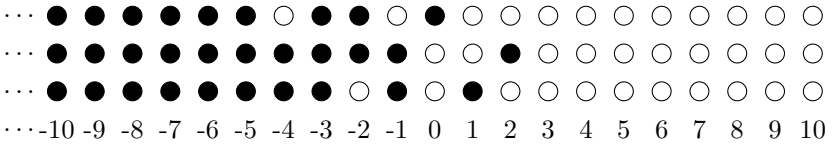
We say that a partition λ is an *e-core partition* if the e-core partition of λ is λ itself. We denote by $\mathrm{Cor}(e)$ the set of e-cores partitions. The discussion above shows that there is a bijection between $\mathrm{Cor}(e)$ and $\mathbb{Z}^e[m]$.

Given a $m \in \mathbb{Z}$, an e-partition $(\lambda^0, \dots, \lambda^{e-1})$ and an e-core λ° partition (or an element in $\mathbb{Z}^l[m]$), there is a unique pair $(\lambda, m) \in \Pi \times \mathbb{Z}$ with e-quotient $\boldsymbol{\lambda}$ and e-core λ°.

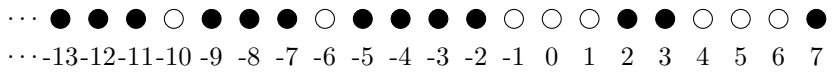
Example 2.8. Take the 3-charge $\mathbf{s} = (0, 1, -1)$ (which is of sum $m = 0$), the abacus associated to $X^{\mathbf{s}}(\emptyset)$ is



and we have $X(\lambda^\circ) = X^{\mathbf{s}}(\emptyset)$ for $\lambda^\circ = (2)$ (which is thus a 3-core). We are interested to find the partition with 3-core (2) and 3-quotient $((2, 1), (2), (2, 1, 1))$. Then, we simply move the black beads in the above abacus to obtain the abacus of $\mathbf{X}^{(0,1,-1)}((2, 1), (2), (2, 1, 1))$. We obtain:



Now, we perform the associated 1-abacus:



which is the abacus of the partition $(8, 5, 5, 2, 2, 2, 2, 1, 1, 1)$. This is the desired partition.

2.6. **Nodes.** Assume that $\boldsymbol{\lambda}^l$ is an l-partition and fix $\mathbf{s}^l = (s_0, \dots, s_{l-1})$. We can define its Young diagram

$$[\boldsymbol{\lambda}^l] := \{(a, b, c) \mid a \geq 1,\ c \in \{0, \dots, l-1\},\ 1 \leq b \leq \lambda^c_a\} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, l\}.$$

The elements in $[\boldsymbol{\lambda}^l]$ are called the *nodes* of $\boldsymbol{\lambda}^l$. The *content* of a node $\gamma = (a, b, c)$ of $\boldsymbol{\lambda}^l$ is the element $b - a + s_c$ of $\mathbb{Z}$ and the residue is the content modulo $e\mathbb{Z}$. We say that:

- the node $(a, b, c) \in [\boldsymbol{\lambda}^l]$ is a *removable j-node* if $[\boldsymbol{\lambda}^l] \setminus \{(a, b, c)\}$ is the Young diagram of a well defined l-partition and if $b - a + s_c \equiv j + e\mathbb{Z}$ that is, the residue of (a, b, c) is $j + e\mathbb{Z}$.
- $(a, b, c) \in \mathbb{N}^2$ is an *addable j-node* if $[\boldsymbol{\lambda}^l] \sqcup \{(a, b, c)\}$ is the Young diagram of a well defined l-partition and if $b - a + s_c \equiv j + e\mathbb{Z}$ that is, the residue of (a, b, c) is $j + e\mathbb{Z}$.

Consider the associated symbol $\mathbf{X} := \mathbf{X}^{\mathbf{s}^l}(\boldsymbol{\lambda}^l)$ and the associated l-abacus. Then note that

- A removable i-node γ in $\boldsymbol{\lambda}^l$ in component c is canonically associated to a black bead in the runner c numbered by an element j such that $j-1$ is occupied with a white bead. We have $j \equiv i + e\mathbb{Z}$. The abacus associated to the l-partition $\boldsymbol{\mu}^l$ obtained by removing this node is obtained by exchanging the white and the black bead.
- An addable i-node γ in $\boldsymbol{\lambda}^l$ in component c is canonically associated to a white bead in the runner c numbered by an element j such that $j+1$ is occupied with a black bead. We have $j \equiv i + e\mathbb{Z}$. The abacus associated to the l-partition $\boldsymbol{\mu}^l$ obtained by adding this node is obtained by exchanging the white and the black bead.

Example 2.9. Keep Example 2.6 with $\boldsymbol{\lambda}^l := ((2,2),(2))$ and $\mathbf{s}^l = (3,4)$ and take $e = 4$. We write the Young diagram with the residue in each box associated with its node:

$$\Big(\begin{array}{|c|c|}\hline 3&0\\\hline 2&1\\\hline\end{array}\,,\begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\Big)$$

We have two removable 1-nodes : $(2,2,0)$ and $(1,2,1)$. They are associated to two black beads numbered by 5 in runner 1 and 3 in runner 0. We have two addable 1-node : $(1,3,0)$ and $(3,1,0)$ which are associated to white beads numbered by 1 and 5 in runner 0. We have also one addable 2-node $(1,3,1)$ and one addable 3 node $(2,1,1)$ associated with white beads numbered by 6 and 3.

3. Multipartitions and combinatorial Level-rank duality

We have already seen how one can associate to each partition its e-core and its e-quotient. In this part, we introduce a variation of these notions, defined by Uglov [16] and we study the case of l-partitions.

3.1. **A variant of the notion of core and quotient.** In [16], Uglov has given a variant of the notion of core and quotient. Assume that λ is a partition. First, Assume that we get the set

$$X^m(\lambda) = (\beta_i)_{i<m}.$$

For all $j < m$, we write

$$\beta_j = c_j + ed_j + elm_i$$

where $c_i \in \{1,\dots,e\}$, $d_i \in \{0,\dots,l-1\}$ and $m_i \in \mathbb{Z}$. Then we set X^d to be the set of increasing integers obtained by reordering the set :

$$\{c_j + em_j \mid d_j = d\}.$$

We obtain an l-symbol

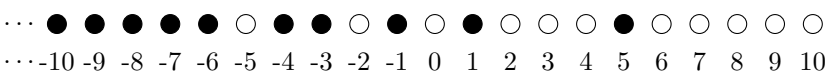
$$(X_{l-1},\dots,X_0),$$

which will be called the *Uglov l-symbol*. From this, we can associate to a partition λ an l-partition $\boldsymbol{\lambda}^l$ together with an l-multicharge $\mathbf{s}^l \in \mathbb{Z}^l[m]$, and of course, one can do the reverse process. We denote $\tau^l(\lambda, m) = (\boldsymbol{\lambda}^l, \mathbf{s}^l)$, τ^l is a thus a bijection from the set of all partitions to the set $\Pi^l \times \mathbb{Z}^l[m]$.

Example 3.1. Let $\lambda = (6,3,2,1,1)$ and $l = 2$ and $e = 3$, we have:

$$X^0(\lambda) = (\dots,-7,-6,-4,-3,-1,1,5).$$

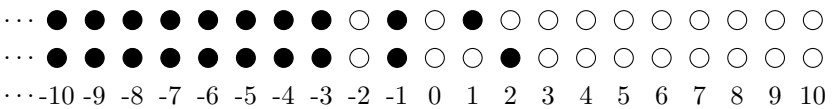
with abacus:



The associated l-symbol is :

$$((\dots,-3,-1,2),(\dots,-3,-1,1))$$

with abacus:

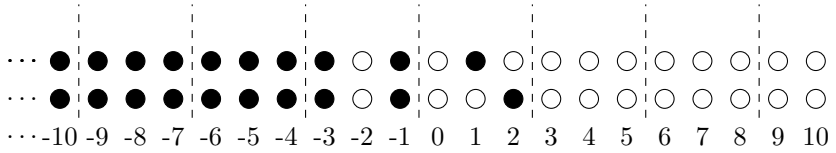


and we thus get $\tau^l(\lambda) = (((3,1),(2,1)),(0,0))$.

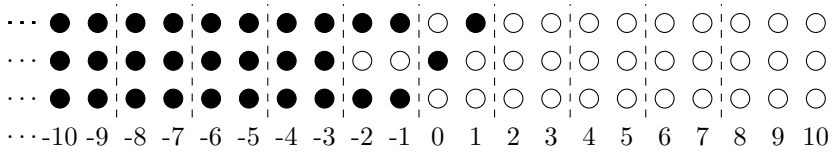
3.2. **Relations between the two notions of quotients.** Assume that we have a partition λ and that $\tau^l(\lambda) = (\boldsymbol{\lambda}^l, \mathbf{s}^l)$, $\tau_e(\lambda) = (\boldsymbol{\lambda}_e, \mathbf{s}_e)$. Then one can go easily from the abacus of $\tau^l(\lambda)$ to the abacus of $\tau_e(\lambda)$ and reciprocally. To do that, start with the abacus associated to $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$

- We define a rectangle on the l-abacus, containing e beads in each abacus. This rectangle starts with the beads numbered with 0 and finishes with the beads numbered with $e-1$. We get a rectangle with el beads, then again define a second rectangle with the beads numbered with e to the beads numbered by $2e-1$ an so on, even with the beads marked with negative integers.
- Rotate each rectangle 90 degree anticlockwise.
- We get a new e-abacus, which is the e-abacus of $(\boldsymbol{\lambda}_e, \mathbf{s}_e)$.

Example 3.2. We keep the above example:



and then after rotation:



which is the 3-abacus of the 3-quotient of λ, as expected.

3.3. **Nodes again.** Let λ be a partition and write $\tau^l(\lambda) = (\boldsymbol{\lambda}^l, \mathbf{s}^l)$. Let $\tau_e(\lambda) = (\boldsymbol{\lambda}_e, \mathbf{s}_e)$ and denote the associated symbol by :

$$(X_0, \dots, X_{e-1}).$$

Assume that there exists $i \in \{0, \dots, e-2\}$ such that $j \in X_i$ and such that $j \notin X_{i+1}$. Let us consider the symbol $\mathbf{Y} = (Y_0, \dots, Y_{e-1})$ such that $Y_k = X_k$ if $k \neq i, i+1$, $Y_i = X_i \setminus \{j\}$ and $Y_{i+1} = X_{i+1} \cup \{j\}$. The associated abacus is obtained by moving the black bead numerated by j from the ith runner to the $i+1$th one. We obtain the e-symbol which is itself associated to the multipartition $\boldsymbol{\mu}^l$ is obtained from $\boldsymbol{\lambda}^l$ by removing a node with residue $i + e\mathbb{Z}$. Reciprocally, adding a node $i + e\mathbb{Z}$ with residue consists in doing the above manipulation on abacus/symbol.

If now $j \in X_{e-1}$ and $j - l \notin X_0$, on can consider $\mathbf{Y} = (X_0 \cup \{j - l\}, \dots, X_{e-1} \setminus \{j\})$. The associated abacus is obtained by moving the black bead numerated by j from the $e+1$th runner to the 0th one at the place $j - l$. Then we have the e-abacus associated to an l-partition $\boldsymbol{\mu}^l$ is obtained from λ^l by removing a node with residue $0 + e\mathbb{Z}$. Reciprocally, removing a node with residue $0 + e\mathbb{Z}$ consists in doing the above manipulation on abacus/symbol.

3.4. **Size.** The following result is implicit in Uglov [16] and Yvonne's papers [18]. We give a formal proof for the convenience of the reader as such a proof seems not to appear in the literature.[3]

Proposition 3.3. *Assume that λ is a partition and that $\tau^l(\lambda) = (\boldsymbol{\lambda}^l, \mathbf{s}^l)$ and $\tau_e(\lambda) = (\boldsymbol{\lambda}_e, \mathbf{s}_e)$. Assume that μ is a partition and that $\tau^l(\mu) = (\boldsymbol{\mu}^l, \mathbf{s}^l)$ and $\tau_e(\lambda) = (\boldsymbol{\mu}_e, \mathbf{s}_e)$. Assume in addition that $|\boldsymbol{\mu}^l| = |\boldsymbol{\lambda}^l|$. Then we have $|\boldsymbol{\mu}_e| = |\boldsymbol{\lambda}_e|$.*

Proof. For $x = l$ or $x = e$ and $0 \leq i \leq e-1$, we set $N_i(\boldsymbol{\nu}, x)$ to be the number of i-nodes for the x-partition $\boldsymbol{\nu}$. Let λ° be the common e-core of λ and μ. By [7, Prop. 2.22], we have:

$$|\lambda| - l|\lambda^\circ| = |\boldsymbol{\lambda}^l| + elN_0(\boldsymbol{\lambda}^l, l),$$

and

$$|\mu| - l|\lambda^\circ| = |\boldsymbol{\mu}^l| + elN_0(\boldsymbol{\mu}^l, l).$$

[3]We thank C.Lecouvey for discussion on this proof

So we have

$$|\mu| - elN_0(\boldsymbol{\mu}^l, l) = |\lambda| - elN_0(\boldsymbol{\lambda}^l, l)$$

Now, as λ° is the e-core of μ and of λ, we have:

$$|\mu| = e|\boldsymbol{\mu}_e| + |\lambda^\circ|$$

and

$$|\lambda| = e|\boldsymbol{\lambda}_e| + |\lambda^\circ|$$

So we obtain that:

$$e|\boldsymbol{\mu}_e| - elN_0(\boldsymbol{\mu}^l, l) = e|\boldsymbol{\lambda}_e| - elN_0(\boldsymbol{\lambda}^l, l)$$

So we need to show that $N_0(\boldsymbol{\mu}, l) = N_0(\boldsymbol{\lambda}, l)$. By [18, (W1)-(W4) p. 57], for all $0 \leq i \leq e-1$, we have:

$$N_i(\boldsymbol{\lambda}^l, \mathbf{s}^l) - N_i(\boldsymbol{\mu}^l, \mathbf{s}^l) = N_0(\boldsymbol{\lambda}_e, \mathbf{s}_e) - N_0(\boldsymbol{\mu}_e, \mathbf{s}_e)$$

So summing over all $0 \leq i \leq e-1$ leads to :

$$|\boldsymbol{\mu}^l| - |\boldsymbol{\lambda}^l| = e(N_0(\boldsymbol{\lambda}_e, \mathbf{s}_e) - N_0(\boldsymbol{\mu}_e, \mathbf{s}_e))$$

We deduce $N_0(\boldsymbol{\lambda}_e, \mathbf{s}_e) = N_0(\boldsymbol{\mu}_e, \mathbf{s}_e)$. By [18, Proof of Prop. 3.24] (see also [8, Lemma 3.4.1]), we have that

$$N_0(\boldsymbol{\lambda}_e, \mathbf{s}_e) - N_0(\boldsymbol{\mu}_e, \mathbf{s}_e) = N_0(\boldsymbol{\lambda}^l, \mathbf{s}^l) - N_0(\boldsymbol{\mu}^l, \mathbf{s}^l)$$

We thus have $N_0(\boldsymbol{\lambda}^l, \mathbf{s}^l) = N_0(\boldsymbol{\mu}^l, \mathbf{s}^l))$, as desired.

□

3.5. **Generalized Core.** We end this section by defining a notion of core for the l-partitions, as done in [7]. We fix $\mathbf{s}^l \in \mathbb{Z}^l[m]$ and an associated l-partition $\boldsymbol{\lambda}^l$, we here see how one can generalize the notion of core and quotient for this multipartition. To do this, we set:

$$\overline{\mathcal{A}}_e^l := \{(s_1, \ldots, s_l) \in \mathbb{Z}^l \mid \forall (i,j) \in \{1, \ldots, l\},\ i < j,\ 0 \leq s_j - s_i \leq e\},$$
$$\mathcal{A}_e^l := \{(s_1, \ldots, s_l) \in \mathbb{Z}^l \mid \forall (i,j) \in \{1, \ldots, l\},\ i < j,\ 0 \leq s_j - s_i < e\}.$$

Definition 3.4. Let $\mathbf{s}^l \in \mathcal{A}_e^l$, we say that the pair $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ is a an e-core if this pair satisfies the following property. Let $(X_0, \ldots, X_l)$ be its associated l-symbol then we have

$$X_0 \subset X_1 \subset \ldots X_{l-1} \subset X_0[e]$$

Remark 3.5. In [7], the definition is a little bit more general by considering arbitrary multicharges.

Of course such an e-core is canonically associated with the partition $(\tau^l)^{-1}(\boldsymbol{\lambda}^l, \mathbf{s}^l)$, but also with a multicharge $\mathbf{s}_e \in \mathbb{Z}^e[m]$ which is defined as $\tau_e(\tau^l)^{-1}(\boldsymbol{\lambda}^l, \mathbf{s}^l) = (\emptyset, \mathbf{s}_e)$

Now, we explain how one can obtain the core of an l-partition together with its multicharge. First, by [7], one can assume that

$$\mathbf{s}^l \in \mathcal{A}_e^l$$

We compute the core of the l-partition $\boldsymbol{\lambda}^l$ as follows. We consider the $(e, \mathbf{s}^l)$-abacus $(X_0, \ldots, X_{l-1})$ of $\boldsymbol{\lambda}^l$. An *elementary operation* on this abacus is defined as a move of one black bead from one runner of the abacus to another satisfying the following rule.

(1) If this black bead is not in the top runner, then we can do such an elementary operation on this black bead only if there is no black bead immediately above (that is in the same position on the runner just above). In this case, we slide the black bead from its initial position, in a runner i, to the runner $i+1$ located above in the same position.
(2) If this black bead is in the top runner in position x, then we can do such an elementary operation only if there is no black bead in position $x - e$ on the lowest runner.

At the end (that is, when we cannot perform any further operations, by construction, we obtain an l-abacus which is canonically associated with an l-symbol

$$(L_0, \ldots, L_{l-1})$$

satisfying:

$$L_0 \subset L_1 \subset \ldots \subset L_{l-1} \subset L_0[e].$$

this corresponds to an l-partition $\boldsymbol{\mu}^l$ and a multicharge $\mathbf{v}^l \in \overline{\mathcal{A}}_e^l$ such that $(\boldsymbol{\mu}^l, \mathbf{v}^l)$ is an e-core. This is called the e-core of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$. By the discussion aboce, we see that doing an elementary operation consists in removing a removable node in the associated e-partition. Thus, if the number $w(\boldsymbol{\lambda}^l)$ which is number of elementary operations needed to reach this core is also the size of this e-partition and this is called the weight of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$. As a consequence a core is a partition (with its multicharge) with weight 0. We have the following proposition which give a nice characterization of cores:

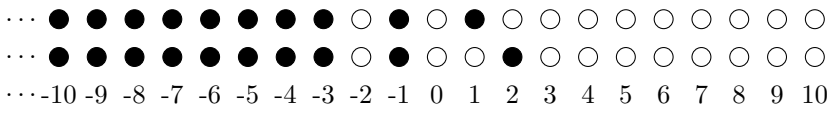
Proposition 3.6. *Let $\mathbf{s}^l \in \mathcal{A}_e^l$ and $\boldsymbol{\lambda}^l \in \Pi^l$ then we have $\boldsymbol{\lambda} \in \mathfrak{C}^l(e, \mathbf{s}^l)$ if and only if for all $i \in \{0, \dots, e-1\}$, we are in one of the following two situation :*

- *$\boldsymbol{\lambda}^l$ admits no addable i-node*
- *$\boldsymbol{\lambda}^l$ admits no removable i-node*

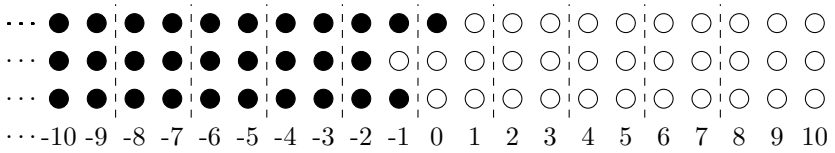
Proof. Take the e-abacus of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$. and assume that $\boldsymbol{\lambda}^l$ admits an addable i-node. Assume that it is on component $c \in \{0, \dots, l-1\}$. Thus, we have a black bead numbered by $j \in \mathbb{Z}$ such that $j \equiv i - 1 + e\mathbb{Z}$ in component c and a white bead in component c numbered by $j+1$. As a consequence, all the bead in component $k > c$ numbered by j and all the beads numbered by $j - te$ with $t > 0$ are blacks. On the other hand, the bead in component $k < c$ numbered by $j+1$ and all the beads numbered by $j+1+te$ with $t > 0$ cannot be black. This implies that there are no removable i-nodes. If $\boldsymbol{\lambda}^l$ admits a removable i-node, the proof is similar.

□

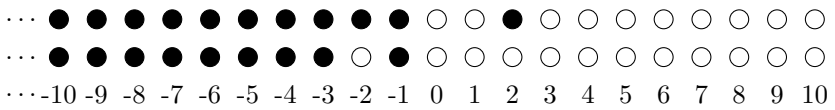
Example 3.7. Tale $\boldsymbol{\lambda} = ((3,1),(2,1)$ with $e = 3$ and $\mathbf{s} = (0,0)$. We have



We have $\boldsymbol{\lambda}_e = (\emptyset, (2), (2))$ thus the weight of $\boldsymbol{\lambda}$ is 3 and the e-core multicharge is $(0, -1, 1)$ with abacus:



The l-abacus of the associated e-core multipartition is :



The e-core multipartition is thus $((1),(2))$ with multicharge $(-1, 1)$.

4. Blocks of Ariki-Koike algebras

In this section, we recall the main objects around the modular representation theory of Ariki-Koike algebras. We refer to [1, 4] for details. Then we use our previous combinatorial definition to describe the notion of blocks for Ariki-Koike algebras as already stated in [7] and explore basic properties around them.

4.1. **Ariki-Koike algebras.** Let $n \in \mathbb{N}$, $l \in \mathbb{N}$, $e \in \mathbb{N}_{>1}$ and let $\mathbf{s}^l = (s_1, \dots, s_l) \in \mathbb{Z}^l$. We set $\eta_e := \exp(2i\pi/e)$. The Ariki-Koike algebra $\mathcal{H}_n^{\mathbf{s}^l}(\eta)$ is the unital associative $\mathbb{C}$-algebra with

- generators $T_1, \dots, T_{n-1}$,

- relations:

$$T_0T_1T_0T_1 = T_1T_0T_1T_0,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}\ (i = 1, ..., n-2),$$
$$T_iT_j = T_jT_i\ (|j-i| > 1),$$
$$(T_0 - \eta^{s_1})(T_0 - \eta^{s_2})...(T_0 - \eta^{s_l}) = 0,$$
$$(T_i - \eta)(T_i + 1) = 0\ (i = 1, ..., n-1).$$

The representation theory of $\mathcal{H}_n^{\mathbf{s}^l}(\eta)$ is controlled by its decomposition matrix. For all $\boldsymbol{\lambda}^l \in \Pi^l$; one can associate a certain finite dimensional $\mathcal{H}_n^{\mathbf{s}^l}(\eta)$-module $S^{\boldsymbol{\lambda}}$ called a Specht module. For each $M \in \mathrm{Irr}(\mathcal{H}_n^{\mathbf{s}^l}(\eta))$, we have the composition factor $[S^{\boldsymbol{\lambda}^l} : M]$. The matrix:

$$\mathcal{D} := ([S^{\boldsymbol{\lambda}^l} : M])_{\boldsymbol{\lambda} \in \Pi^l(n), M \in \mathrm{Irr}(\mathcal{H}_n^{\mathbf{s}^l}(\eta)}$$

is the *decomposition matrix*.

4.2. **Parametrizing the simple modules.** The study of the parametrization of the simple modules for Ariki-Koike algebras have a long story and there are different way to solve this. We here use the concept of basic sets, notion that we quickly recall. There exists a natural pre-ordrer $\prec_{\mathbf{s}^l}$ on the set of l-partitions (see [4]). Then by [4], it can be shown that there exists a subset $\Phi^{\mathbf{s}^l}(n) \subset \Pi^l(n)$ and a bijective map

$$\mathcal{F} : \mathrm{Irr}(\mathcal{H}_n^{\mathbf{s}^l}(\eta)) \to \Phi^{\mathbf{s}^l}(n)$$

such that for all $M \in \mathrm{Irr}(\mathcal{H}_n^{\mathbf{s}^l}(\eta))$ we have

$$[S^{\mathcal{F}(M)} : M] = 1 \text{ and } [S^{\boldsymbol{\mu}} : M_i] \neq 1 \text{ only if } \boldsymbol{\mu} \prec_{\mathbf{s}^l} \mathcal{F}(M)$$

The set $\Phi^{\mathbf{s}^l}(n)$ thus gives a natural indexation for the set of simple modules. These l-partitions are known as Uglov l-partitions. If $\mathbf{s}^l \in \mathcal{A}_e^l$, then they are known as FLOTW l-partitions and they have an easy non recursive definition. This is not the case in general. However, one can go from one parametrization to another thanks to an easy algorithm that we will describe later.

4.3. **Uglov l-partitions.** Let us give a quick definition of the set of Uglov l-partitions. For two nodes, we write $\gamma <_{(\mathbf{s}^l,e)} \gamma'$ if we have $b - a + s_c < b' - a' + s_{c'}$ or if $b - a + s_c = b' - a' + s_{c'}$ and $c < c'$.

Let $\boldsymbol{\lambda}^l$ be an l-partition. We can consider its set of addable and removable i-nodes. Let $w_i(\boldsymbol{\lambda}^l)$ be the word obtained first by writing the addable and removable i-nodes of $\boldsymbol{\lambda}^l$ in increasing order with respect to $\prec_{(e,\mathbf{s}^l)}$ next by encoding each addable i-node by the letter A and each removable i-node by the letter R. Write $\widetilde{w}_i(\boldsymbol{\lambda}^l, \mathbf{s}^l) = A^pR^q$ for the word derived from w_i by deleting as many subwords of type RA as possible. $w_i(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ is called the *i-signature* of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ and $\widetilde{w}_i(\boldsymbol{\lambda}^l)$ the *reduced i-signature* of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$. The addable i-nodes in $\widetilde{w}_i(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ are called the *normal addable i-nodes*. The removable i-nodes in $\widetilde{w}_i(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ are called the *normal removable i-nodes*. If $p > 0$, let γ be the rightmost addable i-node in $\widetilde{w}_i(\boldsymbol{\lambda}^l, \mathbf{s}^l)$. The node γ is called the *good addable i-node*. If $q > 0$, the leftmost removable i-node in $\widetilde{w}_i(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ is called the *good removable i-node*. Note that this notion depends on the order $\prec_{e,\mathbf{s}^l}$ and thus on the choice of $\mathbf{s} \in \mathbb{Z}^l$.

The set of *Uglov l-partitions* is defined recursively as follows.

- We have $\boldsymbol{\emptyset} := (\emptyset, \emptyset, \ldots, \emptyset) \in \Phi^{\mathbf{s}^l}$.
- If $\boldsymbol{\lambda}^l \in \Phi^{\mathbf{s}^l}$ with $\boldsymbol{\lambda}^l \neq \boldsymbol{\emptyset}$, there exist $i \in \{0, \ldots, e-1\}$ and a good removable i-node γ such that if we remove γ from $\boldsymbol{\lambda}^l$, the resulting l-partition is in $\Phi^{\mathbf{s}^l}$. We then denote $\widetilde{f}_i.\boldsymbol{\lambda}^l = \boldsymbol{\mu}^l$, or equivalently $\widetilde{e}_i.\boldsymbol{\mu}^l = \boldsymbol{\lambda}^l$

4.4. **Affine symmetric groups.** Let $\widetilde{\mathfrak{S}}_r$ be the affine symmetric group. This is the Coxeter group with a presentation by

(1) generators : σ_i, $i = 0, \ldots, r-1$,

(2) relations, for all indices i and j (which are read modulo e):

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ (if } i-j \not\equiv 1 + r\mathbb{Z})$$
$$\sigma_i^2 = 1$$

The extended affine symmetric group $\widehat{\mathfrak{S}}_r$ is the semi-direct product $\widetilde{\mathfrak{S}_r} \rtimes \langle\tau\rangle$ where $\langle\tau\rangle \simeq \mathbb{Z}$ where the product is defined by the relation $\tau\sigma_i = \sigma_{i+1}\tau$. Then then $\widehat{\mathfrak{S}}_r$, is generated by τ and the σ_i for $i = 1, \ldots, r-1$. We will now consider two types of action:

(1) We have that $\widehat{\mathfrak{S}}_e$, acts faithfully on $\mathbb{Z}^e$ (at the left) as follows: for any $\mathbf{s}_e = (s_0, \ldots, s_{e-1}) \in \mathbb{Z}^e$, we have that

$$\begin{array}{rcl} \sigma_c.\mathbf{s}_e & = & (s_0, \ldots, s_{c-1}, s_c, \ldots, s_{e-1}) \quad \text{for } c = 1, \ldots, l-1 \text{ and} \\ \tau.\mathbf{s}_e & = & (s_{e-1}+l, s_0, \ldots, s_{e-2}) \end{array}$$

Note that we have

$$\sigma_0.\mathbf{s} = (s_{e-1}+l, s_1, \ldots, s_{e-2}, s_0 - l)$$

(2) We have that $\widehat{\mathfrak{S}}_l$ acts faithfully on $\mathbb{Z}^l$ (at the left) as follows: for any $\mathbf{s}^l = (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l$, we have that

$$\begin{array}{rcl} \mathbf{s}^l.\sigma_c & = & (s_0, \ldots, s_{c-1}, s_c, \ldots, s_l) \quad \text{for } c = 1, \ldots, l-1 \text{ and} \\ \mathbf{s}^l.\tau & = & (s_1, \ldots, s_{l-1}, s_0 + e) \end{array}$$

Note that we have

$$\mathbf{s}^l.\sigma_0 = (s_{e-1} - e, s_1, \ldots, s_{l-2}, s_0 + e)$$

If $\mathbf{s}^l$ and $\mathbf{s}'^l$ are in the same orbit modulo the action of $\widehat{\mathfrak{S}}_l$, then the Ariki-Koike algebras $\mathcal{H}_n^{\mathbf{s}^l}(\eta)$ and $\mathcal{H}_n^{\mathbf{s}'^l}(\eta)$ are isomorphic. This induces a bijection:

$$\Psi^{\mathbf{s}^l \to \mathbf{s}'^l} : \Phi^{\mathbf{s}^l} \to \Phi^{\mathbf{s}'^l}.$$

In fact, this bijection is the restriction of a bijection:

$$\Psi^{\mathbf{s}^l \to \mathbf{s}'^l} : \Pi^l \to \Pi^l$$

which has been described in [6] and may be described just by describing two types of bijections :

$$\Psi^{\mathbf{s}^l \to \mathbf{s}^l.\tau} \text{ and } \Psi^{\mathbf{s}^l \to \mathbf{s}^l.\sigma_i}$$

Proposition 4.1 (J-Lecouvey [7]). *For all $\mathbf{s}^l = (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l$ and $\boldsymbol{\lambda}^l \in \Phi^{\mathbf{s}^l}$ we have*

$$\Psi^{\mathbf{s}^l \to \mathbf{s}^l.\tau}(\boldsymbol{\lambda}^l) = (\lambda^1, \ldots, \lambda^{l-1}, \lambda^0).$$

Now assume that $i \in \{1, \ldots, l-1\}$, then we will describe $\Psi^{\mathbf{s}^l \to \mathbf{s}^l.\sigma_i}$. We have

$$\Psi^{\mathbf{s}^l \to \mathbf{s}^l.\sigma_i}(\boldsymbol{\lambda}) = (\lambda^0, \ldots, \widetilde{\lambda}^i, \widetilde{\lambda}^{i-1}, \ldots, \lambda^{l-1})$$

and we explain now how one can obtain $(\widetilde{\lambda}^i, \widetilde{\lambda}^{i-1})$ from $(\lambda^{i-1}, \lambda^i)$. To do this, consider te symbol associated with $\boldsymbol{\lambda}^l$:

$$(X_0, \ldots, X_{l-1})$$

The description of the bijections essentially rests on the following basic procedure on the pair (X_{i-1}, X_i). Let $X_{i-1} = [x_1, \ldots, x_r]$ and $X_i = [y_1, \ldots, y_s]$. We compute from (X_{i-1}, X_i) a new pair (X'_{i-1}, X'_i) with $X''_i = [x'_1, \ldots, x'_r]$ and $X'_{i-1} = [y'_1, \ldots, y'_s]$ of such sequences by applying the following algorithm :

- Assume $r \geq s$. We associate to y_1 the integer $x_{i_1} \in X_{i-1}$ such that

$$x_{i_1} = \begin{cases} \max\{x \in X_i \mid y_1 \geq x\} \text{ if } y_1 \geq x_1 \\ x_r \text{ otherwise} \end{cases}. \tag{4.1}$$

We repeat the same procedure to the ordered pair $(X_{i-1} \setminus \{x_{i_1}\}, X_i \setminus \{y_1\})$. By induction this yields a subset $\{x_{i_1}, \ldots, x_{i_s}\} \subset X_{i-1}$. Then we define X'_{i-1} as the increasing reordering $\{x_{i_1}, \ldots, x_{i_s}\}$ and X'_i as the increasing reordering of $X_{i-1} \setminus \{x_{i_1}, \ldots, x_{i_s}\} \sqcup X_i$.

- Assume $r < s$. We associate to x_1 the integer $y_{i_1} \in X_i$ such that

$$y_{i_1} = \begin{cases} \min\{y \in X_{i+1} \mid x_1 \leq y\} \text{ if } x_1 \leq y_s \\ y_1 \text{ otherwise} \end{cases} . \tag{4.2}$$

We repeat the same procedure to the ordered sequences $X_{i-1} \setminus \{x_1\}$ and $X_i \setminus \{y_{i_1}\}$ and obtain a subset $\{y_{i_1}, \ldots, y_{i_r}\} \subset X_i$. Then we define X'_i as the increasing reordering $\{y_{i_1}, \ldots, y_{i_r}\}$ and X'_{i-1} as the increasing reordering of $X_i \setminus \{y_{i_1}, \ldots, y_{i_r}\} \sqcup U$.

Proposition 4.2 (J-Lecouvey [7]). *Under the above notation, if $\boldsymbol{\lambda}^l \in \Phi^{\mathbf{s}^l}$, then we have $\boldsymbol{\mu}^l = \Psi^{\mathbf{s}^l \to \mathbf{s}^l.\sigma_i}(\boldsymbol{\lambda}^l)$ where the el-symbol of $\boldsymbol{\mu}^l$ is $(X_0, \ldots, X'_i, X'_{i-1}, \ldots, X_{l-1})$.*

4.5. **Blocks.** By definition, two l-partitions $\boldsymbol{\lambda}^l$ and $\boldsymbol{\mu}^l$ lie in the *same ordinary block* if there exists a sequence $(M_1, \ldots, M_r)$ of simple $\mathbb{F}\mathcal{H}_n^{\mathbf{s}}(\eta)$-modules and a sequence of l-partitions $(\boldsymbol{\lambda}[1], \ldots, \boldsymbol{\lambda}[r+1])$ with $\boldsymbol{\lambda}[1] = \boldsymbol{\lambda}^l$, $\boldsymbol{\lambda}[r+1] = \boldsymbol{\mu}^l$ and for all $i \in \{1, \ldots, r\}$, we have $[S^{\boldsymbol{\lambda}[i]} : M_i] \neq 0$ and $[S^{\boldsymbol{\lambda}[i+1]} : M_i] \neq 0$. The ordinary blocks are the thus equivalence classes under the above equivalence class. Thus, we have a partition of the set of l-partitions by the ordinary blocks:

$$\Pi^l(n) = \mathcal{B}_1 \sqcup \ldots \sqcup \mathcal{B}_r.$$

Similarly, we have a notion of *modular blocks* : two l-partitions in $\Phi^{\mathbf{s}^l}(n)$ are in the same ordinary block if and only there exist a sequence of $(M_1, \ldots, M_r)$ of simple $\mathcal{H}_n^{\mathbf{s}}(\eta)$-modules with $\mathcal{F}(M_1) = \boldsymbol{\lambda}^l$ and $\mathcal{F}(M_r) = \boldsymbol{\mu}^l$ and a sequence of l-partitions $(\boldsymbol{\lambda}[1], \ldots, \boldsymbol{\lambda}[r+1])$ $\boldsymbol{\lambda}[1] = \boldsymbol{\lambda}^l$, $\boldsymbol{\lambda}[r+1] = \boldsymbol{\mu}^l$ such that we have $[S^{\boldsymbol{\lambda}[i]} : M_i] \neq 0$ and $[S^{\boldsymbol{\lambda}[i+1]} : M_i] \neq 0$. Hence a block may be think as a couple $(\mathcal{B}, \mathfrak{B})$ where $\mathcal{B}$ is a set of l-partitions and $\mathfrak{B}$ is a set of Uglov l-partitions. We have $\mathfrak{B} \subset \mathcal{B}$. So the datum of the ordinary block suffices to obtain the modular blocks. We will now see in details how one can describe these blocks. The following is the main Theorem of [7].

Theorem 4.3. *Let $\mathbf{s}^l \in \mathcal{A}_e^l$ then Two l-partitions $\boldsymbol{\lambda}^l$ and $\boldsymbol{\mu}^l$ are in the same block if and only if they have the same e-core. Moreover, all the l-partitions in the same block have the same weight which is thus called the weight of the block.*

This theorem shows that one can also parameterize a block with its core and its weight. A block thus may be denoted $B(\mathbf{s}_e, w)$ where $\mathbf{s}_e \in \mathbb{Z}^e[m]$ denotes the e-core multicharge. We write

$$B(\mathbf{s}_e, w) = (\mathcal{B}, \mathfrak{B})$$

if $\mathcal{B}$ is the associated ordinary block and $\mathfrak{B}$ is the associated modular block.

Example 4.4. Assume that $l = 2$, $\mathbf{s} = (-1, 1)$ and $e = 4$. Thanks to [8], one can compute the associated decomposition matrix. It is given as follows:

$$\Phi^{(0,1)}(4) = \{(\emptyset, (4)), ((1), (2,1)), ((1,1), (1,1)), ((1), (3)), ((1,1), (2)), ((2), (1,1)),$$

$$((2), (2)), ((2,1), (1)), ((2,1,1), \emptyset), ((2,2), \emptyset), ((3), (1)), ((3,1), \emptyset), ((4), \emptyset)\}.$$

$$
\begin{array}{c|c}
((4),\emptyset) & \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . \\
1 & . & 1 & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . \\
1 & . & . & 1 & . & . & 1 & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & 1 \\
. & . & 1 & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & 1 & . & 1 & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & 1 & . & .
\end{pmatrix} \\
(((3),(1)) & \\
(\emptyset,(4)) & \\
((3,1),\emptyset) & \\
(((2),(2)) & \\
((2,2),\emptyset) & \\
((1),(3)) & \\
((2,1),(1)) & \\
((2,1,1),\emptyset) & \\
((2),(1,1)) & \\
((1,1),(2)) & \\
((1),(2,1)) & \\
((1,1),(1,1)) & \\
(\emptyset,(3,1)) & \\
((1,1,1),(1)) & \\
(\emptyset,(2,2)) & \\
((1,1,1,1),\emptyset) & \\
(\emptyset,(2,1,1)) & \\
((1),(1,1,1)) & \\
(\emptyset,(1,1,1,1)) &
\end{array}
$$

We see that we have the following ordinary blocks:

$$\mathcal{B}_1 = \{((4),\emptyset),(3,1),\emptyset),((1),(3)),((1,1),(2)),((2,1,1),\emptyset),((1,1,1),(1)),((1,1,1,1),\emptyset),(\emptyset,(2,1,1))$$
$$,(\emptyset,(1,1,1,1),(\emptyset,(4))),(\emptyset,(3,1))\}$$
$$\mathcal{B}_2 = \{((3),(1)),((2),(2)),(\emptyset,(2,2))\}$$
$$\mathcal{B}_3 = \{((2,2),\emptyset),((1,1),(1,1)),((1),(1,1,1))\}$$
$$\mathcal{B}_4 = \{((1),(2,1)))\}$$

and thus the following modular blocks :

$$\mathcal{B}_1 = \{((4),\emptyset),((3,1),\emptyset),((1),(3)),((1,1),(2)),((2,1,1),\emptyset)(\emptyset,(4)))\}$$
$$\mathcal{B}_2 = \{((3),(1)),((2),(2))\}$$
$$\mathcal{B}_3 = \{((2,2),\emptyset),((1,1),(1,1))\}$$
$$\mathcal{B}_4 = \{((1),(2,1)))\}$$

One can check that:

$$B((1,0,0,0),2)) = (\mathcal{B}_1,\mathfrak{B}_1), B((0,1,1,-1),1) = (\mathcal{B}_2,\mathfrak{B}_2),$$
$$B((2,1,-1,-1),1)) = (\mathcal{B}_3,\mathfrak{B}_3), B((2,0,1,-1),0) = (\mathcal{B}_4,\mathfrak{B}_4),$$

5. Action of the affine Weyl group on blocks

We now see how the affine symmetric group acts on the set of blocks and develop this action thanks to our parametrization of blocks.

5.1. **Action on the set of e-cores.** Assume that $\mathbf{s}^l \in \mathbb{Z}^l[m]$. There is an action of $\widehat{\mathfrak{S}}_e$ on the sets of blocks

$$\{B(\mathbf{s}_e, w) \mid \mathbf{s}_e \in \mathbb{Z}^e[m],\ w \in \mathbb{N}\}$$

Let $\mathbf{s}_e \in \mathbb{Z}^e[m]$, then we define:

$$\sigma.B(\mathbf{s}_e, w) = B(\sigma.\mathbf{s}_e, w)$$

We obtain an action on the set of blocks. In the two following section, we see how one can give two actions on the set of partitions and on the set of l-partitions which are compatible with this action.

5.2. **Action on ordinary blocks.** Let $\boldsymbol{\lambda}^l \in \Pi^l(n)$. Let $j \in \{0, 1 \ldots, e-1\}$. Let

$$\{\gamma_1, \ldots, \gamma_k\},$$

be the set of addable j-nodes of $\boldsymbol{\lambda}^l$. Let

$$\{\eta_1, \ldots, \eta_s\},$$

be the set of removable j-nodes of $\boldsymbol{\lambda}^l$. Then we define

$$\sigma_i.\boldsymbol{\lambda}^l := \boldsymbol{\mu}^l,$$

where

$$[\boldsymbol{\mu}^l] = [\boldsymbol{\lambda}^l] \cup \{\gamma_1, \ldots, \gamma_k\}\} \setminus \{\eta_1, \ldots, \eta_s\}.$$

One can easily perform the computation using the abacus configuration. Let us consider the e-symbol $\mathbf{X}^{\mathbf{s}_e}(\boldsymbol{\lambda}_e) = (X_0, \ldots, X_{e-1})$.

(1) if $1 \leq i \leq e-1$. Then we have

$$\mathbf{X}^{\sigma_i.\mathbf{s}_e}(\boldsymbol{\mu}_e) = (X_0, \ldots, X_i, X_{i-1}, \ldots, X_{e-1})$$

(2) if $i = 0$ then we have:

$$\mathbf{X}^{\sigma_i.\mathbf{s}_e}(\boldsymbol{\mu}_e) = (X_{e-1}[l], \ldots, X_1, \ldots, X_{e-2}, \ldots, X_0[-l])$$

5.3. **Action on modular blocks.** We now give an action of the set of Uglov l-partitions $\Phi^{\mathbf{s}^l}$. This is given as follows. Let $\boldsymbol{\lambda}^l \in \Phi^{\mathbf{s}^l}$. We give the definition of $\sigma_i \star \boldsymbol{\lambda}^l$. We set :

$$\varepsilon_i(\boldsymbol{\lambda}^l) = \max(i \geq 0,\ \widetilde{f}_i^k.\boldsymbol{\lambda}^l \neq 0),$$

$$\varphi_i(\boldsymbol{\lambda}^l) = \max(i \geq 0,\ \widetilde{e}_i^k.\boldsymbol{\lambda}^l \neq 0).$$

Then we set

$$\sigma_i \star \boldsymbol{\lambda}^l = \begin{cases} \widetilde{e}_i^{\varphi(\boldsymbol{\lambda}^l)-\varepsilon(\boldsymbol{\lambda}^l)}\boldsymbol{\lambda}^l & \text{if } \varphi(\boldsymbol{\lambda}^l) \geq \varepsilon(\boldsymbol{\lambda}^l), \\ \widetilde{f}_i^{\varepsilon(\boldsymbol{\lambda}^l)-\varphi(\boldsymbol{\lambda}^l)}\boldsymbol{\lambda}^l & \text{if } \varphi(\boldsymbol{\lambda}^l) \leq \varepsilon(\boldsymbol{\lambda}^l). \end{cases}$$

Contrary to the above case, it is not easy to see that the action is well defined and this comes in fact from a general result proved by Kashiwara [11].

Consider the reduced i-signature of λ^l.

$$\underbrace{A \ldots A}_{\varepsilon_i(\boldsymbol{\lambda}^l)}\underbrace{R \ldots R}_{\varphi_i(\boldsymbol{\lambda}^l)}$$

There exists $m \in \mathbb{N}$ such that the number of black beads in runner $i-1$ is equal to $\varepsilon_i(\boldsymbol{\lambda}^l)+m$ and the number of black beads in runner i is equal to $\varphi_i(\boldsymbol{\lambda}^l) + m$ (the number m corresponds to the number of occurrences "RA" we need to remove to reach the reduced signature from the signature). If we look at the i-signature of $\sigma_i \star \boldsymbol{\lambda}^l$, we now have:

$$\underbrace{A \ldots A}_{\varphi_i(\boldsymbol{\lambda}^l)}\underbrace{R \ldots R}_{\varepsilon_i(\boldsymbol{\lambda}^l)}$$

and the number of black beads in runner $i-1$ is $\varphi_i(\boldsymbol{\lambda}^l) + m$ where as the number of black beads in runner i is $\varepsilon_i(\boldsymbol{\lambda}^l) + m$. We conclude that

$$\sigma_i(\mathcal{B}, \mathcal{B}) = (\sigma_i.\mathcal{B}, \sigma_i \star \mathfrak{B})$$

It is important to note that the above action is not the restriction of the action on partitions to the set of Uglov l-partitions, this is not even true for the set of e-regular partitions ! We see in the action on the partition that an e-regular partition is not necessarily sent to an e-regular partition (for example $\sigma_2(3, 2, 1, 1) = (2, 2, 2, 1, 1)$ for $p = 3$). However, if $\boldsymbol{\lambda}^l$ is an Uglov l-partition then $\sigma_i \star \boldsymbol{\lambda}^l$ and $\sigma_i(\boldsymbol{\lambda}^l)$ have the same size. Thus, the weight of the two partitions are the same and this implies that the action on Uglov l-partitions also preserves the weight.

5.4. **Chuang-Rouquier equivalences.** The above action has an interpretation in terms of the determination of blocks. In [3], Chuang and Rouquier have shown that the action of the affine symmetric group on the set of ordinary blocks induces an equivalence of derived categories between these blocks. This equivalence is even a Morita equivalence in some cases: Assume that we have

$$B(\mathbf{s}_e, w) = (\mathcal{B}, \mathfrak{B})$$

then we say that $B(\mathbf{s}_e, w)$ is an i-Scopes block if for all $\boldsymbol{\mu}^l \in \mathcal{B}$, $\boldsymbol{\mu}^l$ admits no addable i-nodes. If $B(\mathbf{s}_e, w)$ is an i-Scopes block then $B(\mathbf{s}_e, w)$ and $\sigma_i B(\mathbf{s}_e, w)$ are Morita-equivalent. The following result is an easy extension of what happens in the case $l = 1$. Such result has been already studied by Lyle [9], Webster [17] and recently by Li and Tan [14] but without our notion of generalized core.

Proposition 5.1. *Let $\mathbf{s}_e = (s_0, \dots, s_{e-1}) \in \mathbb{Z}^e[m]$. Let $0 \leq i \leq e-1$. Denote $B(\mathbf{s}_e, w) = (\mathcal{B}, \mathfrak{B})$ a block of our Ariki-Koike algebra. The following assertions are equivalent.*

(1) For all $\boldsymbol{\mu}^l \in \mathcal{B}$, $\boldsymbol{\mu}^l$ admits no addable i-nodes.

(2) We have

$$s_i - s_{i-1} \geq \begin{cases} w & \text{if } i = 1, \dots, p-1 \\ w+e & \text{if } i = 0 \end{cases}$$

Proof. Assume that $\mathbf{s}_e = (s_0, \dots, s_{e-1})$ satisfies the conditions in (2). Assume for example that $s_i - s_{i-1} \geq w$. If $\boldsymbol{\mu}^l$ is in $\mathcal{B}$, then the e-abacus of $\boldsymbol{\mu}^l$ is obtained from the associated to $\mathbf{s}_e$ by doing w elementary moves. Because of the above condition, this implies that in this abacus, if we have a bead in runner $i-1$ then we must have one in runner i. As a consequence, we do not have any addable i-node in $\boldsymbol{\mu}^l$. If $s_0 - s_{e-1} \geq w+e$, this is the same proof.

Assume now that $s_i - s_{i-1} < w$ then in $\boldsymbol{\lambda}^l$, one can perform e elementary moves in the runner $i-1$ with the rightmost bead. We see that the resulting partition must have an addable i-node.

□

Remark 5.2. Assume that $\mathcal{B}$ satisfies one of the above property . Let $\mathfrak{B}$ be the associated modular block. Let $\boldsymbol{\lambda}^l \in \mathfrak{B}$ then $\boldsymbol{\lambda}^l$ has no removable i-nodes. So we have $\sigma_i \star \boldsymbol{\lambda} = \sigma_i \boldsymbol{\lambda}$.

5.5. **Explicit description.** One of the main result of this paper is to give an interpretation of the action on the modular blocks via the level-rank duality. In fact, such an interpretation has already been given in [12] in the case $l > 1$ using the crystal basis theory (and for $i \neq 0$). Here, we give a purely combinatorial proof which is also available for $l = 1$ and $i = 0$. We will use the following notation, for an l-partition $\boldsymbol{\lambda}^l$ and $\mathbf{s}_e \in \mathbb{Z}^e$, we set $\boldsymbol{\lambda}^* = (\lambda^{l-1}, \lambda^0, \dots, \lambda^{l-2})$ and $\mathbf{s}_e^* = (s_{e-1} + l, s_0, \dots, s_{e-2})$.

Proposition 5.3. *Assume that $0 \leq i \leq e-1$. Let $\lambda \in \Pi$ and let*

$$\tau^l(\lambda) = (\boldsymbol{\lambda}^l, \mathbf{s}^l), \tau_e(\lambda) = (\boldsymbol{\lambda}_e, \mathbf{s}_e)$$

Assume that $\boldsymbol{\lambda}^l$ is an Uglov l-partition. For $i = 1, \dots, e-1$, we have:

$$(\tau^l)^{-1}(\sigma_i \star \boldsymbol{\lambda}^l) = \tau_e^{-1}(\Psi^{\mathbf{s}_e \to \mathbf{s}_e . \sigma_i}(\boldsymbol{\lambda}_e))$$

For $i = 0$, we have:

$$\Big((\tau^l)^{-1}(\sigma_0 \star \boldsymbol{\lambda}^l\Big))^* = \tau_e^{-1}(\Psi^{\mathbf{s}_e^* \to ((\mathbf{s}_e).\sigma_1)^*}((\boldsymbol{\lambda}_e)^*)$$

Proof. Let

$$\mathbf{X}^l := (X_0, \dots, X_{l-1})$$

be the l-symbol associated to $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$, and consider the e-symbol associated to $(\boldsymbol{\lambda}_e, \mathbf{s}_e)$:

$$\mathbf{Y} = (Y_0, \dots, Y_{e-1}).$$

Recall that one can go from one abacus to the other thanks to the procedure described in §3.2.

Assume that $0 \leq i \leq e-1$. We want to compute $\sigma_i \star \boldsymbol{\lambda}^l$. So we need to look at the set of removable and addable i-nodes of $\boldsymbol{\lambda}^l$. In $\mathbf{X}^l$, the removable i-nodes correspond to elements $\alpha \in X_k$ (for all $0 \leq k \leq l-1$) such that $\alpha \equiv i + e\mathbb{Z}$ and $\alpha - 1 \notin X_k$. The addable i-nodes corresponds to elements $\alpha \in X_k$ (for all $0 \leq k \leq e-1$) such that $\alpha + 1 \equiv i + e\mathbb{Z}$ and $\alpha + 1 \notin X_k$.

Thus, one can also determine the addable and removable i-nodes of $\boldsymbol{\lambda}$ in Y.

- If $i \neq 0$, the removable i-nodes correspond to elements in $\alpha \in X_i$ such that $\alpha \notin X_{i-1}$, the addable i-nodes correspond to the elements $\alpha \in X_{i-1}$ such that $\alpha \notin X_i$.
- If $i = 0$, the removable i-nodes correspond to elements in $\alpha \in X_0$ such that $\alpha - e \notin X_{e-1}$, the addable i-nodes correspond to the elements $\alpha \in X_{e-1}$ such that $\alpha + e \notin X_0$. I

Set $\mathbf{s}_e := (t_1, \dots, t_{e-1})$. Assume that $i \neq 0$. Because both the action of σ_i and $\Psi^{\mathbf{s}_e \to \mathbf{s}_e.\sigma_i}$ are involutive, one can assume that $t_{i-1} \leq t_i$. This means that we have more removable i-nodes than addable. We perform the algorithm for the computation of $\Psi^{\mathbf{s}_e \to \mathbf{s}_e.\sigma_i}(\boldsymbol{\lambda}_e)$. In the associated e-symbol, we consider the couple (X_{i-1}, X_i). Take the i-signature $w_i(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ and read it from left to right. Then one may easily obtain the i-signature of $(\tau^l)^{-1} \circ \tau_e(\Psi^{\mathbf{s}_e \to \mathbf{s}_e.\sigma_i}(\boldsymbol{\lambda}_e))$ from $w_i(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ by applying our algorithm. In this way, one can define pairs between some of the letters R and A exactly as in the algorithm. We will say that a letter R or A is a marked if it is in pair with another letter. Let us first assume that there are now occurence of type RA in this word so that the i-signature is equal to the reduced i-signature:

$$\underbrace{A \dots A}_{\varepsilon_i(\boldsymbol{\lambda}^l)} \underbrace{R \dots R}_{\varphi_i(\boldsymbol{\lambda}^l)}$$

We here have $\varphi_i(\boldsymbol{\lambda}^l) \geq \varepsilon_i(\boldsymbol{\lambda}^l)$. So by our algorithm for computing the bijections, the nodes associated to the letters A are in pairs with the nodes associated to the $\varepsilon_i(\boldsymbol{\lambda}^l)$ rightmost R. Then, if we want to compute $\Psi^{\mathbf{s}_e \to \mathbf{s}_e.\sigma_i}(\boldsymbol{\lambda}_e)$, we need to change the nodes that are not associated with any pairs from removable to addable nodes. The resulting signature is:

$$\underbrace{A \dots A}_{\varphi_i(\boldsymbol{\lambda}^l)} \underbrace{R \dots R}_{\varepsilon_i(\boldsymbol{\lambda}^l)}$$

which is what we wanted. Now, assume that we have an occurence of type RA in the i-signature. We see that the associated letters are marked and that, moreover the other letters in the i-signature are marked if and only if they are marked in the reduced signature. This suffices to conclude by induction on the number of occurences RA. In the case $i = 0$, we keep exactly the same reasoning.

□

6. Orbits of blocks

At the moment, all the main result for the symmetric groups seem to be generalized in the case of Ariki-Koike algebras. However, this is not really the case. When $l = 1$, the action is transitive. When $l > 1$, this is no more the case. This explain the result found in [13] that two blocks of the same weight are non necessarily derived equivalent.

6.1. **Orbits of blocks.** We already know that the action of the affine symmetric group preserves the weight. So one may assume that $B(\mathbf{s}_e, w) = (\mathcal{B}, \mathfrak{B})$ and $B(\mathbf{s}'_e, w)$ are two blocks. We write $\mathbf{s}^l$ and $\mathbf{s}'^l$ for the assoiated two l-multicharges, that is the l-multicharge such that for $\boldsymbol{\lambda}^l \in \mathcal{B}$ we have $\tau^l(\boldsymbol{\lambda}^l, \mathbf{s}^l) = \tau_e(\emptyset; \mathbf{s}_e)$ and the same for $\mathbf{s}'_e$.

First assume that $\mathbf{s}_e$ and $\mathbf{s}'_e$ are in the same orbit modulo the action of the affine symmetric group. For all $i = 0, \dots, l-1$, we have that the l-multicharge associated to $\sigma_i.(\mathbf{s}_e)$ is $\mathbf{s}_e$ so the two l-multicharges associated to $\mathbf{s}_e$ and $\mathbf{s}'_e$ are the same.

Reciprocally, assume that $\mathbf{s}_e$ and $\mathbf{s}'_e$ are such that the two multicharges associated to $\mathbf{s}_e$ and $\mathbf{s}'_e$ are the same. If we look at the l-abacus of $\mathbf{s}_e$, the action of the affine symmetric group only consists in exchanging the beads in the column numbered with c with the one number with $c + 1$ where $c \equiv i + e\mathbb{Z}$. Looking at these abaci, we consider the infinite sequences given by the number of black beads in each column from left to right . There exists σ and σ' such that the sequences associated to $\sigma.\mathbf{s}_e$ and $\sigma'.\mathbf{s}_e$ are both decreasing sequences. As the associated l-multicharges are the same, the associated l-multipartitions must be the same, this implies that $\sigma.\mathbf{s}_e = \sigma'.\mathbf{s}_e$ and thus that $\mathbf{s}_e$ and $\mathbf{s}'_e$ are in the same orbit. So we may conclude:

Proposition 6.1. *$B(\mathbf{s}_e, w)$ and $B(\mathbf{s}'_e, w)$ are in the same orbit if and only if the associated l-multicharges of $\mathbf{s}_e$ and $\mathbf{s}'_e$ are the same*

So now, it is a natural question to ask which types of l-multicharges appear in the above proposition.

6.2. **Orbits.** Assume that we get a multicharge $\mathbf{s}^l \in \mathcal{A}_e^l[m]$, we want to classify the orbits under the action of the affine symmetric group on the set of blocks for the Ariki-Koike algebras associated with this l-multicharge. Assume tat $B(\mathbf{s}_e, w)$ and $B(\mathbf{s}'_e, w')$ are two blocks of the Ariki-Koike algebra $\mathcal{H}_n^{\mathbf{s}^l}(\eta))$. If $B(\mathbf{s}_e, w)$ and $B(\mathbf{s}'_e, w')$ are in the same orbit then we must have $w = w'$. In addition, if $(\boldsymbol{\lambda}, \mathbf{s}^l)$ and $(\boldsymbol{\lambda}', \mathbf{s}'^l)$ are the associated e-core l-partitions, we must have $\mathbf{s}^l = \mathbf{s}'^l$.

Reciprocally, Assume that $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ is an e-core l-partition. Then there exists $\sigma \in \widetilde{\mathfrak{S}}_e$ such that $\sigma.\boldsymbol{\lambda}^l = \emptyset$. This implies that, under the above notation, if $w = w'$ and $\mathbf{s}^l = \mathbf{s}'^l$, then $B(\mathbf{s}_e, w)$ and $B(\mathbf{s}'_e, w')$ are in the same orbit.

So now, our aim is to find all the multicharge $\mathbf{s}'^l$ such that $(\boldsymbol{\lambda}^l, \mathbf{s}'^l)$ appears as a e-core for this Ariki-Koike algebra, for all $n \geq 0$. More precisely, There is a multicharge $\mathbf{s}'^l$ and an l-partition $\boldsymbol{\lambda}^l$ such that $(\tau^l)^{-1}(\boldsymbol{\lambda}^l, \mathbf{s}^l) = (\tau_e)^{-1}(\emptyset, \mathbf{s}_e)$. We want to classify these multicharges $\mathbf{s}'^l$ We claim that this is exacly:

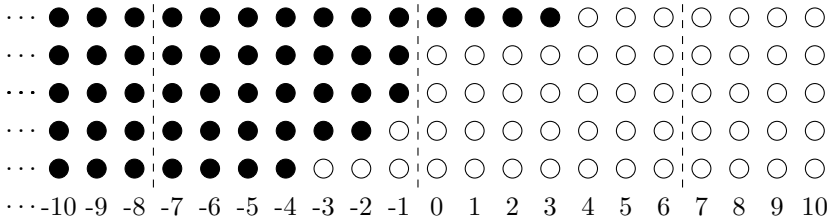
$$\overline{\mathcal{A}}_e^l[m]$$

First, by [7, Prop. 2.11], we know that we indeed have $\mathbf{s}'^l \in \overline{\mathcal{A}}_e^l[m]$. Now assume that we get $\mathbf{s}^l = (s_0, \ldots, s_{l-1})$ in $\overline{\mathcal{A}}_e^l[m]$. We denote $\mathbf{s}'^l = (s_0, \ldots, s_{l-1})$.

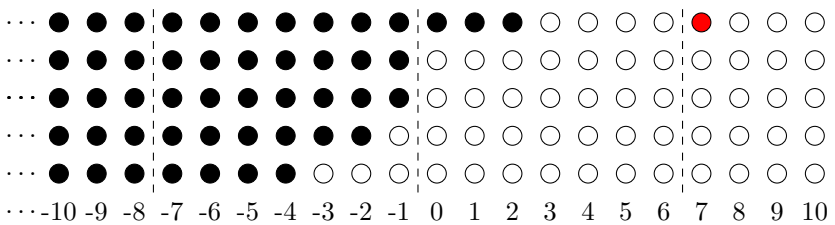
To do this, we start with the empty l-partition together with the multicharge $\mathbf{s}^l$. We need to show that there exists $\boldsymbol{\lambda}^l$ such that the l-multicharge associated to the e-core multicharge of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ is $\mathbf{s}'^l$. To do this, we argue algorithmically. First, we start with the l-abacus of $(\emptyset, \mathbf{s}^l)$ and we move black beads in this abacus to obtain the desired abacus using the following algorithm. We start to study the runner $l-1$: If $s_{l-1} = s'_{l-1}$, then we consider runner $l-2$ and so on. If $s_{l-1} < s'_{l-1}$, Let j be maximal such that $s_j > s'_j$ and $s_{j-1} \neq s_j$ (recall that $\sum_{0 \leq i \leq l-1} s_i = \sum_{0 \leq i \leq l-1} s'_i = m$). we move the rightmost black bead in runner j to a position $s_{l-1} + 1$. Then, we continue this process changing $\mathbf{s}^l$ with the $(s_0, \ldots, s_j - 1, \ldots, s_{l-1} + 1)$.

Assume that $s_{l-1} > s'_{l-1}$ then there exists k such that $s_{l-1} = s_{l-1} = \ldots = s_k > s_{k-1}$. Note that $s_k > s'_k$. Moreover, there exists j such that $s_j < s'_j$. We choose s_j maximal for this property. Then we move the rightmost black bead of runner k to the position $s_j + 1 + e$. We then continue this process bu considering the l-partition $(s_0, \ldots, s_j + 1, \ldots, s_{l-1} + 1)$. At the end, by construction, the l-partition has the desired property.

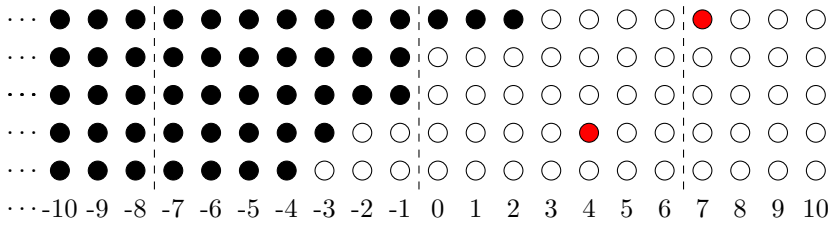
Here is an example: let $l = 5$ and $e = 7$, we take $\mathbf{s}^l = ((-3, -1, 0, 0, 4)$ and $\mathbf{s}^{l'} = (-2, -2, 0, 1, 3)$. We start with the l-abacus of $(\emptyset, \mathbf{s}^l)$:



The algorithm then gives the following abacus:



The new multicharge is $\mathbf{s}^l = ((-3, -1, 0, 1, 3)$



The new multicharge is $\mathbf{s}^l = (-2, -2, 0, 1, 3)$, as desired.

References

[1] S. Ariki, Representations of quantum algebras and combinatorics of Young tableaux. University Lecture Series, 26. American Mathematical Society, Providence, RI, 2002.

[2] M. Fayers, Weights of multipartitions and representations of Ariki-Koike algebras, Adv. Math. 206 (2006), no. 1, 112–144.

[3] J.Chuang and R.Rouquier, Derived equivalences for symmetric groups and sl_2-categorification Annals Math., 245–298, Vol. 167 (2008).

[4] M.Geck and N.Jacon, Representations of Hecke algebras at roots of unity. Algebra and Applications, 15. Springer-Verlag London, Ltd., London, 2011.

[5] N. Jacon An algorithm for the computation of the decomposition matrices for Ariki-Koike algebras, Journal of Algebra (section Computational Algebra) 292 (2005), 100-109,

[6] N. Jacon and C. Lecouvey, Crystal isomorphisms for irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of higher level, Algebras and Representation theory 13, 467-489, 2010.

[7] N. Jacon and C. Lecouvey, Cores of Ariki-Koike algebras Doc. Math. 26, 103-124 (2021)

[8] R.Jenkins, Representations of Rational Cherednik Algebras: Koszulness and Localisation PhD thesis, University of Edinburgh.

[9] S. Lyle, Decomposition numbers for Rouquier blocks of Ariki-Koike algebras I, Journal of Combinatorial Algebra, to appear, 2024.

[10] S. Lyle and A.Mathas Blocks of cyclotomic Hecke algebras Adv. Math. 216 (2007), no. 2, 854–878.

[11] Kashiwara, M., Crystal bases of modified quantized enveloping algebra, Duke Math. J., **73** (1994), 383–413.

[12] C.Lecouvey and T.Gerber, Duality and bicrystals on infinite binary matrices Annales de l'Institut Henri Poincaré D (Combinatorics, Physics and their Interactions)

[13] Y. Li and X. Qi, Representation type of blocks of cyclotomic Hecke algebras of type $G(r, 1, n)$. prepint, **arxiv :2202.06107**

[14] Y. Li and M. Tan, Cores and weights of multipartitions and blocks of Ariki-Koike algebras. Preprint **arXiv:2408.10626**

[15] J. Scopes, Cartan matrices and Morita equivalence for blocks of the symmetric groups, J. Algebra 142 (1991) 441–455.

[16] D. Uglov. Canonical bases of higher-level q-deformed Fock spaces and Kazhdan-Lusztig polynomials. *Progr. Math.*, 191:249–299, 1999.

[17] B. Webster Decomposition numbers for Rouquier blocks of Ariki-Koike algebras I to appear in Journal of Combinatorial Algebra.

[18] X. Yvonne, Bases canoniques d'espaces de Fock de niveau supérieur thèse de doctorat, Université de Caen, 2005, *https://theses.hal.science/tel-00137705*.

IUniversité de Reims Champagne-Ardennes, UFR Sciences exactes et naturelles. Laboratoire de Mathématiques UMR CNRS 9008. Moulin de la Housse BP 1039. 51100 Reims. France.

Email address: david.declercq@unic-reims.fr

Université de Reims Champagne-Ardennes, UFR Sciences exactes et naturelles. Laboratoire de Mathématiques UMR CNRS 9008. Moulin de la Housse BP 1039. 51100 Reims. France.

Email address: nicolas.jacon@univ-reims.fr